1) Evalute $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\, dy \right] \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int_{x=0}^{1} \left[\sin^{-1}(y) \right]_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int_{x=0}^{1} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \left(\frac{\pi}{2} - 0\right) \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2} \left[\sin^{-1}(x) \right]_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1}(1) - \sin^{-1}(0) \right]$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{4}$$

*2) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx\,dy$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx \right] dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2) - x^2}\, dx \right] dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2-y^2}\right)^2 - x^2}\, dx \right] dy$$

W.K.T, $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$$= \int_{y=0}^{a} \left[\frac{x}{2} \sqrt{(a^2-y^2)-x^2} + \frac{a^2-y^2}{2} \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right) \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_{y=0}^{a} \left[0 + \frac{a^2-y^2}{2} \sin^{-1}\left(\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right) - 0 \right] dy \quad (\because \sin^{-1}(0) = 0),$$

$$= \int_{y=0}^{a} \frac{a^2-y^2}{2} \cdot \sin^{-1}(1)\, dy$$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \int_0^a (a^2-y^2)\, dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a$$

$$= \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} - 0 \right] = \frac{\pi}{4} \left(\frac{3a^3 - a^3}{3} \right)$$

$$= \frac{\pi}{\cancel{4}_2} \left(\frac{\cancel{2}^1 a^3}{3} \right) = \frac{\pi a^3}{6} \; //$$

*3) $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy \right] dx$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2}\, dy \right] dx$$

we have $\displaystyle\int \frac{1}{a^2+y^2}\, dy = \frac{1}{a} \tan^{-1}(y/a)$

$$= \int_{x=0}^{1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{1} \left[\frac{1}{\sqrt{1+x^2}} \left\{ \tan^{-1}\left(\frac{\cancel{\sqrt{1+x^2}}}{\cancel{\sqrt{1+x^2}}} \right) - \tan^{-1}(0) \right\} \right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4} - 0 \right) \right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4} \left[\sinh^{-1}(x) \right]_0^1$$

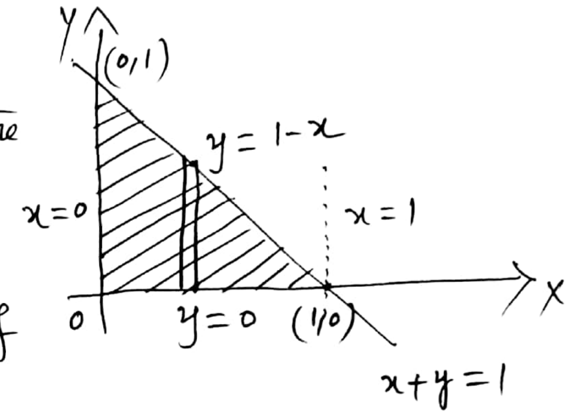
$$= \frac{\pi}{4} \left[\sinh^{-1}(1) - \sinh^{-1}(0) \right]$$

$$= \frac{\pi}{4} \sinh^{-1}(1)$$

$\left(\because \sinh 0 = \frac{e^0 - e^{-0}}{2} = 0 \Rightarrow \sinh^{-1}(0) = 0\right)$

Scanned with CamScanner

*4) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ in the positive quadrant for which $x+y \le 1$



Sol: The line $x+y=1$ intersect the coordinate axes at $(1,0)$ & $(0,1)$. Shaded area is the region of integration.

Now $$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (x^2+y^2)\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{1-y} (x^2+y^2)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x} (x^2+y^2)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1}\left[x^2(1-x)+\frac{1}{3}(1-x)^3-0\right]dx$$

$$= \int_{x=0}^{1}\left[x^2-x^3+\frac{1}{3}(1-x)^3\right]dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{1}{3}\,\frac{(1-x)^4}{-4}\right]_0^1$$

$$= \frac{1}{3}-\frac{1}{4}-0-0+0+\frac{1}{12}(1-0)^4$$

$$= \frac{1}{3}-\frac{1}{4}+\frac{1}{12}$$

$3\,\underline{|\,3,4,12}$
$\quad 1,4,4$

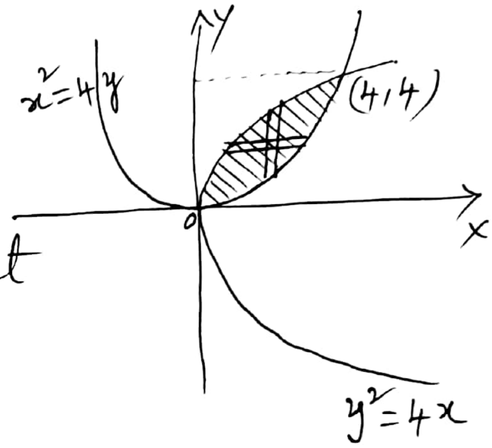
$$= \frac{4-3+1}{12} = \frac{2}{12} = \boxed{\frac{1}{6}}$$

*5) Evaluate i) $\iint_R y\,dx\,dy$

ii) $\iint_R y^2\,dx\,dy$

where R is the region bounded by the parabolas

$y^2 = 4x$ & $x^2 = 4y$



Sol: Given curves $y^2=4x$ & $x^2=4y$
$\Rightarrow y = x^2/4$

solving these two, we get

$$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x$$

$$\Rightarrow \frac{x^4}{16} - 4x = 0$$

$$\Rightarrow x^4 - 64x = 0$$

$$\Rightarrow x(x^3-64) = 0$$

$$x = 0 \text{ (or) } x^3 - 64 = 0$$

$$\Rightarrow x^3 = 64 = 4^3$$

$$\Rightarrow x = 4$$

when $x=0$, $y=0$ & when $x=4$, $y=\sqrt{16}=4$

$\therefore$ Given two curves intersecting at the points $(0,0)$ & $(4,4)$. Shaded area is the region of integration.

i) $$\iint_R y\,dx\,dy = \int_{x=0}^{4}\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dx\,dy = \int_{y=0}^{4}\int_{x=\frac{y^2}{4}}^{2\sqrt{y}} y\,dx\,dy$$

$$= \int_{x=0}^{4}\left(\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\right)dx$$

$$= \int_{x=0}^{4}\left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left(4x - \frac{x^4}{16}\right)dx$$

$$= \frac{1}{2}\left(4\cdot\frac{x^2}{2} - \frac{1}{16}\cdot\frac{x^5}{5}\right)_0^4$$

$$= \frac{1}{2}\left(2\times 16 - \frac{1}{16}\times\frac{4^5}{5} - 0\right)$$

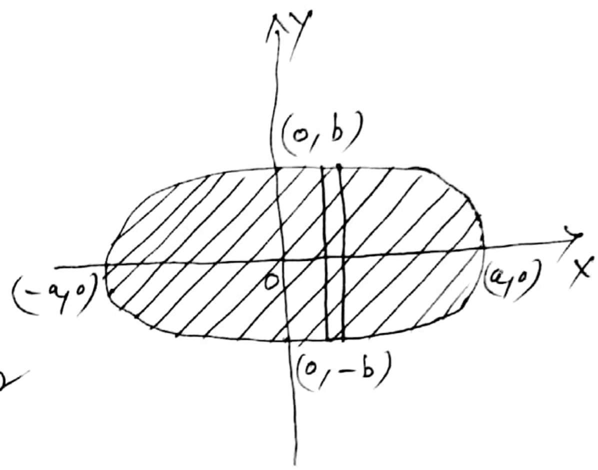
$$= \frac{1}{2}\left(32 - \frac{64}{5}\right)$$

$$= \frac{1}{2}\left(\frac{160-64}{5}\right)$$

$$= \frac{1}{2}\left(\frac{96}{5}\right) = \boxed{\frac{48}{5}}$$

Find $\iint_R (x+y)^2\,dxdy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Sol: Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow (1)$

put $y=0$, we get

$$\frac{x^2}{a^2}=1 \Rightarrow x^2=a^2$$

$$\Rightarrow x=\pm a$$

From (1) $\Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$$\Rightarrow y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$\Rightarrow y=\pm\sqrt{\frac{b^2}{a^2}(a^2-x^2)}$$

$$=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

Shaded area is the region of integration.

