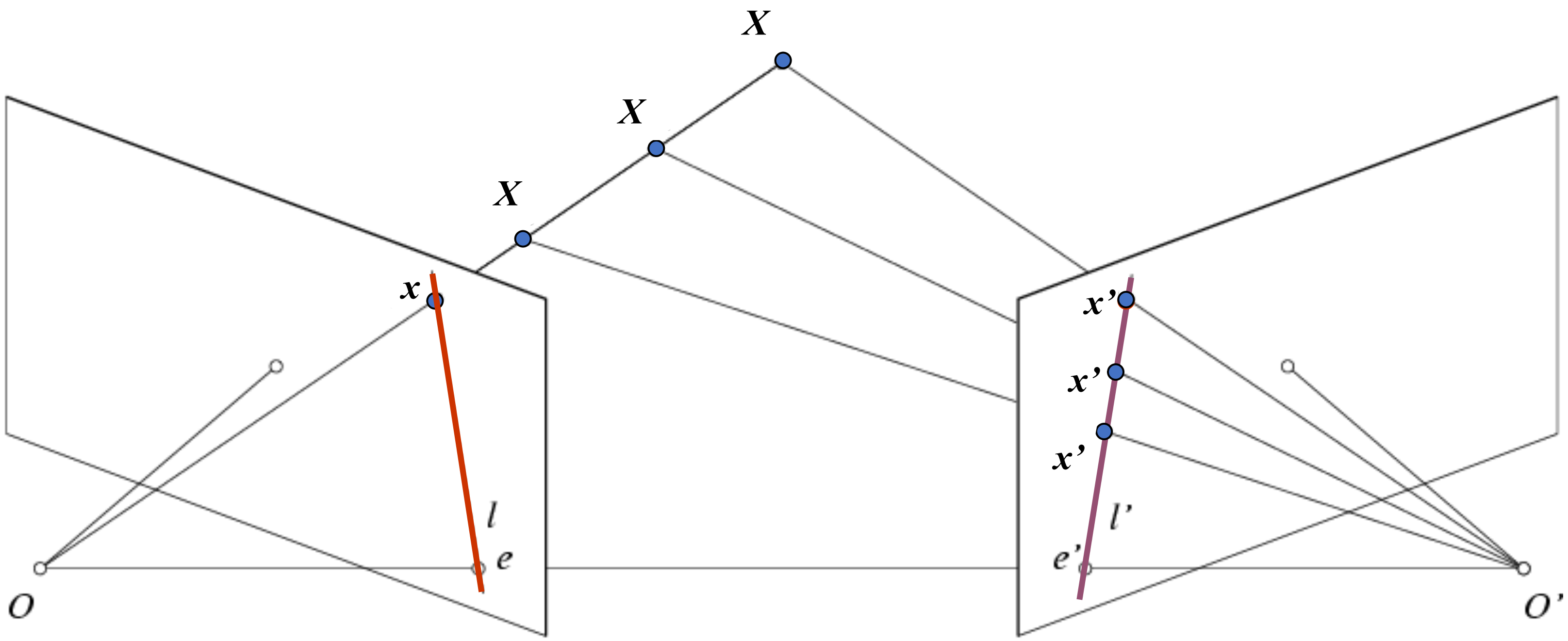
6. Epipolar Geometry

Epipolar constraint, Essential Matrix and Fundamental Matrix

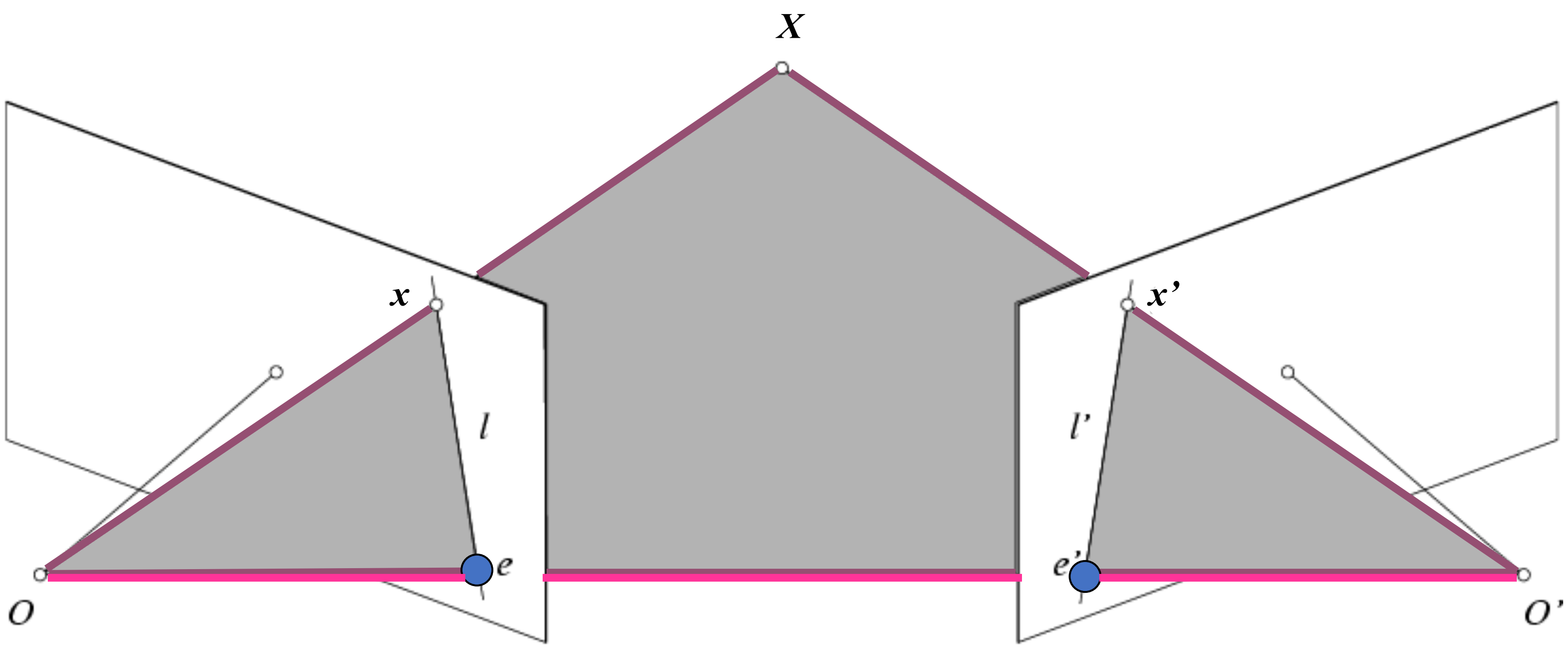
Many Slides from Prof. S. Seitz

Epipolar Geometry



