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JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR

B.Tech – II-I Sem

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19A04303 PROBABILITY THEORY AND STOCHASTIC PROCESSES

Course Objectives:

- To gain the knowledge of the basic probability concepts and acquire skills in handling situations involving more than one random variable and functions of random variables.
- To understand the principles of random signals and random processes.
- To be acquainted with systems involving random signals.
- To gain knowledge of standard distributions that can describe real life phenomena.

Unit I

Probability Introduced Through Sets and Relative Frequency: Experiments and Sample Spaces, Discrete and Continuous Sample Spaces, Events, Probability Definitions and Axioms, Mathematical Model of Experiments, Probability as a Relative Frequency, Joint Probability, Conditional Probability, Total Probability, Bayes' Theorem, Independent Events, Problem Solving.
Definition of a Random Variable, Conditions for a Function to be a Random Variable, Discrete, Continuous, Mixed Random Variable, Distribution and Density functions, Properties, Binomial, Poisson, Uniform, Gaussian, Exponential, Rayleigh, Conditional Distribution, Methods of defining Conditioning Event, Conditional Density, Properties, Problem Solving.

Unit Outcomes:

- Understand the fundamental concepts of probability theory, random variables, and conditional probability. (L1)
- Evaluate the different probability distribution and density functions. (L2)

Unit II

Operations on Single Random Variable: Introduction, Expectation of a random variable, moments-moments about the origin, Central moments, Variance and Skew, Chebyshev's inequality, moment generating function, characteristic function, transformations of random variable.
Multiple Random Variables: Vector Random Variables, Joint Distribution Function, Properties of Joint Distribution, Marginal Distribution Functions, Conditional Distribution and Density – Point Conditioning, Interval conditioning, Statistical Independence, Sum of Two Random

Variables, Sum of Several Random Variables, Central Limit Theorem, (Proof not expected), Unequal Distribution, Equal Distributions.

Unit Outcomes:

- Apply the knowledge to the sum of random variables, central limit theorem in communication system (L2).
- Evaluate the single and multiple random variable concepts to expectation, variance and moments (L4).

Unit III

Operations on Multiple Random Variables: Expected Value of a Function of Random Variables, Joint Moments about the Origin, Joint Central Moments, Joint Characteristic Functions, Jointly Gaussian Random Variables: Two Random Variables case, N Random Variable case, Properties of Gaussian random variables, Transformations of Multiple Random Variables, Linear Transformations of Gaussian Random Variables.

Unit Outcomes:

- Apply the different operations to multiple random variables (L2).
- Understand the concepts of linear transformation of Gaussian random variables (L1).

Unit IV

Random Processes-Temporal Characteristics: The Random Process Concept, Classification of Processes, Deterministic and Nondeterministic Processes, Distribution and Density Functions, concept of Stationarity and Statistical Independence, First-Order Stationary Processes, Second-Order and Wide-Sense Stationarity, N-Order and Strict-Sense Stationarity. Time Averages and Ergodicity, Mean-Ergodic Processes, Correlation-Ergodic Processes, Autocorrelation Function and Its Properties, Cross-Correlation Function and its Properties, Covariance Functions, Gaussian Random Processes, Poisson Random Process.
Random Processes-Spectral Characteristics: The Power Density Spectrum and its Properties, Relationship between Power Spectrum and Autocorrelation Function, The Cross-Power Density Spectrum and its Properties, Relationship between Cross-Power Spectrum and Cross-Correlation Function.

Unit Outcomes:

- Understand and analyze continuous and discrete-time random processes (L1).
- Analyze the concepts and its properties of auto correlation, cross correlation functions and power spectral density (L3).

Unit V

Random Signal Response Of Linear Systems: System Response – Convolution, Mean and Mean squared Value of System Response, autocorrelation Function of Response, Cross-Correlation Functions of Input and Output, Spectral Characteristics of System Response: Power Density Spectrum of Response, Cross-Power Density Spectrums of Input and Output, Band pass, Band Limited and Narrowband Processes, Properties.
Noise Definitions: White Noise, colored noise and their statistical characteristics, Ideal low pass filtered white noise, RC filtered white noise.

Unit Outcomes:

- Describe the theory of stochastic processes to analyze linear systems (L2).
- Apply the knowledge to linear systems; low pass and band pass noise models for random processes (L2).

Course Outcomes:

After completion of the course, student will be able to
CO1: Understanding the concepts of Probability, Random Variables, Random Processes and their characteristics learn how to deal with multiple random variables, conditional probability, joint distribution and statistical independence. (L1)
CO2: Formulate and solve the engineering problems involving random variables and random processes. (L2)
CO3: Analyze various probability density functions of random variables. (L3)
CO4: Derive the response of linear system for Gaussian noise and random signals as inputs. (L3)

TEXT BOOKS:

1. Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", 4th Edition, TMH, 2002.
2. Athanasios Papoulis and S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, PHI, 2002.

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1. Simon Haykin, "Communication Systems", 3rd Edition, Wiley, 2010.
2. Henry Stark and John W.Woods, "Probability and Random Processes with Application to Signal Processing," 3rd Edition, Pearson Education, 2002.
3. George R. Cooper, Clave D. MC Gillem, "Probability Methods of Signal and System Analysis," 3rd Edition, Oxford, 1999.

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Probability Introduced through sets & Relative Frequency

Experiment:

An experiment is a physical action or process whose results are observed and noted. The end result of the experiment is known as outcome.

Experiments are of two types

a) **Deterministic or predictable experiment:**

An experiment whose outcome can be predicted with certainty prior to the experiment is known as deterministic experiment.

eg $P \propto \frac{1}{V}$ provided T remains constant

i.e $PV =$ Constant

b) ~~Non~~ **Undeterministic or Random experiment:**

An experiment whose outcome is not known in advance.

eg: Rolling a single die and observing the numbers that shows up. There are six such numbers & they form all possible outcomes in the experiment.

Random event:

It is the outcome or set of outcomes of a random event.

eg: Consider the experiment of flipping a coin. The result of the exp will be either Tail or head.

Sample space:

The set of all possible outcomes in any given experiment is called sample space. It is denoted by 'S'

eg: 1) when a coin is tossed $S = \{H, T\}$

2) when a die is rolled $S = \{1, 2, 3, 4, 5, 6\}$

Discrete sample space:

If sample space contains finite set of events then it is said to be discrete sample space.

eg: a) when a coin is tossed $S = \{H, T\}$

b) when a die is rolled $S = \{1, 2, 3, 4, 5, 6\}$

Continuous Sample space:

If the sample space contains an infinite number of elements with continuous values within a given range, then it is called a continuous sample space.

eg: a) Consider an experiment of measuring room temperature from t_1 to t_2

$$S = \{t_1 < s < t_2\}$$

Such a sample is called continuous.

b) for an experiment "obtain a number by spinning the pointer on a wheel of chance numbered from 0 to 12"

$$S = \{0 < s \leq 12\}$$

Event:

An event is defined as subset of sample space
An event is a set.

eg: Consider an experiment of rolling a die

then $S = \{1, 2, 3, 4, 5, 6\}$

Let 'A' be an event of getting an even number

$A = \{2, 4, 6\}$

here A is subset of S & A is also a set

A set with N elements can have 2^N subsets.

A discrete sample space has discrete events. when a coin is tossed the event "head" is a discrete and finite event.

Events taken from continuous sample space can be continuous or discrete.

eg: In the exp "choose randomly a number from 6 to 13" the sample space is

$S = \{6 < s \leq 13\}$

Let A be an event that the number falls b/w 7.4 ~~the~~ & 8.5

i.e $A = \{7.4 < a \leq 8.5\}$

B be an event that the number is 9

i.e $B = \{b = 9\}$

An event which never occurs is an impossible event.

A null set is an event with no elements. It is an impossible

~~If~~ ~~Every~~

Mutually exclusive events:

If any two events in an experiment have no common outcomes then the events are said to be mutually exclusive events.

eg: For a die rolling experiment

$S = \{1, 2, 3, 4, 5, 6\}$

event $A = \{\text{even number shows up}\} = \{2, 4, 6\}$

$B = \{\text{odd number shows up}\} = \{1, 3, 5\}$

A & B have no common elements

$\therefore$ A & B are Mutually exclusive events.

Equally likely events:

The events of an experiment are equally likely, if no event occurs in preference with other events.

eg a) when a coin is tossed H & T has equal chance for occurance i.e they are equally likely events.

b) when a die is rolled, all faces of die are equally likely outcomes.

Exhaustive Events:

The total numbers of outcomes that are possible to occur from an experiment are known as exhaustive events.

eg: a) when a die is rolled the total possible exhaustive event is 6

b) when two dices are thrown, then the total no of possible exhaustive events is $6^2 = 36$

Classical or Axiomatic Definition of Probability:

If a trial results in n exhaustive, mutually exclusive & equally likely cases and m of them are favourable to the happening of an event, then the probability 'p' of happening of event 'E' is given by

$$P = P(E) = \frac{m}{n}$$

the no of cases favourable to the non-happening of the event E are $(n-m)$

$$q = P(E') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p = 1 - P(E)$$

$$\therefore\ P + q = 1$$

or $$P(E) + P(E') = 1$$

Probability as Relative Frequency:

If a trial is repeated a number of times under essentially homogenous or identical conditions then the limiting value of trials, as the number of trials become indefinitely large is called the probability of happening of the event.

If in 'n' trials an event 'E' happens 'm' times, then the probability 'p' of the happening of event 'E' is given by $P = P(E) = \lim_{n \to \infty} \frac{m}{n}$

Joint Probability:

Some events are not mutually exclusive because of common elements in the sample space. These elements correspond to the simultaneous or joint occurance of the non exclusive events.

For two events A & B the common elements form the event ~~A&B~~ $A \cap B$.

The probability $P(A \cap B)$ is called the joint probability for two events A & B which intersect in the sample space.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

i.e the prob/: of the union of two events never exceeds the sum of the event prob's, the equality holds only for mutually exclusive events because $A \cap B = \phi$

$$\therefore \; P(A \cap B) = P(\phi) = 0.$$

Conditional Probability:

Given some event B with non zero probability the conditional prob/: of an event A, given B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(A|B)$ means that the probability of an event A may depend on a second event B. If A & B are mutually exclusive then $P(A \cap B) = 0$

$$\therefore \; P(A|B) = 0$$

Axioms of probability:

The probability of an event 'A' denoted by $P(A)$ is so chosen as to satisfy the following axioms.

a) The probability of occurance of an event A in a random experiment may be zero or any positive number and it must not be a -ve number.

i.e $P(A) \geq 0$

b) As the sample space consists of all possible outcomes it should have the highest possible probability i.e one

i.e $P(S) = 1$

c) the probability of the events is equal to the union of any number of mutually exclusive events is equal to the sum of the individual event probabilities.

$$P\left[\bigcup_{n=1}^{N} A_n\right] = \sum_{n=1}^{N} P(A_n) \quad \text{if} \quad A_m \cap A_n = \phi$$

for $m \neq n$, $n = 1, 2 \ldots N$

Mathematical model of experiments:

A real experiment is defined mathematically by three things

1) assignment of sample space.

2) Definition of events of intrest

3) making probability assignments to the events such that the axioms are satisfied.

eg: An experiment consists of observing the sum of numbers showing up when two dices are thrown

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}$$

Let A be an event Sum = 7

$A = \{Sum = 7\} = \{(1,6)\ (6,1)\ (2,5)\ (5,2)\ (3,4)\ (4,3)\}$

event $B = \{8 < Sum \leq 11\} = \{Sum = 9 \text{ or } 10 \text{ or } 11\}$

$= \{(4,5)\ (5,4)\ (6,3)\ (3,6)\ (5,5)\ (6,4)\ (4,6)$
$(5,6)\ (6,5)\}$

event $C = \{10 < sum\} = \{sum = 11, 12\} = \{(6,5)(5,6)(6,6)\}$

$P(A) = \frac{6}{36}$ $\quad P(B) = \frac{9}{36}$ $\quad P(C) = \frac{3}{36}$

Total Probability:

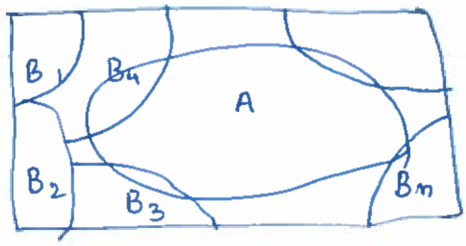
Consider a sample space S that has N mutual exclusive events B_n, $n = 1, 2 \cdots N$ such that

$B_m \cap B_n = \{\phi\}$ for $m \neq n = 1, 2 \cdots N$

The probability of any event A defined on this sample space can be expressed interms of the conditional probabilities of events B_n, This probability is known as the total probability of event A.

i.e $P(A) = \sum_{n=1}^{N} P(A/B_n)\, P(B_n)$

Proof.



$$S = \bigcup_{n=1}^{N} B_n$$

$$A \cap S = A$$

$$A = A \cap \bigcup_{n=1}^{N} B_n$$

$$P(A) = P\left[A \cap \bigcup_{n=1}^{N} B_n\right] = \bigcup_{n=1}^{N} P(A \cap B_n)$$

$$= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$$

$$= P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + \cdots + P(B_n)P(A|B_n)$$

$$P(A) = \sum_{n=1}^{N} P(B_n)\,P(A|B_n)$$

Bayes theorem:

It states that if a sample space S has N mutually exclusive events B_n, $n = 1, 2 \cdots N$ such that $B_m \cap B_n = \{\phi\}$ for $m \neq n = 1, 2 \cdots N$ and any event A is defined on this sample space then the conditional probability of B_n and A can be written as

$$P(B_n|A) = \frac{P(B_n)\,P(A|B_n)}{\sum_{n=1}^{N} P(B_n)\,P(A|B_n)}$$

Proof:
$$P(B_n|A) = \frac{P(A \cap B_n)}{P(A)} = \frac{P(B_n)\,P(A|B_n)}{P(A)}$$

$$= \frac{P(B_n)\,P(A|B_n)}{\sum_{n=1}^{N} P(B_n)\,P(A|B_n)}$$

In Bayes theorem the probabilities $P(B_n)$ are usually referred to as priori probabilities since they apply to the events B_n before the performance of the experiment.

The probs $P(A|B_n)$ are numbers typically known prior to conducting the experiment. The conditional probs are some time called as transition probs in a communication channel.

The probs $P(B_n|A)$ are called posteriori prob's since they apply after the experiments performance when some event A is obtained.

Independent Events:

Two events A & B are said to be statistically independent events if the probability of occurrence of one event is not affected by the occurence of the other event.

i.e $P(A|B) = P(A)$ & $P(B|A) = P(B)$

$$\therefore P(A \cap B) = P(A)\, P(B)$$

the joint probability of two mutually exclusive events is

$$P(A \cap B) = 0$$

$\therefore$ two events cannot be both mutually exclusive & statistically independent.

Multiple Events:

In case of three events A_1, A_2 & A_3 they are said to independent if and only if they are independent by all pairs & also independent as a tripple i.e they must satisfy the following four equations

$$P(A_1 \cap A_2) = P(A_1)\,P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1)\,P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2)\,P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$$

For N events $A_1, A_2 \cdots A_N$ to be called statistically independent we require that all the ~~Continuous~~ Conditions

$$P(A_i \cap A_j) = P(A_i)\,P(A_j)$$
$$P(A_i \cap A_j \cap A_k) = P(A_i)\,P(A_j)\,P(A_k)$$
$$\vdots$$
$$P(A_1 \cap A_2 \cap \cdots \cap A_N) = P(A_1)\,P(A_2)\cdots P(A_N)$$

to be satisfied for all $1 \le i \le j \le k \cdots \le N$

Properties of Independent events:

If N events $A_1, A_2 \cdots A_N$ are independent then any one of them is independent of any event formed by the unions, intersections & Compliments of the others.

For two independent events A_1 & A_2 it results that

A_1 is independent of $\overline{A_2}$

$\overline{A_1}$ is independent of A_2

$\overline{A_1}$ is independent of $\overline{A_2}$

For three independent events A_1, A_2 & A_3 any one is independent of the joint occurrence of the other two

$$P[A_1 \cap (A_2 \cap A_3)] = P(A_1)\,P(A_2 \cap A_3)$$
$$P[A_2 \cap (A_1 \cap A_3)] = P(A_2)\,P(A_1 \cap A_3)$$

$$P[A_3 \cap (A_1 \cap A_2)] = P(A_3)\, P(A_1 \cap A_2)$$

any one event is also independent of the union of the other two.

$$P[A_1 \cap (A_2 \cup A_3)] = P(A_1)\, P(A_2 \cup A_3)$$

$$P[A_2 \cap (A_1 \cup A_3)] = P(A_2)\, P(A_1 \cup A_3)$$

$$P[A_3 \cap (A_1 \cup A_2)] = P(A_3)\, P(A_1 \cup A_2)$$

Problems:

Q: A die is tossed find the prob's of the events
$A = \{\text{odd no shows up}\}$ $B = \{\text{no larger than 3 shows up}\}$
find $P(A)$, $P(B)$, $P(A \cup B)$ & $P(A \cap B)$.

A. $S = \{1, 2, 3, 4, 5, 6\}$ $A = \{1, 3, 5\}$ $B = \{4, 5, 6\}$

$A \cup B = \{1, 3, 4, 5, 6\}$ $A \cap B = \{5\}$

$P(A) = \frac{3}{6}$ $P(B) = \frac{3}{6}$ $P(A \cup B) = \frac{5}{6}$ $P(A \cap B) = \frac{1}{6}$

Q: In a game of dice a "shooter" can win outright if sum of two numbers showing up is either 7 or 11 when two dices are thrown. what is his probability of winning outright.

Sum of 7 = $\{(2,5)\ (5,2)\ (3,4)\ (4,3)\ (1,6)\ (6,1)\}$

Sum of 11 = $\{(5,6)\ (6,5)\}$

A: Prob% of winning outright $= P[\text{Sum} = 7 \text{ or } 11]$

$$= P[\text{Sum} = 7] + P[\text{Sum} = 11]$$

$$= \frac{6}{36} + \frac{2}{36}$$

$$= 8/36$$

Q: A pointer is spun on a fair wheel of chance having its periphery labled from 0 to 100 1.13

a) what is the sample space for this experiment

b) what is the prob/: that the pointer will stop b/w 20 & 35

c) what is the prob/: that ~~wheel~~ pointer will stop on 58.

A: a) $S = \{0 < s \leq 100\}$

b) $P(20 < s \leq 35) = \frac{35-20}{100} = \frac{15}{100}$

c) $P[s = 58] = \frac{1}{100}$

Q: An experiment has a sample space with 10 equally likely elements $S = \{a_1, a_2 \cdots a_{10}\}$ three events are defined ~~by~~ as

$A = \{a_1, a_5, a_9\}$ $B = \{a_1, a_2, a_6, a_9\}$ & $C = \{a_6, a_9\}$

find the probs of a) $A \cup C$ b) $B \cup \bar{C}$ c) $A \cap (B \cup C)$

d) $\overline{A \cup B}$ e) $(A \cup B) \cap C$.

A: a) $A \cup C = \{a_1, a_5, a_6, a_9\}$ $P(A \cup C) = \frac{4}{10}$

b) $B \cup \bar{C} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}$

$P(B \cup \bar{C}) = \frac{10}{10} = 1$

c) $A \cap (B \cup C) = \{a_1, a_9\}$ $P[A \cap (B \cup C)] = \frac{2}{10}$

d) $\overline{A \cup B} = \{a_3, a_4, a_7, a_8, a_{10}\}$ $P(\overline{A \cup B}) = \frac{5}{10}$

e) $(A \cup B) \cap C = \cancel{\{a_3, a_4, a_7},\ \{a_6, a_9\}$ $P((A \cup B) \cap C) = \frac{2}{10}$

Q: let A be an arbitary event. Show that $P(A') = 1 - P(A)$

A. $A \cup A' = S$

$$P(A) + P(A') - P(A \cap A') = P(S)$$

$P(A \cap A') = 0$ Since mutually exclusive

$$P(A') = 1 - P(A).$$

Q: An experiment consists of rolling a single dice. Two events are defined as $A = \{6 \text{ shows up}\}$ $B = \{2 \text{ or } 5 \text{ shows up}\}$ a) find $P(A)$ & $P(B)$

b) define a 3rd event c so that $P(c) = 1 - P(A) - P(B)$.

A: a) $P(A) = P(6) = \frac{1}{6}$

b) $P(B) = P(2 \text{ or } 5) = P(2) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

c) $P(A) + P(B) + P(c) = 1$

If c is exclusive of A & B

$c = \overline{A \cup B} = \{1, 3, 4\}$

$P(c) = \frac{3}{6} = \frac{1}{2}$

Q: In a box there are 500 coloured balls 75 black, 150 green, 175 red, 70 white & 30 blue. what are the prob's of selecting a ball of each colour.

A: $P(\text{black ball}) = \frac{75}{500}$ $P(\text{Green ball}) = \frac{150}{500}$

$P(\text{white ball}) = \frac{70}{500}$ $P(\text{red ball}) = \frac{175}{500}$

$P(\text{blue ball}) = \frac{30}{500}$

Q: A single card is drawn from a 52 card deck

a) what is the prob/: that the card is a Jack

b) what is the prob/: that the card will be 5 or smaller

c) what is the prob/: that the card is a red 10

A: a) $P[\text{Jack}] = \frac{4}{52}$

b) $P[5 \text{ or smaller}] = P[5 \text{ or } 4 \text{ or } 3 \text{ or } 2]$

$= P(5) + P(4) + P(3) + P(2)$

$= \frac{4}{52} + \frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{16}{52}$

c) $P[\text{Red } 10] = \frac{2c_1}{52c_1} = \frac{2}{52}$

Q: A pair of fair dice are thrown in a gambling problem person A wins if the sum of numbers showing up is six or less & one of the two dices shows four. Person B wins if the sum is five or more & one of the dice shows four find a) the prob/: that A wins b) the prob/: of B winning c) the prob/: that both A & B win.

A. a) $P[A \text{ wins}] = P[(2,4)\ (4,2)\ (1,4)\ (4,1)] = \frac{4}{36}$

b) $P[B \text{ wins}] = P[(1,4)\ (4,1)\ (2,4)\ (4,2)\ (3,4)\ (4,3)\ (4,4)$
$(4,5)\ (5,4)\ (4,6)\ (6,4)]$

$= \frac{11}{36}$

c) $P[A \text{ wins } \& \text{ B wins}] = P[A \text{ wins} \cap B \text{ wins}] = P(A \text{ wins})$

Since $A \subset B$

$= \frac{4}{36}$

Q: You (Person A) & two others (B & C) each toss a fair coin in a two step gambling game. In step 1 the person whose toss is not a match to either of the other two is "odd man out" only the remaining two whose coins match go onto step 2 to resolve the ultimate winner.

a) what is the prob/: that you will advance to step 2 after the 1st toss (b) what is the prob/: you will be out after the 1st toss (c) what is the prob/: that no one will be out after the 1st toss.

1. a) P [stay in after 1st toss] = P {(HHT)(HTH)(TTH) (THT)}

$= \frac{4}{8}$

b) P (~~stay~~ out after 1st toss) = P ((HTT)(THH)) = $\frac{2}{8}$

c) P (no odd man out) = P [(HHH) (TTT)] = $\frac{2}{8}$

Q: A particular electronic device is known to contain 10Ω, 22Ω & 48Ω resistors but these resistors are 0.25W, 0.5W, 1W rating depending on how purchases are made to minimize cost. Historically it is found that the probs of the 10Ω resistor being 0.25, 0.5 or 1W are 0.08, 0.1, 0.01 resp for the 22Ω resistors the similar probs are 0.2, 0.26 & 0.05. It is historically found that the probs that any resistors are 0.25, 0.5 & 1W resp 0.4, 0.51 & 0.09 that any resistors are what are the probs that the 48Ω resistors are a) 0.25W (b) 0.5W & c) 1W

	0.25W	0.5W	1W
10Ω	0.08	0.1	0.01
22Ω	0.2	0.26	0.05
48Ω			
	0.4	0.51	0.09

a) $P(48\Omega \cap 0.25W) = 0.4 - 0.08 - 0.2$
$= 0.12$

b) $P(48\Omega \cap 0.5W) = 0.51 - 0.1 - 0.26$
$= 0.15$

c) $P(48\Omega \cap 1W) = 0.09 - 0.01 - 0.05 = 0.03$

Q: when two dices are thrown find the prob's that

a) one die will show 2 & other will show 3 or larger

b) the sum of the two numbers showing up will be 4 or less or will be 10 or more.

A: a) $P[\text{one die is 2 \& other is 3 or more}]$

$$= P[(2,3)\ (3,2)\ (2,4)\ (4,2)\ (2,5)\ (5,2)\ (2,6)\ (6,2)] = \frac{8}{36}$$

b) $P[10 \le \text{sum} \ \& \ \text{sum} \le 4]$

$$= P[(4,6)\ (6,4)\ (5,5)\ (5,6)\ (6,5)\ (6,6)\ (1,3)\ (3,1)\ (1,2)\ (2,1)\ (2,2)\ (1,1)] = \frac{12}{36}$$

Q: In a game two dice are thrown let one die be weighted so that 4 shows up with prob $\frac{2}{7}$ while its other nos all have probs of $\frac{1}{7}$. The same apply to other die except the no 3 is weighted. Determine the prob/: the shooter will win outright by having the sum of the no's showing up be 7. what would be this prob/: for fair dice.

A:

		1/7	1/7	2/7	1/7	1/7	1/7
		1	2	3	4	5	6
1/7	1	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{2}{49}$	1/49	$\frac{1}{49}$	1/49
1/7	2	1/49	1/49	$\frac{2}{49}$	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$
1/7	3	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{2}{49}$	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$
2/7	4	$\frac{2}{49}$	$\frac{2}{49}$	$\frac{4}{49}$	$\frac{2}{49}$	$\frac{2}{49}$	$\frac{2}{49}$
1/7	5	1/49	$\frac{1}{49}$	$\frac{2}{49}$	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$
1/7	6	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{2}{49}$	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$

$P[\text{sum} = 7] = P[(1,6)\ (6,1)\ (5,2)\ (2,5)\ (4,3)\ (3,4)$

$$= \frac{9}{49}$$

$P[\text{sum} = 7 \text{ for fair dice}]$

$$= \frac{6}{36}$$

Q. In a box there are 100 resistors having resistance & tolerances as shown in below table. Let a resistor be selected from the box & assume each resistor has the same likelihood of being chosen. Define three events A as "draw a 47 Ω resistor", B as "draw a resistor with 5% tolerance" and C as "draw a 100 Ω resistor" find all the prob's. 1.18

A.

	5%	10%	Total
22 Ω	10	14	24
47 Ω	28	16	44
100 Ω	24	08	32
Total	62	38	100

$P(A) = P(47\Omega) = \frac{44}{100}$

$P(B) = P(5\%) = \frac{62}{100}$

$P(C) = P(100\Omega) = \frac{32}{100}$

$P(A \cap B) = P(47\Omega \cap 5\%) = \frac{28}{100}$

$P(A \cap C) = P(47\Omega \cap 100\Omega) = 0$

$P(B \cap C) = P(5\% \cap 100\Omega) = \frac{24}{100}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(47\Omega | 5\%) = \frac{28}{62}$

$P(A|C) = \frac{P(A \cap C)}{P(C)} = P(47\Omega | 100\Omega) = 0$

$P(B|C) = \frac{P(B \cap C)}{P(C)} = P(5\% | 100\Omega) = \frac{24}{32}$

Q: For the resistor selection experiment define event D as "draw a 22 Ω resistor" & E as "draw a resistor with 10% tolerance" find $P(D)$, $P(E)$, $P(D \cap E)$, $P(D|E)$ & $P(E|D)$.

A: $P(D) = \frac{24}{100}$ $P(E) = \frac{38}{100}$

$P(D \cap E) = P(22\Omega \cap 10\%) = \frac{14}{100}$

$P(D|E) = P(22\Omega | 10\%) = 14/38$

$P(E|D) = P(10\% | 22\Omega) = 14/24$.

Q: For the resistor selection experiment, define two mutually exclusive events B_1 & B_2 such that $B_1 \cup B_2 = S$

a) use the total prob/: theorem to find the prob/: of the event "select a 22Ω resistor" denoted as D.

b) use bayes theorem to find the prob that the resistor selected had 5% tolerance given it was 22Ω.

A: B_1 = draw a 5% resistor

B_2 = draw a 10% resistor.

a) $P(D) = P(B_1) P(D|B_1) + P(B_2) P(D|B_2)$

$$= \frac{62}{100} \cdot \frac{10}{62} + \frac{38}{100} \cdot \frac{14}{38} = \frac{24}{100}$$

b) $P(5\% | 22\Omega) = P(B_1|D) = \frac{P(B_1 \cap D)}{P(D)}$

$$= \frac{P(B_1) P(D|B_1)}{P(D)} = \frac{\frac{62}{100} \cdot \frac{10}{62}}{24/100} = \frac{10}{24}$$

Q: In three boxes there are capacitors as shown in below table. An experiment consists of 1st randomly selecting a box, assuming each has the same likelihood of selection and then selecting a capacitor from the chosen box.

a) what is the prob/: of selecting a 0.01 μF capacitor given that box 2 is selected.

b) If a 0.01 μF Capacitor is selected what is the prob that it came from box 3.

A:

	box 1	2	3	Total
0.01 μF	20	95	25	140
0.1 μF	55	35	75	165
1 μF	70	80	145	295
	145	210	245	600

a) $P(0.01\,\mu F \mid box\,2) = \dfrac{95}{210}$

b) $P(box\,3 \mid 0.01\,\mu F) = \dfrac{P(box\,3)\,P(0.01\,\mu F \mid box\,3)}{P(box\,1)\,P(0.01\,\mu F \mid box\,1) + P(box\,2)\,P(0.01 \mid box + P(box\,3)\,P(0.01\,\mu F \mid box\,3)}$

$$= \frac{\frac{1}{3}\cdot\frac{25}{245}}{\frac{1}{3}\cdot\frac{20}{145} + \frac{1}{3}\cdot\frac{95}{210} + \frac{1}{3}\cdot\frac{25}{245}} = 0.1474$$

Q. List the nine Conditional probs of capacitor selection, given certain box selections for the previous problem.

A:

$P(0.01\,\mu F \mid box_1) = \dfrac{20}{145}$

$P(0.1\,\mu F \mid box\,1) = \dfrac{55}{145}$

$P(1\,\mu F \mid box\,1) = \dfrac{70}{145}$

$P(0.01\,\mu F \mid box\,3) = \dfrac{25}{245}$

$P(0.1\,\mu F \mid box\,3) = \dfrac{75}{245}$

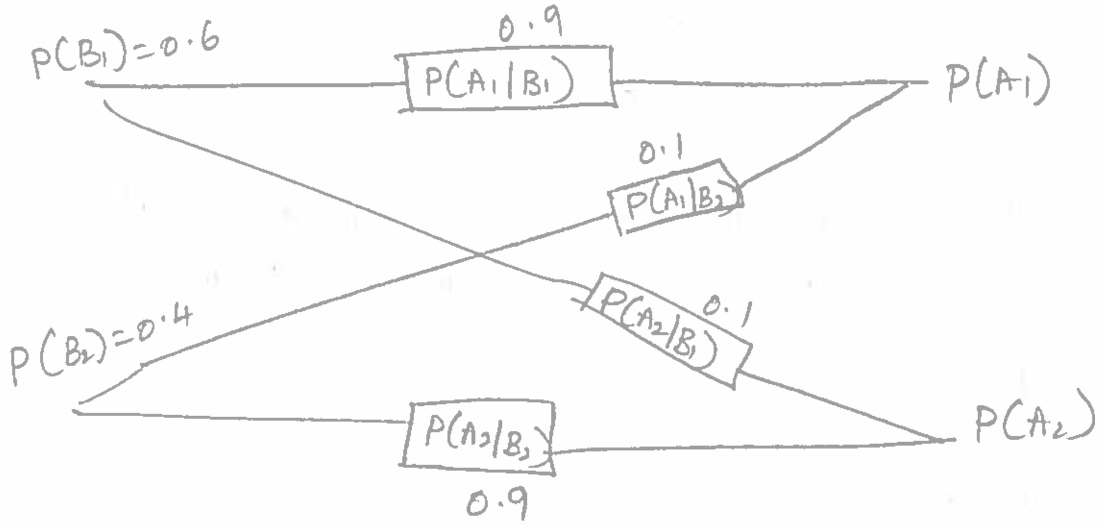
$P(1\,\mu F \mid box\,3) = \dfrac{145}{245}$

$P(0.01\,\mu F \mid box\,2) = \dfrac{95}{210}$

$P(0.1\,\mu F \mid box\,2) = \dfrac{35}{210}$

$P(1\,\mu F \mid box\,2) = \dfrac{80}{210}$

Q. An elementary binary Communication System Consists of a Tx that sends one of two possible symbols (1 or 0) over a channel to Rx. The channel occasionally Causes errors to occur so that 1 shows up at the Rx as 0 and vice versa. The prob's that symbols 1 & 0 are selected for transmission are assumed to be 0.6 & 0.4. The transition prob is 0.9 find 1) the Rxed symbol probs (2) the system prob's for Correction transmission (3) find the error prob's.

A: 

1) Rxed Symbol probs

$$P(A_1) = P(B_1)\,P(A_1|B_1) + P(B_2)\,P(A_1|B_2)$$

$$= 0.6\,(0.9) + 0.4\,(0.1) = 0.58$$

$$P(A_2) = P(B_1)\,P(A_2|B_1) + P(B_2)\,P(A_2|B_2)$$

$$= 0.6\,(0.1) + 0.4\,(0.9) = 0.42$$

2) Correct System transmission of symbols

$$P(B_1|A_1) = \frac{P(B_1 \cap A_1)}{P(A_1)} = \frac{P(B_1)\,P(A_1|B_1)}{P(A_1)}$$

$$= \frac{0.6\,(0.9)}{0.58} = 0.931$$

$$P(B_2|A_2) = \frac{P(B_2 \cap A_2)}{P(A_2)} = \frac{P(B_2)\,P(A_2|B_2)}{P(A_2)} = \frac{0.4\,(0.9)}{0.42} = 0.857$$

3) Probᵇ of system error

$$P(B_1|A_2) = \frac{P(B_1 \cap A_2)}{P(A_2)} = \frac{P(B_1)\,P(A_2|B_1)}{P(A_2)}$$

$$= \frac{0.6\,(0.1)}{0.42} = 0.143$$

$$P(B_2|A_1) = \frac{P(B_2 \cap A_1)}{P(A_1)} = \frac{P(B_2)\,P(A_1|B_2)}{P(A_1)}$$

$$= \frac{0.4\,(0.1)}{(0.58)} = 0.069$$

Q: Rework binary communication system problem if $P(B_1) = 0.6$, $P(B_2) = 0.4$, $P(A_1|B_1) = P(A_2|B_2) = 0.95$
$P(A_2|B_1) = P(A_1|B_2) = 0.05$.

A: $$P(A_1) = P(B_1)\,P(A_1|B_1) + P(B_2)\,P(A_1|B_2)$$

$$= 0.6\,(0.95) + 0.4\,(0.05) = 0.59$$

$$P(A_2) = P(B_1)\,P(A_2|B_1) + P(B_2)\,P(A_2|B_2)$$

$$= 0.6\,(0.05) + 0.4\,(0.9) = 0.41$$

$$P(B_1|A_1) = \frac{P(B_1 \cap A_1)}{P(A_1)} = \frac{P(B_1)\,P(A_1|B_1)}{P(A_1)} = \frac{0.95\,(0.6)}{0.59}$$

$$= 0.966$$

$$P(B_2|A_2) = \frac{P(B_2 \cap A_2)}{P(A_2)} = \frac{P(B_2)\,P(A_2|B_2)}{P(A_2)} = \frac{0.4\,(0.95)}{0.41}$$

$$= 0.927$$

$$P(B_1|A_2) = \frac{P(B_1 \cap A_2)}{P(A_2)} = \frac{P(B_1)\,P(A_2|B_1)}{P(A_2)} = \frac{0.6\,(0.05)}{0.41}$$

$$= 0.073$$

$$P(B_2|A_1) = \frac{P(B_2 \cap A_1)}{P(A_1)} = \frac{P(B_2)\,P(A_1|B_2)}{P(A_1)} = \frac{0.4\,(0.05)}{0.59}$$

$$= 0.034$$

Q: Rework binary Communication System problem if $P(B_1) = 0.7$, $P(B_2) = 0.3$, $P(A_1|B_1) = P(A_2|B_2) = 1$ & $P(A_2|B_1) = P(A_1|B_2) = 0$ what type of channel does this System have.

A. $P(A_1) = P(B_1)\,P(A_1|B_1) + P(B_2)\,P(A_1|B_2)$

$= 0.7\,(1) + 0.3\,(0) = 0.7$

$P(A_2) = P(B_1)\,P(A_2|B_1) + P(B_2)\,P(A_2|B_2)$

$= 0.7\,(0) + 0.3\,(1) = 0.3$

$$P(B_1|A_1) = \frac{P(B_1 \cap A_1)}{P(A_1)} = \frac{P(B_1)\,P(A_1|B_1)}{P(A_1)}$$

$$= \frac{0.7\,(1)}{0.7} = 1$$

$$P(B_2|A_2) = \frac{P(B_2 \cap A_2)}{P(A_2)} = \frac{P(B_2)\,P(A_2|B_2)}{P(A_2)} = \frac{0.3\,(1)}{0.3} = 1$$

$$P(B_1|A_2) = \frac{P(B_1 \cap A_2)}{P(A_2)} = \frac{P(B_1)\,P(A_2|B_1)}{P(A_2)} = \frac{0.7\,(0)}{0.3} = 0$$

$$P(B_2|A_1) = \frac{P(B_2 \cap A_1)}{P(A_1)} = \frac{P(B_2)\,P(A_1|B_2)}{P(A_1)} = \frac{0.3\,(0)}{0.7} = 0$$

The System is ideal or noise free since the prob's of an error in Symbol reception are zero.

Q: A Company sells high fidelity amplifiers capable of generating 10, 25 & 50 W of audio power. It has on hand 100 of the 10 W units of which 15% are defective, 70 of the 25 W units with 10% defective & 30 of 50 W units with 10% defective.

Sol a) what is the prob/: that an amplier sold from the 10 W unit is defective.

b) If each wattage amplifier sells with equal likelihood what is the prob/: of a randomly selected unit being 50 W & defective.

c) what is the prob/: that a randomly selected unit for sale is defective.

A:

Event	10 W	25 W	50 W	total
D = defective	15	7	3	
G = good	85	63	27	
total	100	70	30	

a) $P(D|10W) = \dfrac{15}{100}$

b) $P(D \cap 50W) = P(50W)\,P(D/50W)$

$= \dfrac{1}{3} \cdot \dfrac{3}{30} = \dfrac{1}{30}$

c) $P(D) = P(10W)P(D|10W) + P(25W)P(D|25W) + P(50W)P(D|50W)$

$= \dfrac{1}{3} \cdot \dfrac{15}{100} + \dfrac{1}{3} \cdot \dfrac{7}{70} + \dfrac{1}{3} \cdot \dfrac{3}{30} = 0.1167$

Q. A missile can be accidentaly launched if two relays A & B both have failed. The prob's of A & B failling are known to be 0.01 & 0.03 resp. It is also known that B is more likely to fail (prob of 0.06) if A has failed.

a) what is the prob of an accidental missile launch

b) what is the prob that A will fail if B has failed

c) are the events A fails & B fails statistically independent

1: a) $P(\text{launch}) = P(A \text{ fails} \cap B \text{ fails})$

$= P(A \text{ fails})\, P(B \text{ fails} \mid A \text{ fails})$

$= 0.01\,(0.06) = 0.0006$

b) $P(A \text{ fails} \mid B \text{ fails}) = \dfrac{P(A \text{ fails} \cap B \text{ fails})}{P(B \text{ fails})} = \dfrac{0.0006}{0.03} = 0.02$

c) $P(A \text{ fails}) \cdot P(B \text{ fails}) = 0.01\,(0.03) = 0.0003$

$\neq P(A \text{ fails} \cap B \text{ fails})$

hence the events B fails & A fails are not statistically independent.

Q: A pharmaceutical product consists of 100 pills in a bottle two production lines are used to produce the product are selected with probs 0.45 (line one) & 0.55 (line two). each line can overfill or underfill bottles by atmost 2 pills. Given that line one is observed the prob's are 0.02, 0.06, 0.88 & 0.03 & 0.01 that the no of pills in a bottle will be 102, 101, 100, 99 & 98 resp. for line two the similar probs are 0.03, 0.08, 0.83, 0.04 & 0.02 find (a) the prob/: that a bottle of the product will contain 102 pills repeat for 101, 100, 99 & 98 pills. b) Given that a bottle contains the correct no of pills what is the prob/: it came from line one. (c) what is the prob/: that a purchaser of the product will recieve less than 100 pills.

1:

pills	L1	L2
102	0.02	0.03
101	0.06	0.08
100	0.88	0.83
99	0.03	0.04
98	0.01	0.02

$P(L_1) = 0.45$, $P(L_2) = 0.55$

$P(102 \mid L_1) = 0.02$ $\quad P(102 \mid L_2) = 0.03$

$P(101 \mid L_1) = 0.06$ $\quad P(101 \mid L_2) = 0.08$

$P(100 \mid L_1) = 0.88$ $\quad P(100 \mid L_2) = 0.83$

$P(99 \mid L_1) = 0.03$ $\quad P(99 \mid L_2) = 0.04$

$P(98 \mid L_1) = 0.01$ $\quad P(98 \mid L_2) = 0.02$

a) $P(102 \text{ pills}) = P(L_1)\,P(102|L_1) + P(L_2)\,P(102|L_2)$

$= 0.45\,(0.02) + 0.55\,(0.03) = 0.0255$

$P(101 \text{ pills}) = P(L_1)\,P(101|L_1) + P(L_2)\,P(101|L_1)$

$= 0.45\,(0.06) + 0.55\,(0.08) = 0.071$

$P(100 \text{ pills}) = P(L_1)\,P(100|L_1) + P(L_2)\,P(100|L_2)$

$= 0.45\,(0.88) + 0.55\,(0.83) = 0.8525$

$P(99 \text{ pills}) = P(L_1)\,P(99|L_1) + P(L_2)\,P(99|L_2)$

$= 0.45\,(0.03) + 0.55\,(0.04) = 0.0355$

$P(98 \text{ pills}) = P(L_1)\,P(98|L_1) + P(L_2)\,P(98|L_2)$

$= 0.45\,(0.01) + 0.55\,(0.02) = 0.0155$

b) $P(L_1|100) = \dfrac{P(L_1 \cap 100)}{P(100)} = \dfrac{P(L_1)\,P(100|L_1)}{P(100)}$

$= \dfrac{0.45(0.88)}{0.8525} = 0.4645$

c) $P(\text{pills} < 100) = P(99 \text{ pills}) + P(98 \text{ pills})$

$= 0.0355 + 0.0155 = 0.0510$

Q: A manufacturing plant makes radios that each contain an (IC) integrated ckt supplied by three sources A, B & C. The prob's that the IC in a radio came from one of the sources is $\frac{1}{3}$ and is same for all other sources. IC's are known to be defective with prob's 0.001, 0.003, 0.002 for sources A, B & C

a) what is the prob that any given radio will contain a defective IC

b) If a radio contains a defective IC, find the prob/: that it came from source A. Repeat for sources B & C.

1. A = IC from source A

B = IC from source B $\qquad$ D = defective

C = IC from source C

$P(A) = P(B) = P(C) = \frac{1}{3}$

$P(D|A) = 0.001$, $P(D|B) = 0.003$ $\qquad P(D|C) = 0.002$

a) $P(D) = P(A)\,P(D|A) + P(B)\,P(D|B) + P(C)\,P(D|C)$

$$= \frac{1}{3}\left[0.001 + 0.003 + 0.002\right] = 0.002$$

b) $P(A|D) = \dfrac{P(A)\,P(D|A)}{P(D)} = \dfrac{\frac{1}{3}(0.001)}{0.002} = \dfrac{1}{6}$

$P(B|D) = \dfrac{P(B)\,P(D|B)}{P(D)} = \dfrac{\frac{1}{3}(0.003)}{0.002} = \dfrac{1}{2}$

$P(C|D) = \dfrac{P(C)\,P(D|C)}{P(D)} = \dfrac{\frac{1}{3}(0.002)}{0.002} = \dfrac{1}{3}$

Q: There are three special deck of cards. 1st deck D_1 has all 52 cards of a regular deck, The second D_2 has only the 16 face cards of a regular deck. The third D_3 has only the 36 numbered cards of a regular deck. A random experiment consists of 1st randomly choosing one of the three decks, the second randomly choosing a card from a chosen deck. If $P(D_1) = \frac{1}{2}$, $P(D_2) = \frac{1}{3}$ & $P(D_3) = \frac{1}{6}$ find the prob's of a) drawing an ace (b) drawing 3

c) drawing a red card.

A: a) $P(A = \text{ace}) = P(D_1)P(A|D_1) + P(D_2)P(A|D_2) + P(D_3)P(A|D_3)$

$$= \frac{1}{2}\frac{4}{52} + \frac{1}{3}\frac{4}{16} + \frac{1}{6}(0) = 0.1218$$

b) $P(3) = P(D_1)P(3|D_1) + P(D_2)P(3|D_2) + P(D_3)P(3|D_3)$

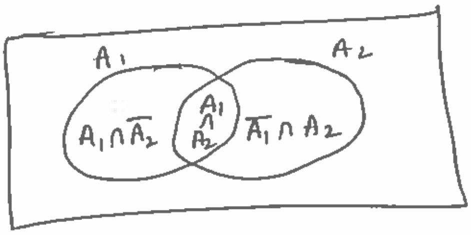
$$= \frac{1}{2}\cdot\frac{4}{52} + \frac{1}{3}\cdot 0 + \frac{1}{6}\cdot\frac{4}{36} = 0.057$$

c) $P(R = \text{red Card}) = P(D_1)\cdot P(R|D_1) + P(D_2)P(R|D_2) + P(D_3)P(R|D_3)$

$$= \frac{1}{2}\cdot\frac{26}{52} + \frac{1}{3}\cdot\frac{8}{16} + \frac{1}{6}\cdot\frac{18}{36} = 0.5$$

Q: Given that two events A_1 & A_2 are statistically independent show that a) A_1 is independent of $\overline{A_2}$ (b) $\overline{A_1}$ is independent of A_2 (c) $\overline{A_1}$ is independent of $\overline{A_2}$.

A



a) $P(A_1 \cap \overline{A_2}) = P(A_1) - P(A_1 \cap A_2)$

$$= P(A_1) - P(A_1)P(A_2)$$

$$= P(A_1)\left[1 - P(A_2)\right]$$

$$= P(A_1)P(\overline{A_2})$$

b) $P(\overline{A_1} \cap A_2) = P(A_2) - P(A_1 \cap A_2)$

$$= P(A_2) - P(A_1)P(A_2) = P(A_2)\left(1 - P(A_1)\right)$$

$$= P(\overline{A_1})P(A_2)$$

c) $P(\overline{A_1} \cap \overline{A_2}) = P(\overline{A_1 \cup A_2}) = 1 - P(A_1 \cup A_2)$

$$= 1 - P(A_1) - P(A_2) + P(A_1 \cap A_2)$$

$$= 1 - P(A_1) - P(A_2) + P(A_1)P(A_2)$$

$$= \left[1 - P(A_1)\right] - P(A_2)\left[1 - P(A_1)\right]$$

$$= \left[1 - P(A_1)\right]\left[1 - P(A_2)\right] = P(\overline{A_1})P(\overline{A_2})$$

Q: Show that there are $2^N - N - 1$ equations required for

$$P(A_1 \cap A_2 \cap \cdots \cap A_N) = P(A_1)\,P(A_2)\cdots P(A_N)$$

A: Combining N things 2 at time $= \binom{N}{2} = {}^{N}C_2$

" " " 3 " " $= \binom{N}{3}$

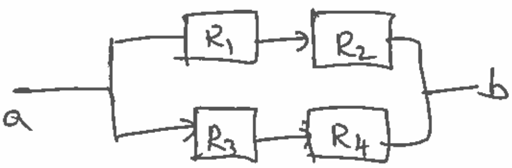
" " " N " " $= \binom{N}{N}$

no of equation $= \sum_{i=2}^{N} \binom{N}{i}$

$$= \sum_{i=0}^{N} \binom{N}{i} - \binom{N}{1} - \binom{N}{0}$$

$$= 2^N - N - 1$$

Q: In a Communication System the Signal is sent from point a to point b arrives by two paths in parallel over each path the Signal passes through two repeaters (in series). each repeater in one path has a prob/ of failing (becoming an o.c) of 0.005. This prob/ is 0.008 for each repeater on the other path. All repeaters fail independently by each other. find the prob/ that the signal will not arrive at point B.



A

P (signal does not arrive)

$$= P(\text{upper path fails} \cap \text{lower path fails})$$

$$= P(\text{upper path fails}) \cdot P(\text{lower path fails})$$

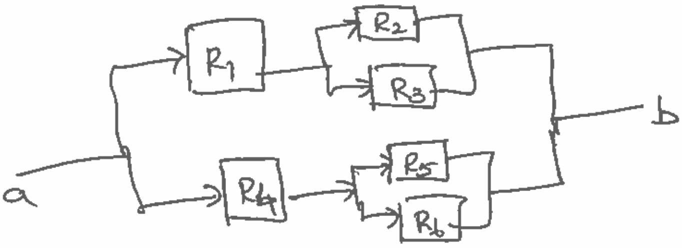
$$= P[R_1 \cup R_2]\; P[R_3 \cup R_4]$$

$$= \left(P(R_1) + P(R_2) - P(R_1)P(R_2)\right)\left(P(R_3) + P(R_4) - P(R_3)\cdot P(R_4)\right)$$

$$= (0.05 + 0.05 - 0.05 \times 0.05)\,[0.08 + 0.08 - 0.08 \times 0.08]$$

Q: work the above problem except assume the path and repeaters as shown in below fig where the probs of the repeaters failing (independently) are $P_1 = P(R_1) = 0.005$

$P_2 = P(R_2) = P(R_3) = P(R_4) = 0.01$, $P_3 = P(R_5) = P(R_6) = 0$



A: P[signal does not arrive]

= P[upper path fails ∩ lower path fails]

= P[upper path fails] P(lower path fails)

$P[\text{upper path fails}] = P[R_1 \cup (R_2 \cap R_3)]$

$= P[R_1] + P(R_2 \cap R_3) - P(R_1 \cap R_2 \cap R_3)$

$= P(R_1) + P(R_2)P(R_3) - P(R_1)P(R_2)P(R_3)$

$= P_1 + P_2^2 - P_1 P_2^2 = 5.0995 \times 10^{-3}$

$P[\text{lower path fails}] = P[R_4 \cup (R_5 \cap R_6)]$

$= P(R_4) + P(R_5 \cap R_6) - P(R_4 \cap R_5 \cap R_6)$

$= P(R_4) + P(R_5)P(R_6) - P(R_4)P(R_5)P(R_6)$

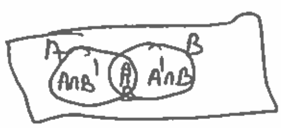
$= P_2 + P_3^2 - P_2 P_3^2 = 12.475 \times 10^{-3}$

$P(\text{signal does not arrive}) = (5.0995 \times 10^{-3})(12.475 \times 10^{-3})$

$= 63.6163 \times 10^{-6}$

Q: For any two events S.T $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A: A & $A' \cap B$ are disjoint



$$A\cup B = A\cup (A'\cap B)$$

$$P(A\cup B) = P[A\cup (A'\cap B)] = P(A) + P(A'\cap B)$$

$$= P(A) + \left(P'(A\cap B) + P(A\cap B)\right) - P(A\cap B)$$

$$= P(A) + P(B) - P(A\cap B)$$

Q: If A, B & C are mutually independent events, Show that $A\cup B$ & C are also independent.

A:
$$P[(A\cup B)\cap C] = P[(A\cap C)\cup (B\cap C)]$$

$$= P(A\cap C) + P(B\cap C) - P(A\cap B\cap C)$$

$$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$$

$$= P(C)[P(A) + P(B) - P(A)P(B)]$$

$$= P(C)\,P(A\cup B).$$

Q: Given that two events A, B & C are pairwise independent & A is independent of $(B\cup C)$. Show that A, B & C are mutually independent.

A
$$P(A\cap B) = P(A)P(B) \qquad P(A\cap (B\cup C)) = P(A)P(B\cup C)$$

$$P(B\cap C) = P(B)P(C)$$

$$P(A\cap C) = P(A)P(C)$$

$$P(A\cap (B\cup C)) = P[(A\cap B)\cup (A\cap C)]$$

$$= P(A\cap B) + P(A\cap C) - P[A\cap B\cap C]$$

$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \rightarrow (1)$$

$$P(A)P(B\cup C) = P(A)[P(B) + P(C) - P(B)P(C)]$$

$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \rightarrow (2)$$

from equ's ① & ②

$P[A \cap B \cap C] = P(A)\, P(B)\, P(C)$

hence A, B & C are mutually independent

Permutations:

For r elements being drawn, the no of possible sequences of r elements from the original n is denoted by P_r^n & is given by

$$P_r^n = \frac{n!}{(n-r)!}$$

this is the number of permutation or sequences of r elements taken from n elements when order of occurence is important.

eg: How many permutations are there when four cards are taken from 52 card deck.

A: $P_4^{52} = \frac{52!}{(52-4)!} = 52 \times 51 \times 50 \times 49 = 6497400$

Combinations:

If r elements have to be taken from n elements without replacement then the resulting no of sequences where order is not important is called the no of combinations of r things taken from n things

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!\, r!}$$

Q. A Coach has 5 athlets from whoom a 3 person team is to be Selected for a Competition. How many such teams Could be selected.

A. no of teams $= {}^5C_3 = \dfrac{5!}{(5-3)!\,3!} = 10$

Q: find the prob/: of three half rupee Coins following all heads up when tossed simultaneously.

A. $P(HHH) = \dfrac{1}{8}$

Q: A Coin is tossed if it turns up heads two balls will be drawn from box A otherwise two balls will be drawn from box B. Box A Contains 3B & 5W balls. Box B Contains 7B & 1W. In both the Cases selections are to be made with replacement. what is the prob/: that box A is used given that both balls drawn are black.

A. $P = \dfrac{1}{2} \times \dfrac{3}{8} \times \dfrac{3}{8} = \dfrac{9}{128}$

Q. Determine the prob of the card being either Red or king. when one Card is drawn from a regular deck of 52 Cards.

A: Let Event E_1 = red Card

E_2 = King Card

$$P(E_1 \text{ or } E_2) = P(\text{Red Card or king Card})$$

$$= P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

Q: what is the prob/: of drawing 3W & 4G balls from a bag containing 5W & 6G balls if 7 balls are drawn simultaneously at random.

A $P[3W \text{ \& } 4G \text{ balls}) = \dfrac{5c_3 \times 6c_4}{52c_7} = \dfrac{5}{11}$

Q. when two dice are thrown find the prob of getting the sum of 10 or 11.

A $P[\text{sum of } 10 \text{ or } 11] = P[\text{sum} = 10] + P[\text{sum} = 11]$

$= P\{(5,5)(4,6)(6,4)\} + P[(5,6)(6,5)]$

$= \dfrac{3}{36} + \dfrac{2}{36} = \dfrac{5}{36}$

Q: A pack contains 4W & 2G pencils another contains 3W & 5G pencils. If one pencil is drawn from each pack. find the prob/: a) both are white (b) one is white & another is G

A. a) $P[\text{both pencils are white}) = \dfrac{4}{6} \times \dfrac{3}{8} = \dfrac{1}{4}$

b) $P[\text{one pencil is white \& another is } G]$

$= P(\text{white from pack1 \& Green from pack2})$

$+ P(\text{white from pack 2 \& Green from pack1})$

$= \dfrac{4}{6} \times \dfrac{5}{8} + \dfrac{2}{6} \times \dfrac{3}{8} = 0.542$

Q: when two dices are thrown simultaneously determine the prob's for the following three events

a) $A = \{\text{sum} = 7\}$ (b) $B = \{8 < \text{sum} \leq 11\}$

c) $C = \{10 < \text{sum}\}$ d) $P(B \cap C)$

A: a) $A = \{Sum = 7\} = \{(1,6)\ (6,1)\ (2,5)\ (5,2)\ (3,4)\ (4,3)\}$

$P(A) = \frac{6}{36} = \frac{1}{6}$

b) $B = \{8 < Sum \le 11\} = \{Sum = 9 \text{ or } 10 \text{ or } 11\}$

$= \{(5,4)\ (4,5)\ (3,6)\ (6,3)\ (5,5)\ (4,6)\ (6,4)\ (5,6)\ (6,5)\}$

$P(B) = \frac{9}{36} = \frac{1}{4}$

c) $C = \{10 < sum\} = \{sum = 11 \text{ or } 12\} = \{(5,6)\ (6,5)\ (6,6)\}$

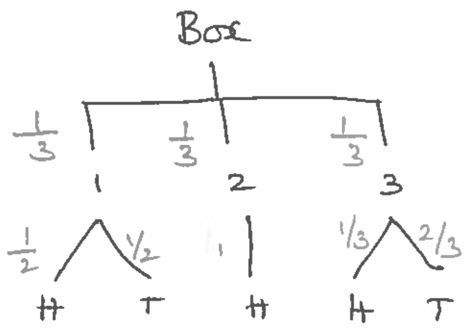
$P(C) = \frac{3}{36} = \frac{1}{12}$

d) $B \cap C = \{Sum = 11\} = \{(5,6)\ (6,5)\}$

$P(B \cap C) = \frac{2}{36}$

Q: A box Contains 3 Coins, one is fair, one is two headed & one is weighted so that the prob of heads appearing is $\frac{1}{3}$. A coin is selected at random & tossed. find the prob/: that head appears.

