unit - II

Mean values Theorems

1. Rolle's mean values theorem: If $f(x)$ is

   (a) continuous in $[a,b]$

   (b) differentiable in (a,b)

   (c) $f(a)=f(b)$ then there exist at least one $c \in (a,b)$ with $\boxed{f'(c)=0}$

2. Lagranges mean value theorem:

   If $f(x)$ (a) continuous in $[a,b]$

   (b) differentiable in (a,b)

   (c) $f(a) \neq f(b)$ then there exist $c \in (a,b)$

   with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$, where $b-a \neq 0$.

3. Cauchy's mean value theorem:

   if $f(x)$ and $g(x)$ are 1. continuous in $[a,b]$

   2. differenciable in (a,b)

   then there exist at least one $c \in (a,b)$ and $g'(c) \neq 0$

   with $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Example problems:-

1. verify rolle's mean value theorem for $f(x)=x^2-5x+6$ in $[2,3]$

Sol: The function $f(x)=x^2-5x+6$ is

1. continuous function in $[2,3]$
2. It is differentiable in $(2,3)$

There exist $c \in (2,3)$ with $f'(c)=0$

$f(x) = x^2-5x+6$

D. w.r to x

$f'(x) = 2x-5$

$x = c$

$f'(c) = 2c-5$

$f'(c) = 0$

$2c - 5 = 0$

$2c = 5$

$\boxed{c = 5/2} \in (2,3)$

Hence rolle's mean value theorem is verified.

2. Verify rolle's mean value theorem for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a, b]$

Sol: The function $f(x)$ 1. It is continuous in $[a, b]$

2. It is differentiable in (a, b)

$f(a) = f(b)$

$x = a \Rightarrow f(a) = \log\left[\frac{a^2+\cancel{a}b}{a(\cancel{a}+b)}\right]$

$= \log 1 = 0$

$x = b \Rightarrow f(b) = \log\left[\frac{b^2+\cancel{a}b}{b(\cancel{a}+b)}\right]$

$= \log 1 = 0$

$\therefore f(a) = f(b)$

$\therefore$ There exist a $c \in (a,b)$ with $f'(c) = 0$

$f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$

$f(x) = \log(x^2+ab) - \log(x(a+b))$

D. w.r.to x

$f'(x) = \frac{1}{x^2+ab}\frac{d}{dx}(x^2+ab) - \frac{1}{x(a+b)}\frac{d}{dx}(x(a+b))$

$f'(x) = \frac{2x}{x^2+ab} - \frac{\cancel{a+b}}{x(\cancel{a+b})}$

$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$

$f'(x) = \frac{2x^2 - (x^2+ab)}{x(x^2+ab)}$

$f'(x) = \frac{2x^2 - x^2 - ab}{x(x^2+ab)}$

$$f'(x) = \frac{x^2-ab}{x(x^2+ab)}$$

$x = c$

$$f'(c) = \frac{c^2-ab}{c(c^2+ab)}$$

$f'(c) = 0$

$$0 = \frac{c^2-ab}{c(c^2+ab)}$$

$c^2-ab = 0$

$c = \sqrt{ab} \in (a,b)$

Hence rolle's theorem verified.

3. verify rolle's theorem for $f(x) = 2x^3+x^2-4x-2$ in $[-\sqrt{2},\sqrt{2}]$

Sol:- The function $f(x) = 2x^3+x^2-4x-2$

1. It is continious function in $[-\sqrt{2},\sqrt{2}]$
2. It is differentiable in $(-\sqrt{2},\sqrt{2})$

$f(x) = 2x^3+x^2-4x-2$

$x = -\sqrt{2} \Rightarrow f(-\sqrt{2}) = 2(-2\sqrt{2}) + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2$

$= -4\sqrt{2}+2+4\sqrt{2}-2$

$= 0$

$x = \sqrt{2} \Rightarrow f(\sqrt{2}) = 4(\sqrt{2})+2-4(\sqrt{2})-2$

$= 0$

$f(-\sqrt{2}) = f(\sqrt{2})$

There exist $c \in (-\sqrt{2},\sqrt{2})$ with $f'(c) = 0$

$f(x) = 2x^3+x^2-4x-2$

D.w.r to x

$f'(x) = 6x^2+2x-4$

$x = c$

$f'(c) = 6c^2+2c-4$

$f'(c) = 0$

$6c^2+2c-4 = 0$

$3c^2+c-2 = 0$

$3c^2+3c-2c-2 = 0$

$3c(c+1)-2(c+1) = 0$

$3c-2=0$ $\qquad \therefore c=-1$

$3c=2$

$c=2/3$

$c=-1, \frac{2}{3} \in (-\sqrt{2}, \sqrt{2})$

Hence rolle's theorem verified.

4. verify rolle's theorem for $f(x)=2x^3+x^2-4x-2$ in $[-\sqrt{3}, \sqrt{3}]$

Sol: The given function $f(x)$ it is continious in $[-\sqrt{3}, \sqrt{3}]$

It is differentiable in $(-\sqrt{3}, \sqrt{3})$

$f(x)=2x^3+x^2-4x+2$

$x=-\sqrt{3} \Rightarrow f(-\sqrt{3})=2(-\sqrt{3})^3+(-\sqrt{3})^2-4(-\sqrt{3})-2$

$=6(-\sqrt{3})+3+4\sqrt{3}-2$

$=-2\sqrt{3}+1$

$x=\sqrt{3} \Rightarrow f(\sqrt{3})=2(\sqrt{3})^3+(\sqrt{3})^2-4(\sqrt{3})-2$

$=6\sqrt{3}+3-4\sqrt{3}-2$

$=2\sqrt{3}+1$

$\therefore f(-\sqrt{3}) \neq f(\sqrt{3})$

Hence rolle's mean value theorem not applicable and not verify

5. Verify rolle's mean value theorem for given $f(x)=(x-a)^m(x-b)^n$ in $[a,b]$

Sol:- The given function $f(x)=(x-a)^m(x-b)^n$

1. it is continious in $[a,b]$

2. it is differerntiable in (a,b)

$f(x)=(x-a)^m(x-b)^n$

$f(a)=f(b)$

(iii) $x=a \Rightarrow f(a)=(a-a)^m(a-b)^n=0$

$x=b \Rightarrow f(b)=(b-a)^m(b-b)^n=0$

$\therefore f(a)=f(b)$.

There exist $c \in (a,b)$ with $f'(c)=0$ $\qquad \frac{d}{dx}u.v=uv'+vu'$

$f(x)=(x-a)^m(x-b)^n$

D.w.r.to x

$f'(x)=(x-a)^m\frac{d}{dx}(x-b)^n+(x-b)^n\frac{d}{dx}(x-a)^m$

$= (x-a)^m (n(x-b)^{n-1}) \frac{d}{dx}(x-b) + (x-b)^n (m(x-a)^{m-1}) \frac{d}{dx}(x-a)$