Relation between x and x' if they project the point X

Epipolar geometry: notation

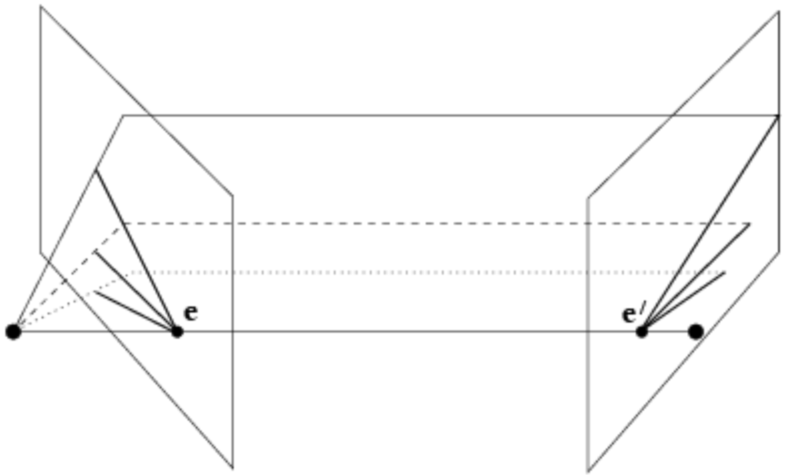


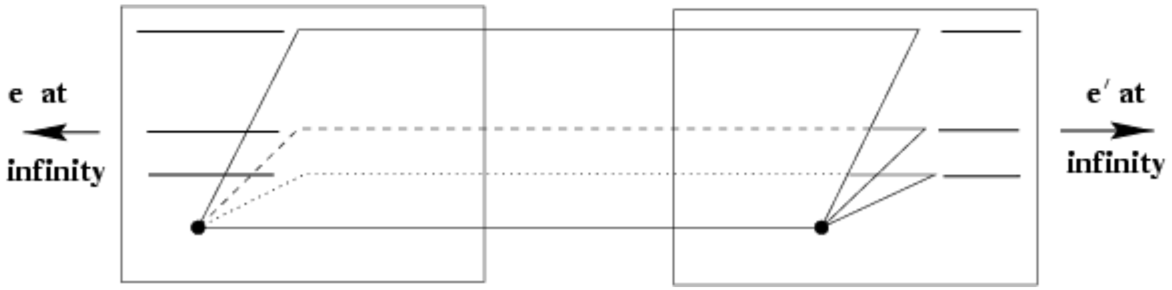
- **Baseline** – line connecting the two camera centers
- **Epipoles**