$\therefore$ $x \rightarrow -a$ to a

$y \rightarrow -\frac{b}{a}\sqrt{a^2-x^2}$ to $+\frac{b}{a}\sqrt{a^2-x^2}$

Now $\iint_R (x+y)^2\,dxdy = \iint_R (x^2+y^2+2xy)\,dx\,dy$

$$=\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2)\,dxdy+\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}2xy\,dxdy$$

we have $\int_{x=-a}^{a} f(x)\,dx = \begin{cases} 2\int_{x=0}^{a} f(x)\,dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd}\end{cases}$

Scanned with CamScanner

$$= 2\times 2 \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy + 0$$

since x^2+y^2 is an even function

xy is an odd function

$$= 4\int_{x=0}^{a}\left[\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\right]dx$$

$$= 4\int_{x=0}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\int_{x=0}^{a}\left[x^2\cdot\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3-0\right]dx$$

$$= 4\int_0^{a}\left[\frac{b}{a}\,x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{\frac{3}{2}}\right]dx$$

put $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

<u>Limits:</u> when $x=0$, $\theta = \sin^{-1}(0) = 0$

when $x = a$, $\theta = \sin^{-1}(1) = \pi/2$

$$= 4\int_{\theta=0}^{\pi/2}\left[\frac{b}{\cancel{a}}\cdot a^{\cancel{2}}\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}+\frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right]\cdot a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left[ab\sin^2\theta\cdot a\sqrt{1-\sin^2\theta}+\frac{b^3}{3a^3}(a^2)^{\frac{3}{2}}(1-\sin^2\theta)^{3/2}\right]\cdot a\cos\theta\, d\theta$$

we have $\sin^2\theta+\cos^2\theta = 1$

$$= 4\int_0^{\pi/2}\left[a^2b\sin^2\theta\sqrt{\cos^2\theta}+\frac{b^3}{3\cancel{a^3}}\cdot \cancel{a^3}(\cos^{\cancel{2}}\theta)^{3/\cancel{2}}\right]\cdot a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left(a^3b\sin^2\theta\cdot\cos^2\theta+\frac{ab^3}{3}\cdot\cos^4\theta\right)d\theta$$

Scanned with CamScanner

$$= 4\int_0^{\pi/2}\left[a^3 b\,(1-\cos^2\theta)\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right]d\theta$$

$$= 4\int_0^{\pi/2}\left[a^3 b\,(\cos^2\theta - \cos^4\theta) + \frac{ab^3}{3}\cos^4\theta\right]d\theta$$

W.K.T, $\int_0^{\pi/2}\cos^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2}$, if n is even

$$= 4\left[a^3 b\left(\frac{2-1}{2}\cdot\frac{\pi}{2} - \frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right) + \frac{ab^3}{3}\left(\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right)\right]$$

$$= 4\left[a^3 b\,\frac{1}{2}\cdot\frac{\pi}{2}\left(1-\frac{3}{4}\right) + \frac{ab^3}{\cancel{3}}\cdot\frac{\pi}{2}\cdot\frac{\cancel{3}}{4}\cdot\frac{1}{2}\right]$$

$$= 4\left[\frac{\pi a^3 b}{4}\cdot\left(\frac{1}{4}\right) + \frac{\pi a b^3}{16}\right]$$

$$= \frac{\pi a^3 b}{4} + \frac{\pi a b^3}{4}$$

$$= \frac{\pi}{4}ab\,(a^2+b^2)$$

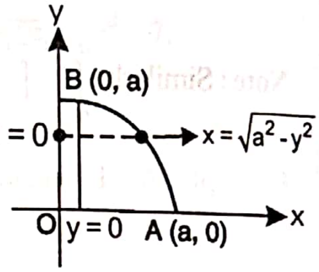
Example 17 : Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

Solution : Consider $\iint_R xy\,dx\,dy = \int\left(\int x\,dx\right)y\,dy$ over R.

$x^2 + y^2 = a^2$ is a circle with centre at (0,0) and radius a units. The given region R of integration is bounded by $OABO$. Let us fix y. For a fixed y, to be in the region, we have to vary x from 0 to $\sqrt{a^2 - y^2}$. However, we will be within the region only if we vary y from 0 to a.

Hence the given integral

$$= \int_{y=0}^{a} \left(\int_{x=0}^{\sqrt{(a^2-y^2)}} x\, dx \right) y\, dy = \int_{y=0}^{a} \left[\frac{x^2}{2} \right]_0^{\sqrt{a^2-y^2}} y\, dy$$

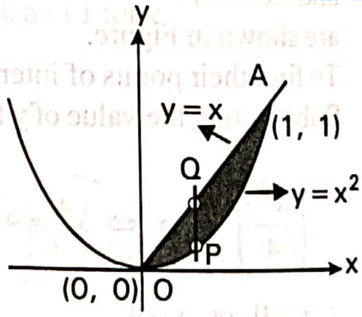


$$= \int_{y=0}^{a} \left[\frac{a^2 - y^2}{2} \right] y\, dy = \int_{y=0}^{a} \left(\frac{a^2 y - y^3}{2} \right) dy$$

$$= \frac{1}{2} \left(\frac{a^2 y^2}{2} - \frac{y^4}{4} \right)_{y=0}^{a} = \frac{1}{2} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \frac{a^4}{8}.$$

Example 18 : Evaluate $\iint xy(x+y)\, dx\, dy$ over the region R bounded by $y = x^2$ and $y = x$.

Solution : $y = x^2$ is a parabola through (0,0) symmertic about y-axis. $y = x$ is a straight line through (0, 0) with slope 1. Let us find their points of intersection.



Solving $y = x^2, y = x$ we get $x^2 = x \Rightarrow x = 0, 1.$ Hence $y = 0, 1.$ The points of intersection of the curves are (0, 0), (1, 1).

Hence the region is as in figure.

Consider $\iint_R xy(x+y) dy\, dx$. For the evaluation of the integral, we have to fix x first. For a fixed x, y varies from x^2 to x. Then to be in the region, we can vary x from 0 to 1. Hence the given integral is equal to

$$\int_{x=0}^{1} \left[\int_{y=x^2}^{x} xy(xy + y)\, dy \right] dx = \int_{x=0}^{1} \left[\int_{y=x^2}^{x} (x^2 y + xy^2)\, dy \right] dx$$

$$= \int_{x=0}^{1} \left(x^2 \frac{y^2}{2} + \frac{xy^3}{3} \right)_{x^2}^{x} dx = \int_{x=0}^{1} \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right] dx$$

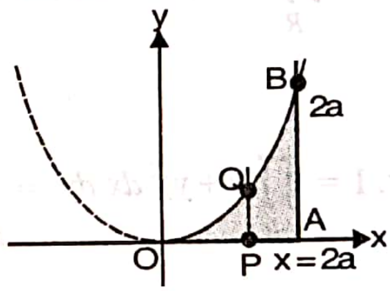
$$= \int_{x=0}^{1} \left(\frac{5}{6} x^4 - \frac{x^6}{2} - \frac{x^7}{3} \right) dx = \left(\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right)_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{9}{168} = \frac{3}{56}.$$

Example 19 : Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x- axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$. **[JNTU (A) June 2010 (Set No. 3)]**

Solution : Let us draw the parabola $x^2 = 4ay$, the line $x = 2a$ and identify the region R of integration. It is as in figure. The integral $\iint_R xy dx\,dy$ is same as $\iint_R xy\,dy\,dx$.

Let us consider a fixed x (Draw a line $x = k$ in the region). Now for this fixed x, y varies from 0 to $x^2/4a$. To be in the region, we have to vary x from 0 to $2a$.



Hence the given integral $= \int_{x=0}^{2a} \int_{y=0}^{x^2/4a} xy\,dy\,dx$

$$= \int_{x=0}^{2a} \left[\int_{y=0}^{x^2/4a} y\,dy \right] x\,dx$$

$$= \int_{x=0}^{2a} \left[\frac{y^2}{2} \right]_{y=0}^{x^2/4a} x\,dx$$

$$= \int_{x=0}^{2a} \frac{x^4}{32a^2} x\,dx = \frac{1}{32a^2} \int_{x=0}^{2a} x^5 dx$$

$$= \frac{1}{32a^2} \left(\frac{x^6}{6} \right)_{x=0}^{2a} = \frac{64\,a^6}{32.\,a^2.6} = \frac{a^4}{3}$$

Example 27 : Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by the line $x+2y=2$, lying in the first quadrant. **[JNTU (A) June 2009 (Set No. 1)]**

Solution : The region R is bounded by the lines $y=0, y=\frac{1}{2}(2-x), x=0$ and $x=2$.

Hence $$\iint_R xy\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{\frac{1}{2}(2-x)} xy\,dy\,dx$$

$$= \int_{x=0}^{2} x.\left(\frac{y^2}{2}\right)_0^{\frac{1}{2}(2-x)} dx = \frac{1}{2}\int_0^2 \frac{x}{4}.(2-x)^2\,dx$$

$$= \frac{1}{8}\int_0^2 (4x-4x^2+x^3)\,dx = \frac{1}{8}\left(4.\frac{x^2}{2}-4\frac{x^3}{3}+\frac{x^4}{4}\right)_0^2$$

$$= \frac{1}{8}\left(8-\frac{32}{3}+4\right) = \frac{1}{8}\left(12-\frac{32}{3}\right) = \frac{1}{24}(4) = \frac{1}{6}$$

Double integrals in polar co-ordinates :—

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$, $r=r_2$.

First we integrate w.r.t. r between the limits $r=r_1$ and $r=r_2$ keeping θ fixed. The resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Geometrically, AB and CD are the curves $r_1 = f_1(\theta)$ & $r_2 = f_2(\theta)$ bounded by the straight lines $\theta=\theta_1$ & $\theta=\theta_2$.