$= n(x-a)^m (x-b)^{n-1} + m(x-b)^n (x-a)^{m-1}$

$= (x-a)^m (x-b)^n [n \cdot (x-b)^{-1} + m(x-a)^{-1}]$

$= (x-a)^m (x-b)^n \left[\frac{n}{x-b} + \frac{m}{x-a}\right]$

$x = c$

$f'(c) = (c-a)^m (c-b)^n \left[\frac{n}{c-b} + \frac{m}{c-a}\right]$

$f'(c) = 0$

$0 = (c-a)^m (c-b)^n \left[\frac{n(c-a)+m(c-b)}{(c-a)(c-b)}\right]$

$0 = n(c-a) + m(c-b)$

$0 = nc - na + mc - mb$

$0 = c(n+m) - (an+bm)$

$an+bm = c(n+m)$

$c = \frac{an+bm}{n+m}$

(6) verify rolle's theorem for $f(x) = (x+2)^3 (x-3)^4$ in $[-2,3]$

Sol: The given function $f(x) = (x+2)^3 (x-3)^4$

1. It is continious in $[-2,3]$
2. It is differentiable in $(-2,3)$

$f(x) = (x+2)^3 (x-3)^4$

iii) $f(a) = f(b)$

$x = -2 \Rightarrow f(-2) = (-2+2)^3 (-2-3)^4 = 0(-5)^4 = 0$

$x = 3 \Rightarrow f(3) = (3+2)^3 (3-3)^4 = 125 \times 0 = 0$

$\therefore f(a) = f(b)$

There exist $c \in (-2,3)$ with $f'(c) = 0$

$f(x) = (x+2)^3 (x-3)^4$

D. w.r.to x $\qquad \frac{d}{dx}uv = uv' + vu'$

$f'(x) = (x+2)^3 \frac{d}{dx}(x-3)^4 + (x-3)^4 \frac{d}{dx}(x+2)^3$

$= (x+2)(4(x-3)^3 \frac{d}{dx}(x-3)) + (x-3)^4 (3(x+2)^2 \frac{d}{dx}(x+2))$

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$= 4(x+2)^3(x-3)^3 + 3(x-3)^4(x+2)^2$

$= \cancel{(x+2)^3 (x-3)^4 \left[4(x-3)^3 + 3(x+2)^2\right]}$

$= (x+2)^2 (x-3)^3 \left[4(x+2) + 3(x-3)\right]$

$x = c$

$f'(c) = (c+2)^2(c-3)^3 \left[4(c+2) + 3(c-3)\right]$

$f'(c) = 0$

$0 = (c+2)^2(c-3)^3 \left[4c+8+3c-9\right]$

$4c+8+3c-9 = 0$

$7c - 1 = 0$

$7c = 1$

$c = 1/7 \in [-2, 3]$

Hence rolle's theorem verified.

(7) verify rolle's theorem for the following functions.

(i) $f(x) = \tan x$ in $[0, \pi]$

(ii) $f(x) = x^3$ in $[1, 3]$

(iii) $f(x) = \frac{\sin x}{e^x}$ or $\sin x . e^{-x}$ in $[0, \pi]$

(iv) $f(x) = x^{2/3} - 2x^{1/3}$, in $[0, 8]$

(v) $f(x) = \frac{x^2 - x - 6}{x - 1}$ in $[2, 6]$

(i) $f(x) = \tan x$ in $[0, \pi]$

Sol: Given $f(x) = \tan x$ in $[0, \pi]$

The function $f(x)$ is continuous in $[0, \pi]$

It is differentiable in $(0, \pi)$

$f(a) = f(b)$

$f(x) = \tan x$ in $[0, \pi]$

$x = 0 \Rightarrow f(0) = \tan 0 = 0$

$x = \pi \Rightarrow f(\pi) = \tan(180) = 0$

$f(a) = f(b)$

$f(0) = f(\pi)$

There exist $c \in (0, \pi)$ with $f'(c) = 0$

$f(x) = \tan x$

D.w.r.to x.

$f'(x) = \sec^2 x$

$f'(c) = 0$

$\sec^2 c = 0$

$\sec c = 0$

$c = \sec^{-1}(0)$

$c = \sec^{-1}(\sec \frac{1}{90})$

$c = \frac{1}{90}$ (or) $\frac{2}{\pi}$

(ii) $f(x) = x^3$

Given $f(x) = x^3$ in $[1,3]$

The function $f(x)$ is continious in $[1,3]$

It is differentiable in $(1,3)$

$f(a) = f(b)$

$x = 1 \Rightarrow f(1) = 1^3 = 1$

$x = 3 \Rightarrow f(3) = 3^3 = 81$

$\therefore f(1) \neq f(3)$

$\therefore$ The function $f(x)$ is not follows 3^{rd} condition

$\therefore c \notin (1,3)$

(iii) $f(x) = \frac{\sin x}{e^x}$

Sol:- Given $f(x) = \frac{\sin x}{e^x}$ (or) $f(x) = e^{-x} \sin x$ in $[0,\pi]$

$f(x) = e^{-x} \sin x$ in $[0,\pi]$

The given function $f(x)$ 1. It is continious in $[0,\pi]$

2. It is differentable in $(0,\pi)$

3. $f(a) = f(b)$

$f(x) = e^{-x} \sin x$ in $(0,\pi]$

$x = 0 \Rightarrow f(0) = e^{-0} \sin 0 = 0$

$x = \pi \Rightarrow f(\pi) = e^{-\pi} \sin \pi = 0$

$\therefore f(0) = f(\pi)$

$\therefore$ There exist $c \in (a,b) \ni f'(c) = 0$

$f(x) = e^{-x} \sin x$

D.w.r.to x

$$f'(x) = e^{-x} \frac{d}{dx} \sin x + \sin x \frac{d}{dx} e^{-x}$$

$$= e^{-x} \cos x + \sin x (-e^{-x})$$

$$= -e^{-x} \sin x + e^{-x} \cos x$$

$$= e^{-x} (\cos x - \sin x)$$

$\Rightarrow x = c$

$\therefore f'(c) = e^{-c} (\cos c - \sin c)$

$\therefore f'(c) = 0$

$e^{-c} (\cos c - \sin c) = 0$

$\cos c - \sin c = 0$

$\sin c = \cos c$

$\sin c = \sin(\frac{\pi}{2} - c)$

$c = \frac{\pi}{2} - c$

$2c = \pi/2$

$c = \pi/4$

$\therefore c = \frac{\pi}{4} \in [0, \pi]$

Hence rolle's mean value theorem verified.

(iv) $f(x) = x^{2/3} - 2x^{1/3}$ in $[0, 8]$

Sol:- Given $f(x) = x^{2/3} - 2x^{1/3}$ in $[0, 8]$.

The given function $f(x)$ 1. It is continious in $[0, 8]$

2. It is differentiable in $(0, 8)$

$f(a) = f(b)$

$f(x) = x^{2/3} - 2x^{1/3}$

$x = 0 \Rightarrow f(0) = 0 - 2(0) = 0$

$x = 8 \Rightarrow f(8) = (8)^{2/3} - 2(8)^{1/3}$

$= (2^3)^{2/3} - 2(2^3)^{1/3}$

$= 4 - 4 = 0$

$f(0) = f(8)$

$\therefore$ There exist $c \in [0, 8] \ni f'(c) = 0$

$f(x) = x^{2/3} - 2x^{1/3}$

D.w.r to x

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3}$$

$$= \frac{2}{3}\left(x^{-1/3} - x^{-2/3}\right)$$

$x = c$

$$f'(c) = \frac{2}{3}\left(c^{-1/3} - c^{-2/3}\right)$$

$\therefore f'(c) = 0$

$$\frac{2}{3}\left(c^{-1/3} - c^{-2/3}\right) = 0$$

$$c^{-1/3} - c^{-2/3} = 0$$

$$c^{-\frac{1}{3} - \left(-\frac{2}{3}\right)} = 0$$

$$c = -\frac{1}{3} + \frac{2}{3}$$

$$c = \frac{1}{3} \in (0,8)$$

(v) $f(x) = \dfrac{x^2 - x - 6}{x-1}$ in $[2,6]$

sol:- Given $f(x) = \dfrac{x^2 - x - 6}{x-1}$ in $[2,6]$

The given function $f(x)$ 1. It is continious in $[2,6]$

2. It is differentiable in $(2,6)$

$f(a) = f(b)$

$f(x) = \dfrac{x^2 - x - 6}{x-1}$ in $[2,6]$

$x = 2 \Rightarrow f(2) = \dfrac{4-2-6}{2-1} = \dfrac{4-8}{1} = -4$

$x = 6 \Rightarrow f(6) = \dfrac{36-6-6}{6-1} = \dfrac{36-12}{5} = \dfrac{24}{5}$

$f(2) \neq f(6)$

Hence rolle's mean value theorem is not verified.