  = intersections of baseline with image planes

  = projections of the other camera center
- **Epipolar Plane** – plane containing baseline (1D family)

Epipolar geometry: notation



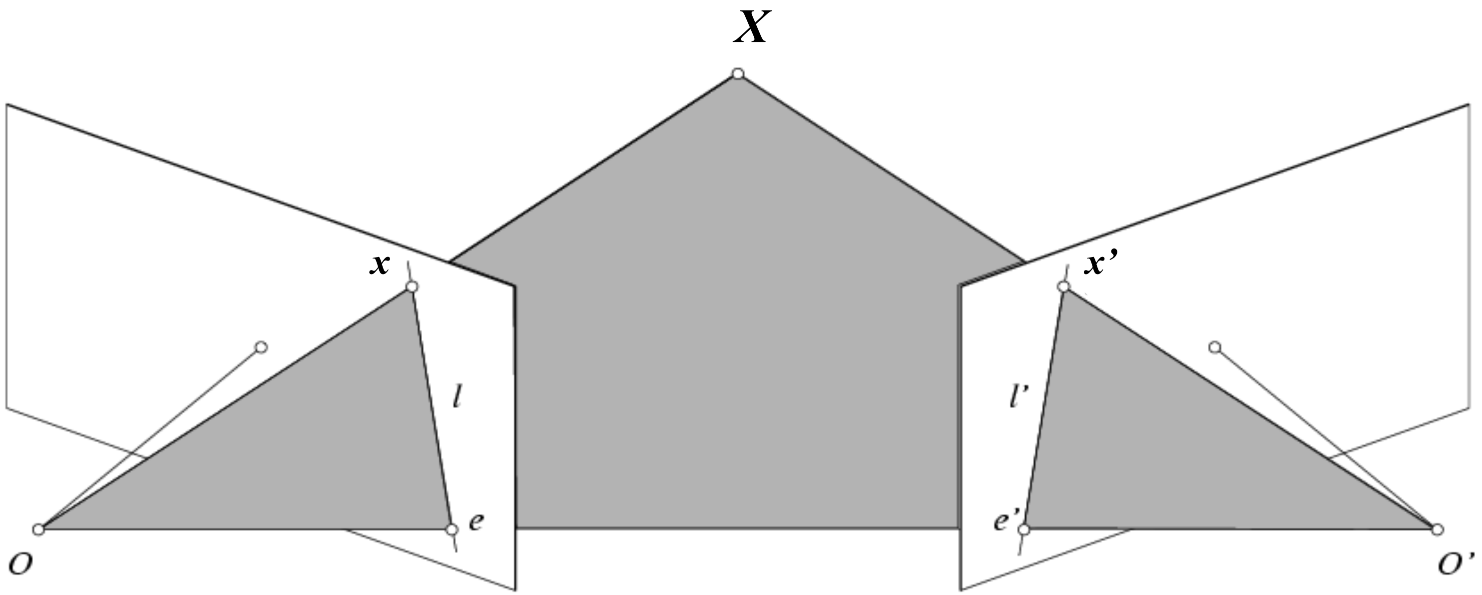


Converging lines:
Motion non-parallel to image plane

Parallel lines:
Motion parallel to image plane

What happens when camera moves forward/backward?

Epipolar constraint: Calibrated case



Given the intrinsic parameters of the cameras:

1. Convert to normalized coordinates by pre-multiplying all points with the inverse of the calibration matrix; set first camera's coordinate system to world coordinates

$$\hat{x} = K^{-1} x = X$$

Homogeneous 2d point (3D ray towards X)

2D pixel coordinate (homogeneous)