A: $P[H\ H\ H] = P(H \text{ from box } 1)$
$+ P(H \text{ from box } 2)$
$+ P(H \text{ from box } 3)$



$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{3} = \frac{11}{18}$

Q: If a three digit decimal no is chosen at random find the prob/: that exactly k digits are greater than or
[illegible]

1: for $k=0$ $\quad P_1 \; P_2 \; P_3 \quad <5 \quad <5 \quad <5 \quad = {}^4c_1 \cdot {}^5c_1 \cdot {}^5c_1$

$= 100$

$$P[k=0) = \frac{100}{900}$$

$\underline{k=1}$

$P_1 \; P_2 \; P_3 \quad \geq 5 \quad <5 \quad <5 \quad = {}^5c_1 \cdot {}^5c_1 \cdot {}^5c_1 = 125$

$P_1 \; P_2 \; P_3 \quad <5 \quad \geq 5 \quad <5 \quad = {}^4c_1 \cdot {}^5c_1 \cdot {}^5c_1 = 100$

$P_1 \; P_2 \; P_3 \quad <5 \quad <5 \quad \geq 5 \quad = {}^4c_1 \cdot {}^5c_1 \cdot {}^5c_1 = 100$

$$P(k=1] = \frac{125+100+100}{900} = \frac{325}{900}$$

$\underline{k=2}$

$P_1 \; P_2 \; P_3 \quad \geq 5 \quad \geq 5 \quad <5 \quad = {}^5c_1 \cdot {}^5c_1 \cdot {}^5c_1 = 125$

$P_1 \; P_2 \; P_3 \quad \geq 5 \quad <5 \quad \geq 5 \quad = {}^5c_1 \cdot {}^5c_1 \cdot {}^5c_1 = 125$

$P_1 \; P_2 \; P_3 \quad <5 \quad \geq 5 \quad \geq 5 \quad = {}^4c_1 \cdot {}^5c_1 \cdot {}^5c_1 = 100$

$$P[k=2] = \frac{125+125+100}{900} = \frac{350}{900}$$

$\underline{k=3}$

$P_1 \; P_2 \; P_3 \quad \geq 5 \quad \geq 5 \quad \geq 5 \quad = {}^5c_1 \cdot {}^5c_1 \cdot {}^5c_1 = 125$

$$P(k=3) = \frac{125}{900}$$

$P(0 \leq k \leq 3) =$ overall prob $= P(k=0] + P(k=1) + P(k=2] + P(k=3)$

$$= \frac{100}{900} + \frac{325}{900} + \frac{350}{900} + \frac{125}{900}$$

$$= \frac{900}{900} = 1$$

Q: A Jar Contains two white & 3 balack balls. A Sample of Size 4 is made. what is the prob/: that the Sample is in the order $P[W\,B\,W\,B]$.

A: $P[W\,B\,W\,B] = \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} = \frac{1}{10} = 0.1$

Q. A Jar Contains 52 badges numbered from 1 to 52. Suppose that the numbers 1 to 13 are Considered lucky. A Sample of Size 2 is drawn from the jar with replacement. what is the prob/: that a) both badges drawn will be lucky (b) neither badges will be lucky (c) exactly one of the badges drawn will be lucky (d) atleast one of the badges will be lucky.

A:

a) $P[\text{both badges lucky}] = \frac{13c_1}{52c_1} \cdot \frac{13c_1}{52c_1} = 0.0625$

b) $P[\text{neither badges lucky}] = \frac{39c_1}{52c_1} \cdot \frac{39c_1}{52c_1} = 0.5625$

c) $P[\text{one badge lucky \& other unlucky}] = \frac{13c_1}{52c_1} \cdot \frac{39c_1}{52c_1} = 0.375$

d) $P[\text{atleast one badge lucky}] = 1 - P(\text{neither badges lucky})$

$= 1 - 0.5625 = 0.4375$

Q: In a factory there are four machines the machines produce 10%, 20%, 30%, 40% of an item respectively. The defective items produced by each machine are 5%, 4%, 3% & 2% resp. Now an item is selected which is to be defective. what is the prob/: of it being from 2nd machine.

A: Let B_1, B_2, B_3 & B_4 be the events of producing an item by the four machines.

$P(B_1) = 0.1, \quad P(B_2) = 0.2, \quad P(B_3) = 0.3, \quad P(B_4) = 0.4$

Let D = producing defective item

$P(D|B_1) = 0.05$, $P(D|B_2) = 0.04$, $P(D|B_3) = 0.03$

$P(D|B_4) = 0.02$

$P(D) = P(B_1)P(D|B_1) + P(B_2)P(D|B_2) + P(B_3)P(D|B_3) + P(B_4)P(D|B_4)$

$= 0.03$

Prob/: of selecting a defective item from the second machine is

$$P(B_2|D) = \frac{P(B_2)P(D|B_2)}{P(D)} = \frac{0.2\,(0.04)}{0.03} = 0.267$$

Q. In a bolt factory machines A, B & C manufacture 30%, 30, & 40% of the total o/p resp. from their o/p's 4, 5, 3 percents are defective bolts. A bolt is drawn at random and found to be defective what are the prob's that it was manufac-ured by machines A, B & C.

A. $P(A) = 0.3$, $P(B) = 0.3$, $P(C) = 0.4$

D = defective $P(D|A) = 0.04$, $P(D|B) = 0.05$

$P(D|C) = 0.03$

$P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C) = 0.039$

$$P(A|D) = \frac{P(A)P(D|A)}{P(D)} = 0.3071 = \frac{\overset{0.3}{\cancel{0.04}}\,(0.04)}{0.039}$$

$$P(B|D) = \frac{P(B)P(D|B)}{P(D)} = \frac{0.3\,(0.05)}{0.039} = 0.3846$$

$$P(C|D) = \frac{P(C)P(D|C)}{P(D)} = \frac{0.4\,(0.03)}{0.039} = 0.3071$$

Q: A letter is known to have come from either ~~two~~ TATA NAGAR or CALCUTTA on the envelop only the two consecutive letters 'TA' are vissible. find the prob: that the letter has come from CALCUTTA.

A. let A be the event that the letter ~~two~~ is from Calcutta

B " " " " " " " TATANAGAR

$P(A) = P(B) = \frac{1}{2}$ $\qquad P(C|A) = \frac{1}{7}$

C = " two consecutive letters TA' $\qquad P(C|B) = \frac{2}{7}$

$$P(A|C) = \frac{P(A)\,P(C|A)}{P(C)}$$

$$= \frac{P(A)\,P(C|A)}{P(A)\,P(C|A) + P(B)\,P(C|B)} = \frac{\frac{1}{7} \times \frac{1}{2}}{\frac{1}{7} \times \frac{1}{2} + \frac{1}{7} \times \frac{2}{7}}$$

$= 0.33$

Q: A letter is taken at random out of "ASSISTANT" and out of "STATISTICS" what is the chance that they are the same letters.

A. Let event 1 be the letter from ASSISTANT

event 2 " " " " STATISTICS

Prob of letter A from event 1 $\quad P(A_1) = \frac{2}{9}$

" " " S " " 1 $\quad P(S_1) = \frac{3}{9}$

" " " I " " 1 $\quad P(I_1) = \frac{1}{9}$

" " " T " " 1 $\quad P(T_1) = \frac{2}{9}$

" " " N " " 1 $\quad P(N_1) = 1/9$

Illy for event 2 $P(A_2) = \frac{1}{10}$, $P(S_2) = \frac{3}{10}$

$P(I_2) = \frac{2}{10}$, $P(T_2) = \frac{3}{10}$ & $P(C_2) = \frac{1}{10}$

Since the events 1 & 2 are independent

$$P = P(A_1 A_2) + P(S_1 S_2) + P(I_1 I_2) + P(T_1 T_2)$$

$$= P(A_1)P(A_2) + P(S_1)P(S_2) + P(I_1)P(I_2) + P(T_1)P(T_2)$$

$$= \left(\frac{2}{9} \times \frac{1}{10}\right) + \left(\frac{3}{9} \times \frac{3}{10}\right) + \left(\frac{1}{9} \times \frac{2}{10}\right) + \left(\frac{2}{9} \times \frac{3}{10}\right)$$

$$= 0.211$$

Q. what is the prob/: of drawing 3W & 4G balls from a bag that contains 5W & 6G balls if 7 balls are drawn simultaneously at random.

A. $$P(WWWGGGG) = \frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \times \frac{5}{7} \times \frac{4}{6} \times \frac{3}{5}$$

$$= 0.013$$

Q. Show that the chances of throwing six with 4, 3, 2 dices respectively are 1 : 6 : 18.

A. when 4 dices are thrown the favourable out comes for getting a sum of 6 are

$\{1113, 1131, 1311, 3111, 2211, 2121, 2112, 1122, 1221, 1212\}$

$$P[\text{sum} = 6] = \frac{10}{6^4} = P_1$$

when 3 dices are thrown the favourable out comes are

for 1 2 3 the no of ways are $3! = 6$

for 1 1 4 " " " are $\frac{3!}{2!} = 3$

for 2 2 2 " " " " $3!/3! = 1$

total no of ways = $6+3+1 = 10$

$\therefore P[\text{sum} = 6] = \frac{10}{6^3} = P_2$

when two dices are thrown the favourable out comes are

$\{(1,5)\ (5,1)\ (2,4)\ (4,2)\ (3,3)\}$

$P[\text{Sum} = 6] = \frac{5}{6^2} = P_3$

$P_1 : P_2 : P_3 = \frac{10}{6^4} : \frac{10}{6^3} : \frac{5}{6^2} = \frac{10}{36} : \frac{10}{6} : 5$

$= \frac{2}{36} : \frac{2}{6} : 1 = 2 : 12 : 36$

$= 1 : 6 : 18$

Q: A box contains 4 bad and 6 good tubes. The tubes are checked by drawing a tube at random, testing and repeating the process untill all 4 bad tubes are located. what is the prob/: that the 4th bad tube will be located a) on the 5th test (b) on the 10th test.

A. Let B = bad tube G = Good tube.

one possible way of locating 4th bad tube on 5th test is G B B B $\underline{B}$

$P(G\ B\ B\ B\ B) = \frac{6c_1 \cdot 4c_1 \cdot 3c_1 \cdot 2c_1 \cdot {}^1c_1}{10c_5}$

$= \frac{6 \times 4 \times 3 \times 2 \times 1}{10 \times 9 \times 8 \times 7 \times 6} = \frac{1}{210}$

no of ways $= 4c_3 = 4$ $\qquad {}^{(n-1)}c_{(n-1)}$

required prob/: $= 4 \times \frac{1}{210} = 0.019$

b) 4th bad tube located on 10th test

G G G G G G B B B B

$$\text{Prob}/: = \frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}\cdot\frac{4}{4}\cdot\frac{3}{3}\cdot\frac{2}{2}\cdot\frac{1}{1}$$

$$= \frac{1}{210}$$

no of favourable ways $= {}^{(10-1)}C_{(4-1)} = {}^{9}C_{3} = 84$

$\therefore$ required prob $= 84 \times \frac{1}{210} = 0.4$

Q. find the prob/: of 3 Coins Showing all heads when tossed Simultaneously.

A. $P[H_1\, H_2\, H_3] = P[H_1]\, P[H_2]\, P[H_3] = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$

Q. From a pack of 52 Cards define event A as "drawing a king Card", event B as "drawing a J or Q" and C as "drawing a heart Card" then find whether A, B & C are independent.

A. $P(A) = \frac{4}{52}$, $P(B) = P[J \text{ or } Q] = P(J) + P(Q)$

$= \frac{8}{52}$

$P(C) = \frac{13}{52}$

$P(A \cap C) = P[K \cap \text{heart Card}] = \frac{1}{52}$

$P(A \cap B) = P[K \cap J \text{ or } Q] = 0$

$P(B \cap C) = P(J \text{ or } Q \cap \text{heart Card}) = \frac{2}{52}$

$P(A)\, P(B) = \frac{4}{52}\cdot\frac{8}{52} \neq P(A \cap B)$

$\therefore$ A & B are not independent

but A & B are mutually exclusive

$$P(B)\,P(C) = \frac{8}{52} \cdot \frac{13}{52} = \frac{1}{26} = P(B \cap C)$$

$\therefore$ B & C are independent

$$P(A)\,P(C) = \frac{4}{52} \cdot \frac{13}{52} = \frac{1}{52} = P(A \cap C)$$

$\therefore$ A & C are independent

Q: A pack contains 4 W & 2 G pencils, another contains 3 W & 5 G pencils. If one pencil is drawn from each pack find the proby. that a) both are white & (b) one is white and another is G.

A. a) P (both are white) = P (W from 1st pack) · P(W from 2nd pack)

$$= \frac{4}{6} \cdot \frac{3}{8} = 0.25$$

b) P (1 white & another is Green) $= \frac{4}{6} \cdot \frac{5}{8} + \frac{2}{6} \cdot \frac{3}{8}$

$$= 0.417$$

Bernoulli Trials:

Consider an experiment in which there are only two possible outcomes. Assume that these events are statistically independent for every trial. Let an event occur on any given trial with probl. $P(A) = P$. If the experiment is repeated for N trials, called Bernoulli trials then the probl. of event A occuring exactly K times in N independent trails $(K < N)$ is given by

$$P[A \text{ occurs exactly } k \text{ times}) = {}^{N}C_{k}\, p^{k}\, (1-p)^{N-k}$$

where $${}^{N}C_{k} = \frac{N!}{(k!)\,(N-k)!}$$

Poisson's Approximation:

$$P(A \text{ occurs exactly } k \text{ times}) = \frac{(NP)^{k}\, e^{-NP}}{k!}$$

for N very large & P very small

eg:: An aircraft is used to fire at target. It will be successful if 2 or more bombs hit the target. If the aircraft fires 3 bombs and the prob of the bomb hitting the target is 0.4, Then what is the prob/: that the target is hit.

A: $N = 3$, $P = 0.4$ Bernoulli's problem

$$P(k=2) = {}^{3}C_{2}\,(0.4)^{2}\,(0.6)^{3-2} = 0.228$$

$$P(k=3) = {}^{3}C_{3}\,(0.4)^{3}\,(0.6)^{3-3} = 0.064$$

Prob of target is hit $= P(k=2) + P(k=3) = 0.352$

Q. A PC board contains 205 components. Each component has the prob of not failing as 0.9996. All components are required not to fail if the PC board is not to fail. Each component is independent of all others as far as failure mechanisms are concerned. Find ① the prob of exactly one component failing ② The prob/: of PC board not failing ③ The prob of atmost one component failing

1: Given $N = 205$, $P =$ prob of Component failing

$P = 1 -$ Prob of Component not failing $= 1 - 0.9996 = 0.0004$

N is very large & P is very small

using poisson's approximation

$$NP = 0.082$$

$$P(K) = \frac{(NP)^K e^{-NP}}{K!}$$

a) Prob of exactly one component failing $K=1$

$$P[K=1] = \frac{(0.082)^1 \times e^{-0.082}}{1!} = 0.0755$$

b) Prob of the PC board not failing

= Prob of no component failing $K=0$

$$P[K=0] = \frac{(0.082)^0 e^{-0.082}}{0!} = 0.9213$$

c) Prob/: of atmost one component failing

= Prob/: of ~~any~~ one or more component failing

$$P[K>0] = 1 - P(0) = 1 - 0.9213 = 0.0787$$

Q: A Coin is tossed infinite no of times. Show that the prob/: that K heads are observed at the N^{th} toss is equal to $^{(N-1)}C_{(K-1)} \, p^K q^{N-K}$.

A: a Coin is tossed infinite no of times

let event A be "a head observed" $P(A) = p$

Prob of A not occuring $= q = 1 - p$

Prob of event A occurs exactly K no of times $= P[K \leq N]$

$$= N_{C_k}\, p^k q^{N-k}$$

for N^{th} toss, the prob of getting k heads are $= \frac{k_{C_1}}{N_{C_1}} = \frac{k}{N}$

$\therefore$ required prob $= P = \frac{k}{N} P[k \geq N]$

$$= \frac{k}{N} N_{C_k}\, p^k q^{N-k}$$

$$= \frac{k}{N} \frac{N!}{(N-k)!\,k!} p^k q^{N-k}$$

$$= \frac{(N-1)!}{[(N-1)-(k-1)]!\,(k-1)!} p^k q^{N-k}$$

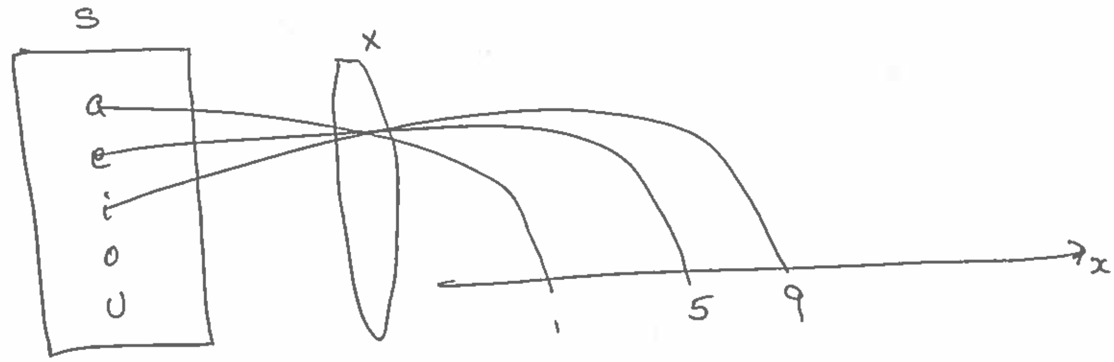
$$= (N-1)_{C_{(k-1)}}\, p^k q^{N-k}$$

<u>Random Variable:</u>

A rv is defined as a real fn of the elements of sample space S. RV's are represented by a capital letters such as W, X, Y etc and any particular value of the rv by a lower case letter such as w, x or y.

A rv X can be considered to be a fn that maps all elements of the sample space into points on a real line.

<u>eg:</u>



Conditions for Function to be RV:

The required Conditions for a fn to be RV are

1) Every point in the Sample Space must correspond to only one value of the RV.

   i.e RV must not be multivalued — eg $X(\lambda_1) = 2, 3$ is not valid

2) The set $[X \leq x]$ shall be an event for any real number x. The prob/. of this event is equal to the sum of the prob's of all elementary events corresponding to $[X \leq x]$. This is denoted as $P[X \leq x]$.

3) The prob's of events $[X = \infty]$ and $[X = -\infty]$ are zero

   i.e $P(X = -\infty) = 0$ & $P[X = \infty] = 0$

Discrete RV:

The values of a dRV are only the discrete values in a given sample space. The sample space for a dRV can be continuous, discrete or even both. They may also be finite or infinite.

Let X be a dRV with integer events $X = \{x_1, x_2 \cdots x_n\}$. The prob/. of X at any event is a fn of x_i. It is given by

$$P[X = x_i] = f_X(x_i) \qquad i = 1, 2, 3 \cdots$$

eg: A sample space is defined by the set $S = \{1, 2, 3, 4\}$ a RV ~~X(s)~~ X is defined by $X = X(s) = s^3$

Since s is a dRV its points map to dRV X in the set $\{1, 8, 27, 64\}$. If the prob's of elements of S are

$$P(1) = \frac{4}{24}, \quad P(2) = \frac{3}{24}, \quad P(3) = \frac{7}{24} \quad \& \quad P(4) = \frac{10}{24}$$

then the prob/. of the RV's values become $P[X=1] = \frac{4}{24}$

$P[X=8] = \frac{3}{24}$, $P[X=27] = \frac{7}{24}$ & $P(X=64) = \frac{10}{24}$

because of one-to-one mapping of discrete pts

Continuous r.v:

The values of a continuous r.v are continuous on a given continuous sample space. A continuous sample space has an infinite range of values.

eg: the experiment "wheel of chance" has a continuous sample space but we could define a d.r.v as having the value '1' for the set of outcomes $\{0 \le s \le 6\}$ & -1 for $\{6 < s \le 12\}$

the result is a d.r.v defined on a continuous sample space

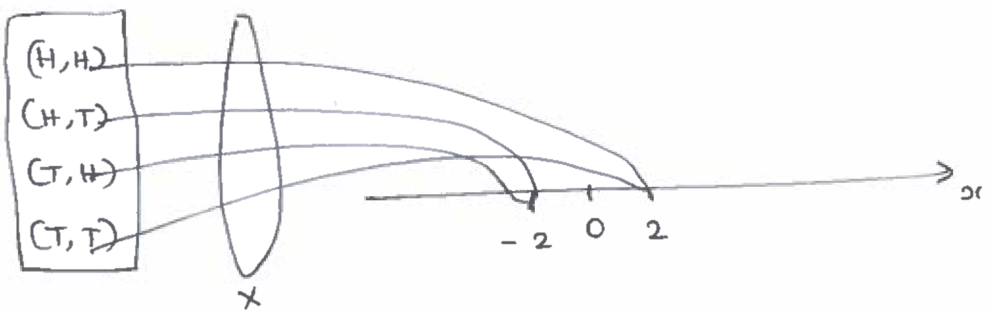
Mixed r.v:

The values of a mixed r.v are both continuous and discrete in a given sample space.

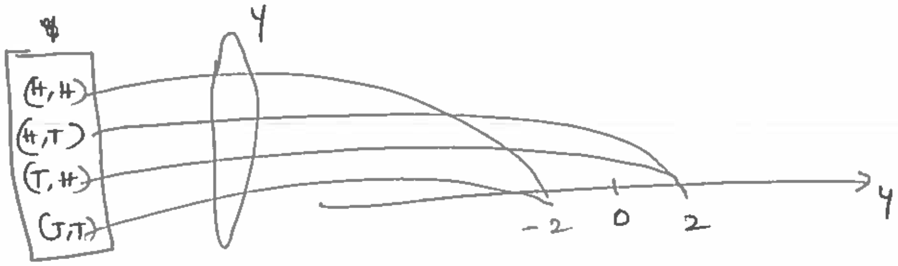
eg: $\{10, 12, 14, 15 \le x \le 20, 22, 24\}$

Q: A man matches coin flips with a friend. He ~~Hsher~~ wins \$2 if coins match & loses \$2 if they do not match. Sketch the sample space showing possible outcomes for this exp & illustrate how the pts map onto the real line x that defines the values of the r.v X = "dollar won on a trial". Show a 2nd mapping for a r.v Y = "dollar won by the friend on a trial".

A: a)

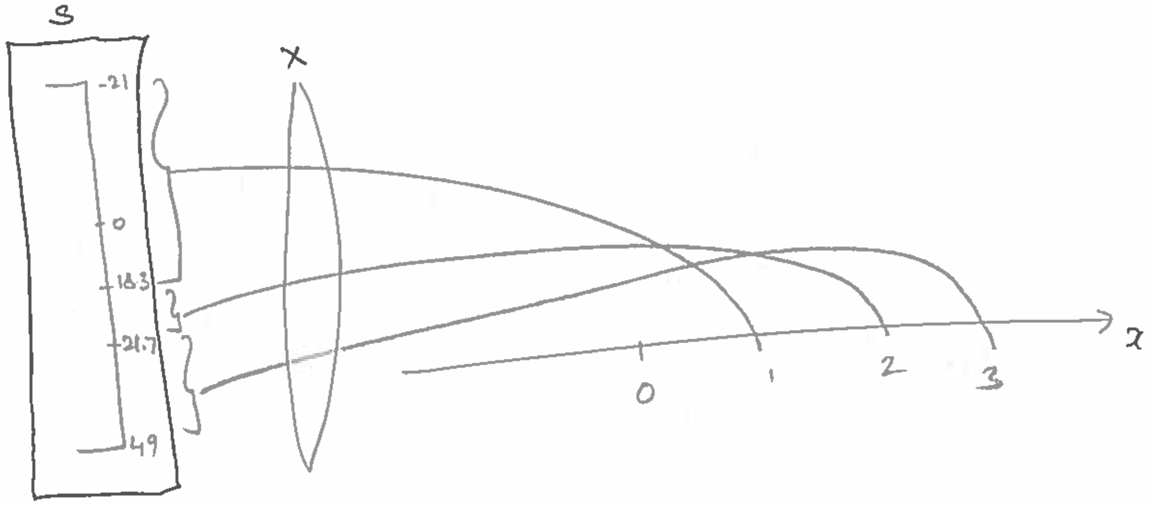


b)



Q: Temperature in a given city varies randomly during any year from -21 to $49^\circ c$. A house in the city has a thermostat that assumes only three positions (1) represents "call for heat below $18.3^\circ c$" (2) represents "dead or idle zone" & (3) represents "call for air conditioning above $21.7^\circ c$. Draw a sample space for this problem showing the mapping neccessary to define a rv x = "thermostat setting".

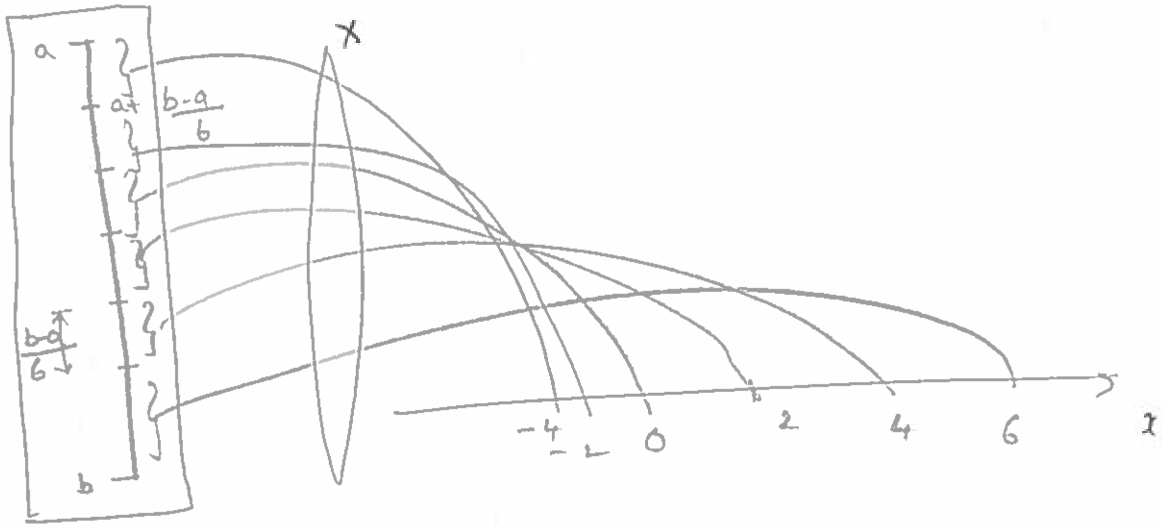
A: $$S = \{-21^\circ c < s \le 49^\circ c\}$$



Q. A random voltage can have any value defined by the set $S = \{a \le s \le b\}$. A quantizer divides S into 6 equal sized contiguous subsets & generates a voltage rv x having value $\{-4, -2, 0, 2, 4, 6\}$ each value of x is equal to the midpoint of the subset of S from which it is mapped. a) sketch the sample space & the mapping to the line x that defines the values of x

b) find a & b.

A:



$$\frac{a + a + \frac{b-a}{6}}{2} = -4 \Rightarrow a + \frac{(b-a)}{12} = -4 \longrightarrow (1)$$

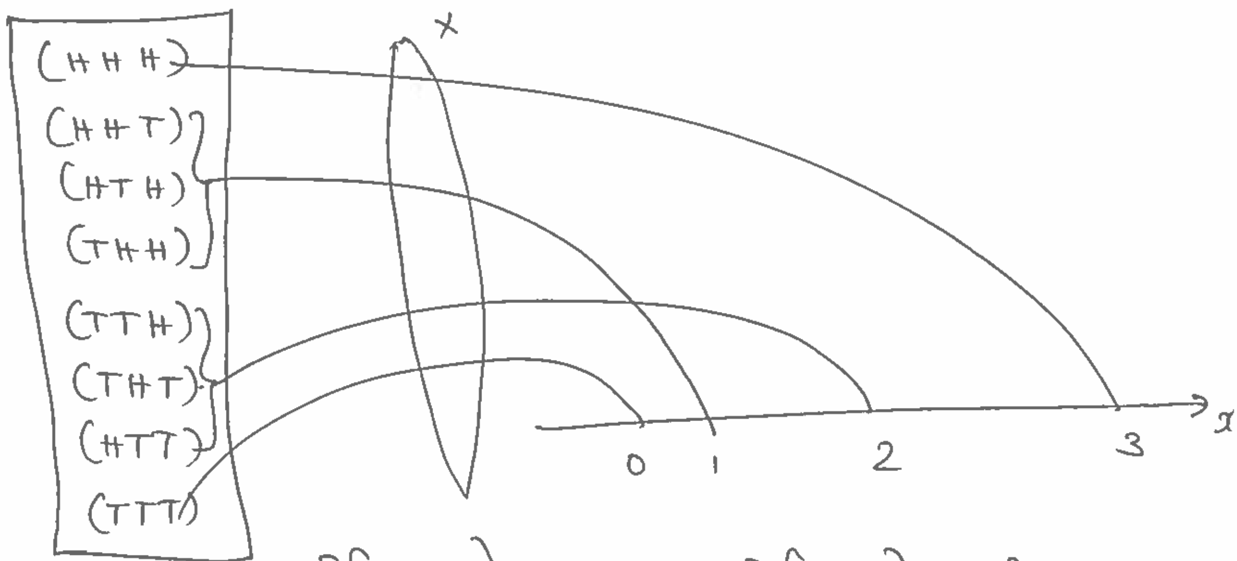
$$\frac{b-a}{6} = 2 \longrightarrow (2)$$

Solving (1) & (2) we get

$a = -5$, $b = 7$

Q: An honest coin is tossed 3 times (a) sketch the applicable sample space showing all possible elements. Let x be a rv that has values representing the no of heads on any tripple toss & sketch the mapping of S onto the real line denoted by defining x (b) find the prob's of values of x.

A.



b) $P[x=0] = \frac{1}{8}$, $P[x=1] = \frac{3}{8}$

$P[x=2] = 3/8$ & $P[x=3] = 1/8$

Q: work the above problem for a biased coin for which $P(H) = 0.6$

1. Bernouli trial problem $p = 0.6,\ q = 0.4,\ n = 3$

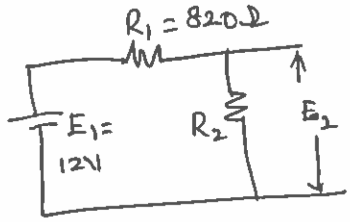
$$P[X=0] = {}^3C_0\,(0.6)^0\,(0.4)^{3-0} = 0.064$$

$$P[X=1] = {}^3C_1\,(0.6)^1\,(0.4)^{3-1} = 0.288$$

$$P[X=2] = {}^3C_2\,(0.6)^2\,(0.4)^{3-2} = 0.432$$

$$P[X=3] = {}^3C_3\,(0.6)^3\,(0.4)^{3-3} = 0.216$$

Q: Resistor R_2 shown in below fig is randomly selected from a box of resistors containing 180 Ω, 470 Ω, 1000 Ω & 2200 Ω. All resistors values have the same likelihood of being selected. The voltage E_2 is a drv find the set of values E_2 can have & give their probs.



1. $$E_2 = \left(\frac{R_2}{820 + R_2}\right) \times 12$$

for 180 Ω $\quad E_2 = 12 \times \left(\frac{180}{180+820}\right) = 2.16V$

for 470 Ω $\quad E_2 = 12 \times \left(\frac{470}{820+470}\right) = 4.372V$

for 1000 Ω $\quad E_2 = 12 \times \left(\frac{1000}{820+1000}\right) = 6.593V$

for 2200 Ω $\quad E_2 = 12 \times \left(\frac{2200}{820+2200}\right) = 8.741V$

thus $E_2 = \{2.16,\ 4.372,\ 6.593,\ 8.741\}$

Since all resistor values are equally likely so all the voltage values are equally probable.

thus $P = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$

1.52

Q: A Sample space is defined by $S = \{1, 2 \le s \le 3, 4, 5\}$ a RV is defined by $X = 2$ for $0 \le s \le 2.5$, $X = 3$ for $2.5 < s \le 3.5$ & $X = 5$ for $3.5 \le s \le 6$.

a) Is X discrete, Continuous or mixed

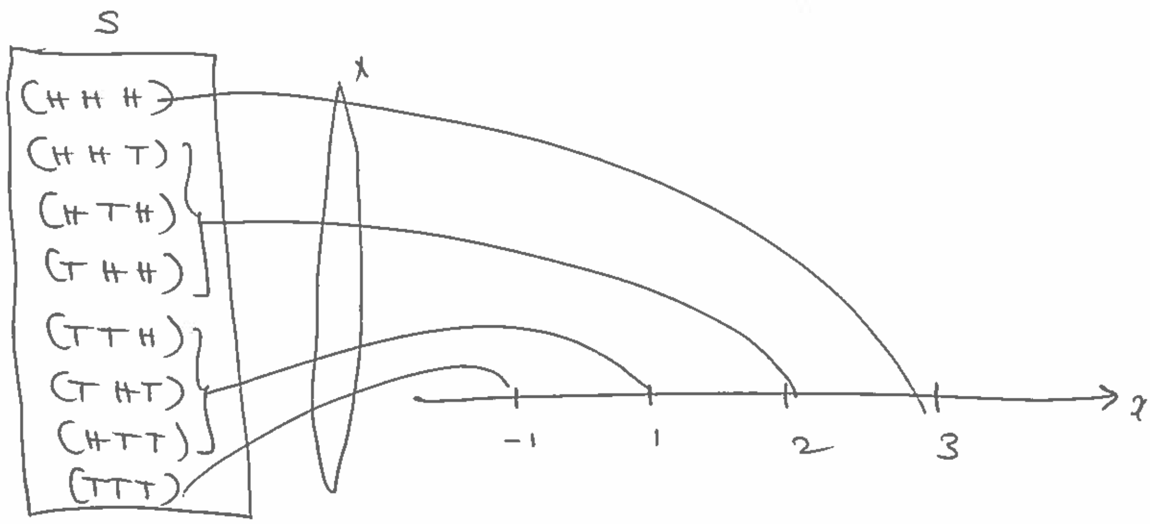
b) give a set that defines the values X can take.

A a) X is a discrete RV

b) $X = \{2, 3, 5\}$

Q: A Gambler flips a fair coin 3 times (a) draw a sample space S for this exp. A RV X representing his winning is defined as follows: He loses \$1 if he gets no heads in 3 flips, he wins \$1, \$2, \$3 if he obtains 1, 2 or 3 heads resp. a) Show how elements of S map to values of x (b) what are the probs of the various values of x.

A. a)



b) $P[X = -1] = \frac{1}{8}$, $P[X = 1] = \frac{3}{8}$

$P[X = 2] = \frac{3}{8}$, $P[X = 3] = \frac{1}{8}$

Distribution Function:

The prob/: distribution fn (PDF) describes the probabilistic behaviour of a r.v.

The prob/: $P[X \leq x]$ is the prob/: of the event $\{X \leq x\}$. It is a fn of x. It is also known as cummulative prob/: distribution fn of r.v X and denoted as $F_X(x)$ which is a fn of x. Thus

$$\boxed{F_X(x) = P[X \leq x]}$$

where x is any real number in the range $-\infty \leq x \leq \infty$

Properties:

1) $F_X(-\infty) = 0$

1: $F_X(-\infty) = P[X \leq -\infty]$

Since there are no real numbers less than $-\infty$

$\therefore P[X \leq -\infty] = 0$

$\therefore F_X(-\infty) = 0$

2) $F_X(\infty) = 1$

$F_X(\infty) = P[X \leq \infty] = P(S) = 1$

3) $0 \leq F_X(x) \leq 1$

4) If $x_1 < x_2$ then $F_X(x_1) \leq F_X(x_2)$

1. The event $\{X \leq x_1\}$ is a subset of the event $\{X \leq x_2\}$

$\therefore P[X \leq x_1] \leq P[X \leq x_2]$

$\therefore F_X(x_1) \leq F_X(x_2)$

5) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

1: $\{X \leq x_1\}$ & $\{x_1 < X < x_2\}$ are mutually exclusive events

$$[X \le x_2] = [X \le x_1] + [x_1 < X \le x_2]$$

$$P[X \le x_2] = P[X \le x_1] + P[x_1 < X \le x_2]$$

$$F_X(x_2) = F_X(x_1) + P[x_1 < X \le x_2]$$

$$\therefore \; P[x_1 < X \le x_2] = F_X(x_2) - F_X(x_1)$$

6) $F_X(x^+) = F_X(x)$ where $x^+ = x + \epsilon$ & $\epsilon > 0$ is infinitely small i.e $\epsilon \to 0$

7) $P(X = x) = F_X(x) - F_X(x^-)$

1: $$P[x_1 < X \le x_2] = F_X(x_2) - F_X(x_1)$$

If $x_1 = x - \epsilon$ & $x_2 = x$

as $\epsilon \to 0$ $x_1 = x^-$

$$P[x - \epsilon < X \le x] = F_X(x) - F_X(x - \epsilon)$$

as $\epsilon \to 0$

$$P[X = x] = F_X(x) - F_X(x^-)$$

8) $P[x_1 \le X \le x_2] = F_X(x_2) - F_X(x_1^-)$

1. $$[x_1 \le X \le x_2] = [x_1 < X \le x_2] + [X = x_1]$$

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) + P[X = x_1]$$

$$= F_X(x_2) - F_X(x_1) + F_X(x_1) - F_X(x_1^-)$$

$$= F_X(x_2) - F_X(x_1^-)$$

9) $P[X > x] = 1 - F_X(x)$

1: $[X \le x]$ & $(X > x)$ are two mutually exclusive events

$$[X \le x] + [X > x] = S$$

$$P[X \le x] + P[X > x] = P(S)$$

$$P[X > x] = 1 - F_X(x)$$

PDF of d.r.v:

If x is d.r.v then

$$F_X(x) = \sum_{i=1}^{N} P[x = x_i]\, u(x - x_i)$$

where $u(x)$ is a unit step fn defined by

$$u(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$$

$P[X = x_i]$ can also be written as $P(x_i)$

$$\therefore \boxed{F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)}$$

Density Function:

The prob. density fn denoted by $f_X(x)$ is defined as the derivative of the distribution function.

$$\boxed{f_X(x) = \frac{dF_X(x)}{dx}}$$

If x is a d.r.v then the pdf of the r.v is given by

$$\boxed{f_X(x) = \sum_{i=1}^{N} P(x_i)\, \delta(x - x_i)}$$

properties of density function:

1) $f_X(x) \ge 0$ for all x.

since $F_X(x)$ varies from 0 to 1. The differentiation of these values must be a +ve number.

2) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

1. $\int_{-\infty}^{\infty} f_X(x)\, dx = F_X(x)\Big|_{-\infty}^{\infty} = F_X(-\infty) - F_X(\infty) = 1 - 0 = 1$

3) $F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi$.

1: $\int_{-\infty}^{x} f_X(\xi)\, d\xi = F_X(x)\Big|_{-\infty}^{x} = F_X(x) - F_X(-\infty)$

$= F_X(x) - 0 = F_X(x)$

4) $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx$.

1: $\int_{x_1}^{x_2} f_X(x)\, dx = F_X(x)\Big|_{x_1}^{x_2} = F_X(x_2) - F_X(x_1)$

$= P[x_1 < X \leq x_2]$.

Probability Distribution & density function curve for a d.r.v:

Consider an experiment of rolling two dices & let x be the sum of numbers showing up on the dices.

$P[X=0] = 0$

$P[X=1] = 0$

$P[X=2] = \frac{1}{36}$

$P[X=3] = \frac{2}{36}$

$P[X=4] = \frac{3}{36}$

$P[X=5] = \frac{4}{36}$

$P[X=6] = \frac{5}{36}$

$P[X=7] = \frac{6}{36}$

$P[X=8] = \frac{5}{36}$

$P[X=9] = \frac{4}{36}$

$P[X=10] = \frac{3}{36}$

$P[X=11] = \frac{2}{36}$

$P[X=12] = \frac{1}{36}$

$F_X(x) = P[X \leq x]$

$F_X(0) = P[X \leq 0]$

$= 0$

$F_X(1) = P[X \le 1] = 0$

$F_X(2) = P[X \le 2] = \frac{1}{36}$

$F_X(3) = P[X \le 3] = \frac{3}{36}$

$F_X(4) = P[X \le 4] = \frac{6}{36}$

$F_X(5) = P[X \le 5] = \frac{10}{36}$

$F_X(6) = P[X \le 6] = \frac{15}{36}$

$F_X(7) = \frac{21}{36} = P[X \le 7]$

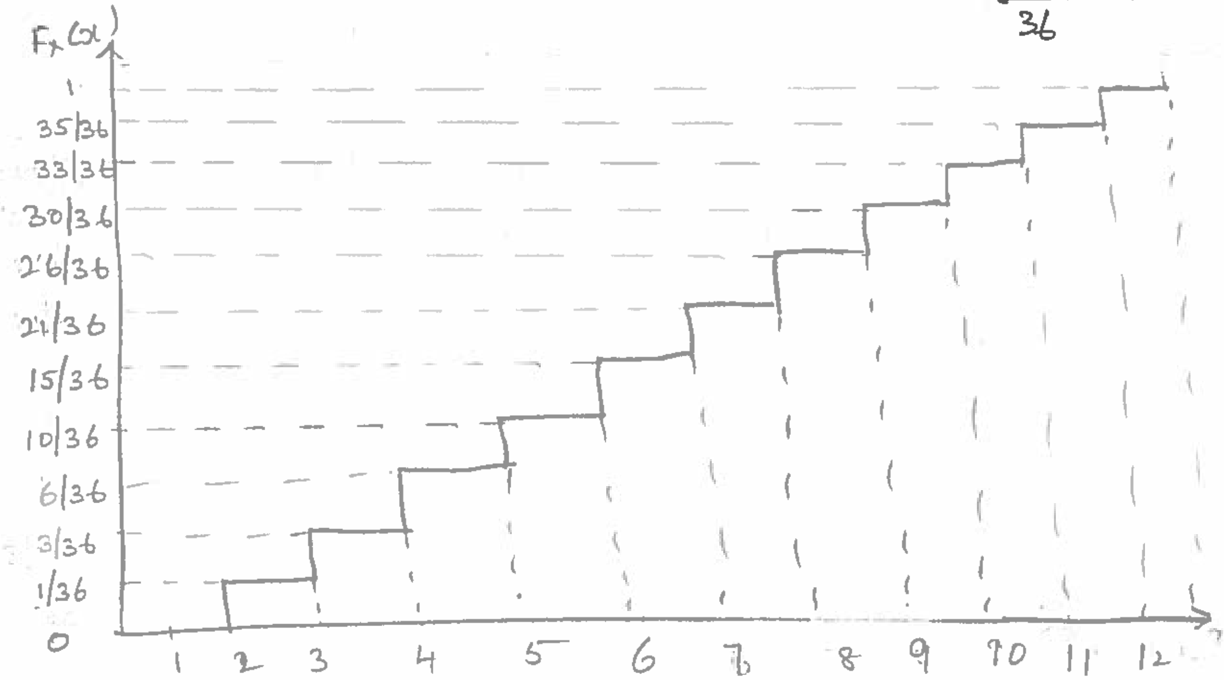
$F_X(8) = P[X \le 8] = \frac{26}{36}$

$F_X(9) = P[X \le 9] = \frac{30}{36}$

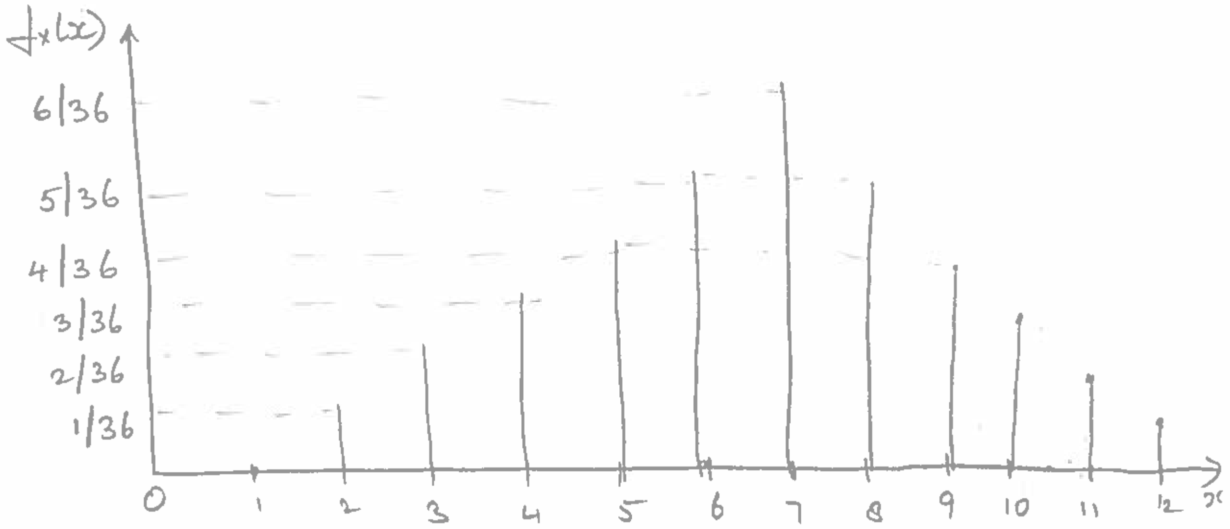
$F_X(10) = P(X \le 10) = \frac{33}{36}$

$F_X(11) = P[X \le 11] = \frac{35}{36}$

$F_X(12) = P(X \le 12) = \frac{36}{36} = 1$



from the above plot it is clear that the PDF $F_X(x)$ must have a step fn for a dRV. Since the derivative of unit step fn is unit impulse fn therefore pdf $f_X(x)$ for a dRV is impulse fn.



Probability Distribution &

Density fn Curve for a CRV:

Consider an experiment where a pointer on a wheel of chance is spun. assume that the wheel is numbered from 1 to 12

$F_X(0) = P[X \le 0] = 0$

$F_X(1) = P[X \le 1] = \frac{1}{12}$

$F_X(2) = P[X \le 2] = \frac{2}{12}$

$F_X(3) = P[X \le 3] = \frac{3}{12}$

$F_X(4) = P[X \le 4] = \frac{4}{12}$

$F_X(5) = P[X \le 5] = \frac{5}{12}$

$F_X(6) = P[X \le 6] = \frac{6}{12}$

$F_X(7) = P[X \le 7] = \frac{7}{12}$

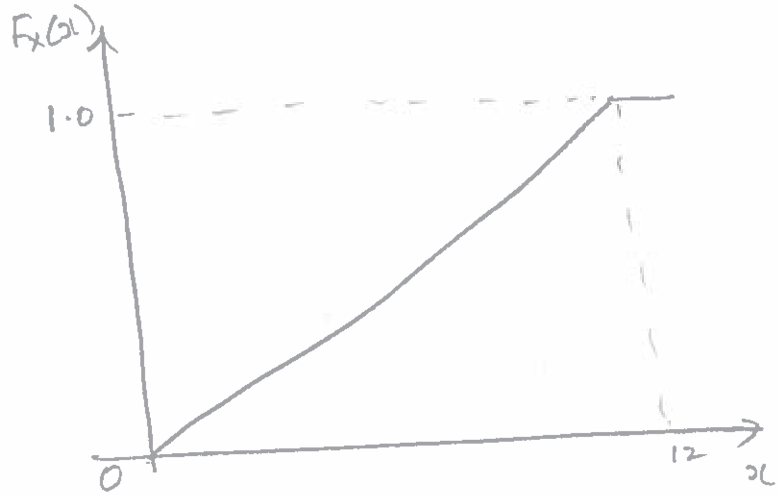
$F_X(8) = P[X \le 8] = \frac{8}{12}$

$F_X(9) = P[X \le 9] = \frac{9}{12}$

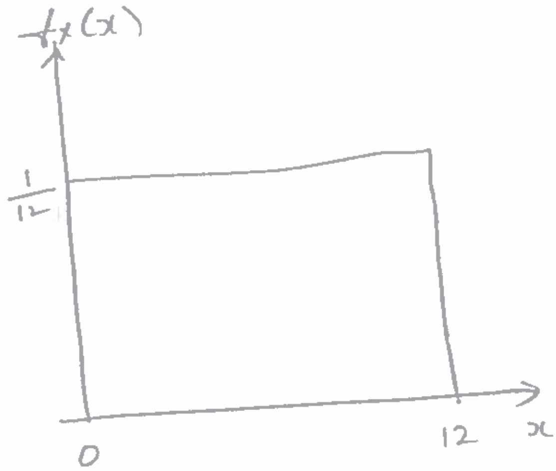
$F_X(10) = P[X \le 10] = \frac{10}{12}$

$F_X(11) = P[X \le 11] = \frac{11}{12}$

$F_X(12) = P[X \le 12] = \frac{12}{12}$



The PDF of a CRV is a ramp fn



The pdf of a CRV is unit step fn.

Q: A RV X is known to have a distribution fn $F_X(x) = u(x)\left[1 - e^{-\frac{x^2}{b}}\right]$ where $b > 0$ is a constant find its density fn.

A: $f_X(x) = \dfrac{dF_X(x)}{dx}$

$$= u(x)\left[-e^{-x^2/b}\frac{(-2x)}{b}\right] + \delta(x)\left[1-e^{-x^2/b}\right]$$

$$= \frac{2x}{b}u(x)\,e^{-\frac{x^2}{b}} + \delta(x)\left[1-e^{-\frac{x^2}{b}}\right]$$

Q: Determine which of the following are valid distribution fns.

a) $G_X(x) = 1 - e^{-x^2/2} \quad x \geq 0$

$\quad\quad\quad\quad 0 \quad\quad x < 0$

A: $G_X(+\infty) = 1 - e^{-\infty} = 1 - 0 = 1$

$G_X(-\infty) = 0$

$\therefore$ $G_X(x)$ is a valid distribution fn.

b) $G_X(x) = 0 \quad\quad x < 0$

$= 0.5 + 0.5 \sin\dfrac{\pi(x-1)}{2} \quad\quad 0 \leq x \leq 2$

$= 1 \quad\quad x \geq 2$

$G_X(\infty) = 1$ & $G_X(-\infty) = 0$

$\therefore$ $G_X(x)$ is a valid distribution fn

c) $G_X(x) = \dfrac{x}{a}\left[u(x-a) - u(x-2a)\right]$

A $G_X(x) = \dfrac{x}{a} \quad\quad a \leq x \leq 2a$

$G_X(-\infty) = 0$ & $G_X(\infty) \neq 1$

$\therefore$ $G_X(x)$ is not a valid distribution fn

Q: If the fn $G_x(x) = K\sum_{n=1}^{N} n^3 u(x-n)$ must be a valid prob/: distribution fn determine K to make it valid 1.60

A:
$$G_x(\infty) = 1$$

$$K\sum_{n=1}^{N} n^3 u(\infty - n) = 1$$

$$K\sum_{n=1}^{N} n^3 = 1 \implies K\frac{N^2(N+1)^2}{4} = 1$$

$$\therefore K = \frac{4}{N^2(N+1)^2}$$

Q: A RV X has the distribution fn $F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n)$ find the prob's $P[-\infty < x \leq 6.5]$, $P[x > 4]$ & $P[6 < x \leq 9]$

A: $P(-\infty < x \leq 6.5) = F_X(6.5) - F_X(-\infty) = F_X(6.5)$

$$= \sum_{n=1}^{12} \frac{n^2}{650} u(6.5-n)$$

$$= \frac{1}{650}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{650} = 0.14$$

$$P[x > 4] = 1 - P[x \leq 4] = 1 - F_X(4)$$

$$= 1 - \frac{1}{650}[1^2+2^2+3^2+4^2] = 0.9538$$

$$P[6 < x \leq 9] = F_X(9) - F_X(6) = \sum_{n=1}^{9}\frac{n^2}{650} - \sum_{n=1}^{6}\frac{n^2}{650}$$

$$= \frac{1}{650}[7^2+8^2+9^2] = 0.2985$$

Q: determine the real constant a for any arbitrary real constant m, 0 < b such that $f_X(x) = a e^{-|x-m|/b}$ is a valid density fn.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$2\int_{m}^{\infty} a\, e^{-\frac{(x-m)}{b}}\, dx = 1$$

$$\frac{x-m}{b} = \xi$$

$$dx = b\,d\xi$$

$$2ab\int_{0}^{\infty} e^{-\xi}\, d\xi = 1$$

$$2ab\,\underbrace{\left[e^{-\xi}\right]_0^{\infty}}_{-1} = 1 \Rightarrow 2ab = 1$$

$$a = \frac{1}{2b}$$

Q: An intercom system master station provides music to six hospital rooms. The prob/: that any one room will be switched on and draw power at any time is 0.4, when ON, a room draws 0.5W

a) find & plot the density & distribution fn for the rv "power delivered by the master station".

b) If the master-station amplifier is overloaded when more than 2W is demanded, what is the prob/: of overload'

A. $N = 6, \quad P = 0.4, \quad q = 0.6$

$$P[x = 0W] = 6c_0\,(0.4)^0\,(0.6)^{6-0} = 0.0467$$

$$P(x = 0.5W) = 6c_1\,(0.4)^1\,(0.6)^{6-1} = 0.1866$$

$$P[x = 1W] = 6c_2\,(0.4)^2\,(0.6)^{6-2} = 0.311$$

$$P[x = 1.5W] = 6c_3\,(0.4)^3\,(0.6)^{6-3} = 0.2765$$

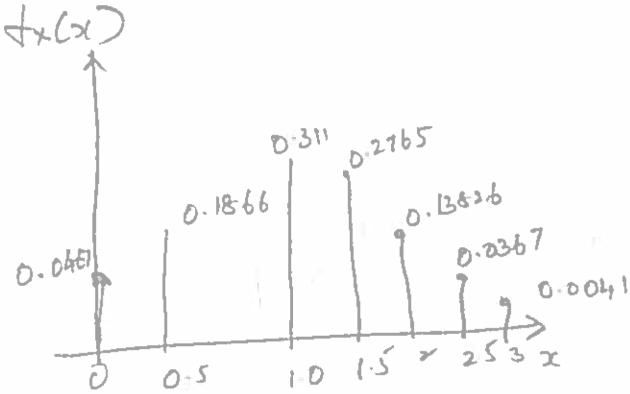
$$P[x = 2W] = 6c_4\,(0.4)^4\,(0.6)^{6-4} = 0.1382$$

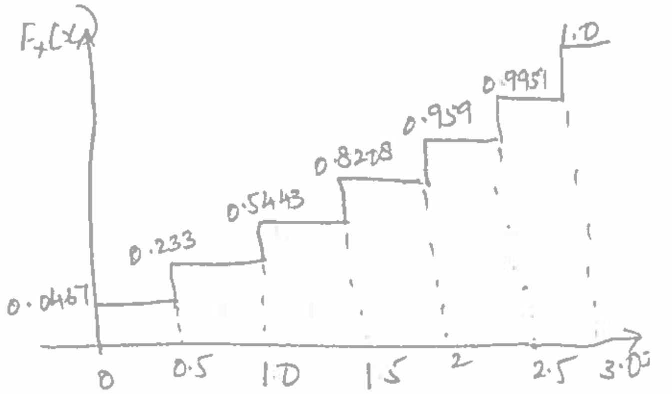
$$P[x = 2.5W] = 6c_5\,(0.4)^5\,(0.6)^{6-5} = 0.0369$$

$$P[x = 3W] = 6c_6\,(0.4)^6\,(0.6)^{6-6} = 0.0041$$

a) $f_X(x) = 0.0467\,\delta(x) + 0.1866\,\delta(x-0.5) + 0.311\,\delta(x-1)$
$+ 0.2765\,\delta(x-1.5) + 0.13826\,\delta(x-2) + 0.0367$
$\delta(x-2.5) + 0.0041\,\delta(x-3)$

$F_X(x) = 0.0467\,U(x) + 0.1866\,U(x-0.5) + 0.311\,U(x-1)$
$+ 0.2765\,U(x-1.5) + 0.13826\,U(x-2) + 0.0367$
$U(x-2.5) + 0.0041\,U(x-3)$





b) $P[\text{over load}] = P[x > 2] = P[x = 2.5\,\Omega] + P[x = 3\,\Omega]$

$= 0.041$

Q. find a constant $b > 0$ so that the fn $f_X(x) = \frac{e^{3x}}{4}$
$0 < x \leq b$ is a valid pdf.

A $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\int_0^b \frac{e^{3x}}{4}\,dx = 1 \Rightarrow \frac{1}{4}\frac{\left(e^{3x}\right)_0^b}{3} = 1$

$\frac{1}{12}\left[e^{3b} - 1\right] = 1$

$e^{3b} - 1 = 12 \Rightarrow e^{3b} = 13$

$b = \frac{1}{3}\log(13)$

Q. Give the fn $g_X(x) = 4\cos\left(\frac{\pi x}{2b}\right)\text{rect}\left(\frac{x}{2b}\right)$
find a value of b so that $g_X(x)$ is a valid

prob/: density fn

A:

$$\int_{-\infty}^{\infty} f_x(x)\, dx = 1$$

$$\int_{-b}^{b} 4 \cos\left(\frac{\pi x}{2b}\right) dx = 1$$

$$4 \frac{\sin \frac{\pi x}{2b} \Big|_{-b}^{b}}{\frac{\pi}{2b}} = 1 \Rightarrow \frac{8b}{\pi}\left[1+1\right] = 1$$

$$b = \pi/16$$

Q: A RV X has the density fn $f_x(x) = \frac{1}{2} u(x) e^{-x/2}$ define events $A = (1 < x \leq 3)$ $B = [x \leq 2.5]$ $C = A \cap B$ find the prob's of events A, B & C.