Imp

(1) verify Lagranges mean value theorem of $f(x) = \sin x$ in $[0, \frac{\pi}{2}]$

Sol: Given $f(x) = \sin x$ in $[0, \frac{\pi}{2}]$

The function $f(x)$ 1. It is continious in $[0, \frac{\pi}{2}]$

2. It is differentiable in $(0, \frac{\pi}{2})$

3. $f(a) \neq f(b)$

$f(x) = \sin x$

$x = 0 \Rightarrow f(0) = \sin 0 = 0$

$x = \frac{\pi}{2} \Rightarrow f(\pi/2) = \sin\frac{\pi}{2} = 1$

$\therefore f(0) \neq f(\frac{\pi}{2})$

$\therefore$ There exist $c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

$f(x) = \sin x$

D.w.r to x.

$f'(x) = \cos x$

$x = c$

$f'(c) = \cos c$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(\pi/2) - f(0)}{\frac{\pi}{2} - 0}$$

$$f'(c) = \frac{1 - 0}{\frac{\pi}{2} - 0}$$

$$f'(c) = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$\cos c = 2/\pi$

$c = \cos^{-1}(2/\pi) \in [0, \frac{\pi}{2}]$

Hence lagranges mean value theorem verified.

(2) verify lagranges mean value theorem $f(x) = \log_e^x$ in $[1, e]$

Sol: Given $f(x) = \log_e^x$

The function $f(x)$ It is continious in $[1, e]$

It is differentiable in $(1, e)$

3. $f(a) \neq f(b)$

$f(x) = \log_e^x$ in $[1, e]$

$x = 1 \Rightarrow f(1) = \log_e^1 = 0$

$x = e \Rightarrow f(e) = \log_e^e = 1$

$\therefore f(a) \neq f(b)$

$\therefore$ There exist $c \in (1, e) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f'(c) = \frac{1 - 0}{e - 1}$$

$$f'(c) = \frac{1}{e-1}$$

$$f(x) = \log_e^x$$

D. w.r to x

$$f'(x) = \frac{1}{x}$$

$$x = c$$

$$f'(c) = \frac{1}{c}$$

$$\frac{1}{c} = \frac{1}{e-1}$$

$$c = e - 1 \in [1, e]$$

(3) verify lagranges mean value theorem $f(x) = \cos x$ in $[0, \frac{\pi}{2}]$.

Sol:- Given $f(x) = \cancel{\log_e x}\ \cos x$

The function $f(x)$ is continious in $[0, \frac{\pi}{2}]$

it is differentiable in $(0, \frac{\pi}{2})$

3. $f(a) \neq f(b)$

$f(x) = \cos x$ in $[0, \frac{\pi}{2}]$

$x = 0 \Rightarrow f(0) = \cos 0 = 1$

$x = \frac{\pi}{2} \Rightarrow f(\pi/2) = \cos \frac{\pi}{2} = 0$

$\therefore f(0) \neq f(\pi/2)$

$\therefore$ There exist $c \in (0, \frac{\pi}{2}) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f'(c) = \frac{0 - 1}{\frac{\pi}{2} - 0} = \frac{-2}{\pi}$$

$$f(x) = \cos x$$

D. w.r to x.

$f'(x) = -\sin x$

$x = c$

$f'(c) = -\sin c$

$-\sin c = -\frac{2}{\pi}$

$\sin c = \frac{2}{\pi}$

$c = \sin^{-1}\left(\frac{2}{\pi}\right) \in \left[0, \frac{\pi}{2}\right]$

$= \cancel{\sin^{-1}(\sin}$

Hence lagranges mean value theorem verified.

(4) verify lagranges mean value theorem $f(x) = x(x-2)(x-3)$ in $[0,4]$.

Sol:- Given $f(x) = x(x-2)(x-3)$

The function $f(x)$ it is continious in $[0,4]$

It is differentiable in $(0,4)$

3. $f(a) \neq f(b)$

$f(x) = x(x-2)(x-3)$ in $[0,4]$

$x = 0 \Rightarrow f(0) = 0(0-2)(0-3) = 0$

$x = 4 \Rightarrow f(4) = 4(4-2)(4-3) = 4(2)(1) = 8$

$f(0) \neq f(4)$

$\therefore$ There exist $c \in (0,4) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$f'(c) = \frac{8-0}{4-0} = 2$

$f(x) = x(x-2)(x-3)$

$f(x) = x(x^2-5x+6)$

$f(x) = x^3-5x^2+6x$

$f'(x) = 3x^2-10x+6$

$x = c$

$f'(c) = 3c^2-10c+6$

$2 = 3c^2-10c+6$

$3c^2-10c+6-2 = 0$

$3c^2 - 10c + 4 = 0$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$c = \frac{10 \pm \sqrt{100 - 4(3)(4)}}{2 \times 3}$$

$$c = \frac{10 \pm \sqrt{100 - 48}}{6}$$

$$c = \frac{10 \pm \sqrt{52}}{6}$$

$$c = \frac{10 \pm 4\sqrt{13}}{6}$$

$$c = \frac{5 \pm 2\sqrt{13}}{3} \in [0,4].$$

Hence lagranges mean value theorem is verified.

(5) find the c value by using lagranges mean value theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$

Sol: Given $f(x) = (x-1)(x-2)(x-3)$

The function $f(x)$ is continious in $[0,4]$

It is differentiable in $(0,4)$

3. $f(a) \neq f(b)$

$f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$

$x = 0 \Rightarrow f(0) = (0-1)(0-2)(0-3) = (-1)(-2)(-3) = -6$

$x = 4 \Rightarrow f(4) = (4-1)(4-2)(4-3) = 3(2)(1) = 6$

$f(0) \neq f(4)$

$\therefore$ There exist $c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b-a}$

$$f'(c) = \frac{6+6}{4-0} = 3$$

$f(x) = (x-1)(x-2)(x-3)$

$= (x-1)(x^2 - 5x + 6)$

$= x^3 - 5x^2 + 6x - x^2 + 5x - 6$

$f(x) = x^3 - 6x^2 + 11x - 6$

D. w.r.to x

$f'(x) = 3x^2 - 12x + 11$

$x = c$

$f'(c) = 3c^2 - 12c + 11$

$3c^2 - 12c + 11 = 3$

$3c^2 - 12c + 11 - 3 = 0$

$3c^2 - 12c + 8 = 0$

$$c = \frac{12 \pm \sqrt{144 - 4(3)(8)}}{2(3)}$$

$$c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$c = \frac{12 \pm \sqrt{48}}{6}$$

$$c = \frac{\cancel{12}^{6} \pm \cancel{4}^{2}\sqrt{3}}{\cancel{6}_{3}}$$

$$c = \frac{6 \pm 2\sqrt{3}}{3}$$

(6) verify lagranges mean value theorem.

(i) $f(x) = x^3 - 2x^2$ in $[2,5]$

(ii) $f(x) = 2x^2 - 7x + 10$, $a = 2$, $b = 5$ in $[2,5]$

(iii) $f(x) = 2x^2 + 7x - 10$ in $[0,5]$

i. sol: Given $f(x) = x^3 - 2x^2$ in $[2,5]$

The function $f(x)$ is continious in $[2,5]$

it is differentiable in $(2,5)$

3. $f(a) \neq f(b)$

$f(x) = x^3 - 2x^2$

$x = 2 \Rightarrow f(2) = 8 - 2(2)^2 = 8 - 8 = 0$

$x = 5 \Rightarrow f(5) = 125 - 2(25) = 125 - 50 = 75$

$f(2) \neq f(5)$

$\therefore$ There exist $c \in (2,5) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f'(c) = \frac{75 - 0}{5 - 2} = \frac{75}{3} = 25$$

$f(x) = x^3 - 2x^2$

$f'(x) = 3x^2 - 4x$

$\Rightarrow x = c$

$f'(c) = 3c^2 - 4c$

$25 = 3c^2 - 4c$

$3c^2 - 4c - 25 = 0$

$a = 3,\ b = -4,\ c = -25$

$$c = \frac{4 \pm \sqrt{16 - 4(3)(-25)}}{2(3)}$$

$$c = \frac{4 \pm \sqrt{16 + 300}}{6}$$

$$c = \frac{4 \pm \sqrt{316}}{6}$$

$$c = \frac{\cancel{4}^{2} \pm \cancel{2}^{1}\sqrt{79}}{\cancel{6}_{3}}$$

$$c = \frac{2 \pm 1\sqrt{79}}{3} \in [2,5]$$

Hence lagranges mean value theorem is verified.