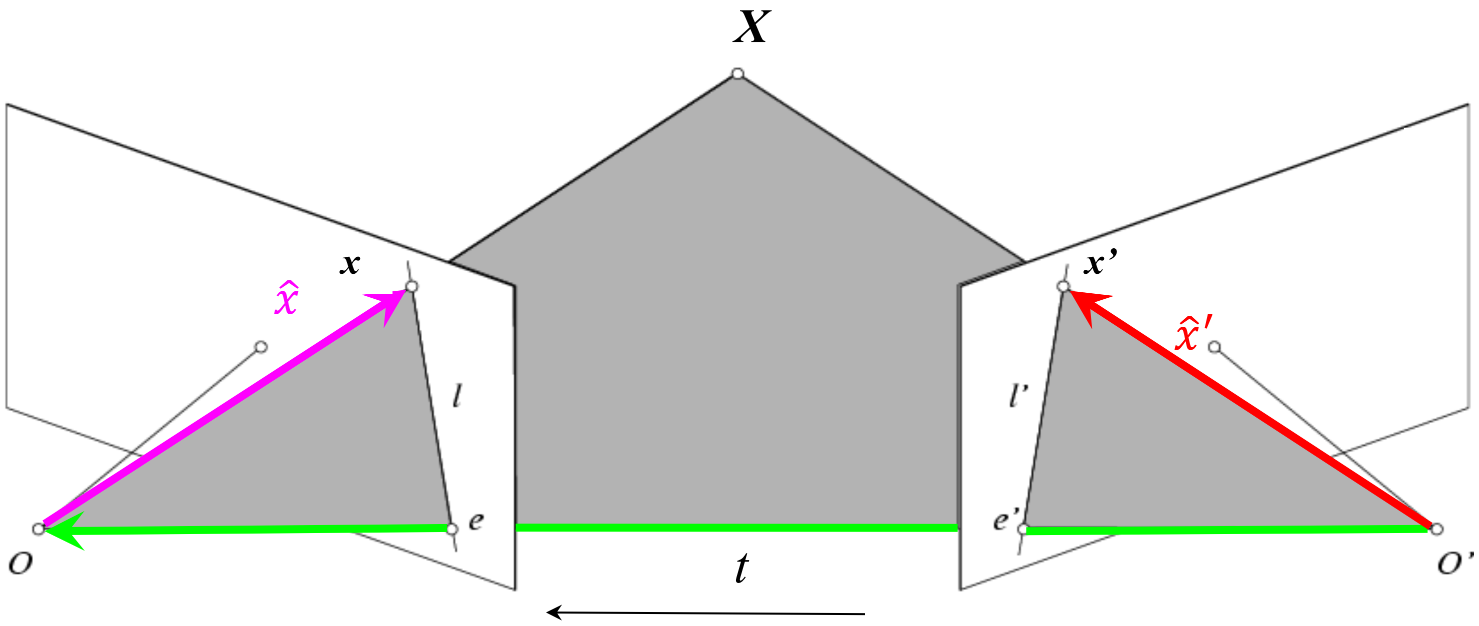
3D scene point

$$\boxed{\hat{x} = R\hat{x}' + t}$$

$$\hat{x}' = K'^{-1} x' = X'$$

3D scene point in 2nd camera's 3D coordinates

Epipolar constraint: Calibrated case



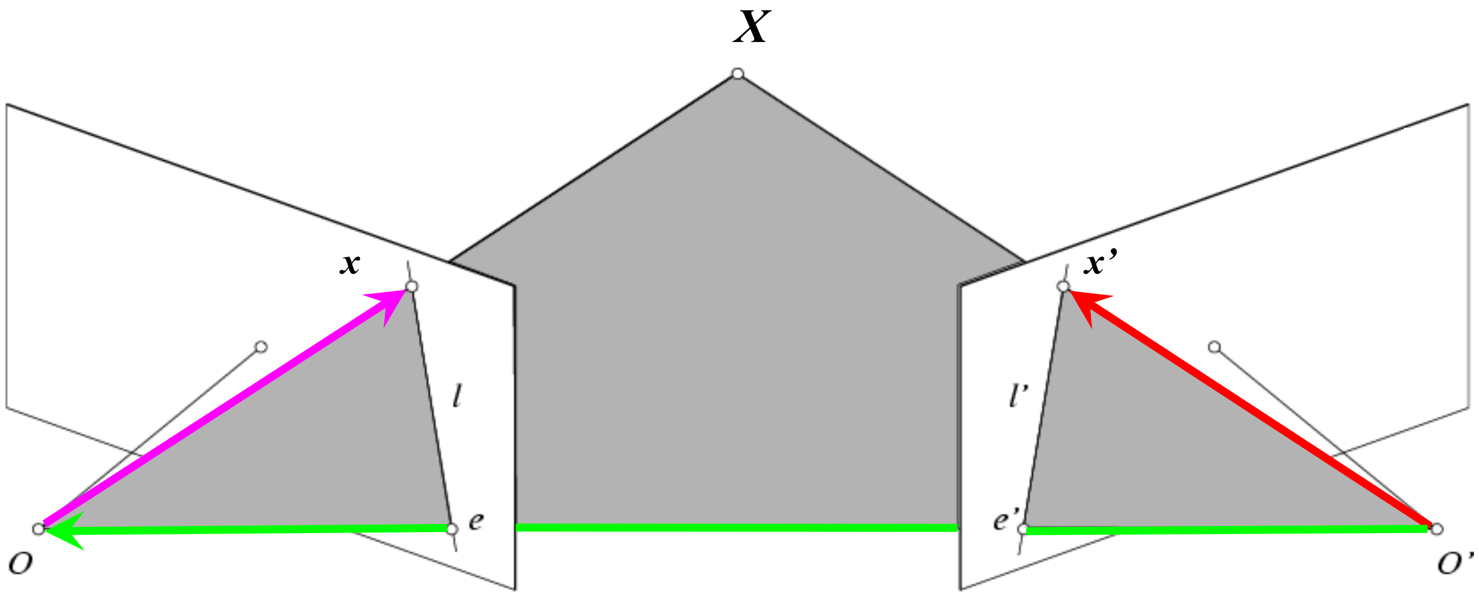
$$\hat{x} = K^{-1}x = X \qquad \hat{x}' = K'^{-1}x' = X'$$