So that ABCD is the region of integration.

$$\therefore \iint_R f(r,\theta)\,dr\,d\theta = \int_{\theta=\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)\,dr\,d\theta = \int_{r=r_1}^{r_2}\int_{\theta=f_1(r)}^{f_2(r)} f(r,\theta)\,d\theta\,dr$$

Ex: 1) Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Sol: $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[\pi - 0\right] = \frac{a^2\pi}{4}$$

2) Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$

Sol: $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r\,dr\right]d\theta$

$$= \int_0^{\pi}\left[\left\{\frac{r^2}{2}\right\}_0^{a(1+\cos\theta)}\right]d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(2\cos^2\theta/2\right)^2 d\theta$$

$$= 2a^2\int_0^{\pi}\cos^4\theta/2\,d\theta$$

put $\theta/2 = t$

$d\theta = 2dt$

$$= 2a^2\int_0^{\pi/2}\cos^4 t\,(2dt)$$

$$= 4a^2\ \frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\pi/2$$

$$= \cancel{4}a^2\cdot\frac{3}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{4}$$

*3) Evaluate $\displaystyle\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol: $\displaystyle\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = \int_0^{\pi/4}\left[\int_0^{a\sin\theta}\frac{r\,dr}{\sqrt{a^2-r^2}}\right]d\theta .$

$$= -\frac{1}{2}\int_0^{\pi/4} 2\left[\sqrt{a^2-r^2}\right]_0^{a\sin\theta} d\theta$$

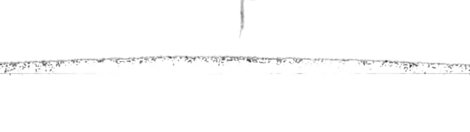
$$= -\int_0^{\pi/4}\left[\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2}\right]d\theta$$

$$= (-a)\int_0^{\pi/4}(\cos\theta-1)\,d\theta$$

$$= (-a)\left[\sin\theta-\theta\right]_0^{\pi/4}$$

$$= (-a)\left[\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right] = a\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right)$$

4) Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r=a(1-\cos\theta)$ above the initial line.



Sol: The cardioid $r = a(1-\cos\theta)$ is symmetrical about the initial line and it passes through the pole O when $\theta = 0$

The region of integration R above the initial line is covered by radial strips whose ends are $r = 0$ and $r = a(1-\cos\theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$

$$\therefore \iint_R r\sin\theta \, dr\, d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1-\cos\theta)} r\sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \, \frac{a^2(1-\cos\theta)^2}{2} \, d\theta$$

$$\cancel{= \frac{a^2}{2}\int_0^{\pi} \sin\theta \cdot (2\sin^2\theta/2)^2 \, d\theta}$$

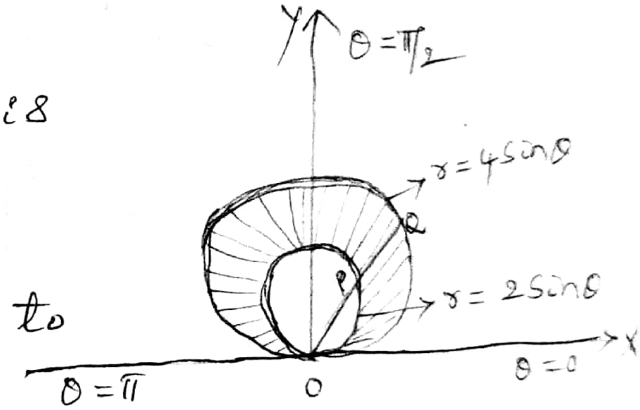
$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta \, (1-\cos\theta)^2 \, d\theta.$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi}$$

$$= \frac{a^2}{6}(2^3 - 0) = \frac{4a^2}{3} //$$

5) Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol: The region of integration R is the shaded area.



Here r varies from $r = 2\sin\theta$ to $r = 4\sin\theta$ and to cover the whole region θ varies from 0 to π.

$$\therefore \iint r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

Scanned with CamScanner

$$= \int_{\theta=0}^{\pi} \frac{1}{4}\left(256 \sin^4\theta - 16 \sin^4\theta\right) d\theta$$

$$\begin{array}{r} 256 \\ 16 \\ \hline 240 \end{array}$$

$$= 60 \int_{\theta=0}^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 120 \left[\frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2} \right]$$

$$= \cancel{120}^{\cancel{30}\,15} \times \frac{3}{\cancel{4}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{2}$$

$$= \frac{45\pi}{2} \; //$$

* Evaluate $\iint r \sin\theta \, dr \, d\theta$ over the Cardioid $r = a(1+\cos\theta)$ above the initial line. Ans: $4a^2/3$

Change of variables in double integral :—

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between the old variables x, y with the new variables u, v of the new coordinate system.

Then $$\iint_R F(x,y)\, dx\, dy = \iint_R F(f,g)\, |J|\, du\, dv \quad \to (1)$$

where $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

which is called the Jacobian of the coordinate transformation.

1) change of variables from cartesian to polar coordinates:

In this case, we have $u = r$ & $v = \theta$

$x = r\cos\theta$, $y = r\sin\theta$

Now $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

Scanned with CamScanner

Hence eqn (1) becomes

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

This corresponds to $\iint_R F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\, r\,dr\,d\theta$

* Ex: 1) evaluate the following integral by transforming into polar coordinates $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$

Sol: $0 \le x \le a$, $0 \le y \le \sqrt{a^2-x^2}$

$x=0,\ x=a$; $y=0,\ y=\sqrt{a^2-x^2}$

$\Rightarrow x^2+y^2=a^2$

put $x = r\cos\theta$, $y = r\sin\theta$

we have $x^2+y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$

The limits for r: 0 to a and for θ: 0 to $\pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^a (r\sin\theta)\cdot r\,(r\,dr\,d\theta)$$

$$= \int_0^{\pi/2}\int_0^a r^3 \sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^a r^3\,dr\right]\sin\theta\,d\theta$$

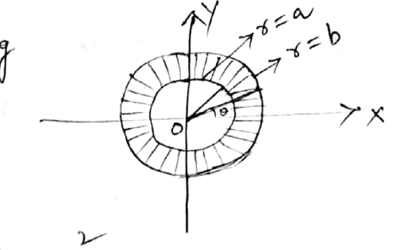
$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \sin\theta\,d\theta$$

$$= \frac{a^4}{4}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{4}(0+1) = \frac{a^4}{4}$$

2) By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annulus region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$

Sol: change to polar coordinates by putting



$x = r\cos\theta\,,\quad y = r\sin\theta$

$\therefore\ dx\,dy = r\,dr\,d\theta$

Now $x^2+y^2 = a^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$\Rightarrow r^2 = a^2 \Rightarrow r = a$

$x^2+y^2 = b^2 \Rightarrow r^2 = b^2 \Rightarrow r = b$

and θ varies from 0 to 2π

$$\therefore \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\cos^2\theta \cdot r^2\sin^2\theta}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\sin^2\theta\left[\frac{r^4}{4}\right]_a^b d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,(b^4-a^4)\,d\theta$$

$$= \frac{b^4-a^4}{4}\int_0^{2\pi}\frac{\sin^2 2\theta}{4}\,d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi}\left(\frac{1-\cos4\theta}{2}\right)d\theta$$

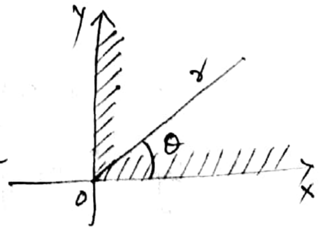
$$= \frac{b^4-a^4}{32}\left[\theta - \frac{\sin4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{b^4-a^4}{32}\left[2\pi - 0\right]$$

$$= \frac{\pi(b^4-a^4)}{16}$$

3) Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates.

Sol: Given $x \to 0$ to ∞, $y \to 0$ to ∞.



Therefore the region of integration is the first quadrant of the xy-plane.

changing to polar coordinates, by putting

$x = r\cos\theta\,,\quad y = r\sin\theta$

$\Rightarrow dx\,dy = r\,dr\,d\theta$

In the region of integration r varies from 0 to ∞ and θ varies from 0 to $\pi/2$.

$$\therefore \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

put $r^2 = t$

$2r\,dr = dt$

$$= \int_0^{\pi/2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[-e^{-t}\right]_0^{\infty} d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2}(0+1)\,d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4} \,//$$

Change of order of Integration :-

The change of order of integration is nothing but changes the limits of integration.

For instance, to interchanged the order of integration is

$$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx = \int_{y=a}^{b}\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$

we first sketch the region of integration followed by taking up of horizontal strip (instead of vertical strip). Thus the new limits are

$y = a$ to b and $x = f_1(y)$ to $f_2(y)$ after changing order of integration.