(ii) $f(x) = 2x^2 - 7x + 10$; $a = 2, b = 5$ in $[2,5]$

Sol: Given $f(x) = 2x^2 - 7x + 10$

The function $f(x)$ continious in $[2,5]$

it is differentiable in $(2,5)$

3. $f(a) \neq f(b)$

$f(x) = 2x^2 - 7x + 10$ in $[2,5]$

$x = 2 \Rightarrow f(2) = 2(2)^2 - 7(2) + 10 = 8 - 14 + 10 = 4$

$x = 5 \Rightarrow f(5) = 2(5)^2 - 7(5) + 10 = 50 - 35 + 10 = 25$

$f(2) \neq f(5)$

$\therefore$ There exist $c \in (2,5) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f'(c) = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7$$

$f(x) = 2x^2 - 7x + 10$

D. w.r.to x.

$f'(x) = 4x - 7$

$\Rightarrow x = c$

$f'(c) = 4c - 7$

$4c - 7 = 7 \Rightarrow 4c - 7 - 7 = 0$

$\Rightarrow 4c - 14 = 0$

$\Rightarrow 2c - 7 = 0$

$12c - 21 = 1$

$12c - 21 - 1 = 0 \Rightarrow 2c = 7$

$12c - 22 = 0 \Rightarrow c = 7/2 \in (2,5)$

$12c = 22$

$c = \frac{22}{12}$

$c = \frac{11}{6} \in (2,5)$

(iii) $f(x) = 2x^2 + 7x - 10$ in $[0,5]$

Sol: Given $f(x) = 2x^2 + 7x - 10$ in $[0,5]$

The function $f(x)$ continious in $[0,5]$

It is differentiable in $(0,5)$

$f(a) \neq f(b)$

$f(x) = 2x^2 + 7x - 10$ in $[0,5]$

$x = 0 \Rightarrow f(0) = 2(0)^2 + 7(0) - 10 = -10$

$x = 5 \Rightarrow f(5) = 2(5)^2 + 7(5) - 10 = 50 + 35 - 10 = 25$

$f(0) \neq f(5)$

$\therefore$ There exist $c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

$f'(c) = \frac{25 + 10}{5} = 7$

$f(x) = 2x^2 + 7x - 10$

D. w. r to x.

$f'(x) = 4x + 7$

$x = c$

$f'(c) = 4c + 7$

$4c + 7 = 7$

$4c = 7 - 7$

$4c = 0$

$c = 0 \in (0,4)$

Hence lagranges mean value theorem is verified.

a) prove if $a < b$, $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$

by using lagranges mean value theorem and hence show that

b)

i. $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$.

ii. $\frac{\pi}{4} + \frac{1}{5} < \tan^{-1} 2 < \frac{\pi}{4} + \frac{1}{2}$

sol:- Let $f(x) = \tan^{-1} x$ in $[a,b]$

The function $f(x)$ is continious in $[a,b]$

It is differentiable in (a,b)

$f(a) \neq f(b)$

$f(x) = \tan^{-1} x$

$x = a \Rightarrow f(a) = \tan^{-1}(a)$

$x = b \Rightarrow f(b) = \tan^{-1}(b)$

$\therefore f(a) \neq f(b)$,

These exist $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$.

$$f'(c) = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$$

$f(x) = \tan^{-1} x$

w.r.to x.

$f'(x) = \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}$

$x = c$

$f'(c) = \frac{1}{1+c^2}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$$

$a < c < b$

$a^2 < c^2 < b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2} \Rightarrow \frac{b-a}{1+a^2} > \tan^{-1} b - \tan^{-1} a > \frac{b-a}{1+b^2}$

$$\frac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+a^2}$$

(i) $a=1$, $b=\frac{4}{3}$

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < Tan^{-1}\frac{4}{3} - Tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{\frac{1}{3}}{\frac{25}{93}} < Tan^{-1}\frac{4}{3} - Tan^{-1}1 < \frac{\frac{1}{3}}{2}$$

$$\frac{3}{25} < Tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < Tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

(ii) $a=1$, $b=2$

$$\frac{2-1}{1+4} < Tan^{-1}2 - Tan^{-1}1 < \frac{2-1}{1+1}$$

$$\frac{1}{5} < Tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\frac{\pi}{4} + \frac{1}{5} < Tan^{-1}2 < \frac{\pi}{4} + \frac{1}{2}$$

Q) prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}$ $a=\frac{1}{2}, b=\frac{3}{5}$

Sol:- Given $f(x) = \sin^{-1}x$ in $[a,b]$.

$f(x)$ is continious in $[a,b]$

$f(x)$ is differentiable in (a,b)

$f(a) \neq f(b)$

$\Rightarrow x=a \Rightarrow f(a) = \sin^{-1}a$

$\Rightarrow x=b \Rightarrow f(b) = \sin^{-1}b$

$f(a) \neq f(b)$

$\therefore$ There exist $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$f'(c) = \frac{\sin^{-1}b - \sin^{-1}a}{b-a}$$

$f(x) = \sin^{-1}x$

$f'(x) = \frac{d}{dx}\sin^{-1}x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$x = c$

$$f'(c) = \frac{1}{\sqrt{1-c^2}}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b-a}$$

$$a < c < b$$

$$a^2 < c^2 < b^2$$

$$1-a^2 < 1-c^2 < 1-b^2$$

$$\frac{1}{1-a^2} > \frac{1}{1-c^2} > \frac{1}{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} > \frac{1}{\sqrt{1-c^2}} > \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} > \frac{\sin^{-1}b - \sin^{-1}a}{b-a} > \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} > \frac{\sin^{-1}b - \sin^{-1}a}{\cancel{b-a}} > \frac{b-a}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-b^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-a^2}}$$

(i) $a = \frac{1}{2}$, $b = \frac{3}{5}$

$$\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{9}{25}}} < \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2} < \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}$$

$$\frac{\frac{6-5}{10}}{\sqrt{\frac{16}{25}}} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{\frac{6-5}{10}}{\sqrt{\frac{3}{4}}}$$

$$\frac{\frac{1}{\cancel{10}}}{\frac{4}{\cancel{8}}} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{\frac{1}{\cancel{10}\,5}}{\frac{\sqrt{3}}{2}}$$

$$\frac{1}{8} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{1}{5\sqrt{3}}$$

$$\frac{1}{8} + \frac{\pi}{6} < \sin^{-1}\frac{3}{5} < \frac{1}{5\sqrt{3}} + \frac{\pi}{6} \Rightarrow \frac{\pi}{6} + \frac{1}{5\sqrt{3}} > \sin^{-1}\frac{3}{5} > \frac{1}{8} + \frac{\pi}{6}$$

1Q) prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$ by using lagranges mean value theorem.