$$\hat{x} = R\hat{x}' + t \;\Rightarrow\; \hat{x} \cdot [t \times (R\hat{x}')] = 0$$

(because $\hat{x}$, $R\hat{x}'$, and t are co-planar)

$$t \times \hat{x} = t \times (R\,\hat{x}' + t) = \mathrm{t} \times (R\,\hat{x}') \;\Rightarrow\; \hat{x} \cdot (t \times \hat{x}) = \hat{x}[\,\mathrm{t} \times (R\,\hat{x}')] = 0$$

Essential matrix



$$\hat{x} \cdot [t \times (R\hat{x}')] = 0 \quad \Rightarrow \quad \hat{x}^T E \hat{x}' = 0 \quad \text{with} \quad E = [t]_\times R$$

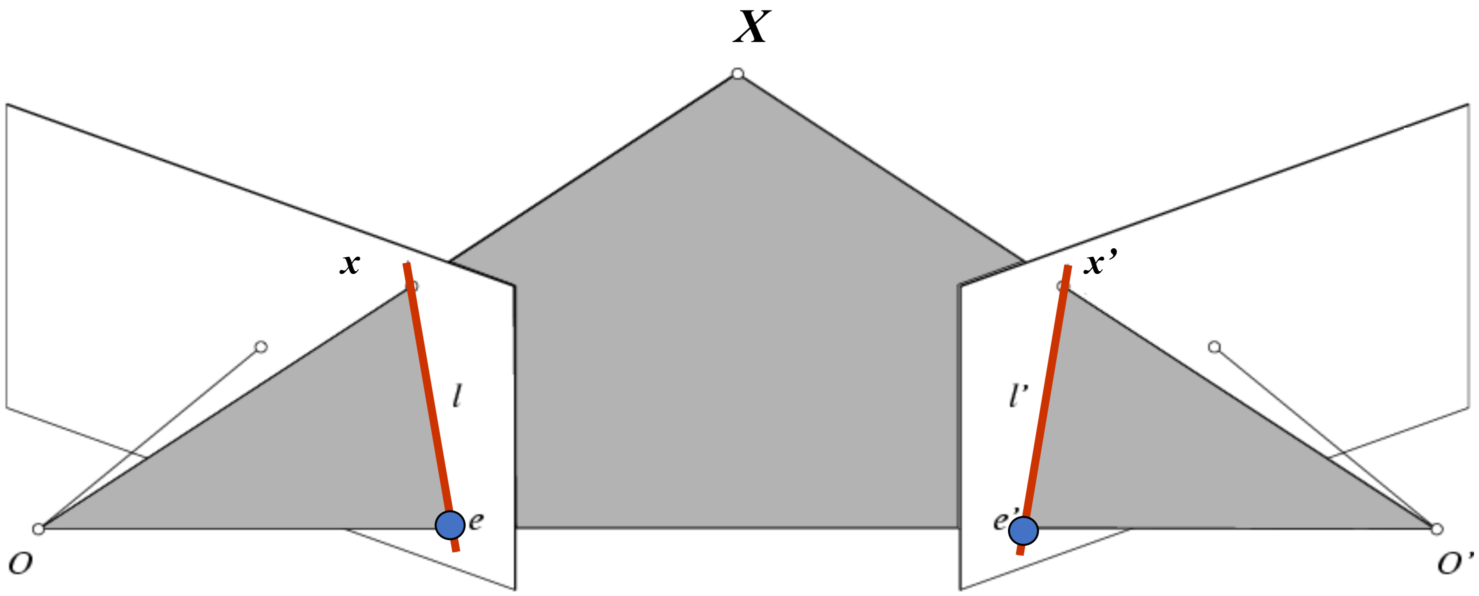
$$\mathbf{a} = (a_1\ a_2\ a_3)^\mathrm{T}$$

$$\mathbf{b} = (b_1\ b_2\ b_3)^\mathrm{T}$$

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad \mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b}$$

