9. Structure from Shading (SfS)

A bit of differential geometry, Estimating 3D structure using shading as a cue

Lines and Surfaces

- A line (in 2D): $y = mx+c$, (m,c) are fixed, x is a variable
- A plane (in 2D): ??

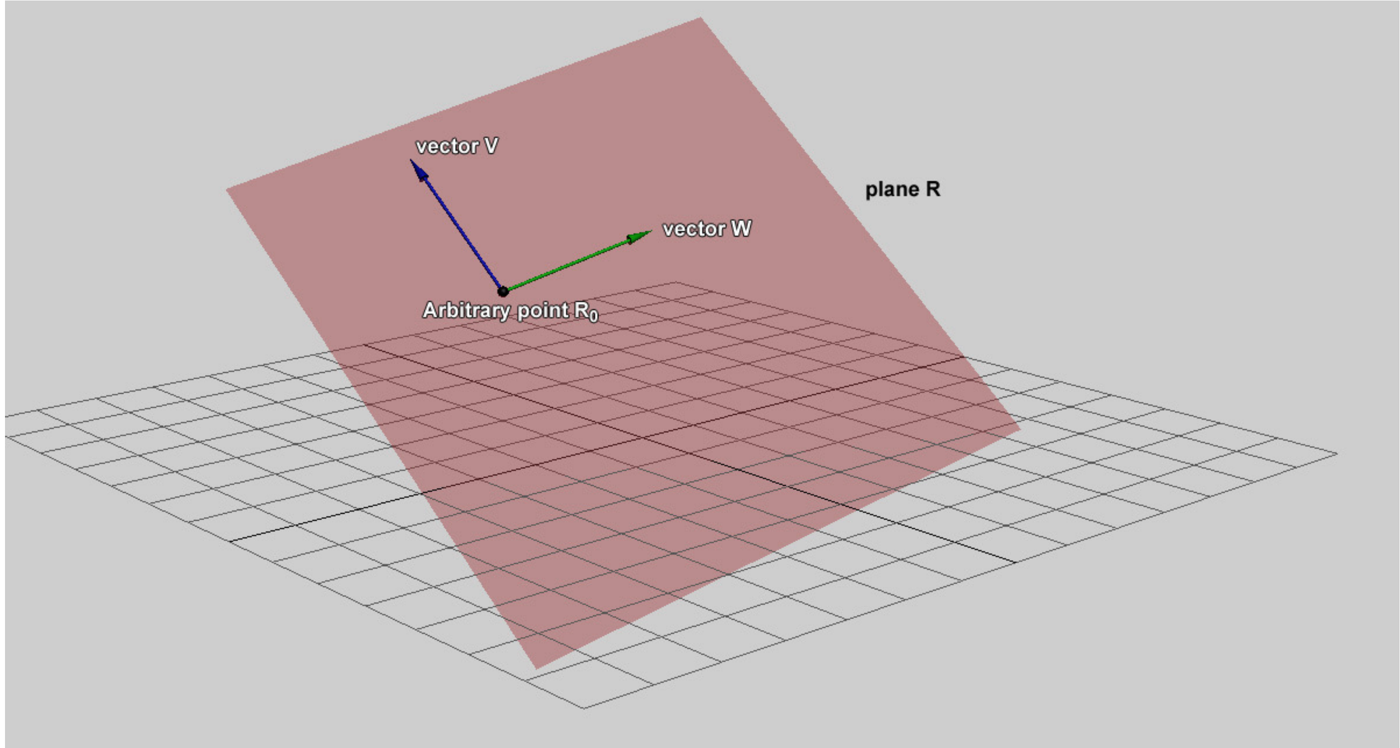
Lines and Surfaces

- A line (in 2D): $y = mx+c$, (m,c) are fixed, x is a variable
- A plane (in 2D): only canonical 2D plane exists
- You can lift this 2D plane (in 3D) and you can have several distinct planes out of it

Distinction: Distinct liftings (or transformations) will end up in unique intersection with canonical one

Lines and Surfaces

Distinction: Distinct liftings (or transformations) will end up in unique intersection with canonical one (z=0)



Lines and Surfaces

What did we learn?

You need only 1 parameter to express a line in 2D

Line in 3D: ??

Lines and Surfaces

What did we learn?

You need only 1 parameter to express a line in 2D

In 3D: ax+by+cz+d=0. It is seen as an intersection of 2 planes

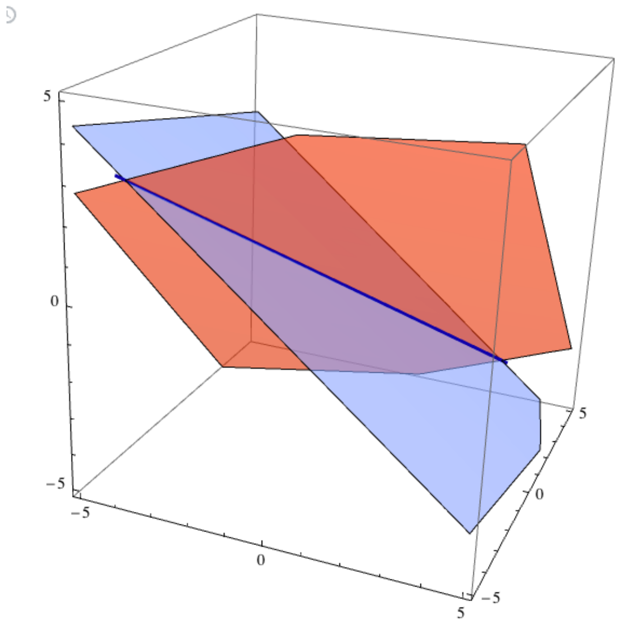
But x,y,z have a co-dependency on a common variable

x= x0 +at, y= y0 +bt, z=z0+ct

(x0,y0,z0)- point passing through line

(a,b,c)- known

t- variable



Lines and Surfaces

What did we learn?

You need only 1 parameter to express a line in 2D (or higher)

Q: What is $y=mx+c$ in 3D??

What did we learn?