A: $C = A \cap B = (1 < x \leq 2.5)$

$$F_x(x) = \int_0^x \frac{1}{2} \cancel{u(x)}\, e^{-x/2}\, dx = \frac{1}{2} \frac{e^{-x/2}}{-1/2}\Big|_0^x$$

$$= \left[1 - e^{-x/2}\right] u(x).$$

$$P[A] = P(1 < x \leq 3) = F_x(3) - F_x(1)$$

$$= (1 - e^{-1.5}) - (1 - e^{-0.5}) = e^{-0.5} - e^{-1.5} = 0.3834$$

$$P(B) = P(x \leq 2.5) = F_x(2.5) = 1 - e^{-1.25} = 0.7135$$

$$P(C) = P[A \cap B] = P[1 < x \leq 2.5) = F_x(2.5) - F_x(1)$$

$$= (1 - e^{-1.25}) - (1 - e^{-0.5}) = e^{-0.5} - e^{-1.25} = 0.32$$

Q: For real constants $b > 0$, $c > 0$ & any a find a condition on constant a & a realtionship b/w c & a such that the fn $f_x(x) = a\left[1 - \frac{x}{b}\right]$ $0 \leq x \leq c$ is a valid pdf.

A: $f_X(x) = a\left(1 - \frac{x}{b}\right) \geq 0$ for $b > 0$ & $x > 0$

if $a > 0$ & $0 \leq x \leq b$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \Rightarrow \int_0^c a\left(1 - \frac{x}{b}\right) dx = 1$$

$$a\left[x - \frac{x^2}{2b}\right]_0^c = 1 \Rightarrow a\left(c - \frac{c^2}{2b}\right) = 1$$

$$\frac{a\left(2bc - c^2\right)}{2b} = 1 \Rightarrow 2abc - ac^2 = 2b$$

$$ac^2 - 2abc + 2b = 0$$

this occurs for $c = b \pm b\sqrt{1 - \frac{2}{ab}}$

only root with -ve correspond to $\frac{c}{b} < 1$ so final conditions are $a > 0$, $c < b$ & $\frac{c}{b} = 1 - \sqrt{1 - \frac{2}{ab}}$

Q: find a value for constant A such that

$$f_X(x) = \begin{cases} 0 & x < -1 \\ A(1-x^2)\cos\frac{\pi x}{2} & -1 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$$ is a valid pdf.

A. $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$A \int_{-1}^{1} (1-x^2)\cos\frac{\pi x}{2}\, dx = 1$$

$$A\left[\int_{-1}^{1} \cos\frac{\pi x}{2}\, dx - \int_{-1}^{1} x^2 \cos\frac{\pi x}{2}\, dx\right] = 1$$

$$A\left[\frac{4}{\pi} - \frac{8}{\pi^3}\left(\frac{\pi^2}{2} - 4\right)\right] = 1$$

$$A \cdot \frac{32}{\pi^3} = 1 \Rightarrow A = \frac{\pi^3}{32} = 0.9$$

1.65

Q: The R.V X has discrete values in the set $\{-1, -0.5, 0.7, 1.5, 3\}$ the corresponding probs are assumed to be $\{0.1, 0.2, 0.1, 0.4, 0.2\}$ find & plot the distribution fn.

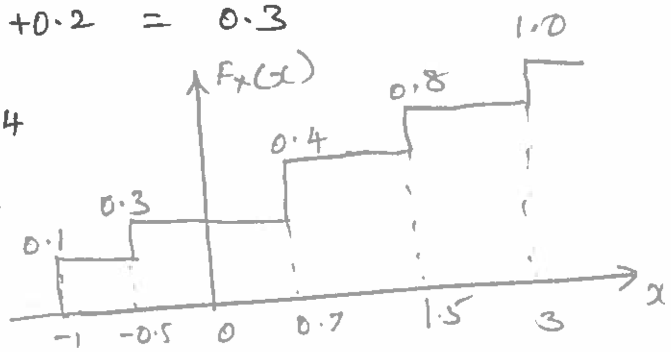
A: $F_X(-1) = P[X \le -1] = 0.1$

$F_X(-0.5) = P[X \le -0.5] = 0.1 + 0.2 = 0.3$

$F_X(0.7) = P[X \le 0.7] = 0.4$

$F_X(1.5) = P[X \le 1.5] = 0.8$

$F_X(3) = P[X \le 3] = 1.0$



Q: The probl: density fn of x is given by the following

x	0	1	2	3	4	5	6
$P(x)$	K	3K	5K	7K	9K	11K	13K

find $P(x<4)$, $P[x \ge 5]$, $P(3 < x \le 6]$ find the min value m so that $P[x \le m] > 0.3\}$.

A: $\sum P(x) = 1 \Rightarrow K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$49K = 1 \Rightarrow K = \frac{1}{49}$

$P(x < 4) = K + 3K + 5K + 7K = 16K = \frac{16}{49}$

$P[x \ge 5] = 11K + 13K = 24K = \frac{24}{49}$

$P[3 < x \le 6] = P[x = 4, 5, 6] = 9K + 11K + 13K = 33K = \frac{33}{49}$

$P[x \le m > 0.3$

Let $m = 3$ $P[x \le 3] = \frac{16}{49} = 0.326 > 0.3$

$\therefore$ min value of m is 3

Q: Verify that the following is valid distribution fn or not 1.66

$$F_X(x) = 0 \qquad x < -a$$
$$= \frac{1}{2}\left(\frac{x}{a}+1\right) \qquad -a \le x \le a$$
$$= 1 \qquad x > a$$

A:
$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{1}{2a} \qquad -a \le x \le a$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-a}^{a} \frac{1}{2a}\,dx = \frac{1}{2a}\left(x\right)_{-a}^{a} = 1$$

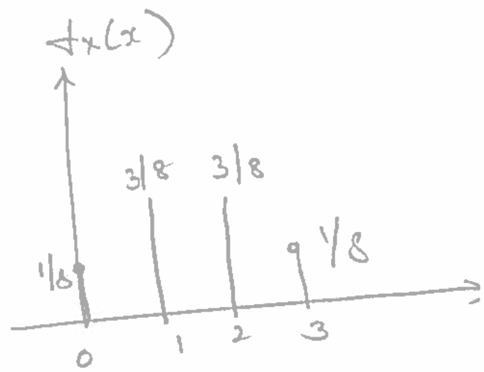
Since $f_X(x)$ is a valid density fn

$\therefore$ $F_X(x)$ is a valid distribution fn.

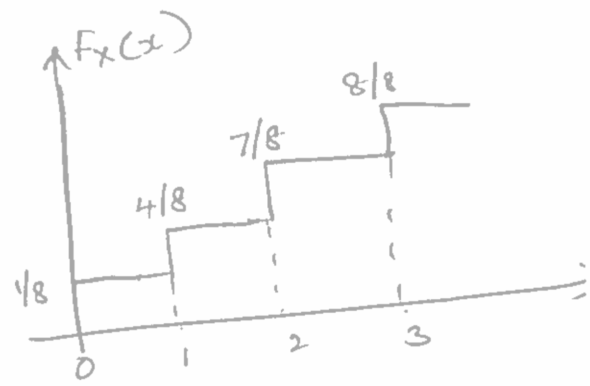
Q: A fair coin is tossed 3 times & the faces showing up are observed. write the sample space discription. If x is the no of heads in each of the outcomes of this experiment. find the prob/: fn. find & sketch the pdf & CDF.

A: $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

x	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



$$f_X(x) = \frac{1}{8}\delta(x) + \frac{3}{8}\delta(x-1) + \frac{3}{8}\delta(x-2) + \frac{1}{8}\delta(x-3)$$



$$F_X(0) = P[X \le 0] = \frac{1}{8}$$
$$F_X(1) = P[X \le 1] = \frac{4}{8}$$
$$F_X(2) = P[X \le 2] = \frac{7}{8}$$
$$F_X(3) = P[X \le 3] = \frac{8}{8}$$

Q: Consider the exp of tossing four coins. The r.v x 1.67 is associated with no of tails showing. Compute & sketch the CDF of x.

A. $S = \{$TTTT, TTTH, TTHT, HTTT, THTT, HHTT, HTHT, HTTH, TTHH, THHT, THTH, HHHT, HHTH, HTHH, THHH, HHHH$\}$

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

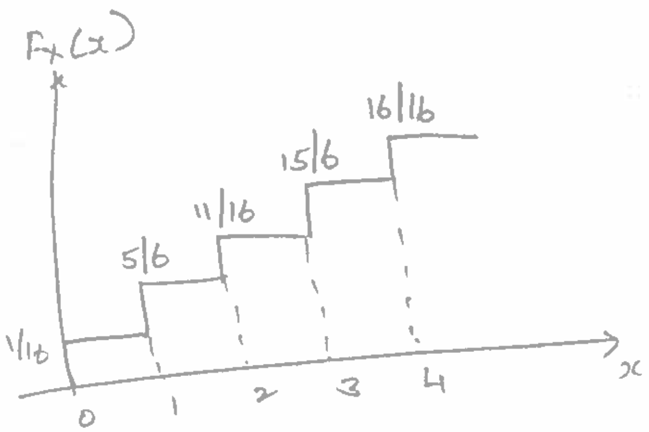
$F_x(0) = P(x \le 0) = \frac{1}{16}$

$F_x(1) = P(x \le 1) = \frac{5}{16}$

$F_x(2) = P[x \le 2] = \frac{11}{16}$

$F_x(3) = P(x \le 3) = \frac{15}{16}$

$F_x(4) = P[x \le 4] = \frac{16}{16}$



Q: A r.v x has the following prob. distribution

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find k, $P(x \le 3)$, min value of m such that $P[x \le m] > 0.5$.

A: $\Sigma P(x) = 1$

$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0$

$k = \frac{1}{10}, -1$ since k should not be negative

$k = \frac{1}{10}$

$P[x < 3] = 0 + k + 2k + 2k = 5k = \frac{5}{10} = 0.5$

$P[x \leq m] > 0.5$

let $m=4$ $\quad P[x \leq 4] = 8K = \frac{8}{10} = 0.8 > 0.5$

$\therefore$ min value of m is 4

Q: Consider the pdf $f_x(x) = ae^{-b|x|}$ where x is a rv whose allowable values range from $-\infty$ to ∞ find CDF, relation b/w a & b, the prob that x lies b/w 1 & 2.

A: $$F_x(x) = \int_{-\infty}^{x} f_x(\xi)\, d\xi = \int_{-\infty}^{0} a e^{b\xi} d\xi + \int_0^x a e^{-b\xi} d\xi$$

$\underline{x \geq 0}$

$$= a\frac{\left(e^{b\xi}\right)_{-\infty}^{0}}{b} + \frac{a}{-b}\left(e^{-b\xi}\right)_0^x$$

$$= \frac{a}{b}(1-0) + \frac{a}{-b}\left[e^{-bx}-1\right]$$

$$= \frac{a}{b}\left(2-e^{-bx}\right)$$

$\underline{x < 0}$ $\quad F_x(x) = \int_{-\infty}^{x} a e^{b\xi}\, d\xi = \frac{a}{b}\left[e^{b\xi}\right]_{-\infty}^{x}$

$$= \frac{a}{b} e^{bx}$$

$$\int_{-\infty}^{\infty} a e^{-b|x|}\, dx = 1$$

$$\int_{-\infty}^{0} a e^{bx}\, dx + \int_0^{\infty} a e^{-bx}\, dx = 1$$

$$\frac{a}{b}\left[e^{bx}\right]_{-\infty}^{0} + \frac{a}{-b}\left[e^{-bx}\right]_0^{\infty} = 1$$

$$\frac{a}{b} + \frac{a}{b} = 1 \implies \frac{2a}{b} = 1$$

$a = b/2$ $\quad P(1<x<2) = \int_1^2 a e^{-bx}\, dx$

$$= -\frac{a}{b}\left[e^{-bx}\right]_1^2 = \frac{1}{2}\left[e^{-b} - e^{-2b}\right]$$

Q: A CRV x has a pdf $f_x(x) = 3x^2$ $0 \le x \le 1$ find a & b such that 1) $P[x \le a] = P(x > a)$ 2) $P[x > b) = 0.05$.

A: 1) $P[x \le a] = P[x > a] = \frac{1}{2}$

$$\int_0^a 3x^2\, dx = \frac{1}{2} \Rightarrow 3\frac{x^3}{3}\Big|_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt[3]{2}} = 0.793$$

2) $P[x > b) = 0.05 = \frac{1}{20}$

$$\int_b^1 3x^2\, dx = \frac{1}{20} \Rightarrow [x^3]_b^1 = \frac{1}{20}$$

$$1 - b^3 = \frac{1}{20} \Rightarrow 1 - \frac{1}{20} = b^3$$

$$\frac{19}{20} = b^3 \Rightarrow b = \left(\frac{19}{20}\right)^{1/3} = 0.983$$

Q: find the value of the constant k so that $f_x(x) = kx^2(1-x^3)$ $0 < x < 1$ is a proper pdf.

A
$$\int_{-\infty}^{\infty} f_x(x)\, dx = 1$$

$$\int_0^1 k\, x^2(1-x^3)\, dx = 1 \Rightarrow k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1$$

$$k\left[\frac{1}{3} - \frac{1}{6}\right] = 1$$

$$\therefore k = 6$$

Q: If the pdf of a RV is given by $f_x(x) = x$ $0 < x$ … , $(2-x)$ $1 < x <$ … find the prob's that the RV having its value b/w a) 0.2 & 0.8 b) 0.6 & 1.2.

A: a) $P[0.2 < x < 0.8] = \int_{0.2}^{0.8} x\, dx = \frac{x^2}{2}\Big|_{0.2}^{0.8} = 0.3$

b) $P[0.6 < x < 1.2] = \int_{0.6}^{1} x\, dx + \int_1^{1.2} (2-x)\, dx$

$$= \frac{x^2}{2}\Big|_{0.6}^{1} + \left[2x - \frac{x^2}{2}\right]_1^{1.2} = 0.5$$

Q: If a mass fn is given by $p(x) = \begin{cases} Ax & x = 1, 2 \dots 50 \\ A(100-x) & x = 51, 52 \dots 100 \\ 0 & \text{else} \end{cases}$

find ① A that makes the fn a prob/: mass fn

2) find $P[X > 50]$, $P[X < 50]$, $P[25 < X < 75]$, $P[x = \text{odd no}]$

3) If the events in ② are A, B, C & D find $P(A/B)$, $P(A|C)$, $P(A|D)$, $P(C|D)$. Are the pairs A, B; A, C; A, D; C, D independent events.

1.- 1) $\sum_{x=1}^{50} Ax + \sum_{x=51}^{100} A(100-x) = 1$

$$A\left[\sum_{x=1}^{50} x + \sum_{x=51}^{100} (100-x)\right] = 1$$

$$A\left[(1+2+\dots+50) + (49+48+\dots+1)\right] = 1$$

$$A\left[\frac{50 \times 51}{2} + \frac{49 \times 50}{2}\right] = 1$$

$$A \times \frac{50}{2}(51 + 49) = 1 \Rightarrow A = \frac{1}{2500}$$

2) a) $P[X > 50] = \sum_{x=51}^{100} A(100 - x)$

$$= \frac{1}{2500}[49 + 48 + \dots + 1] = \frac{1}{2500}\frac{49 \times 50}{2} = 0.49$$

b) $P[X < 50] = \sum_{x=1}^{49} \frac{x}{2500} = \frac{1}{2500}[1 + 2 + \dots + 49]$

$$= \frac{1}{2500}\frac{49 \times 50}{2} = 0.49$$

c) $P(X = 50) = \frac{x}{2500} = \frac{50}{2500} = \frac{1}{50} = 0.02$

d) $P[25 < x < 75] = \sum_{x=26}^{50} \frac{x}{2500} + \sum_{x=51}^{74} \frac{100-x}{2500}$

$= \frac{1}{2500}\left[(26+27+\cdots+50) + (49+48+\cdots+26)\right]$

$= \frac{2}{2500}\left[(26+27+\cdots+49)\right] + \frac{50}{2500}$

$= \frac{2}{2500}\left[\frac{49\times50}{2} - \frac{25\times26}{2}\right] + \frac{50}{2500} = 0.74$

e) $P(\text{odd no}) = \sum_{x=1,3\ldots}^{49} Ax + \sum_{x=51,53}^{99} A(100-x)$

$= \frac{1}{2500}\left[(1+3+5+\cdots 49) + (49+47+\cdots+1)\right]$

$= \frac{2}{2500}(1+3+5+\cdots+49) = \frac{2}{2500}\sum_{m=1}^{25}(2m-1)$

$= \frac{2}{2500}\left[2\sum_{m=1}^{25} m - \sum_{m=1}^{25} 1\right]$

$= \frac{2}{2500}\left(2\frac{(25\times26)}{2} - 25\right) = 0.5$

3) $P(A) = P[x > 50] = 0.49$

$P(B) = P[x < 50] = 0.49$

$P(C) = P[x = 50] = 0.02$

$P(D) = P[25 < x < 75] = \frac{74}{100} = 0.74$

$P(E) = P[\text{odd no}] = \frac{1}{2} = 0.5$

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P[x>50 \cap x<50]}{P(B)} = 0$

$\therefore$ A & B are mutually exclusive events & not independent events

$$P(A|C) = \frac{P(A\cap C)}{P(C)} = \frac{P[x>50 \cap x=50]}{P(C)} = 0$$

$\therefore$ A & C are mutually exclusive events.

$$P(A|D) = \frac{P(A\cap D)}{P(D)} = \frac{P(x>50 \cap 25<x<75)}{P(25<x<75)}$$

$$= \frac{P[51 \le x \le 74]}{0.74} = \frac{1}{0.74}\sum_{x=51}^{74}\frac{100-x}{2500}$$

$$= \frac{1}{0.74(2500)}[49+48+\cdots+26]$$

$$= 0.486$$

$$P(A\cap D) = \frac{36}{100}, \quad P(A) = \frac{49}{100}, \quad P(D) = \frac{74}{100}$$

Since $P(A\cap D) \neq P(A)\,P(D)$

$\therefore$ A & D are not independent events

$$P(C|D) = \frac{P(C\cap D)}{P(D)} = \frac{P(x=50 \cap 25<x<75)}{P(D)}$$

$$= \frac{P[x=50]}{P(D)} = \frac{1/50}{0.74} = \frac{2}{74}$$

Since $P(C\cap D) \neq P(C)\cdot P(D)$

$\therefore$ C & D are not independent events

Q: If the pdf of a RV X is given by $f_X(x) = c\,e^{-x/4}$, $0<x<1$
find the value that c must have & find $F_X(0.5)$

1. $$\int_0^1 c\,e^{-x/4}\,dx = 1 \Rightarrow \frac{c\,e^{-x/4}}{-1/4}\Big|_0^1 = 1$$

$$-4c\left[e^{-1/4} - 1\right] \Rightarrow c = \frac{1}{4(1-e^{-0.25})} = 1.13$$

$$F_X(0.5) = P(x \le 0.5) = \int_0^{0.5} 1.13\, e^{-x/4}\, dx$$

$$= 1.13 \frac{e^{-x/4}\Big|_0^{0.5}}{-1/4} = -4.52\left[e^{-0.5/4} - 1\right) = 0.5311$$

Q: The CDF ~~cdf~~ of a RV Y is $F_Y(y) = 1 - e^{-0.4\sqrt{y}}$ $y \geq 0$ find $P[2.5 < y \le 6.2]$

A: $P[2.5 < y \le 6.2] = F_Y(6.2) - F_Y(2.5)$

$$= \left(1 - e^{-0.4\sqrt{6.2}}\right] - \left[1 - e^{-0.4\sqrt{2.5}}\right]$$

$$= e^{-0.4\sqrt{2.5}} - e^{-0.4\sqrt{6.2}} = 0.1616$$

Q: A RV X has a pdf $f_X(x) = c(1-x^4)$ $-1 < x \le 1$ find c & $P(|x| < \frac{1}{2}]$

A. $$\int_{-1}^{1} c(1-x^4)\, dx = 1$$

$$c\left(x - \frac{x^5}{5}\right]_{-1}^{1} = 1 \Rightarrow c = 5/8$$

$$P[|x| < \tfrac{1}{2}] = \int_{-1/2}^{1/2} \frac{5}{8}(1-x^4)\, dx = \frac{5}{8}\left(x - \frac{x^5}{5}\right]_{-1/2}^{1/2}$$

$$= 0.617.$$

Q: Let X be a CRV with density fn $f_X(x) = \frac{x}{9} + k$ $0 \le x \le 6$ find the value of K & find $P[2 < x \le 5)$.

A $$\int_0^6 \left(k + \frac{x}{9}\right) dx = 1$$

$$\left(kx + \frac{x^2}{18}\right]_0^6 = 1 \Rightarrow k = -1/6$$

$$P(2 < x \le 5) = \int_2^5 \left(\frac{x}{9} - \frac{1}{6}\right) dx = \frac{x^2}{18} - \frac{x}{6}\Big|_2^5 = 0.667$$

Q: The pdf of a rv x is given by $f_x(x) = k \quad a \le x \le b$
$= 0 \quad$ else

where k is a constant. 1) determine the value of k

2) let a=1 & b=2 Calculate $P[|x| \le c]$ for c=0.5

A: 1) $\int_a^b k\, dx = 1 \Rightarrow k\,[x]_a^b = 1 \Rightarrow k = \frac{1}{b-a}$

2) $a = 1, \quad b = 2 \Rightarrow k = 1$

$\therefore f_x(x) = 1 \quad 1 \le x \le 2$
$= 0 \quad$ else

$P[|x| \le c] = P[|x| \le 0.5] = \int_{-0.5}^{0.5} f_x(x)\, dx = 0$

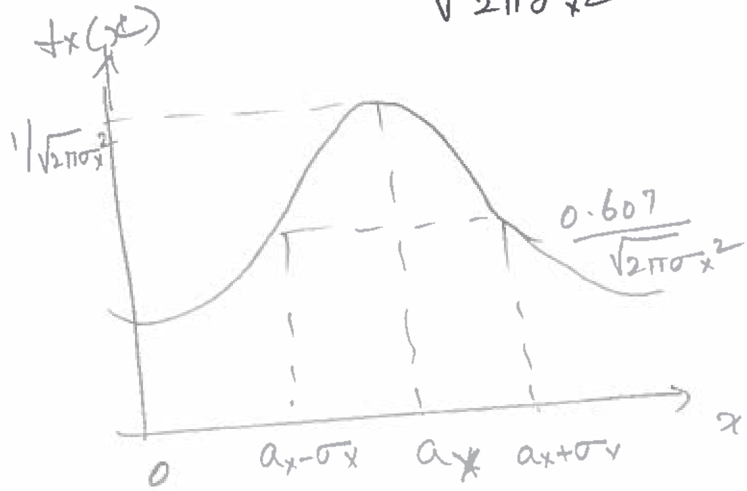
$= \int_{-0.5}^{0} f_x(x)\, dx + \int_0^{0.5} f_x(x)\, d$

Gaussian Density Function:

A rv x is called Gaussian if its density fn has the form

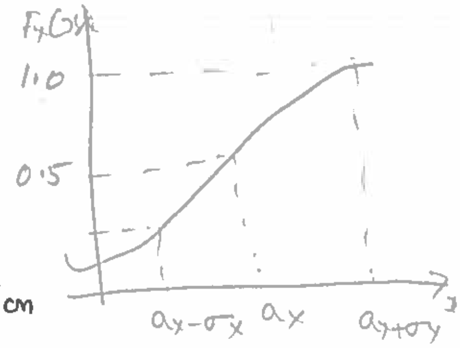
$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \longrightarrow (1)$$

where $\sigma_x > 0$
$-\infty < a_x < \infty$ are real constants.



The Gaussian density fn gives accurate description of many practical random quantities. It is possible to eliminate noise in communication systems by knowing its behaviour using Gaussian density fn.

The Gaussian distribution fn is given by



$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \, dx \rightarrow ②$$

for a normalized Gaussian distribution fn $a_x = 0$ & $\sigma_x = 1$ & $F_X(x) = F(x)$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-x^2}{2}} \, dx \longrightarrow ③$$

$$F(-x) = 1 - F(x)$$

let $x = \frac{x - a_x}{\sigma_x}$ substituting x in ② we get

$$dx = \frac{dx}{\sigma_x}$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} e^{-x^2/2} \sigma_x \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} e^{-x^2/2} \, dx$$

$$= F\left(\frac{x - a_x}{\sigma_x}\right)$$

$$\therefore \; F_X(x) = F\left(\frac{x - a_x}{\sigma_x}\right)$$

Q: find the prob of the event $\{X \le 5.5\}$ for a Gaussian RV having $a_x = 3$ & $\sigma_x = 2$.

A. $P[X \le 5.5] = F_X(5.5) = F\left(\frac{5.5-3}{2}\right) = F(1.25)$

$= 0.8944$

Q: assume that the height of clouds above ground at some pt is a Gaussian rv x with $a_x = 1830$ m & $\sigma_x = 460$ m find the prob that clouds will be higher than 2750 m. 1.76

A $P[x > 2750] = 1 - P[x \leq 2750] = 1 - F_x(2750)$

$$= 1 - F\left(\frac{2750 - 1830}{460}\right) = 1 - F(2.0)$$

$$= 1 - 0.9772 = 0.0228$$

Q: For a Gaussian rv x, $a_x = 7$ & $\sigma_x = 0.5$ find the prob/: of the event $[x \leq 7.3]$.

A. $P(x \leq 7.3) = F_x(7.3) = F\left(\frac{7.3 - 7}{0.5}\right) = F(0.6)$

$$= 0.7264$$

Q. A rv x is Gaussian with $a_x = 0$ & $\sigma_x = 1$

1) what is the prob/: that $|x| > 2$

2) " " " " " $x > 2$.

A. 1) $P(|x| > 2) = P[2 < x] + P[x < -2]$

$$= 1 - P[x \leq 2] + P[x \leq -2]$$

$$= 1 - F(2) + F(-2)$$

$$= 1 - F(2) + 1 - F(2)$$

$$= 2 - 2F(2) = 2 - 2(0.9772)$$

$$= 0.0456$$

2) $P[x > 2] = 1 - P[x \leq 2] = 1 - F(2)$

$$= 1 - 0.9772 = 0.0228$$

Q: For the Gaussian density fn S.T $\int_{-\infty}^{\infty} x\, f_X(x)\, dx = a_X$.

A: $$\int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_X^2}}\, e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}\, dx$$

$$= \int_{-\infty}^{\infty} (x-a_X) \cdot \frac{1}{\sqrt{2\pi}\,\sigma_X}\, e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}\, dx + a_X \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}}\, e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}\, dx$$

$$= \int_{-\infty}^{\infty} \xi \sqrt{2}\,\sigma_X \cdot \frac{1}{\sqrt{2\pi}\,\sigma_X}\, e^{-\xi^2}\, d\xi\, \sqrt{2}\,\sigma_X \qquad \text{let } \xi = \frac{x-a_X}{\sqrt{2}\sigma_X}$$

$$+ a_X (1)$$

$$= \frac{2\sigma_X^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \xi\, e^{-\xi^2}\, d\xi}_{\text{odd fn } = 0} + a_X (1)$$

$$= a_X$$

Q: for the Gdf show that $\int_{-\infty}^{\infty} (x-a_X)^2 f_X(x) = \sigma_X^2$.

A: $$\int_{-\infty}^{\infty} (x-a_X)^2 f_X(x)\, dx = \int_{-\infty}^{\infty} (x-a_X)^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma_X}\, e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}\, dx$$

Let $\xi = \frac{x-a_X}{\sqrt{2}\,\sigma_X} \Rightarrow d\xi = \frac{dx}{\sqrt{2}\,\sigma_X}$

$$\int_{-\infty}^{\infty} (x-a_X)^2 f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{2\sigma_X^2}{\sqrt{\pi}}\, \xi^2 e^{-\xi^2}\, d\xi$$

$$= \frac{4\sigma_X^2}{\sqrt{\pi}} \int_{0}^{\infty} \xi^2 e^{-\xi^2}\, d\xi = \frac{4\sigma_X^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{4}$$

$$= \sigma_X^2$$

Q. A production line manufactures 1000 Ω resistors that must satisfy a 10% tolerance.

a) If resistance is adequately described by a Gaussian rv x for which $a_x = 1000\,\Omega$ & $\sigma_x = 40\,\Omega$ what fraction of the resistors is to be rejected.

b) If a machine is not properly adjusted, the product resistance change to the case where $a_x = 1050$ (5% shift) what fraction is now rejected.

A: a) P[resistors rejected] $= P[x \le 900] + P(x > 1100)$

$$= F_x(900) + 1 - F_x(1100)$$

$$= F\left(\frac{900-1000}{40}\right) + 1 - F\left(\frac{1100-1000}{40}\right)$$

$$= 1 + F(-2.5) - F(2.5)$$

$$= 1 + 1 - F(2.5) - F(2.5)$$

$$= 2 - 2F(2.5) = 2 - 2(0.9938)$$

$$= 0.0124 = 1.24\% \text{ rejected}$$

b) P(resistors rejected) $= P[x \le 900) + P(x > 1100]$

$$= F_x(900) + 1 - F_x(1100)$$

$$= F\left(\frac{900-1050}{40}\right) + 1 - F\left(\frac{1100-1050}{40}\right)$$

$$= 1 - F(3.75) + 1 - F(1.25)$$

$$= 2 - F(3.75) - F(1.25)$$

$$= 2 - 0.9999 - 0.8944$$

$$= 0.1057 = 10.57\% \text{ rejected}$$

Q. A Gaussian rv has $a_x = 2$ & $\sigma_x = 2$ find $P[x > 1]$ & $P[x \le -1]$.

A. $P(x > 1) = 1 - P[x \le 1] = 1 - F_x(1) = 1 - F\left(\frac{1-2}{2}\right)$

$$= 1 - F(-0.5) = 1 - 1 + F(0.5) = 0.6915$$

$P[x \le -1] = F_x(-1) = F\left(\frac{-1-2}{2}\right) = F(-1.5)$

$= 1 - F(1.5) = 1 - 0.9332 = 0.0668.$

Q: In a certain junior olympics, javelin throw distances are well approximated by a gaussian distribution for which $a_x = 30m$ & $\sigma_x = 5m$. In a qualifying round contestants must throw farther than 26 m to qualify. In the main event the record thrown is 42 m.

a) what is the prob/: of being disqualified in the qualifying round.

b) In the main event what is the prob/: the record will be broken.

A: a) $P[x \le 26m] = F_x(26) = F\left(\frac{26-30}{5}\right) = F(-0.8)$

$= 1 - F(0.8) = 1 - 0.7881 = 0.2119$

b) $P[x > 42] = 1 - F_x(42) = 1 - F\left(\frac{42-30}{5}\right)$

$= 1 - F(2.4) = 1 - 0.9918 = 0.0082$

Q: suppose height to bottom of clouds is a Gaussian RV X for which $a_x = 4000m$ & $\sigma_x = 1000m$. A person bets that cloud ht tommorow will fall in the set $A = (1000 < x \le 3300)$ while 2nd person bets that height will be satisfied by $B = (2000 < x \le 4200m)$. A third person bets both are correct. find the prob/: that each person will win the bet.

A. a) $P[A] = P(1000 < x \le 3300] = F_x(3300) - F_x(1000)$

$= F\left(\frac{3300-4000}{1000}\right) - F\left(\frac{1000-4000}{1000}\right)$

$= F(-0.7) - F(-3)$

$= 1 - F(0.7) - 1 + F(3) = 0.9987 - 0.7580 = 0.2407$

b) $P(B) = P[2000 < x \leq 4200] = F_x(4200) - F_x(2000)$

$= F\left(\frac{4200-4000}{1000}\right) - F\left(\frac{2000-4000}{1000}\right)$

$= F(0.2) - F(-2) = F(0.2) - 1 + F(2)$

$= 0.5793 - 1 + 0.9772 = 0.5565$

c) $P[\text{both Correct}] = P(A \cap B) = P[2000 < x \leq 3300]$

$= F_x(3300) - F_x(2000)$

$= F\left(\frac{3300-4000}{1000}\right) - F\left(\frac{2000-4000}{1000}\right)$

$= F(-0.7) - F(-2) = 1 - F(0.7) - 1 + F(2)$

$= 1 - 0.7580 - 1 + 0.9772$

$= 0.2192$

Q: The o/p voltage x from the Rxr in a particular binary digital Communication System when a binary zero is being Rxed is Gaussian as defined by $a_x = 0$ & $\sigma_x = 0.3$ when a binary one is being Rxed it is also gaussian but as defined by $a_x = 0.9$ & $\sigma_x = 0.3$. The Rxr decision logic specifies that at the end of binary interval if $x > 0.45$ a binary one is being Rxed. If $x \leq 0.45$ a binary zero is decided. If it is given that a binary zero is truly being Rxed find the prob that a) a binary one (mistake) will be decided & (b) binary zero is decided (Correct decision).

A: a) $P[\text{mistake}] = P[x > 0.45] = 1 - P[x \leq 0.45]$

$= 1 - F_x(0.45) = 1 - F\left(\frac{0.45-0}{0.3}\right) = 1 - F(1.5)$

$= 0.0668$

b) $P[\text{Correct decision}] = P[x \leq 0.45] = F_x(0.45)$

$= F\left(\frac{0.45-0}{0.3}\right) = F(1.5) = 0.9332$

Q: For the Gaussian RV show that the curve's point of inflection (where the 1st derivative of the pdf wrt x has zero slope) occurs at $a_x \pm \sigma_x$.

A: $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$$

$$\frac{df_X(x)}{dx} = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \left[-\frac{(x-a_x)}{\sigma_x^2}\right]$$

$$= -\frac{(x-a_x)}{\sigma_x^2} f_X(x)$$

$$\frac{d^2 f_X(x)}{dx^2} = 0$$

$$-\frac{(x-a_x)}{\sigma_x^2}\frac{df_X(x)}{dx} + f_X(x)\left(-\frac{1}{\sigma_x^2}\right) = 0$$

$$+\frac{(x-a_x)^2}{\sigma_x^4} f_X(x) - \frac{1}{\sigma_x^2} f_X(x) = 0$$

$$(x-a_x)^2 - \sigma_x^2 = 0 \Rightarrow x = a_x \pm \sigma_x$$

Q: A RV X is known to be Gaussian with $a_x = 1.6$, $\sigma_x = 0.4$. find a) $P[1.4 < X \leq 2]$ (b) $P[-0.6 < (X-1.6) \leq 0.6]$

A: a) $$P(1.4 < X \leq 2) = F_X(2) - F_X(1.4)$$

$$= F\left(\frac{2-1.6}{0.4}\right) - F\left(\frac{1.4-1.6}{0.4}\right)$$

$$= F(1) - F(-0.5) = F(1) - 1 + F(0.5)$$

$$= 0.8413 + 0.6915 - 1 = 0.5328$$

b) $$P[-0.6 < (X-1.6) \leq 0.6] = P[1.0 < X \leq 2.2]$$

$$= F_X(2.2) - F_X(1.0) = F\left(\frac{2.2-1.6}{0.4}\right) - F\left(\frac{1-1.6}{0.4}\right)$$

$= F(1.5) - F(-1.5) = 2F(1.5) - 1 = 2(0.9332) - 1.0$

$= 0.8664$

~~Q. The Radial distance~~ +

Q. assume that the time of arrival of birds at a particular place on a migratory route as measured in days from the 1st of the year (Jan 1st) is approximated as a grv X with $a_X = 200$ & $\sigma_X = 20$ days. a) what is the prob/s that the birds arrive after 160 days but on or before the 210th day. (b) what is the prob the birds will arrive after 231st day.

A. a) $P[160 < X \leq 210] = F_X(210) - F_X(160)$

$$= F\left(\frac{210-200}{20}\right) - F\left(\frac{160-200}{20}\right)$$

$$= F(0.5) - F(-2)$$

$$= F(0.5) - 1 + F(2)$$

$$= 0.6915 - [1 - 0.9772] = 0.6687$$

b) $P[X > 231] = 1 - P[X \leq 231] = 1 - F_X(231)$

$$= 1 - F\left(\frac{231-200}{20}\right) = 1 - F(1.55)$$

$$= 1 - 0.9394 = 0.0606$$

Binomial Density & Distribution fn:

This is applicable to drvs only where there are two possible outcomes.

The binomial density fn of a rv X is given by

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

where $\binom{N}{k} = N_{C_k} = \dfrac{N!}{(N-k)!\,k!}$

the binomial Distribution fn is given by

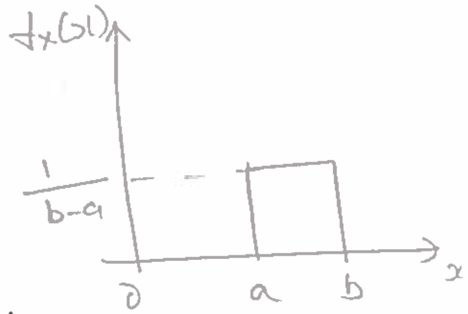
$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \, u(x-k)$$

It applies to many games of chance, detection problems in radar & sonar and many experiments having only two possible outcomes in any given trial.

Uniform density & Distribution fn:

The uniform density fn of a rv x is given by

$$f_X(x) = \frac{1}{b-a} \qquad a \le x \le b$$
$$0 \qquad \text{else}$$



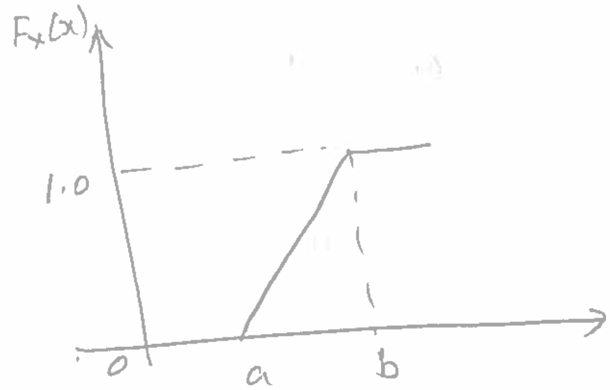
& the distribution fn is given by

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= 0 \qquad x < a$$

$$= \frac{x-a}{b-a} \qquad a \le x \le b$$

$$= 1 \qquad b \le x$$



applications:

1) The round distribution of errors in the round-off process are uniformly distributed.

2) In digital communications, when a sample of signal is rounded off to its nearest level or in a game when a real no is converted into an integer, the pdf of

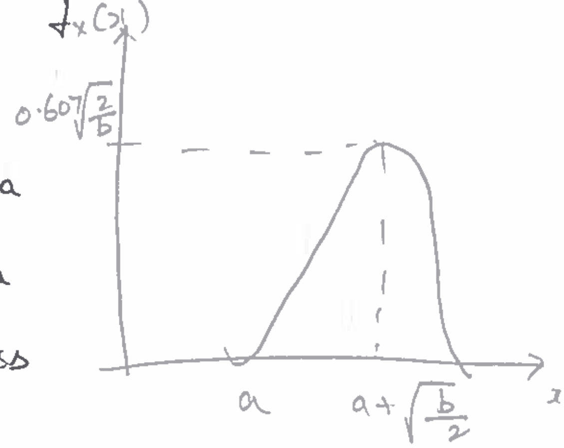
errors are uniformly distributed.

Rayleigh density & distribution fn:

The Rayleigh density fn of a RV x is defined as

$$f_X(x) = \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}} \qquad x \geq a$$

$$0 \qquad x < a$$

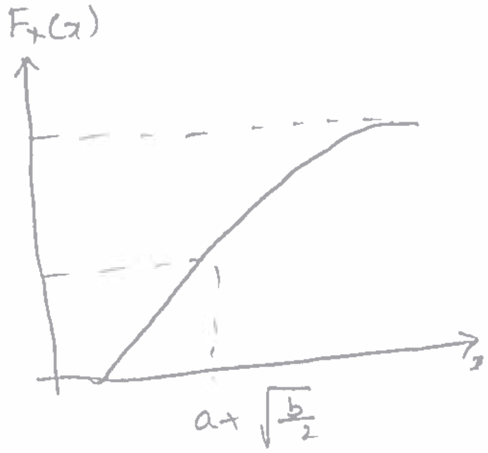


where a & b are real constants

$-\infty < a < \infty$ & $b > 0$

The distribution fn is

$$F_X(x) = \begin{cases} 0 & x < a \\ 1 - e^{-\frac{(x-a)^2}{b}} & x \geq a \\ 1 & x = \infty \end{cases}$$



Applications:

1) It describes the envelope of white noise, when the noise is passed through a band pass filter

2) The Rayleigh's density fn has a relationship with the Gaussian density fn.

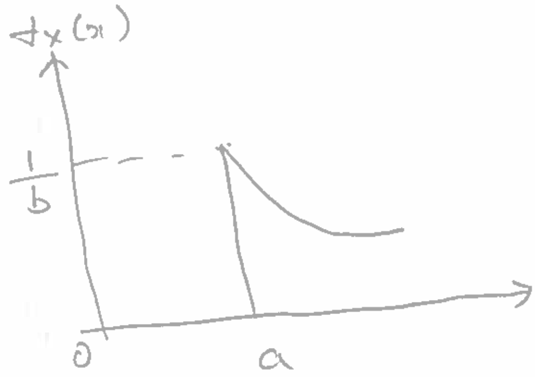
3) Some types of signal fluctuations Rxed by the Rxr are modelled as Rayleigh's distribution.

Exponential distribution & density fn:

The exponential density fn of a rv x is defined by

$$f_x(x) = \frac{1}{b} e^{\frac{-(x-a)}{b}} \qquad x > a$$

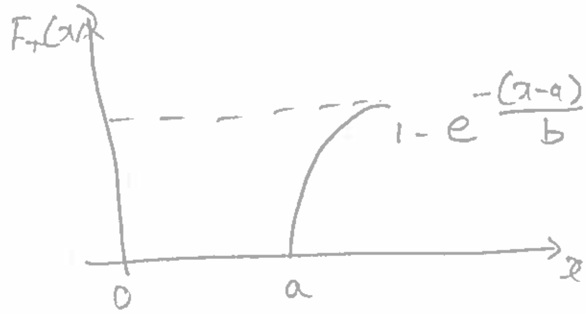
$$0 \qquad x < a$$



the Distribution fn is given by

$$F_x(x) = 1 - e^{\frac{-(x-a)}{b}} \qquad x > a$$

$$0 \qquad x < a$$



Applications:

1) It approximately describes the fluctuations in signal strength recieved by radar from certain types of aircraft.
2) It is useful in describing raindrop sizes when a large no of rainstrom measurements are made.

Poisson density & distribution fn:

The poisson density fn of a rv x is given by

$$f_x(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

where $b > 0$ is a real constant

& Distribution fn is given by

$$F_x(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

Applications:

It is mostly applied to counting type problems. It describes.

1) the no of telephone calls made during a period of

1) the no of telephone calls made during a period of time
2) the no of defective elements in a given sample.
3) the no of electrons emitted from a cathode in a given time interval.
4) the no of items waiting in a queue etc.

Q: The power reflected from an aircraft of a complicated shape that is Rxed by a radar can be described by an exponential rv P. The density of P is therefore

$f_P(p) = \frac{1}{P_0} e^{-\frac{p}{P_0}}$ $p > 0$ where P_0 is the average amount of power Rxed. At some given time P may have a value different from its average value. What is the prob/: that Rxed power is larger than the power Rxed on the average.

A: $$F_P(p) = \int_0^p \frac{1}{P_0} e^{-P/P_0}\, dP = \frac{1}{P_0} \frac{\left(e^{-P/P_0}\right)_0^p}{-1/P_0}$$

$$= 1 - e^{-p/P_0}$$ exponential density fn

$$P(P > P_0] = 1 - P[P \le P_0] = 1 - F_P(P_0)$$

$$= 1 - \left[1 - e^{-P_0/P_0}\right] = e^{-1} = 0.368$$

Q. Assume automobile arrivals at a gasoline station is poisson distributed and occur at an average rate of 50/hour. The station has only one gasoline pump. If all cars are assumed to required one minute to obtain fuel what is the prob/: that a waiting line will occur at the pump.

1: For a poisson distribution

$$F_x(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \qquad b = \lambda T$$

$\lambda = 50$ per hour. $\quad T = 1$ min

$$b = \lambda T = \frac{50}{60} \times 1 = \frac{5}{6}$$

Prob/: of waiting line = P [two or more cars arrive in any 1 min duration]

$$= P[x \geq 2]$$

$$= 1 - P[x \leq 1]$$

$$= 1 - F_x(1) = 1 - e^{-5/6} \sum_{k=0}^{1} \frac{(5/6)^k}{k!}$$

$$= 1 - e^{-5/6}\left[1 + \frac{5}{6}\right] = 0.2032$$

Q: use the exponential density fn & solve for I_2 defined by $I_2 = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx$ & solve for I_1 defined by $I_1 = \int_{-\infty}^{\infty} x f_x(x)\, dx$ verify that I_1 & I_2 satisfy the equ $I_2 - I_1^2 = b^2$

A: $$f_x(x) = \frac{1}{b} e^{-\frac{(x-a)}{b}} \qquad x > a$$

$$I_1 = \int_{-\infty}^{\infty} x f_x(x)\, dx = \int_a^{\infty} x \frac{1}{b} e^{-\frac{(x-a)}{b}}\, dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} x \cdot e^{-x/b}\, dx = \frac{e^{a/b}}{b}\left[e^{-x/b}\left[\frac{x}{-1/b} - \frac{1}{(-1/b)^2}\right]\right]_a^{\infty}$$

$$= -\frac{1}{b}\left[-ab - b^2\right] = \frac{ab + b^2}{b} = a + b$$

$$I_2 = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx = \int_a^{\infty} x^2 \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}}\, dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} x^2 \, e^{-x/b} \, dx$$

$$= \frac{e^{a/b}}{b} \left[e^{-x/b} \left(\frac{x^2}{-1/b} - \frac{2x}{(-1/b)^2} + \frac{2}{(-1/b)^3} \right) \right]_a^{\infty}$$

$$= \frac{-1}{b} \left[-a^2 b - 2ab^2 - 2b^3 \right]$$

$$= a^2 + 2ab + 2b^2$$

$$I_2 - I_1^2 = a^2 + 2ab + 2b^2 - (a+b)^2 = b^2$$

Q. Verify that the max value of $f_X(x)$ for a Rayleigh density fn occurs at $x = a + \sqrt{\frac{b}{2}}$ & is equal to $\sqrt{\frac{2}{b}}\, e^{-1/2} = 0.607 \sqrt{\frac{2}{b}}$ this value of x is called the mode of the R.V.

A. $$f_X(x) = \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}} \qquad x \geq a$$

$$\frac{d f_X(x)}{dx} = 0$$

$$\frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}} \left[-\frac{2(x-a)}{b} \right] + \frac{2}{b}\, e^{-\frac{(x-a)^2}{b}} = 0$$

$$-\frac{2(x-a)^2}{b} + 1 = 0$$

$$\therefore \quad x = a + \sqrt{\frac{b}{2}}$$

$$f_X(x) \Big|_{x = a + \sqrt{\frac{b}{2}}} = \frac{2}{b}\left(\sqrt{\frac{b}{2}}\right) e^{-\frac{\left(\sqrt{\frac{b}{2}}\right)^2}{b}}$$

$$= \sqrt{\frac{2}{b}}\, e^{-1/2} = 0.607 \sqrt{\frac{2}{b}}$$

Q: The lifetime of a System expressed in weeks is a Rayleigh r.v x for which $f_x(x) = \frac{x}{200} e^{-\frac{x^2}{400}}$ $x \geq 0$

a) what is the prob/: that the System will not last a full week.

b) what is the prob/: that the System lifetime will exceed one year.

A.
$$F_x(x) = 1 - e^{-\frac{(x-a)^2}{b}}$$

$$f_x(x) = \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}} \qquad x \geq a$$

$$f_x(x) = \frac{x}{200}\, e^{-\frac{x^2}{400}} \qquad x \geq 0$$

$\therefore\ a = 0,\ b = 400 \qquad \therefore\ F_x(x) = 1 - e^{-\frac{x^2}{400}}$

a) $P[x \leq 1] = F_x(1) = 1 - e^{-1/400} = 0.0025$

b) $P[x > 52] = 1 - P[x \leq 52] = 1 - F_x(52)$

$$= 1 - \left[1 - e^{-(52)^2/400}\right] = e^{-(52)^2/400} = 0.00116$$

Q: The Cauchy r.v has the pdf $f_x(x) = \dfrac{b/\pi}{b^2 + (x-a)^2}$

for real no's $0 < b$ & $-\infty < a < \infty$

S.T the distribution fn of x is $F_x(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{b}\right)$

A: $F_x(x) = \int_{-\infty}^{x} f_x(u)\, du = \int_{-\infty}^{x} \frac{b/\pi}{b^2 + (u-a)^2}\, du$

$$\xi = u - a \Rightarrow d\xi = du$$

$$F_x(x) = \frac{b}{\pi}\int_{-\infty}^{x-a} \frac{d\xi}{b^2 + \xi^2} = \frac{b}{\pi}\left[\frac{1}{b}\tan^{-1}\left(\frac{\xi}{b}\right)\right]_{-\infty}^{x-a}$$

$$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{b}\right)$$

Q: The no of cars arriving at a certain bank drive in window during any 10 min period is a poisson r.v x with $b=2$ find a) the prob/: that more than 3 cars will arrive during any 10 min period.

b) the prob/: that no cars will arrive.

A.
$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

a) $P[x>3] = 1-P[x\le 3]$

$$= 1 - e^{-2} \sum_{k=0}^{\infty} \frac{2^k}{k!} u(3-k)$$

$$= 1 - e^{-2}\left[1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!}\right] = 0.1429$$

b) $P[x=0] = f_X(0) = e^{-2} = 0.1353$

Q: Let x be a Rayleigh r.v with $a=0$ find the Prob/: that x will have values higher than its mode.

A.
$$F_X(x) = 1 - e^{-\frac{(x-a)^2}{b}} = 1 - e^{-\frac{x^2}{b}} \quad \text{Given } a=0$$

mode is $x = a + \sqrt{\frac{b}{2}} = \sqrt{\frac{b}{2}}$

$$P\left[x > \sqrt{\tfrac{b}{2}}\right] = 1 - P\left[x \le \sqrt{\tfrac{b}{2}}\right] = 1 - \left[1 - e^{-\frac{\left(\sqrt{\frac{b}{2}}\right)^2}{b}}\right]$$

$$= e^{-1/2} = 0.6065$$

Q: A certain large city averages three murders per week their occurrances follow a poisson distribution.

a) what is the prob/: that there will be five or more murders in a given week.

b) on the average how many weeks a year in the city expect to have no murders.

c) how many weeks per year can the city expect the no of murders per week equal to or exceed the average no per week.

A:

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k) \qquad b = 3$$

a) $P[5 \text{ or more}] = P(X \geq 5) = 1 - P[X \leq 4]$

$$= 1 - F_X(4) = 1 - e^{-3} \sum_{k=0}^{\infty} \frac{3^k}{k!} u(4-k)$$

$$= 1 - e^{-3} \left[1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}\right) = 0.1847$$

b) $P[X=0] = e^{-3} = 0.0498$

average number of weeks, the city has no murders

$= 52 \times 0.0498 = 2.5889$ weeks.

c) $P(3 \text{ or more}] = P[X \geq 3] = 1 - P(X \leq 2]$

$$= 1 - F_X(2) = 1 - e^{-3} \sum_{k=0}^{\infty} \frac{3^k}{k!} u(2-k)$$

$$= 1 - e^{-3} \left(1 + 3 + \frac{3^2}{2!}\right] = 0.5768$$

$\therefore$ average no of weeks in a year when the number of murders exceeds the average value $= 52 \times 0.5768$

$= 29.9941$ weeks.

Q. A computer undergoes down time if a certain critical component fails, this component is known to fail at an average rate of once per four weeks. No significant down time occurs if replacement components are on hand because repairs can be made rapidly. There are three

Components on hand and ordered replacements are not due for six weeks.

a) What is the prob/- of significant down time occuring before the ordered components arrive.

b) If the significant shipment is delayed two weeks what is the prob/- of significant down time occuring before the shipment arrives.

Sol: a) P[significant down time]

$$= P[\text{more than 3 failures in 6 weeks}]$$

$$= P[X>3] = 1 - P[X\le 3]$$

$$= 1 - F_X(3) = 1 - e^{-6/4}\left[1 + \frac{6}{4} + \frac{(6/4)^2}{2!} + \frac{(6/4)^3}{3!}\right]$$

$$= 0.0656$$

b) P[significant down time

$$= P[\text{more than 3 failures in 8 weeks}]$$

$$= P[X>3] = 1 - P[X\le 3]$$

$$= 1 - F_X(3) = 1 - e^{-\frac{8}{4}}\left(1 + \frac{(8/4)}{1!} + \frac{(8/4)^2}{2!} + \frac{(8/4)^3}{3!}\right)$$

$$= 0.1429$$

Conditional Distribution:

Let A be identified as the event $[X\le x]$ for the r.v X. If B is given then the conditional distribution fn of the r.v is given by

$$F_X(x|B) = P[X\le x|B] = \frac{P[X\le x \cap B]}{P(B)}$$

Conditional distribution applies to discrete, continuous & mixed r.v's

properties of Conditional distribution fn:

1) $F_X(-\infty|B) = 0$

2) $F_X(\infty|B) = 1$

3) $0 \leq F_X(x|B) \leq 1$

4) If $x_1 < x_2$ then $F_X(x_1|B) \leq F_X(x_2|B)$

5) $P[x_1 < X \leq x_2 | B] = F_X(x_2|B) - F_X(x_1|B)$

6) $F_X(x^+|B) = F_X(x|B)$

Conditional density:

The Conditional density fn of the RV X is defined as the derivative of Conditional distribution fn

$$f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

properties of Conditional density:

1) $f_X(x|B) \geq 0$

2) $\int_{-\infty}^{\infty} f_X(x|B)\, dx = 1$

3) $F_X(x|B) \geq \int_{-\infty}^{x} f_X(\xi|B)\, d\xi$

4) $P[x_1 < X \leq x_2|B] = \int_{x_1}^{x_2} f_X(x|B)\, dx$

Q: Two boxes R, G, B balls in them the no of balls of each colour is given in below table. A box is selected at random a ball is to be selected from the box. one box is slightly larger than the other, causing it to be selected more frequentl Let B_2 be the event "select the larger box" while B_1 is the event "select the smaller box" assume $P(B_1) = \frac{2}{10}$, $P(B_2) = \frac{8}{10}$

define a d.r.v x to have values x = 1, 2, 3 when 1.94
a R, G or B ball is selected. Let B be an events equal to either B_1 or B_2. find the Conditional distribution & density fn.

1.

a		B_1	B_2	total
1	R	5	80	85
2	G	35	60	95
3	B	60	10	75
	total	100	150	250

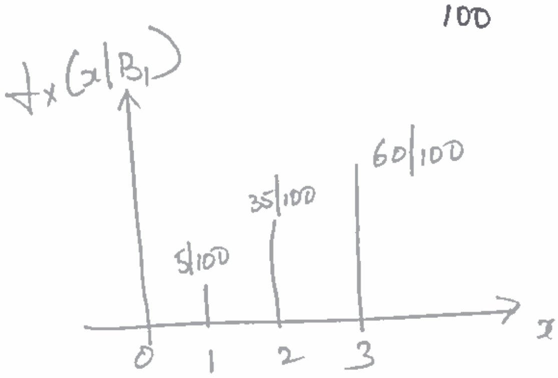
$P[x=1 \mid B=B_1] = 5/100$, $P[x=2 \mid B=B_1] = 35/100$

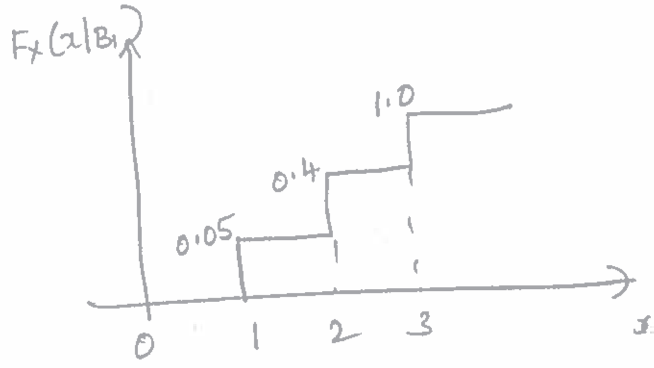
$P[x=3 \mid B=B_1] = 60/100$

Conditional density fn is

$$f_x(x \mid B_1) = \frac{5}{100}\delta(x-1) + \frac{35}{100}\delta(x-2) + \frac{60}{100}\delta(x-3)$$

$$F_x(x \mid B_1) = \frac{5}{100}u(x-1) + \frac{35}{100}u(x-2) + \frac{60}{100}u(x-3)$$





Methods of Defining Conditioning Event:

The event B can be defined from some characteristics of the physical experiment. It may be defined interms of the r.v 'x' or some other r.v other than x.

1) If the event B is defined interms of X let $B = [X \le b]$ where b is some real number $-\infty < b < \infty$

$$F_X(x \mid X \le b) = P[X \le x \mid X \le b] = \frac{P[X \le x \cap X \le b]}{P(X \le b)}$$

If $b \le x$ then $P[X \le x \cap X \le b] = P[X \le b]$

$$\therefore F_X(x \mid X \le b) = \frac{P(X \le b)}{P[X \le b]} = 1$$

If $x < b$

$$P[(X \le x) \cap (X \le b)] = P[X \le x]$$

$$F_X(x \mid X \le b) = \frac{P(X \le x)}{P[X \le b]} = \frac{F_X(x)}{F_X(b)}$$

$$\therefore F_X(x \mid X \le b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & x \ge b \end{cases}$$

$$\therefore f_X(x \mid X \le b) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx} = \dfrac{f_X(x)}{F_X(b)} & x < b \\ 0 & x \ge b \end{cases}$$

Questions

1) Define the following with a suitable example
   a) trial   c) random event
   b) event   d) sample space
   e) discrete & continuous sample space
   f) discrete & continuous event.

2) State the axioms of probability.

3) State & prove Bayes theorem.

4) Define independent & mutually exclusive events.

5) State the properties of independent events.

6) State & prove the properties of prob/: distribution fn.