Essential Matrix
(Longuet-Higgins, 1981)

Properties of the Essential matrix



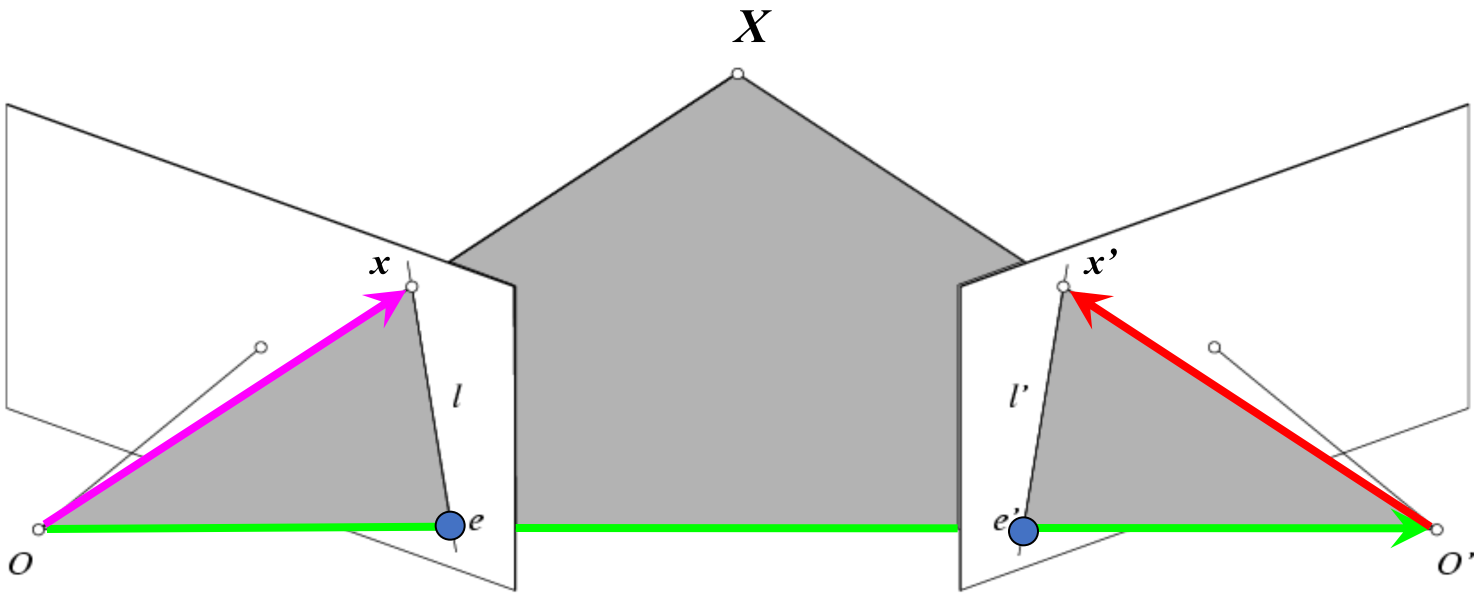
$$\hat{x} \cdot [t \times (R\hat{x}')] = 0 \quad \Rightarrow \quad \hat{x}^T E \hat{x}' = 0 \quad \text{with} \quad E = [t]_\times R$$

Drop ^ below to simplify notation

- $E\,x'$ is the epipolar line associated with x' ($l = E\,x'$)
- $E^T x$ is the epipolar line associated with x ($l' = E^T x$)
- $E\,e' = 0$ and $E^T e = 0$
- E is singular (rank two)
- E has five degrees of freedom
  - (3 for R, 2 for t because it's up to a scale)