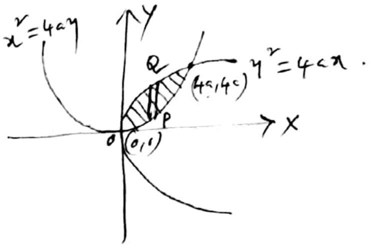
Ex:1) change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol: Given $0 \le x \le 4a$ & $\frac{x^2}{4a} \le y \le 2\sqrt{ax}$

$$\frac{x^2}{4a} = y \quad , \quad y = 2\sqrt{ax}$$

$$\Rightarrow x^2 = 4ay \quad , \quad y^2 = 4ax .$$

The region of integration is the shaded region in figure.



changing the order of integration, we must fix y first.

For a fixed y, x varies from $y^2/4a$ to $2\sqrt{ay}$ and then y varies from 0 to 4a.

In this case the vertical strip slides as a horizontal strip.

Hence the integral is equal to

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right] dy$$

$$= \int_{y=0}^{4a} \left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_{y=0}^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left[\frac{2\sqrt{a}\,y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16\times\cancel{4}\times a^3}{\cancel{12}_3\,a}$$

$$= \frac{32a^2 - 16a^2}{3} = \frac{16a^2}{3}$$

Scanned with CamScanner

*2) change the order of integration and evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$$

Sol: Given limits are

$0 \le x \le 1$ & $x^2 \le y \le 2-x$

x varies from 0 to 1. $y = x^2 \to (1)$, $y = 2-x$

$\Rightarrow x + y = 2 \to (2)$

solving (1) & (2), we get

$x + x^2 = 2$

$\Rightarrow x^2 + x - 2 = 0$ $2x - 1 = -2$

$x^2 + 2x - x - 2 = 0$ $2 - 1 = 1$

$x(x+2) - 1(x+2) = 0$

$(x-1)(x+2) = 0$

$x = 1$ or $x = -2$

If $x = 1$, $y = 1$

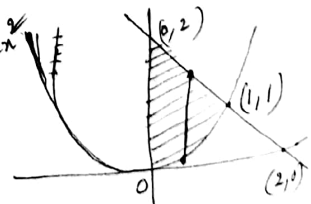
$x = -2$, $y = 4$

Hence the points of intersection of the curves are $(1,1)$ & $(-2,4)$.

The line $x + y = 2$ passes through $(0,2)$, $(2,0)$.

we shall draw the curves $y = x^2$ & $y = 2-x$

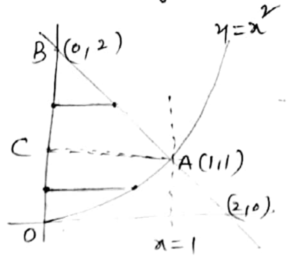
The shaded region in the figure is the region of integration.



Suppose we change the order of integration.

In this case the vertical strip slides as a horizontal strip.

In the changed order we have to take two horizontal strips since during sliding one edge of the strip remains on $x = 0$ but the other edge of the strip doesn't remain on a single curve.



$\therefore$ The region is

Area OAB = Area OAC + Area CAB

We shall fix y first.

For the region OACO, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1.

For the region CABC, x varies from 0 to $2-y$ and y varies from 1 to 2

Hence $$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OACO} xy\,dx\,dy + \iint_{CABC} xy\,dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy + \int_{y=1}^{2} \left[\frac{x^2 y}{2}\right]_0^{2-y} dy$$

$$= \int_{y=0}^{1} \left(\frac{y^2}{2}\right) dy + \int_{y=1}^{2} \frac{(2-y)^2 y}{2}\,dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2\,dy + \frac{1}{2}\int_{y=1}^{2} (4y + y^3 - 4y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[2y^2 + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{120 - 112 - 3}{12}\right]$$

3
28×4
112

$$= \frac{1}{6} + \frac{1}{2} \cdot \frac{5}{12} = \frac{4+5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}$$

Scanned with CamScanner

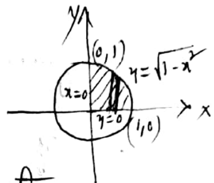
*3) By changing the order of integration, evaluate
$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$$

Sol: Given $0 \le x \le 1$ & $0 \le y \le \sqrt{1-x^2}$

$y = 0,\ y = \sqrt{1-x^2}$

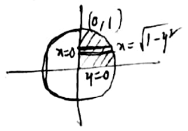
$\Rightarrow x^2 + y^2 = 1$

using these limits we can draw the region of integration.



For changing the order of integration, the vertical strip slides as a horizontal strip.

we shall fix y first, x varies from 0 to $\sqrt{1-y^2}$ and y varies from 0 to 1.



$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\,dy$$

$$= \int_{y=0}^{1} \left[xy^2\right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \int_{y=0}^{1} \left(\sqrt{1-y^2}\; y^2\right) dy$$

put $y = \sin\theta$

$dy = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \cos\theta \cdot \sin^2\theta\,(\cos\theta\, d\theta)$$

$$= \int_0^{\pi/2} \cos^2\theta\, \sin^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\,(1-\cos^2\theta)\, d\theta.$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{4}\left(1 - \frac{3}{4}\right) = \frac{\pi}{16}$$

4) change the order of integration and evaluate $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy$

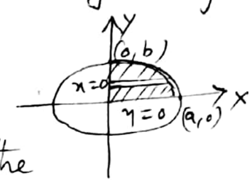
Sol: Given $0 \le y \le b$ & $0 \le x \le \frac{a}{b}\sqrt{b^2-y^2}$

$y=0$ & $y=b$, $x=0$ & $x = \frac{a}{b}\sqrt{b^2-y^2}$

$\Rightarrow b^2x^2 = a^2b^2 - a^2y^2$

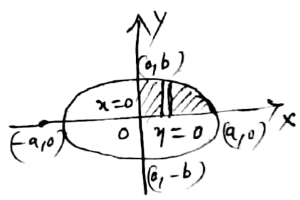
$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

using these limits we can draw the region of integration.



on changing the order of integration, the horizontal strip slides as a vertical strip.

we shall fix x first, y varies from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$ and x varies from 0 to a.



$$\therefore \int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_{x=0}^{a}\left[\frac{xy^2}{2}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a}\left[\frac{x}{2}(a^2-x^2)\frac{b^2}{a^2}\right] dx$$

$$= \frac{b^2}{2a^2}\int_{x=0}^{a}(a^2x - x^3)\,dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{4}\right) = \frac{a^2b^2}{8} \;//$$

5) Evaluate the integral by changing the order of integration $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ **Ans:** $\frac{241}{60}$

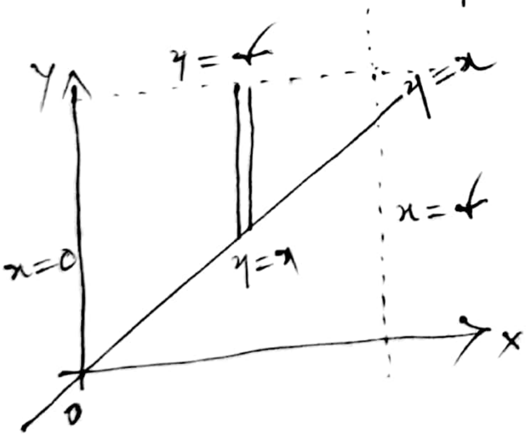
Evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dx\,dy$, by changing the order of integration

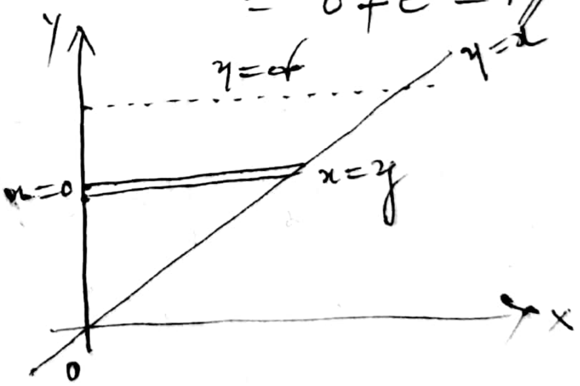
$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}\,[x]_0^y\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}(y-0)\,dy = \left[-e^{-y}\right]_0^{\infty}$$

$$= 0 + e^0 = 1$$





Triple integrals:

Let $f(x,y,z)$ be a function defined over a 3-dimensional finite region V.

The triple integral of f over the volume V and is represented by $\iiint_V f\,dv$ or $\int_V f\,dv$

If the region V is bounded by the surfaces

$x=x_1$, $x=x_2$; $y=y_1$, $y=y_2$; $z=z_1$, $z=z_2$ then

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$$

Evaluation of Triple integrals:—

Case i): If x_1, x_2; y_1, y_2; z_1, z_2 are all constants, then the order of integration is not important provided the limits of integration are changed accordingly.