2m Q) Imp Explain why lagranges mean value theorem doe's not hold for $f(x) = x^{2/3}$ in $[-1,1]$

Sol:- Given $f(x) = x^{2/3}$ in $[-1,1]$

$f(x)$ is continious in $[-1,1]$

The function $f(x)$ it is not differentiable in $(-1,1)$

$\therefore$ lagranges mean value theorem is not apply for

$f(x) = x^{2/3}$ in $[-1,1]$

1Q) $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$

Sol:- Given $f(x) = \cos^{-1}x$

$f(x)$ is continious in $[a,b]$

$f(x)$ is diffierentiable in (a,b)

3 $f(a) \neq f(b)$

$f(x) = \cos^{-1}x$ in $[a,b]$

$x = a \rightarrow f(a) = \cos^{-1}a$

$x = b \Rightarrow f(b) = \cos^{-1}b$

$\therefore$ $f(a) \neq f(b)$

$\therefore$ There exist $c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b-a}$

$$f'(c) = \frac{\cos^{-1}b - \cos^{-1}a}{b-a}$$

$f(x) = \cos^{-1}x$

D.w.r.to x

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$x = c$

$$f'(c) = \frac{-1}{\sqrt{1-c^2}}$$

$$\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}b - \cos^{-1}a}{b-a}$$

$a < c < b$

$$a^2 < c^2 < b^2$$

$$1-a^2 < 1-c^2 < 1-b^2$$

$$\frac{1}{1-a^2} > \frac{1}{1-c^2} > \frac{1}{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} > \frac{1}{\sqrt{1-c^2}} > \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1}b-\cos^{-1}a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-(b-a)}{\sqrt{1-a^2}} > \cos^{-1}b-\cos^{-1}a > \frac{-(b-a)}{\sqrt{1-b^2}}$$

$$\frac{-(b-a)}{\sqrt{1-b^2}} < \cos^{-1}b-\cos^{-1}a < \frac{-(b-a)}{\sqrt{1-a^2}}$$

$$a=\frac{1}{2},\ b=\frac{3}{5}$$

$$\frac{-\left(\frac{3}{5}-\frac{1}{2}\right)}{\sqrt{1-\left(\frac{3}{5}\right)^2}} < \cos^{-1}\frac{3}{5}-\cos^{-1}\frac{1}{2} < \frac{-\left(\frac{3}{5}-\frac{1}{2}\right)}{\sqrt{1-\left(\frac{1}{2}\right)^2}}$$

$$-\frac{\left(\frac{6-5}{10}\right)}{\sqrt{\frac{25-9}{25}}} < \cos^{-1}\frac{3}{5}-\frac{\pi}{3} < \frac{-\left(\frac{6-5}{10}\right)}{\sqrt{\frac{4-1}{4}}}$$

$$-\frac{\frac{1}{\cancel{10}\,2}}{\frac{4}{\cancel{5}}} < \cos^{-1}\frac{3}{5}-\frac{\pi}{3} < \frac{-\frac{1}{\cancel{10}\,5}}{\frac{\sqrt{3}}{\cancel{2}}}$$

$$-\frac{1}{8} < \cos^{-1}\frac{3}{5}-\frac{\pi}{3} < -\frac{1}{5\sqrt{3}}$$

$\frac{\pi}{3} - \frac{1}{8} < \cos^{-1}\frac{3}{5} < \frac{\pi}{3} - \frac{1}{5\sqrt{3}}$

$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} < \cos^{-1}\frac{3}{5} < \frac{\pi}{3} - \frac{1}{8}$

Q) calculate appoximately $\sqrt[5]{245}$ by using lagranges mean value theorem. in the $[243, 245]$

Sol:- Let $a = 243$

$b = 245$

$\therefore$ Let $f(x) = \sqrt[5]{x}$

$f(x) = x^{1/5}$

$f'(x) = \frac{1}{5} x^{-4/5}$

(i) The given function $f(x)$ is continious in $[243, 245]$

(ii) $f(x)$ is differientiable in $(243, 245)$

(iii) $f(a) \neq f(b)$

$\therefore$ ~~The~~ $f(243) \neq f(245)$.

$\therefore$ There exist $c \in (243, 245) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{5} c^{-4/5} = \frac{\sqrt[5]{245} - \sqrt[5]{243}}{245-243}$$

$$\frac{1}{5} c^{-4/5} = \frac{\sqrt[5]{245} - \sqrt[5]{243}}{2}$$

$$\frac{2}{5} c^{-4/5} + \sqrt[5]{243} = \sqrt[5]{245}$$

Let $c = 243$

$$\therefore \frac{2}{5}(243)^{-4/5} + (243)^{1/5} = \sqrt[5]{245}$$

$$(243)^{1/5}\left[\frac{2}{5}(243)^{-1} + 1\right] = \sqrt[5]{245}$$

$$3.0049 = \sqrt[5]{245}$$

Note: where c is considered as 243 because the c valu and $\sqrt{243}$ value are nearly equal

8) find the region in which $f(x) = 1-4x-x^2$ is increasing and the region in which it is decreasing by using lagranges mean value theorem.

Sol: Given $f(x) = 1-4x-x^2$

It is continious in $[a,b]$, It is differentiable in (a,b)

$f(x) = 1-4x-x^2$

D.w.r to x

$f'(x) = -4-2x$

if $f'(x) = 0$

$2x+4=0$

$2x=-4$

$x = -4/2 = -2$

$\therefore$ (i) for $x<-2$, $f'(x)>0$

(ii) If $x>-2$, $f'(x)<0$

$\therefore$ $f(x)$ is increasing in $(-\infty,-2]$

and decreasing in $(-2,\infty)$

9) use lagranges mean value theorem through the following

(1) for $0<a<b$, prove that $\frac{1-a}{b} < \log\frac{b}{a} < \frac{b}{a}-1$ and hence show that $\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$

(ii) prove that $\frac{\pi}{6}+\frac{1}{2\sqrt{3}} < \sin^{-1}\frac{3}{4} < \frac{\pi}{6}+\frac{1}{\sqrt{7}}$

(iii) At $\sqrt[4]{65}$

1 sol:- Given $f(x) = \log x$

The function $f(x)$ is continious in $[a,b]$

it is differentiable in (a,b)

$f(a) \neq f(b)$

$x=a \Rightarrow f(a) = \log a$

$x=b \Rightarrow f(b) = \log b$

$f(a) \neq f(b)$

$\therefore$ There exist $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$f'(c) = \frac{\log b - \log a}{b-a}$

$f(x) = \log x$

$f'(x) = \frac{1}{x}$

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$$f'(c) = \frac{1}{c}$$

$$\frac{1}{c} = \frac{\log b - \log a}{b-a}$$

8)

m

Sol:-

$$\boxed{\frac{b-a}{c} = \log b - \log a}$$

$$a < c < b$$

$$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{1}{a} > \frac{\log b - \log a}{b-a} > \frac{1}{b}$$

$$\frac{b-a}{a} > \log b - \log a > \frac{b-a}{b}$$

$$\frac{b}{a} - 1 < \log\frac{b}{a} < 1 - \frac{a}{b}$$

$a = 5$, $b = 6$

$$\frac{6}{5} - 1 < \log\frac{6}{5} < 1 - \frac{5}{6}$$

$$\frac{1}{5} < \log\frac{6}{5} < \frac{1}{6}$$

$$\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$$

(ii) ~~f(x) =~~ $\frac{\pi}{6} + \frac{1}{2\sqrt{3}} < \sin^{-1}\frac{3}{4} < \frac{\pi}{6} + \frac{1}{\sqrt{7}}$

$$f(x) = \sin^{-1}x$$

It is continious in $[a, b]$ and differentiable in (a,

$$f(a) \neq f(b)$$

$x = a \Rightarrow f(a) = \sin^{-1}a$

$x = b \Rightarrow f(b) = \sin^{-1}b$

$\therefore f(a) \neq f(b)$

$\therefore$ There exist $c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b-a}$

$$f'(c) = \frac{\sin^{-1}b - \sin^{-1}a}{b-a}$$

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$f(x) = \sin x$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ $\therefore x = c$