7) State & prove the properties of pdf.

8) State the properties of conditional distribution fn.

9) State the properties of conditional density fn.

10) If the events A_1 & A_2 are independent then show that A_1 is independent of $\overline{A_2}$, $\overline{A_2}$ is independent of $\overline{A_1}$, $\overline{A_1}$ is independent of A_2.

11) Define prob/: using axiomatic approach or classical method & using relative frequency method.

12) Show that two events can not be both mutually exclusive & statistically independent.

13) Define the following density fn's

a) Gaussian ⓑ uniform ⓒ binomial Q-2

d) exponential e) poisson ⓕ Rayleigh.

Additional Problems

1) An ordinary 52 card deck, is thoroughly shuffled. You are dealt four cards up. what is the prob/: that all four cards are sevens.

2) Determine the prob/: of the card being either Red or a king when one card is drawn from a regular deck of 52 cards.

3) Three newspapers A, B & C are published in a city and a survey of readers indicates the following
20% read A, 16% read B, 14% read C, 8% read A & B, 5% read A & C
2% read A, B & C.
For one reader chosen at random, compute the prob/: that
a) he reads none of the papers.
b) he reads exactly one of the papers.
c) he reads A & B if it is known that he reads atleast one of the papers published.

4) A shipment of component consists of three identical boxes. one box contains 2000 components of which 25% are defective, the second box contains 5000 components of which 20% are defective and third box contains 2000 components of which 600 are defective. A box is selected at random and a component is removed at random from the box. what is the prob/: that this component is defective? what is the prob/:, that it came from the second box.

5) A letter is known to have come either from LONDON or CLIFTON. on the post mark only the two consecutive letters 'ON' are legible. what is the chance that it came from LONDON.

6) Two cards are drawn from a 52 card deck (the first is not replaced) (a) Given that the first card is

queen, what is the prob/: that the second is also a queen.

b) Repeat part ⓐ but replace the first card with a queen and the second card with a 7

c) what is the prob/: that both cards will be a queen.

7) An analog signal recieved at the detector (measured in microvolts) may be modelled as a Gaussian rv $N(200, 256)$ at a fixed point in time. what is the prob/: that the signal is larger than 240 µV given that it ~~was~~ is larger than 210 µV.

8) The lifetime of IC chips manufactured by a semiconductor manufacturer is approximately normally distributed with a mean $= 5\times10^6$ hrs and standard deviation of 5×10^6 hrs. A mainframe manufacturer requires that atleast 95% of a batch should have a lifetime greater that 4×10^6 hrs. will the deal be made.

9) A rv x has a pdf $f_x(x) = c(1-x^4) \quad -1 \le x \le 1$

$\qquad\qquad\qquad\qquad 0 \quad$ elsewhere

find a) c ⓑ $P[|x| < \frac{1}{2}]$

10)

Multiple choice Questions

1) The probability any event is ________ [ ]

a) positive (b) Negative (c) both a & b (d) none

2) The probability of sample space is ________ [ ]

a) 1 (b) 0 (c) >1 (d) <100

3) If A & B are mutually exclusive then $P(A\cap B)=$ ____ [ ]

a) 0 (b) $P(A)P(B)$ (c) $P(A)+P(B)$ (d) none

4) If A & B are mutually exclusive then $P(A\cup B)=$ ____ [ ]

a) 0 (b) $P(A)P(B)$ (c) $P(A)+P(B)$ (d) none

5) If A & B are independent then $P(A\cap B)=$ ____ [ ]

a) $P(A)P(B)$ (b) $P(A)+P(B)$ (c) 0 (d) none

6) If A & B are independent then $P(A|B)=$ ____ [ ]

a) $P(A)$ b) $P(B)$ (c) $P(A)P(B)$ (d) $P(A\cap B)$

7) If A & B are independent then $P(B/A)=$ ____ [ ]

a) $P(B)$ b) $P(B)$ c) 0 (d) none

8) According to total probability $P(A)=$ ________ [ ]

a) $\sum_{n=1}^{N} P(A|B_n)P(B_n)$ (b) $\sum_{n=1}^{N}$ ~~P(~~ $P(A_n|B)P(B)$

c) $\sum_{n=1}^{N} P(A+B_n)$ (d) none

9) According to Bayes theorem $P(B_n|A) =$ ______ [ ]

a) $\dfrac{P(B_n)\,P(A|B_n)}{\sum_{n=1}^{N} P(B_n)\,P(A|B_n)}$ (b) $\dfrac{P(A_n)\,P(A_n|B)}{\sum_{n=1}^{N} P(B_n)\,P(A|B_n)}$

c) $P(B_n)\,P(A)$ (d) none

10) If A_1 & A_2 are independent then ( )

a) $\overline{A_1}$ is independent of A_2

b) A_1 is independent of $\overline{A_2}$

c) $\overline{A_1}$ is independent of $\overline{A_2}$

d) all the above are correct

11) If A_1, A_2 & A_3 are independent then one event is independent of ______ other two events [ ]

a) union (b) Intersection (c) both a & b (d) none

12) $P[x = -\infty] =$ ______ [ ]

a) 0 (b) $-\infty$ (c) 1 (d) ∞

13) $P[x = \infty] =$ ______ [ ]

a) 0 (b) ∞ (c) 1 (d) $-\infty$

14) For a rv the mapping should be ______ [ ]

a) one to one mapping

b) many to one mapping

c) one to many mapping

(d) both a & b

15) A d.r.v can be defined on ________ [ ]

a) Continuous Sample space
b) discrete Sample space
c) mixed Sample space
d) All the above.

16) $P[X \le x) =$ ________ [ ]

a) $f_X(x)$ b) $F_X(x)$ c) both a & b d) none.

17) $F_X(-\infty) =$ ________ [ ]

a) $-\infty$ b) 1 c) 0 d) ∞

18) $F_X(\infty) =$ ________ [ ]

a) $-\infty$ b) 1 c) 0 d) ∞

19) $P[x_1 < X \le x_2] =$ ________ [ ]

a) $F_X(x_1) - F_X(x_2)$ b) $F_X(x_2) - F_X(x_1)$
c) $F_X(x_1) \cdot F_X(x_2)$ d) $F_X(x_1) * F_X(x_2)$

20) $P(X > x) =$ ________ [ ]

a) $1 - F_X(x)$ b) $F_X(x)$ c) $1 + F_X(x)$ d) none

21) $f_X(x)$ is ________ [ ]

a) ≥ 0 b) ≤ 0 c) both a & b d) none

22) The area under probl. density fn Curve is ____ [ ]

a) 1 b) 0 c) ∞ d) none

23) The PDF Curve for a drv is ______ [ ]

a) step fn (b) ramp fn (c) impulse fn (d) none

24) The pdf Curve for a drv is ______ [ ]

a) Step fn (b) ramp fn (c) impulse fn (d) none.

25) The PDF Curve for a crv is ______ [ ]

a) Step fn (b) ramp fn (c) impulse fn (d) none

26) The pdf Curve for a crv is ______ [ ]

a) step fn (b) ramp fn (c) impulse fn (d) none

27) For a normalized Gaussian rv x $F(-x) =$ ______ [ ]

a) $1 - F(x)$ (b) $-F(x)$ (c) $F(x)$ (d) none

28) For a Gaussian density fn $F_X(x) =$ ______ [ ]

a) $F\left(\frac{x+a_X}{\sigma_X}\right)$ (b) $F\left(\frac{x-a_X}{\sigma_X}\right)$ (c) $F(x)$ (d) none

29) $F_X(x|B) =$ ______ [ ]

a) $P[X \le x | B]$ (b) $P[X \le x]$ (c) $P(B)$ (d) none

30) $F_X(-\infty|B) =$ ______ [ ]

a) 0 (b) $-\infty$ (c) 1 (d) ∞

31) $F_X(\infty|B) =$ ______ [ ]

a) 0 (b) $-\infty$ (c) 1 (d) ∞

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TABLE B-1
Values of $F(x)$ for $0 \le x \le 3.89$ in steps of 0.01

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000

Although a closed-form solution for $Q(x)$ is not known, an excellent approximation is

$$Q(x) \approx \left[\frac{1}{0.661x + 0.339\sqrt{x^2 + 5.51}}\right]\frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad x \ge 0 \qquad \text{(B-8)}$$

which is due to Börjesson and Sundberg, 1979. The maximum absolute relative error in the approximation for $Q(x)$ is given as 0.27 percent for any $x \ge 0$. By using the approximation (B-8) for $Q(x)$ in (B-7), an excellent approximation for $F(x)$ is realized.

Unit - 2

Operations on Single RV:

Expectation:

The averaging process when applied to a RV is called the expectation. It is denoted by $E(X)$ & read as the expected value of X or mean value of X or the statistical average of X.

Expected value of a Random Variable:

If X is a dRV which takes on values in a finite set $x_1, x_2 \ldots x_N$ with prob's $P(x) = P[X = x_i]$ for $i = 1, 2 \ldots N$

then $$E[X] = \bar{X} = m = \sum_{i=1}^{N} x_i P(x_i)$$

If X is a Continuous RV then

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

provided $\int_{-\infty}^{\infty} f_X(x)\, dx < \infty$

If X is a dRV with N possible values x_i

then $$f_X(x) = \sum_{i=1}^{N} P(x_i)\, \delta(x - x_i)$$

Expected Value of a Function of RV:

Consider a RV X with Pdf $f_X(x)$. If $g(x)$ is a real function of x, then the expected value of $g(x)$ for a Continuous RV X is defined as

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

If X is a drv then

$$E(g(x)) = \sum_{i=1}^{N} g(x_i)\, P(x_i)$$

Conditional Expectation of a rv :-

If a rv X has a Conditional pdf $f_X(x|B)$ where B is any event defined in the sample space then the Conditional expectation of X is defined as

$$E[X|B] = \int_{-\infty}^{\infty} x\, f_X(x|B)\, dx \quad \rightarrow (1)$$

Let the rv $B = [X \le b]$ $\quad -\infty < b < \infty$

then

$$f_X(x|B) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\, dx} & x < b \\ 0 & \text{else} \end{cases} \quad \rightarrow (2)$$

Substituting (2) in (1) we get

$$E[X|\cancel{X} \le b] = \frac{\int_{-\infty}^{\infty} x\, f_X(x)\, dx}{\int_{-\infty}^{b} f_X(x)\, dx}$$

Properties of Expectation:

1) If a rv x is a Constant i.e $x = a$ then $E[a] = a$ where a is a Constant.

Sol: $E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$

$$E(a) = \int_{-\infty}^{\infty} a\, f_x(x)\, dx = a \int_{-\infty}^{\infty} f_x(x)\, dx = a$$

2) If a is any Constant, then $E[ax] = a\,E[x]$

Sol: $E[ax] = \int_{-\infty}^{\infty} ax\, f_x(x)\, dx = a \int_{-\infty}^{\infty} x\, f_x(x)\, dx$

$$= a\, E[x]$$

3) If a & b are any two Constants, then

$$E[ax+b] = a\,E(x) + b$$

Sol: $E[ax+b] = \int_{-\infty}^{\infty} (ax+b) f_x(x)\, dx$

$$= a \int_{-\infty}^{\infty} x\, f_x(x)\, dx + b \int_{-\infty}^{\infty} f_x(x)\, dx$$

$$= a\, E(x) + b$$

4) If $x \geq 0$ then $E[x] \geq 0$

Sol: If x is a continuous rv such that $x \geq 0$

then $E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \int_{-\infty}^{0} x\, f_x(x)\, dx + \int_{0}^{\infty} x\, f_x(x)\, dx$

$$= 0 + \int_{0}^{\infty} x\, f_x(x)\, dx \geq 0$$

5) If x is any r.v then the inequality

$$|E[x]| \leq E[|x|]$$

Sol: wkt $x \leq |x|$ & also $-x \leq |x|$

then $E[x] \leq E[|x|]$ & $E[-x] \leq E[|x|]$ or

$$-E[x] \leq E[|x|]$$

$$\therefore \; |E[x]| \leq E[|x|]$$

6) If $g_1(x)$ & $g_2(x)$ are two functions of a r.v. then

$$E(g_1(x) + g_2(x)) = E[g_1(x)] + E[g_2(x)]$$

Sol: $$E[g_1(x) + g_2(x)] = \int_{-\infty}^{\infty} [g_1(x) + g_2(x)] f_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} g_1(x) f_x(x)\, dx + \int_{-\infty}^{\infty} g_2(x) f_x(x)\, dx$$

$$= E[g_1(x)] + E[g_2(x)]$$

7) Mathematical expectation of the product of a no of independent r.v's is equal to the product of their individual expectations

i.e $E[xy] = E[x]\, E[y]$

Sol: $$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{xy}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_x(x)\, f_y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\, f_x(x)\, dx \int_{-\infty}^{\infty} y\, f_y(y)\, dy$$

$$= E[x]\, E[y]$$

Q: A d.r.v x has possible values $x_i = i^2$, $i = 1, 2, 3, 4, 5$ which occur with prob's 0.4, 0.25, 0.15, 0.1 & 0.1 resp find the mean value of x.

A: $\bar{x} = E[x] = \sum_{i=1}^{5} x_i P(x_i) = \sum_{i=1}^{5} i^2 P(x_i)$

$= 1^2(0.4) + 2^2(0.25) + 3^2(0.15) + 4^2(0.1) + 5^2(0.1)$

$= 6.85$

Q: The natural numbers are the possible values of a r.v x i.e $x_n = n$, $n = 1, 2 \ldots$ these numbers occur with prob's $P(x_n) = \left(\frac{1}{2}\right)^n$ find the expected value of x.

A: $E[x] = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{n}{2^n}$

note

$\boxed{\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \text{ if } x < 1}$

$= \frac{1/2}{(1-1/2)^2} = 2$

Q: If the prob in the above problem are $P(x_n) = p^n$ $0 < p < 1$ show that $p = \frac{1}{2}$ is only value of p that is allowed for the problem as formulated.

A: $\sum_i P(x_i) = 1$

$\sum_{n=1}^{\infty} p^n = 1$

$\frac{p}{1-p} = 1$ is satisfied only when

$p = 1/2$

Q: find the expected value of the fn $g(x) = x^3$ where x is a RV defined by the density fn $f_X(x) = \frac{1}{2}u(x)e^{-x/2}$

A: $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{\infty} x^3 \cdot \frac{1}{2} u(x) e^{-x/2} dx$

$$= \frac{1}{2}\int_0^{\infty} x^3 e^{-x/2}\, dx$$

$$= \frac{1}{2} e^{-x/2}\left[\frac{x^3}{-1/2} - \frac{3x^2}{(-1/2)^2} + \frac{6x}{(-1/2)^3} - \frac{6}{(-1/2)^4}\right]\Bigg|_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{6}{(1/2)^4}\right] = 48$$

Q: A RV X has a prob. density fn $f_X(x) = \frac{1}{2}\cos x$ $\quad -\frac{\pi}{2} < x \le \frac{\pi}{2}$ find the mean value of

a) $g(x) = 4x^2$ $\quad$ (b) $g(x) = 4x^4$

A: a) $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\pi/2}^{\pi/2} 4x^2 \frac{\cos x}{2} dx$

$$= 2\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx$$

$$= 2\left[x^2 \sin x + 2x\cos x - 2\sin x\right]_{-\pi/2}^{\pi/2}$$

$$= 2\left[\frac{\pi^2}{4}(1) + \frac{2\pi}{2}(0) - 2(1) - \frac{\pi^2}{4}(-1) - 2\left(-\frac{\pi}{2}\right)(0) + 2(-1)\right]$$

$$= 2\left[\frac{\pi^2}{4} - 2 + \frac{\pi^2}{4} - 2\right] = 4\left[\frac{\pi^2}{4} - 2\right]$$

$$= \pi^2 - 8 = 1.8696$$

b) $E[g(x)] = \int_{-\pi/2}^{\pi/2} 4x^4 \cdot \frac{1}{2} \cos x \; dx$

$= 2 \int_{-\pi/2}^{\pi/2} x^4 \cos x \; dx$

$= 2 \left[(4x^3 - 24x) \cos x + (x^4 - 12x^2 + 24) \sin x \right]_{-\pi/2}^{\pi/2}$

$= 2 \left[\frac{\pi^4}{16} - 12\frac{\pi^2}{4} + 24 + \frac{\pi^4}{16} - 12\frac{\pi^2}{4} + 24 \right]$

$= \frac{\pi^4}{16} - \frac{12\pi^2}{4} + 24 \quad = 1.917$

Q: The power (mw) returned to a radar from a certain class of aircraft has the prob. density fn $f_P(P) = \frac{1}{10} e^{-P/10}$ $u(P)$. Suppose a given aircraft belongs to this class but is known to not produce a power larger than 15 mw.

a) find the prob. density fn of P conditional on $P \leq 15$ mw

~~b) find~~ b) find the conditional mean value of P.

A: a) $\int_{-\infty}^{\infty} f_P(P)\, dP = \int_0^{15} \frac{e^{-P/10}}{10} \, dp = 1 - e^{-15/10}$

$= 0.777$ mw

b) $E[P \mid P \leq 1.5] = \dfrac{\int_0^{15} P \cdot \frac{e^{-P/10}}{10} \, dP}{[1 - e^{-15/10}]} = \dfrac{10[1 - 2.5e^{-1.5}]}{[1 - e^{-1.5}]}$

$= 5.692$ mw

Q: A RV X is uniformly distributed on the interval $(-5, 15)$ another RV $Y = e^{-x/5}$ is formed. find $E(Y)$

A: $y = g(x) = e^{-x/5}$

$$f_X(x) = \frac{1}{15-(-5)} = \frac{1}{20} \qquad -5 \le x \le 15$$

$$E[Y] = \int_{-5}^{15} \frac{1}{20} e^{-x/5}\, dx = \frac{1}{20} \left. \frac{e^{-x/5}}{-1/5} \right|_{-5}^{15}$$

$$= \frac{1}{4}\left[e^{1} - e^{-3}\right] = 0.667$$

Q: A gaussian voltage RV X has mean value $\bar{X} = a_x = ($ & variance $\sigma_x^2 = 9$ the voltage X is applied to a square law full wave diode detector with a transfer characteristic $Y = 5x^2$ find the mean value of the o/p voltage Y.

A: $$E[Y] = \int_{-\infty}^{\infty} 5x^2 \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx$$

$$= 5\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\cdot 3} e^{-\frac{x^2}{6}} dx \qquad \xi = \frac{x}{\sigma_x}$$

$dx = d\xi\sigma$

$= d\xi$

$$= 5\sigma_x^2$$

$$= 5 \times 9$$

$$= 45$$

Moments:

There are two types of moments for a function of a rv x.

a) moments about the origin

b) moments about the mean or central moments

moments about the origin:

For a rv x the nth moment of continuous rv about the origin is given by

$$m_n = E[x^n] = \int_{-\infty}^{\infty} x^n f_x(x)\, dx$$

$$m_0 = \int_{-\infty}^{\infty} x^0 f_x(x)\, dx = \int_{-\infty}^{\infty} f_x(x)\, dx = 1$$

$$m_1 = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \overline{X} = E[x] = \text{mean of } x$$

for a d rv x

$$m_n = \sum_{i=1}^{N} x_i^n\, P(x_i)$$

$$m_2 = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx = \overline{x^2} = E(x^2) = \text{mean square value of } x$$

$= \text{rms value of the signal}$

moments about the mean:

moments about the mean value of x are called central moments.

$$\mu_n = E[(x-\overline{x})^n] = \int_{-\infty}^{\infty} (x-\overline{x})^n f_x(x)\, dx$$

$$\mu_0 = E\left[(x-\bar{x})^0\right] = \int_{-\infty}^{\infty} f_x(x)\, dx = 1 = m_0$$

$$\mu_1 = E\left[(x-\bar{x})^1\right] = E(x) - \bar{x} = \bar{x} - \bar{x} = 0$$

If x is a d r v then

$$\mu_n = \sum_{i=1}^{N} (x_i - \bar{x}_i)^n \, P(x = x_i)$$

Variance:

The Variance of the density function $f_x(x)$ for a r v x is defined as the second central moment μ_2 of x.

$$\mu_2 = \sigma_x^2 = E\left[(x-\bar{x})^2\right] = E\left\{x^2 + \bar{x}^2 - 2x\bar{x}\right\}$$

$$= E\left[x^2\right] + \bar{x}^2 - 2E(x)\,\bar{x}$$

$$= E\left[x^2\right] + \bar{x}^2 - 2\bar{x}\,\bar{x}$$

$$= E\left[x^2\right] - \bar{x}^2 = m_2 - m_1^2$$

$$= E(x^2) - E(x)^2$$

standard deviation:

The Standard deviation of a r v is defined as the square root of the Variance.

$$\sigma_x = \sqrt{E\left[(x-\bar{x})^2\right]}$$

Skew & Coefficient of Skewness:

The skew of the density function $f_X(x)$ for a r.v. X is defined as the third central moment μ_3 of X. It is given by

$$\mu_3 = E\left[(X-\overline{X})^3\right] = \cancel{E\left\{X^3 - \overline{X}^3 - 3\overline{X}X^2 + 3X^2\overline{X}\right\}}$$

$$= \cancel{\overline{X^3} - \overline{X}^3 - 3\overline{X}\,\overline{X^2} + 3\overline{X^2}\,\overline{X}}$$

$$= E\left[X^3 - 3\overline{X}X^2 + 3\overline{X}^2X - \overline{X}^3\right]$$

$$= \overline{X^3} - 3\overline{X}\,\overline{X^2} + 3\overline{X}^2\,\overline{X} - \overline{X}^3$$

$$= \overline{X^3} - 3\overline{X}\,\overline{X^2} + 3\overline{X}^3 - \overline{X}^3$$

$$= \overline{X^3} - 3\overline{X}\,\overline{X^2} + 2\overline{X}^3$$

$$= \overline{X^3} - 3\overline{X}\left[\sigma_X^2 + \overline{X}^2\right] + 2\overline{X}^3$$

$$= \overline{X^3} - 3\overline{X}\sigma_X^2 - 3\overline{X}^3 + 2\overline{X}^3$$

$$= \overline{X^3} - 3\overline{X}\sigma_X^2 - \overline{X}^3$$

Skew is a measure of the asymmetry of $f_X(x)$ about its mean. It is the amount of deviation of symmetry from the mean value.

The normalized third central moment $\dfrac{\mu_3}{\sigma_X^3}$ is known as skewness of the density function or coefficient of skewness.

Properties of Variance:

1) The variance of a constant is zero i.e if k is a constant then $Var(k) = 0$.

$E[k] = \bar{k} = k$

Sol: $Var(k) = E[(k-\bar{k})^2] = E[(k-k)^2] = E[0] = 0$

2) If k is a constant then for a r.v x

$Var(kx) = k^2 Var(x)$

Sol. $Var(kx) = E[(kx - \overline{kx})^2]$

$= E[(kx - k\bar{x})^2] = k^2 E[(x-\bar{x})^2]$

$= k^2 Var(x)$

3) If x is a r.v and a & b are real constants then $Var(ax+b) = a^2 Var(x)$

Sol: $Var(ax+b) = E[(ax+b - \overline{ax+b})^2]$

$= E[(ax+b - a\bar{x} - b)^2]$

$= E[a^2(x-\bar{x})^2]$

$= a^2 E[(x-\bar{x})^2] = a^2 Var(x)$

4) If two r.v's x_1 & x_2 are independent then

$Var(x_1+x_2) = Var(x_1) + Var(x_2)$

& $Var(x_1-x_2) = Var(x_1) + Var(x_2)$

A: $Var(x_1 + x_2) = E\left[(x_1 + x_2 - \overline{x_1 + x_2})^2\right]$

$$= E\left[\{(x_1 - \overline{x_1}) + (x_2 - \overline{x_2})\}^2\right]$$

$$= E\left[(x_1 - \overline{x_1})^2\right] + E\left[(x_2 - \overline{x_2})^2\right]$$

$$+ E\left[(x_1 - \overline{x_1})(x_2 - \overline{x_2})\right]$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 + \underset{\downarrow 0}{E\left[(x_1 - \overline{x_1})\right]} \; \underset{\downarrow 0}{E\left[(x_2 - \overline{x_2})\right]}$$

$$= Var(x_1) + Var(x_2)$$

$Var(x_1 - x_2) = E\left[(x_1 - x_2 - \overline{x_1 - x_2})^2\right]$

$$= E\left[(x_1 - \overline{x_1})^2 + (x_2 - \overline{x_2})^2 - 2(x_1 - \overline{x_1})(x_2 - \overline{x_2})\right]$$

$$= E\left[(x_1 - \overline{x_1})^2\right] + E\left[(x_2 - \overline{x_2})^2\right] - 2\,\underset{\downarrow 0}{E\left[(x_1 - \overline{x_1})\right]}\,\underset{\downarrow 0}{E\left[(x_2 - \overline{x_2})\right]}$$

$$= Var(x_1) + Var(x_2)$$

Q: find the mean, variance, skew & coefficient of skewness for an exponential density fn.

A: exponential density fn is given by

$$f_X(x) = \frac{1}{b} e^{-\frac{(x-a)}{b}} \qquad x \geq a$$

$$\text{mean} = \bar{X} = E(X) = m_1 = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_a^{\infty} x \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}}\, dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} x \; e^{-x/b} \, dx$$

$$= \frac{e^{a/b}}{b} \left[e^{-x/b} \left(\frac{x}{-1/b} - \frac{1}{(-1/b)^2} \right] \right]_a^{\infty}$$

$$= \frac{e^{a/b}}{b} \left(e^{-a/b} \left[ab + b^2 \right] \right) = \frac{ab+b^2}{b} = a+b$$

$$m_2 = E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x) \, dx = \int_a^{\infty} x^2 \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}} dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} x^2 \, e^{-x/b} \, dx$$

$$= \frac{e^{a/b}}{b} \left[e^{\frac{-x}{b}} \left[\frac{x^2}{(-1/b)} - \frac{2x}{(-1/b)^2} + \frac{2}{(-1/b)^3} \right] \right]_a^{\infty}$$

$$= \frac{1}{b} \left[a^2 b + 2ab^2 + 2b^3 \right)$$

$$= a^2 + 2ab + 2b^2$$

$$\text{Variance} = \sigma_x^2 = m_2 - m_1^2 = a^2 + 2ab + 2b^2 - (a+b)^2$$

$$= b^2$$

$$m_3 = \overline{x^3} = \int_{-\infty}^{\infty} x^3 f_x(x) \, dx = \int_a^{\infty} x^3 \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}} dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} x^3 \cdot e^{-\frac{x}{b}} \, dx$$

$$= \frac{e^{a/b}}{b} \left(e^{-\frac{x}{b}} \left[\frac{x^3}{(-1/b)} - \frac{3x^2}{(-1/b)^2} + \frac{6x}{(-1/b)^3} - \frac{6}{(-1/b)^4} \right] \right]_a^{\infty}$$

$$= \frac{1}{b}\left[a^3 b + 3a^2 b^2 + 6ab^3 + 6b^4\right]$$

$$= a^3 + 3a^2 b + 6ab^2 + 6b^3$$

$$\mu_3 = \text{Skew} = \overline{x^3} - 3\bar{x}\,\sigma_x^2 - \bar{x}^3$$

$$= a^3 + 3a^2 b + 6ab^2 + 6b^3 - 3(a+b)b^2 - (a+b)^3$$

$$= 2b^3$$

$$\text{Coefficient of Skewness} = \frac{\mu_3}{\sigma_x^3} = \frac{2b^3}{b^3} = 2$$

Q: find the mean & Variance of uniform density fn.

A: the uniform density fn is given by

$$f_x(x) = \frac{1}{b-a} \quad a \le x \le b$$
$$0 \quad \text{else}$$

$$m_1 = \bar{x} = \int_{-\infty}^{\infty} x\, f_x(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

$$m_2 = \overline{x^2} = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$\sigma_x^2 = m_2 - m_1^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$

Q: find the mean Variance of a gaussian rv x.

A: gaussian rv x is defined by

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \qquad -\infty < x < \infty$$

$$m_1 = E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx \qquad \text{let } a_x = m$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-m)^2}{2\sigma_x^2}}\, dx$$

let $t = \frac{x-m}{\sigma_x} \Rightarrow x = m + \sigma_x t$

$dx = \sigma_x dt$

$$E(x) = \int_{-\infty}^{\infty} (m+\sigma_x t) \cdot \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{t^2}{2}} \sigma_x\, dt$$

$$= m \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{1} + \sigma_x \underbrace{\int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{0}$$

$$= m + 0$$

$$= m = a_x$$

$$m_2 = E[x^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-m)^2}{2\sigma_x^2}} dx$$

let $\frac{x-m}{\sigma_x} = t \Rightarrow x = m + t\sigma_x$

$dx = dt\,\sigma_x$

$$E(x^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma_x t + m)^2 e^{-\frac{t^2}{2}} dt$$

$$= \sigma_x^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 \cdot e^{-\frac{t^2}{2}} dt + m^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$+ 2m\sigma_x \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= \sigma_x^2 + m^2 + 2m\sigma_x(0)$$

$$= \sigma_x^2 + m^2$$

$$Var(x) = m_2 - m_1^2 = \sigma_x^2 + m^2 - m^2 = \sigma_x^2$$

Q: find the mean & Variance of Rayleigh density fn?

A:
$$f_x(x) = \frac{2}{b}(x-a) e^{-\frac{(x-a)^2}{b}} \qquad x \geq a$$

$$m_1 = E(x) = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

$$= \int_a^{\infty} x \cdot \frac{2}{b}(x-a) e^{-\frac{(x-a)^2}{b}} dx$$

$$= 2\int_a^{\infty} \frac{(x-a)^2}{b} e^{-\frac{(x-a)^2}{b}} dx + 2a \int_a^{\infty} \frac{(x-a)}{b} e^{-\frac{(x-a)^2}{b}} dx$$

In 1st integral let $\xi = \dfrac{x-a}{\sqrt{b}} \Rightarrow dx = \sqrt{b}\, d\xi$

$(x-a)^2 = b\xi^2$

In 2nd integral let $\eta = \dfrac{(x-a)^2}{b} \Rightarrow d\eta = \dfrac{2(x-a)}{b}\, dx$

$$E(x) = \int_0^\infty \frac{2}{b} \cdot b\xi^2 e^{-\xi^2} \sqrt{b}\, d\xi + a\int_0^\infty e^{-\eta}\, d\eta$$

$$= 2\sqrt{b} \int_0^\infty \xi^2 e^{-\xi^2} d\xi + a \left. \frac{e^{-\eta}}{-1} \right|_0^\infty$$

$$= 2\sqrt{b}\, \frac{\sqrt{\pi}}{4} + a$$

$$= a + \frac{\sqrt{b\pi}}{2}$$

$$E(x^2) = \int_a^\infty x^2 \cdot \frac{2}{b}(x-a) e^{-\frac{(x-a)^2}{b}} dx$$

$$= \int_a^\infty \frac{2}{b}(x-a)[x-a+a]^2 e^{-\frac{(x-a)^2}{b}} dx$$

$$= \frac{2}{b}\int_a^\infty (x-a)^3 e^{-\frac{(x-a)^2}{b}} dx + \frac{4a}{b}\int_a^\infty (x-a)^2 e^{-\frac{(x-a)^2}{b}} dx$$

$$+ \frac{2a^2}{b}\int_a^\infty (x-a) e^{-\frac{(x-a)^2}{b}} dx$$

In 1st integral let $\xi = \dfrac{(x-a)^2}{b} \Rightarrow d\xi = \dfrac{2(x-a)}{b}\, dx$

2nd integral $\xi = \dfrac{x-a}{\sqrt{b}} \Rightarrow d\xi = \dfrac{dx}{\sqrt{b}}$

3rd integral $\quad \xi = \frac{(x-a)^2}{b} \Rightarrow d\xi = \frac{2(x-a)}{b} dx$

$$E(x^2) = b\int_0^\infty \xi e^{-\xi} d\xi + 4a\sqrt{b}\int_0^\infty \xi^2 e^{-\xi^2} d\xi + a^2\int_0^\infty e^{-\xi} d\xi$$

$$\Rightarrow \cancel{b(0)} = b(1) + 4a\sqrt{b}\frac{\sqrt{\pi}}{4} + a^2(1)$$

$$= a^2 + a\sqrt{b\pi} + b$$

$$\sigma_x^2 = E(x^2) - E(x)^2 = a^2 + a\sqrt{b\pi} + b - \left(a + \frac{\sqrt{b\pi}}{2}\right)^2$$

$$= a^2 + a\sqrt{b\pi} + b - a^2 - \frac{b\pi}{4} - a\sqrt{b\pi}$$

$$= b - \frac{b\pi}{4} = b\left(\frac{4-\pi}{4}\right)$$

Q: For a bionomial density fn S.T $E[x] = \bar{x} = NP$ &
$\sigma_x^2 = NPq = NP(1-P)$

1: $$f_x(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

$$\bar{x} = E[x] = \sum_{i=1}^{N} x_i P(x_i) = \sum_{x=0}^{N} x \binom{N}{x} p^x q^{N-x} \delta(x-k)$$

$$= \sum_{k=0}^{N} k \binom{N}{k} p^k q^{N-k}$$

$$= \sum_{k=1}^{N} k \cdot \frac{N!}{(N-k)!\, k!} p^k q^{N-k}$$

$$= \sum_{k=1}^{N} k \frac{N(N-1)!}{\ldots} p\, p^{k-1} q^{(N-1)-(k-1)}$$

$$= Np \sum_{k=1}^{N} \frac{(N-1)!}{(k-1)!\,[(N-1)-(k-1)]!}\, p^{k-1} q^{(N-1)-(k-1)}$$

$$= Np \sum_{k=1}^{N} \binom{N-1}{k-1} p^{k-1} q^{(N-1)-(k-1)}$$

$$= Np\,(p+q)^{N-1} = Np$$

$$E(x^2) = \sum_{x=0}^{N} x^2 \binom{N}{x} p^x q^{N-x}\, \delta(x-k)$$

$$= \sum_{k=0}^{N} k^2 \binom{N}{k} p^k q^{N-k}$$

$$= \sum_{k=1}^{N} k^2 \binom{N}{k} p^k q^{N-k}$$

$$= \sum_{k=1}^{N} [k(k-1)+k] \binom{N}{k} p^k q^{N-k}$$

$$= \sum_{k=1}^{N} k(k-1) \binom{N}{k} p^k q^{N-k} + \sum_{k=1}^{N} k \binom{N}{k} p^k q^{N-k}$$

$$= \sum_{k=2}^{N} k(k-1) \binom{N}{k} p^k q^{N-k} + Np$$

$$= \sum_{k=2}^{N} k(k-1) \frac{N!}{k!\,(N-k)!} p^k q^{N-k} + Np$$

$$= \sum_{k=2}^{N} k(k-1) \frac{N(N-1)(N-2)!}{k(k-1)(k-2)!\,[(N-2)-(k-2)]!}\, p^2 p^{k-2} q^{(N-2)-(k-2)} + Np$$

$$= \sum_{k=2}^{N} \frac{(N-2)!}{(k-2)!\,[(N-2)-(k-2)]!}\, N(N-1)\, p^2 p^{k-2} q^{(N-2)-(k-2)} + Np$$

$$= N(N-1)p^2 \sum_{k=2}^{N} \binom{N-2}{k-2} p^{k-2} q^{(N-2)-(k-2)} + NP$$

$$= N(N-1)p^2 (p+q)^{N-2} + NP$$

$$= N(N-1)p^2 + NP$$

$$= N^2p^2 - Np^2 + NP = N^2p^2 + NP(1-P)$$

$$= N^2p^2 + Npq$$

$$\sigma_x^2 = E(x^2) - E(x)^2 = N^2p^2 + Npq - (NP)^2$$

$$= Npq$$

Q: For a poisson rv show that mean value & Variance is b.

A. $$f_x(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

$$E(x) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-b} b^k}{k!} = b \sum_{k=1}^{\infty} \frac{b^{k-1} e^{-b}}{(k-1)!}$$ let $j = k-1$

$$= b \sum_{j=0}^{\infty} \frac{e^{-b} b^j}{j!} = b$$

$$E(x^2) = \sum_{k=0}^{\infty} k^2 \frac{e^{-b} b^k}{k!} = b \sum_{k=1}^{\infty} \frac{k^2 e^{-b} b^{k-1}}{k!}$$

$$= b \sum_{k=1}^{\infty} \frac{k e^{-b} b^{k-1}}{(k-1)!} = b \sum_{k=1}^{\infty} [k-1+1] \frac{e^{-b} b^{k-1}}{(k-1)!}$$

$$= b \sum_{k=1}^{\infty} \frac{(k-1) e^{-b} b^{k-1}}{(k-1)!} + b \sum_{k=1}^{\infty} \frac{e^{-b} b^{k-1}}{(k-1)!}$$

$$= b\sum_{k=2}^{\infty} b\,\frac{b^{k-2}\,e^{-b}}{(k-2)!} + b\sum_{k=1}^{\infty}\frac{b^{k-1}\,e^{-b}}{(k-1)!}$$

$$= b^2\sum_{j=0}^{\infty}\frac{e^{-b}\,b^j}{j!} + b\sum_{j=0}^{\infty}\frac{b^j\,e^{-b}}{j!} \qquad k-1=j \quad k-2=j$$

$$= b^2 + b$$

$$\sigma_x^2 = E(x^2) - E(x)^2 = b^2+b - b^2 = b$$

Q: find the mean value & variance for a r.v with laplace density fn $f_x(x) = \frac{1}{2b} e^{-\frac{|x-m|}{b}}$ where b & m are constants $b>0$, $-\infty < m < \infty$.

A: Since $f_x(x)$ is symmetric about $x=m$

$$\therefore E(x) = m = \bar{x}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty}(x-\bar{x})^2 f_x(x)\,dx$$

$$= \int_{-\infty}^{\infty}(x-m)^2 \cdot \frac{1}{2b}\, e^{-\frac{|x-m|}{b}}\,dx \qquad \xi = \frac{x-m}{b} \quad d\xi = \frac{dx}{b}$$

$$= \int_{-\infty}^{\infty} b^2\xi^2 \cdot \frac{1}{2}\, e^{-|\xi|}\,d\xi$$

$$= \frac{b^2}{2}\int_{-\infty}^{\infty}\xi^2 e^{-|\xi|}\,d\xi = b^2\int_0^{\infty}\xi^2 e^{-\xi}\,d\xi$$

$$= b^2\left[\xi^2\frac{e^{-\xi}}{-1}\Big|_0^{\infty} + \int_0^{\infty} e^{-\xi}\cdot \underbrace{2\xi\,d\xi}_{2}\right]$$

$$= 2b^2$$

Q: find the expression for all moments about the origin & Central moments for uniform density fn.

A. $m_n = \int_{-\infty}^{\infty} x^n f_x(x)\, dx = \int_a^b x^n \cdot \frac{1}{b-a}\, dx$

$= \frac{1}{b-a} \frac{\left[x^{n+1}\right]_a^b}{(n+1)} = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$

$\mu_n = \int_{-\infty}^{\infty} (x-\bar{x})^n\, dx = \int_a^b (x-\bar{x})^n \cdot \frac{1}{b-a}\, dx$

$= \frac{1}{b-a} \frac{(x-\bar{x}]^{n+1}\Big|_a^b}{(n+1)} = \frac{\left(b - \frac{(b+a)}{2}\right]^{n+1} - \left[a - \frac{(a+b)}{2}\right]^{n+1}}{(n+1)(b-a)}$

$= \frac{(b-a)^{n+1} - (a-b)^{n+1}}{2^{n+1}(n+1)(b-a)}$

Q: Define a fn $g(\cdot)$ of a RV x by $g(x) = \begin{cases} 1 & x \geq x_0 \\ 0 & x < x_0 \end{cases}$

where x_0 is a real no. $-\infty < x_0 < \infty$

S.T $E[g(x)] = 1 - F_x(x_0)$.

A: $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx = \int_{-\infty}^{x_0} 0 \cdot f_x(x)\, dx$

$+ \int_{x_0}^{\infty} 1 \cdot f_x(x)\, dx$

$= 1 - \int_{-\infty}^{x_0} f_x(x)\, dx = 1 - F_x(x)\Big|_{-\infty}^{x_0}$

$= \ldots F_x(x_0)$

Q. Prove that Central moments μ_n are related m_k about the origin by

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-\bar{x})^{n-k} m_k .$$

A: $$\mu_n = E\left[(x-\bar{x})^n\right] = E\left[\sum_{k=0}^{n} \binom{n}{k} x^k (-\bar{x})^{n-k}\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} E\left[x^k\right] (-\bar{x})^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} m_k (-\bar{x})^{n-k}$$

Q: A RV x has a density fn $f_x(x)$ & moments m_n. If the density is shifted higher in x by an amount $\alpha > 0$ to a new origin. Show that moments of the shifted denoted by m_n' are related to the moments m_n by

$$m_n' = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} m_k$$

A: $$m_n' = \int_{-\infty}^{\infty} x^n f_x(x-\alpha)\, dx$$

Let $\xi = x - \alpha$
$d\xi = dx$
$x = \alpha + \xi$

$$= \int_{-\infty}^{\infty} (\alpha+\xi)^n f_x(\xi)\, d\xi$$

$$= \int_{-\infty}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \xi^k \alpha^{n-k} f_x(\xi)\, d\xi$$

$$= \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \int_{-\infty}^{\infty} \xi^n f_x(\xi)\, d\xi$$

$$= \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} m_k$$

Q: For any drv x with values x_i having prob's of occurence $P(x_i)$. Show that the moments of x are $m_n = \sum_{i=1}^{N} x_i^n P(x_i)$ & $\mu_n = \sum_{i=1}^{N} (x_i - \bar{x})^n P(x_i)$ where N may be infinite for some x.

A:
$$f_X(x) = \sum_{i=1}^{N} P(x_i)\, \delta(x - x_i)$$

$$m_n = \int_{-\infty}^{\infty} x^n f_X(x)\, dx = \int_{-\infty}^{\infty} x^n \sum_{i=1}^{N} P(x_i)\, \delta(x - x_i)\, dx$$

$$= \sum_{i=1}^{N} P(x_i)\, x_i^n$$

Q: find the skew & coefficient of skewness for a Rayleigh rv for which $a = 0$

A
$$\bar{X} = a + \sqrt{\frac{b\pi}{4}} = \sqrt{\frac{b\pi}{4}}$$

$$\sigma_X^2 = b\left(1 - \frac{\pi}{4}\right)$$

$$\overline{X^2} = \sigma_X^2 + \bar{X}^2 = b\left(1 - \frac{\pi}{4}\right) + \frac{b\pi}{4} = b$$

$$\overline{X^3} = \int_0^{\infty} x^3 \cdot \frac{2}{b} x\, e^{-\frac{x^2}{b}}\, dx = \int_0^{\infty} \frac{2}{b} x^4 e^{-\frac{x^2}{b}}\, dx$$

Let $v = \frac{x^2}{b}$

$2x \frac{dx}{b} = d$

$$= b^{3/2} \int_0^{\infty} v^{3/2} e^{-v}\, dv = \frac{3\sqrt{\pi}}{4} b^{3/2}$$

$$\text{skew} = \mu_3 = \overline{X^3} - 3\bar{X}\,\overline{X^2} + 2\bar{X}^3$$

$$= b^{3/2} \frac{3\sqrt{\pi}}{4} - 3 b^{1/2} \frac{\sqrt{\pi}}{2} \cdot b + \frac{2 b^{3/2} \pi \sqrt{\pi}}{8}$$

$$= \frac{\sqrt{\pi}}{4} (\pi - 3)\, b^{3/2}$$

$$= \frac{\mu_3}{\;} = \frac{2\pi(\pi - 3)}{(\;\;\pi)^{3/2}} = 0.6311$$

Q: A r.v X has the density function $f_X(x) = \frac{3}{32}(-x^2+8x-12)$, $2 \le x \le 6$; find the following moments m_0, m_1, m_2, μ_2

A. $$m_n = \int_2^6 x^n \cdot \frac{3}{32}\left[-x^2+8x-12\right] dx$$

$$= \frac{3}{32}\left[-\frac{x^{n+3}}{n+3} + 8\frac{x^{n+2}}{n+2} - 12\frac{x^{n+1}}{n+1}\right]_2^6$$

$m_0 = 1$, $m_1 = 4$

$m_2 = 16.8$ $\quad \mu_2 = m_2 - m_1^2 = 16.8 - 16 = 0.8$

Q: A r.v X has $\overline{X} = -3$, $\overline{X^2} = 11$, $\sigma_X^2 = 2$ for new r.v $Y = 2X-3$ find $\overline{Y}$, $\overline{Y^2}$, σ_Y^2

A: $\overline{Y} = 2\overline{X} - 3 = -9$

$\overline{Y^2} = E[Y^2] = E[4x^2+9-12x] = 4\overline{X^2} - 12\overline{X} + 9 = 89$

$\sigma_Y^2 = \overline{Y^2} - \overline{Y}^2 = 89 - 81 = 8$

Q: A r.v X has a probl. density fn $f_X(x) = \frac{5}{4}(1-x^4)$ for $0 \le x \le 1$ find $E(x)$, $E(4x+2)$, $E(x^2)$

A: $$E(x) = \int_0^1 x \cdot \frac{5}{4}(1-x^4)\, dx = \frac{5}{4}\left[\frac{x^2}{2} - \frac{x^6}{6}\right]_0^1 = \frac{5}{12}$$

$$E(4x+2) = 4E(x) + 2 = 4\left(\frac{5}{12}\right) + 2 = \frac{11}{3}$$

$$E(x^2) = \int_0^1 x^2 \cdot \frac{5}{4}(1-x^4)\, dx = \frac{5}{4}\left[\frac{x^3}{3} - \frac{x^7}{7}\right]_0^1 = \frac{5}{21}$$

Functions For moments:

To calculate the nth moments of a r.v x, the functions used are

a) Characteristic Function

b) Moment Generating function

Characteristic Function:

The expected value of the function $e^{j\omega x}$ is called the characteristic function. It is expressed by

$$\phi_x(\omega) = E\left[e^{j\omega x}\right] = \int_{-\infty}^{\infty} f_x(x)\, e^{j\omega x}\, dx \qquad -\infty < \omega < \infty$$

& $$f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_x(\omega)\, e^{-j\omega x}\, d\omega$$

for a d.r.v $\phi_x(\omega) = \sum_i e^{j\omega x_i} p(x_i)$

the nth moment of x is given by

$$\boxed{m_n = (-j)^n \left.\frac{d^n \phi_x(\omega)}{d\omega^n}\right|_{\omega=0}}$$

wkt $\phi_x(\omega) = E\left[e^{j\omega x}\right]$

$$= E\left[1 + j\omega x + \frac{(j\omega x)^2}{2!} + \frac{(j\omega x)^3}{3!} + \cdots\right]$$

$$= 1 + j\omega E(x) - \frac{\omega^2}{2} E(x^2) - j\frac{\omega^3}{3!} E\left[x^3\right] + \cdots$$

$$= m_0 + j\omega m_1 - \frac{\omega^2}{2} m_2 - \frac{j\omega^3}{3!} m_3 + \cdots$$

at $\omega = 0$

$$\phi_x(0) = m_0 = 1$$

$$\frac{d\phi_x(\omega)}{d\omega} = jm_1 - \omega m_2 - \frac{j3\omega^2}{3!} m_3 - \cdots$$

at $\omega = 0$ i.e $\left.\frac{d\phi_x(\omega)}{d\omega}\right|_{\omega=0} = jm_1$

$$m_1 = (-j)\left.\frac{d\phi_x(\omega)}{d\omega}\right|_{\omega=0}$$

$$\frac{d^2\phi_x(\omega)}{d\omega^2} = -m_2 - j\omega m_3 - \cdots$$

$$\left.\frac{d^2\phi_x(\omega)}{d\omega^2}\right|_{\omega=0} = -m_2$$

$$\therefore \quad m_2 = (-j)^2 \left.\frac{d^2\phi_x(\omega)}{d\omega^2}\right|_{\omega=0}$$

$$|||^{ly} \quad m_n = (-j)^n \left.\frac{d^n\phi_x(\omega)}{d\omega^n}\right|_{\omega=0}$$

Properties of Characteristic Function:

1) The characteristic f^n is unity at $\omega=0$

i.e $\left.\phi_x(\omega)\right|_{\omega=0} = \phi_x(0) = 1$

Pf:
$$\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx$$

$$\left.\phi_x(\omega)\right|_{\omega=0} = \phi_x(0) = \int_{-\infty}^{\infty} e^{j0x} f_x(x)\,dx$$

$$= \int_{-\infty}^{\infty} f_x(x)\,dx = 1$$

2) The maximum amplitude of the characteristic fn is unity at $\omega = 0$

i.e $|\phi_X(\omega)| \leq \phi_X(0) \leq 1$

Sol: $\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$

$$|\phi_X(\omega)| = \left| \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx \right| \leq \int_{-\infty}^{\infty} |e^{j\omega x}|\, |f_X(x)|\, dx$$

$$\leq \int_{-\infty}^{\infty} |f_X(x)|\, dx \leq 1$$

3) $\phi_X(\omega)$ is a continuous fn of ω in the range $-\infty < \omega < \infty$.

Sol. $\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$

Since ω is continuous, $e^{j\omega x}$ is also continuous. Hence $\phi_X(\omega)$ is a continuous function.

4) $\phi_X(-\omega)$ & $\phi_X(\omega)$ are continuous fns.

$\phi_X(-\omega) = \phi_X^*(\omega)$ & $\phi_X^*(-\omega) = \phi_X(\omega)$

Sol: $\phi_X(\omega) = E[e^{j\omega x}]$

$$\phi_X(-\omega) = E[e^{-j\omega x}] = E[e^{j\omega x}]^* = \phi_X^*(\omega)$$

$$\phi_X^*(-\omega) = E[e^{-j\omega x}]^* = E[e^{j\omega x}] = \phi_X(\omega).$$

5) If $\phi_X(\omega)$ is a characteristic fn of a RV X then the characteristic function of $Y = ax + b$ is given by

$\phi_Y(\omega) = e^{j\omega x} \phi_X(a\omega)$ where a & b are real constants

A: $Y = ax + b$

$$\phi_Y(\omega) = E[e^{j\omega Y}] = E(e^{j\omega(ax+b)}) = E(e^{j\omega ax} \cdot e^{j\omega b})$$

$$= e^{j\omega b} E(e^{j(a\omega)x}) = e^{j\omega b} \phi_X(a\omega)$$

6) If $\phi_X(\omega)$ is a characteristic fn of a RV X then

$\phi_X(c\omega) = \phi_{cX}(\omega)$ where c is a real constant

A: $\phi_X(\omega) = E[e^{j\omega x}]$

$$\phi_X(c\omega) = E[e^{jc\omega x}] = E[e^{j\omega(cx)}] = \phi_{cx}(\omega)$$

7) If X_1 & X_2 are two independent RV's then

$\phi_{X_1+X_2}(\omega) = \phi_{X_1}(\omega) \cdot \phi_{X_2}(\omega)$.

A $\phi_X(\omega) = E(e^{j\omega x}]$

then $\phi_{X_1+X_2}(\omega) = E(e^{j\omega(X_1+X_2)}) = E[e^{j\omega X_1} \cdot e^{j\omega X_2}]$

$$= E[e^{j\omega X_1}] E(e^{j\omega X_2}] = \phi_{X_1}(\omega) \phi_{X_2}(\omega).$$

Q: find the characteristic fn of the exponential density fn.

A: $\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx = \int_a^{\infty} e^{j\omega x} \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}} dx$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} e^{-(\frac{1}{b} - j\omega)x} dx$$

$$= \frac{e^{a/b}}{b} \frac{\left[e^{-(\frac{1}{b}-j\omega)x} \right]_a^{\infty}}{-(\frac{1}{b}-j\omega)} = \frac{e^{j\omega a}}{1 - bj\omega}$$

Q: Find the characteristic for uniform density fn.

A: $\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_x(x)\, dx = \int_a^b e^{j\omega x} \cdot \frac{1}{b-a}\, dx$

$$= \frac{1}{b-a} \frac{\left[e^{j\omega x}\right]_a^b}{j\omega} = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$$

Q: Show that the characteristic fn of a rv having the binomial density fn is $\phi_x(\omega) = (1-p+pe^{j\omega})^N$

A: $\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx$

$$= \int_{-\infty}^{\infty} e^{j\omega x} \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k}\, \delta(x-k)\, dx$$

$$= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \int_{-\infty}^{\infty} e^{j\omega x}\, \delta(x-k)\, dx$$

$$= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k}\, e^{j\omega k}$$

$$= \sum_{k=0}^{N} \binom{N}{k} (pe^{j\omega})^k (1-p)^{N-k}$$

$$= (1-p+pe^{j\omega})^N$$

Q: Show that the characteristic fn of a poisson rv is $\phi_x(\omega) = e^{-b[1-e^{j\omega}]}$

A. $\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx = \int_{-\infty}^{\infty} e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!}\, \delta(x-k)\, e^{j\omega x}\, dx$

$$= e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \int_{-\infty}^{\infty} e^{j\omega x} \, \delta(x-k) \, dx$$

$$= e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} e^{j\omega k} = e^{-b} \sum_{k=0}^{\infty} \frac{(b e^{j\omega})^k}{k!}$$

$$= e^{-b} \cdot e^{b e^{j\omega}} = e^{-b + b e^{j\omega}} = e^{-b\left[1 - e^{j\omega}\right]}$$

Q: The erlang r.v x has a characteristic fn $\phi_x(\omega) = \left(\frac{a}{a - j\omega}\right)^N$ for $a > 0$, & $N = 1, 2 \ldots$

Show that $\overline{X} = \frac{N}{a}$ & $\overline{x^2} = \frac{N(N+1)}{a^2}$ & $\sigma_x^2 = \frac{N}{a^2}$

A: $$m_1 = \overline{x} = -j \frac{d\phi_x(\omega)}{d\omega}\Bigg|_{\omega=0} = -j\, a^N \left[\frac{-N(-j)}{(a - j\omega)^{N+1}}\right]_{\omega=0}$$

$$= \frac{N}{a}$$

$$m_2 = \overline{x^2} = -\frac{d^2\phi_x(\omega)}{d\omega^2}\Bigg|_{\omega=0} = -jN\, a^N \left[\frac{-(N+1)(-j)}{(a - j\omega)^{N+2}}\right]_{\omega=0}$$

$$= \frac{N(N+1)}{a^2}$$

$$\sigma_x^2 = m_2 - m_1^2 = \frac{N(N+1)}{a^2} - \frac{N^2}{a^2} = \frac{N}{a^2}$$

Q: The characteristic fn for a g r.v x having a mean value of '0' is $\phi_x(\omega) = e^{-\frac{\sigma_x^2 \omega^2}{2}}$ find all the moments of x using $\phi_x(\omega)$.

1: $\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx = \int_{-\infty}^{\infty} f_x(x) \sum_{n=0}^{\infty} \frac{(j\omega x)^n}{n!}\, dx$

$$= \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} \int_{-\infty}^{\infty} f_x(x)\, x^n\, dx$$

$$\boxed{\text{Note } e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}}$$

$$= \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} m_n$$

$$= \sum_{n=0}^{\infty} (j)^n \frac{m_n}{n!} \omega^n \longrightarrow (1)$$

$$\phi_x(\omega) = e^{-\frac{\sigma_x^2 \omega^2}{2}} = \sum_{k=0}^{\infty} \frac{\left(-\frac{\sigma_x^2 \omega^2}{2}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\sigma_x^{2k} \omega^{2k}}{2^k k!} \longrightarrow (2)$$

equ (1) = (2)

we get $m_n = 0$ $\quad n =$ odd

$n =$ even $\quad k = \frac{n}{2}$

So $\frac{(+j)^n m_n}{n!} = \frac{\sigma_x^n (-1)^{n/2}}{2^{n/2} \left(\frac{n}{2}\right)!}$

$$m_n = \frac{n!\, \sigma_x^n}{2^{\frac{n}{2}} \left(\frac{n}{2}\right)!}$$

Moment Generating Function:

The MGF of a rv is used to generate the n^{th} moments about the origin.

The MGF of RV X with prob. density fn $f_X(x)$ is defined by

$$M_X(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_X(x)\, dx$$

for a d RV X

$$M_X(v) = \sum_i e^{vx_i} P(x_i)$$

The main disadvantage of the MGF is that it may not exist for all RV's and all values of v. but the characteristic fn exists for all values of X & ω.

$$M_X(v) = E[e^{vx}]$$

$$= E\left[1 + vx + \frac{(vx)^2}{2!} + \frac{(vx)^3}{3!} + \cdot\right]$$

$$= E(1) + vE[x] + \frac{v^2}{2} E[x^2] + \frac{v^3}{3!} E[x^3] + \cdots$$

$$= m_0 + v m_1 + \frac{v^2}{2} m_2 + \frac{v^3}{3!} m_3 + \cdots$$

$$M_X(v)\Big|_{v=0} = M_X(0) = m_0 = 1$$

$$\frac{dM_X(v)}{dv} = m_1 + v\, m_2 + \frac{3v^2}{3!} m_3 + \cdots$$

$$\frac{dM_X(v)}{dv}\Big|_{v=0} = m_1$$

$$\frac{d^2 M_X(v)}{dv^2} = m_2 + v m_3 + \cdots$$

$$\frac{d^2 M_X(v)}{dv^2}\Big|_{v=0} = m_2$$

$$\boxed{\therefore \quad m_n = \left.\frac{d^n M_x(v)}{dx^n}\right|_{v=0}}$$

Properties of MGF:

1) The MGF at $v=0$ is unity

i.e ~~MG~~ $M_x(v)\big|_{v=0} = 1$

Δ: $M_x(v) = E\left[e^{vx}\right]$

$$M_x(v)\big|_{v=0} = E\left(e^{0x}\right] = E[1] = 1$$

2) Let x be a r.v with MGF $M_x(v)$ then the MGF for $y = ax+b$ is $M_y(v) = e^{bv} M_x(av)$

Δ. $M_y(v) = E\left[e^{vy}\right] = E\left[e^{v(ax+b)}\right]$

$$= E\left[e^{vax} \cdot e^{bv}\right] = e^{bv} E\left(e^{v(ax)}\right]$$

$$= e^{bv} M_x(av)$$

3) If $M_x(v)$ is the MGF of a r.v x then $M_x(cv) = M_{cx}(v)$ where c is a real constant

Δ: $M_x(v) = E\left[e^{vx}\right]$

$$M_x(cv) = E\left[e^{cvx}\right] = E\left[e^{v(cx)}\right] = M_{cx}(v)$$

4) If x_1 & x_2 are two independent r.v's with MGF $M_{x_1}(v)$ & $M_{x_2}(v)$ then

$$M_{x_1+x_2}(v) = M_{x_1}(v)\, M_{x_2}(v)$$

A: $M_X(v) = E[e^{vx}]$

$$M_{X_1+X_2}(v) = E[e^{v(x_1+x_2)}] = E[e^{vx_1} \cdot e^{vx_2}]$$

$$= E[e^{vx_1}]\, E[e^{vx_2}] = M_{X_1}(v)\, M_{X_2}(v)$$

Q: find the MGF of the exponential density fn.