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx = \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

Case ii): If z_1, z_2 are functions of x & y and y_1, y_2 are functions of x while x_1 & x_2 are constants. Then the integration must be performed first w.r.t z then w.r.t. y and finally w.r.t. x.

$$\iiint_V f(x,y,z)\,dv = \int_{x=a}^{b} \boxed{\int_{y=f_1(x)}^{y=f_2(x)} \boxed{\int_{z=f_2(x,y)}^{z=g_2(x,y)} f(x,y,z)\,dz}\,dy}\,dx$$

Ex: 1) Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2z \, dz\, dy\, dx$

Sol: $\int_0^2 \int_1^3 \int_1^2 xy^2z \, dz\, dy\, dx = \int_0^2 \int_1^3 \left[\frac{xy^2z^2}{2}\right]_1^2 dy\, dx$

$= \int_0^2 \int_1^3 \left(\frac{4xy^2}{2} - \frac{xy^2}{2}\right) dy\, dx$

$= \int_0^2 \int_1^3 \frac{3xy^2}{2} \, dy\, dx$

$= \frac{3}{2} \int_0^2 \left[\frac{xy^3}{3}\right]_1^3 dx$

$= \frac{1}{2} \int_0^2 (27x - x)\, dx$

$= \frac{1}{2} \left[\frac{26x^2}{2}\right]_0^2$

$= \frac{13}{2}(4) = 26$

*2) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dx\, dy\, dz$

Sol: $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dx\, dy\, dz = \int_0^a \int_0^x e^{x+y} \left[\int_0^{x+y} e^z \, dz\right] dx\, dy$

$= \int_0^a \int_0^x e^{x+y} \left[e^z\right]_0^{x+y} dx\, dy$

$= \int_0^a \int_0^x e^{x+y} \left(e^{x+y} - 1\right) dx\, dy$

$= \int_0^a \int_0^x \left[e^{2(x+y)} - e^{x+y}\right] dx\, dy$

$= \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx.$

$= \int_0^a \left\{\left(\frac{e^{4x}}{2} - e^{2x}\right) - \left(\frac{e^{2x}}{2} - e^{x}\right)\right\} dx$

$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x}\right]_0^a$

$= \left(\frac{e^{4a}}{8} - \frac{e^{2a}}{2} - \frac{e^{2a}}{4} + e^{a}\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right)$

$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}$

*3) Evaluate the triple integral $\int_0^1 \int_y^1 \int_0^{1-x} x \, dz \, dx \, dy$

<u>Sol:</u> $\int_0^1 \int_y^1 \int_0^{1-x} x \, dz \, dx \, dy = \int_0^1 \int_y^1 x \, [z]_0^{1-x} \, dx \, dy$

$$= \int_0^1 \int_y^1 (x - x^2) \, dx \, dy.$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_y^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{3} - \frac{y^2}{2} + \frac{y^3}{3} \right) dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3} \right) dy$$

$$= \left[\frac{y}{6} - \frac{y^3}{6} + \frac{y^4}{12} \right]_0^1$$

$$= \cancel{\frac{1}{6}} - \cancel{\frac{1}{6}} + \frac{1}{12} = \frac{1}{12} \,//$$

*4) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$

<u>Sol:</u> $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy \, dx$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy (1 - x^2 - y^2) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3 y - xy^3) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{xy^2}{2} - \frac{x^3 y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} \right] dx$$

$$= \frac{1}{8} \int_0^1 \left(2x - 2x^3 - 2x^3 + 2x^5 - (x - x^5 + 2x^3) \right) dx$$

$$= \frac{1}{8} \int_0^1 (x^5 - 2x^3 + x) \, dx$$

Scanned with CamScanner

$$= \frac{1}{8}\left[\frac{x^6}{6} - \frac{2x^4}{4} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{8}\left(\frac{1}{6} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}\right) = \frac{1}{48}$$

*5) Evaluate $\int_0^{\pi/2}\int_x^{\pi/2}\int_0^{xy} \cos(z/x)\, dz\, dy\, dx$

Sol: $\int_0^{\pi/2}\int_x^{\pi/2}\int_0^{xy} \cos(z/x)\, dz\, dy\, dx = \int_0^{\pi/2}\int_x^{\pi/2}\left[\frac{\sin(z/x)}{1/x}\right]_0^{xy} dy\, dx$

$$= \int_0^{\pi/2}\int_x^{\pi/2} (x \sin y)\, dy\, dx$$

$$= \int_0^{\pi/2} \left[-x\cos y\right]_x^{\pi/2} dx$$

$$= \int_0^{\pi/2} (0 + x\cos x)\, dx$$

$$= \left[x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx$$

$$= \frac{\pi}{2} + \left[\cos x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + 0 - 1$$

$$= \frac{\pi}{2} - 1$$

6) Evaluate $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

Sol: $\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z} (x+y+z)\, dx\, dy\, dz = \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy + \frac{y^2}{2} + yz\right]_{x-z}^{x+z} dx\, dz$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} \left\{\left[x(x+z) + \frac{(x+z)^2}{2} + (x+z)z\right] - \left[x(x-z) + \frac{(x-z)^2}{2} + (x-z)z\right]\right\} dx\, dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} \left\{\cancel{x^2} + xz + \frac{1}{2}(\cancel{x^2} + \cancel{z^2} + 2xz) + xz + z^2 - \cancel{x^2} + \cancel{xz} - \frac{1}{2}(\cancel{x^2} + \cancel{z^2} - 2xz) - \cancel{xz} + z^2\right\} dx\, dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} (4xz + 2z^2)\, dx\, dz$$

$$= \int_{z=-1}^{1} \left(2x^2 z + 2xz^2\right)_0^z dz$$

$$= \int_{z=-1}^{1} (2z^3 + 2z^3)\, dz$$

$$= \left[\frac{4z^4}{4}\right]_{-1}^{1}$$

$$= 1 - 1 = 0$$

7) Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{(a^2-r^2)/a} r\,dr\,d\theta\,dz$

Sol: $\displaystyle\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\int_{z=0}^{(a^2-r^2)/a} r\,dr\,d\theta\,dz = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r\,[z]_0^{\frac{a^2-r^2}{a}}\,dr\,d\theta$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r\left(\frac{a^2-r^2}{a}\right) dr\,d\theta$$

$$= \frac{1}{a}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} (a^2 r - r^3)\, dr\,d\theta$$

$$= \frac{1}{a}\int_{\theta=0}^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_{\theta=0}^{\pi/2}\left(\frac{a^4\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right) d\theta$$

$$= \frac{a^3}{4}\int_{\theta=0}^{\pi/2} (2\sin^2\theta - \sin^4\theta)\, d\theta$$

$$= \frac{a^3}{4}\left[\cancel{2}\cdot\frac{2-1}{\cancel{2}}\cdot\frac{\pi}{2} - \frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{a^3}{4}\left(\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{\pi a^3}{8}\left(1 - \frac{3}{8}\right)$$

$$= \frac{\pi a^3}{8}\cdot\left(\frac{5}{8}\right) = \frac{5\pi a^3}{64}$$

8) Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

Sol: $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - (z)^2}}\, dz\right] dy\,dx$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\right) - 0\right] dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \left[y\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$$

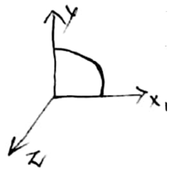
$$= \frac{\pi}{2}\left[0 + \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{\pi^2}{8}$$

* 9) Evaluate $\iiint xyz\, dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2 = a^2$

Sol: Given sphere is $x^2+y^2+z^2 = a^2 \Rightarrow z = \sqrt{a^2-x^2-y^2}$

The projection of the sphere on the xy-plane is the circle $x^2+y^2 = a^2$

So, this circle is covered as y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a.



$$\therefore \iiint xyz\, dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\, dx\,dy\,dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dx\,dy$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2)\, dx\,dy$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\, dx\,dy$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_{y=0}^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[\frac{a^2x}{2}(a^2-x^2) - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[\frac{a^4x - a^2x^3}{2} - \frac{a^2x^3 - x^5}{2} - \frac{a^4x + x^5 - 2a^2x^3}{4}\right]dx$$

$$= \frac{1}{8}\int_{x=0}^{a}\left[2a^4x - 2a^2x^3 - \cancel{2a^2x^3} + 2x^5 - a^4x - x^5 + \cancel{2a^2x^3}\right]dx$$

$$= \frac{1}{8}\int_{x=0}^{a}\left[a^4x - 2a^2x^3 + x^5\right]dx$$

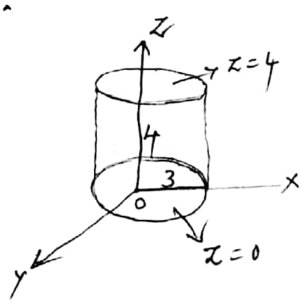
$$= \frac{1}{8}\left[\frac{a^4x^2}{2} - \frac{2a^2x^4}{4} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[\cancel{\frac{a^6}{2}} - \cancel{\frac{2a^6}{4_2}} + \frac{a^6}{6}\right]$$

$$= \frac{a^6}{48} \,//$$

*

10) Find the volume of the solid enclosed by $x^2+y^2=9$, $z=0$ and $z=4$ by using triple integral.