You need only 1 parameter to express a line in 2D

You need only 2 parameters to express a surface in 3D (or higher)

3D surfaces

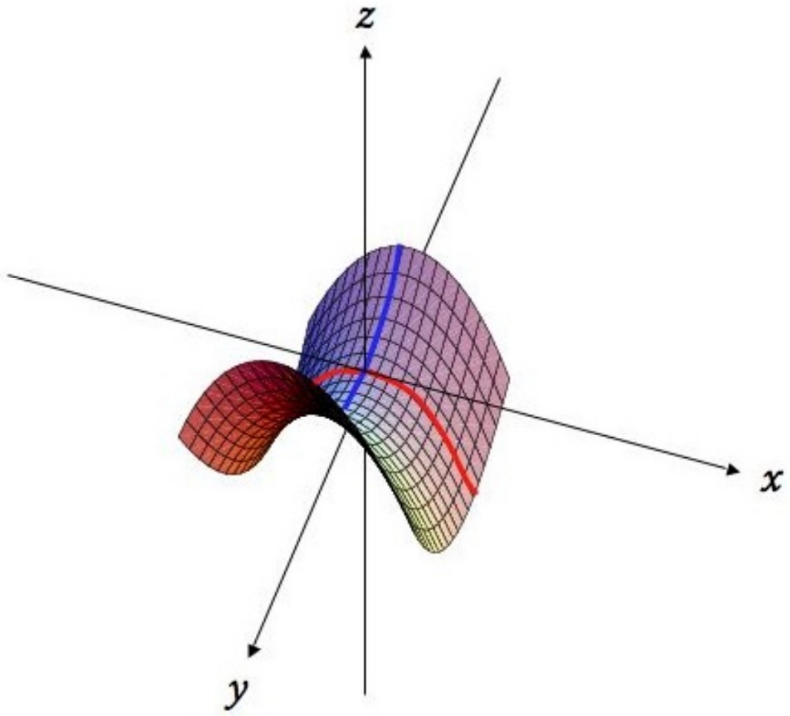
Parametric representation: $S(u,v) = [X(u,v), Y(u,v), Z(u,v)]$

Basically, a 2D entity embedded in 3D

Developable: can be unwrapped

onto a sheet of paper

Non-developable: can't be unwrapped



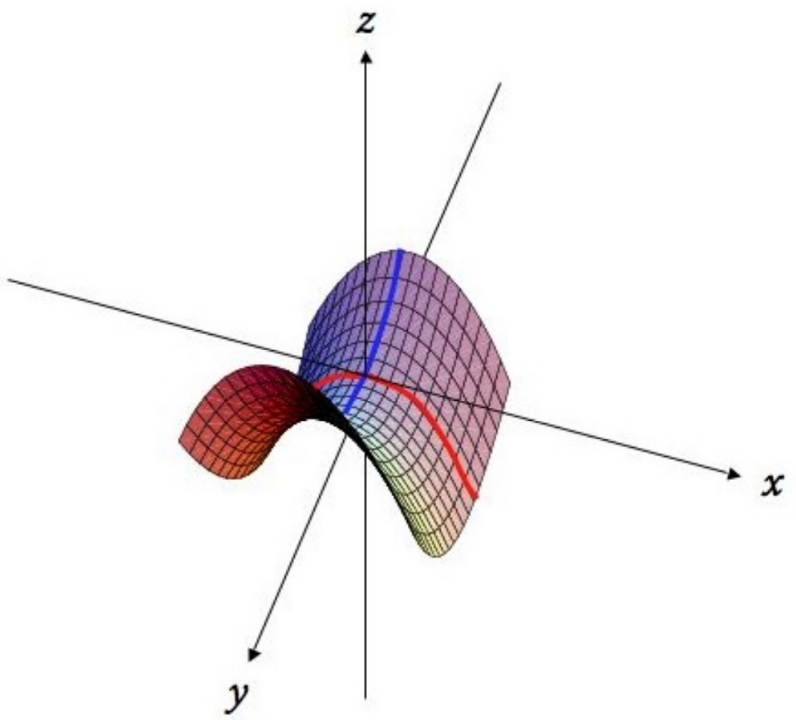
3D surfaces

Parametric representation: $S(u,v) = [X(u,v),Y(u,v),Z(u,v)]$

Basically, a 2D entity embedded in 3D

Developable: can be unwrapped
onto a sheet of paper
(cylinder, for example)

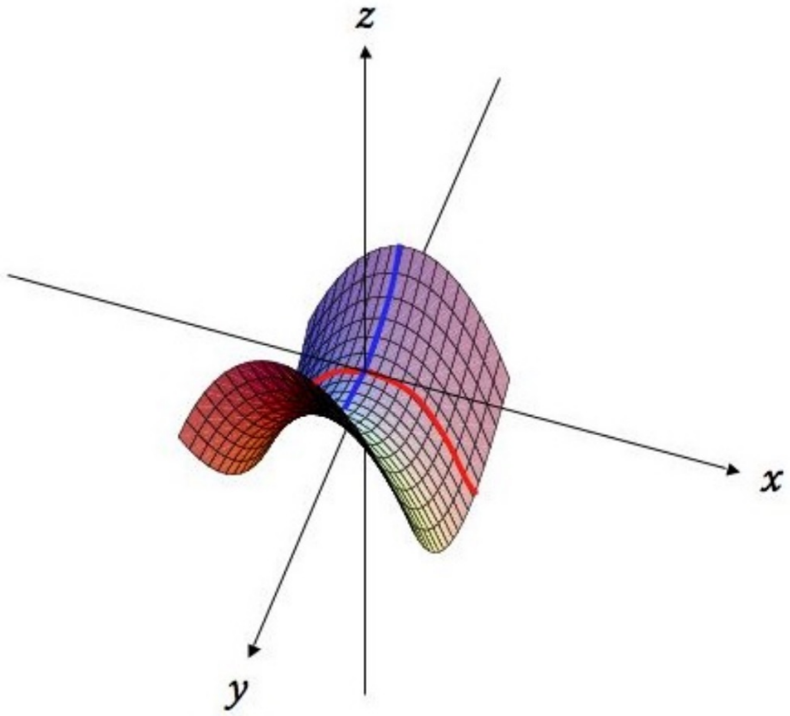
Non-developable: can't be unwrapped
(sphere, for example)



Jacobian

Parametric representation: $\mathbf{S}(u,v) = [X(u,v), Y(u,v), Z(u,v)]$

$$\mathbf{J}_S = \begin{pmatrix} dX(u,v)/du & dX(u,v)/dv \\ dY(u,v)/du & dY(u,v)/dv \\ dZ(u,v)/du & dZ(u,v)/dv \end{pmatrix} = [\mathbf{S}(u,v)_u \ \ \mathbf{S}(u,v)_v]$$



Su, **S**v: are tangent vectors

Su x **S**v: Normal vector

What do Jacobians tell?

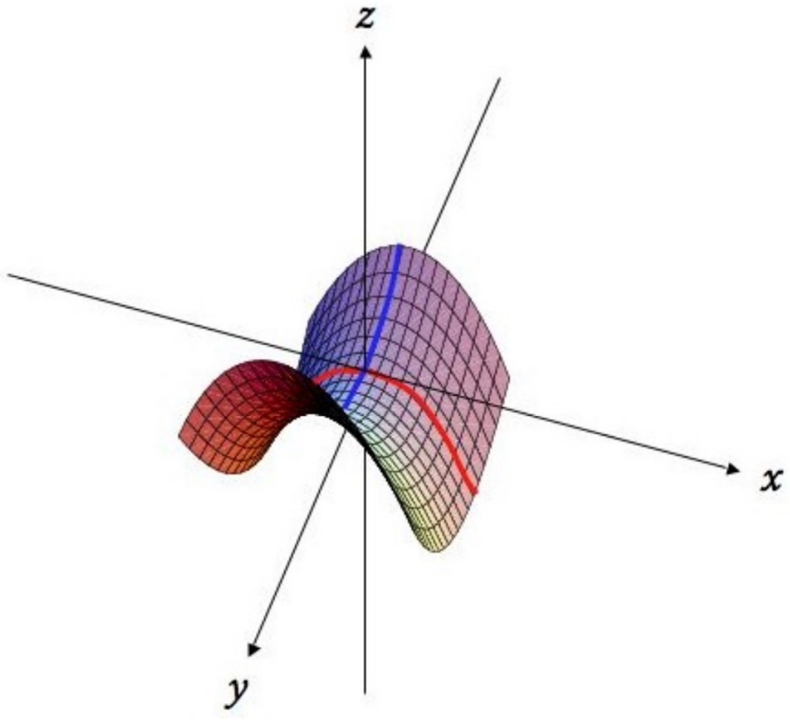
How the space is transforming in the neighborhood of a point:

How much breakage is expected at a point locally while unwarping

Basically, it tells us what is the transformation

Can be used to measure lengths, areas, angles

-> look into $\mathbf{J}^T\mathbf{J}$



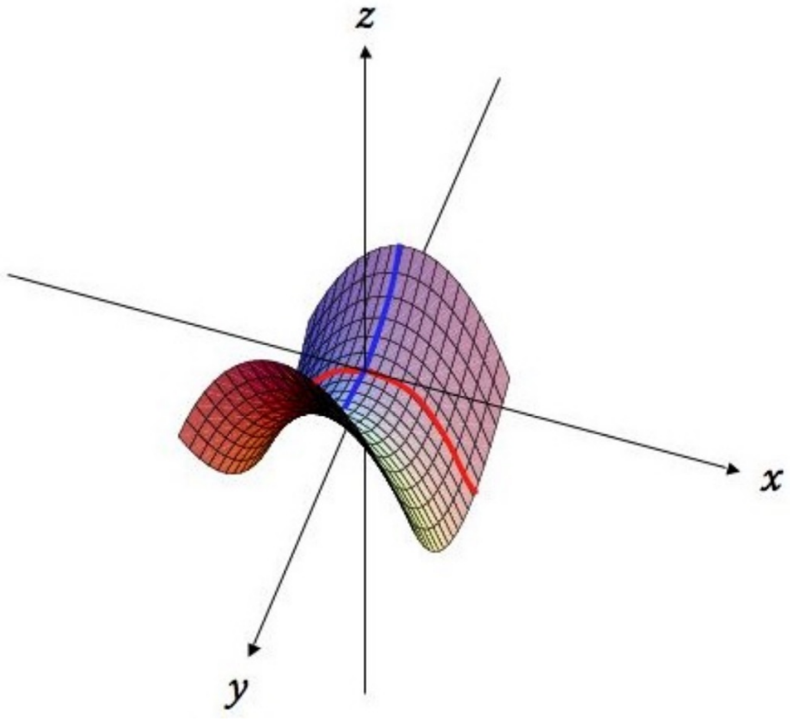
Measuring lengths, angles, areas

$\mathbf{J}^T\mathbf{J}$: also known as metric tensor

Tells you how near-Euclidean the space is, around a point

$ds^2 = [du\ dv]\ \mathbf{J}^T\mathbf{J}\ [du\ dv]^T$

If $\mathbf{J}^T\mathbf{J} = \mathbf{I}$, what does it mean?



Measuring lengths, angles, areas

$\mathbf{J}^T\mathbf{J}$: also known as metric tensor

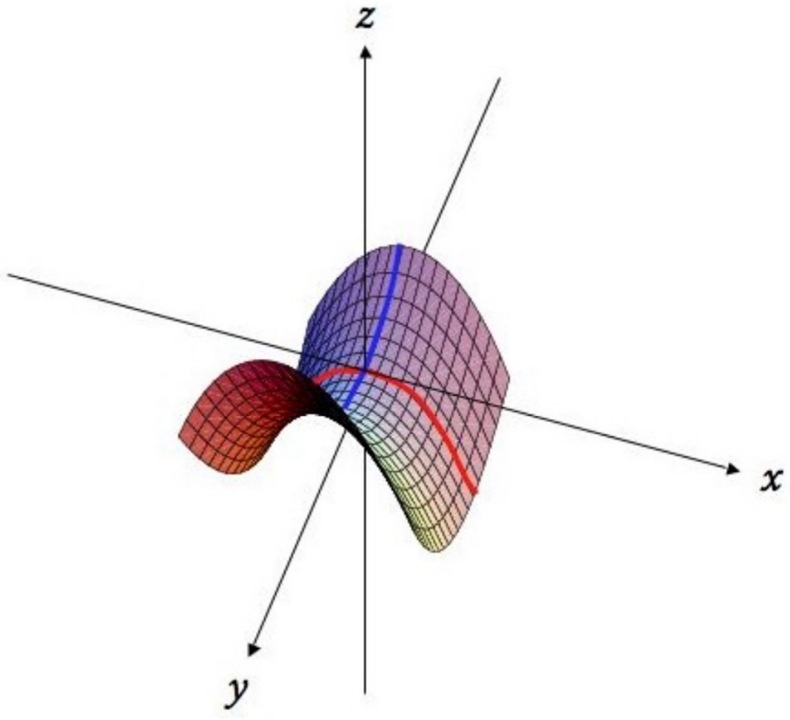
Tells you how near-Euclidean the space is, around a point

$ds^2 = [du\ dv]\ \mathbf{J}^T\mathbf{J}\ [du\ dv]^T$

If $\mathbf{J}^T\mathbf{J} = \mathbf{I}$,

lengths, angles areas are preserved

Locally, the surface transforms rigidly



Measuring lengths, angles, areas

Consider a cylindrical surface surface

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

Consider rho = 1

J = ? , $\mathbf{J}^{\mathrm{T}}\mathbf{J} = \mathbf{I}$

Measuring lengths, angles, areas

Consider a spherical surface

$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta; \\
y &= \rho \sin\varphi \sin\theta; \\
z &= \rho \cos\varphi.
\end{aligned}
$$