Skew-symmetric matrix

Epipolar constraint: Uncalibrated case



- If we don't know *K* and *K'*, then we can write the epipolar constraint in terms of *unknown* normalized coordinates:

$$\hat{x}^T E \hat{x}' = 0 \qquad\qquad x = K\hat{x}, \quad x' = K'\hat{x}'$$

The Fundamental Matrix

Without knowing K and K', we can define a similar relation using *unknown* normalized coordinates

$$\hat{x}^T E \hat{x}' = 0$$

$$\hat{x} = K^{-1} x$$

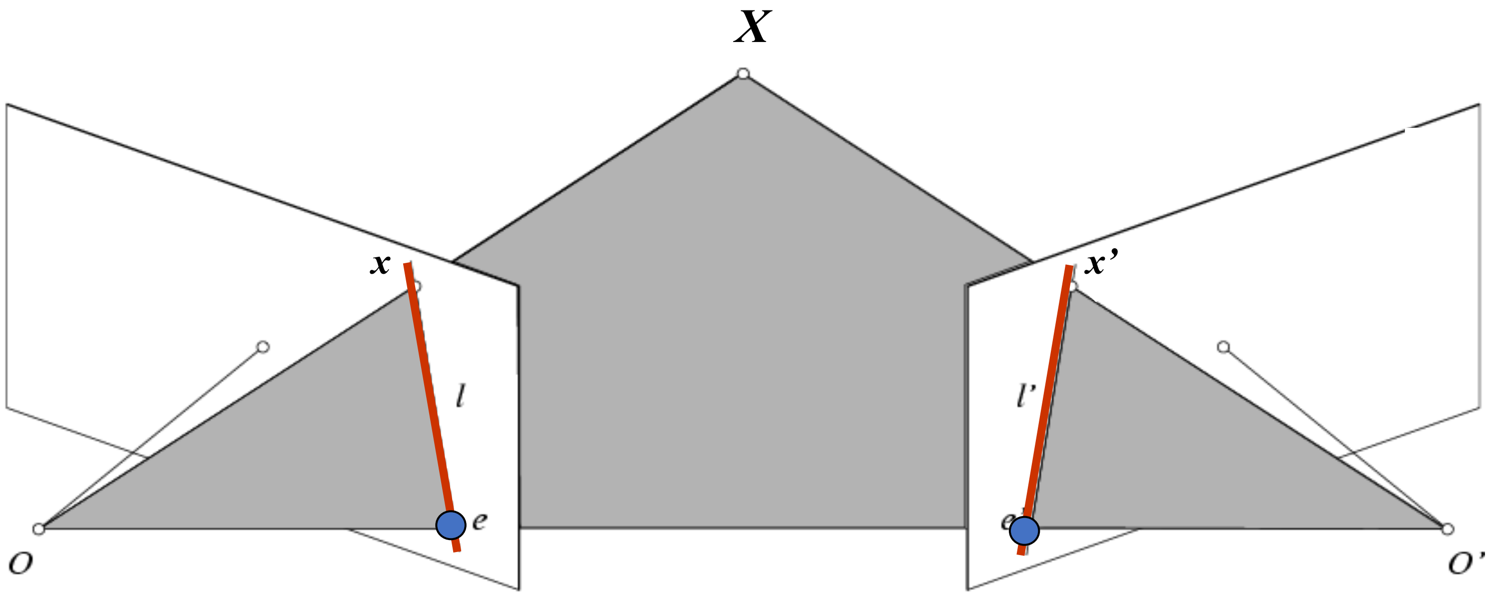
$$\hat{x}' = K'^{-1} x'$$

$$x^T F x' = 0 \quad \text{with} \quad F = K^{-T} E K'^{-1}$$

Fundamental Matrix
(Faugeras and Luong, 1992)

- Estimating the fundamental matrix is known as "weak calibration"
- If we know the calibration matrices of the two cameras, we can estimate the essential matrix: $E = K^T F K'$
- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters

Properties of the Fundamental matrix



$$x^T F x' = 0 \quad \text{with} \quad F = K^{-T} E K'^{-1}$$

- $F x'$ is the epipolar line associated with x' $(l = F x')$
- $F^T x$ is the epipolar line associated with x $(l' = F^T x)$
- $F e' = 0$ and $F^T e = 0$
- F is singular (rank two): det(F)=0
- F has seven degrees of freedom: 9 entries but defined up to scale, det(F)=0

Estimating F: 8-point algorithm

1. Solve a system of homogeneous linear equations
   a. Write down the system of equations
   b. For det(F) = 0, use SVD: S(3,3) = 0

$$\mathbf{x}^T F \mathbf{x'} = 0$$

$$uu'f_{11} + uv'f_{12} + uf_{13} + vu'f_{21} + vv'f_{22} + vf_{23} + u'f_{31} + v'f_{32} + f_{33} = 0$$

$$\mathrm{A}\boldsymbol{f} = \begin{bmatrix} u_1u_1{}' & u_1v_1{}' & u_1 & v_1u_1{}' & v_1v_1{}' & v_1 & u_1{}' & v_1{}' & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_v' & u_nv_n{}' & u_n & v_nu_n{}' & v_nv_n{}' & v_n & u_n{}' & v_n{}' & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ \vdots \\ f_{33} \end{bmatrix} = \mathbf{0}$$