Sol: Given $\underbrace{x^2+y^2=9}_{(1)}$, $z=0$ & $z=4$

$\Rightarrow y = \pm\sqrt{9-x^2}$

y varies from $-\sqrt{9-x^2}$ to $+\sqrt{9-x^2}$

put $y=0$ in (1) $\Rightarrow x = \pm 3$

x varies from -3 to $+3$.

$$\therefore \text{volume of the solid} = \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=0}^{4} dz\,dy\,dx$$

$$= \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\,dy\,dx$$

$$= 4\int_{x=-3}^{3}\left(\sqrt{9-x^2}+\sqrt{9-x^2}\right)dx$$

$$= 8\int_{x=-3}^{3}\sqrt{3^2-x^2}\,dx$$

$$= 8\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}(x/3)\right]_{-3}^{3}$$

$$= 8\left[0 + \frac{9}{2}\left(\frac{\pi}{2}\right) - 0 - \frac{9}{2}\left(-\frac{\pi}{2}\right)\right]$$

$$= 8\left(\frac{18\pi}{4}\right) = 36\pi$$

11) Evaluate $\int_{0}^{\pi/4}\int_{0}^{\log\sec z}\int_{-\infty}^{2y} e^{x}\,dx\,dy\,dz$

Sol: $$\int_{z=0}^{\pi/4}\int_{y=0}^{\log\sec z}\int_{x=-\infty}^{2y} e^{x}\,dx\,dy\,dz = \int_{z=0}^{\pi/4}\int_{y=0}^{\log\sec z}\left[e^{x}\right]_{-\infty}^{2y} dy\,dz$$

$$= \int_{z=0}^{\pi/4}\int_{y=0}^{\log\sec z}\left(e^{2y}-0\right) dy\,dz$$

$$= \int_{z=0}^{\pi/4}\left[\frac{e^{2y}}{2}\right]_{0}^{\log\sec z} dz$$

$$= \frac{1}{2}\int_{z=0}^{\pi/4}\left(e^{2\log\sec z}-1\right) dz$$

$$= \frac{1}{2}\int_{z=0}^{\pi/4}\left(\sec^2 z-1\right) dz$$

$$= \frac{1}{2}\left[\tan z - z\right]_{0}^{\pi/4}$$

$$= \frac{1}{2}\left(1-\frac{\pi}{4}\right) = \frac{4-\pi}{8}$$

12) Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ taken over the volume bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$ and $z=0$, $z=1$

Sol: $$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(x+y+z)\,dx\,dy\,dz = \int_{x=0}^{1}\int_{y=0}^{1}\left[xz+yz+\frac{z^2}{2}\right]_{0}^{1} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left(x+y+\frac{1}{2}\right) dy\,dx$$

$$= \int_{x=0}^{1}\left(xy+\frac{y^2}{2}+\frac{y}{2}\right)_{0}^{1} dx$$

$$= \int_{x=0}^{1}\left(x+\frac{1}{2}+\frac{1}{2}\right) dx$$

$$= \left(\frac{x^2}{2}+\frac{x}{2}+\frac{x}{2}\right)_{0}^{1}$$

$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 3/2$$

13) Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$, where the domain V is bounded by the plane $x+y+z=a$ $(a>0)$ and the coordinate planes. Ans: $\frac{a^4}{8}$

Sol: $$\iiint (x+y+z)\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)\,dx\,dy\,dz$$

Scanned with CamScanner

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \left[(x+y) z + \frac{z^2}{2} \right]_0^{a-x-y} dx\,dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \left[(x+y)(a-x-y) + \frac{(a-x-y)^2}{2} \right] dx\,dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \left[x(a-x-y) + (a-x)y - y^2 + \frac{(a-x-y)^2}{2} \right] dx\,dy$$

$$= \int_{x=0}^{a} \left[\frac{x(a-x-y)^2}{-2} + (a-x)\frac{y^2}{2} - \frac{y^3}{3} + \frac{(a-x-y)^3}{-6} \right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} \left[0 + \frac{(a-x)^3}{2} - \frac{(a-x)^3}{3} + 0 + \frac{x(a-x)^2}{2} + \frac{(a-x)^3}{6} \right] dx$$

$$= \left[\frac{(a-x)^4}{-8} + \frac{(a-x)^4}{12} + \frac{(a-x)^4}{-24} + \frac{1}{2}\left(\frac{a^2x^2}{2} + \frac{x^4}{4} - \frac{2ax^3}{3} \right) \right]_0^a$$

$$= \frac{1}{2}\left(\frac{a^4}{2} + \frac{a^4}{4} - \frac{2a^4}{3} \right) + \frac{a^4}{8} - \frac{a^4}{12} + \frac{a^4}{24}$$

$$= \frac{3a^4}{8} - \frac{a^4}{3} + \frac{a^4}{8} - \frac{a^4}{12} + \frac{a^4}{24}$$

$$= \frac{(9-8+3-2+1)a^4}{24} = \frac{(13-10)a^4}{24} = \frac{a^4}{8}$$

* 14) Evaluate $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2+z^2)\, dx\,dy\,dz$ **Ans:** $\frac{8abc}{3}(a^2+b^2+c^2)$

15) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz$ **Ans:** $1/6$

2014 1) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy = \frac{1}{4}\left(e^2 - 8e + 13\right)$

2) Evaluate the triple integral $\iiint xy^2z \, dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Sol: The eqns of the sphere in the first octant are

$$x^2+y^2+z^2=a^2, \quad x \geq 0, y \geq 0, z \geq 0$$

Ans: $\frac{a^7}{105}$.

$$= \frac{8c}{b}\int_{x=0}^{a} \left[0 + \frac{p^2}{2}\sin^{-1}(1) - 0\right] dx$$

$$= \frac{4c}{\cancel{b}}\int_{x=0}^{a} \frac{\pi}{2} \cdot b^{\cancel{2}}\left(1-\frac{x^2}{a^2}\right) dx$$

$$= 2\pi bc \int_{x=0}^{a} \left(1-\frac{x^2}{a^2}\right) dx$$

$$= 2\pi bc \left(x - \frac{x^3}{3a^2}\right)_0^a$$

$$= 2\pi bc \left(a - \frac{a^3}{3a^2} - 0\right)$$

$$= 2\pi bc \, \frac{2a^{\cancel{3}}}{3\cancel{a^2}} = \frac{4\pi}{3} abc //$$

3) Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dx\,dy\,dz = \frac{8}{3}\log 2 - \frac{19}{9}$

Scanned with CamScanner

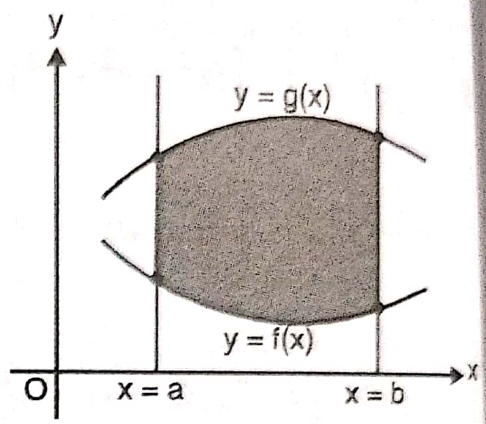
Applications of Multiple Integrals:

8.7 AREA ENCLOSED BY A PLANE CURVE

Consider the area enclosed by the curves $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ in the xy plane.

The area of the region R bounded by the given curves is given by

$$\iint_R dx\, dy \quad \text{or} \quad \iint_R dy\, dx = \int_{x=a}^{b} \int_{y=f(x)}^{g(x)} dy\, dx$$



If the region is represented through polar coordinates, then the area is given by $\iint_R r\, dr\, d\theta$.

Scanned with CamScanner

SOLVED EXAMPLES

Example 1 : Find the area enclosed by the parabolas $x^2 = y$ and $y^2 = x$.

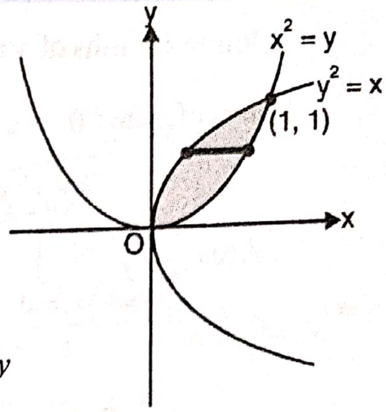
Solution : Given curvers are $x^2 = y$ ... (1) and $y^2 = x$ ... (2)

To find their points of intersection, solve (1) and (2).

Squaring on both sides of (1), $x^4 = y^2 = x$, using (2)

$\Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1$

$\therefore$ Given parabolas intersect at the points O(0, 0) and P(1, 1).



$$\therefore \text{The required area} = \iint_R dx\,dy = \int_{y=0}^{1}\left(\int_{x=y^2}^{\sqrt{y}} dx\right) dy = \int_{y-0}^{1}\left(\sqrt{y} - y^2\right) dy$$

$$= \left(\frac{2}{3}y^{3/2} - \frac{y^3}{3}\right)_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq.units.}$$

Example 2 : Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Solution : Given curves are $y^2 = 4ax$ ... (1) and $x^2 = 4ay$ ... (2)

To find their points of intersection, solve (1) and (2).