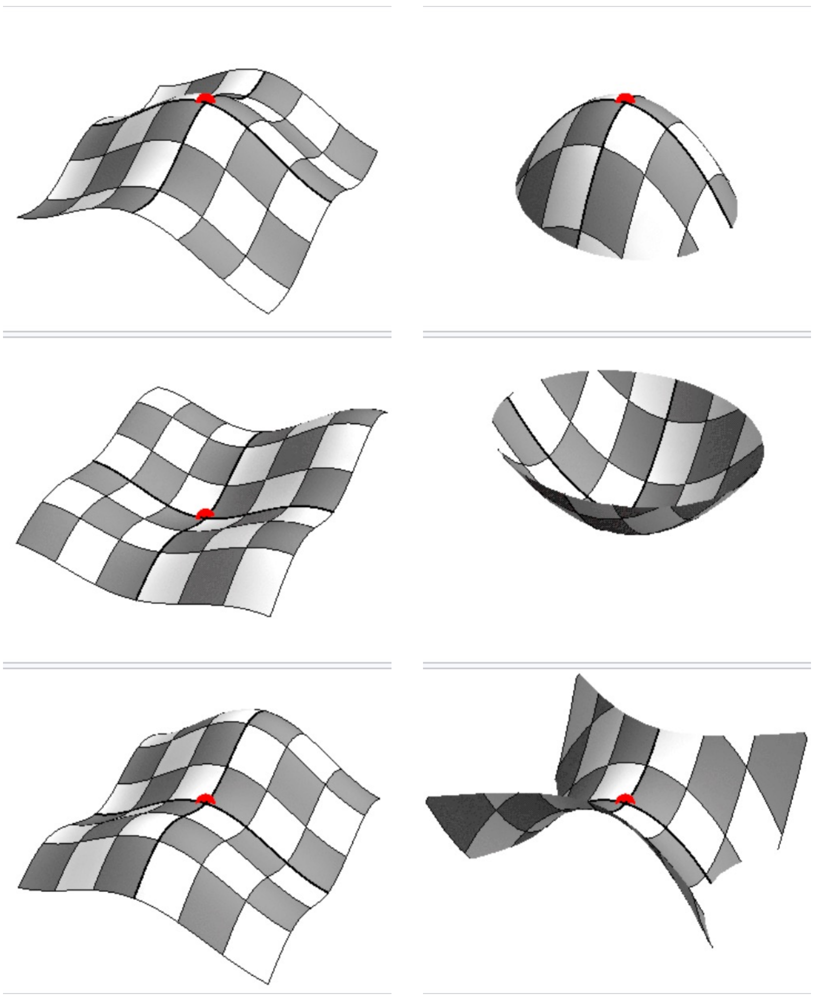
$$
\mathbf{J} = \begin{bmatrix} \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ -\rho\sin\varphi & 0 \end{bmatrix}
$$

, $\mathbf{J}^{\mathrm{T}}\mathbf{J} =$?

What do Jacobians tell?

What if you don't slide?

How to differentiate between these?



Hessians

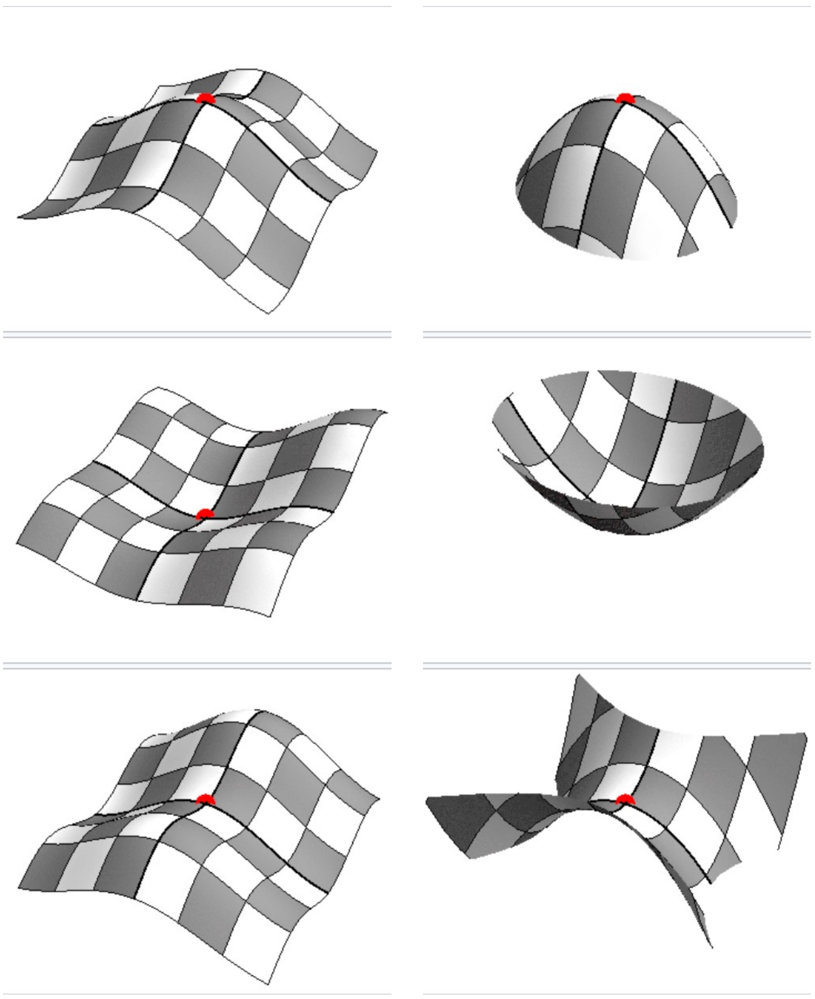
$\mathbf{S}(u,v) = [X(u,v), Y(u,v), Z(u,v)]$

$\mathbf{J_S} = [\mathbf{S}(u,v)_u \;\; \mathbf{S}(u,v)_v]$

$\mathbf{H}_u = [\mathbf{S}(u,v)_{uu} \;\; \mathbf{S}(u,v)_{vu}]$

$\mathbf{H}_v = [\mathbf{S}(u,v)_{vu} \;\; \mathbf{S}(u,v)_{vv}]$

$\mathbf{H} = \mathbf{J}(\mathbf{J_S})$



Hessians

It tells you if you are dropped on the surface (you might remain stable), what happens if you are pushed into some direction

It tells you how quickly a surface deviates from its tangent plane in a local neighborhood