8-point algorithm

1. Solve a system of homogeneous linear equations
   a. Write down the system of equations
   b. Solve **f** from A**f**=**0** using SVD
2. Resolve det(F) = 0 constraint by SVD

Notes:

- Use RANSAC to deal with outliers (sample 8 points)
  - How to test for outliers?

$|x'F\ x\ | < threshold?$

- Solve in normalized coordinates
  - mean=0
  - standard deviation ~= (1,1,1)
  - just like with estimating the homography for stitching

Homography vs Fundamental Matrix

Assume we have matched points x $\Leftrightarrow$ x’ with outliers

Homography (No Translation)

- Correspondence Relation

$$\mathbf{x}' = \mathbf{H}\mathbf{x} \Rightarrow \mathbf{x}' \times \mathbf{H}\mathbf{x} = \mathbf{0}$$

1. Normalize image coordinates

$$\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x} \quad \tilde{\mathbf{x}}' = \mathbf{T}'\mathbf{x}'$$

2. RANSAC with 4 points
   - Solution via SVD
3. De-normalize: $\mathbf{H} = \mathbf{T}'^{-1}\tilde{\mathbf{H}}\mathbf{T}$

Fundamental Matrix (Translation)

- Correspondence Relation

$$\mathbf{x}'^{T}\mathbf{F}\mathbf{x} = 0$$

1. Normalize image coordinates

$$\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x} \quad \tilde{\mathbf{x}}' = \mathbf{T}'\mathbf{x}'$$

2. RANSAC with 8 points
   - Initial solution via SVD
   - Enforce $\det(\tilde{\mathbf{F}}) = 0$ by SVD
3. De-normalize: $\mathbf{F} = \mathbf{T}'^{T}\tilde{\mathbf{F}}\mathbf{T}$

$\boxed{\textbf{In our case, } \mathbf{F} = \mathbf{T}^{T}\mathbf{F}\mathbf{T}'}$

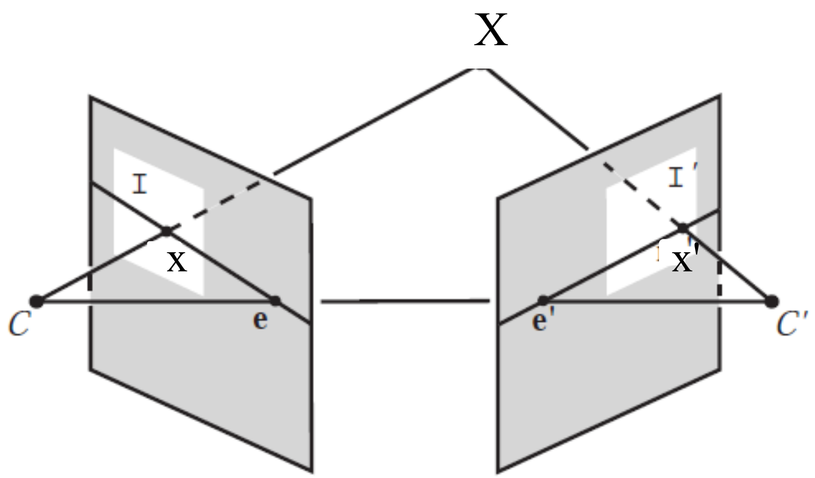
Estimating the Fundamental Matrix

- 8-point algorithm
  - Least squares solution using SVD on equations from 8 pairs of correspondences
  - Enforce det(F)=0 constraint using SVD on F
- 7-point algorithm
  - Use least squares to solve for null space (two vectors) using SVD and 7 pairs of correspondences
  - Solve for linear combination of null space vectors that satisfies det(F)=0
- Minimize reprojection error
  - Non-linear least squares

Note: estimation of F (or E) is degenerate for a planar scene.

“Gold standard” algorithm

- Use 8-point algorithm to get initial value of F
- Use F to solve for P and P’ (discussed later)
- Jointly solve for 3d points **X** and **F** that minimize the squared re-projection error



See Algorithm 11.2 and Algorithm 11.3 in Hartley and Zissermann (pages 284-285) for details

Let’s recap…

- Fundamental matrix song