Squaring (2), we get

$x^4 = 16a^2y^2 = 16a^2(4ax)$, using (1)

$\therefore x^4 = 64a^3x \Rightarrow x[x^3 - (4a)^3] = 0 \Rightarrow x = 0, 4a$

When $x = 0, y = 0$ and when $x = 4a, y = 4a$.

Hence the two parabolas intersect at O(0, 0) and P(4a, 4a).

$\therefore$ Area, $A = \iint_R dx\,dy$

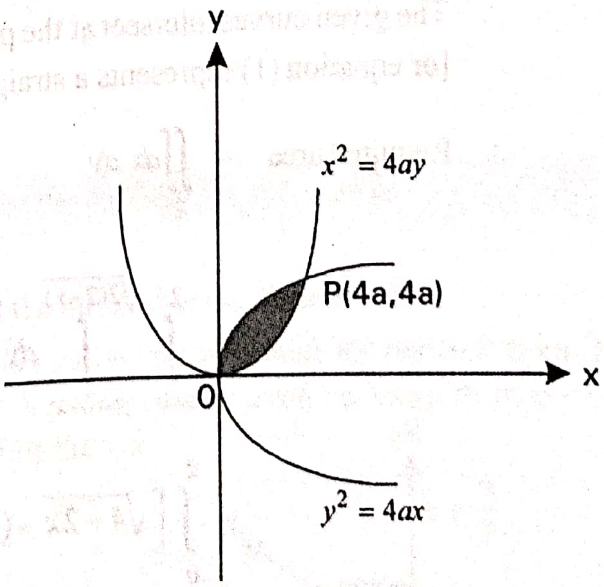
The region R can be covered by varying x from the upper curve $x = y^2/4a$ to the lower curve $x = 2\sqrt{ay}$, while y varies from 0 to $4a$.

$$\text{Thus A} = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$$

$$= \int_{y=0}^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{a}\sqrt{y}} dy$$

$$= \int_0^{4a}\left(2\sqrt{a}\sqrt{y} - \frac{y^2}{4a}\right) dy$$

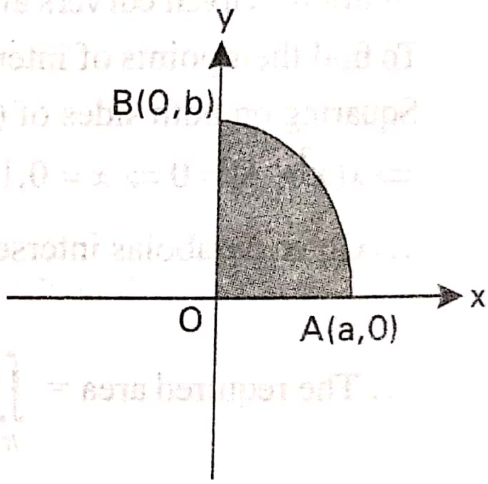
$$= \left[\frac{2\sqrt{a}y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{16a^2}{3}$$



Example 3 : Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution : Limits of y are : $0 \to b\cdot\sqrt{1-\frac{x^2}{a^2}}=\frac{b}{a}\sqrt{a^2-x^2}$

Limits of x are : $0 \to a$



$$\therefore \text{Area} = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\, dx = \int_{x=0}^{a}[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx = \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)\right] = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4} \text{ sq.units.}$$

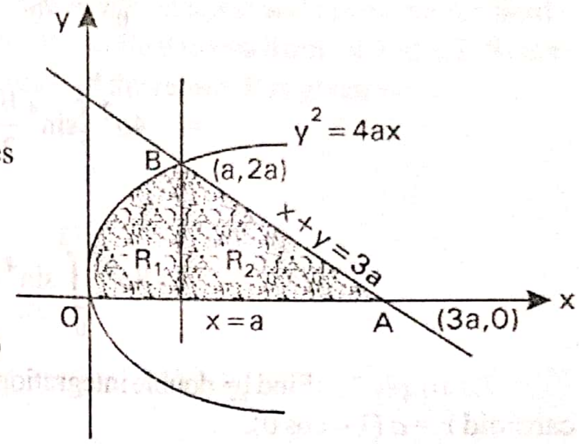
Example 5 : Using double integration determine the area of the region bounded by the curves $y^2=4ax, x+y=3a$ and $y=0$.

Solution : Given curves are

$y^2=4ax$ ...(1)

$x+y=3a$ ...(2)

$y=0$ ...(3)



To find the points of intersection of the two curves $y^2=4ax$ and $x+y=3a$, solve (1) and (2).

Substituting the value of y from eqn. (2) in eqn. (1), we get

$$(3a-x)^2=4ax$$

i.e. $9a^2+x^2-6ax=4ax$ or $x^2-10ax+9a^2=0$

or $(x-a)(x-9a)=0$

$\therefore x=a, x=9a$

Substituting $x=a$ in (2), we get

$$y=2a$$

$\therefore$ The curves (1) and (2) intersect at the point B $(a, 2a)$.

Similarly the curves (2) and (3) meet at the point A $(3a, 0)$.

Hence required area, A $= \iint_R dx\,dy = \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{4ax}} dy\,dx + \int_{x=a}^{3a}\int_{y=0}^{3a-x} dy\,dx$$

$$= \int_{x=0}^{a} (y)_0^{\sqrt{4ax}}\,dx + \int_a^{3a} (y)_0^{3a-x}\,dx = 2\sqrt{a}\int_0^a \sqrt{x}\,dx + \int_a^{3a}(3a-x)dx$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\left(x^{3/2}\right)_0^a + \left[\frac{(3a-x)^2}{-2}\right]_a^{3a} = \frac{4\sqrt{a}}{3}(a^{3/2}) - \frac{1}{2}[0-4a^2]$$

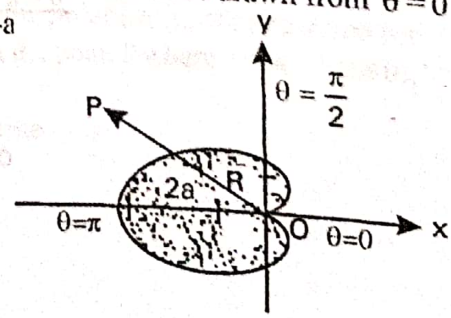
$$\therefore A = \frac{4a^2}{3}+2a^2 = \frac{10a^2}{3} \text{ sq. units.}$$

Example 6 : Using double integral, find the area of the cardioid $r=a(1-\cos\theta)$.

Solution :

The cardioid $r=a(1-\cos\theta)$ is symmetrical about the initial line *i.e.* about $\theta=0$.

To determine the polar limits of integration, imagine a radius vector through the region R from O which emerges at the point P where $r=a(1-\cos\theta)$. Such radii vectors can be drawn from $\theta=0$ to $\theta=\pi$. The region R is made into two equal parts by the x-a



Hence required area $= 2\iint_R r\,dr\,d\theta$

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta = a^2\int_0^{\pi}(1-\cos\theta)^2 d\theta$$

$$= 4a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\,d\theta = 4a^2\int_0^{\pi/2}\sin^4\phi\cdot 2d\phi \quad \left[\text{Putting } \frac{\theta}{2}=\phi\right]$$

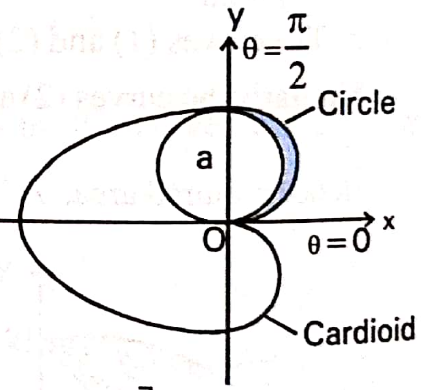
$$= 8a^2\int_0^{\pi/2}\sin^4\phi\,d\phi = 8a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

Example 7 : Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Solution : Here the outer curve is the circle $r = a\sin\theta$ and the inner curve is the cardioid $r = a(1-\cos\theta)$ in the shaded region.

The two curves meet at $\theta = 0$ and $\theta = \pi/2$.

Hence the required area



$$= \int_{\theta=0}^{\pi/2}\int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi/2}[\sin^2\theta-(1-\cos\theta)^2]\,d\theta = \frac{a^2}{2}\int_0^{\pi/2}\left[-1+(\sin^2\theta-\cos^2\theta)+2\cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[-1-\cos 2\theta+2\cos\theta]\,d\theta = \frac{a^2}{2}\left[-\theta-\frac{\sin 2\theta}{2}+2\sin\theta\right]_0^{\pi/2} = \frac{a^2}{2}\left(-\frac{\pi}{2}+2\right) = a^2\left(1-\frac{\pi}{4}\right).$$

8.9 VOLUME AS A TRIPLE INTEGRAL

Suppose a three dimensional solid is cut into elemental rectangular parallelopipeds by drawing planes parallel to the coordinate planes. The volume of an elemental parallelopiped δV is $\delta x\ \delta y\ \delta z$.