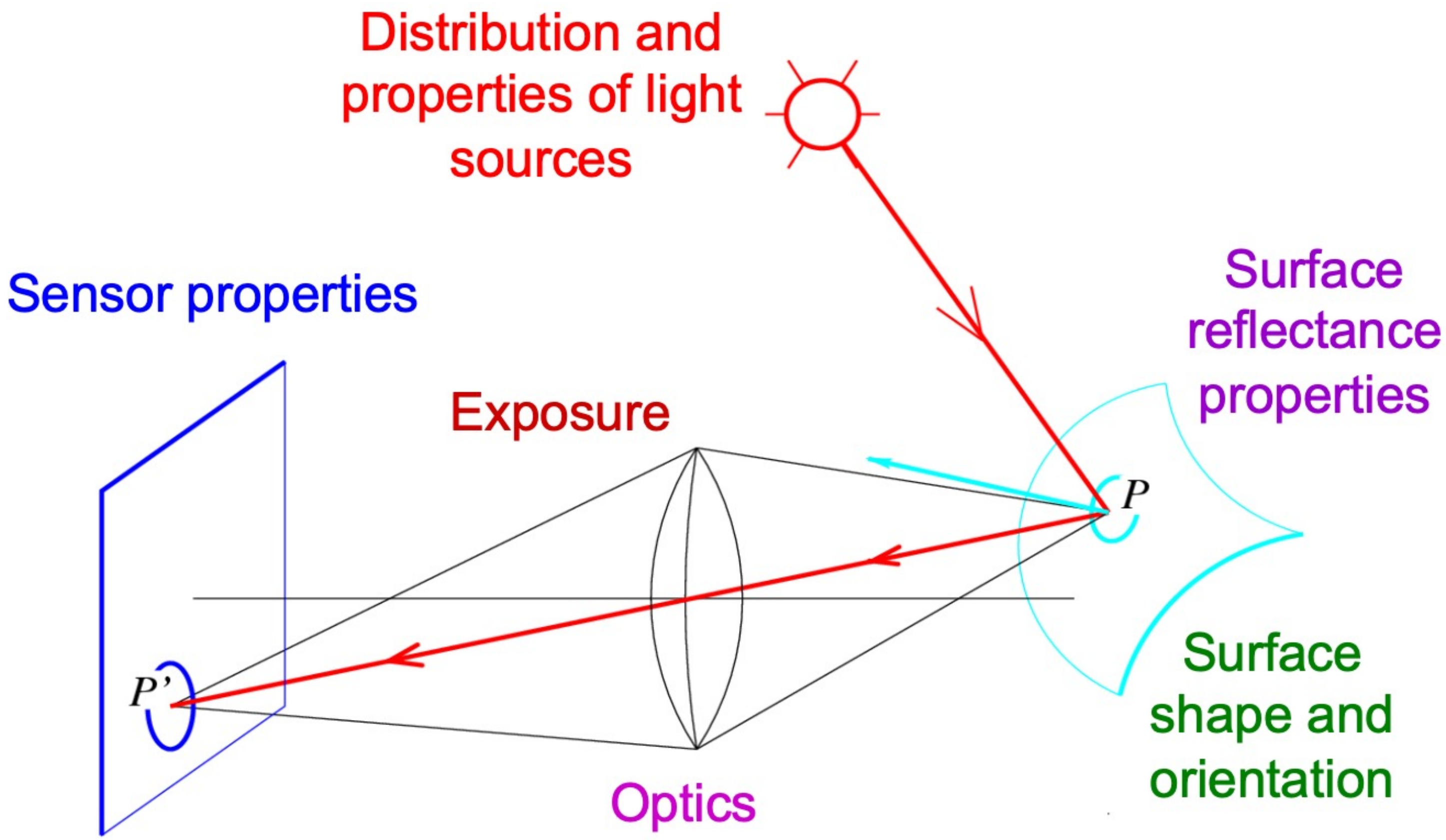
Shape from Shading (SfS)

Light falling on surfaces gives an indication of relative depths

P. Claesz, Still Life with a Skull and a Writing Quill, 1628

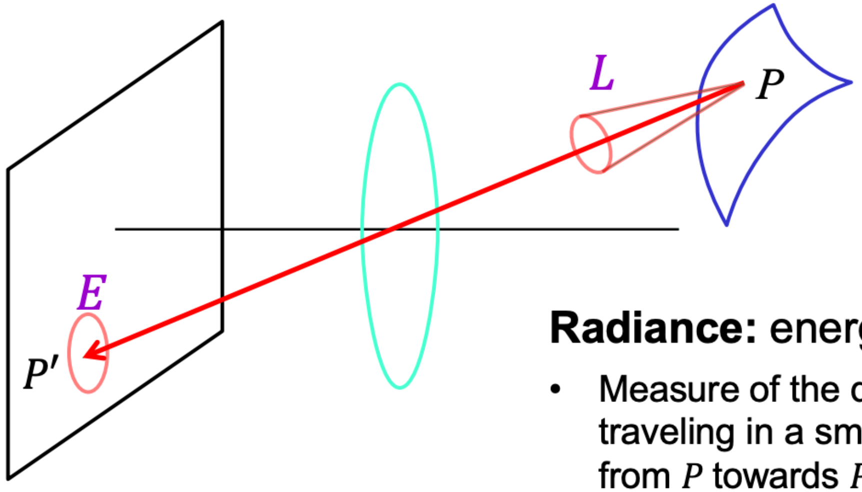
Shape from Shading (SfS)

What impacts pixel brightness?



Slide by L. Fei-Fei

Pixel Brightness



Irradiance: energy arriving at a surface

- Incident power per unit area (not foreshortened)
- Units: Watts per square meter

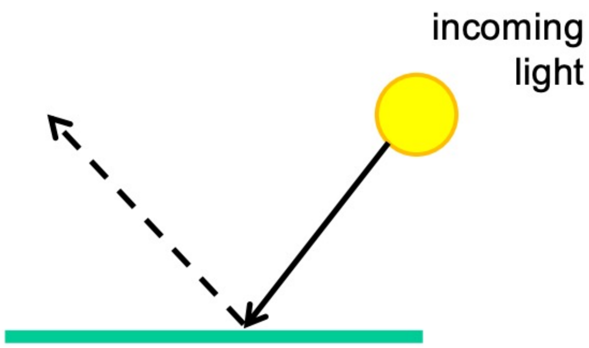
Radiance: energy carried by a ray

- Measure of the density of photons traveling in a small cone of directions from P towards P'
- Power per unit area perpendicular to the direction of travel, per unit solid angle
- Units: Watts per square meter per steradian

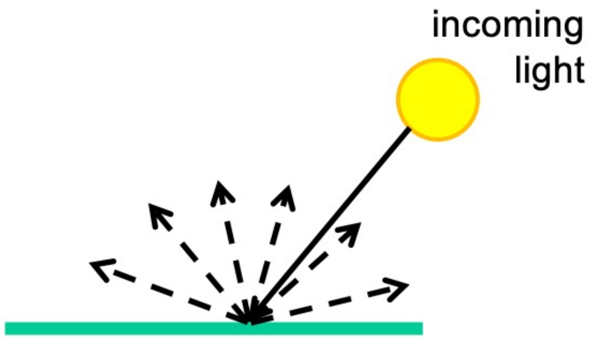
E and L are linearly related. E = a L (a depends on lens area, focal length and angle between viewing ray and optical axis)

Reflection models

- **Specular reflection:** light is reflected about the surface normal

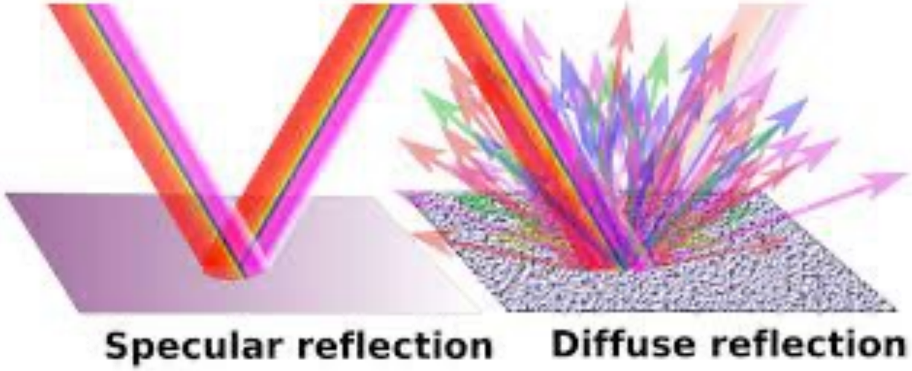


- **Diffuse reflection:** light scatters equally in all directions



Slide from D. Hoiem

Reflection models



Bidirectional reflectance distribution function (BRDF)