A: $M_X(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_X(x)\, dx$

$$= \int_a^{\infty} e^{vx} \cdot \frac{1}{b} e^{-\frac{(x-a)}{b}}\, dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} e^{(v-\frac{1}{b})x}\, dx = \frac{e^{a/b}}{b} \left. \frac{e^{-(\frac{1}{b}-v)x}}{-(\frac{1}{b}-v)} \right|_a^{\infty}$$

$$= \frac{e^{av}}{1-bv}$$

$$m_1 = \text{mean} = \left. \frac{dM_X(v)}{dv} \right|_{v=0} = \left. \frac{e^{av}[a(1-bv)+b]}{(1-bv)^2} \right|_{v=0}$$

$$= a+b$$

Q: find the MGF for uniform density fn.

A: $M_X(v) = \int_{-\infty}^{\infty} e^{vx} f_X(x)\, dx = \int_a^b e^{vx} \cdot \frac{1}{b-a}\, dx$

$$= \frac{1}{b-a} \frac{[e^{vx}]_a^b}{v} = \frac{e^{bv} - e^{av}}{v(b-a)}$$

Q: The MGF of a g r v is $M_X(v) = e^{\frac{\sigma_x^2 v^2}{2}}$

find m_n.

A: $M_X(v) = \int_{-\infty}^{\infty} e^{vx} f_X(x)\,dx = \int_{-\infty}^{\infty} f_X(x) \sum_{n=0}^{\infty} \frac{(vx)^n}{n!}\,dx$

$$= \sum_{n=0}^{\infty} \frac{v^n}{n!} \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

$$= \sum_{n=0}^{\infty} \frac{v^n}{n!} m_n \longrightarrow (1)$$

$$M_X(v) = e^{\frac{\sigma_x^2 v^2}{2}} = \frac{\sum_{k=0}^{\infty} \left(\frac{\sigma_x^2 v^2}{2}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\sigma_x^{2k} v^{2k}}{2^k k!} \longrightarrow (2)$$

equ (1) = (2) $\quad m_n = 0 \quad n$ is odd

for n = even $\quad \frac{m_n}{n!} = \frac{\sigma_x^n}{2^{\frac{n}{2}} \left(\frac{n}{2}\right)!}$

$$\therefore m_n = \frac{n!\,\sigma_x^n}{(2)^{n/2} \left(\frac{n}{2}\right)!}$$

Q: The characteristic fn of the laplace density fn is given by $\phi_X(\omega) = \frac{e^{jm\omega}}{[1+(b\omega)^2]}$ find the mean 2nd moment & variance of the r v X.

A: $m_1 = E(X) = -j \frac{d\phi_X(\omega)}{d\omega}\Big|_{\omega=0}$

$$= (-j)\frac{\left[\left[1+(b\omega)^2\right]e^{j\omega m}(jm) - e^{j\omega m}2(b\omega).b\right]}{\left(1+(b\omega)^2\right)^2}\Bigg|_{\omega=0}$$ 2.3

$$= m$$

$$m_2 = (-j)^2 \frac{d^2\phi_x(\omega)}{d\omega^2}\Bigg|_{\omega=0}$$

$$= -\frac{\left[\left[1+(b\omega)^2\right]e^{j\omega m}(jm)^2 + e^{j\omega m}2(b\omega)b(jm)\right]}{\left[1+(b\omega^2)\right]^4}\Bigg|_{\omega=0}$$

$$+ \frac{\left[1+(b\omega^2)\right]^2\left[2b^2\omega e^{j\omega m}(j m) + e^{j\omega m}2b^2\right] - b^2\omega.2e^{j\omega m}.2\left(1+(b\omega)^2\right)2(b\omega)b\Big)}{\left(1+(b\omega^2)\right)^4}\Bigg|_{\omega=0}$$

$$= 2b^2 + m^2$$

$$\sigma_x^2 = m_2 - m_1^2 = 2b^2 + m^2 - m^2 = 2b^2$$

Q: The Chi-Square density fn has a characteristic fn $\phi_x(\omega) = \frac{1}{(1-j2\omega)^{N/2}}$ find mean & 2nd moment.

A: $$\overline{X} = -j\frac{d\phi_x(\omega)}{d\omega}\Bigg|_{\omega=0} = -j\left(-\frac{N}{2}\right)\left[\frac{-j2}{(1-j2\omega)^{\frac{N}{2}+1}}\right]_{\omega=0}$$

$$\overline{x^2} = (-j)^2 \frac{d^2\phi_x(\omega)}{d\omega^2}\Bigg|_{\omega=0} = -\frac{d}{d\omega}\left[\frac{jN}{(1-j2\omega)^{\frac{N}{2}+1}}\right]_{\omega=0}$$

$$= \frac{2N\left(\frac{N}{2}+1\right)}{(1-j2\omega)^{\frac{N}{2}+2}}\Bigg|_{\omega=0} = N^2+2N$$

$$\sigma_x^2 = \overline{x^2} - \overline{x}^2 = N^2+2N-N^2 = 2N$$

Chebychev's Inequality:

For a given r.v x with mean value $\overline{x}$ & variance σ_x^2 it states that

$$P\{|x-\overline{x}| \geq \epsilon\} \leq \frac{\sigma_x^2}{\epsilon^2}$$ where ϵ is very small positive number.

Proof: wkt $P(x \leq x) = F_x(x) = \int_{-\infty}^{x} f_x(x)\,dx$

$$P[|x-\overline{x}| \geq \epsilon] = P[(x-\overline{x}) \leq -\epsilon] + P[(x-\overline{x}) \geq \epsilon]$$

$$= P[x \leq \overline{x}-\epsilon] + P[x \geq \overline{x}+\epsilon]$$

$$= \int_{-\infty}^{\overline{x}-\epsilon} f_x(x)\,dx + \int_{\overline{x}+\epsilon}^{\infty} f_x(x)\,dx$$

$$= \int_{|x-\overline{x}| \geq \epsilon}^{\infty} f_x(x)\,dx$$

wkt $\sigma_x^2 = \int_{-\infty}^{\infty} (x-\overline{x})^2 f_x(x)\,dx$

$$= \int_{-\infty}^{|x-\bar{x}|\geq \epsilon} (x-\bar{x})^2 f_x(x)\,dx + \int_{(x-\bar{x})\geq \epsilon}^{\infty} (x-\bar{x})^2 f_x(x)\,dx$$

$$\geq \int_{|x-\bar{x}|\geq \epsilon}^{\infty} (x-\bar{x})^2 f_x(x)\,dx$$

If $x-\bar{x} = \epsilon$

then $\sigma_x^2 \geq \int_{|x-\bar{x}|\geq \epsilon}^{\infty} \epsilon^2 f_x(x)\,dx \geq \epsilon^2 \int_{|x-\bar{x}|\geq \epsilon}^{\infty} f_x(x)\,dx$

$$\sigma_x^2 \geq \epsilon^2 P\left[|x-\bar{x}| \geq \epsilon\right]$$

$$\therefore P\left[|x-\bar{x}| \geq \epsilon\right] \leq \frac{\sigma_x^2}{\epsilon^2}$$

||ly $P\left[|x-\bar{x}| < \epsilon\right] \geq 1 - \frac{\sigma_x^2}{\epsilon^2}$

If $\sigma_x^2 \to 0$ then $P\left[|x-\bar{x}| < \epsilon\right] \geq 1$ for any ϵ

i.e if the variance of a RV tends to zero then the prob/: at mean value tends to one.

If $\epsilon = k\sigma_x$ where k is any real no. then

$$P\left[|x-\bar{x}| \geq k\sigma_x\right] \leq \frac{1}{k^2}$$

$$\& \quad P\left[|x-\bar{x}| < k\sigma_x\right] \geq 1 - \frac{1}{k^2}$$

Markov's Inequality:

If x is a cRV with pdf $f_x(x)$ & if $f_x(x) = 0$ for $x < 0$ then Markov's inequality states that

$$P[x \geq a] \leq \frac{\bar{x}}{a} \quad \text{for } a > 0$$

Chernoff's Inequality and Bound:

Consider a cRV x with pdf $f_x(x)$ & MGF $M_x(v)$ then Chernoff's inequality states that

$$P[x \geq a] \leq e^{-va} M_x(v) \quad \text{for any real } v > 0$$

Q: Find the largest prob. that any RV's values are smaller than its mean by 4 standard deviations or larger than its mean by the same amount.

A. The prob. of x being smaller than $\bar{x} - 4\sigma_x$ or the prob. of x being larger than $\bar{x} - 4\sigma_x$ is given by

$$P[x \geq \bar{x} + 4\sigma_x] + P[x \leq \bar{x} - 4\sigma_x] = P[|x - \bar{x}| \geq 4\sigma_x]$$

$\bar{x}$ = mean & σ_x = SD using chebychev's inequality

$$P[|x - \bar{x}| \geq \epsilon] = \frac{\sigma_x^2}{\epsilon^2} \qquad \text{here } \epsilon = 4\sigma_x$$

$$P[|x - \bar{x}| \geq 4\sigma_x] \leq \frac{\sigma_x^2}{4^2 \sigma_x^2} = \frac{1}{16} = 6.25\%$$

Transformation of a RV:

Transformation is used to convert a given RV x into another RV y.

i.e $y = T[x]$

Monotonic transformation of a CRV:

If $y = T[x]$

& $x = T^{-1}(y)$

for a monotonic transformation, either increasing or decreasing the density fn of y is

$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$$

Q: Show that the linear transformation of G RV produces another GRV.

A: Consider a GRV x. Let $y = ax+b = T[x]$

$$x = \frac{y-b}{a}$$

$$\frac{dx}{dy} = \frac{1}{a}$$

$$f_y(y) = \left|\frac{dx}{dy}\right| f_x(x) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{\left[\frac{y-b}{a} - a_x\right]^2}{2\sigma_x^2}}$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{\left[\frac{y-(a_x+b)}{a}\right]^2}{2\sigma_x^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi (a\sigma_x)^2}} e^{-\frac{[y-(a a_x+b)]^2}{2(a\sigma_x)^2}}$$

this a gdf with variance $\sigma_y^2 = a^2\sigma_x^2$ &

mean $y = a a_x + b$

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-a_y)^2}{2\sigma_y^2}}$$

Non-Monotonic transformation of a crv:

Consider that a rv Y is a non-monotonic transformation of a rv X.

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + f_X(x_3)\left|\frac{dx_3}{dy}\right| + \ldots$$

Q: find the pdf of a rv Y for the square law transformation of a rv X with pdf $f_X(x)$

A: the square law transformation is

$Y = Ax^2$ where $A > 0$ is a real constant

$$x = \pm\sqrt{\frac{y}{A}}$$

$$x_1 = \sqrt{\frac{y}{A}} \quad \& \quad x_2 = -\sqrt{\frac{y}{A}}$$

$$\frac{dx_1}{dy} = \frac{1}{2\sqrt{Ay}} \quad \& \quad \frac{dx_2}{dy} = -\frac{1}{2\sqrt{Ay}}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= f_X\left(\sqrt{\frac{y}{A}}\right)\frac{1}{2\sqrt{Ay}} + f_X\left(-\sqrt{y/A}\right)\frac{1}{2\sqrt{Ay}}$$

$$= \frac{1}{\ldots}\left(f_X(\sqrt{y/A}) + f_X(-\sqrt{y/A})\right) \quad y > 0$$

Transformation of a Discrete r.v:

If x is a d r.v & the transformation is monotonic.

$$y = T[x_n]$$

$$\& \quad P(y_n) = P(x_n)$$

$$\therefore \quad f_Y(y) = \sum_n P(y_n)\,\delta(y - y_n)$$

$$\& \quad F_Y(y) = \sum_n P(y_n)\,u(x - x_n)$$

If the transformation is non monotonic, there may exist more than one value x_n which corresponds to y_n.

$\therefore$ $P(y_n)$ equals the sum of all prob value of x_n for which $y_n = T[x_n]$

Q: A d r.v X with pdf is given below

x	0	1	2	3	4
$P(x)$	0.2	1.5	0.3	0.15	0.2

find the density fn of Y for the transformation

$$y = 3x^3 - 3x^2 + 2$$

at $x = 0$ $\quad y = 2$

$x = 1$ $\quad y = 2$

$x = 2$ $\quad y = 14$

$x = 3$ $\quad y = 56$

$x = 4$ $\quad y = 730$

$$P[y=2] = P[x=0] + P[x=1] = 0.2 + 1.5 = 0.35$$

$$f_Y(y) = 0.35\,\delta(y-2) + 0.3\,\delta(y-14) + 0.1\,\delta(y-56) + 0.2\,\delta(y-730)$$

Q: If $f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ find the density fn for $y = \frac{x^2}{9}$

A: $x^2 = 9y \Rightarrow x = \pm 3\sqrt{y}$

$x_1 = 3\sqrt{y}$ $\qquad x_2 = -3\sqrt{y}$

$\left|\frac{dx_1}{dy}\right| = \frac{3}{2\sqrt{y}}$ $\qquad \left|\frac{dx_2}{dy}\right| = \frac{3}{2\sqrt{2y}}$

$$f_y(y) = f_x(x_1)\left|\frac{dx_1}{dy}\right| + f_x(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= \frac{3}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(3\sqrt{y})^2}{2}} + \frac{3}{2\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(-3\sqrt{y})^2}{2}}$$

$$= \frac{3}{\sqrt{2\pi y}} e^{-\frac{9y}{2}}$$

Q. The pdf of a RV X is given by $f_x(x) = \frac{x}{20}$ $\quad 2 \leq x \leq 5$

find the pdf of $y = 3x - 5$.

A. $y = 3x - 5 \Rightarrow x = \frac{y+5}{3}$

$y = 3x - 5$

$x = 2, \quad y = 1$

$x = 5 \quad y = 10$

$\left|\frac{dx}{dy}\right| = \frac{1}{3}$

$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right| = \frac{1}{3} f_x\left(\frac{y+5}{3}\right)$$

$$= \frac{1}{3} \frac{(y+5)/3}{20} = \frac{y+5}{180} \qquad 1 \leq y \leq 10$$

$\qquad 0 \qquad$ else

Q: A grf with variance 10 & mean 5 is transformed to $y = e^x$ find the pdf of y.

A. $\sigma_x^2 = 10, \quad m_x = 5$

1: $f_X(x) = \frac{1}{\sqrt{2\pi(10)}} e^{-\frac{(x-5)^2}{2(10)}}$

$= \frac{1}{\sqrt{20\pi}} e^{-\frac{(x-5)^2}{20}}$

$y = e^x$

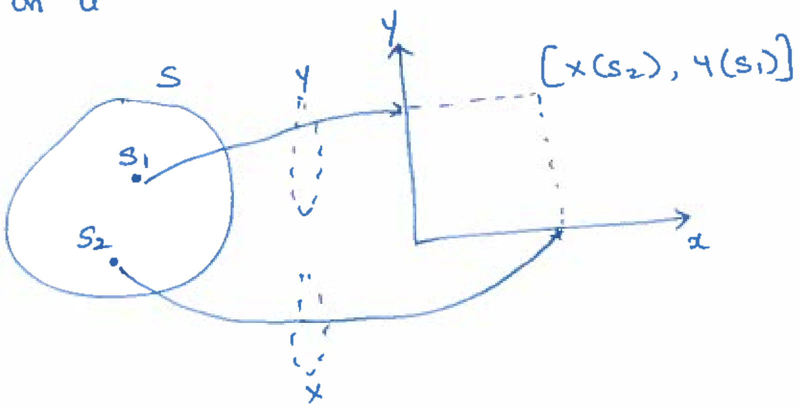
$x = \log y \qquad \left|\frac{dx}{dy}\right| = \frac{1}{y}$

$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{y} f_X(\log y)$

$= \frac{1}{y} \cdot \frac{1}{\sqrt{20\pi}} e^{-\frac{[\log y - 5]^2}{20}}$

Vector Random Variables:

Consider a sample space S, and let X & Y are two r.v's on it.



Let the specific values of X & Y are denoted by x & y respectively. Then any ordered pair of numbers (x, y) is considered to be random point in the xy-plane. This random point may be taken as a specific value of a vector r.v or random vector.

Joint Probability Distribution Function:

Consider two r.v's X & Y with elements {x} and {y} in the xy plane

The ordered pair of numbers (x, y) is called random vector in the two dimensional product space or joint sample space.

Let the two events $A = [X \le x]$ & $B = [Y \le y]$

then the joint PDF for the event $\{X \le x, Y \le y\}$ is defined as

$$F_{XY}(x, y) = P[X \le x, Y \le y] = P(A \cap B)$$

For Discrete RV's if $X = \{x_1, x_2 \cdots x_N\}$ and $Y = \{y_1, y_2 \cdots y_N\}$ with joint Prob's $P(x_n, y_m) = P[X = x_n, Y = y_m]$

then the joint PDF is

$$F_{XY}(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_n)\, u(y - y_m)$$

for N RV's X_n $n = 1, 2 \cdots N$ the joint distribution function is given as

$$F_{X_1 X_2 \cdots X_N}(x_1, x_2 \cdots x_N) = P[X_1 \le x_1, X_2 \le x_2, X_3 \le x_3 \cdots X_N \le x_N]$$

Properties of Joint Distribution Function:

For two RV's X & Y

1) a) $F_{XY}(-\infty, -\infty) = 0$

A: $F_{XY}(-\infty, -\infty) = P[X \le -\infty \cap Y \le -\infty] = P[\phi] = 0$

b) $F_{XY}(-\infty, y) = 0$

$F_{XY}(-\infty, y) = P[X \le -\infty \cap Y \le y] = P[\phi] = 0$

c) $\cancel{P\{X \le x, Y \le -\infty\} = 0}$ $F_{XY}(x, -\infty) = 0$

$\cancel{F_{XY}(x}$ $P\{\cancel{X \le x}, Y \cancel{\le \infty}\} \cancel{= P\{X \le}$

$F_{XY}(x, -\infty) = P[X \le x \cap Y \le -\infty] = P[\phi] = 0$

2) $F_{XY}(\infty, \infty) = 1$

$F_{XY}(\infty, \infty) = P[X \le \infty, Y \le \infty] = P[S \cap S] = P(S) = 1$

(3) $0 \le F_{XY}(x, y) \le 1$

from properties (1) & (2) it is clear that $F_{XY}(x, y)$ lies between 0 & 1

(4) $F_{X,Y}(x, y)$ is a monotonic and non decreasing fn of x & y.

(5) $P[x_1 < X \le x_2, y_1 < Y \le y_2]$

$$= F_{XY}(x_2, y_2) + F_{XY}(x_1, y_1) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2)$$

(6) $F_{X,Y}(x, \infty) = F_X(x)$

$$F_{XY}(x, \infty) = P[X \le x \cap Y \le \infty] = P[X \le x \cap S]$$
$$= P[X \le x] = F_X(x)$$

$$F_{XY}(\infty, y) = P[X \le \infty \cap Y \le y] = P[S \cap Y \le y]$$
$$= P[Y \le y] = F_Y(y)$$

Joint Probability Density Function:

The joint pdf of two r.v's X & Y is defined as the second derivative of the joint distribution fn. It can be expressed as

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$$

If X & Y are discrete r.v's then

$$f_{XY}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, \delta(x-x_n)\, \delta(y-y_m)$$

for N r.v's X_n, $n=1,2 \ldots N$, the joint density fn becomes the N-fold partial derivative of the N dimensional distribution fn

i.e $$f_{X_1X_2\ldots X_N}(x_1, x_2 \ldots x_N) = \frac{\partial^N F_{X_1X_2\ldots X_N}(x_1,x_2 \ldots x_N)}{\partial x_1\, \partial x_2 \ldots \partial x_N}$$

Properties of joint Density function:

1) $f_{XY}(x,y) \geq 0$

2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

3) $F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy$

4) $P[x_1 < X \leq x_2,\ y_1 < Y \leq y_2] = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\, dx\, dy$

5) $F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = F_{XY}(x,\infty)$

$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = F_{XY}(\infty,y)$

6) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \dfrac{dF_X(x)}{dx}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \dfrac{dF_Y(y)}{dy}$

Marginal Distribution function:

The marginal distribution function of X & Y is given by

$F_X(x) = F_{XY}(x,\infty)$

$F_Y(y) = F_{XY}(\infty,y)$

The marginal density fn of X & Y is

$f_X(x) = \dfrac{dF_X(x)}{dx}$ & $f_Y(y) = \dfrac{dF_Y(y)}{dy}$

Conditional Distribution & Density functions:

The Conditional distribution fn of a RV X given some event B is defined as

$$F_X(x|B) = P[X \le x | B] = \frac{P[X \le x \cap B]}{P(B)}$$

point Conditioning:

Let B be defined at a specified value of y (point Condition) given by

$$B = \{y - \Delta y < y \le y + \Delta y\}$$

where Δy is a very small qty and $\Delta y \to 0$

then

$$F_X(x \mid y-\Delta y < y \le y+\Delta y) = \frac{P[X \le x \cap (y-\Delta y \le y \le y+\Delta y)]}{P[y-\Delta y \le y \le y+\Delta y]}$$

$$= \frac{F_{XY}(x, (y-\Delta y < y \le y+\Delta y))}{F_Y(y-\Delta y < y \le y+\Delta y)}$$

$$= \frac{\int_{y-\Delta y}^{y+\Delta y} \int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy}{\int_{y-\Delta y}^{y+\Delta y} f_Y(y)\, dy}$$

Since Δy is very small qty

$$F_X(x \mid y-\Delta y \le y+\Delta y) = \frac{\int_{-\infty}^{x} f_{XY}(x,y)\, dx \; (2\Delta y)}{f_Y(y)\cdot(2\Delta y)}$$

as $\Delta y \to 0$

$$F_X(x \mid y=y) = \frac{\int_{-\infty}^{x} f_{XY}(x,y)\, dx}{f_Y(y)}$$

differentiating w.r.t x

$$f_X(x \mid y=y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

or $$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

||ly $$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Interval Conditioning:

Let $B = [y_1 < Y \leq y_2]$

$$F_X(x|B) = F_X(x \mid y_1 < Y \leq y_2)$$

$$= \frac{P\{X \leq x \cap y_1 < Y \leq y_2\}}{P[y_1 < Y \leq y_2]}$$

$$= \frac{F_{XY}(x, y_2) - F_{XY}(x, y_1)}{F_Y(y_2) - F_Y(y_1)}$$

$$= \frac{\int_{y_1}^{y_2} \int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy}{\int_{y_1}^{y_2} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy}$$

$$f_X(x|B) = \frac{\int_{y_1}^{y_2} f_{XY}(x,y)\, dy}{\int_{y_1}^{y_2} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy}$$

Statistical Independence of r.v's:

Consider two r.v's X & Y with events $A = [X \le x]$, $B = [Y \le y]$ for two real no's x & y

Two r.v's are said to be statistically independent if & only if the joint prob is equal to the product of the individual prob's.

$$P[X \le x, Y \le y] = P[X \le x] \, P[Y \le y]$$

$$F_{XY}(x, y) = F_X(x) \, F_Y(y)$$

or

$$f_{XY}(x, y) = f_X(x) \, f_Y(y)$$

$$F_X(x|y) = \frac{F_{XY}(x,y)}{F_Y(y)} = \frac{F_X(x) F_Y(y)}{F_Y(y)} = F_X(x)$$

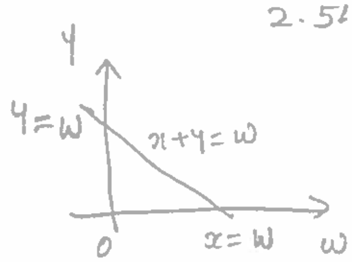
& $$F_Y(y|x) = F_Y(y)$$

$$f_X(x|y) = f_X(x) \quad \& \quad f_Y(y|x) = f_Y(y)$$

thus two r.v's X & Y are statistically independent if & only if their joint distribution or density function is equal to the product of their marginal distribution or density functions.

Sum of two r.v's:

Let W be a r.v equal to the sum of two independent r.v's X & Y



$$W = X + Y$$

$$F_W(w) = P[W \le w] = P(x + y \le w)$$

$$= \int_{-\infty}^{\infty} \int_{x=-\infty}^{w-y} f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \left[\int_{x=-\infty}^{w-y} f_X(x)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy = f_X(x) * f_Y(y)$$

thus the density fn of the sum of two statistically independent r.v's is the convolution of their individual density fns.

Sum of several r.v's:

Consider N statistically independent r.v's X_n $n = 1, 2 \ldots N$. If

$$Y = X_1 + X_2 \ldots + X_N \quad \text{Then}$$

$$f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) \ldots * f_{X_N}(x_N)$$

Central Limit theorem:

It states that the prob density function of a sum of N independent r.v's approaches the gaussian density function as N tends to infinity.

unequal distributions:

Let $\overline{X_i}$ & $\sigma_{X_i}^2$ be the mean & variances of

N r.v's x_i , $i = 1, 2 \ldots N$

The central limit theorem states that the sum of the r.v's

$$Y_N = x_1 + x_2 \ldots + x_N$$ which has mean

$$\overline{Y_N} = \overline{x_1} + \overline{x_2} + \ldots + \overline{x_N}$$ and variances

$$\sigma_{Y_N}^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \ldots + \sigma_{x_N}^2$$ has a prob/: distribution fn asymptotically approaching a Gaussian distribution as N tends to infinity.

Equal Distributions:

Consider N C.R.V's x_n , $n = 1, 2 \ldots N$ having the same distribution & density fns.

Let $y = x_1 + x_2 \ldots + x_N$

Let W be a normalized r.v i.e

$$W = \frac{y - \bar{y}}{\sigma_y} \quad \text{where } y = \sum_{n=1}^{N} x_n$$

$$\bar{y} = \sum_{n=1}^{N} \overline{x_n}$$

& $$\sigma_{y}^2 = \sum_{n=1}^{N} \sigma_{x_n}^2$$

$$W = \frac{\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} \overline{x_n}}{\left[\sum_{n=1}^{N} \sigma_{x_n}^2 \right]^{1/2}}$$

Since all r.v's have the same distribution

$$\sigma_{x_n}^2 = \sigma_x^2$$

$$\overline{x_n} = \overline{x}$$

$$W = \frac{\sum_{i=1}^{N}(x_n - \bar{x})}{(N\sigma_x^2)^{1/2}} = \frac{\sum_{i=1}^{N}(x_n - \bar{x})}{\sqrt{N}\,\sigma_x}$$

The characteristic fn of W is

$$\phi_W(\omega) = E\left[e^{j\omega W}\right]$$

$$= E\left[e^{j\omega\left(\frac{1}{\sqrt{N}\sigma_x}\sum_{n=1}^{N}(x_n-\bar{x})\right)}\right]$$

Since all x_n, $n = 1, 2 - N$ are independent & have equal distribution

$$\phi_\omega(W) = E\left[e^{\frac{j\omega}{\sqrt{N}\sigma_x}(x_n-\bar{x})}\right]^N$$

wkt $E\left[e^{\frac{j\omega}{\sqrt{N}\sigma_x}(x_n-\bar{x})}\right] = E\left[1 + \frac{j\omega}{\sqrt{N}\sigma_x}(x_n-\bar{x}) + \left(\frac{j\omega}{\sqrt{N}\sigma_x}\right)^2 \frac{\left[(x_n-\bar{x})^2\right]}{2!} + \frac{R_N}{N}\right]$

where $\frac{R_N}{N}$ is the remainder term.

$$E\left[e^{\frac{j\omega}{\sqrt{N}\sigma_x}(x_n-\bar{x})}\right] = 1 + \frac{j\omega}{\sqrt{N}\sigma_x}E(x_n-\bar{x}) - \frac{\omega^2}{N\sigma_x^2}\frac{E\left[(x_n-\bar{x})^2\right]}{2} + \frac{E[R_N]}{N}$$

$$= 1 - \frac{\omega^2}{N\sigma_x^2}\cdot\frac{\sigma_x^2}{2} + \frac{E[R_N]}{N}$$

$$= 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}$$

$$\therefore \phi_W(\omega) = \left[1 - \frac{\omega^2}{2N} + E\left[\frac{R_N}{N}\right]\right]^N$$

$$\log \phi_W(\omega) = N \log\left[1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}\right]$$

wkt

$$\log(1-z) = -\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right] \qquad |z|<1$$

Let $z = \dfrac{\omega^2}{2N} - \dfrac{E[R_N]}{N}$

$$\log \phi_W(W) = -N\left[\frac{\omega^2}{2N} - \frac{E(R_N)}{N} + \frac{1}{2}\left(\frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right)^2 + \cdots\right]$$

as $N \to \infty$

$$\lim_{N\to\infty} \log \phi_W(W) = -\frac{\omega^2}{2} + 0 + \cdots = -\frac{\omega^2}{2}$$

$$\lim_{N\to\infty} \phi_W(W) = e^{-\frac{\omega^2}{2}}$$

The above expression represents the characteristic fn of a normalised Gaussian RV. Hence at large values of N, the RV W follows a Gaussian distribution.

Q: A joint sample space for 2 RV's X & Y has 4 elements (1, 1) (2, 2) (3, 3) (4, 4) prob's of these elements are 0.1, 0.35, 0.05 & 0.5 determine

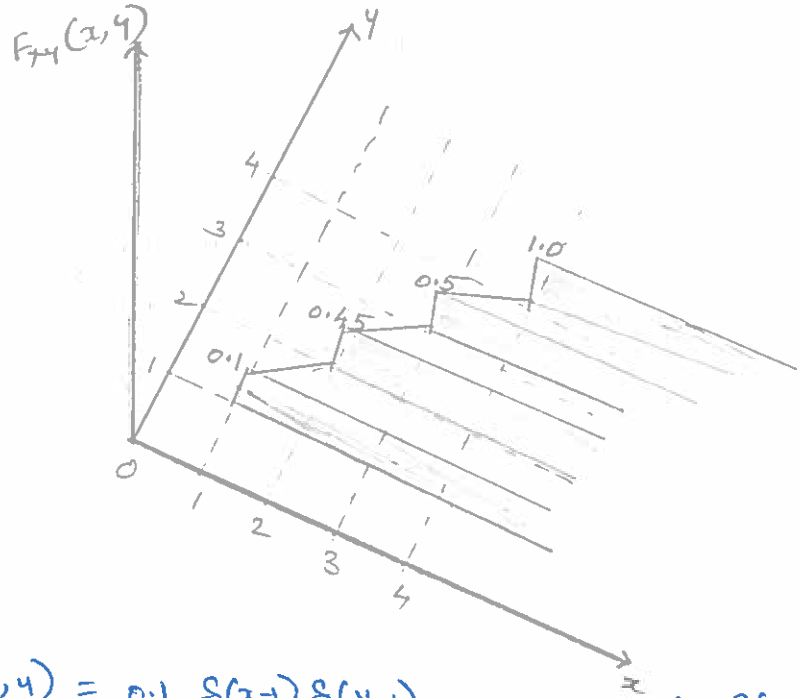
1) Prob distribution fn $F_{XY}(x, y)$

2) Prob density fn $f_{XY}(x, y)$

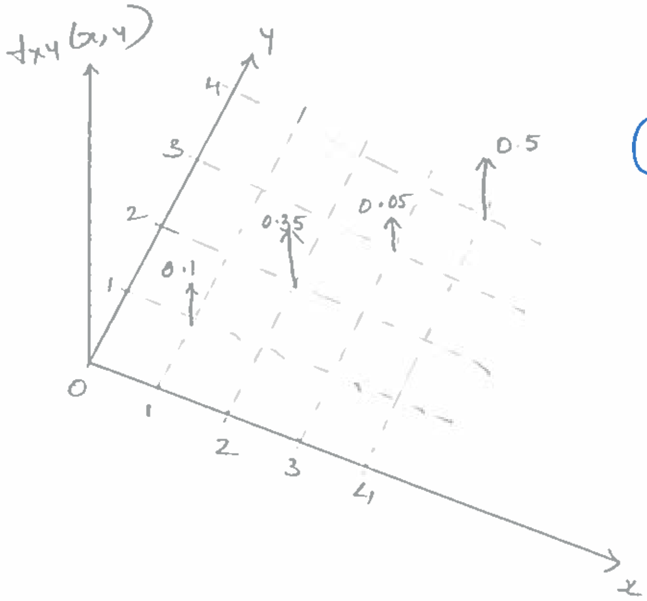
3) $P[X \le 2.5, Y \le 6]$ (4) $P[X \le 3]$

5) find the marginal distribution fn's of X & Y

A: 1) $F_{XY}(x, y) = 0.1\, u(x-1)\, u(y-1) + 0.35\, u(x-2)\, u(y-2)$
$+ 0.05\, u(x-3)\, u(y-3) + 0.5\, u(x-4)\, u(y-4)$



2) $f_{XY}(x, y) = 0.1\, \delta(x-1)\, \delta(y-1) + 0.35\, \delta(x-2)\, \delta(y-2) + 0.05\, \delta(x-3)$
$\delta(y-3) + 0.5\, \delta(x-4)\, \delta(y-4)$



(3) $P[X \le 2.5, Y \le 6] = F_{XY}(2.5, 6)$

$= 0.1(1)(1) + 0.35(1)(1)$
$+ 0.05(0)(1) + 0.5(0)(1)$

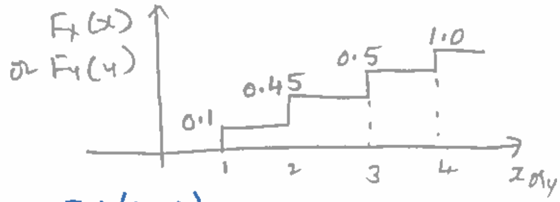
$= 0.45$

(4) $P[X \le 3.0] = F_{XY}(3, \infty)$

$= 0.1(1)(1) + 0.35(1)(1)$
$+ 0.05(1)(1) + 0.5(0)(1)$

(5) $F_X(x) = F_{X,Y}(x, \infty)$

$= 0.1\,U(x-1) + 0.35\,U(x-2) + 0.05\,U(x-3) + 0.05\,U(x-4)$



$F_Y(y) = F_{X,Y}(\infty, y)$

$= 0.1\,U(y-1) + 0.35\,U(y-2) + 0.05\,U(y-3) + 0.05\,U(y-4)$

Q: DRV's X & Y have a joint distribution fn

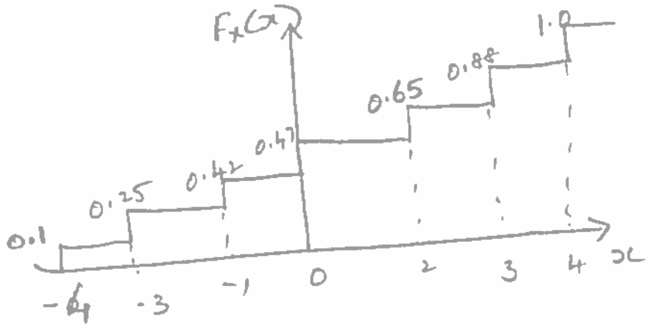
$F_{XY}(x,y) = 0.1\,U(x+4)\,U(y-1) + 0.15\,U(x+3)\,U(y+5) + 0.17\,U(x+1)\,U(y-3) + 0.05\,U(x)\,U(y-1) + 0.18\,U(x-2)\,U(y+2) + 0.23\,U(x-3)\,U(y-4) + 0.12\,U(x-4)\,U(y+3)$

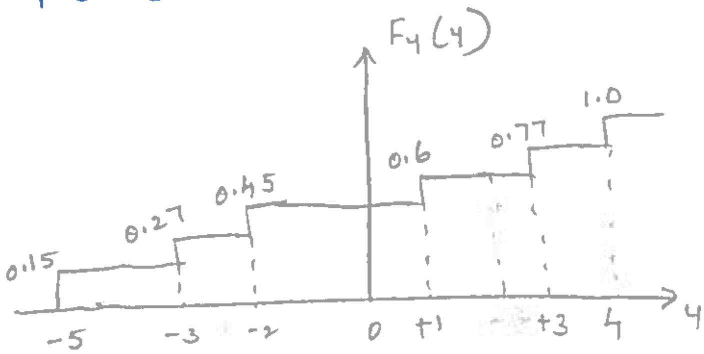
find the marginal distributions $F_X(x)$ & $F_Y(y)$ & also find $\bar{X}$ & $\bar{Y}$, $P[-1 < X \le 4, -3 < Y \le 3]$

A: $F_X(x) = F_{XY}(x,\infty) = 0.1\,U(x+4) + 0.15\,U(x+3) + 0.17\,U(x+1) + 0.05\,U(x) + 0.18\,U(x-2) + 0.23\,U(x-3) + 0.12\,U(x-4)$

$F_Y(y) = F_{XY}(\infty, y) = 0.1\,U(y-1) + 0.15\,U(y+5) + 0.17\,U(y-3) + 0.05\,U(y-1) + 0.18\,U(y+2) + 0.23\,U(y-4) + 0.12\,U(y+3)$

$= 0.15\,U(y+5) + 0.12\,U(y+3) + 0.18\,U(y+2) + 0.15\,U(y-1) + 0.17\,U(y-3) + 0.23\,U(y-4)$





2) $\cancel{\bar{x} = \sum_{i=1}^{7} x_i \, p(x_i)}$

$= 0.1(-4)(1) + 0.15(-3)(-5) + 0.17(-1)(3) + 0.05(0)(1)$
$+ 0.18(2)(-2) + 0.23(3)(4) + 0.12(4)(-3)$

2) $\bar{x} = 0.1(-4) + 0.15(-3) + 0.17(-1) + 0.05(0) + 0.18(+2)$
$+ 0.23(3) + 0.12(4) = 0.51$

$\bar{y} = 0.15(-1) + 0.12(-3) + 0.18(-2) + 0.15(1) + 0.17(3)$
$+ 0.23(4) = 0.11$

3) $P(-1 < x \leq 4, -3 < y \leq 3) = F_{xy}(4, 3) + F_{xy}(-1, -3)$
$- F_{xy}(-1, 3) - F_{xy}(4, -3)$

$= 0.05 + 0.18 = 0.23$

Q: The joint distribution fn for 2 R.V's x & y is

$$F_{xy}(x, y) = u(x)\, u(y) \left[1 - e^{-ax} - e^{-ay} + e^{-a(x+y)}\right]$$

find the joint & marginal density fn's of x & y and also find the marginal distribution of x & y.

A: $$f_{xy}(x, y) = \frac{\partial^2 F_{xy}(x, y)}{\partial x \, \partial y}$$

$$= \frac{\partial}{\partial x}\left[u(x)\left(1 - e^{-ax}\right)\right] \frac{\partial}{\partial y}\left(u(y)\left(1 - e^{-ay}\right)\right)$$

$$= \left\{ u(x)\, a e^{-ax} + \delta(x)(1-e^{-ax}) \right\} \left\{ u(y)\, a e^{-ay} + \delta(y)(1-e^{-ay}) \right\}$$

$$= u(x)\, u(y)\, a^2\, e^{-a(x+y)}$$

$$F_X(x) = F_{XY}(x, \infty) = u(x)(1-e^{-ax})$$

$$F_Y(y) = F_{XY}(\infty, y) = u(y)(1-e^{-ay})$$

$$f_X(x) = \frac{dF_X(x)}{dx} = u(x)\, a e^{-ax}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = u(y)\, a e^{-ay}$$

Q) R.V's X & Y have the joint distribution fn

$$F_{XY}(x,y) = \begin{cases} \frac{5}{4}\left[\frac{x + e^{-(x+1)y^2}}{x+1} - e^{-y^2}\right] u(y) & 0 \le x \le 4 \\ 0 & x < 0,\ y < 0 \\ 1 + \frac{1}{4} e^{-5y^2} - \frac{5}{4} e^{-y^2} & 4 \le x,\ y \ge 0 \end{cases}$$

find

1) the marginal distribution fn of x & y

2) $P[3 < x \le 5,\ 1 < y \le 2]$

A: $F_X(x) = F_{XY}(x, \infty) = \begin{cases} 0 & x < 0 \\ \frac{5x}{4(x+1)} & 0 < x \le 4 \\ 1 & x \ge 4 \end{cases}$

$$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} 0 & y < 0 \\ 1 + \frac{1}{4}e^{-5y^2} - \frac{5}{4}e^{-y^2} & y \geq 0 \end{cases}$$

$$P\left[3 < x \leq 5,\ 1 < y \leq 2\right] = F_{XY}(5,2) + F_{XY}(3,1) - F_{XY}(3,2) - F_{XY}(5,1)$$

$$= \left[1 + \frac{1}{4}e^{-5(2)^2} - \frac{5}{4}e^{-(2)^2}\right] + \frac{5}{4}\left[\frac{3 + e^{-(3+1)(1)^2}}{3+1} - e^{-(1)^2}\right]$$

$$- \frac{5}{4}\left[\frac{3 + e^{-(3+1)(2)^2}}{3+1} - e^{-(2)^2}\right] - \left[1 + \frac{1}{4}e^{-5(1)^2} - \frac{5}{4}e^{-(1)^2}\right]$$

$$= 0.004039$$

Q: The fn $F_{XY}(x,y) = a\left(\frac{\pi}{2} + \tan^{-1}\frac{x}{2}\right)\left[\frac{\pi}{2} + \tan^{-1}\frac{y}{3}\right]$ is a valid distribution fn for r.v's x & y if the Constant 'a' is chosen properly. what should be the value of 'a'.

S.

$$F_{XY}(\infty, \infty) = 1$$

$$a\left(\frac{\pi}{2} + \frac{\pi}{2}\right)\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

$$a\pi^2 = 1 \quad \Rightarrow \quad a = \frac{1}{\pi^2}$$

2. $F_{XY}(x,y) = a\left(\frac{\pi}{2} + \frac{\sqrt{3}\,x}{3+x^2} + \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right)\left[\frac{\pi}{2} + \frac{\sqrt{5}\,y}{5+y^2} + \tan^{-1}\left(\frac{y}{\sqrt{5}}\right)\right]$

$$F_{XY}(\infty, \infty) = 1$$

$$a\left[\frac{\pi}{2} + 0 + \frac{\pi}{2}\right]\left[\frac{\pi}{2} + 0 + \frac{\pi}{2}\right] = 1$$

$a\pi^2 = 1 \quad \Rightarrow \quad a = \frac{1}{\pi^2}$

Q: suppose that a pair of random nos generated by a computer are represented as values of rv X & Y having the joint distribution fn:

$$F_{XY}(x,y) = \begin{cases} 0 & x<0,\ y<0 \\ \frac{27}{26}x\left(1-\frac{x^2}{27}\right) & 0<x<1,\ 1\le y \\ \frac{27}{26}y\left(1-\frac{y^2}{27}\right) & 1\le x,\ 0<y\le 1 \\ \frac{27}{26}xy\left(1-\frac{x^2y^2}{27}\right) & 0<x\le 1,\ 0<y\le 1 \\ 1 & 1\le x,\ 1\le y \end{cases}$$

a) determine the marginal distribution fns of X & Y

b) find the prob of the event $[0<X\le 0.5,\ 0<Y\le 0.25]$

A: a) $F_X(x) = F_{XY}(x,\infty) = \begin{cases} 0 & x<0 \\ \frac{27}{26}x\left(1-\frac{x^2}{27}\right) & 0<x\le 1 \\ 1 & 1\le x \end{cases}$

$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} 0 & y<0 \\ \frac{27}{26}y\left[1-\frac{y^2}{27}\right] & 0<y\le 1 \\ 1 & 1\le y \end{cases}$

b) $P[0<X\le 0.5,\ 0<Y\le 0.25]$

$= F_{XY}(0.5, 0.25) + F_{XY}(0,0) - F_{XY}(0, 0.25) - F_{XY}(0.5, 0$

Q: A fair coin is tossed twice define r.v.s x = no of heads on the 1st toss, y = no of heads on the 2nd toss find ① the joint densit fn of x & y

② the joint distribution fn of x & y.

A: 1) $$f_{xy}(x,y) = \frac{1}{4}\delta(x)\delta(y) + \frac{1}{4}\delta(x)\delta(y-1) + \frac{1}{4}\delta(x-1)\delta(y) + \frac{1}{4}\delta(x-1)\delta(y-1)$$

2) $$F_{xy}(x,y) = \frac{1}{4}u(x)u(y) + \frac{1}{4}u(x)u(y-1) + \frac{1}{4}u(x-1)u(y) + \frac{1}{4}u(x-1)u(y-1)$$

Q: find the Constant b so that the fn $f_{xy}(x,y) = be^{-(x+y)}$ $0<x<a$, $0<y<\infty$ is a valid jdf & also find the joint distribution fn & find $P[0.5a < x \le 0.75a]$, marginal density fns of x & y.

A: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = 1$$

$$b\int_0^{\infty} e^{-y} \int_0^a e^{-x}\,dx = 1$$

$$b\underbrace{\left[e^{-y}\right]_0^{\infty}}_{1}\underbrace{\left(e^{-x}\right)_0^a}_{-1} = 1 \implies b = \frac{1}{1-e^{-a}}$$

$$F_{xy}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{xy}(x,y)\,dx\,dy \qquad x<a,\ 0<y$$

$$= \int_0^y b\,e^{-y}\,dy \int_0^x e^{-x}\,dx$$

$\underline{\underline{x<a}}$ $$= \frac{1}{1-e^{-a}}\left[1-e^{-y}\right]\left[1-e^{-x}\right]$$

$x \geq a$

$$F_{XY}(x,y) = b\int_0^y \int_a^\infty e^{-x}\, e^{-y}\, dx\, dy$$

$$= b\int_0^y e^{-y}\, dy \int_a^\infty e^{-x}\, dx$$

$$= \frac{1}{(1-e^{-a})}(1-e^{-y})(1-e^{-a}) = (1-e^{-y})$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = \int_0^\infty u(x)\, b\, e^{-(x+y)}\, dy$$

$$= b\, e^{-x}\, u(x) \int_0^\infty e^{-y}\, dy$$

$$= b\, e^{-x}\, u(x) \qquad a \leq x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_0^a u(y)\, b e^{-(x+y)}\, dx$$

$$= b\, u(y) e^{-y} \int_0^a e^{-x}\, dx = b\, u(y)\left(1-e^{-a}\right] e^{-y}$$

$$P\left(0.5a < x \leq 0.75a\right] = \int_{0.5a}^{0.75a} f_X(x)\, dx$$

$$= b\int_{0.5a}^{0.75} e^{-x}\, dx = b\left[e^{-0.5a} - e^{-0.75a}\right)$$

Q: Determine the Constant b such that each of the following are valid jdf

1) $f_{XY}(x,y) = 3xy \qquad 0<x<1,\ 0<y<b$

$\qquad\qquad 0 \qquad$ else

A: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$$

$$3\int_0^b y\,dy \int_0^1 x\,dx = 1$$

$$3\,\frac{\left[y^2\right]_0^b}{2}\,\frac{\left[x^2\right]_0^1}{2} = 1$$

$$b^2 = \frac{4}{3} \Rightarrow b = \frac{2}{\sqrt{3}}$$

2) $f_{XY}(x,y) = bx(1-y) \qquad 0 < x < 0.5,\ 0 < y < 1$

$$\int_0^1 \int_0^{0.5} b\,x\,(1-y)\,dx\,dy = 1$$

$$b\int_0^1 (1-y)\,dy \int_0^{0.5} x\,dx = 1$$

$$b\left(y - \frac{y^2}{2}\right]_0^1 \frac{\left[x^2\right]_0^{0.5}}{2} = 1$$

$$b\left[1-\frac{1}{2}\right]\frac{\left[0.25-0\right]}{2} = 1$$

$$b = 16$$

3) $f_{XY}(x,y) = b\left(x^2 + 4y^2\right) \qquad 0 < |x| \le 1$

$0 < y < 2$

$$\int_0^2 \int_{-1}^1 b\left(x^2 + 4y^2\right)dx\,dy = 1$$

$$b\int_0^2 \frac{x^3}{3} + 4xy^2\Big|_{-1}^1 dy = 1$$

$$b\int_0^2 \frac{2}{3} + 8y^2\,dy = 1$$

$$b\left[\frac{2}{3}y + \frac{8y^3}{3}\right]_0^2 = 1 \Rightarrow b = \frac{3}{68}$$

Q: given the fn $f_{XY}(x,y) = \dfrac{x^2+y^2}{8\pi}$ $\quad x^2+y^2<b$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \quad$ else

a) find the constant b, so that this a valid jdf

b) find $P[0.5b < x^2+y^2 \le 0.8b]$

A. $x^2+y^2<b$ represents the area of the xy plane within a circle of radius $\sqrt{b}$ with centre at the origin.

Converting (x,y) into polar coordinates

$$r^2 = x^2+y^2 \implies r = \sqrt{b}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \& \qquad dx\,dy = r\,dr\,d\theta$$

a) $\displaystyle\iint f_{XY}(x,y)\,dx\,dy = 1 \implies \iint \frac{x^2+y^2}{8\pi}\,dx\,dy = 1$

$$\int_0^{2\pi}\int_{r=0}^{\sqrt{b}} \frac{r^2}{8\pi}\, r\,dr\,d\theta = 1$$

$$\frac{1}{8\pi}\int_0^{2\pi} d\theta \int_{r=0}^{\sqrt{b}} r^3\,dr = 1$$

$$\frac{1}{8\pi}\left[\theta\right]_0^{2\pi} \frac{\left[r^4\right]_0^{\sqrt{b}}}{4} = 1$$

$$\frac{1}{8\pi}(2\pi)\frac{b^2}{4} = 1 \implies b^2 = 16$$

$$\& \quad b = 4$$

b) $\displaystyle P\left(\sqrt{0.2b} < r \le \sqrt{0.6b}\right) = \int_0^{2\pi}\int_{\sqrt{0.5b}}^{\sqrt{0.8b}} \frac{r^3}{8\pi}\,dr\,d\theta$

$$= \frac{1}{8\pi}\left[\theta\right]_0^{2\pi} \left[r^4\right]_{\sqrt{0.5b}}^{\sqrt{0.8b}} = \cancel{\frac{1}{8\pi}(2\pi)(0.34b^2 - 0.045}$$

$$= \frac{1}{16}\left[0.64b^2 - 0.25b^2\right] = \frac{0.39b^2}{16} = 0.39$$

Q: Given the fn $f_{xy}(x,y) = b(x+y)^2 \quad \begin{matrix} -2<x<2 \\ -3<y<3 \end{matrix}$

find the Constant b such that this is a valid joint density fn. Determine the marginal density fn's $f_x(x)$ & $f_y(y)$.

A: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = 1$$

$$\int_{-3}^{3}\int_{-2}^{2} b(x+y)^2\,dx\,dy = 1$$

$$\frac{b}{3}\int_{-3}^{3} \frac{(x+y)^3}{1}\Big|_{-2}^{2}\,dy = 1$$

$$\frac{b}{3}\int_{-3}^{3} (2+y)^3 - (-2+y)^3\,dy = 1$$

$$\frac{b}{3}\left[\frac{(2+y)^4}{4} - \frac{(-2+y)^4}{4}\right]_{-3}^{3} = 1$$

$$104\,b = 1 \implies b = \frac{1}{104}$$

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_{-3}^{3} b(x+y)^2\,dy$$

$$= \frac{b}{3}(x+y)^3\Big|_{-3}^{3} = \frac{(x+3)^3 - (x-3)^3}{312} \quad -2<x<2$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_{-2}^{2} b(x+y)^2\,dx$$

$$= \frac{b}{3}\left[(x+y)^3\right]_{-2}^{2} = \frac{(2+y)^3 - (-2+y)^3}{312} \qquad -3<y<3$$

Q: find the value of the constant b so that the fn

$f_{XY}(x,y) = b x y^2 e^{-2xy} u(x-2)\, u(y-1)$ is a valid jdf.

1. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\int_{y=1}^{\infty}\int_{x=2}^{\infty} b\,x\,y^2 e^{-2xy}\,dx\,dy = 1$$

$$\int_{y=1}^{\infty} b\,y^2 \int_{x=2}^{\infty} x\cdot e^{-2xy}\,dx\,dy = 1$$

$$b\int_{1}^{\infty} y^2 \left[e^{-2xy}\left(\frac{x}{-2y} - \frac{1}{(-2y)^2}\right)\right]_2^{\infty} dy = 1$$

$$b\int_{1}^{\infty} e^{-4y}\left(y+\frac{1}{4}\right)dy = 1$$

$$b\left[\int_1^{\infty} e^{-4y}\,y\,dy + \frac{1}{4}\int_1^{\infty} e^{-4y}\,dy\right] = 1$$

$$b\left[e^{-4y}\left(\frac{y}{-4} - \frac{1}{(-4)^2}\right)\Big|_1^{\infty} + \frac{1}{4}\,\frac{e^{-4y}}{-4}\Big|_1^{\infty}\right] = 1$$

$$\frac{3be^{-4}}{8} = 1$$

$$b = \frac{8}{3}e^{-4}$$

Q: Two r.v.s' X & Y have a joint density fn

$$f_{XY}(x,y) = \frac{10}{4}\left(u(x) - u(x-4)\right) e^{-(x+1)y^2} y^3 \, u(y)$$

find the marginal density & distribution fn of x & y.

A.
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \frac{10}{4}\left[u(x) - u(x-4)\right] \int_0^{\infty} y^3 e^{-(x+1)y^2}\, dy$$

$y^2 = t$

$y\,dy = \frac{dt}{2}$

$$= \frac{10}{4}\left[u(x) - u(x-4)\right] \int_0^{\infty} t \,.\, e^{-(x+1)t} \frac{dt}{2}$$

$$= \frac{10}{8}\left[u(x) - u(x-4)\right] \left[e^{-(x+1)t}\left[\frac{t}{-(x+1)} - \frac{1}{(-(x+1))^2}\right]\right]_0^{\infty}$$

$$= \frac{10}{8}\left[u(x) - u(x-4)\right) \cdot \frac{1}{(x+1)^2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \frac{10}{4} u(y)\, y^3 e^{-y^2} \int_0^4 e^{-xy^2} dx$$

$$= \frac{10}{4} u(y)\, y^3 e^{-y^2} \frac{\left[e^{-xy^2}\right]_0^4}{-y^2}$$

$$= \frac{10}{4} u(y)\, y \left[e^{-y^2} - e^{-5y^2}\right]$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \frac{10}{8}\int_0^x \frac{dx}{(x+1)^2} = \frac{10}{8}\left[\frac{x}{x+1}\right]$$

$0 \le x \le 4$

$$F_Y(y) = \frac{10}{4} \int_0^y y\left[e^{-y^2} - e^{-5y^2}\right] dy$$

$$= \frac{10}{4}\left[\int_0^y y \cdot e^{-y^2} dy - \int_0^y y \cdot e^{-5y^2} dy\right]$$

$y^2 = t$

$y\,dy = \frac{dt}{2}$

$$= \frac{10}{4}\left(\int_0^{y^2} e^{-t} \frac{dt}{2} - \int_0^{y^2} e^{-5t} \frac{dt}{2}\right]$$

$$= \frac{10}{8}\left(\frac{\left[e^{-t}\right]_0^{y^2}}{-1} - \frac{\left[e^{-5t}\right]_0^{y^2}}{-5}\right]$$

$$= \frac{10}{8}\left(1 - e^{-y^2} - (1 - e^{-5y^2})\right]$$

$$= \frac{10}{8}\left[e^{-5y^2} - e^{-y^2}\right]$$

Q. find the marginal density fn of x & y from the joint density fn $f_{XY}(x,y) = 2u(x)\,u(y)\,e^{-\left(4y + \frac{x}{2}\right)}$. Are x & y independent.

A. $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = 2u(x)\,e^{-x/2} \int_0^{\infty} e^{-4y}\,dy$

$$= \frac{1}{2} u(x)\, e^{-x/2}$$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = 2u(y)\,e^{-4y} \int_0^{\infty} e^{-x/2}\,dx$

$$= 4\, u(y)\, e^{-4y}$$

$f_X(x)\, f_Y(y) = \frac{1}{2} u(x)\, e^{-\frac{x}{2}} \cdot 4\, u(y)\, e^{-4y}$

$= 2\,u(x)\,u(y)\,e^{-\left[\frac{x}{2} + 4y\right]} = f_{XY}(x,y)$

∴ x & y statistically independent

2.72

Q: The joint density of 2 r.v.'s x & y is

$$f_{xy}(x,y) = 0.1\,\delta(x)\,\delta(y) + 0.12\,\delta(x-4)\,\delta(y) + 0.05\,\delta(x)\delta(y-1)$$
$$+ 0.25\,\delta(x-2)\,\delta(y-1) + 0.3\,\delta(x-2)\,\delta(y-3) + 0.18\,\delta(x-4)\delta(y-3)$$

find & plot the marginal distributions of x & y.