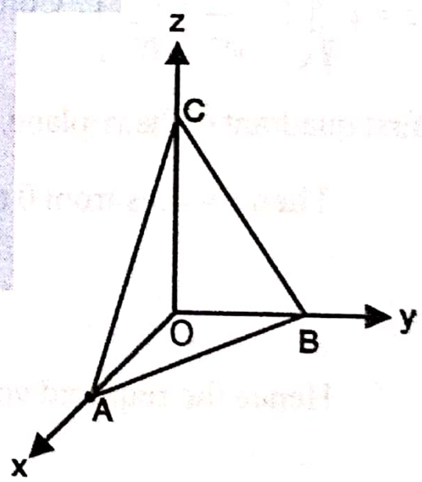
Hence the total volume of the solid is $\iiint\limits_V dv = \iiint\limits_V dx\ dy\ dz$ where the integration is carried over the entire volume.

SOLVED EXAMPLES

Example 1 : Find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

(or) Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes by triple integral.

[JNTU (H) June 2009 (Set No.3)]

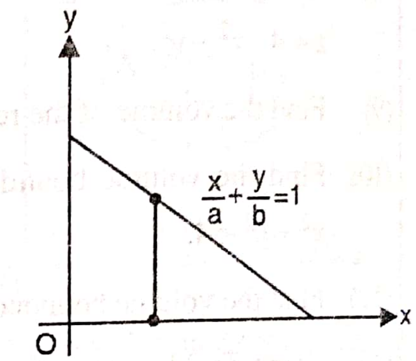


Solution : The required volume $= \iiint\limits_V dxdydz$

On the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,\ z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Hence for a fixed (x,y) on the xy plane within the ΔOAB, z varies from 0 to $c\left(1-\dfrac{x}{a}-\dfrac{y}{b}\right)$ within the solid. Then for a fixed x within the ΔOAB, y varies from 0 to $b\left(1-\dfrac{x}{a}\right)$. Then x varies from 0 to a.

$\therefore$ The required volume of the tetrahedron



$$= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

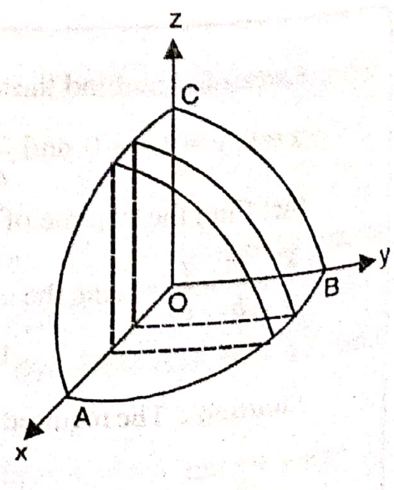
$$= \int_{x=0}^{a}\left[c\,y\left(1-\frac{x}{a}\right)-\frac{c}{b}\frac{y^2}{2}\right]_{y=0}^{b\left(1-\frac{x}{a}\right)}dx$$

$$= \int_{x=0}^{a}\left[cb\left(1-\frac{x}{a}\right)^2-\frac{cb}{2}\left(1-\frac{x}{a}\right)^2\right]dx = \left[\frac{cb}{2}\left(1-\frac{x}{a}\right)^3\cdot\frac{1}{3\left(-\frac{1}{a}\right)}\right]_{x=0}^{a} = \frac{abc}{6}\,c.\text{units}.$$

Example 2 : Find the volume of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. **[JNTU 1998]**

(or) Find the volume of the greatest rectangular parallelopied that can be inscribed in an ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. **[JNTU 1999, (H) June, Dec. 2010, 2011 (Set No. 4)]**

Solution : The solid figure $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ is cut into 8 equal pieces by the three coordinate planes. Hence the volume of the solid is equal to 8 times the volume of the solid bounded by $x=0$, $y=0$, $z=0$ and the surface $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.



For a fixed point (x, y) on the xy plane, z varies from $z=0$ to $z=c\sqrt{\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right)}$. Consider the quadrant of the ellipse in the first quadrant of the xy plane. For a fixed x, y varies from 0 to $b\sqrt{1-\dfrac{x^2}{a^2}}$

Then x varies from 0 to a.

$$\text{Hence the required volume} = 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}} dz\,dy\,dx$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx \qquad \ldots(1)$$

Write $1-\dfrac{x^2}{a^2}=\dfrac{p^2}{b^2}$

$\therefore$ The required volume

$$=8\int_{x=0}^{a}\int_{y=0}^{p}\frac{c}{b}\sqrt{p^2-y^2}\,dy\,dx = 8\frac{c}{b}\int_{x=0}^{a}\left[\int_{y=0}^{p}\sqrt{(p^2-y^2)}dy\right]dx \qquad \ldots(2)$$

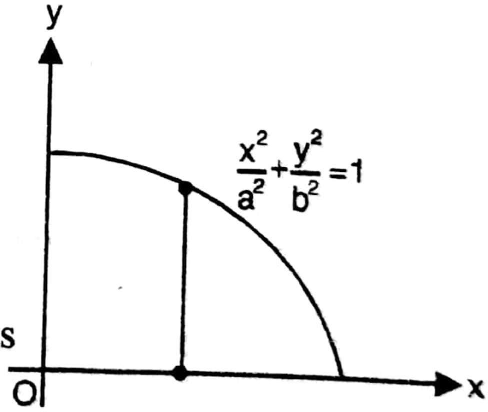
But $\displaystyle\int_{y=0}^{p}\sqrt{p^2-y^2}\,dy = \int_{0}^{\pi/2} p\cos\theta . p\cos\theta d\theta$ $\quad\left[\begin{array}{l} Put\ y = p\sin\theta \Rightarrow dy = p\cos\theta\ d\theta \\ if\ y=0, \theta=0 \text{ and } if\ y=p, \theta=\pi/2 \end{array}\right.$

$$= p^2\int_{\theta=0}^{\pi/2}\cos^2\theta d\theta = p^2.\frac{\pi}{4}=\frac{\pi}{4}b^2\left(1-\frac{x^2}{a^2}\right) \qquad \ldots(3)$$

Using (3) in (2), the required volume

$$=\frac{8c}{b}.\frac{\pi}{4}b^2\int_{x=0}^{a}\left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$=2\pi bc\left[a-\frac{a}{3}\right]=2\pi bc.\frac{2a}{3}=\frac{4\pi}{3}abc \text{ cubic units}$$



Note : Putting $a=b=c$, we obtain volume of the sphere $x^2+y^2+z^2=a^2$ as $\dfrac{4\pi a^3}{3}$.

Example 4 : Find the volume bounded by the xy plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.

Solution : Given surfaces are

$$z=0 \quad ...(1)$$

$$x^2+y^2=1 \quad ...(2)$$

and $x+y+z=3 \Rightarrow z=3-x-y$ ...(3)

The projection of the volume on the xy-plane is the region R enclosed by the circle $x^2+y^2=1$.

$\therefore$ The required volume can be covered as follows :

z : From 0 to $3-x-y$

y : From $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

x : From -1 to 1

Thus the volume bounded by the given surfaces,

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{3-x-y} dz\, dy\, dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [z]_0^{3-x-y}\, dy\, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3-x-y)dy\, dx = \int_{-1}^{1}\left[3y-xy-\frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1}\left\{3\cdot 2\sqrt{1-x^2}-x\cdot 2\sqrt{1-x^2}-\frac{1}{2}(0)\right\}dx$$

$$= \int_{-1}^{1}[6\cdot\sqrt{1-x^2}-2x\sqrt{1-x^2}]dx = 6\cdot 2\int_{0}^{1}\sqrt{1-x^2}\,dx-2(0)$$

[$\because$ Since integrand is odd function in second integral]

$$= 12\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = 12\left[\frac{1}{2}\sin^{-1}(1)\right]$$

$$= 6\cdot\frac{\pi}{2} = 3\pi \text{ cubic units.}$$

Example 5 : Evaluate $\iiint (x+y+z)dx\, dy\, dz$ over the tetrahedron bounded by the co-ordinate planes and the plane $x+y+z=1$.

Solution : The region of integration is given by

z : From 0 to $1-x-y$

y : From 0 to $1-x$

x : From 0 to 1

$$\therefore \iiint_V (x+y+z)dx\, dy\, dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (x+y+1-x-y)^2\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} dy\, dx = \frac{1}{2}\int_0^1 [y]_0^{1-x}\, dx$$

$$= \frac{1}{2}\int_0^1 (1-x)dx = \frac{1}{2}\left[\frac{(1-x)^2}{-2}\right]_0^1$$

$$= -\frac{1}{4}(0-1) = \frac{1}{4}.$$

Exampl...

Scanned with CamScanner