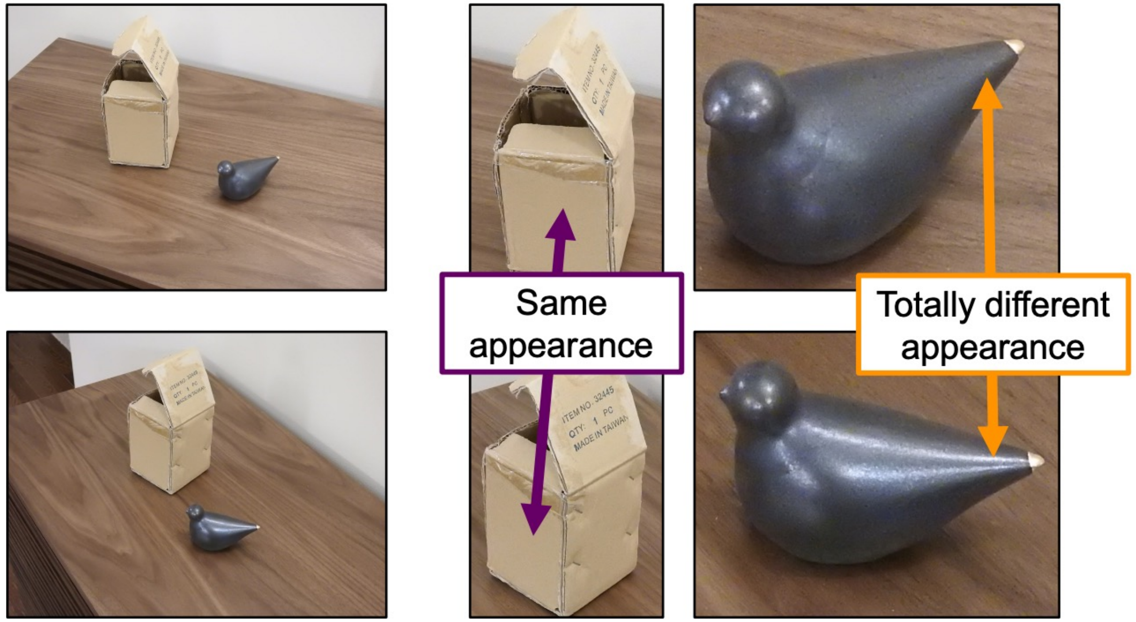
Ratio of the radiance in the emitted direction to irradiance in the incident direction

It tells how bright a surface appears when viewed from one direction when light falls on it from another

Bidirectional reflectance distribution function (BRDF)

On specular surfaces, it changes with light source position

On diffused surfaces, it doesn't

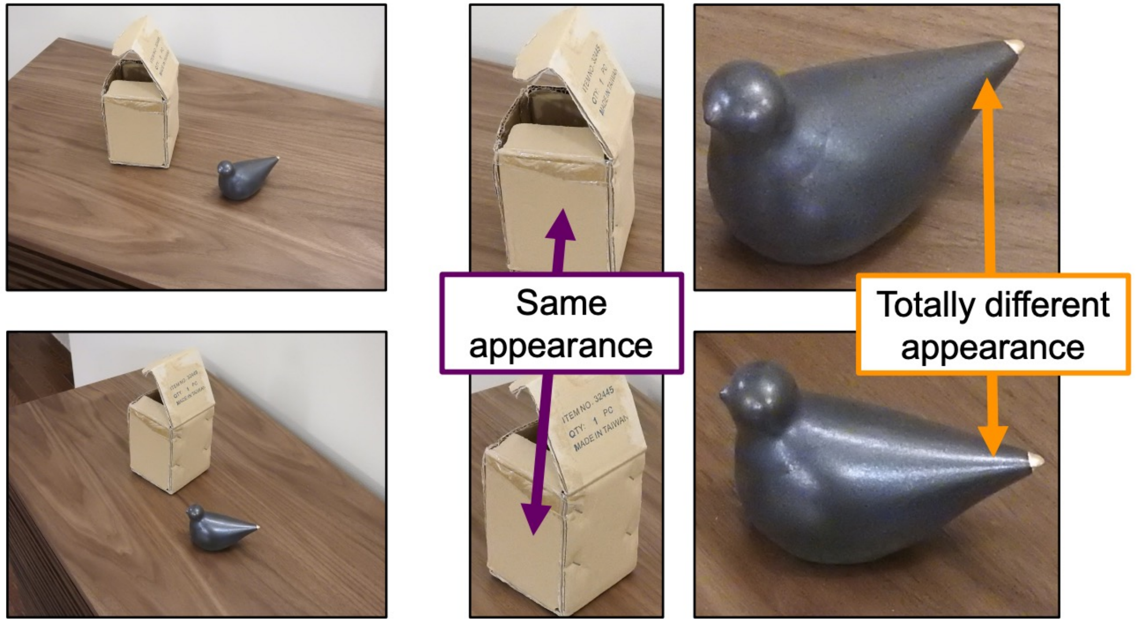


Source: J. Johnson and D. Fouhey

Bidirectional reflectance distribution function (BRDF)

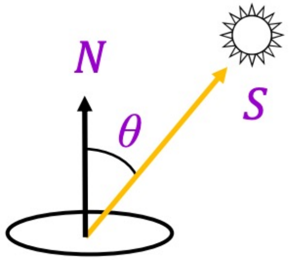
On specular surfaces, it changes with light source position

On diffused surfaces, it doesn't

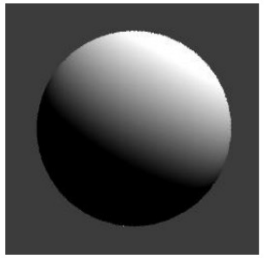


Source: [J. Johnson and D. Fouhey]

Lambert's Law on Diffused Surfaces



$$
\boxed{\begin{aligned} I &= \rho\,(S \cdot N) \\ &= \rho \|S\| \cos\theta \end{aligned}}
$$



I: reflected intensity (technically: *radiosity*, or total power leaving the surface per unit area)

ρ: albedo (fraction of incident irradiance reflected by the surface)

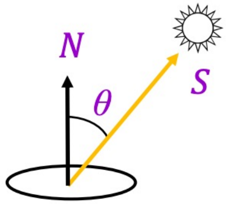
S: direction of light source (magnitude proportional to intensity of the source)

N: unit surface normal

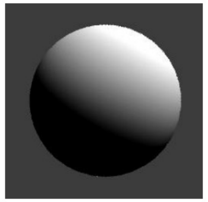
Slide by Prof. Svetlana Lazebnik

Shape from Shading (SfS)