S.

$$F_{xy}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy$$

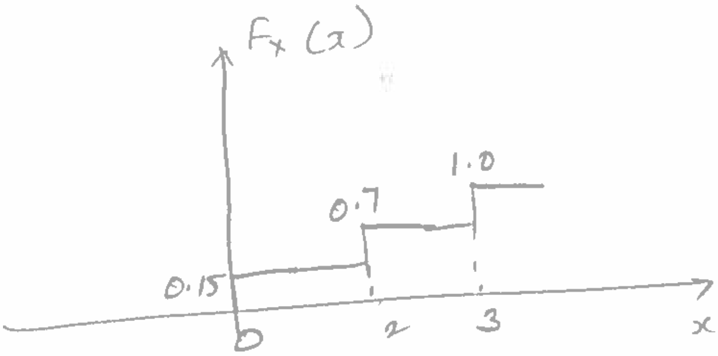
$$= 0.1\,U(x)\,U(y) + 0.12\,U(x-4)\,U(y) + 0.05\,U(x)\,U(y-1)$$
$$+ 0.25\,U(x-2)\,U(y-1) + 0.3\,U(x-2)\,U(y-3) + 0.18\,U(x-4)\,U(y-3)$$

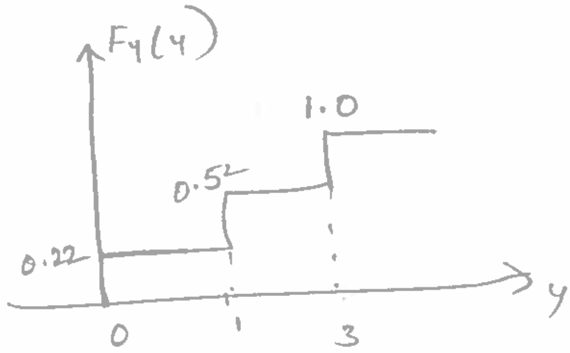
$$F_x(x) = F_{xy}(x,\infty)$$
$$= 0.1\,U(x) + 0.12\,U(x-4) + 0.05\,U(x) + 0.25\,U(x-2)$$
$$+ 0.3\,U(x-2) + 0.18\,U(x-4)$$

$$= 0.15\,U(x) + 0.55\,U(x-2) + 0.3\,U(x-4)$$

$$F_y(y) = F_{xy}(\infty, y)$$
$$= 0.1\,U(y) + 0.12\,U(y) + 0.05\,U(y-1) + 0.25\,U(y-1)$$
$$+ 0.3\,U(y-3) + 0.18\,U(y-3)$$

$$= 0.22\,U(y) + \cancel{0.05}\,0.30\,U(y-1) + 0.48\,U(y-3)$$





Questions

1) State & Prove the properties of Variance of a r.v.
2) Define Variance & Skew
3) State & Prove the properties of the Characteristic fn of a rv X.
4) Define the mgf of a r.v.
5) explain the Concept of transformation of a rv X.
6) Define the joint distribution & joint density fn of two rv's X & Y.
7) Define the Conditional Distribution & density fn of two rv's X & Y.
8) State the properties of Condition density function
9) State & Prove the Central limit theorem
10) State the properties of joint distribution & density fn.
11) what is the prob/: distribution fn of Sum of two rvs.
12) Explain point Conditioning & Interval Conditioning.

Additional Problems

1) A Gaussian RV with Variance 10 & mean 5 is transformed to $Y = e^x$ find the Pdf of Y.

2) If $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ find the density fn for

$Y = \frac{x^2}{9}$

3) The Pdf of a RV x is given by $f_X(x) = \frac{x}{20}$ $2 < x \leq 5$ find the pdf of $Y = 3x - 5$.

4) S.T the characteristic fn of a Gaussian RV with zero mean & Variance σ^2 is

$$\phi_X(\omega) = e^{-\frac{\sigma^2 \omega^2}{2}}$$

5) find the density fn of the RV X if the characteristic fn is $\phi_X(\omega) = \begin{cases} 1 - |\omega| & |\omega| \leq 1 \\ 0 & \text{else.} \end{cases}$

6) Show that the distribution fn for which the characteristic fn is $e^{-|\omega|}$ has the density fn

$$f_X(x) = \frac{1}{\pi(1+x^2)} \quad -\infty < x < \infty.$$

7) A RV x has pdf $f_X(x) = \frac{1}{2^x}$ $x = 1, 2 \ldots n$ find the mgf.

8) find the mgf of the RV whose moments are given by $m_n = (n+1)!\, 2^n$

9) A RV θ is uniformly distributed in the interval (θ_1, θ_2) where θ_1 & θ_2 are real and satisfy $\theta_1 \le \theta \le \theta_2 < 1$ find the pdf of the RV $y = \cos\theta$.

10) A RV x is uniformly distributed in $(0, 6)$ if x is transformed to a new RV $y = 2(x-3)^2 - 4$ find density of y, $\bar{y}$ & σ_y^2.

11) Let x be continuous RV with pdf $f_x(x) = \frac{x}{12}$ $1 < x < 5$ find the pdf of $y = 2x - 3$.

12) The joint pdf x & y is $f_{xy}(x,y) = \frac{1}{4} e^{-|x|-|y|}$

a) are x & y statistically independent RV's

b) Calculate the prob/: for $x \le 1$ & $y \le 0$

13) Statistically Independent RVs x & y have densities $f_x(x) = 5u(x)e^{-5x}$, $f_y(y) = 2u(y)e^{-2y}$ find the density of the sum $W = X + Y$.

14) A RV x has the density fn $f_x(x) = \frac{1}{a} e^{-b|x|}$ $-\infty \le x \le \infty$ find $E(x)$, $E(x^2)$ & σ_x^2.

15) Let x be a RV defined by the density fn $f_x(x) = \frac{5}{4}(1-x^4)$ $0 \le x \le 1$ find $E(x)$, $E(x^2)$ & σ_x^2

16) If x & y are independent RV's P.T $Var(xy) = Var(x) = Var(y)$ if $E(x) = E(y) = 0$.

17) The density fn of a RV x is $g_x(x) = 5e^{-5x}$ $0 \le x \le \infty$

find 1) $E(x)$ 2) $E[(x-1)^2]$ 3) $E(3x-1)$

18) find the expected value of the fn $g(x)=x^2$ where x is rv defined by the density $f_x(x) = a\,e^{-ax}\,u(x)$ where a is Constant.

19) If the mean & Variance of binomial distribution are 6 & 1.5 resp find $E[x - P(x \geq 3)]$

20) find the density fn of a rv x whose characteristic fn is $\phi_x(\omega) = \frac{1}{2} e^{-|\omega|}$ $-\infty \leq \omega \leq \infty$.

21) A gaussian rv x having a mean value of zero & Variance one is transformed to an another rv Y by a square law transformation. find the density fn of Y.

22) Let x be a rv defined by the density fn $f_x(x) = \frac{\pi}{16} \cos \frac{\pi x}{8}$ $-4 \leq x \leq 4$ find $E[3x]$ & $E[x^2]$

23) A rv x is uniformly distributed in the interval $(-5, 15)$ another rv $Y = e^{-x/5}$ is formed find $E(Y)$ & $f_Y(y)$

24) find the expected value of the number on a die when thrown.

25) when two dices are thrown Simultaneously find the expected value of the sum of number points on them.

26) find the mgf of a) $y = ax+b$ (b) $y = \frac{x+a}{b}$

27) If the RV x has mgf $M_x(t) = \frac{2}{2-t}$ determine the Variance of x.

28) The joint pdf of two RV's x & y is given by
$f(x,y) = a(2x+y^2)$ $0 \le x \le 2,\ 2 \le y \le 4$
find a) value of a (b) $P[x \le 1, y > 3]$

29) Let x & y be jointly Continuous RV's with joint pdf $f_{xy}(x,y) = x^2 + \frac{xy}{3}$ $0 \le x \le 1,\ 0 < y \le 2$
find $f_x(x)$, $f_y(y)$, are x & y independent, $f_x(x/y)$
$f_y(y/x)$

30) find the density of $\omega = x+y$ where the densities of x & y are assumed to be $f_x(x) = u(x) - u(x-1)$
& $f_y(y) = u(y) - u(y-1)$

31) Let x & y be independent RV's each $N(0,1)$
find the mean & Variance of $z = (x^2+y^2)^{1/2}$.

32) The joint pdf of two RV's x & y is given by
$f(x,y) = c(2x+y)$ $0 \le x \le 1,\ 0 < y \le 2$ find (1) Value of
c (2) marginal distribution fns of x & y.

33) Two rvs x & y have joint Pdf $f_{xy}(x,y) = Ae^{-(2x+y)}$ $x, y \geq 0$

find A, marginal density & distribution fns of x & y

find the joint CDF

34) Consider a linear amplifier defined by i/p – o/p relation $Y = ax+b$ where a & b are Constants. The i/p x is a Gaussian rv with mean m & Variance σ^2 determine the Pdf of the rv y at the amplifier o/p.

35) find the density fn of $w = x+y$ where the densities of x & y are assumed to be

$f_x(x) = \frac{1}{a}\left[u(x) - u(x-a)\right]$ $\quad f_y(y) = \frac{1}{b}\left[u(y) - u(y-b)\right]$

where $0 < a < b$.

Multiple - Choice Questions

1) The characteristic fn $\phi_x(\omega)$ at $\omega = 0$ is [ ]

a) ∞ b) 1 c) 0 d) -1

2) The normalised third central moment is known as [ ]

a) mean function b) skewness of density fn

c) Standard deviation d) Variance

3) The second central moment is known as [ ]

a) mean b) stand and deviation

c) Variant d) skewness

4) The max magnitude of characteristic fn is [ ]

a) 0 b) -1 c) ∞ d) 1

5) The nth central moment μ_n = ( )

a) $\int_0^{\infty} (x-\bar{x})^n \, dx$ b) $\int_{-\infty}^{\infty} (x-\bar{x})^{n+1} f_x(x) \, dx$

c) $\int_{-\infty}^{\infty} (x-\bar{x})^n f_x(x) \, dx$ d) $\int_{-\infty}^{\infty} (x-\bar{x})^{n+1} \, dx$

6) The mgf of x, $M_x(v)$ is expressed as [ ]

a) $E[e^x]$ b) $E[e^{vx}]$ c) $e^v x$ d) $E[x^2]$

7) If $y = T(x)$ then $F_Y(y)$ = [ ]

a) $f_x(y) \left|\frac{dx}{dy}\right|$ b) $f(y) \left|\frac{dx}{dy}\right|$ c) $f_x(x) \left|\frac{dy}{dx}\right|$ d) $F_x(x) \left|\frac{dx}{dy}\right|$

8) If $y = ax+b$ then σ_y^2 is ( ) MCQ-2

a) $a\sigma_x$ (b) σ_x^2 (c) $a\sigma_x^2$ (d) $a+b$

9) A transformation T is said to monotonically decreasing if ( )

a) $T(x_1) > T(x_2)$ for $x_1 < x_2$ (b) $T(x_1) = T(x_2)$ for $x_1 < x_2$

c) $T(x_1) < T(x_2)$ for $x_1 < x_2$ (d) $T(x_1) > T(x_2)$ for $x_1 < x_2$

10) If x is d.r.v then $E[g(x)]$ is

a) $\sum_{i=1}^{N} P(x_i)$ (b) $g(x)P(x)$ (c) $\sum_{i=1}^{N} g(x_i)P(x_i)$ (d) $\sum_{i=1}^{N} g(x_i)$

11) If x is a c.r.v and $y = ax+b$ then the expected value of y is [ ]

a) $aE(x)$ b) $aE(x)+b$ c) $a\,Var(x)+b$ (d) $a^2 Var(x)$

12) $F_{XY}(-\infty, -\infty) =$ ______ [ ]

a) 0 (b) ∞ (c) 1 (d) none

13) $F_{XY}(\infty, \infty) =$ ______ [ ]

a) 0 (b) ∞ (c) 1 (d) none

14) $F_{XY}(\infty, y) =$ ______ [ ]

a) $F_X(x)$ (b) $F_Y(y)$ (c) ∞ (d) 1

15) $F_{XY}(x, \infty) =$ ______ [ ]

a) ∞ (b) 1 (c) $f_X(x)$ (d) $F_Y(y)$

16) If $Y = x_1 + x_2 + \cdot + x_N$ where x_1, x_2 & x_N are Statistically independent RV's then $f_Y(y) =$ ______ [ ]

a) $f_{x_1}(x_1) * f_{x_2}(x_2) * \cdots * f_{x_N}(x_N)$

b) $f_{x_1}(x_1) + f_x(x_2) + \cdots + f_{x_N}(x_N)$

c) $f_{x_1}(x_1) \cdot f_{x_2}(x_2) \cdot \cdots \cdot f_{x_N}(x_N)$

d) none

17) The distribution fn of one RV X Conditioned by a second RV Y with interval $\{y_a < y \leq y_b\}$ is known as [ ]

a) mgf b) point Conditioning c) expectation

d) interval Conditioning.

18) If X & Y are statistically independent then [ ]

$F_{XY}(x, y) =$ ______

a) $F_X(x) + F_Y(y)$ b) $F_X(x) \cdot F_Y(y)$

c) $F_X(x) * F_Y(y)$ d) $F_X(x) - F_Y(y)$

19) If X & Y are statistically independent then [ ]

$f_{XY}(x, y) =$ ______

a) $f_X(x) * f_Y(y)$ b) $f_X(x) \cdot f_Y(y)$ c) $f_X(x) + f_Y(y)$

d) $f_X(x) - f_Y(y)$

20) If $Y = X_1 + X_2 \cdots + X_N$ then $\overline{Y} =$ ______ [ ]

a) $\sum_{i=1}^{N} \overline{Y_i}$ b) $\sum_{i=1}^{N} \overline{X_i}$ c) $\sum_{i=1}^{\infty} \overline{x_i}$ d) none

21) The PDF of a sum of large no of independent RV's approaches ______ [ ]

a) Rayleigh Distribution b) uniform distribution
c) Gaussian Distribution d) poisson distribution

22) central limit theorem is mostly applicable to statistically ______ [ ]

a) dependent RV's (b) Independent RV's
c) all RV's d) any RV's.

23) For N RV's the sum $Y_N = x_1 + x_2 \cdots + x_N$ has Gaussian RV as N tends to ______ [ ]

a) finite (b) infinity (c) unity (d) zero.

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Unit - 3

Operations on Multiple Random Variables

Expected value of a fn of RV:

When more than one single RV is involved expectation must be taken wrt all the variables involved.

If $g(x,y)$ is some fn of two RV's X & Y then the expected value of $g(x,y)$ is given by

$$\bar{g} = E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\, dx\, dy$$

For N RV's $X_1, X_2 \dots X_N$ & some fn of these variables defined by $g(x_1, x_2 \dots x_N)$ the expected value of the fn becomes

$$\bar{g} = E[g(x_1, x_2, \dots x_N)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(x_1, x_2 \dots x_N) f_{X_1 X_2 \dots X_N}(x_1, x_2 \dots x_N)\, dx_1 dx_2 \dots dx_N$$

Joint moments about the origin:

Joint moments about the origin are denoted by m_{nk} & are defined by

$$m_{nk} = E[x^n y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\, dx\, dy$$

the sum $n+k$ is called order of the moments

$$m_{n0} = E[x^n] = m_n = \text{moments of } x$$

$$m_{0k} = E[y^k] = m_k = \text{moments of } y$$

m_{02}, m_{20} & m_{11} are all second order moments of x & y.

the first order moments are

$$m_{01} = E[y] = \bar{y}$$
$$m_{10} = E(x] = \bar{x}$$

$m_{11} = E(xy)$ is called the correlation of x & y

$$R_{xy} = E(xy) = m_{11} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{xy}(x,y)\, dx\, dy$$

If $R_{xy} = E[x]\,E[y]$ then x & y are said to be uncorrelated or independent.

If $R_{xy} = 0$ then x & y are orthogonal.

eg: let the r.v x has mean value $\bar{x} = 3$ & $\sigma_x^2 = 2$ find the 2nd moment of x about the origin. If $y = -6x + 22$ find $\bar{y}$, R_{xy}.

A: $E(x^2) = m_{20} = \sigma_x^2 + \bar{x}^2 = 2 + 9 = 11$

$$y = -6x + 22$$

$$\bar{y} = E(y) = E[-6x + 22] = -6\bar{x} + 22 = 4$$

$$R_{xy} = m_{11} = E(xy) = E[x(-6x + 22)]$$
$$= E[-6x^2 + 22x] = -6\overline{x^2} + 22\bar{x} = 0$$

Since $R_{xy} = 0$ $\therefore$ x & y are orthogonal

$R_{xy} \neq E(x)\,E(y) = 12$ so x & y are not uncorrelated

Two rvs can be orthogonal even though correlated when one y is related to other x by the linear fn $y = ax+b$

x & y are alway correlated if $|a| \neq 0$ regardless of the value of b.

they are uncorrelated if $a=0$

orthogonality occurs when $b = -a \dfrac{E[x^2]}{E(x)}$ $\quad E(x) \neq 0$

If $E(x)=0$, then x & y can not be orthogonal for any value of a except at $a=0$

For N rv's $x_1, x_2 \ldots x_N$ the $(n_1+n_2 \ldots + n_N)$ order joint moments are defined by

$$m_{n_1 n_2 \ldots n_N} = E\left[x_1^{n_1} x_2^{n_2} \ldots x_N^{n_N}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{n_1} x_2^{n_2} \ldots x_N^{n_N} f_{x_1 x_2 \ldots x_N}(x_1, x_2 \ldots x_N)\, dx_1\, dx_2 \ldots dx_N$$

where $n_1, n_2 \ldots n_N$ are all integers

Joint Central Moments:

For two rv's x & y the joint central moments

$$\mu_{nk} = E\left[(x-\bar{x})^n (y-\bar{y})^k\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^k f_{xy}(x,y)\, dx\, dy$$

the second central moments

$$\mu_{20} = E\left[(x-\bar{x})^2\right] = \sigma_x^2$$

$$\mu_{02} = E\left[(y-\bar{y})^2\right] = \sigma_y^2$$

$$\mu_{11} = E\left[(x-\bar{x})(y-\bar{y})\right] = E\left[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}\right]$$

$$= E(xy) - E[x]\bar{y} - \bar{x}E[y] + \bar{x}\bar{y}$$

$$C_{xy} = R_{xy} - \bar{x}\bar{y}$$

If x & y are independent & uncorrelated then

$$R_{xy} = \bar{x}\,\bar{y}$$

$$\therefore C_{xy} = 0$$

If x & y are orthogonal then $R_{xy} = 0$

$$\therefore C_{xy} = -\bar{x}\,\bar{y}$$

the normalised 2nd order moment correlation co efficient of x & y is given by

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\,\mu_{02}}} = \frac{C_{xy}}{\sigma_x\,\sigma_y} = \frac{R_{xy} - \bar{x}\bar{y}}{\sigma_x\,\sigma_y}$$

$$-1 \le \rho \le 1$$

for N rvs $x_1, x_2 \ldots x_N$ the $(n_1 + n_2 \cdots + n_N)$ order joint central moment is defined by

$$\mu_{n_1 n_2 \cdots n_N} = E\left[(x_1-\bar{x}_1)^{n_1}(x_2-\bar{x}_2)^{n_2} \cdots (x_N-\bar{x}_N)^{n_N}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (x_1-\bar{x}_1)^{n_1}(x_2-\bar{x}_2)^{n_2} \cdots (x_N-\bar{x}_N)^{n_N}$$

$$f_{x_1 x_2 \cdots x_N}(x_1, x_2 \cdots x_N)\, dx_1\, dx_2 \cdots dx_N$$

Q: Find the mean & variance of weighted sum of N R.V's.

A. Let $g(x_1, x_2 \ldots x_N) = \sum_{i=1}^{N} \alpha_i x_i$

$$E[g(x_1, x_2 \ldots x_N)] = E\left(\sum_{i=1}^{N} \alpha_i x_i\right)$$

$$= \alpha_i \sum_{i=1}^{N} E(x_i)$$

$$= \sum_{i=1}^{N} \alpha_i \overline{x_i}$$

The mean value of weighted sum of R.V's equals the weighted sum of mean values.

Let $X = \sum_{i=1}^{N} \alpha_i x_i$

$$\overline{X} = E(X) = \sum_{i=1}^{N} \alpha_i E[x_i] = \sum_{i=1}^{N} \alpha_i \overline{x_i}$$

$$X - \overline{X} = \sum_{i=1}^{N} \alpha_i (x_i - \overline{x_i})$$

$$\sigma_X^2 = E[(X-\overline{X})^2] = E\left[\sum_{i=1}^{N} \alpha_i (x_i - \overline{x_i}) \sum_{j=1}^{N} \alpha_j (x_j - \overline{x_j})\right]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j E[(x_i - \overline{x_i})(x_j - \overline{x_j})]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j C_{x_i x_j}$$

~~For~~ thus the variance of a weighted sum of N R.V's X_i equals the weighted sum of all their covariances $C_{x_i x_j}$

for uncorrelated R.V's $\quad C_{x_i x_j} = 0 \quad i \neq j$

$$= \sigma_{x_i}^2 \quad i = j$$

$$\therefore \sigma_x^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{x_i}^2$$

i.e the variance of a weighted sum of uncorrelated RV's equals the weighted sum of variances of the RV's

Q: RV's x & y have the jdf $f_{xy}(x,y) = \frac{1}{24}$ $0<x<6, 0<y<$
what is the expected value of the fn $g(x,y) = (xy)^2$

A: $$E[(xy)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy)^2 f_{xy}(x,y)\,dx\,dy$$

$$= \int_0^4 \int_0^6 x^2 y^2 \cdot \frac{1}{24}\,dx\,dy$$

$$= \frac{1}{24}\left[\frac{x^3}{3}\right]_0^6 \left[\frac{y^3}{3}\right]_0^4 = 64$$

Q: The density fn of two RV's x & y is
$f_{xy}(x,y) = u(x)\,u(y)\,16e^{-4(x+y)}$ find the mean value of the fn

$$g(x,y) = \begin{cases} 5 & 0<x\le\frac{1}{2},\ 0<y\le\frac{1}{2} \\ -1 & \frac{1}{2}<x,\ \frac{1}{2}<y \\ 0 & \text{else} \end{cases}$$

A: $$E[g(x,y)] = \int_0^{1/2}\int_0^{1/2} 5\cdot 16\, e^{-4(x+y)}\,dx\,dy - \int_{1/2}^{\infty}\int_{1/2}^{\infty} 16\, e^{-4(x+y)}\,dx\,dy$$

$$= 80\, \frac{e^{-4x}}{-4}\Big|_0^{1/2} \frac{e^{-4y}}{-4}\Big|_0^{1/2} - 16\,\frac{e^{-4x}}{-4}\Big|_{1/2}^{\infty} \frac{e^{-4y}}{-4}\Big|_{1/2}^{\infty}$$

$$= 5\left[e^{-2}-1\right]\left[e^{-2}-1\right] - \left[-e^{-2}\right]\left[-e^{-2}\right]$$

$$= 5\left[e^{-4}+1-2e^{-2}\right] - e^{-4} = 4e^{-4} - 10e^{-2} + 5$$

Q: for the rv's x & y the jdf is given by $f_{XY}(x,y) = u(x)\,u(y)\,16\,e^{-4(x+y)}$ find the mean value of the fn $g(x,y) = e^{-(2x^2+2y^2)}$

A: $$E[g(x,y)] = \int_0^\infty\int_0^\infty 16\,e^{-4(x+y)}\,e^{-(2x^2+2y^2)}\,dx\,dy$$

$$= 16\int_0^\infty e^{-(2x^2+4x)}\,dx \int_0^\infty e^{-(2y^2+4y)}\,dy$$

$$= 16e^4\int_0^\infty e^{-(2x^2+4x+2)}\,dx \int_0^\infty e^{-(2y^2+4y+2)}\,dy$$

$$= 16e^4\int_0^\infty e^{-2(x+1)^2}\,dx \int_0^\infty e^{-2(y+1)^2}\,dy$$

let $\xi = 2(x+1) \Rightarrow d\xi = 2\,dx$

$$E[g(x,y)] = 16e^4\int_2^\infty e^{-\frac{\xi^2}{2}}\,\frac{d\xi}{2}\int_2^\infty e^{-\frac{\xi^2}{2}}\,d\xi/2$$

$$= 8\pi e^4\left[\frac{1}{\sqrt{2\pi}}\int_2^\infty e^{-\frac{\xi^2}{2}}\,d\xi\right]^2$$

$$= 8\pi e^4\left[P[x>2]\right]^2$$

$$= 8\pi e^4\left(1-P[x\le 2]\right)$$

$$= 8\pi e^4\left(1-F(2)\right) = 8\pi e^4\left(1-0.9772\right)^2$$

$$= 0.713$$

Q: Three statistically independent rv's x_1, x_2 & x_3 have mean values $\bar{x}_1 = 3$, $\bar{x}_2 = 6$ & $\bar{x}_3 = -2$ find the mean value of the following fn:

a) $g(x_1,x_2,x_3) = x_1 + 3x_2 + 4x_3$

b) $g(x_1,x_2,x_3) = x_1\,x_2\,x_3$

c) $g(x_1, x_2, x_3) = -2x_1x_2 - 3x_1x_3 + 4x_2x_3$ 3.8

d) $g(x_1, x_2, x_3) = x_1 + x_2 + x_3$.

1. a) $E\left[g(x_1, x_2, x_3)\right] = E\left[x_1 + 3x_2 + 4x_3\right]$

$= \bar{x}_1 + 3\bar{x}_2 + 4\bar{x}_3 = 3 + 18 - 8 = 13$

b) $E\left[x_1x_2x_3\right] = \bar{x}_1\,\bar{x}_2\,\bar{x}_3 = -36$

c) $E\left[g(x_1, x_2, x_3)\right] = E\left(-2x_1x_2 - 3x_1x_3 + 4x_2x_3\right)$

$= -2\bar{x}_1\bar{x}_2 - 3\bar{x}_1\bar{x}_3 + 4\bar{x}_2\bar{x}_3$

$= -66$

d) $E\left[g(x_1, x_2, x_3)\right] = E\left[x_1 + x_2 + x_3\right] = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 7$

Q: find the mean value of the fn $g(x, y) = x^2 + y^2$ where x & y are RV's defined by the density fn

$f_{XY}(x, y) = \dfrac{e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{2\pi\sigma^2}$ where σ^2 is constant

A: $$E\left[g(x, y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)\,\frac{e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{2\pi\sigma^2}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\,dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{y^2}{2\sigma^2}}\,dy$$

$$+ \int_{-\infty}^{\infty} y^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{y^2}{2\sigma^2}}\,dy \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= \sigma^2(1) + \sigma^2(1) = 2\sigma^2$$

Q: Two statistically independent r.v's x & y have mean values $\bar{x} = E(x) = 2$, $\bar{y} = 4$, $\overline{x^2} = 8$, $\overline{y^2} = 25$ find the mean value, 2nd moment & variance of the r.v $w = 3x - y$.

A: $\bar{w} = 3\bar{x} - \bar{y} = 6 - 4 = 2$

$$\overline{w^2} = E[(3x-y)^2] = E[9x^2 - 6xy + y^2]$$

$$= 9\overline{x^2} - 6\bar{x}\bar{y} + \overline{y^2} = 49$$

$$\sigma_w^2 = \overline{w^2} - \bar{w}^2 = 49 - 4 = 45$$

Q: Two r.v's x & y have means $\bar{x} = 1$, $\bar{y} = 2$, $\sigma_x^2 = 4$, $\sigma_y^2 = 1$ & correlation coefficient $\rho_{xy} = 0.4$ new r.v's are defined by $V = -x + 2y$, $w = x + 3y$, find mean, variance, correlation, correlation coefficient of r.v's V & w.

A: $\bar{V} = -\bar{x} + 2\bar{y} = 3$ $\qquad \bar{w} = \bar{x} + 3\bar{y} = 7$

$$\sigma_V^2 = E[(v-\bar{v})^2] = E[\{(-x+2y) - (-\bar{x}+2\bar{y})\}^2]$$

$$= E[\{-(x-\bar{x}) + 2(y-\bar{y})\}^2]$$

$$= E[(x-\bar{x})^2] + 4E[(y-\bar{y})^2] - 4E[(x-\bar{x})(y-\bar{y})]$$

$$= \sigma_x^2 + 4\sigma_y^2 - 4C_{xy}$$

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} \Rightarrow C_{xy} = \rho_{xy}\sigma_x\sigma_y = 0.8$$

$$\sigma_V^2 = 4 + 4 - 4(0.8) = 4.8$$

$$\sigma_w^2 = E[(w-\bar{w})^2] = E[\{(x-\bar{x}) + 3(y-\bar{y})\}^2]$$

$$= E[(x-\bar{x})^2] + 9E[(y-\bar{y})^2] + 6E[(x-\bar{x})(y-\bar{y})]$$

$$= \sigma_x^2 + 9\sigma_y^2 + 6C_{xy} = 17.8$$

$$R_{vw} = E[vw] = E[(-x+2y)(x+3y)]$$

$$= E(-x^2 - xy + 6y^2)$$

$$= -E(x^2) - R_{xy} + 6E(y^2)$$

$$= -(\sigma_x^2 + \bar{x}^2) - R_{xy} + 6(\sigma_y^2 + \bar{y}^2)$$

$$= -(4+1) - (C_{xy} + \bar{x}\bar{y}) + 6(1+4)$$

$$= -5 - (0.8+2) + 6(5) = 22.2$$

$$\rho_{vw} = \frac{C_{vw}}{\sigma_v \sigma_w} = \frac{R_{vw} - \bar{v}\,\bar{w}}{\sigma_v \sigma_w} = \frac{22.2 - 21}{\sqrt{4.8}\sqrt{17.8}} = 0.1298$$

Q: Two r.v's x & y are related by the expression $y = ax+b$ where a & b are any real nos

a) S.T their correlation coefficients $\rho = 1$ if $a>0$ for any b
$= -1$ if $a<0$ for any b

b) S.T their covariance is $C_{xy} = a\sigma_x^2$ where σ_x^2 is variance of x.

A: a) $$\rho = \frac{E[(x-\bar{x})(y-\bar{y})]}{\sigma_x \sigma_y}$$

$$y = ax+b$$

$$\bar{y} = a\bar{x}+b$$

$$(y-\bar{y}) = a(x-\bar{x})$$

$$\sigma_y^2 = a^2 E[(x-\bar{x})^2] = a^2\sigma_x^2$$

$$\sigma_y = |a|\sigma_x$$

$$\rho = \frac{E[(x-\bar{x})(y-\bar{y})]}{\sigma_x \sigma_y} = \frac{E[(x-\bar{x})\,a(x-\bar{x})]}{\sigma_x \sigma_y}$$

$$= \frac{a\,E\left[(x-\bar{x})^2\right]}{\sigma_x\,|a|\,\sigma_x} = \frac{a\,\sigma_x^2}{|a|\,\sigma_x^2} = \frac{a}{|a|}$$

$$= 1 \quad \text{if } a>0 \text{ for any } b$$
$$\quad -1 \quad \text{if } a<0 \text{ for any } b$$

$$C_{xy} = E\left[(x-\bar{x})(y-\bar{y})\right] = E\left[(x-\bar{x})\,a\,(x-\bar{x})\right]$$
$$= a\,E\left[(x-\bar{x})^2\right] = a\,\sigma_x^2$$

Q. S.T the correlation coefficient satisfies the expression

$$|\rho| = \frac{|\mu_{11}|}{\sqrt{\mu_{02}\,\mu_{20}}} \leq 1$$

A: Since $E\left[\{(x-\bar{x})\alpha + (y-\bar{y})\}^2\right] \geq 0$ for any real no α

$$E\left[\alpha^2(x-\bar{x})^2 + 2\alpha(x-\bar{x})(y-\bar{y}) + (y-\bar{y})^2\right] \geq 0$$

$$\alpha^2\sigma_x^2 + 2\alpha\mu_{11} + \sigma_y^2 \geq 0 \qquad b^2-4ac \leq 0$$

$$(2\mu_{11})^2 - 4\sigma_x^2\sigma_y^2 \leq 0$$

$$4\mu_{11}^2 \leq 4\sigma_x^2\sigma_y^2 \qquad \sigma_x = \sqrt{\mu_{20}}, \quad \sigma_y = \sqrt{\mu_{02}}$$

$$\frac{\mu_{11}^2}{\sigma_x^2\sigma_y^2} \leq 1 \implies \frac{|\mu_{11}|}{\sigma_x\sigma_y} \leq 1$$

$$|\rho| = \frac{|\mu_{11}|}{\sqrt{\mu_{02}\,\mu_{20}}} \leq 1$$

Q. RVs x & y have the jdf $f_{XY}(x,y) = \frac{(x+y)^2}{40}$ $\quad -1<x<1$, $-3<y<3$

find 1) 2nd order moments of x & y

2) σ_x^2, σ_y^2 ③ ρ ④ 3rd order moments.

A. 1) $$m_{20} = \int_{-3}^{3}\int_{-1}^{1} x^2 \frac{(x+y)^2}{40}\,dx\,dy = \frac{9}{25} = 0.36$$

$$m_{02} = \int_{-3}^{3}\int_{-1}^{1} y^2 \frac{(x+y)^2}{40}\,dx\,dy = \frac{129}{25} = 5.16$$

$$m_{11} = \int_{-3}^{3}\int_{-1}^{1} xy \frac{(x+y)^2}{40}\,dx\,dy = \frac{6}{10} = 0.6$$

2) $$\sigma_x^2 = m_{20} - m_{10}^2$$

$$m_{10} = \int_{-3}^{3}\int_{-1}^{1} x \frac{(x+y)^2}{40}\,dx\,dy = 0$$

$$m_{01} = \int_{-3}^{3}\int_{-1}^{1} y \frac{(x+y)^2}{40}\,dx\,dy = 0$$

$$\sigma_x^2 = 0.36 \quad , \quad \sigma_y^2 = 5.16$$

3) $$\rho = \frac{C_{xy}}{\sigma_x\sigma_y} = \frac{m_{11} - m_{10}m_{01}}{\sigma_x\sigma_y} = 0.44$$

4) $$m_{03} = \int_{-3}^{3}\int_{-1}^{1} y^3 \frac{(x+y)^2}{40}\,dx\,dy = 0$$

$$m_{03} = \int_{-3}^{3}\int_{-1}^{1} x^3 \frac{(x+y)^2}{40}\,dx\,dy = 0$$

$$m_{12} = \int_{-3}^{3}\int_{-1}^{1} xy^2 \frac{(x+y)^2}{40}\,dx\,dy = 0$$

$$m_{21} = \int^{3}\int_{-1}^{1} x^2y \frac{(x+y)^2}{}\,dx\,dy = 0$$

Q: The density fn $f_{XY}(x,y) = \frac{xy}{9}$ $0<x<2,\ 0<y<3$ applies to 2 R.V.s X & Y

1) S.T X & Y are uncorrelated
2) " " X & Y are also statistically Independent.

A: 1) $R_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$

$$= \int_0^3\int_0^2 xy \cdot \frac{xy}{9}\, dx\, dy = \frac{1}{9}\frac{[x^3]_0^2}{3}\frac{[y^3]_0^3}{3} = \frac{8}{3}$$

$$E(X) = \int_0^3\int_0^2 x\cdot\frac{xy}{9}\, dx\, dy = \frac{4}{3}$$

$$E(Y) = \int_0^3\int_0^2 y\cdot\frac{xy}{9}\, dx\, dy = 2$$

Since $R_{XY} = E(X)\,E(Y) = \frac{8}{3}$

$\therefore$ X & Y are uncorrelated.

Q: Two R.V X & Y have the pdf $f_{XY}(x,y) = \frac{2}{43}(x+0.5y)^2$ $0<x<2,\ 0<y<3$ find ① all 1st & 2nd order moments ② find the covariance ③ are X & Y uncorrelated.

A: 1) $m_{10} = E(X) = \int_0^3\int_0^2 x\cdot\frac{2}{43}(x+0.5y)^2\, dx\, dy = 1.326$

$m_{01} = E(Y) = \int_0^3\int_0^2 y\cdot\frac{2}{43}(x+0.5y)^2\, dx\, dy = 1.866$

$m_{20} = E(X^2) = \int_0^3\int_0^2 x^2\cdot\frac{2}{43}(x+0.5y)^2\, dx\, dy = 2.009$

$m_{02} = E(Y^2) = \int_0^3\int_0^2 y^2\cdot\frac{2}{43}(x+0.5y)^2\, dx\, dy = 4.130$

$m_{11} = E(XY) = \int_0^3\int_0^2 xy\cdot\frac{2}{43}(x+0.5y)^2\, dx\, dy = 2.424$

2) $C_{XY} = R_{XY} - \bar{X}\bar{Y} = m_{11} - m_{10}m_{01} = -0.0502$

3) Since $C_{xy} \neq 0$

$\therefore$ x & y are not uncorrelated.

Q: Define r.v's v & w by $v = x + ay$ & $w = x - ay$ where a is a real no, x & y are r.v's determine a in terms of moments of x & y such that v & w are orthogonal.

1.
$$R_{vw} = E[vw] = E[\{x+ay\}\{x-ay\}] = E(x^2 - a^2y^2)$$
$$= E(x^2) - a^2 E(y^2)$$
$$R_{vw} = 0 \Rightarrow E(x^2) - a^2 E(y^2) = 0$$
$$|a| = \sqrt{E(x^2)/E(y^2)} = \sqrt{m_{20}/m_{02}}$$

Q: In the above problem S.T w & v are statistically independent if $a^2 = \sigma_x^2/\sigma_y^2$

1:
$$\bar{v} = \bar{x} + a\bar{y} \quad , \quad \bar{w} = \bar{x} - a\bar{y}$$
$$C_{vw} = E[\{(v-\bar{v})(w-\bar{w})\}]$$
$$= E[\{(x-\bar{x}) + a(y-\bar{y})\}\{(x-\bar{x}) - a(y-\bar{y})\}]$$
$$= E[(x-\bar{x})^2 - a^2(y-\bar{y})^2]$$

If v & w are independent then $C_{vw} = 0$

$$E[(x-\bar{x})^2] - a^2 E[(y-\bar{y})^2] = 0$$
$$\sigma_x^2 = a^2 \sigma_y^2 \Rightarrow \boxed{a^2 = \frac{\sigma_x^2}{\sigma_y^2}}$$

Q: three uncorrelated r.v's x_1, x_2, x_3 have means $\bar{x}_1 = 1$, $\bar{x}_2 = -3$, $\bar{x}_3 = 1.5$, 2nd moment $E(x_1^2) = 2.5$, $E(x_2^2) = 11$

$E(x_3^2) = 3.5$ Let $y = x_1 - 2x_2 + 3x_3$ be a new RX find the mean value & Variance of y.

A. $E(y) = E(x_1 - 2x_2 + 3x_3) = \bar{x}_1 - 2\bar{x}_2 + 3\bar{x}_3 = 11.5$

$$\sigma_y^2 = E[(y-\bar{y})^2] = E\left[\{(x_1-\bar{x}_1) - 2(x_2-\bar{x}_2) + 3(x_3-\bar{x}_3)\}^2\right]$$

$$= E[(x_1-\bar{x}_1)^2] + 4E[(x_2-\bar{x}_2)^2] + 9E[(x_3-\bar{x}_3)^2]$$
$$- 4E[(x_1-\bar{x}_1)(x_2-\bar{x}_2)] + 6E[(x_1-\bar{x}_1)(x_3-\bar{x}_3)]$$
$$-12E[(x_2-\bar{x}_2)(x_3-\bar{x}_3)]$$

$$= \sigma_{x_1}^2 + 4\sigma_{x_2}^2 + 9\sigma_{x_3}^2 - 4C_{x_1x_2} + 6C_{x_1x_3} - 12C_{x_2x_3}$$

since x_1, x_2 & x_3 are uncorrelated $C_{x_1,x_2} = C_{x_1x_3} = C_{x_2x_3} = 0$

$$\therefore \sigma_y^2 = \sigma_{x_1}^2 + 4\sigma_{x_2}^2 + 9\sigma_{x_3}^2$$

$$= [2.5 - 1^2] + 4[11 - (-3)^2] + 9[(3.5) - (1.5)^2]$$

$$= 20.75$$

Q. Given $w = (ax+3y)^2$ where x & y are zero mean RVs with variances $\sigma_x^2 = 4$, $\sigma_y^2 = 16$ & their correlation coefficient is $\rho = -0.5$ a) find the value for the parameter that minimizes the mean value of w (b) find the min mean value.

A. $E(w) = E[a^2x^2 + 9y^2 + 6axy]$

$E(x) = E(y) = 0$
$E(x^2) = \sigma_x^2 = 4$
$E(y^2) = \sigma_y^2 = 16$
$R_{xy} = \rho\sigma_x\sigma_y$
$= -4$

$$\bar{w} = 4a^2 + 144 - 24a$$

$$\frac{d\bar{w}}{da} = 0 \Rightarrow 8a - 24 = 0$$

$$\therefore a = 3$$

$$\bar{w}_{min}\Big|_{a=3} = 4(3)^2 + 144 - 24(3) = 108$$

Q: For RV's x & y having $\bar{x} = 1$, $\bar{y} = 2$, $\sigma_x^2 = 6$, $\sigma_y^2 = 9$ $\rho = -2/3$ find C_{xy}, R_{xy} & m_{20} & m_{02}.

A. $C_{xy} = \rho\,\sigma_x\,\sigma_y = -2\sqrt{6}$

$R_{xy} = C_{xy} + \bar{x}\,\bar{y} = 2 - 2\sqrt{6}$

$m_{20} = \sigma_x^2 + \bar{x}^2 = 7$, $m_{02} = \sigma_y^2 + \bar{y}^2 = 13$

Q: If $\bar{x} = \frac{1}{2}$, $\overline{x^2} = \frac{5}{2}$, $\bar{y} = 2$, $\overline{y^2} = \frac{19}{2}$, $C_{xy} = -\frac{1}{2\sqrt{3}}$ for RVs x & y find σ_x^2, σ_y^2, R_{xy} & ρ

what is the mean value of the RV $\omega = (x+3y)^2 + 2x + 3$.

A. $\sigma_x^2 = \overline{x^2} - \bar{x}^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$

$\sigma_y^2 = \overline{y^2} - \bar{y}^2 = \frac{19}{2} - 4 = \frac{11}{2}$

$R_{xy} = C_{xy} + \bar{x}\,\bar{y} = 1 - \frac{1}{2\sqrt{3}}$

$\rho = \frac{C_{xy}}{\sigma_x \sigma_y} = -\frac{1}{3}\sqrt{\frac{2}{33}}$ $\qquad \overline{xy} = R_{xy}$

$\bar{\omega} = \overline{x^2} + 9\overline{y^2} + 6\overline{xy} + 2\bar{x} + 3 = 98 - \sqrt{3}$

Q: Let x & y be statistically independent RV's with $\bar{x} = \frac{3}{4}$, $\overline{x^2} = 4$, $\bar{y} = 1$, $\overline{y^2} = 5$ for a RV $\omega = x - 2y + 1$ find R_{xy}, $R_{x\omega}$, $R_{y\omega}$, C_{xy}, are x & y uncorrelated.

A. $R_{xy} = \bar{x}\,\bar{y} = \frac{3}{4}$

$R_{y\omega} = E[y(x - 2y + 1)]$

$= R_{xy} - 2\overline{y^2} + \bar{y}$

$= -8.25$

$R_{x\omega} = E[x(x - 2y + 1)]$

$= \overline{x^2} - 2\overline{xy} + \bar{x}$

$= \overline{x^2} - 2R_{xy} + \bar{x} = 3.25$

$C_{xy} = R_{xy} - \bar{x}\,\bar{y} = 0$

$\therefore$ x & y are uncorrelated

Q: statistically independent r.v's x & y have moments $m_{10} = 2$, $m_{20} = 14$, $m_{02} = 12$, $m_{11} = -6$ find the moment μ_{22}.

s: $\mu_{22} = E[(x-\bar{x})^2 (y-\bar{y})^2] = E[(x-\bar{x})^2]\, E[(y-\bar{y})^2]$

$= \sigma_x^2 \sigma_y^2 = \mu_{20}\, \mu_{02}$

$m_{11} = \overline{xy} = \bar{x}\,\bar{y} = -6$

$\bar{y} = \frac{-6}{\bar{x}} = \frac{-6}{m_{10}} = \frac{-6}{2} = -3$

$\mu_{20} = \sigma_x^2 = m_{20} - m_{10}^2 = 14 - 4 = 10$

$\mu_{02} = \sigma_y^2 = m_{02} - m_{01}^2 = 12 - (-3)^2 = 3$

$\mu_{22} = 10 \times 3 = 30$

Q: In a control system a r.v x is known to have a mean value $\bar{x} = -2V$, $\overline{x^2} = 9V^2$. If the voltage x is amplified by an amplifier that given an o/p $y = -1.5x + 2$
$= -\frac{3}{2}x + 2$
find σ_x^2, σ_y^2, $\bar{y}$, $\overline{y^2}$, R_{xy}.

s. $\sigma_x^2 = \overline{x^2} - \bar{x}^2 = 9 - 4 = 5$

$\bar{y} = -1.5\bar{x} + 2 = 5$

$\overline{y^2} = \frac{9}{4}\overline{x^2} + 4 - 6\bar{x} = \frac{145}{4}$

$R_{xy} = \overline{xy} = -\frac{3}{2}\overline{x^2} + 2\bar{x} = -\frac{35}{2}$

$\sigma_y^2 = \overline{y^2} - \bar{y}^2$

$= \frac{145}{4} - 25$

$= 45/4$

Q: Two r.v's x & y are defined by $\bar{x} = 0$, $\bar{y} = -1$, $\overline{x^2} = 2$, $\overline{y^2} = 4$, $R_{xy} = -2$ two new r.v's w & v are $w = 2x + y$, $v = -x - 3y$ find $\bar{w}$, $\bar{v}$, $\overline{w^2}$, $\overline{v^2}$, R_{wv}, σ_x^2 & σ_y^2.

s. $\bar{w} = 2\bar{x} + \bar{y} = -1$ $\qquad \bar{v} = -\bar{x} - 3\bar{y} = 3$ $\qquad R_{xy} = \overline{xy}$

$$\overline{w^2} = \overline{(2x+y)^2} = 4\overline{x^2} + \overline{y^2} + 4\overline{xy} = 4$$

$$\overline{v^2} = \overline{(-x-3y)^2} = \overline{x^2} + 9\overline{y^2} + 6\overline{xy} = 26$$

$$R_{wv} = \overline{wv} = \overline{(2x+y)(-x-3y)}$$

$$= -2\overline{x^2} - 7\overline{xy} - 3\overline{y^2} = -2$$

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = 2$$

$$\sigma_y^2 = \overline{y^2} - \bar{y}^2 = 3$$

Q: Statistically independent r.v's x & y have means $\bar{x} = 1$, $\bar{y} = -\frac{1}{2}$, $\overline{x^2} = 4$, $\overline{y^2} = \frac{11}{4}$ another r.v is defined by $w = 3x^2 + 2y + 1$ find σ_x^2, σ_y^2, R_{xy}, C_{xy}, $\bar{w}$ & R_{wy}.

A:

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = 4 - 1 = 3$$

$$\sigma_y^2 = \overline{y^2} - \bar{y}^2 = 5/2$$

$$R_{xy} = \bar{x}\,\bar{y} = -\frac{1}{2}$$

$$C_{xy} = R_{xy} - \bar{x}\,\bar{y} = 0$$

$$\bar{w} = 3\overline{x^2} + 2\bar{y} + 1 = 12$$

$$R_{wy} = \overline{wy} = 3\overline{x^2}\,\bar{y} + 2\overline{y^2} + \bar{y} = -1$$

Joint characteristic Function:

The joint characteristic fn of two r.v's x & y is defined by

$$\phi_{xy}(\omega_1, \omega_2) = E\left[e^{j\omega_1 x + j\omega_2 y}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x, y)\, e^{j\omega_1 x + j\omega_2 y}\, dx\, dy$$

where

$$f_{xy}(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{xy}(\omega_1, \omega_2)\, e^{-j\omega_1 x - j\omega_2 y}\, d\omega_1\, d\omega_2$$

by setting $\omega_1 = 0$ or $\omega_2 = 0$ the characteristic fns of y or x are obtained. They are called marginal characteristic fns.

$$\phi_x(\omega_1) = \phi_{xy}(\omega_1, 0) \quad \& \quad \phi_y(\omega_2) = \phi_{xy}(0, \omega_2)$$

the joint moments m_{nk} can be found from the joint characteristic fn given by

$$m_{nk} = (-j)^{n+k} \left. \frac{\partial^{n+k} \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k} \right|_{\omega_1 = \omega_2 = 0}$$

Q: two R.v's x & y have the joint characteristic fn $\phi_{xy}(\omega_1, \omega_2) = e^{-2\omega_1^2 - 8\omega_2^2}$ S.T x & y are both zero mean R.v's & they are uncorrelated.

A: $$\bar{x} = m_{10} = (-j)^{1+0} \left. \frac{\partial \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1} \right|_{\omega_1 = 0 = \omega_2}$$

$$= -j \left(e^{-2\omega_1^2 - 8\omega_2^2} (-4\omega_1) \right)_{\omega_1 = \omega_2 = 0} = 0$$

$$\bar{y} = m_{01} = (-j)^{0+1} \left. \frac{\partial \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_2} \right|_{\omega_1 = \omega_2 = 0}$$

$$= -j \left(e^{-2\omega_1^2 - 8\omega_2^2} (-16\omega_2) \right)_{\omega_1 = \omega_2 = 0} = 0$$

$$R_{xy} = m_{11} = (-j)^{1+1} \left. \frac{\partial \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1 \, \partial \omega_2} \right|_{\omega_1 = \omega_2 = 0}$$

$$= (-j)^2 \frac{\partial}{\partial \omega_1} \left(e^{-2\omega_1^2} \right) \frac{\partial}{\partial \omega_2} \left(e^{-8\omega_2^2} \right) \Bigg|_{\omega_1 = \omega_2 = 0}$$

$$= -\left[e^{-2\omega_1^2} (-4\omega_1) \right] \left[e^{-8\omega_2^2} (-16\omega_2) \right]_{\omega_1 = \omega_2 = 0} = 0$$

Since $C_{XY} = R_{XY} = 0$

$\therefore$ X & Y are uncorrelated & orthogonal.

The joint characteristic fn of N rvs $X_1, X_2 \dots X_N$ is defined by

$$\phi_{X_1 X_2 \dots X_N}(\omega_1, \omega_2 \dots \omega_N) = E\left[e^{j\omega_1 x_1 + j\omega_2 x_2 + \dots + j\omega_N x_N}\right]$$

joint moments

$$m_{n_1 n_2 \dots n_N} = (-j)^R \left. \frac{\partial^R \phi_{X_1 X_2 \dots X_N}(\omega_1, \omega_2 \dots \omega_N)}{\partial \omega_1^{n_1} \partial \omega_2^{n_2} \dots \partial \omega_N^{n_N}} \right|_{\text{all } \omega_i = 0}$$

where $R = n_1 + n_2 + \dots + n_N$

joint characteristic fn is useful when the pdf is needed for the sum of N statistically independent rvs.

Q: Let the rv $Y = X_1 + X_2 \dots + X_N$ be the sum of N statistically independent rvs X_i, $i = 1, 2 \dots N$. Find the density fn of Y.

A: $\phi_{X_1 X_2 \dots X_N}(\omega_1, \omega_2 \dots \omega_N)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{X_1 X_2 \dots X_N}(x_1, x_2 \dots x_N)\, e^{j\omega_1 x_1 + j\omega_2 x_2 \dots + j\omega_N x_N}\, dx_1\, dx_2 \dots dx_N$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{i=1}^{N} f_{X_i}(x_i)\, e^{j\sum_{i=1}^{N} \omega_i x_i}\, dx_i$$

$$= \prod_{i=1}^{N} \int_{-\infty}^{\infty} f_X(x_i)\, e^{j\omega_i x_i}\, dx_i = \prod_{i=1}^{N} \phi_{X_i}(\omega_i)$$

$$\phi_Y(\omega) = E\left[e^{j\omega Y}\right] = E\left[e^{j\sum_{i=1}^{N}\omega x_i}\right]$$

$$= \phi_{x_1x_2 - x_N}(\omega, \omega - - \omega) = \prod_{i=1}^{N} \phi_{x_i}(\omega)$$

$$f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_Y(\omega)\, e^{-j\omega y}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod_{i=1}^{N} \phi_{x_i}(\omega)\, e^{-j\omega y}\, d\omega$$

If x_i are identically distributed then

$$\phi_{x_i}(\omega) = \phi_x(\omega)$$

$$\therefore f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\phi_x(\omega)\right]^N e^{-j\omega y}\, d\omega$$

Q: For N RVs S.T $\left|\phi_{x_1x_2-x_N}(\omega_1, \omega_2 - \omega_N)\right| \le \phi_{x_1x_2-x_N}(0, 0 - - 0) = 1$

A: $\left|\phi_{x_1x_2-x_N}(\omega_1, \omega_2 - \omega_N)\right| = \left|E\left[e^{j\sum_{i=1}^{N}\omega_i x_i}\right]\right|$

$$= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f_{x_1x_2-x_N}(x_1, x_2 - x_N)\, e^{j\sum_{i=1}^{N}\omega_i x_i}\, dx_1\, dx_2 - - dx_N\right|$$

$$\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} - - \int_{-\infty}^{\infty} \left|f_{x_1x_2-x_N}(x_1, x_2 - x_N)\right| \left|e^{j\sum_{i=1}^{N}\omega_i x_i}\right| dx_1\, dx_2 \cdots dx_N$$

$$\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} - - \int_{-\infty}^{\infty} f_{x_1x_2-x_N}(x_1, x_2 - x_N)\, dx_1\, dx_2 - dx_N$$

$$\le \phi_{x_1x_2-x_N}(0, 0, 0 \cdots 0) \le 1$$

Q: RVs x_1 & x_2 have the joint characteristic fn

$$\phi_{x_1x_2}(\omega_1, \omega_2) = \left[(1 - j2\omega_1)(1 - j2\omega_2)\right]^{-N/2} \quad \text{where } N > 0$$

is an integer a) find the correlation & moments m_{20} & m_{02} b) determine the means of x_1 & x_2

c) what is the Correlation Coefficient.

Sol..
$$\frac{\partial \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_1} = \frac{1}{(1-j2\omega_2)^{N/2}}\left[\frac{jN}{(1-j2\omega_1)^{\frac{N}{2}+1}}\right]$$

$$\frac{\partial^2 \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_1^2} = \frac{1}{(1-j2\omega_2)^{N/2}}\left[\frac{-N(N+2)}{(1-j2\omega_1)^{\frac{N}{2}+2}}\right]$$

$$\frac{\partial \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_2} = \frac{1}{(1-j2\omega_1)^{N/2}}\left[\frac{jN}{(1-j2\omega_2)^{\frac{N}{2}+1}}\right]$$

$$\frac{\partial^2 \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_2^2} = \frac{1}{(1-j2\omega_1)^{\frac{N}{2}}}\left[\frac{-N(N+2)}{(1-j2\omega_2)^{\frac{N}{2}+2}}\right]$$

$$m_{20} = (-j)^2 \frac{\partial^2 \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_1^2}\bigg|_{\omega_1=\omega_2=0} = N(N+2)$$

$$m_{02} = (-j)^2 \frac{\partial^2 \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_2^2}\bigg|_{\omega_1=\omega_2=0} = N(N+2)$$

$$R_{x_1x_2} = m_{11} = (-j)^2 \frac{\partial^2 \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_1 \partial \omega_2}\bigg|_{\omega_1=\omega_2=0} = N^2$$

$$m_{10} = \frac{-j\partial \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_1}\bigg|_{\omega_1=\omega_2=0} = N$$

$$m_{01} = -j\frac{\partial \phi_{x_1x_2}(\omega_1,\omega_2)}{\partial \omega_2}\bigg|_{\omega_1=\omega_2=0} = N$$

$$\rho = \frac{C_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}} = \frac{R_{x_1x_2} - \overline{x_1}\,\overline{x_2}}{\ldots} = \frac{m_{11} - m_{10}m_{01}}{\sqrt{\ldots}} = 0$$

Joint moment Generating Function:

The joint mgf of two rv's x & y is defined by as the expected value of the joint function

$$g(x,y) = e^{v_1 x + v_2 y}$$

$$M_{xy}(v_1, v_2) = E\left(e^{v_1 x + v_2 y}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{v_1 x + v_2 y} f_{xy}(x,y)\,dx\,dy$$

the joint moments m_{nk} is given by

$$m_{nk} = \left.\frac{\partial^{n+k} M_{xy}(v_1, v_2)}{\partial v_1^n \, \partial v_2^k}\right|_{v_1 = v_2 = 0}$$

Jointly Gaussian Random Variables:

two Random Variables:

Two rv's x & y are said to be jointly gaussian if their jdf is of the form

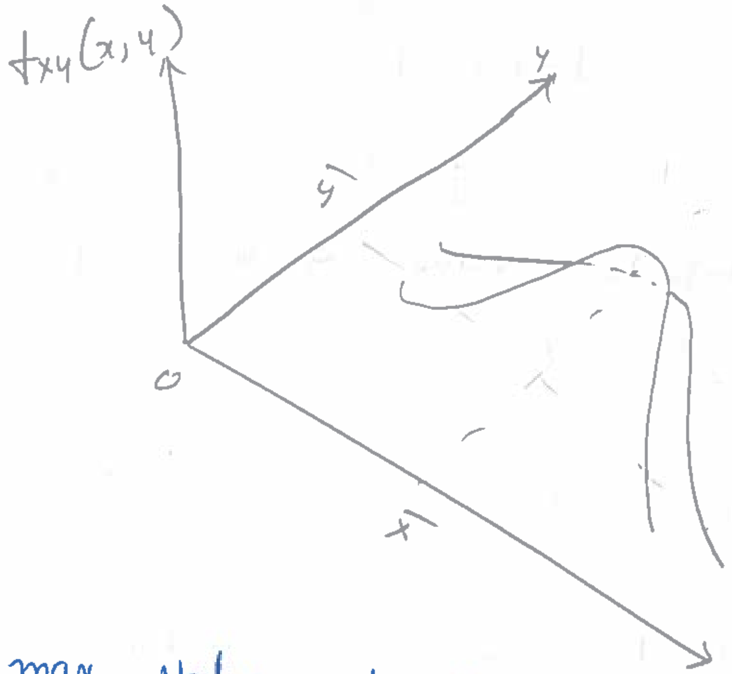
$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\bar{x})^2}{\sigma_x^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})^2}{\sigma_y^2}\right]}$$

this is also called as bivariate Gaussian density fn

where $\bar{x} = E(x)$, $\bar{y} = E(y)$

$$\sigma_x^2 = E\left((x-\bar{x})^2\right], \quad \sigma_y^2 = E\left[(y-\bar{y})^2\right]$$

$$\rho = \frac{E\left[(x-\bar{x})(y-\bar{y})\right]}{\sigma_x\sigma_y} = \frac{C_{xy}}{\sigma_x\sigma_y}$$



The max value of the density fn is located at $(\bar{x}, \bar{y})$ & it is

$$f_{XY}(x,y)\Big|_{(\bar{x},\bar{y})} = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

N Random Variables:

N RV's $X_1, X_2 \ldots X_N$ are called jointly gaussian if their density fn can be written as

$$f_{X_1,X_2 \ldots X_N}(x_1, x_2 \ldots x_N) = \frac{1}{(2\pi)^{N/2}\,|[C_X]|^{1/2}}\, e^{-\frac{[x-\bar{X}]^t [C_X]^{-1}[x-\bar{X}]}{2}}$$

where
$$[C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix} \qquad [x-\bar{X}] = \begin{bmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \\ \vdots \\ x_N-\bar{x}_N \end{bmatrix}$$

$$C_{ij} = E\left[(x_i-\bar{x}_i)(x_j-\bar{x}_j)\right]$$
$$= C_{x_i x_j}$$

if $i=j$ $\quad C_{ij} = E\left[(x_i-\bar{x}_i)^2\right] = \sigma_{x_i}^2$

$i \neq j$ $\quad C_{ij} = C_{x_i x_j}$

The covariance matrix for $N=2$ is

$$[C_x] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{x_1}^2 & C_{12} \\ C_{21} & \sigma_{x_2}^2 \end{bmatrix}$$

Properties of Gaussian random variables:

1) Gaussian rvs are completely defined by their means, variances and covariances.
2) If the Gaussian rvs are uncorrelated, then they are statistically independent.
3) All marginal density fns derived from N-variate Gaussian density fns are Gaussian.
4) All conditional density fns are also Gaussian.
5) The linear transformations of Gaussian random variables are Gaussian.