$f'(c) = \frac{1}{\sqrt{1-c^2}}$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1} b - \sin^{-1} a}{b-a}$$

$a^2 < c^2 < b^2$

$1-a^2 > 1-c^2 > 1-b^2$

$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$

$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$

$\therefore a = \frac{1}{2},\ b = 3/4$

$\frac{\frac{3}{4} - \frac{1}{2}}{\sqrt{1-\frac{1}{4}}} < \sin^{-1}\frac{3}{4} - \sin^{-1}\frac{1}{2} < \frac{\frac{3}{4} - \frac{1}{2}}{\sqrt{1-\frac{9}{16}}}$

$\frac{\frac{3-2}{4}}{\frac{\sqrt{3}}{2}} < \sin^{-1}\frac{3}{4} - \frac{\pi}{6} < \frac{\frac{3-2}{4}}{\frac{\sqrt{7}}{4}}$

$\frac{+1}{2\sqrt{3}} < \sin^{-1}\frac{3}{4} - \frac{\pi}{6} < \frac{1}{\sqrt{7}}$

$\frac{\pi}{6} + \frac{1}{2\sqrt{3}} < \sin^{-1}\frac{3}{4} < \frac{\pi}{6} + \frac{1}{\sqrt{7}}$

(iii) $\sqrt[6]{65}$

Let $a = 63$

$b = 65$

$\therefore f(x) = \sqrt[6]{x}$

$f(x) = (x)^{1/6} \Rightarrow f'(x) = \frac{1}{6} x$

(i) The given function is continious in [63, 65]

it is differentiable in (63, 65)

$f(a) \neq f(b)$

$\therefore f(63) \neq f(65)$

$\therefore$ There exist $c \in (63, 65) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{6} c^{-5/6} = \frac{\sqrt[6]{65} - \sqrt[6]{63}}{65-63}$$

$$\frac{1}{6} c^{-5/6} = \frac{\sqrt[6]{65} - \sqrt[6]{63}}{2}$$

$$\frac{2}{6} c^{-5/6} = \sqrt[6]{65} - \sqrt[6]{63}$$

$$\frac{2}{6} c^{-5/6} + \sqrt[6]{63} = \sqrt[6]{65}$$

Let $c = 63$

$$\frac{2}{6}(63)^{-5/6} + (63)^{1/6} = \sqrt[6]{65}$$

$$(63)^{1/6}\left(\frac{1}{3}(63)^{-5}+1\right) = \sqrt[6]{65}$$

$$\cancel{\begin{array}{l} 47.93\ [0.33 \times 1.0076+1] \\ 47.93\ [0.33231+1] \\ 47.62\ [1.33231] \\ 63.444 \end{array}}$$

$\cancel{47.93\left(0.33(-992,436,543)+1\right)}$

$$1.994 + \frac{1}{3}(0.0316) = \sqrt[6]{65}$$

$$1.994 + 0.1053$$

$$2.0045$$

Q) find c of cauchy's mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$, where $0 < a < b$

sol:- Given $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

$g(x) = \frac{1}{\sqrt{x}} \Rightarrow g'(x) = \frac{-1}{2x\sqrt{x}}$

f(x) and g(x) are continious functions in $[a, b]$,

it is differentiable in (a, b)

$\therefore$ There exist $c \in (a, b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}$$

$$-c = \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}}$$

$$-c = \frac{\cancel{\sqrt{b} - \sqrt{a}}}{\frac{-(\cancel{\sqrt{b} - \sqrt{a}})}{\sqrt{ab}}}$$

$$-c = -\sqrt{ab}$$

$$c = \sqrt{ab}$$

Hence cauchy's mean value theorem verified.

Q) find c value by using cauchy's mean value theorem in $[a, b]$ where $f(x) = e^x$, $g(x) = e^{-x}$

3, 7

2, 6

$(a, b > 0)$

(3, 7)

(2, 6)

sol:- Given $f(x) = e^x \rightarrow f'(x) = e^x$

$g(x) = e^{-x} \rightarrow g'(x) = -e^{-x}$

f(x) and g(x) is continious function in $[a, b]$,

f(x) and g(x) is differentiable function in (a, b)

f(x) and g(x) is ... $\therefore$ There exist $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$-e^c \cdot e^c = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-e^{2c} = \frac{e^b - e^a}{\frac{e^a - e^b}{e^b e^a}}$$

$$-e^{2c} = \frac{\cancel{e^b - e^a}}{\frac{-\cancel{(e^b - e^a)}}{e^{a+b}}}$$

$$e^{2c} = e^{a+b}$$

$$2c = a+b$$

$$c = \frac{a+b}{2} \in (a,b)$$

Hence Cauchy's mean value theorem verified

(ii) $a = 3, b = 7$

$$c = \frac{3+7}{2} = \frac{10}{2} = 5$$

(iii) $a = 2, b = 6$

$$c = \frac{2+6}{2} = \frac{8}{2} = 4$$

(ii) Given $f(x) = e^x \Rightarrow f'(x) = e^x$

$g(x) = e^{-x} \Rightarrow g'(x) = -e^{-x}$

$f(x)$ and $g(x)$ is continious in $[3,7]$ and it is differentiable in $(3,7)$

$\therefore$ There exist $c \in (3,7)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^7 - e^3}{e^{-7} - e^{-3}}$$

$$-e^{c}\cdot e^{c} = \frac{e^{7}-e^{3}}{\frac{1}{e^{7}}-\frac{1}{e^{3}}}$$

$$-e^{2c} = \frac{e^{7}-e^{3}}{\left(\frac{e^{3}-e^{7}}{e^{7}e^{3}}\right)}$$

$$-e^{2c} = \frac{\cancel{e^{7}-e^{3}}}{\frac{-\cancel{(e^{7}-e^{3})}}{e^{7}e^{3}}}$$

$$\cancel{-}e^{2c} = \cancel{-}e^{7+3}$$

$$2c = 10$$

$$c = 10/2$$

$$c = 5 \in (3,5)$$

Hence Cauchy's mean value theorem verified.

(ii) $f(x) = e^{x}$ and $g(x) = e^{-x}$ in $[2,6]$

Sol: Given $f(x) = e^{x} \Rightarrow f'(x) = e^{x}$

$g(x) = e^{-x} \Rightarrow g'(x) = -e^{-x}$

$f(x)$ and $g(x)$ is continious in $[2,6]$ and it is differentiable in $(2,6)$

$\therefore$ There exist $c \in (2,6)$ Such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^{c}}{-e^{-c}} = \frac{e^{6}-e^{2}}{e^{-6}-e^{-2}}$$

$$-e^{c}\cdot e^{c} = \frac{e^{6}-e^{2}}{\frac{1}{e^{6}}-\frac{1}{e^{2}}}$$

$$-e^{2c} = \frac{\frac{e^{-e}}{e^2-e^6}}{e^6 e^2}$$

$$-e^{2c} = \frac{\frac{\cancel{e^6-e^2}}{-\cancel{(e^6-e^2)}}}{e^6 e^2}$$

$$\cancel{-}e^{2c} = \cancel{-}e^{6+2}$$

$$2c = 8$$

$$c = 8/2$$

$$c = 4 \in (2,6)$$

Hence Cauchy's mean value theorem verified.