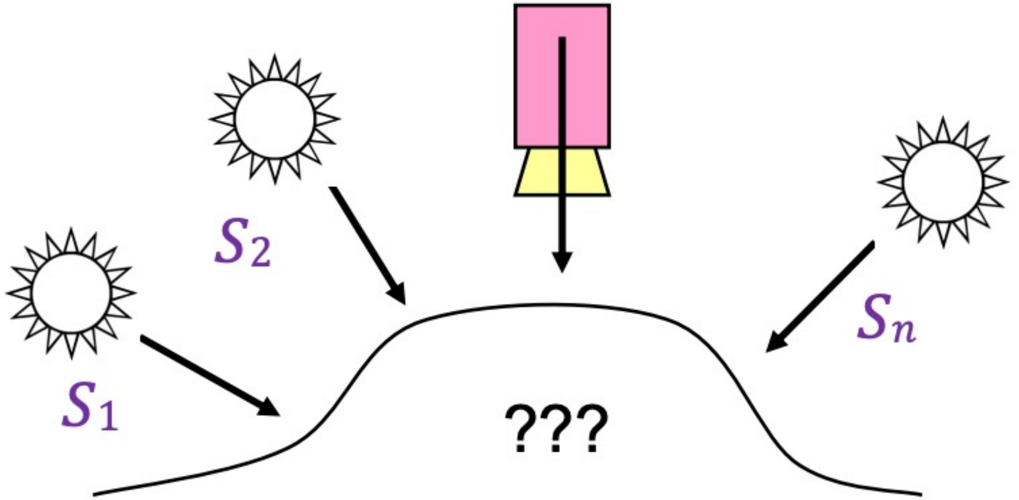
Given I, can we recover N, S?



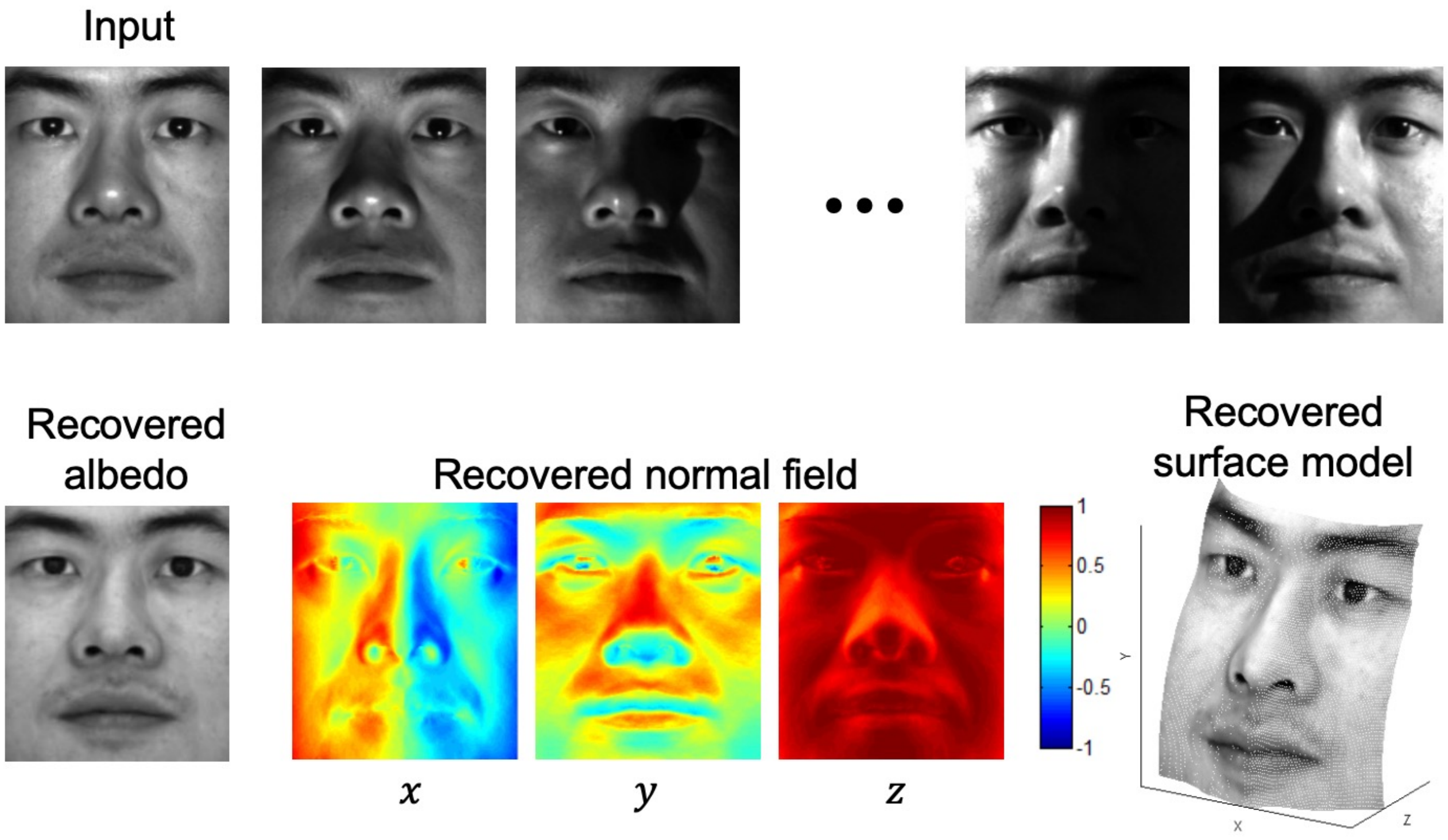
$$\begin{aligned} I &= \rho\,(S\cdot N) \\ &= \rho \|S\| \cos\theta \end{aligned}$$



SfS: Take multiple images by moving light source to recover normals



Shape from Shading (SfS)



Slide by Prof. Svetlana Lazebnik

Shape from Shading (SfS)

Assume orthographic projection: $Z = f(x,y)$

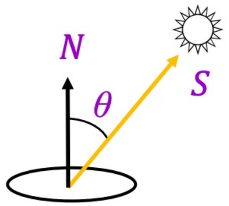
What is normal?

Slide by Prof. Svetlana Lazebnik

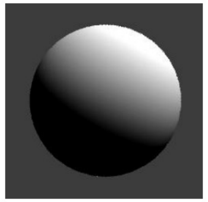
Shape from Shading (SfS)

Assume orthographic projection: Z = f(x,y)

N = $[f_x; f_y; 1]/\mathrm{sqrt}(1+f_x^2+f_y^2)$



$$\boxed{\begin{aligned} I &= \rho\,(S \cdot N) \\ &= \rho\|S\|\cos\theta \end{aligned}}$$



Knowns: I and S

Unknowns: N and albedo

Shape from Shading (SfS)

- **Known:** source vectors S_j and pixel values $I_j(x, y)$
- **Unknown:** surface normal $N(x, y)$ and albedo $\rho(x, y)$
- Assume that the response function of the camera is a linear scaling by a factor of k
- Lambert's law:

$$
\begin{aligned}
I_j(x, y) &= k\,\rho(x, y)\big(N(x, y) \cdot S_j\big) \\
&= \big(\rho(x, y)N(x, y)\big) \cdot \big(k\,S_j\big) \\
&= g(x, y) \cdot V_j
\end{aligned}
$$