Eg Consider two rvs x_1 & y_1 related to two other rvs x & y by the coordinate rotation

$$x_1 = +x\cos\theta + y\sin\theta \quad , \quad y_1 = y\cos\theta - x\sin\theta$$

$$C_{x_1y_1} = E\left[(x_1-\bar{x}_1)(y_1-\bar{y}_1)\right]$$

$$= E\Big[\{x\cos\theta + y\sin\theta - \bar{x}\cos\theta - \bar{y}\sin\theta\}$$
$$\{y\cos\theta - x\sin\theta - \bar{y}\cos\theta + \bar{x}\sin\theta\}\Big]$$

$$= E\left[\{(x-\bar{x})\cos\theta + (y-\bar{y})\sin\theta\}\{(y-\bar{y})\cos\theta - (x-\bar{x})\sin\theta\}\right]$$

$$= E\left((x-\bar{x})(y-\bar{y})\right)\cos^2\theta - E\left[(x-\bar{x})^2\right]\cos\theta\sin\theta$$
$$+ E\left[(y-\bar{y})^2\right]\sin\theta\cos\theta - E\left((x-\bar{x})(y-\bar{y})\right)\sin^2\theta$$

$$= C_{xy}\cos^2\theta - \sigma_x^2 \frac{\sin 2\theta}{2} + \sigma_y^2 \frac{\sin 2\theta}{2} - C_{xy}\sin^2\theta$$

If x_1, y_1 are uncorrelated then $C_{x_1y_1} = 0$

$$C_{xy}\left[\cos^2\theta - \sin^2\theta\right] - \frac{1}{2}\sin 2\theta\left(\sigma_y^2 - \sigma_x^2\right) = 0$$

$$C_{xy}\cos 2\theta + \frac{1}{2}\sin 2\theta\left(\sigma_y^2 - \sigma_x^2\right) = 0$$

$$C_{xy}\cos 2\theta = \frac{1}{2}\left(\sigma_y^2 - \sigma_x^2\right)\sin 2\theta$$

$$\frac{2C_{xy}}{\sigma_y^2 - \sigma_x^2} = \tan 2\theta \qquad C_{xy} = \rho\sigma_x\sigma_y$$

$$\frac{2\rho\sigma_x\sigma_y}{\sigma_y^2 - \sigma_x^2} = \tan 2\theta$$

$$\boxed{\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right]}$$

A coordinate rotation through an angle $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right)$ is sufficient to convert correlat rv's x & y having variances σ_x^2 & σ_y^2 & correlation coeff coefficient ρ into two statistically independent Gaussian r.v.

Q: Two gaussian rvs x & y have variances $\sigma_x^2 = 9$, $\sigma_y^2 = 4$ & It is known that a coordinate rotation by an angle $-\frac{\pi}{8}$ results in new variables y_1 & y_2 that are uncorrelated what is ρ.

A.

$$\rho = \frac{\sigma_x^2 - \sigma_y^2}{2\sigma_x\sigma_y}\tan 2\theta = \frac{9-4}{2\sqrt{9}\sqrt{4}}\tan^{-1}\left(-\frac{\pi}{4}\right) = -\frac{5}{12}$$

Q: Let x & y be jointly gaussian r.v's where $\sigma_x^2 = \sigma_y^2$ & $\rho = -1$ find the transformation matrix such that the new r.v's Y_1 & Y_2 are statistically independent

A: $$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2-\sigma_y^2}\right) = \frac{1}{2}\tan^{-1}(-\infty) = -\frac{\pi}{4}$$

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Q: Gaussian r.v.s x & y have 1st & 2nd order moments $\overline{x} = -1$, $\overline{x^2} = 1.16$, $\overline{y^2} = 2.89$, $\overline{y} = 1.5$, $R_{xy} = 1.724$

find C_{xy}, ρ & θ

A: $$C_{xy} = R_{xy} - \overline{x}\,\overline{y} = -0.224$$

$$\sigma_x^2 = \overline{x^2} - \overline{x}^2 = 0.16$$

$$\sigma_y^2 = \overline{y^2} - \overline{y}^2 = 0.64$$

$$\rho = \frac{C_{xy}}{\sigma_x\sigma_y} = -0.7$$

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2-\sigma_y^2}\right] = 0.3755 \text{ rad}$$

Q: Suppose the annual snowfalls for two near by resorts are adequately represented by jointly gaussian r.v's x & y for which $\rho = 0.82$, $\sigma_x = 1.5$ m, $\sigma_y = 1.2$ m & $R_{xy} = 81.746\ m^2$. If the avg snowfall at one resort is 10 m. what is the average snow fall at the other resort.

A: $$\rho = \frac{C_{xy}}{\sigma_x\sigma_y} = \frac{R_{xy} - \overline{x}\,\overline{y}}{\sigma_x\sigma_y}$$

$$\overline{y} = \frac{R_{xy} - \rho\sigma_x\sigma_y}{\overline{x}} = 8.0 \text{ m}$$

Q: Two gaussian rv's x & y have a correlation coefficient $\rho = 0.25$. The standard deviation of x is 1.9. A linear transformation $(\theta = \frac{\pi}{6})$ is known to transform x & y to a new rvs that are statistically independent what is σ_y

A.
$$\tan 2\theta = \frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$$

$$\sqrt{3} = \frac{2(0.25)1.9\,\sigma_y}{(1.9)^2 - \sigma_y^2}$$

$$\sigma_y^2 + \frac{1.9}{2\sqrt{3}}\sigma_y - (1.9)^2 = 0$$

$$\sigma_y = \frac{1.9}{4\sqrt{3}}\left(-1 \pm \sqrt{49}\right)$$

$$\sigma_y = \frac{1.9}{4\sqrt{3}}(-1+7) = 1.6454$$

Q. Gaussian rv's x_1 & x_2 for which $\bar{x}_1 = 2$, $\sigma_{x_1}^2 = 9$, $\bar{x}_2 = -1$, $\sigma_{x_2}^2 = 4$ & $C_{x_1x_2} = -3$ are transformed to new rv's y_1 & y_2 according to $y_1 = -x_1 + x_2$, $y_2 = -2x_1 - 3x_2$ find $\overline{x_1^2}$, $\rho_{x_1x_2}$, $\sigma_{y_1}^2$, $\sigma_{y_2}^2$, $C_{y_1y_2}$.

A.
$$\overline{x_1^2} = \sigma_{x_1}^2 + \bar{x}_1^2 = 9 + 4 = 13$$

$$\overline{x_2^2} = \sigma_{x_2}^2 + \bar{x}_2^2 = 4 + 1 = 5$$

$$\rho_{x_1x_2} = \frac{C_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}} = \frac{-3}{2(3)} = -1/2$$

$$[C_y] = [T][C_x][T]^t$$

$$= \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 19 & 3 \\ 3 & 36 \end{bmatrix}$$

$$\sigma_{y_1}^2 = 19, \quad \sigma_{y_2}^2 = 36, \quad C_{y_1y_2} = 3$$

Transformation of Random Variables:

Let N RVs X_n, $n = 1, 2, 3, \ldots N$ be continuous or discrete or mixed. Define another set of RVs Y_n, $n = 1, 2 \ldots N$ by the transformation of X_n

i.e $Y_n = T_n[x_1, x_2 \ldots x_N]$ $\quad n = 1, 2 \ldots N$

T_n may be linear, non-linear, continuous, segmented etc

It is also assumed that a set of 'inverse' fns T_j^{-1} exists such that the old variables may be expressed as single valued continuous fns of the new variables.

i.e $X_j = T_j^{-1}[y_1, y_2 \ldots y_N]$ $\quad j = 1, 2 \ldots N$

i.e the transformation has a one to one correspondence b/w the set of RV's X & Y.

for 2 RVs transformation b/w (x_1, x_2) & (y_1, y_2) i.e $N = 2$ the jdf is

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(x_1, x_2) \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

for N RVs the jdf is

$$f_{Y_1 Y_2 \ldots Y_N}(y_1, y_2 \ldots y_N) = |J| \, f_{X_1 X_2 \ldots X_N}(x_1 = T_1^{-1}, x_2 = T_2^{-1} \ldots x_N = T_N^{-1})$$

where $|J| = \begin{vmatrix} \frac{\partial T_1^{-1}}{\partial y_1} & \frac{\partial T_1^{-1}}{\partial y_2} & \cdots & \frac{\partial T_1^{-1}}{\partial y_N} \\ \vdots & & & \\ \frac{\partial T_N^{-1}}{\partial y_1} & \frac{\partial T_N^{-1}}{\partial y_2} & \cdots & \frac{\partial T_N^{-1}}{\partial y_N} \end{vmatrix}$

eg let $y_1 = T(x_1, x_2) = ax_1 + bx_2$

$y_2 = T(x_1, x_2) = cx_1 + dx_2$

where a, b, c & d are Constant

$$x_1 = T_1^{-1}(y_1, y_2) = \frac{dy_1 - by_2}{ad - bc}$$

$$x_2 = T_2^{-1}(y_1, y_2) = \frac{-cy_1 + ay_2}{ad - bc}$$

$$|J| = \begin{vmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{vmatrix} = \frac{1}{(ad-bc)}$$

$$\therefore f_{y_1y_2}(y_1, y_2) = |J| \, f_{x_1x_2}(x_1, x_2)$$

$$= \frac{1}{(ad-bc)} f_{x_1x_2}\left(\frac{dy_1 - by_2}{ad-bc}, \frac{-cy_1 + ay_2}{ad-bc}\right)$$

Q: RVs x & y have the jdf $f_{xy}(x,y) = \frac{8}{3} u(x-2) u(y-1) x y^2 e^{(4-2xy)}$ undergo a transformation $[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ to generate new rvs y_1 & y_2 find the jdf of y_1 & y_2.

A. $a = 1, \; b = 1, \; c = 1, \; d = -1$

$$f_{y_1y_2}(y_1, y_2) = \frac{1}{|-1-1|} f_{x_1x_2}\left(\frac{-y_1 - y_2}{-2}, \frac{-y_1 + y_2}{-2}\right)$$

$$= \frac{1}{2} f_{x_1x_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{4}{3} \, u\left(\frac{y_1 + y_2}{2} - 2\right) u\left(\frac{y_1 - y_2}{2} - 1\right) (y_1 + y_2) \left(\frac{y_1 - y_2}{2}\right)^2$$

$$\cdot\ e^{4-2\left(\frac{y_1+y_2}{2}\right)\left(\frac{y_1-y_2}{2}\right)}$$

$$= \frac{8}{6}\, u\left(\frac{y_1+y_2-4}{2}\right) u\left(\frac{y_1-y_2-2}{2}\right)(y_1-y_2)(y_1^2-4y_2^2).$$

$$e^{4-\frac{(y_1^2-y_2^2)}{2}}$$

Q: Three r.v's x_1, x_2 & x_3 represent samples of random noise voltage taken at 3 times. Their covariance matrix is defined by
$$[C_x] = \begin{bmatrix} 3 & 1.8 & 1.1 \\ 1.8 & 1.1 & 1.8 \\ 1.1 & 1.8 & 3 \end{bmatrix}$$

A transformation $[T] = \begin{bmatrix} 4 & -1 & -2 \\ 2 & 2 & 1 \\ -3 & -1 & 3 \end{bmatrix}$ converts the variables to new r.v's y_1, y_2, y_2 find the covariance matrix of new r.v's.

A: $$[C_y] = [T]\,[C_x]\,[T]^t$$

$$= \begin{bmatrix} 4 & -1 & -2 \\ 2 & 2 & 1 \\ -3 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1.8 & 1.1 \\ 1.8 & 1.1 & 1.8 \\ 1.1 & 1.8 & 3 \end{bmatrix}\begin{bmatrix} 4 & 2 & -3 \\ -1 & 2 & -1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 38.2 & 13.8 & -34.8 \\ 13.8 & 53 & -17.1 \\ -34.8 & -17.1 & 37.2 \end{bmatrix}$$

Linear Transformation of Gaussian Random Variables:

Consider N Gaussian r.v's y_n $n = 1, 2 \dots N$ having a linear transformation with the set of N Gaussian r.v's x_n, $n = 1, 2 \dots N$

i.e $\quad y_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1N}x_N$

$\quad y_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2N}x_N$

$\quad \vdots \qquad \dots + a_{NN}x_N$

where the coefficients a_{ij} $i \& j = 1, 2 \ldots N$ are real no's

$$[T] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \qquad [y] = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$[\bar{y}] = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_N \end{bmatrix} \qquad [x] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad [\bar{x}] = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_N \end{bmatrix}$$

$$[y] = [T][x] \qquad [y - \bar{y}] = [T][x - \bar{x}]$$

$$[x] = [T]^{-1}[y] \qquad [x - \bar{x}] = [T]^{-1}[y - \bar{y}]$$

$$x_i = T_i^{-1}(y_1, y_2 \ldots y_N) = a^{i1} y_1 + a^{i2} y_2 + \ldots + a^{iN} y_N$$

$$\frac{\partial x_i}{\partial y_j} = \frac{\partial T_i^{-1}}{\partial y_j} = a^{ij}$$

$$|J| = |[T]^{-1}| = \frac{1}{|[T]|}$$

$$C_{x_i x_j} = E\left[(x_i - \bar{x}_i)(x_j - \bar{x}_j)\right]$$

$$= \sum_{k=1}^{N} a^{ik} \sum_{m=1}^{N} a^{jm} E\left[(y_k - \bar{y}_k)(y_m - \bar{y}_m)\right]$$

$$= \sum_{k=1}^{N} a^{ik} \sum_{m=1}^{N} a^{jm} C_{y_k y_m}$$

$C_{x_i x_j}$ is the ijth element of the covariance matrix $[C_x]$

$$[C_x] = [T]^{-1} [C_y] \left[[T]^t\right]^{-1}$$

$$[C_y] = [T] [C_x] [T]^t$$

N variate Gaussian density fn is

$$f_{x_1 x_2 \ldots x_N}(x_1, x_2 \ldots x_N) = \frac{1}{(2\pi)^{N/2} |[C_x]|^{1/2}} e^{-\frac{[x - \bar{x}]^t [C_x]^{-1} [x - \bar{x}]}{2}}$$

$$\therefore f_{y_1 y_2 \cdots y_N}(y_1, y_2 \cdots y_N) = \frac{|[C_Y]^{-1}|^{1/2}}{(2\pi)^{N/2}} e^{\frac{-[y-\bar{y}]^t [C_Y]^{-1} [y-\bar{y}]}{2}}$$

The linear transformation of Gaussian r.v's produces Gaussian r.v's.

Q: Two Gaussian r.v's X_1 & X_2 have zero means & variances $\sigma_{X_1}^2 = 4$, $\sigma_{X_2}^2 = 9$ their covariance $C_{X_1X_2} = 3$. If x_1 & x_2 are linearly transformed to new r.v's y_1 & y_2 according to $y_1 = x_1 - 2x_2$, $y_2 = 3x_1 + 4x_2$ find the mean, variance & covariance of y_1 & y_2.

A.
$$[T] = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \qquad [C_X] = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$$

$$[C_Y] = [T][C_X][T]^t = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix} \qquad \sigma_{y_1}^2 = 28, \quad \sigma_{y_2}^2 = 252$$

$$C_{y_1 y_2} = -66$$

Q: Zero mean Gaussian r.v's X_1, X_2 & X_3 having a covariance matrix $[C_X] = \begin{bmatrix} 4 & 2.05 & 1.05 \\ 2.05 & 4 & 2.05 \\ 1.05 & 2.05 & 4 \end{bmatrix}$ are transformed to new variables

$$y_1 = 5x_1 + 2x_2 - x_3, \quad y_2 = -x_1 + 3x_2 + x_3, \quad y_3 = 2x_1 - x_2 + 2x_3$$

find the covariance matrix of y_1, y_2, y_3.

A.
$$[C_Y] = [T][C_X][T]^t = \begin{bmatrix} 5 & 2 & -1 \\ -1 & 3 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2.05 & 1.05 \\ 2.05 & 4 & 2.05 \\ 1.05 & 2.05 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 142.3 & 30.9 & 40.6 \\ 30.9 & 41.9 & 12.6 \\ [illegible] & [illegible] & [illegible] \end{bmatrix}$$

Q: Two Gaussian R.V's x_1 & x_2 are defined by mean & covariance matrices $[\bar{x}] = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $[C_x] = \begin{bmatrix} 5 & -2/\sqrt{5} \\ -2/\sqrt{5} & 4 \end{bmatrix}$ two new R.V's y_1 & y_2 are formed using the transformation $[T] = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ find the matrix $[\bar{y}]$ $[C_y]$ & $\rho_{y_1 y_2}$. 3.34

A: $[\bar{y}] = [T][\bar{x}] = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$

$$[C_y] = [T]\,[C_x]\,[T]^t = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 5 & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & 4 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5.1 & 3.38 \\ 3.38 & 4.35 \end{bmatrix}$$

$$\rho = \frac{C_{y_1 y_2}}{\sigma_{y_1} \sigma_{y_2}} = \frac{3.38}{\sqrt{(5.1)(4.35)}} = 0.707$$

Q: Two R.V's X & Y have means 1 & 2 resp & variances 4 & 1 resp. The correlation coefficient is 0.4 new R.V's ω & V are defined by $V = -x+2y$, $\omega = x+3y$ find the mean, variance & correlation $R_{V\omega}$.

A: $\bar{V} = -\bar{x} + 2\bar{y} = 3$, $\bar{\omega} = \bar{x} + 3\bar{y} = 7$

$\sigma_V^2 = E[\{(x-\bar{x}) + 2(y-\bar{y})\}^2]$ $C_{xy} = \rho_{xy}\sigma_x\sigma_y = 0.8$

$= \sigma_x^2 + 4\sigma_y^2 - 4C_{xy} = 4.8$

$\sigma_\omega^2 = E[\{(x-\bar{x}) + 3(y-\bar{y})\}^2] = \sigma_x^2 + 9\sigma_y^2 + 6C_{xy}$

$= 17.8$

$R_{V\omega} = E[V\omega] = -\overline{x^2} - R_{xy} + 6\overline{y^2}$

$R_{xy} = C_{xy} + \bar{x}\,\bar{y} = 2.8$

$\overline{x^2} = \sigma_x^2 + \bar{x}^2 = 5$, $\overline{y^2} = \sigma_y^2 + \bar{y}^2 = 5$

$R_{vw} = -5 - 2.8 + 6(5) = 22.2$

$$\rho_{vw} = \frac{C_{vw}}{\sigma_v \sigma_w} = \frac{R_{vw} - \bar{v}\,\bar{w}}{\sigma_v \sigma_w} = 0.12$$

Q. S.T two RVs X_1 & X_2 with jdf $f_{x_1 x_2}(x_1, x_2) = \frac{1}{16}$ $\begin{matrix} |x_1| < 4 \\ 2 < x_2 < 4 \end{matrix}$ are independent & orthogonal.

A.- $$R_{x_1 x_2} = \int_{x_1=-4}^{4} \int_{x_2=2}^{4} x_1 x_2 \cdot \frac{1}{16}\, dx_1\, dx_2$$

$$= \frac{1}{16} \frac{\left[x_1^2\right]_{-4}^{4}}{2} \frac{\left[x_2^2\right]_2^4}{2} = 0$$

$\therefore$ X_1 & X_2 are orthogonal.

Q. Two RVs X & Y has the following jdf

$$f_{XY}(x, y) = 2 - x - y \quad \begin{matrix} 0 \le y \le 1 \\ 0 \le x \le 1 \end{matrix}, \qquad = 0 \quad \text{else}$$

find $V(x)$ & $V(y)$

A. $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^1 (2 - x - y)\, dy$$

$$= 2y - xy - \frac{y^2}{2} \Big|_0^1 = \frac{3}{2} - x \qquad 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 (2 - x - y)\, dx = \frac{3}{2} - y \qquad 0 \le y \le 1$$

$$E(x) = \int_0^{\infty} x\, f_X(x) = \int_0^1 x\left(\frac{3}{2} - x\right) dx$$

$$= \frac{3}{2} \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = 5/12$$

$$E(x^2) = \int_0^1 x^2 f_x(x)\,dx = \int_0^1 x^2\left(\tfrac{3}{2}-x\right)dx$$

$$= \frac{3}{2}\,\frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$V(x) = E(x^2) - E(x)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$\text{||ly } V(y) = 11/144$$

Q: For 2 zero mean grv x & y s.t their joint characteristic fn is

$$\phi_{xy}(\omega_1,\omega_2) = e^{-\frac{1}{2}\left[\sigma_x^2\omega_1^2 + 2P\sigma_x\sigma_y\omega_1\omega_2 + \sigma_y^2\omega_2^2\right]}$$

s. $$\phi_x(\omega_1) = e^{-\frac{1}{2}\sigma_x^2\omega_1^2}$$

$$\bar{x} = -j\frac{\partial\phi_x(\omega_1)}{\partial\omega_1}\Big|_{\omega_1=0} = 0$$

$$\phi_y(\omega_2) = e^{-\frac{1}{2}\sigma_y^2\omega_2^2}$$

$$\bar{y} = -j\frac{\partial\phi_y(\omega_2)}{\partial\omega_2}\Big|_{\omega_2=0} = 0$$

since $\bar{x}$ & $\bar{y}$ are zero

$\therefore$ $\phi_{xy}(\omega_1,\omega_2)$ is the joint characteristic fn of zero mean Gaussian rv's x & y

Q: For 2 rvs x & y $f_{xy}(x,y) = 0.5\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2) + 0.4\,\delta(x-1)\,\delta(y+2) + 0.2\,\delta(x-1)\,\delta(y-1) + 0.5\,\delta(x-1)\,\delta(y-3)$ find the correlation, cov, P_{xy} & are x & y uncorrelated or orthogonal.

3.37

1. $R_{xy} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{xy}(x,y)\,dx\,dy$

$= 0.5(-1)(0) + 0.1(0)(0) + 0.1(0)(2) + 0.4(1)(-2)$
$\quad + 0.2(1)(1) + 0.5(1)(3)$

$= 0.9$

$C_{xy} = R_{xy} - E(x)E(y)$

$f_x(x) = 0.5\,\delta(x+1) + 0.2\,\delta(x) + 1.1\,\delta(x-1)$

$E(x) = \int_{-\infty}^{\infty} x f_x(x)\,dx = 0.5(-1) + 0.2(0) + 1.1(1)$

$= 0.6$

$f_y(y) = 0.6\,\delta(y) + 0.1\,\delta(y-2) + 0.4\,\delta(y+2) + 0.2\,\delta(y-1)$
$\quad + 0.5\,\delta(y-3)$

$E(y) = \int_{-\infty}^{\infty} y f_y(y)\,dy$

$= 0.6(0) + 0.1(2) + 0.4(-2) + 0.2(1) + 0.5(3)$

$= 1.0$

$C_{xy} = R_{xy} - \bar{x}\,\bar{y} = 0.9 - (0.6)(1) = 0.3$

$\rho = \dfrac{C_{xy}}{\sigma_x \sigma_y}$ $\qquad E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$

$= 0.5(-1)^2 + 0.2(0)^2 + 1.1(1)^2 = 1.6$

$E(y^2) = \int_{-\infty}^{\infty} y^2 f_y(y)\,dy = 0.6(0)^2 + 0.1(2)^2 + 0.4(-2)^2$
$\quad + 0.2(1)^2 + 0.5(3)^2 = 6.7$

$\sigma_x^2 = 1.6 - (0.6)^2 = 1.24$ $\qquad \rho_{xy} = \dfrac{C_{xy}}{\sigma_x \sigma_y} = \dfrac{0.24}{\sqrt{(1.24)(5.7)}}$

$\sigma_y^2 = 6.7 - (1)^2 = 5.7$ $\qquad = 0.0919$

Since $C_{xy} \neq 0$ $\therefore$ x & y are not uncorrelated

Since $R_{xy} \neq 0$ $\therefore$ x & y are not orthogonal.

Q: If x & y are two independent R.Vs such that
$E(x) = \lambda_1$, $\sigma_x^2 = \sigma_1^2$, $E(y) = \lambda_2$, $\sigma_y^2 = \sigma_2^2$

P.T $\sigma_{xy}^2 = \sigma_1^2 \sigma_1^2 + \lambda_1^2 \sigma_2^2 + \sigma_1^2 \lambda_2^2$

A: $E(x) = \lambda_1$, $E(y) = \lambda_2$ $\quad$ x & y independent

$\sigma_x^2 = \sigma_1^2$, $\sigma_y^2 = \sigma_2^2$

$$\sigma_{xy}^2 = E\left[(xy)^2\right] - \left(E(xy)\right)^2$$

$$= E(x^2)\,E(y^2) - E(x)^2\,E(y)^2$$

$$= (\sigma_1^2 + \lambda_1^2)(\sigma_2^2 + \lambda_2^2) - \lambda_1^2 \lambda_2^2$$

$$= \sigma_1^2 \sigma_2^2 + \sigma_1^2 \lambda_2^2 + \sigma_2^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_2^2$$

$$= \sigma_1^2 \sigma_2^2 + \sigma_1^2 \lambda_2^2 + \sigma_2^2 \lambda_1^2$$

Questions

1) Define the expected value of a fn of two RV's.

2) Define the joint moments of RVs X & Y

3) Define the central moments of RVs X & Y.

4) Define joint characteristic fn.

5) Define joint mgf.

6) Explain the Gaussian density fn for N RV's

7) Explain the linear transformation of Gaussian RVs.

8) Define the covariance & correlation coefficient of two RV's X & Y.

9) Show that the variance of a weighted sum of uncorrelated RV's equals the weighted sum of the variances of the RV's.

10) State the properties of Gaussian RV's.

Additional Problems

1) Let $y = x_1 + x_2 + \cdots + x_N$ be the sum of N statistica[l] independent r.v's x_i, $i = 1, 2 \cdots N$. If x_i are identically distributed then find the density fn of y.

2) Two r.vs x & y have a uniform density on a circular region defined by

$$f_{xy}(x,y) = \begin{cases} \dfrac{1}{\pi r^2} & x^2+y^2 \le r^2 \\ 0 & \text{else} \end{cases}$$

find the mean value of the fn $g(x,y) = x^2+y^2$.

3) If x, y & z are uncorrelated and independent variab[les] with the same variance σ^2 and zero mean. find the correlation co efficient b/w $(x+y)$ & $(y+z)$.

4) A joint density is given as

$$f_{xy}(x,y) = \begin{cases} x(y+1.5) & 0<x<1 \ \& \ 0<y<1 \\ 0 & \text{else} \end{cases}$$

find all the moment m_{nk} where n & $k = 0, 1, 2$

5) For two r.v's x & y S.T $Var(ax+by) = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\,C_{xy}$ where a & b are constants.

6) State & Prove the properties of joint characteristic fn.

7) State & Prove the properties of Covariance.

Multiple - choice Questions

1) If $g(x,y)$ is a fn of two r.v's x & y, the expected value of $g(x,y)$ is [ ]

a) $\int_{-\infty}^{\infty} g(x,y)\,dx$

b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy$

c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy$

d) $\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} f_{xy}(x,y)\,dx\,dy$

2) The $(n+k)^{th}$ order joint moment of two r.v's x & y is defined as m_{nk} [ ]

a) $\int_{-\infty}^{\infty} x^n f_{xy}(x,y)\,dx$

b) $\int_{-\infty}^{\infty} y^k f_{xy}(x,y)\,dy$

c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy$

d) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{xy}(x,y)\,dx\,dy$

3) The $(n+k)^{th}$ order joint central moment of the r.v's x & y is defined as m_{nk} [ ]

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{xy}(x,y)\,dx\,dy$

b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy$

c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-m_x)^n f_{xy}(x,y)\,dx\,dy$

d) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-m_x)^n (y-m_y)^k f_{xy}(x,y)\,dx\,dy$

4) $\text{Cov}(x,y)$ for r.v's x & y is [ ]

a) $E(xy)$

b) $E(xy) - E(x)$

c) $E(xy) - E(x)E(y)$

d) $E(xy) + E(x)E(y)$

5) The joint moments m_{nk} can be found from the joint characteristic fn as $m_{nk} =$ [ ]

a) $(-j)^{n+k} \left.\dfrac{\partial^{n+k} \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k}\right|_{\omega_1=\omega_2=0}$ (b) $(j)^{n+k} \left.\dfrac{\partial^{n+k} \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k}\right|_{\omega_1=\omega_2=0}$

c) $\left.\dfrac{\partial^{n+k} \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k}\right|_{\omega_1=\omega_2=0}$ (d) $-\left.\dfrac{\partial^{n+k} \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k}\right|_{\omega_1=\omega_2=0}$

6) which of the following is correct [ ]

a) $\rho > 1$ (b) $-\infty < \rho < \infty$ (c) $0 < \rho \leq 1$ (d) $-1 \leq \rho \leq 1$

7) To convert correlated RV's x & y into two statistically independent Gaussian RV's, the coordinate rotation through an angle θ is [ ]

a) $\frac{1}{2} \tan^{-1}\left(\dfrac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right)$ (b) $\tan^{-1}\left[\dfrac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right]$

c) $\frac{1}{2} \tan^{-1}\left(\dfrac{\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right)$ (d) $\tan^{-1}\left[\dfrac{\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right]$

8) Two RV's x & y with identical mgf are ________ distributed

(a) identically (b) differently (c) both (d) none

9) The joint moments of two statistically independent RV's is equal to ________________

a) Sum of the individual moments

b) Product of the individual moments

c) both (d) none.

10) The covariance of two independent r.v.s is ( )

a) zero (b) one (c) two (d) none.

11) Mgf of a r.v is used to generate moments about __________ of the r.v. [ ]

a) mean (b) origin (c) infinite (d) none

12) characteristic function is used to find moments about __________ of a r.v [ ]

a) origin (b) mean (c) both (d) none.

13) If the PDF of a r.v is symmetric about the the origin then its mgf is __________ [ ]

a) asymmetric (b) general (c) symmetric (d) none

14) m_{n0} = __________ [ ]

a) n^{th} moment of x (c) both

b) n^{th} moment of y (d) none

15) m_{11} is called as [ ]

a) correlation (b) covariance (c) variance (d) none

16) If $R_{xy} = 0$ then x & y are [ ]

a) independent b) orthogonal c) both d) none

17) If $R_{xy} = E[x]\,E[y]$ then x & y are [ ]

a) Independent b) orthogonal c) both d) none

18) μ_{11} is called as [ ]

a) Correlation of x & y b) covariance of x & y

c) Variance of x d) Variance of y

18) If x & y are independent RVs then the density of $z = x + y$ is the ______ of the individual densities of x & y [ ]

a) Convolution b) Fourier transform

c) Addition d) multiplication.

19) If x & y are two Gaussian RVs with variances σ_x^2 & σ_y^2 then RVs $V = x + Ky$, $W = x - Ky$ are statistical independent for K equal to

a) $\sigma_x \sigma_y$ b) $\frac{\sigma_x}{\sigma_y}$ c) $\frac{\sigma_y}{\sigma_x}$ d) $\sigma_x + \sigma_y$

20) x & y are Gaussian RVs with the same variance and $\rho_{xy} = -1$ the angle θ of a coordinate rotation that generate ~~non~~ RV's that are statistically independent is [ ]

a) $\frac{\pi}{2}$ b) $-\frac{\pi}{2}$ c) $-\frac{\pi}{4}$ d) none

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Unit - 4

Random Process - Temporal characteristics

Random Process:

The random process is defined as an ensemble a collection of functions of time, with prob/. measure associated. It is usually denoted by $x(t, \lambda)$ or simple $x(t)$

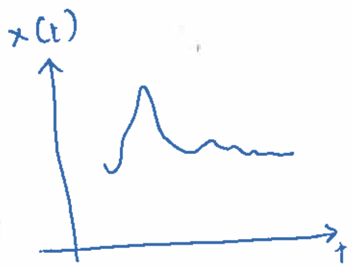
If λ is fixed, the r.p is a fn of time only. It is called single time fn.

If t is fixed, the rp is a fn of λ only & hence it will represent a rv at time t.

If both t & λ are fixed then $x(t, \lambda)$ is a merly a number

Classification of Random process:

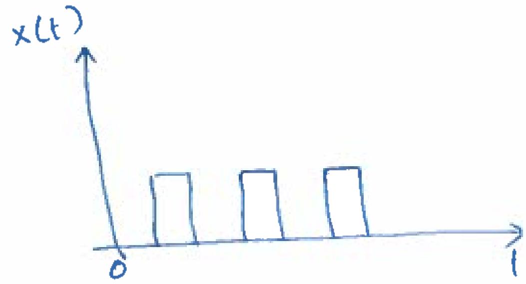
Continuous rp:



A rp is said to be continuous if both the rv 'x' & time 't' are continuous over the entire time.

eg The fluctuations of a noise voltage in any n/w is a continuous process.

Discrete rp:



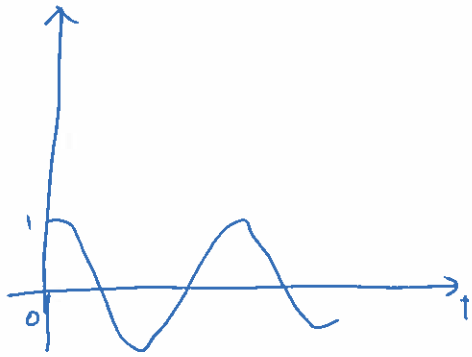
In a discrete rp, the rv 'x' has only discrete values while time 't' is continuous.

eg : The voltage available at one end of the switch because of random opening & closing of the switch is a discrete

Deterministic R.P:

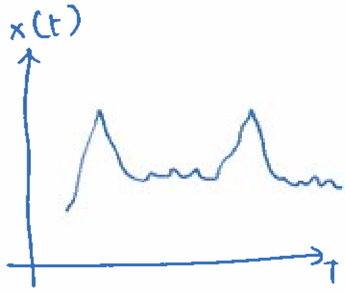
If the future values of any sample fn can be predicted from a knowledge of the past values, then the R.P is called deterministic R.P

$x(t) = A \cos \omega t$



Non deterministic R.P:

If the future values of a sample fn cannot be predicted from the knowledge of the past values then the R.P is called non deterministic R.P



eg: the fluctuations of noise voltage in any n/w

Distribution & density Functions:

For any given time t_1, the distribution fn associated with a R.P X_1 is given by

$$F_X(x_1, t_1) = P[X(t_1) \leq x_1]$$

This is called as first order distribution function or distribution fn of the R.P $X(t_1)$

For two R.P $X(t_1)$ & $X(t_2)$ the second order joint distribution fn is given by

$$F_X(x_1, x_2, t_1, t_2) = P[X(t_1) \leq x_1, X(t_2) \leq x_2]$$

The Nth order joint distribution fn of N - number of R.P $X(t_1), X(t_2) \dots X(t_N)$ is given by

$$F_x(x_1, x_2 \ldots x_N, t_1, t_2 \ldots t_N) = P[x(t_1) \leq x_1, x(t_2) \leq x_2 \ldots x(t_N) \leq x_N]$$

The derivative of the first order distribution fn wrt x_1 is called as first order density fn.

$$f_x(x_1, t_1) = \frac{d F_x(x_1, t_1)}{dx_1}$$

For two RP $x(t_1)$ & $x(t_2)$ the second order joint density is given by

$$f_x(x_1, x_2, t_1, t_2) = \frac{\partial^2 F_x(x_1, x_2, t_1, t_2)}{\partial x_1 \partial x_2}$$

the N^{th} order joint density fn is written as

$$f_x(x_1, x_2 \ldots x_N, t_1, t_2 \ldots t_N) = \frac{\partial^n F_x(x_1, x_2 \ldots x_N, t_1, t_2 \ldots t_N)}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

Statistical Independance:

Two RP $x(t)$ & $y(t)$ are said to be statistically independent process if the group of RV's $x(t_1), x(t_2) \ldots x(t_N)$ are independent of the group of RV's $y(t_1'), y(t_2') \ldots y(t_M')$ for any choice of times $t_1, t_2 \ldots t_N$ & $t_1', t_2' \ldots t_M'$

i.e $f_{xy}(x_1, x_2 \ldots x_N, y_1, y_2 \ldots y_M; t_1, t_2 \ldots t_N, t_1', t_2' \ldots t_M')$

$$= f_x(x_1, x_2 \ldots x_N, t_1, t_2 \ldots t_N) f_y(y_1, y_2 \ldots y_M, t_1', t_2' \ldots t_M')$$

Stationary R.P:

A R.P $x(t)$ is said to stationary if all the statistical properties of the R.P (i.e mean, moments & Variance) are not affected by a shift in the time.

eg. This mean that the R.P $x(t)$ & $x(t+\tau)$ posses the same statistical properties for any value of τ.

First order stationary process:

A R.P is called stationary to first order if its first order density fn does not change with a shift in the time origin.

i.e $f_x(x_1, t_1) = f_x(x_1, t_1+\tau)$ for any value of τ

∴ the condition for a process to first order stationary is that its mean value must be constant at any time

i.e $E[x(t)] = \bar{X} =$ Constant.

Second order stationary process:

A R.P is called stationary to order two, if its second order density fn does not change with shift in the time

i.e $f_x(x_1, x_2, t_1, t_2) = f_x(x_1, x_2, t_1+\tau, t_2+\tau)$ for all values of t_1, t_2 & τ

The condition for a process to be second order stationary is that its autocorrelation fn ~~depts~~ should depend only on time differences and not on absolute time.

i.e $R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$

$\tau = t_2 - t_1$

$t_1 = t$

$$R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)] = R_{xx}(\tau)$$

All second order stationary process are also first order stationary r.p.

Wide Sense stationary process:

A r.p $x(t)$ is called wide-sense stationary (wss) process or weak sense stationary process if it satisfies the following condition.

1) the mean value of the process is constant

i.e $E[x(t)] = \bar{x} = \text{Constant}$

2) Its auto correlation fn depends only on τ and not on time 't'.

i.e $E[x(t)\, x(t+\tau)] = R_{xx}(\tau)$ is independent of t

Two r.p $x(t)$ & $y(t)$ are said to be jointly wss if

1) $E[x(t)] = \bar{x} = \text{Constant}$

2) $E[y(t)] = \bar{y} = \text{Constant}$

3) $E[x(t)\, y(t+\tau)] = R_{xy}(\tau)$ is independent of t

Strict Sense Stationary process:

A Rp $x(t)$ is said to be strict sense stationary if its N^{th} order joint density fn does not change with time or shift in time value.

i.e

$$f_x(x_1, x_2 \ldots x_N, t_1, t_2 \ldots t_N) = f_x(x_1, x_2 \ldots x_N, t_1+\tau, t_2+\tau, \ldots t_N+\tau)$$

for all $t_1, t_2 \ldots t_N$ & τ.

A strict sense stationary is also called an N^{th} order stationary process or stationary to order N.

A strict sense stationary (SSS) is also WSS process but not vice versa.

Statistical averages:

The two statistical averages mostly used in the description of a Rp are mean & auto correlation fn.

The mean $m(t)$ of the Rp $x(t)$ is the expected value of the RV 'x' at time & is given by

$$m(t) = E[x(t)] = \int_{-\infty}^{\infty} x \, f_x(x, t) \, dx$$

the auto correlation $R_{x_1 x_2}(t_1, t_2)$ of Rp $x(t)$ is the expected value of the Product $x(t_1)\, x(t_2)$ & is given by

$$R_{x_1 x_2}(t_1, t_2) = E[x(t_1)\, x(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \, f_{x_1 x_2}(x_1, x_2, t_1, t_2) \, dx_1$$

Time averages:

The time averaged mean is defined as

$$\langle m_x \rangle = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T/2}^{T/2} x(t)\, dt$$

& the time averaged auto correlation fn is given by

$$\langle R_{xx}(\tau) \rangle = \underset{T\to\infty}{Lt} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$$

the time averaged cross correlation fn is given by

$$\langle R_{xy}(\tau) \rangle = \underset{T\to\infty}{Lt} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y(t+\tau)\, dt$$

Ergodic RP:

A RP is said to be ergodic if the time averages of the process are equal to the ensemble averages.

Mean Ergodic:

A RP $x(t)$ is said to be mean ergodic or ergodic in mean if the time average of any sample fn $x(t)$ is equal to its statistical average $\bar{x}$ which is constant and the prob of all other sample fns is equal to one.

i.e $$E[x(t)] = \bar{x} = \underset{T\to\infty}{Lt} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

Correlation Ergodic process:

A stationary rp $x(t)$ is said to be correlation ergodic process if the statistical autocorrelation is equal to the time average auto correlation fn.

i.e $R_{xx}(\tau) = E[x(t)\,x(t+\tau)] = \langle R_{xx}(\tau) \rangle$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$$

Auto Correlation Function:

The autocorrelation provides a measure of similar b/w a given signal & the replica of the same signal delayed by a particular time.

Consider two r.v.s $x(t_1)$ & $x(t_2)$ obtained by observing a rp $x(t)$ at times t_1 & t_2 then the auto correlation $R_{x_1x_2}(t_1, t_2)$ of the rp $x(t)$ is defined as the expected value of the product $x(t_1) \cdot x(t_2)$

$$R_{XX}(t_1, t_2) = E[x(t_1)\, x(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2, t_1, t_2)\, dx_1 dx_2$$

Let $t_1 = t$ & $t_2 - t_1 = \tau \Rightarrow t_2 = t + \tau$

$$R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$$

For a stationary to order two or more the autocorrelation fn depends only τ & not on the time t

i.e $R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)] = R_{xx}(\tau)$

Properties of Auto Correlation:

The auto correlation of a wss RP has the following properties:

1) The mean square value of $x(t)$ is

$$E[x^2(t)] = R_{xx}(0)$$

It is equal to the power of the process $x(t)$.

Pf: $R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

If $\tau = 0$ $\quad R_{xx}(0) = E(x(t)\, x(t)) = E[x^2(t)]$

2) The auto correlation fn $R_{xx}(\tau)$ is an even fn of τ i.e

$$R_{xx}(\tau) = R_{xx}(-\tau).$$

Pf: $R_{xx}(\tau) = E(x(t)\, x(t+\tau))$

$R_{xx}(-\tau) = E[x(t)\, x(t-\tau)]$ $\quad$ let $t-\tau = a$

$= E[x(\tau+a)\, x(a)]$ $\quad$ let $a = t$

$= E(x(t)\, x(t+\tau)) = R_{xx}(\tau)$

3) The auto correlation fn evaluated at the origin $(\tau=0)$ will be its max magnitude & it will be equal to or greater than evaluated at any other time i.e

i.e $\quad R_{xx}(0) \geq |R_{xx}(\tau)|$

Pf: Consider $E\left[\left[x(t_1) \pm x(t_2)\right]^2\right] \geq 0$

$$E\left[x^2(t_1) + x^2(t_2) \pm 2x(t_1)\, x(t_2)\right] \geq 0$$

$$E\left(x^2(t_1)\right) + E\left(x^2(t_2)\right) \pm 2E\left[x(t_1)\, x(t_2)\right] \geq 0$$

$$R_{xx}(0) + R_{xx}(0) \pm 2R_{xx}(t_1, t_2) \geq 0$$

$t_1 = t$
$t_2 - t_1 = \tau$
$t_2 = t + \tau$

$$2R_{xx}(0) \pm 2R_{xx}(t, t+\tau) \geq 0$$

$$2R_{xx}(0) \pm 2R_{xx}(\tau) \geq 0$$

$$\therefore \quad R_{xx}(0) \leq \mp R_{xx}(\tau)$$

$$R_{xx}(0) \leq |R_{xx}(\tau)|$$

4) the auto correlation fn of a rp $x(t)$ is a finite energy fn i.e

$$R_{xx}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, d\tau < \infty$$

5) the auto correlation of a rp cannot have an arbitary shape.

6) If $x(t)$ is periodic then its ~~od~~ auto correlation is also periodic.

1. If $x(t)$ is periodic with period T_0

i.e $x(t) = x(t \pm T_0)$

$$x(t+\tau) = x(t+\tau \pm T_0)$$

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

$$R_{xx}(\tau \pm T_0) = E[x(t)\, x(t+\tau \pm T_0)]$$

$$= E[x(t)\, x(t \pm T_0 + \tau)]$$

$$= E[x(t)\, x(t+\tau)]$$

$$= R_{xx}(\tau)$$

7) If a r.p $x(t)$ has no periodic component & if $E[x(t)] = \bar{x} \neq 0$ then $\lim_{\tau \to \infty} R_{xx}(\tau) = \bar{x}^2$

Sol: $R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

as $\tau \to \infty$ $x(t)$ & $x(t+\tau)$ are independent

$$\lim_{\tau \to \infty} R_{xx}(\tau) = \lim_{\tau \to \infty} E[x(t)\, x(t+\tau)]$$

$$= E[x(t)]\, E[x(t+\tau)]$$

$$= \bar{x} \cdot \bar{x} = \bar{x}^2$$

8) If the r.p $x(t)$ is ergodic, zero mean & has no periodic component then

$$\lim_{\tau \to \infty} R_{xx}(\tau) = 0$$

Sol: $$\lim_{\tau \to \infty} R_{xx}(\tau) = \lim_{\tau \to \infty} E[x(t)\, x(t+\tau)] = \lim_{\tau \to \infty} E[x(t)]\, E[x(t+\tau)]$$

$$= 0$$

Cross Correlation Function:

The cross correlation fn of two r.p is defined as the measure of similarity b/w a given signal & a time delayed version of second signal.

If $x(t)$ & $y(t)$ are two r.p then the cross correlation $R_{xy}(t_1, t_2)$ is the expected value of the product $x(t_1)\, y(t_2)$

i.e $$R_{xy}(t_1, t_2) = E[x(t_1)\, y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{xy}(x, y, t_1, t_2)\, dx\, dy$$

let $t_1 = t$ $\quad t_2 - t_1 = \tau \Rightarrow t_2 = t + \tau$

$$R_{xy}(t, t+\tau) = E[x(t)\, y(t+\tau)]$$

If $x(t)$ & $y(t)$ are jointly wss RP then

$$R_{xy}(t, t+\tau) = R_{xy}(\tau)$$

$$\therefore R_{xy}(\tau) = E[x(t)\, y(t+\tau)].$$

Properties of Cross Correlation:

1) The cross correlation fn is not generally an even fn of τ but it has symmetry relationship of

$$R_{xy}(\tau) = R_{yx}(-\tau)$$

A: $R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

$$R_{yx}(\tau) = E[y(t)\, x(t+\tau)]$$

$$R_{yx}(-\tau) = E[y(t)\, x(t-\tau)] \qquad \text{let } t-\tau = a$$

$$= E[y(a+\tau)\, x(a)] \qquad a = t$$

$$= E[x(t)\, y(t+\tau)] = R_{xy}(\tau)$$

2) The cross correlation fn of two RP $x(t)$ & $y(t)$ does not have a max value at its origin i.e at $\tau = 0$.

3) If $x(t)$ & $y(t)$ are two RP & $R_{xx}(\tau)$ & $R_{yy}(\tau)$ are their respective auto correlation fns then the relation

$$|R_{xy}(\tau)| \le \sqrt{R_{xx}(0)\, R_{yy}(0)}$$ holds good.

A. $$E\left[\left\{\frac{x(t_1)}{\sqrt{R_{xx}(0)}} - \frac{y(t_2)}{\sqrt{R_{yy}(0)}}\right\}^2\right] \geq 0$$

$$E\left[\frac{x^2(t_1)}{R_{xx}(0)}\right] + \frac{E\left[y^2(t_2)\right]}{R_{yy}(0)} - \frac{2E\left[x(t_1)\,y(t_2)\right]}{\sqrt{R_{xx}(0)\,R_{yy}(0)}} \geq 0$$

$$\frac{R_{xx}(0)}{R_{xx}(0)} + \frac{R_{yy}(0)}{R_{yy}(0)} - \frac{2R_{xy}(t_1,t_2)}{\sqrt{R_{xx}(0)\,R_{yy}(0)}} \geq 0$$

$$2 \geq \frac{2R_{xy}(t_1,t_2)}{\sqrt{R_{xx}(0)\,R_{yy}(0)}}$$

$t_1 = t$

$t_2 - t_1 = \tau$

$t_2 = t+\tau$

$$\sqrt{R_{xx}(0)\,R_{yy}(0)} \geq R_{xy}(t,t+\tau) \quad \text{for WSS}$$

$$\sqrt{R_{xx}(0)\,R_{yy}(0)} \geq R_{xy}(\tau)$$

$$\therefore \left|R_{xy}(\tau)\right| \leq \sqrt{R_{xx}(0)\,R_{yy}(0)}$$

4) If $x(t)$ & $y(t)$ are two r.p ~~the~~ & if $R_{xx}(\tau)$ $R_{yy}(\tau)$ are their respective auto correlation fn's then the relation

$$\left|R_{xy}(\tau)\right| \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]$$ holds good.

1: Geometric mean of two +ve quantities cannot exceed their arithmetic mean

i.e $$\sqrt{R_{xx}(0)\,R_{yy}(0)} \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]$$

$$\therefore \left|R_{xy}(\tau)\right| \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]$$

5) If the r.p $x(t)$ & $y(t)$ are independent to each other then their cross correlation ~~statis~~ satisfy the relation

$$R_{xy}(\tau) = R_{yx}(\tau) = \bar{x}\,\bar{y}.$$

S: $R_{xy}(\tau) = E[x(t)\,y(t+\tau)] = E[x(t)]\,E[y(t+\tau)] = \bar{x}\cdot\bar{y}$

$R_{yx}(\tau) = E[y(t)\,x(t+\tau)] = E[y(t)]\,E[x(t+\tau)] = \bar{y}\cdot\bar{x}$

$\therefore\ R_{xy}(\tau) = R_{yx}(\tau) = \bar{x}\cdot\bar{y}$

6) If the RP $x(t)$ & $y(t)$ have periodic component with a period T_0 then their cross correlation fns $R_{xy}(\tau)$ & $R_{yx}(\tau)$ will also contain the same periodic components with the same period T_0.

S: $x(t) = x(t \pm T_0)$ $\qquad y(t) = y(t \pm T_0)$

$x(t+\tau) = x(t+\tau \pm T_0)$ $\qquad y(t+\tau) = y(t+\tau \pm T_0)$

$R_{xy}(\tau) = E[x(t)\,y(t+\tau)]$

$R_{xy}(\tau \pm T_0) = E[x(t)\,y(t+\tau \pm T_0)]$

$= E[x(t)\,y(t \pm T_0 + \tau)] = E[x(t)\,y(t+\tau)]$

$= R_{xy}(\tau)$

$R_{yx}(\tau \pm T_0) = E[y(t)\,x(t+\tau)]$

$= E[y(t)\,x(t+\tau \pm T_0)]$

$= E[y(t)\,x(t \pm T_0 + \tau)] = E[y(t)\,x(t \pm T_0$

$= R_{yx}(\tau)$

7) If the mean of two RP $x(t)$ & $y(t)$ are zero then $\lim_{\tau \to \infty} R_{xy}(\tau) = \lim_{\tau \to \infty} R_{yx}(\tau) = 0$

1: $\underset{\tau\to\infty}{\mathrm{Lt}}\ R_{xy}(\tau) = \underset{\tau\to\infty}{\mathrm{Lt}}\ E[x(t)\, y(t+\tau)]$

$$= \underset{\tau\to\infty}{\mathrm{Lt}}\ E[x(t)]\ E[y(t+\tau)] = \bar{x}\,\bar{y} = 0$$

$$\underset{\tau\to\infty}{\mathrm{Lt}}\ R_{yx}(\tau) = \underset{\tau\to\infty}{\mathrm{Lt}}\ E[y(t)\, x(t+\tau)]$$

$$= \underset{\tau\to\infty}{\mathrm{Lt}}\ E[y(t)]\ E[x(t+\tau)] = \bar{y}\cdot\bar{x} = 0$$

$$\therefore \underset{\tau\to\infty}{\mathrm{Lt}}\ R_{xy}(\tau) = \underset{\tau\to\infty}{\mathrm{Lt}}\ R_{yx}(\tau) = 0$$

Auto Covariance:

The auto covariance $C_{xx}(t_1, t_2)$ of the r.p $x(t)$ is defined as the covariance of the r.v.s $x(t_1)$ & $x(t_2)$

$$C_{xx}(t_1, t_2) = E\Big[\{x(t_1) - E[x(t_1)]\}\{x(t_2) - E[x(t_2)]\}\Big]$$

$$= E[x(t_1)\, x(t_2)] - E[x(t_1)]\, E[x(t_2)]$$

$$= R_{xx}(t_1, t_2) - E[x(t_1)]\, E[x(t_2)]$$

Let $t_1 = t, \quad t_2 - t_1 = \tau$

$$C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - E[x(t)]\, E[x(t+\tau)]$$

$\tau = 0$

$$C_{xx}(t, t) = R_{xx}(t, t) - E[x(t)]\, E[x(t)]$$

$$= E[x^2(t)] - E[x(t)]^2$$

$$= \mathrm{Var}[x(t)]$$

Cross Covariance:

The cross covariance for two RP $x(t)$ & $y(t)$ is defined as

$$C_{xy}(t, t+\tau) = E\{[x(t) - \overline{x(t)}][y(t) - \overline{y(t+\tau)}]\}$$

$$= R_{xy}(t, t+\tau) - E[x(t)]\,E[y(t+\tau)]$$

for a WSS RP

$$C_{xy}(\tau) = R_{xy}(\tau) - \bar{x}\,\bar{y}$$

If $C_{xy}(\tau) = 0$ then x & y are uncorrelated & independent

If x & y are orthogonal then $R_{xy}(\tau) = 0$

$$\therefore C_{xy}(\tau) = -\bar{x}\,\bar{y}.$$

Gaussian Random Process:

Consider a continuous RP $x(t)$ & let N RV's be defined $x_1 = x(t_1)$, $x_2 = x(t_2)$ ... $x_i = x(t_i)$ $x_N = x(t_N)$ corresponding to N time instants $t_1, t_2 \ldots t_N$

If for any $N = 1, 2 \ldots$ and any times $t_1, t_2 \ldots t_N$ these RV's are jointly gaussian i.e they have jointly gaussian i.e they have a joint density given by

$$f_{x_1, x_2 \ldots x_N}(x_1, x_2 \ldots x_N) = \frac{|[C_x]^{-1}|^{1/2}}{(2\pi)^{N/2}} e^{\frac{-[x-\bar{x}]^t [C_x]^{-1} [x-\bar{x}]}{2}}$$

$$[x - \bar{x}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{x}_N \end{bmatrix} \quad \& \quad [C_x] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

$$\bar{x_i} = E[x_i] = E[x(t_i)]$$

$$C_{ik} = C_{x_i x_k} = E[(x_i - \bar{x_i})(x_k - \bar{x_k})]$$

$$= E[\{x(t_i) - E[x(t_i)]\}\{x(t_k) - E[x(t_k)]\}]$$

$$= C_{xx}(t_i, t_k)$$

$$C_{xx}(t_i, t_k) = R_{xx}(t_i, t_k) - E[x(t_i)]E[x(t_k)]$$

for a WSS R P mean will be constant

$$x_i = E[x(t_i)] = \bar{x} = \text{Constant}$$

& $C_{xx}(t_i, t_k) = C_{xx}(t_k - t_i)$

$$R_{xx}(t_i, t_k) = R_{xx}(t_k - t_i)$$

A WSS gaussian process is also strictly stationary

Poisson Random process:

An important example of DRP is poisson process. It describes the no of times that some event has occured as fn of time, where the event occurs at random times.