Q) If $f(x)=\log x$, $g(x)=x^2$ in $[a,b]$ with $b>a>1$ us Cauchy's mean value theorem prove that $\frac{\log b-\log a}{b-a}$:

Sol:- Given $f(x)=\log x \Rightarrow f'(x)=\frac{1}{x}$

$g(x)=x^2 \Rightarrow g'(x)=2x$

$f(x)$ and $g(x)$ is continious in $[a,b]$

it is differentiable in (a,b)

$\therefore$ There exist $c\in(a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{1}{c}}{2c} = \frac{\log b-\log a}{b^2-a^2}$$

$$\frac{1}{2c^2} = \frac{\log b-\log a}{b^2-a^2}$$

$$\frac{1}{2c^2} = \frac{\log b-\log a}{(a+b)(b-a)}$$

$$\frac{a+b}{2c^2} = \frac{\log b-\log a}{b-a}$$

(8) (i) $f(x) = x^2$ and $g(x) = x^3$ in $[1, 2]$

(ii) $f(x) = \sin x$ and $g(x) = \cos x$ in $[0, \frac{\pi}{2}]$

(iii) $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$ in $[a, b]$

(i) Sol: Given $f(x) = x^2 \Rightarrow f'(x) = 2x$

$g(x) = x^3 \Rightarrow g'(x) = 3x^2$

$f(x)$ and $g(x)$ is continious in $[1,2]$ and it is differentiable in $(1,2)$

$\therefore$ There exist $c \in (1,2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{2c}{3c^2} = \frac{4-1}{8-1}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$$9c = 14$$

$$c = 14/9 \in (1,2)$$

Hence cauchy's mean value theorem verified

(ii) Given $f(x) = \sin x$ and $g(x) = \cos x$ in $[0, \frac{\pi}{2}]$

Given $f(x) = \sin x \Rightarrow f'(x) = \cos x$

$g(x) = \cos x \Rightarrow g'(x) = -\sin x$

$f(x)$ and $g(x)$ is continious in $[0, \frac{\pi}{2}]$ and it is differentiable in $(0, \pi/2)$

$\therefore$ There exist $c \in (0, \pi/2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\cos c}{-\sin c} = \frac{\sin\frac{\pi}{2} - \sin 0}{\cos\frac{\pi}{2} - \cos 0}$$

$$-\cot c = \frac{1-0}{0-1}$$

$$\cot c = 1$$

$$c = \cot^{-1}\left(\cot\frac{\pi}{4}\right)$$

$$c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right]$$

Hence cauchy's mean value theorem verified.

(iii) $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$ in $[a,b]$

Sol: Given $f(x) = \frac{1}{x^2} \Rightarrow f'(x) = \frac{-2}{x^3}$

$$g(x) = \frac{1}{x} \Rightarrow g'(x) = \frac{-1}{x^2}$$

$f(x)$ and $g(x)$ is continious in $[a,b]$ and

it is differentiable in (a,b)

$\therefore$ There exist $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{-\frac{2}{c^{\cancel{3}}}}{-\frac{1}{\cancel{c^2}}} = \frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\frac{2}{c} = \frac{\frac{b^2-a^2}{a^2b^2}}{\frac{a-b}{ab}}$$

$$\frac{2}{c} = \frac{\frac{(a+b)\cancel{(a-b)}}{(ab)^{\cancel{2}}}}{\frac{\cancel{(a-b)}}{\cancel{ab}}}$$

$$\frac{2}{c} = \frac{a+b}{ab}$$

$$2ab = c(a+b)$$

$$c = \frac{2ab}{a+b} \in (a,b)$$

Hence cauchy's mean value theorem verified.

8) verify cauchy's mean value theorem for
$f(x)$ and $f'(x)$ in $[1, e]$ where $f(x) = \log x$

Sol: Given $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$

$g(x) = f'(x) = \frac{1}{x} \Rightarrow g'(x) = \frac{-1}{x^2}$

The function $f(x)$ is continious $[a,b]$ $[1,e]$

$f(x)$ is differentiable in (a,b)

$\therefore$ There exist $c \in (1,e)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{1}{c}}{\frac{-1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e} - \frac{1}{1}}$$

$$-\frac{\frac{1}{c}}{\frac{1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e} - 1}$$

$$-\frac{c^2}{c} = \frac{1-0}{\frac{1}{e}-1}$$

$$-c = \frac{1}{\frac{1-e}{e}}$$

$$-c = \frac{1}{\frac{-(e-1)}{e}}$$

$$c = \frac{e}{e-1} \in (1,e)$$

Hence cauchy's mean value theorem is verify

Taylor's Theorem:

* If $f : [a,b] \longrightarrow R$ such that

i) f^{n-1} is continious on $[a,b]$

(ii) f^{n-1} is differentiable on (a,b) or f^n exist on (a,b)
and $p \in Z^+$ such that there exist a point $c \in (a,b)$

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$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}$

$f^{n-1}(a) + R_n$

where $R_n = \frac{(b-a)^p (b-c)^{n-p} f^n(c)}{(n-1)!\, p}$

It is called taylor's theorem.

R_n is called schlomilch-roche! form of a remainder

Maclaurin ! Theorem:

If $f : [0,x] \to R$ such that

i) f^{n-1} is continious in $[0,x]$

ii) f^{n-1} is differentiable in $(0,x)$

Then their exist areal number $\theta \in (0,1)$ such that f

$f(x) = f(0) + xf'(0) + \frac{f''(0)}{2!} + f'''(3) + \cdots + R_n$

where $r^n = \frac{x^n (1-\theta)^{n-p} f^{(n)}(\theta x)}{p(n-1)!}$

Q) find the taylor's series expansion of $\sin(x)$ in powers of $x - \frac{\pi}{4}$.

Sol:- The taylor's series expansion of $f(x)$ in powers of $x-a$ is given by

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$

$f(x) = \sin x$

$f'(x) = \cos x \Rightarrow f'(\pi/4) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''(\pi/4) = -\sin\pi/4 = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x \Rightarrow f'''(\pi/4) = -\cos\pi/4 = -\frac{1}{\sqrt{2}}$

$a = \pi/4$

$f(x) = f(\pi/4) + \left(\frac{x-\pi/4}{1!}\right) f'(\pi/4) + \frac{(x-\frac{\pi}{4})^2}{2!} f''(\pi/4) + \frac{(x-\frac{\pi}{4})^2}{3!}$

$f'''\left(\frac{\pi}{4}\right) + \cdots$

$\sin x = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{(x-\pi/4)^2}{2}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^2}{6}\left(\frac{-1}{\sqrt{2}}\right) + \cdots$

$\sin x = \frac{1}{\sqrt{2}} + \frac{(x-\frac{\pi}{4})}{\sqrt{2}} - \frac{(x-\frac{\pi}{4})^2}{2\sqrt{2}} - \frac{(x-\frac{\pi}{4})^3}{6\sqrt{2}} + \cdots$

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use taylor's theorem express the polynomial $f(x) = 2x^3 + 7x^2 + x - 6$ in powers of $x-1$

The taylor's series expansion of $f(x)$ in powers of $x-a$ is.

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-3)^3}{3!} f'''(a) + \cdots$$

$f(a) = 2x^3 + 7x^2 + x - 6$

$f'(a) = 6x^2 + 14x + 1 \Rightarrow f'(a) = 6a^2 + 14a + 1 = 21$

$f''(a) = 12x + 14 \Rightarrow f''(a) = 12a + 14 = 26$

$f'''(a) = 12 \Rightarrow f'''(a) = 12 = 12$

$\therefore a = 1$

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^2}{3!} f'''(a) + \cdots$$

$$= 2a^3 + 7a^2 + a - 6 + \frac{x-1}{1!}(6a^2 + 14a + 1) + \frac{(x+1)^2}{2!}(12a + 14) + \frac{(x+1)^3}{6}(12) + \cdots$$

$$= 2a^3 + 7a^2 + a - 6 + \frac{x-1}{1}(6a^2 + 14a + 1) + \frac{(x-1)^2}{2}(12a + 14) + \frac{(x-1)^3}{6}(12) + \cdots$$