Slide by Prof. Svetlana Lazebnik

Shape from Shading (SfS)

- For each pixel, set up a linear system:

$$\underbrace{\begin{bmatrix} V_1^T \\ V_2^T \\ \vdots \\ V_n^T \end{bmatrix}}_{\substack{n \times 3 \\ \text{known}}} \underbrace{g(x, y)}_{\substack{3 \times 1 \\ \text{unknown}}} = \underbrace{\begin{bmatrix} I_1(x, y) \\ I_2(x, y) \\ \vdots \\ I_n(x, y) \end{bmatrix}}_{\substack{n \times 1 \\ \text{known}}}$$

- Obtain least-squares solution for $g(x, y)$, which we defined as $\rho(x, y)N(x, y)$
- Since $N(x, y)$ is the *unit* normal, $\rho(x, y)$ is given by the magnitude of $g(x, y)$
- Finally, $N(x, y) = \frac{1}{\rho(x,y)} g(x, y)$

Slide by Prof. Svetlana Lazebnik

Shape from Shading (SfS)

$N = [f_x; f_y; 1]/\text{sqrt}(1+f_x^2+f_y^2)$

Write the estimated vector g as

$$g(x,y) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

Then we obtain values for the partial derivatives of the surface:

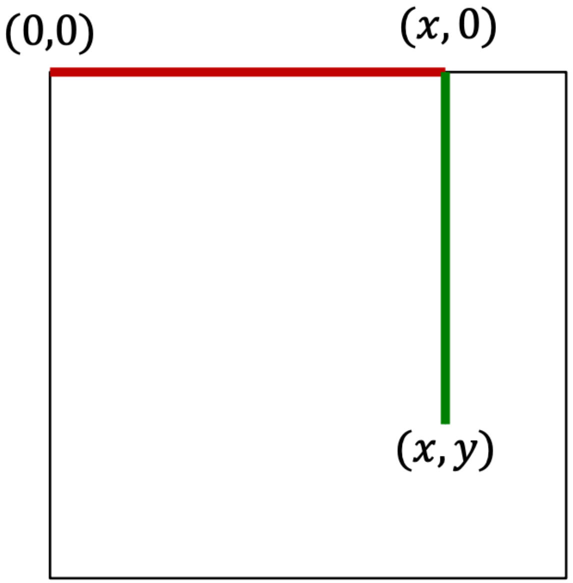
$$f_x(x,y) = \frac{g_1(x,y)}{g_3(x,y)}$$

$$f_y(x,y) = \frac{g_2(x,y)}{g_3(x,y)}$$

Recovering surfaces from normals

We can now recover the surface height at any point by integration along some path, e.g.

$$f(x,y) = \int_0^x f_x(s,0)ds + \int_0^y f_y(x,t)dt + C$$



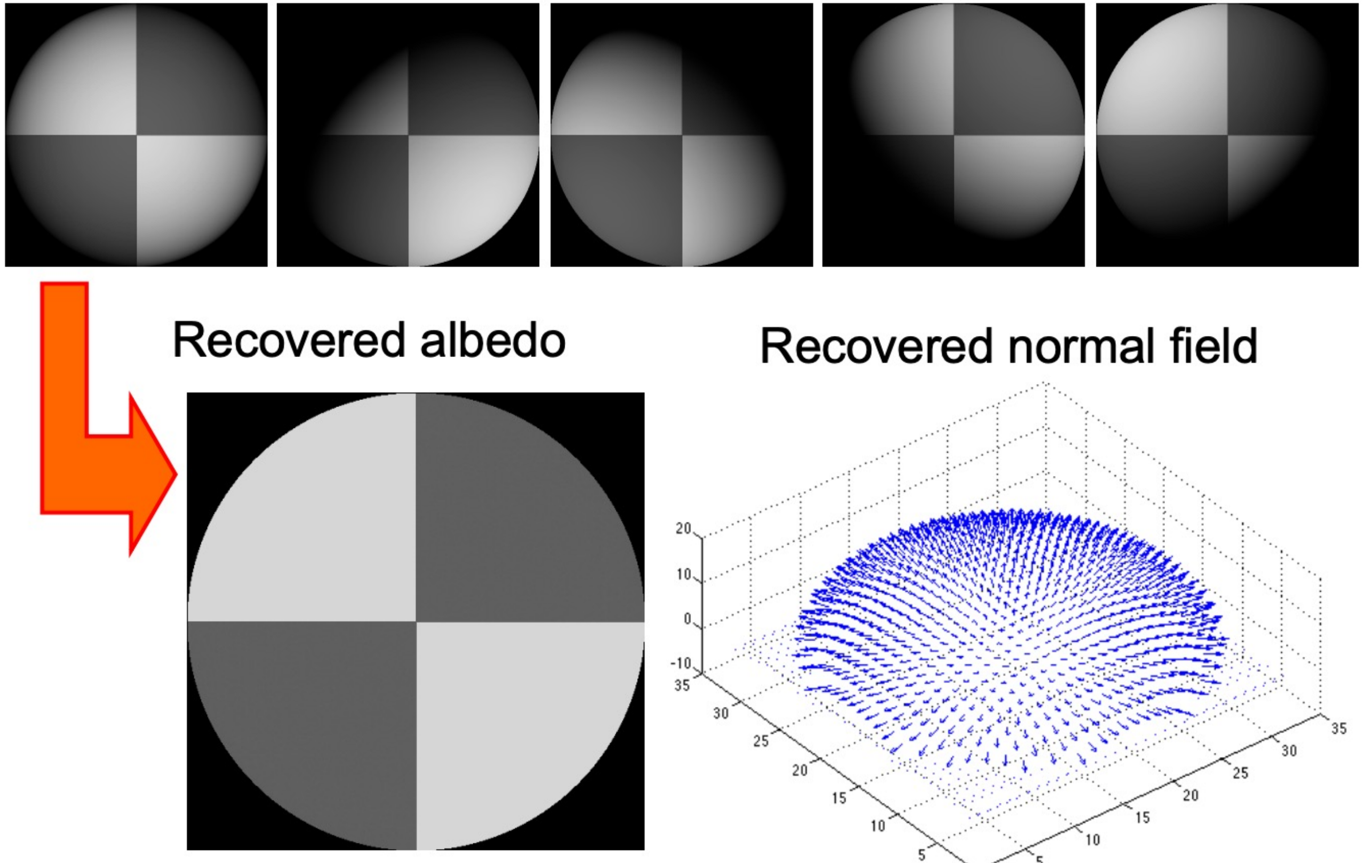
Enforce:

$$\frac{\partial}{\partial y}\left(\frac{g_1(x,y)}{g_3(x,y)}\right) = \frac{\partial}{\partial x}\left(\frac{g_2(x,y)}{g_3(x,y)}\right)$$

For robustness, it is better to take integrals over many different paths and average the results

Slide by Prof. Svetlana Lazebnik

SfS



F&P 2nd ed., sec. 2.2.4

Slide by Prof. Svetlana Lazebnik