3) find the polynomial $f(x) = x^3 + 3x^2 + x - 4$

Sol:- Given $f(x) = x^3 + 3x^2 + x - 4 \Rightarrow f(1) = 1 + 3 + 1 - 4 = 1$

$f'(x) = 3x^2 + 6x + 1 \Rightarrow f'(1) = 3 + 6 + 1 = 10$

$f''(x) = 6x + 6 \Rightarrow f''(1) = 6 + 6 = 12$

$f'''(x) = 6 \Rightarrow f'''(1) = 6$

Given $f(x)$ is continious and differentiable in the given interval $\therefore$ The Taylor's Series expansion

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

Given $x - 1 = 0$
$x = 1 = a$

$$f(x) = f(1) + \frac{(x-1)}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \cdots$$

$$f(x) = 1 + \frac{(x-1)}{1!} 10 + \frac{(x-1)^2}{2} 12 + \cdots$$

$f(x) = 1 + 10x - 10 + 6(x-1)^2 + \cdots$

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$f(x) = 10x + 6(x-1)^2 - 9 + \cdots$

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Q) using Taylor's series expansion expand $f(x) = \log_e x$ in terms of $x-1$ and hence find the value of $\log_e 1.1$

Sol: Given $f(x) = \log_e x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f''(x) = \frac{-1}{x^2} \Rightarrow f'''(x)$

$x = 1 \Rightarrow f(1) = \log_e 1 = 0$

$f'(1) = 1$

$f''(1) = -1$

$f'''(1) = 2$

$\therefore$ By using Taylor series expansion

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + f''(a)\frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} f'''(x) +$$

$x - 1 = 0 \Rightarrow x = 1 = a$

$$f(x) = f(1) + \frac{x-1}{1!} f'(1) + \frac{(x-1)^2}{2} f''(1) + \frac{(x-1)^3}{6} f'''(1) + \cdots$$

$$f(x) = 0 + \frac{x-1}{1} 1 + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6_3} \cancel{2}(\cancel{1}) + \cdots$$

$$f(x) = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$$

$x = 1.1 \Rightarrow \log_e 1.1$

$$f(1.1) = \frac{(1.1-1)}{1} - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} + \cdots$$

$$f(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \cdots$$

sept-2021 Imp.

Q) Expand $f(x) = e^x \cos y$ in a taylor's series expansion about the point $(1, \frac{\pi}{4})$

Sol: Given $f(x) = e^x \cos y$

$f'(x) = e^x . -\sin y \frac{dy}{dx} + \cos y . e^x$

$f'(x) = e^x -(\sin y\, y') + \cos y . e^x$

$f''(x) = -\left[e^x . \frac{d}{dx}(\sin y\, y') + y' \sin y . \frac{d}{dx}(e^x)\right]$

$+\left[e^x . \frac{d}{dx}(\cos y) + \cos y . e^x\right]$

$f''(x) = -\left[e^x(\sin y . y'' + y' \cos y) + e^x y' \sin y\right]$

$+ e^x(-\sin y) y' + e^x \cos y]$

$\therefore x = 1, \quad y = \frac{\pi}{4}$

$(x) = e^1 \cos\frac{\pi}{4} = e\cdot\frac{1}{\sqrt2}$

$(x) = 0 + e^1 \cdot \cos\frac{\pi}{4} = e\cdot\frac{1}{\sqrt2}$

$(x) = e^1 \cdot \cos\frac{\pi}{4} = e\cdot\frac{1}{\sqrt2}$

By using Taylor's series expansion,

$x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$

$a = 1$

$x) = f(1) + \frac{(x-1)}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \cdots$

$(x) = e\cdot\frac{1}{\sqrt2} + (x-1)e\cdot\frac{1}{\sqrt2} + \frac{(x-1)^2}{2!} e\cdot\frac{1}{\sqrt2} + \cdots$

$= \frac{e}{\sqrt2}\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots\right)$

Verify Taylors theorem $f(x) = (1-x)^{5/2}$ with lagranges form of remainder upto two terms in $[0,1]$

Given $f(x)$ is continious and deriviable in $[0,1]$

The taylor's theorem with lagranges form of remainder is

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x), \quad 0 < \theta < 1$

Given $f(x) = (1-x)^{5/2}$

$f'(x) = \frac{5}{2}(1-x)^{5/2 - 1}\cdot(-1) = -\frac{5}{2}(1-x)^{3/2}$

$f''(x) = -\frac{5}{2}\cdot\frac{3}{2}(1-x)^{3/2-1}\cdot(-1) = \frac{15}{4}(1-x)^{1/2}$

$\therefore$ By using taylor's series expansion

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta\cdot x)$

$= 1 - \frac{5}{2}x + \frac{x^2 15}{24}(1-\theta x)^{1/2}$

$= \frac{4 - x10 + 15x^2(1-\theta x)^{1/2}}{4}$

$= \frac{9(1-\theta x)^{1/2}}{4}$

$= 1 - \frac{5}{2}x + \frac{x^2}{2}\frac{15}{4}(1-\theta x)^{1/2}$

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Substitute $x=1$ that $f(x)=0$.

$$0 = 1 - \frac{5}{2} + \frac{1}{2}\frac{15}{4}(1-\theta)^{1/2}$$

$$+\frac{3}{2} = \frac{15}{8}(1-\theta)^{1/2}$$

$$\frac{3}{2} \times \frac{8}{15} = (1-\theta)^{1/2}$$

$$\frac{4}{5} = (1-\theta)^{1/2}$$

Sq on b.s

$$\frac{16}{25} = 1-\theta$$

$$\frac{16}{25} - 1 = -\theta$$

$$\frac{16-25}{25} = -\theta$$

$$\frac{-9}{25} = -\theta$$

$$\theta = 9/25$$

Q) find the expansion of the following by using maclaurin's Series.

1. $f(x) = e^x$
2. $f(x) = \cos x$
3. $f(x) = \sin x$
4. $f(x) = \cosh x$
5. $f(x) = \sinh x$.

1. $f(x) = e^x$

Sol: Given $f(x) = e^x$

By maclaurin's series the expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^n(0)$$

$x=0$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$

$e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{6}(1) + \cdots$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

2. $f(x) = \cos x$

Sol: Given $f(x) = \cos x$

By maclourin's series the expansion of $f(x)$ is

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$

$x = 0$

$f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{4}(x) = \cos x \Rightarrow f^4(0) = 1$

$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

3) $f(x) = \sin x$

Sol: Given $f(x) = \sin x$

By maclourin's series the expansion of $f(x)$ is

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f(0) = 1$

$f''(x) = -\sin x \Rightarrow f(0) = 0$

$f'''(x) = -\cos x \Rightarrow f(0) = -1$

$f^4(x) = \sin x \Rightarrow f(0) = 0$

$f^5(x) = \cos x \Rightarrow f(0) = 1$

$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$

$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

(4) $f(x) = \cosh x$.

Sol: Given $f(x) = \cosh x$.

By maclaurin's series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{4}(0) + \cdots$$

$f(x) = \cosh x \Rightarrow f(0) = \cosh(0) = 1$

$f'(x) = \sinh x \Rightarrow f'(0) = 0$

$f''(x) = \cosh x \Rightarrow f''(0) = 1$

$f'''(x) = \sinh x \Rightarrow f'''(0) = 0$

$f^{4}(x) = \cosh x \Rightarrow f^{4}(0) = 1$

$$\cosh x \; \cancel{f(x)} = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!} + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

(5) $f(x) = \sinh x$

Sol: Given $f(x) = \sinh x$

By maclaurin's series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{4}(0) + \cdots$$

$f(x) = \sinh x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = 1$

$f''(x) = +\sinh x \Rightarrow f''(0) = 0$

$f'''(x) = \cosh x \Rightarrow f'''(0) = 1$

$f^{4}(x) = \sinh x \Rightarrow f^{4}(0) = 0$

$$\sinh x = \frac{x}{1!}(1) + \frac{x^3}{3!}(1) + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$