Poisson process is also known as poisson counting process

To define poisson process we require that

1) the event occurs only once in any vanishingly small interval of time.

2) no of events that occur in any given time interval is independent of the no of events in any other non overlapping time intervals

The Prob of exactly k occurences over a time interval $(0,t)$ is

$$P\left[x(t)=k\right] = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad k=0,1,2,\ldots$$

& the prob density fn is

$$f_x(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\,\delta(x-k)$$

mean value of poisson density fn is

$$E\left[x(t)\right] = \int_{-\infty}^{\infty} x\, f_x(x)\,dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\,\delta(x-k)$$

$$= \sum_{k=0}^{\infty} \frac{k\,(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

$$E\left[x^2(t)\right] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\,\delta(x-k)$$

$$= \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$= \lambda t + (\lambda t)^2 = \lambda t + \lambda^2 t^2$$

$$\mathrm{Var}\left[x(t)\right] = E\left[x^2(t)\right] - E\left[x(t)\right]^2 = \lambda t + \lambda^2 t^2 - (\lambda t)^2$$

$$= \lambda t$$

The joint density fn for the poisson process at times $0 < t_1 < t_2$ is

$$f_x(x_1,x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(k_1,k_2)\,\delta(x-k_1)\,\delta(x-k_2)$$

$$P(k_1, k_2) = \frac{(\lambda t_1)^{k_1} \left[\lambda (t_2 - t_1)\right]^{k_2 - k_1} e^{-\lambda t_2}}{k_1! \, (k_2 - k_1)!}$$

Q: Prove the r.p $x(t) = A \cos(\omega_c t + \theta)$ is WSS if it is assumed that A & ω_c are constants & θ is a uniformly distributed on the interval $(0, 2\pi)$.

A: For a r.p to WSS it has to satisfy two conditions

a) $E[x(t)]$ is a constant

b) $R_{xx}(\tau)$ depends on τ & not on t.

$$f(\theta) = \frac{1}{2\pi - 0} \quad 0 \le \theta \le 2\pi$$

$$= \frac{1}{2\pi} \quad 0 \le \theta \le 2\pi$$

$$E[x(t)] = \int_{-\infty}^{\infty} x \, f_x(x) \, dx$$

$$= \int_0^{2\pi} \frac{1}{2\pi} A \cos(\omega_c t + \theta) \, d\theta$$

$$= \frac{A}{2\pi} \sin(\omega_c t + \theta) \Big|_0^{2\pi}$$

$$= \frac{A}{2\pi} \left[\sin \omega_c t - \sin \omega_c t\right] = 0 = \text{Constant}$$

$$R_{xx}(t, t+\tau) = E[x(t) \, x(t+\tau)]$$

$$= E\left[A \cos(\omega_c t + \theta) \, A \cos(\omega_c t + \omega_c \tau + \theta)\right]$$

$$= \frac{A^2}{2} E\left[\cos(2\omega_c t + \omega_c \tau + 2\theta) + \cos \omega_c \tau\right]$$

$$= \frac{A^2}{2} \left[\int_0^{2\pi} \frac{1}{2\pi} \cos(2\omega_c t + \omega_c \tau + 2\theta) \, d\theta + \int_0^{2\pi} \frac{1}{2\pi} \cos \omega_c \tau \, d\theta\right]$$

$$= \frac{A^2}{4\pi} \cos \omega_c \tau \, [\theta]_0^{2\pi}$$

$$= \frac{A^2}{} \cos \omega_c \tau$$

thus $R_{xx}(\tau)$ depends only on τ & not on time 't'

thus the second condition is also satisfied

$\therefore$ the given rp is a wss rp.

Q: Consider a rp $x(t) = \cos(\omega t + \theta)$ where ω is a constant & θ is a rv with a probl. density $P(\theta) = \frac{1}{2\pi}$ $0 \le \theta \le 2\pi$; 0 else

Prove that $x(t)$ is an ergodic r.p.

A: $m(t)$ = mean = $E[x(t)] = 1$

auto correlation = $R_{xx}(\tau) = \frac{1}{2}\cos\omega_c\tau$ } Ensemble averages

Time averages:

$$\langle m_T \rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos(\omega t + \theta)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left.\frac{\sin(\omega t + \theta)}{\omega}\right|_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \frac{\sin(\omega T + \theta) - \sin(-\omega T + \theta)}{\omega}$$

$$= 0$$

$$\langle R_{xx}(\tau) \rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{\cos(2\omega t + \omega\tau + 2\theta) + \cos\omega\tau}{2}\, dt$$

$$= \lim_{T\to\infty} \frac{1}{4T}\int_{-T}^{T} \cos(2\omega t + \omega\tau + 2\theta)\, dt$$

$$+ \lim_{T\to\infty} \frac{1}{4T}\int_{-T}^{T} \cos\omega\tau\, dt$$

$$= \lim_{T\to\infty} \frac{1}{4T}\cos\omega\tau\, [t]_{-T}^{T} = \frac{1}{2}\cos\omega\tau$$

Since the ensemble averages & time averages are equal the given rp is an ergodic rp.

Q: Prove that the rp $x(t) = A \cos(\omega_c t + \theta)$ is not stationary if it is assumed that A & ω_c are constants & θ is a uniformly distributed variable on the interval $(0, \pi)$

A $E[x(t)] = \int_0^{\pi} \frac{1}{\pi} A \cos(\omega_c t + \theta)\, d\theta$

$$= \frac{A}{\pi} \int_0^{\pi} \cos(\omega_c t + \theta)\, d\theta$$

$$= \frac{A}{\pi} \sin(\omega_c t + \theta) \Big|_0^{\pi}$$

$$= \frac{A}{\pi} \left[-\sin \omega_c t - \sin \omega_c t \right]$$

$$= -\frac{2A}{\pi} \sin \omega_c t$$

Since $E[x(t)]$ is a fn of 't' the rp $x(t)$ is not a stationary rp.

Q: find the mean & Variance of a stationary rp whose auto correlation fn is given by $R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}$

A: $\lim_{\tau \to \infty} R_{xx}(\tau) = \bar{x}^2$

$$\bar{x}^2 = \lim_{\tau \to \infty} 18 + \frac{2}{6+\tau^2} = 18$$

$$\bar{x} = \sqrt{18} = 3\sqrt{2}$$

$$E[x^2(t)] = \overline{x^2} = R_{xx}(0) = 18 + \frac{2}{6} = \frac{55}{3}$$

$$\sigma^2 = \overline{x^2} - \bar{x}^2 = \frac{55}{3} - 18 = 1/3$$

Q: The autocorrelation fn of a WSS RP is given by $R_{xx}(\tau) = \dfrac{4\tau^2+100}{\tau^2+4}$ find the mean & variance of this process.

A: $\lim_{\tau\to\infty} R_{xx}(\tau) = \bar{X}^2$

$$\bar{X} = \sqrt{\lim_{\tau\to\infty} \frac{4\tau^2+100}{\tau^2+4}} = \sqrt{4} = 2$$

$$\overline{X^2} = R_{xx}(0) = \frac{100}{4} = 25$$

$$\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 25 - 4 = 21$$

Q: Assume that $x(t)$ is a WSS RP with auto correlation $R_{xx}(\tau) = e^{-\alpha|\tau|}$ determine the 2nd moment of the RV $x(8)-x(5)$.

A: $E\left[\{x(8) - x(5)\}^2\right] = E\left[x^2(8)\right] + E\left[x^2(5)\right] - 2E\left[x(5)x(5\right.$

$$= R_{xx}(0) + R_{xx}(0) - 2R_{xx}(3)$$

$$= 1+1+-2e^{-\alpha|3|} = 2\left(1-e^{-\alpha|3|}\right)$$

Q: If $x(t)$ is a RP with mean 3 & auto correlation of $9+4e^{-0.2|t_1-t_2|}$ determine the mean, variance & covariance of RV's $z = x(5)$ & $\omega = x(8)$.

A. $\bar{z} = E[z] = E[x(5)] = 3$

$$\bar{\omega} = E[\omega] = E[x(8)] = 3$$

$$E[z^2] = E[x(5)x(5)] = R_{xx}(5,5) = 9+4e^{-0.2|5-5|}$$

$$= 9+4 = 13$$

$$E[w^2] = E[x(8)\,x(8)] = R_{xx}(8,8) = 9+4e^{-0.2|8-8|}$$

$$= 9+4 = 13$$

$$\sigma_z^2 = \overline{z^2} - \bar{z}^2 = 13-(3)^2 = 4$$

$$\sigma_w^2 = \overline{w^2} - \bar{w}^2 = 13-(3)^2 = 4$$

$$R_{zw} = E[zw] = E[x(5)\,x(8)] = 9+4e^{-0.2|8-5|}$$

$$= 9+4e^{-0.6}$$

$$\text{cov} = R_{zw} - \bar{z}\,\bar{w} = 9+4e^{-0.6} - (3)(3) = 4e^{-0.6} = 2.195$$

Q: If the RP $x(t) = \sin(\omega t + y)$ where y is a RV uniformly distributed in the interval $(0, 2\pi)$. P.T for the process $x(t)$ $\quad C(t_1,t_2) = R(t_1,t_2) = \dfrac{\cos\omega(t_1-t_2)}{2}$

S: $$E[x(t_1)] = \int_0^{2\pi} \frac{1}{2\pi}\sin(\omega t_1 + y)\,dy$$

$$= -\frac{1}{2\pi}\cos(\omega t_1 + y)\Big|_0^{2\pi}$$

$$= -\frac{1}{2\pi}\left[\cos(\omega t_1 + 2\pi) - \cos\omega t_1\right]$$

$$= -\frac{1}{2\pi}\left[\cos\omega t_1 - \cos\omega t_1\right] = 0$$

Illy $E[x(t_2)] = 0$

$$R_{xx}(t_1,t_2) = E[x(t_1)\,x(t_2)]$$

$$= E[\sin(\omega t_1 + y)\sin(\omega t_2 + y)]$$

$$= \frac{1}{2}E\left[\cos\omega(t_1-t_2) - \cos(\omega t_1 + \omega t_2 + 2y)\right]$$

$$= \frac{1}{2}\int_0^{2\pi}\frac{1}{2\pi}\cos\omega(t_1-t_2)\,dy - \frac{1}{2}\int_0^{2\pi}\frac{1}{2\pi}\cos(\omega t_1 + \omega t_2 + 2y)\,dy$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} \cos \omega (t_1 - t_2) \quad [4]_0^{2\pi}$$

$$= \frac{1}{2} \cos \omega (t_1 - t_2)$$

$$\text{Cov}(t_1, t_2) = R_{xx}(t_1, t_2) - E[x(t_1)]\, E[x(t_2)]$$

$$= \frac{1}{2} \cos \omega (t_1 - t_2)$$

Q. Consider a r.p $x_c(t) = A\,x(t) \cos(\omega_c t + \theta)$ where $x(t)$ is zero mean, stationary r.p $x(t)$ & θ are assumed to be independent A & ω_c are Constants. θ is a r.v uniformly distributed in the interval $(-\pi, \pi)$. Show that $x_c(t)$ is a wss r.p.

A. $E[x_c(t)] = E[A\,x(t) \cos(\omega_c t + \theta)]$

$$= A\, E[x(t)]\, E(\cos(\omega_c t + \theta))$$

$$= 0$$

$$R_{x_c x_c}(t, t+\tau) = E[x(t)\, x(t+\tau)]$$

$$= E[A\,x(t) \cos(\omega_c t + \theta) \cdot A\, x(t+\tau) \cos(\omega_c t + \omega_c \tau + \theta)]$$

$$= \frac{A^2}{2} E[x(t)\, x(t+\tau)]\, E[\cos \omega_c \tau + \cos(2\omega_c t + \omega_c \tau + 2\theta)]$$

$$= \frac{A^2}{2} R_{xx}(\tau) \cos \omega_c \tau$$

Since mean & auto correlation do not depend on 't' the process $x_c(t)$ is stationary in the wide sense.

Q: Consider two r.p $x(t) = A \cos \omega t + B \sin \omega t$ & $y(t) = B \cos \omega t - A \sin \omega t$ where A & B are uncorrelated zero mean r.v.s with the same variance & ω is a

4.25

Constant. Show that $x(t)$ & $y(t)$ are jointly WSS process.

A: $E[AB]=0 \qquad E[A^2] = E[B^2] = \sigma^2$

$R_{xy}(t,t+\tau) = E[x(t)\, y(t+\tau)]$

$= E\left[\left(A \cos\omega t + B \sin\omega t\right)\left(B \cos(\omega t+\omega\tau) - A \sin(\omega t+\omega\tau)\right)\right]$

$= E(AB) \cos\omega t \cos(\omega t+\omega\tau) - E(A^2) \cos\omega t \sin(\omega t+\omega\tau)$
$+ E(B^2) \sin\omega t \cos(\omega t+\omega\tau) - E(AB) \sin\omega \sin(\omega t+\omega\tau)$

$= \sigma^2 \left[\sin\omega t \cos(\omega t+\omega\tau) - \cos\omega t \sin(\omega t+\omega\tau)\right]$

$= \sigma^2 \sin(\omega t - \omega t - \omega\tau)$

$= -\sigma^2 \sin\omega\tau$

Thus the cross correlation fn $R_{xy}(t,t+\tau)$ depends only on the term 'τ' and not on 't'

$\therefore$ two RP $x(t)$ & $y(t)$ are jointly WSS RP.

Q: For the RP $x(t) = A\cos\omega t + B\sin\omega t$ where A & B are RV's with $E[A] = E[B] = 0$, $E[A^2] = E[B^2] = 0$, $E[AB]=0$ prove that the process is mean ergodic.

A: $m(t) = E(x(t)) = E\left[A \cos\omega t + B \sin\omega t\right)$

$= E(A)\, E(\cos\omega t) + E(B)\, E(\sin\omega t) = 0$

$\langle m_T \rangle = \frac{1}{2T} \lim_{T\to\infty} \int_{-T}^{T} A \cos\omega t + B \sin\omega t \, dt$

$= \lim_{T\to\infty} \frac{1}{2T}\, 2\int_0^T A \cos\omega t \, dt = \lim_{T\to\infty} \frac{A}{T} \sin\omega t \Big|_0^T$

$= \lim \quad A \quad \sin\omega T \quad = 0$

Since $m(t) = \langle m_T \rangle$

$\therefore$ the given process $x(t)$ is mean-ergodic.

Q: Given that the RP $x(t) = 10\cos(100t + \phi)$ where ϕ is a uniformly distributed RV in the interval $(-\pi, \pi)$ S.T the process is correlation ergodic.

A: $R_{xx}(\tau) = E\left(x(t)\, x(t+\tau)\right)$

$$= E\left[10\cos(100t+\phi)\, 10\cos(100t + 100\tau + \phi)\right]$$

$$= 50\, E\left[\cos(200t + 100\tau + 2\phi) + \cos 100\tau\right]$$

$$= 50\cos 100\tau$$

$$\langle R_{xx}(\tau)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 10\cos(100t+\phi)\cdot 10\cos(100t + 100\tau + \phi)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 50\cos(200t + 100\tau + 2\phi)\, dt$$

$$+ 50 \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos 100\tau\, dt$$

$$= 50\cos 100\tau$$

Since $R_{xx}(\tau) = \langle R_{xx}(\tau)\rangle$

$\therefore$ the process is correlation ergodic

Q: Consider a RP $x(t) = \cos(\omega t + \theta)$ where ω is a real constant & θ is a uniformly distributed RV in $(0, \frac{\pi}{2})$ Show that $x(t)$ is not a WSS process and also find the average

1. $E[x(t)] = \int_0^{\pi/2} \frac{2}{\pi} \cos(\omega t+\theta)\, d\theta$

$= \frac{2}{\pi} \sin(\omega t+\theta)\Big|_0^{\pi/2}$

$= \frac{2}{\pi}\left[\cos\omega t - \sin\omega t\right]$

Since $E[x(t)]$ depends on time 't' $\therefore$ $x(t)$ is not WSS

$E[x^2(t)] = E\left(\cos^2(\omega t+\theta)\right) = \frac{E\left(1+\cos(2\omega t+2\theta)\right)}{2}$

$= \frac{1}{2} + \frac{1}{2}\cdot\frac{2}{\pi}\int_0^{\pi/2} \cos(2\omega t+2\theta)\, d\theta$

$= \frac{1}{2} + \frac{1}{\pi} \sin(2\omega t+2\theta)\Big|_0^{\pi/2}$

$= \frac{1}{2} - \frac{\sin 2\omega t}{\pi}$

$P_{xx} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E(x^2(t))\, dt$

$= \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-T}^{T} \frac{1}{2} - \frac{\sin 2\omega t}{\pi}\, dt\right]$

$= \lim_{T\to\infty} \frac{1}{2T}\,\frac{1}{2}\int_{-T}^{T} dt = \lim_{T\to\infty} \frac{1}{2T}\cdot\frac{1}{2}\,(t)_{-T}^{T}$

$= \frac{1}{2}$

Q: Consider two RP $x(t) = 3\cos(\omega t+\theta)$ & $y(t) = 2\cos(\omega t+\theta-\frac{\pi}{2})$ where θ is a RV with uniform density fn $\left(\frac{2\pi}{\omega}\right)\left(\frac{\omega}{2\pi}\right)$ P.T $R_{xx}(0)\,R_{yy}(0) \geq |R_{xy}(\tau)|$ also find the Condition under which $R_{xy}(\tau)$ is maximum.

A: $R_{xx}(\tau) = E[x(t)\,x(t+\tau)] = E[3\cos(\omega t+\theta)\cdot 3\cos(\omega t+\omega\tau+\theta)]$

$$= \frac{9}{2} \cos \omega\tau$$

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)] = E\left(2\cos\left(\omega t+\theta-\frac{\pi}{2}\right)\cdot 2\cos\left(\omega t+\omega\tau+\theta-\frac{\pi}{2}\right)\right)$$

$$= 2\cos\omega\tau$$

$$R_{XX}(0) = \frac{9}{2} \qquad R_{YY}(0) = 2$$

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E\left(3\cos(\omega t+\theta)\cdot 2\cos(\omega t+\omega\tau+\theta)\right)$$

$$= 3\int_0^{2\pi/\omega} \frac{\omega}{2\pi}\left(\cos(2\omega t+\omega\tau+2\theta)+\cos\omega\tau\right)d\theta$$

$$= 3\cos\omega\tau$$

$$|R_{XY}(\tau)| = |3\cos\omega\tau| \le \sqrt{\frac{9}{2}\cdot 2} \qquad \le 3$$

$$|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,R_{YY}(0)}$$

the cross correlation fn $R_{XY}(\tau)$ is maximum if

$$3\cos\omega\tau = 3 \Rightarrow \cos\omega\tau = 1$$

$$\omega\tau = 2\pi n \Rightarrow \tau = \frac{2\pi n}{\omega}$$

Q. A RP is given as $X(t) = A$ where A is a cRV uniformly distributed on $(0,1)$ show that $X(t)$ is stationary process.

A. $f(A) = 1 \qquad 0 \le A \le 1$

$$E[X(t)] = \int_0^1 X(t)\,f(A)\,dA = \int_0^1 A\,dA = \left.\frac{A^2}{2}\right|_0^1 = \frac{1}{2}$$

$$= \text{Constant}$$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \int_0^1 A\cdot A\,dA = \left.\frac{A^3}{3}\right|_0^1 = \frac{1}{3}$$

$\therefore X(t)$ is a stationary process

Q: Let N(t) be a zero mean WSS noise process for which $R_{NN}(\tau) = \frac{N_0}{2}\delta(\tau)$, $N_0 > 0$ determine whether N(t) is mean ergodic

A:
$$E[N(t)] = 0 \qquad E[N^2(t)] = R_{NN}(0) = \frac{N_0}{2}$$

$$\sigma_N^2 = E[N^2(t)] - E[N(t)]^2 = \frac{N_0}{2} - 0 = \frac{N_0}{2}$$

A RV N(t) is said to be mean ergodic if $\sigma_N^2 = 0$

$\therefore$ N(t) is not mean ergodic.

Q: Given $\bar{X} = 6$ & $R_{XX}(t, t+\tau) = 36 + 25e^{-|\tau|}$ for a RP X(t)
Indicate which of the following statements are true based on what is known with certainty X(t)

a) is 1st order stationary
b) has total average power of 61W
c) is ergodic
d) is WSS
e) has a periodic component
f) has an AC power of 36W.

A:
$$\bar{X} = 6, \quad R_{XX}(\tau) = 36 + 25e^{-|\tau|}$$

a) Since mean $\bar{X} = 6$ = Constant

X(t) is first order stationary

b) $P_{XX} = R_{XX}(0) = 36 + 25 = 61\,W$

Hence it is true

c) the process ergodic because the ensemble averages are equal to the time averages.

d) Since mean is constant & auto correlation depends only τ

$\therefore x(t)$ is WSS

e) Since $\lim_{\tau \to \infty} R_{xx}(\tau) = \bar{x}^2 = 36$

the process has no periodic component

f) $E[x^2(t)] = R_{xx}(0) = 61$

AC power $= \sigma_x^2 = E[x^2(t)] - E[x(t)] = 61 - 6^2 = 25\,W$

Q: A stationary process has an autocorrelation fn given by

$R(\tau) = \dfrac{25\tau^2 + 36}{6.25\tau^2 + 4}$ find the mean value, ms value & variance of the process

A: $R(0) = \overline{x^2} = \dfrac{36}{4} = 9$

$\bar{x}^2 = \lim_{\tau \to \infty} R(\tau) = \dfrac{25}{6.25} = 4 \Rightarrow \bar{x} = 2$

$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = 9 - 4 = 5$

Q: Assume that an ergodic RP $x(t)$ has an autocorrelation fn $R_{xx}(\tau) = 18 + \dfrac{2}{6+\tau^2}[1 + 4\cos 2\tau]$ find $\bar{x}$

does this process have a periodic component, what is the average power in $x(t)$.

A: $\bar{x}^2 = \lim_{\tau \to \infty} R_{xx}(\tau) = 18 \Rightarrow \bar{x} = \sqrt{18} = 4.2426$

The process has periodic component

avg power $= E[x^2] = R_{xx}(0) = 18 + \dfrac{10}{6} = 19.667\,W$

Q: A Stationary ergodic r.p has an auto correlation fn with the periodic component as $R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$ find the mean & Variance of $x(t)$

A: $\bar{x}^2 = \lim_{\tau\to\infty} R_{xx}(\tau) = \lim_{\tau\to\infty} 25 + \frac{4}{1+6\tau^2} = 25$

$\bar{x} = 5$

$E(x^2) = R_{xx}(0) = 25+4 = 29$

$\sigma_x^2 = E(x^2) - E(x)^2 = 29 - 25 = 4$

Q: A r.p $Y(t) = x(t) - x(t+\tau)$ is defined interms of a process $x(t)$ which is WSS

a) S.T mean value of $y(t)$ is zero even if $x(t)$ has a non zero mean value.

b) S.T $\sigma_y^2 = 2[R_{xx}(0) - R_{xx}(\tau)]$

c) ~~S.T~~ If $y(t) = x(t) + x(t+\tau)$ find $\bar{y}$ & σ_y^2

A: a) $E[y(t)] = E[x(t) - x(t+\tau)] = E[x(t)] - E[x(t+\tau)]$

$= \bar{x} - \bar{x} = 0$

b) $\sigma_y^2 = E[y^2(t)] - E(y(t))^2 = E[y^2(t)]$

$= E[(x(t) - x(t+\tau))^2] = E[x^2(t)] + E[x^2(t+\tau)] - 2E[x(t)x(t+\tau)]$

$= R_{xx}(0) + R_{xx}(0) - 2R_{xx}(\tau)$

$= 2[R_{xx}(0) - R_{xx}(\tau)]$

c) $y(t) = x(t) + x(t+\tau)$

$E(y(t)) = E[x(t) + x(t+\tau)] = E[x(t)] + E[x(t+\tau)]$

$= \bar{x} + \bar{x} = 2\bar{x}$

$$\sigma_y^2 = E\left[y^2(t)\right] - E\left[y(t)\right]^2$$

$$= E\left[x^2(t)\right] + E\left[x^2(t+\tau)\right] + 2E\left(x(t)\,x(t+\tau)\right) - 4\bar{x}^2$$

$$= R_{xx}(0) + R_{xx}(0) + 2R_{xx}(\tau) - 4\bar{x}^2$$

$$= 2\left(R_{xx}(0) + R_{xx}(\tau)\right] - 4\bar{x}^2$$

Q: Determine if whether the R.P. $x(t) = A$ where A is a R.V with mean $\bar{A}$ & variance σ_A^2 is mean ergodic

A. If $x(t)$ is mean ergodic then

$$m(t) = \langle m_T \rangle$$

$$m(t) = E\left[x(t)\right] = A$$

$$\langle m_T \rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\,dt$$

$$= A \lim_{T\to\infty} \frac{1}{2T}\left(t\right)_{-T}^{T} = A$$

Since $m(t) = \langle m_T \rangle$ $x(t)$ is mean ergodic

Q. Consider a R.P. $x(t) = A\cos(\omega_1 t + \theta)$ & $y(t) = B\cos(\omega_2 t + \phi)$ where A & B & ω_1 & ω_2 are constants while θ & ϕ are statistically independent R.V's uniformly distributed on $(0, 2\pi)$

a) S.T $X(t)$ & $Y(t)$ are jointly WSS

b) If $\theta = \phi$ S.T $x(t)$ & $y(t)$ are not jointly WSS unless $\omega_1 = \omega_2$.

A.
$$f(\theta) = \frac{1}{2\pi} \quad 0 \le \theta \le 2\pi \qquad f(\phi) = \frac{1}{2\pi} \quad 0 \le \phi \le 2\pi$$

$$E[x(t)] = \int_0^{2\pi} A\cos(\omega_1 t+\theta)\cdot\frac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi} \sin(\omega_1 t+\theta)\Big|_0^{2\pi} = 0$$

$$E[y(t)] = \int_0^{2\pi} B\cos(\omega_2 t+\phi)\cdot\frac{1}{2\pi}\, d\phi$$

$$= \frac{B}{2\pi} \sin(\omega_2 t+\phi)\Big|_0^{2\pi} = 0$$

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)] = \frac{A^2}{2}\int_0^{2\pi} \frac{1}{2\pi}\cos\omega_1\tau\, d\theta$$

$$+ \frac{A^2}{2}\int_0^{2\pi} \frac{1}{2\pi} \cos(2\omega_1 t+\omega_1\tau+2\theta)\, d\theta$$

$$= \frac{A^2}{2}\cos\omega_1\tau$$

$$R_{yy}(\tau) = \frac{B^2}{2}\cos\omega_2\tau$$

Since independent

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)] = E[x(t)]\; E[y(t+\tau)]$$

$$= \frac{A}{2\pi}\int_0^{2\pi}\cos(\omega_1 t+\theta)\, d\theta \cdot \frac{B}{2\pi}\int_0^{2\pi}\cos(\omega_2 t+\omega_2\tau+\phi)\, d\phi$$

$$= \frac{AB}{4\pi^2}(0)\,(0) = 0$$

Since mean values of $x(t)$ & $y(t)$ are constant

& $R_{xy}(\tau) = 0$

$\therefore$ $x(t)$ & $y(t)$ are jointly WSS.

2) $\theta = \phi$

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$$

$$= E\left[A\cos(\omega_1 t+\theta)\, B\cos(\omega_2 t+\omega_2\tau+\theta)\right]$$

$$= \frac{AB}{2} \; E\left[\cos\left[t(\omega_2-\omega_1)+\omega_2\tau\right] + \cos\left(\omega_1 t+\omega_2 t+\omega_2\tau+2\right.\right.$$

$$= \frac{AB}{2} \int_0^{2\pi} \frac{1}{2\pi} \cos\left(t(\omega_2-\omega_1)+\omega_2\tau\right) d\theta$$

$$= \frac{AB}{2} \cos\left((\omega_2-\omega_1)t+\omega_2\tau\right)$$

thus the cross correlation fn depends on both t & τ

$\therefore$ to be jointly WSS we require

$$\omega_2 = \omega_1$$

then $R_{xy}(\tau) = \frac{AB}{2} \cos \omega_2 \tau$

Q: A RP is described by $x(t) = A^2 \cos^2(\omega_c t + \theta)$ where A & ω_c are constants & θ is a RV uniformly distributed b/w $\pm\pi$. Is $x(t)$ WSS.

A. $f(\theta) = \frac{1}{2\pi} \qquad -\pi \le \theta \le \pi$

$$E[x(t)] = \int_{-\pi}^{\pi} \frac{1}{2\pi} A^2 \cos^2(\omega_c t+\theta)\, d\theta$$

$$= \frac{A^2}{4\pi} \int_{-\pi}^{\pi} 1 + \cos(2\omega_c t + 2\theta) \; d\theta$$

$$= \frac{A^2}{4\pi} \int_{-\pi}^{\pi} d\theta = \frac{A^2}{2} = \text{Constant}$$

$$R_{xx}(\tau) = E\left[x(t)\, x(t+\tau)\right]$$

$$= E\left[A^2 \cos^2(\omega_c t+\theta)\, A^2 \cos^2(\omega_c t+\omega_c\tau+\theta)\right]$$

$$= \frac{A^4}{(2\pi)} \left\{ \int_{-\pi}^{\pi} \frac{\left(1+\cos(2\omega_c t+\theta)\right)}{2} \frac{\left(1+\cos(2\omega_c t+2\omega_c\tau+2\theta)\right)}{2} d\theta \right.$$

$$= \frac{A^4}{8\pi}\left[\theta\right]_{-\pi}^{\pi} + \frac{A^4}{8\pi}\int_{-\pi}^{\pi} \cos(2\omega_c t+\theta)\, d\theta \quad (\downarrow 0)$$

$$+ \frac{A^4}{8\pi}\int_{-\pi}^{\pi} \cos(2\omega_c t + 2\omega_c \tau + 2\theta)\, d\theta \quad (\downarrow 0)$$

$$+ \frac{A^4}{8\pi}\int_{-\pi}^{\pi} \cos(2\omega_c t + 2\theta)\cos(2\omega_c t + 2\omega_c\tau + 2\theta)\, d\theta$$

$$= \frac{A^4}{4} + \frac{A^4}{16\pi}\int_{-\pi}^{\pi}\left[\cos 2\omega_c\tau + \cos(4\omega_c t + 2\omega_c\tau + 4\theta)\right] d\theta$$

$$= \frac{A^4}{4} + \frac{A^4}{16\pi}\left[\theta\right]_{-\pi}^{\pi} \cos 2\omega_c\tau$$

$$= \frac{A^4}{4} + \frac{A^4}{8}\cos 2\omega_c\tau = \frac{A^4}{4}\left[1 + \frac{1}{2}\cos 2\omega_c\tau\right]$$

Q. Consider a R.P $x(t) = A\cos\omega t$, where ω is a constant & A is a R.V uniformly distributed over (0, 1) find the auto correlation & auto covariance of $x(t)$.

S. $f(A) = 1 \quad 0 \le A \le 1$

$$E[x(t)] = \int_0^1 A\cos\omega t\, dA = \cos\omega t \left.\frac{[A^2]}{2}\right|_0^1 = \frac{1}{2}\cos\omega t$$

$$E[x(t+\tau)] = \int_0^1 A\cos(\omega t+\omega\tau)\, dA = \left.\frac{A^2}{2}\right|_0^1 \cos(\omega t+\omega\tau)$$

$$= \frac{1}{2}\cos(\omega t+\omega\tau)$$

$$R_{xx}(t, t+\tau) = \int_0^1 A\cos\omega t\; A\cos(\omega t+\omega\tau)\, dA$$

$$= \left.\frac{A^3}{3}\right|_0^1 \cos\omega t\cos(\omega t+\omega\tau) = \frac{1}{3}\cos\omega t\cos(\omega t+\omega\tau)$$

$$C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - E[x(t)]\, E[x(t+\tau)]$$

$$= \frac{\cos\omega t\cos(\omega t+\omega\tau)}{} - \frac{1}{}\cos\omega t\cos(\omega t+\omega\tau)$$

$$= \frac{\cos \omega t \ \cos(\omega t + \omega \tau)}{12}$$

Q: A rp is defined by $x(t) = x_0 + vt$ where x_0 & v are statistically independent rv, uniformly distributed in interval (x_{01}, x_{02}) & (v_1, v_2) find mean, auto correlation, auto covariance of $x(t)$. Is $x(t)$ stationary in any sense if so state the type.

A: $f(x_0) = \frac{1}{x_{02} - x_{01}} \qquad x_{01} < x_0 \leq x_{02}$

$f(v) = \frac{1}{v_2 - v_1} \qquad v_1 < v \leq v_2$

$$E[x(t)] = E[x_0] + t\, E[v]$$

$$= \frac{1}{x_{02} - x_{01}} \int_{x_{01}}^{x_{02}} x_0 \, dx_0 + \frac{t}{v_2 - v_1} \int_{v_1}^{v_2} v \, dv$$

$$= \frac{x_0^2 \Big|_{x_{01}}^{x_{02}}}{2(x_{02} - x_{01})} + \frac{t}{(v_2 - v_1)} \frac{v^2}{2} \Big|_{v_1}^{v_2}$$

$$= \frac{(x_{02}^2 - x_{01}^2)}{2(x_{02} - x_{01})} + \frac{t}{(v_2 - v_1)} \frac{(v_2^2 - v_1^2)}{2}$$

$$= \frac{1}{2}\left[x_{02} + x_{01} + t(v_2 + v_1)\right]$$

$$R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)] = E\left[x_0^2 + x_0 vt + x_0 v\tau + x_0 vt + v^2 t + v^2 t\tau\right]$$

$$= E[x_0^2] + (2t+\tau) E[x_0 v] + (t^2 + t\tau) E[v^2]$$

$$E(x_0^2) = \frac{1}{x_{02} - x_{01}} \int_{x_{01}}^{x_0^2} x_0^2 \, dx_0 = \frac{x_0^3 \Big|_{x_{01}}^{x_{02}}}{3(x_{02} - x_{01})} = \frac{x_{02}^3 - x_{01}^3}{3(x_{02} - x_{01})}$$

$$= \frac{x_{02}^2 + x_{01}^2 + x_{01}x_{02}}{3}$$

$$E\left(v^2\right) = \int_{v_1}^{v_2} v^2 f(v)\, dv = \frac{1}{(v_2 - v_1)} \int_{v_1}^{v_2} v^2\, dv$$

$$= \frac{v^3 \Big|_{v_1}^{v_2}}{3(v_2 - v_1)} = \frac{v_2^3 - v_1^3}{3(v_2 - v_1)} = \frac{v_2^2 + v_1^2 + v_1 v_2}{3}$$

$$E\left[x_0 v\right] = E\left[x_0\right] E\left[v\right]$$

$$= \frac{1}{(x_{02} - x_{01})} \int_{x_{01}}^{x_{02}} x_0\, dx_0 + \frac{1}{(v_2 - v_1)} \int_{v_1}^{v_2} v\, dv$$

$$= \frac{x_{02}^2 - x_{01}^2}{2(x_{02} - x_{01})} + \frac{v_2^2 - v_1^2}{2(v_2 - v_1)}$$

$$= \frac{(x_{02} + x_{01})(v_2 + v_1)}{4}$$

$$R_{xx}(t, t+\tau) = \frac{x_{02}^2 + x_{01}^2 + x_{01}x_{02}}{3} + \frac{(2t+\tau)(x_{02} + x_{01})(v_2 + v_1)}{4}$$

$$+ \frac{t(t+\tau)(v_2^2 + v_1^2 + v_1 v_2)}{4}$$

$$E\left[x(t+\tau)\right] = \frac{1}{2}\left[(x_{02} + x_{01}) + (t+\tau)(v_2 + v_1)\right]$$

$$C_{xx}(t, t+\tau) = R_{xx}(t, t+\tau) - E\left[x(t)\right] E\left[x(t+\tau)\right]$$

$$= \frac{x_{02}^2 + x_{01}^2 + x_{01}x_{02}}{3} + \frac{(2t+\tau)(x_{02} + x_{01})(v_2 + v_1)}{4}$$

$$+ \frac{t(t+\tau)(v_2^2 + v_1^2 + v_1 v_2)}{4} - \frac{1}{2}\left[x_{02} + x_{01} + t(v_1 + v_2)\right].$$

$$\frac{1}{2}\left[x_{02} + x_{01} + (t+\tau)(v_1 + v_2)\right]$$

$$= \frac{x_{02}^2 + x_{01}^2 - 2x_{02}x_{01}}{12} + \frac{t(t+\tau)}{12}\left[v_2^2 + v_1^2 - 2v_1v_2\right]$$

the mean of $x(t)$ is not constant

auto correlation fn depends on time t &

auto covariance fn depends on time t

hence the given rp is not stationary.

Q: air craft arrival at an airport according to a poisson process is at a rate of 12 per hour. All air crafts are handled by one air traffic controller. If the controller takes a 2 min coffee break, what is the prob/. that he will miss 1 or more arriving air crafts.

A: $\lambda = 12$, $t = 2$ min $\quad \lambda t = 24$

$$P(x(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad k = 0, 1, 2 \ldots$$

Prob/. that he will miss one or more air crafts

= 1 − Prob/. that he does not miss any air craft

$= 1 - P[x(t) = 0]$

$$= 1 - \frac{(24)^0 e^{-24}}{0!} = 1 - e^{-24}$$

Q: Telephone calls are initiated through an exchange mean average rate of 75 per min and are described by a poisson process. Find the prob/. that more than 3 calls are initiated in any 15 sec period.

4:39

A: $\lambda = 75$ per min $= \dfrac{75}{60}$ per sec

$t = 5$ sec

$$\lambda t = \frac{75}{60} \times 5 = 6.25$$

$$P[x(t) > 3] = 1 - P[x(t) \le 3]$$

$$= 1 - \sum_{k=0}^{3} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$= 1 - e^{-6.25} \sum_{k=0}^{3} \frac{(6.25)^k}{k!}$$

$$= 1 - e^{-6.25}\left[1 + 6.25 + \frac{(6.25)^2}{2!} + \frac{(6.25)^3}{3!}\right)$$

$$= 1 - 0.1302 = 0.869$$

Q: Let $x(t)$ be a WSS process with auto correlation $R_{xx}(\tau) = e^{-a|\tau|}$ where a is a +ve constant. If $x(t)$ amplitude modulates a carrier $\cos(\omega_0 t + \theta)$ where ω_0 is constant & θ is a r.v uniformly distributed on $(-\pi, \pi)$ find the auto correlation of $y(t)$.

A: $x(t) \longrightarrow \otimes \longrightarrow y(t) = X(t)\cos(\omega_0 t + \theta)$

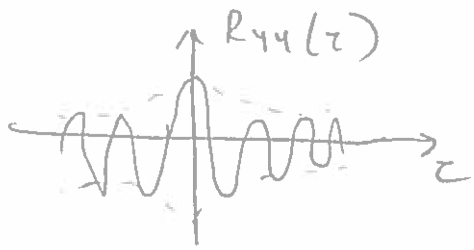
$\uparrow \cos(\omega_0 t + \theta)$

$$R_{yy}(\tau) = E[y(t)\,y(t+\tau)] = E[x(t)\cos(\omega_0 t + \theta)\, x(t+\tau)\cos(\omega_0 t + \omega_0\tau + \theta)]$$

$$= \frac{1}{2} E[x(t)\,x(t+\tau)]\, E[\cos\omega_0\tau + \cos(2\omega_0 t + \omega_0\tau + 2\theta)]$$

$$= \frac{1}{2} R_{xx}(\tau)\cos\omega_0\tau$$

$$= \frac{1}{2} e^{-a|\tau|}\cos\omega_0\tau$$



Q: Let $x(t)$ be a stationary r.p. that is differentiable. Let $\dot{x}(t)$ be its derivation

a) S.T $E[\dot{x}(t)] = 0$ (b) find $R_{\dot{x}\dot{x}}(\tau)$ interms of $R_{xx}(\tau)$

A: a) given $x(t)$ is a stationary r.p

$$\dot{x}(t) = \underset{\Delta\to\infty}{\mathrm{lt}} \frac{x(t+\Delta) - x(t)}{\Delta}$$

$$E[\dot{x}(t)] = \underset{\Delta\to\infty}{\mathrm{lt}} \frac{E[x(t+\Delta)] - E[x(t)]}{\Delta}$$

$$= \underset{\Delta\to\infty}{\mathrm{lt}} \frac{\bar{X} - \bar{X}}{\Delta} = 0$$

$$R_{\dot{x}\dot{x}}(\tau) = E[\dot{x}(t)\,\dot{x}(t+\tau)] = E\left(\frac{dx(t)}{dt}\,\frac{dx(t+\tau)}{dt}\right)$$

$$= \frac{\partial^2}{\partial t^2} E[x(t)\,x(t+\tau)] = \frac{\partial^2}{\partial t^2} R_{xx}(t, t+\tau)$$

$$= \frac{\partial^2}{\partial t^2} R_{xx}(\tau)$$

Q: statistically Independent zero mean r.p $x(t)$ & $y(t)$ have auto correlation fns $R_{xx}(\tau) = e^{-|\tau|}$ & $R_{yy}(\tau) = \cos(2\pi\tau)$

a) find the auto correlation fn of the sum $w_1(t) = x(t) + y(t)$

b) find the acf of the difference $w_2(t) = x(t) - y(t)$

c) find the ccf of $w_1(t)$ & $w_2(t)$

A: a) $R_{w_1w_1}(\tau) = E[w_1(t)\, w_1(t+\tau)]$

$$= E[\{x(t) + y(t)\}\{x(t+\tau) + y(t+\tau)\}]$$

$$= E(x(t)\,x(t+\tau)) + E[x(t)\,y(t+\tau)] + E[y(t)\,x(t+\tau)]$$

$$= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

Since $x(t)$ & $y(t)$ are independent $\quad \bar{x} = 0 = \bar{y}$

$$R_{xy}(\tau) = R_{yx}(\tau) = \bar{x}\cdot\bar{y} = 0$$

$$\therefore R_{w_1w}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) = e^{-|\tau|} + \cos 2\pi\tau$$

(b) $w_2(t) = x(t) - y(t)$

$$R_{w_2w_2}(\tau) = E\left[w_2(t)\, w_2(t+\tau)\right]$$

$$= E\left[\{x(t)-y(t)\}\{x(t+\tau) - y(t+\tau)\}\right]$$

$$= E\left[x(t)\, x(t+\tau)\right] - E\left[x(t)\, y(t+\tau)\right]$$

$$- E\left[y(t)\, x(t+\tau)\right] + E\left[y(t)\, y(t+\tau)\right]$$

$$= R_{xx}(\tau) - R_{xy}(\tau) - R_{yx}(\tau) + R_{yy}(\tau)$$

$$= R_{xx}(\tau) + R_{yy}(\tau) = e^{-|\tau|} + \cos 2\pi\tau$$

(c) $R_{w_1w_2}(\tau) = E\left[w_1(t)\, w_2(t+\tau)\right]$

$$= E\left[\{x(t) + y(t)\}\{x(t+\tau) - y(t+\tau)\}\right]$$

$$= E\left[x(t)\, x(t+\tau)\right] - E\left[x(t)\, y(t+\tau)\right]$$

$$+ E\left[y(t)\, x(t+\tau)\right] - E\left[y(t)\, y(t+\tau)\right]$$

$$= R_{xx}(\tau) - R_{xy}(\tau) + R_{yx}(\tau) - R_{yy}(\tau)$$

$$= R_{xx}(\tau) - R_{yy}(\tau)$$

$$= e^{-|\tau|} - \cos(2\pi\tau)$$

Q: Given a WSS gaussian rp $x(t_i)$, $t_i = 1, 2, 3$ the auto correlation fn is $R_{xx}(\tau) = 16 e^{-2|\tau|}$ with mean $\bar{x} = 3$ find the covariance matrix.

A: $R_{xx}(\tau) = 16\, e^{-2|\tau|}$ & $\bar{x} = 3$

$\tau = t_k - t_i$

$$R_{xx}(\tau) = 16\, e^{-2|t_k - t_i|} \quad i, k = 1, 2, 3$$

$$[R_x] = \begin{bmatrix} 16 & 16e^{-2} & 16e^{-4} \\ 16e^{-2} & 16 & 16e^{-2} \\ 16e^{-4} & 16e^{-2} & 16 \end{bmatrix} = \begin{bmatrix} 16 & 2.16 & 0.29 \\ 2.16 & 16 & 2.16 \\ 0.29 & 2.16 & 16 \end{bmatrix}$$

$$C_{xx}(t_k - t_i) = R_{xx}(\tau) - \bar{x}^2$$

$$\therefore [C_x] = \begin{bmatrix} 16-9 & 16e^{-2}-9 & 16e^{-4}-9 \\ 16e^{-2}-9 & 16-9 & 16e^{-2}-9 \\ 16e^{-4}-9 & 16e^{-2}-9 & 16-9 \end{bmatrix} = \begin{bmatrix} 7 & -6.83 & -8.71 \\ -6.83 & 7 & -6.83 \\ -8.71 & -2.83 & 7 \end{bmatrix}$$

Power Density Spectrum:

The amplitude of the rp, when it varies randomly with time, does not satisfy Dirchlets Condition. Therefore it is not possible to apply fourier transform directly on the rp for freq domain analysis

The acf of a WSS rp is used to study spectral characteristics such a power density spectrum or Power spectral density (PSd)

The PSd of a rp $x(t)$ is defined as

$$S_{xx}(\omega) = \lim_{T \to \infty} \frac{E\left[|X_T(\omega)|^2 \right]}{2T}$$ where $X_T(\omega)$ is the F.T of $x(t)$ in the interval

Consider a r.p $x(t)$ & one of its sample fn $x(t)$. Let $x_T(t)$ be defined as the portion of $x(t)$ b/w the limits $(-T, T)$

i.e $\quad x_T(t) = x(t) \quad -T < t < T$

$\quad\quad\quad\quad\;\; 0 \quad\quad$ else

$$X_T(\omega) = \int_{-T}^{T} x_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

the energy contained in $x(t)$ in the interval $(-T, T)$ is

$$E(T) = \int_{-T}^{T} x_T^2(t)\, dt = \int_{-T}^{T} x^2(t)\, dt$$

According to parsevals theorem

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2\, d\omega$$

the average power $P(T)$ in the r.p $X_T(t)$ is given by

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega$$

the average power P_{xx} in the r.p $x(t)$ is given by

$$P_{xx} = \lim_{T \to \infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E\left[|X_T(\omega)|^2 \right]}{2T}\, d\omega \right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \lim_{T \to \infty} \frac{E\left[|X_T(\omega)|^2 \right]}{2T} \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$$

where $S_{xx}(\omega) = \lim_{T\to\infty} \dfrac{E\left[|X_T(\omega)|^2\right]}{2\pi}$ is known as PSD

Relationship b/w PSD & ACF:

The ACF & PSD form a fourier transform pair

i.e $S_{xx}(\omega) \xleftrightarrow{F} R_{xx}(\tau)$

i.e $S_{xx}(\omega) = F\left[R_{xx}(\tau)\right] = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$

& $R_{xx}(\tau) = F^{-1}\left[S_{xx}(\omega)\right] = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$

1: Consider a R.P $x(t)$ & one of its sample fn $x(t)$.

Let $x_T(t) = x(t) \quad -T < t < T$

PSD of the process $x(t)$ is given by

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{E\left[X_T(\omega)\, X_T^*(\omega)\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1)\, e^{+j\omega t_1}\, dt_1 \int_{-T}^{T} x(t_2) e^{-j\omega t_2}\, dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E\left[x(t_1)\, x(t_2)\right] e^{-j\omega(t_2-t_1)}\, dt_1\, dt_2$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1, t_2)\, e^{-j\omega(t_2-t_1)}\, dt_1\, dt_2$$

let $t_1 = t$
$t_2 - t_1 = \tau$
$t_2 = t+\tau$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T-t}^{T-t} \int_{-T}^{T} R_{xx}(t, t+\tau) e^{-j\omega\tau} \, dt \, d\tau$$

$$= \int_{-\infty}^{\infty} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t, t+\tau) \, dt \right] e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} A\left[R_{xx}(t, t+\tau) \right] e^{-j\omega\tau} d\tau$$

where $A\left[R_{xx}(t, t+\tau)\right]$ represents the time averaged acf

$$A\left(R_{xx}(t, t+\tau)\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t, t+\tau) \, dt$$

for a wss rp

$$A\left[R_{xx}(t, t+\tau)\right] = R_{xx}(\tau)$$

$$\therefore \quad S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = F\left[R_{xx}(\tau)\right]$$

$$\therefore \quad R_{xx}(\tau) = F^{-1}\left[S_{xx}(\omega)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega$$

$$\therefore \quad R_{xx}(\tau) \xleftrightarrow{F} S_{xx}(\omega)$$

this relation is called as Wiener-Khintchine relation

Properties of PDS:-

1) The zero freq value of PSD of a wss rp equals the total area under the graph of auto correlation fn

i.e $S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau) \, d\tau$

1: wkt $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

at $\omega=0$ $S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau$ = area under the graph of auto correlation fn

2) The mean square value of a wss rp equals the total area under the graph of psd.

i.e $E[x^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$

1: wkt $R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega$

at $\tau=0$ $R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$

$\therefore E[x^2(t)] = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$

3) The psd of a wss rp is always non negative

i.e $S_{xx}(\omega) \geq 0$ for all ω

1. from the above property

$E[x^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega \geq 0$

$\therefore S_{xx}(\omega) \geq 0$

4) The psd of a real value rp is an even fn of freq i.e $S_{xx}(\omega) = S_{xx}(-\omega)$

1: $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

$S_{xx}(-\omega) = \int^{\infty} R_{xx}(\tau) e^{j\omega\tau} d\tau$

$$S_{xx}(-\omega) = \int_{\infty}^{-\infty} R_{xx}(\tau) e^{-j\omega\tau} (-d\tau)$$ let $\tau = -\tau$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$ Since $R_{xx}(\tau) = R_{xx}(-\tau)$

$$= S_{xx}(\omega)$$

5) If $x(t)$ is a wss RP with PSD $S_{xx}(\omega)$ then the PSD of the derivative of $x(t)$ is equal to ω^2 times the PSD $S_{xx}(\omega)$.

S i.e $S_{\dot{x}\dot{x}}(\omega) = \omega^2 S_{xx}(\omega)$

S: $$S_{\dot{x}\dot{x}}(\omega) = \lim_{T\to\infty} \frac{E\left[|\dot{x}_T(\omega)|^2\right]}{2T}$$

$$x_T(\omega) = \int_{-T}^{T} x(t) e^{-j\omega t} dt$$

$$\dot{x}_T(\omega) = \frac{d x_T(\omega)}{dt} = \int_{-T}^{T} x(t) (-j\omega) e^{-j\omega t} dt$$

$$= -j\omega \int_{-T}^{T} x(t) e^{-j\omega t} dt = -j\omega\, x_T(\omega)$$

$$S_{\dot{x}\dot{x}}(\omega) = \lim_{T\to\infty} \frac{E\left\{|(-j\omega)^2 x_T^2(\omega)|\right]}{2T}$$

$$= \lim_{T\to\infty} \omega^2 \frac{E\left[|x_T(\omega)|^2\right]}{2T}$$

$$= \omega^2 \lim_{T\to\infty} \frac{E\left[|x_T(\omega)|^2\right]}{2T} = \omega^2 S_{xx}(\omega).$$

Cross Power Density Spectrum:

The cross PDS of two RP $x(t)$ & $y(t)$ is

$$S_{xy}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[X_T^*(\omega)\, Y_T(\omega)\right]$$

Relationship between cross PSD & cross correlation fn:

The cross correlation & cross psd of two stationary r.p $x(t)$ & $y(t)$ form a F.T pair

i.e $R_{xy}(\tau) \xleftrightarrow{F} S_{xy}(\omega)$

$$S_{xy}(\omega) = F\left[R_{xy}(\tau)\right] = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{xy}(\tau) = F^{-1}\left[S_{xy}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega$$

1: Consider two r.p $x(t)$ & $y(t)$

Let $x_T(t) = x(t) \quad -T \le t \le T$

$y_T(t) = y(t) \quad -T \le t \le T$

$$S_{xy}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[X_T^*(\omega)\, Y_T(\omega)\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T} x(t_1)\, e^{+j\omega t_1} dt \int_{-T}^{T} y(t_2)\, e^{-j\omega t_2} dt_2\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E\left[x(t_1)\, y(t_2)\right] e^{-j\omega(t_2-t_1)}\, dt_1\, dt_2$$

Let $t_1 = t$, $t_2 - t_1 = \tau \implies t_2 = t+\tau$

$dt_1 = dt \qquad dt_2 = d\tau$

$$S_{xy}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T-t}^{T-t}\int_{-T}^{T} E\left[x(t)\, y(t+\tau)\right] e^{-j\omega\tau}\, dt\, d\tau$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T-t}^{T-t} \int_{-T}^{T} R_{xy}(t, t+\tau)\, e^{-j\omega\tau}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} \left[\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t, t+\tau)\, dt \right] e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \mathcal{C}\left[R_{xy}(t, t+\tau) \right] e^{-j\omega\tau}\, d\tau$$

where $\mathcal{C}\left[R_{xy}(t, t+\tau) \right] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t, t+\tau)\, dt$

for a stationary RP $\mathcal{C}\left[R_{xy}(t, t+\tau) \right] = R_{xy}(\tau)$

$$\therefore S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau = F\left(R_{xy}(\tau) \right)$$

$$R_{xy}(\tau) = F^{-1}\left(S_{xy}(\omega) \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\therefore R_{xy}(\tau) \xleftrightarrow{F} S_{xy}(\omega).$$

Properties of Cross PDS:

1) $S_{xy}(\omega) = S_{yx}(-\omega) = S_{yx}^{*}(\omega)$

1: $$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_{yx}(-\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{j\omega\tau}\, d\tau \qquad \text{let } \tau = -\tau$$

$$= \int_{+\infty}^{-\infty} R_{yx}(-\tau)\, e^{-j\omega\tau}(-d\tau)$$

$$= \int^{\infty} R_{yx}(\ldots)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau = S_{xy}(\omega)$$

$$S_{yx}^{*}(\omega) = \left(\int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j\omega\tau}\, d\tau\right)^{*} = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{j\omega\tau}\, d\tau$$

let $\tau = -\tau$

$$= \int_{+\infty}^{-\infty} R_{yx}(-\tau)\, e^{-j\omega\tau}\, (-d\tau)$$

$$= \int_{-\infty}^{\infty} R_{yx}(-\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= S_{xy}(\omega)$$

2) The real part of $S_{xy}(\omega)$ & $S_{yx}(\omega)$ are even fn's of ω.

Sol:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R.P\left[S_{xy}(\omega)\right] = \int_{-\infty}^{\infty} R_{xy}(\tau)\, \cos\omega\tau\, d\tau$$

$$R.P\left[S_{xy}(-\omega)\right] = \int_{-\infty}^{\infty} R_{xy}(\tau)\, \cos(-\omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)\, \cos\omega\tau\, d\tau$$

$$= R.P\left[S_{xy}(\omega)\right]$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R.P\left[S_{yx}(\omega)\right] = \int_{-\infty}^{\infty} R_{yx}(\tau)\, \cos\omega\tau\, d\tau$$

$$R.P\left[S_{yx}(-\omega)\right] = \int_{-\infty}^{\infty} R_{yx}(\tau)\, \cos(-\omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau)\, \cos\omega\tau\, d\tau = R.P\left[S_{yx}(\omega)\right]$$

3) The imaginary ~~odd~~ part of $S_{xy}(\omega)$ & $S_{yx}(\omega)$ are odd fns of ω.

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

$$I.P\left[S_{xy}(\omega)\right] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \sin \omega\tau \, d\tau$$

$$I.P\left[S_{xy}(-\omega)\right] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \sin(-\omega\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \sin \omega\tau \, d\tau$$

$$= -\left(I.P \text{ of } S_{xy}(\omega)\right)$$

$$I.P\left(S_{yx}(\omega)\right) = -\int_{-\infty}^{\infty} R_{yx}(\tau) \sin \omega\tau \, d\tau$$

$$I.P\left[S_{yx}(-\omega)\right] = -\int_{-\infty}^{\infty} R_{yx}(\tau) \sin(-\omega\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau) \sin \omega\tau \, d\tau$$

$$= -I.P\left[S_{yx}(\omega)\right]$$

$\therefore$ I.P $S_{xy}(\omega)$ & $S_{yx}(\omega)$ is an odd fn of ω

4) $S_{xy}(\omega) = 0 = S_{yx}(\omega)$ if $x(t)$ & $y(t)$ are orthogonal.

1: If $x(t)$ & $y(t)$ are orthogonal then

$$R_{xy}(\tau) = R_{yx}(\tau) = 0$$

$$F\left[R_{xy}(\tau)\right] = F\left(R_{yx}(\tau)\right] = F(0)$$

$$S_{xy}(\omega) = S_{yx}(\omega) = 0$$

5) If $x(t)$ & $y(t)$ are uncorrelated & have constant means $\bar{x}$ & $\bar{y}$ then $S_{xy}(\omega) = S_{yx}(\omega) = 2\pi \bar{x}\bar{y}\,\delta(\omega)$

A: If $x(t)$ & $y(t)$ are uncorrelated then

$$R_{xy}(\tau) = R_{yx}(\tau) = \bar{x}\,\bar{y}$$

$$F\left[R_{xy}(\tau)\right] = F\left(R_{yx}(\tau)\right] = F\left[\bar{x}\,\bar{y}\right]$$

$$S_{xy}(\omega) = S_{yx}(\omega) = \bar{x}\cdot\bar{y}\;F\left[1\right]$$

$$= \bar{x}\cdot\bar{y}\cdot 2\pi\,\delta(\omega)$$

$$= 2\pi\,\bar{x}\,\bar{y}\;\delta(\omega)$$

Q: For a RP $x(t) = A\cos(\omega t+\theta)$ where A is a real constant, ω is a RV with density fn $f(\omega)$ & θ is a RV uniformly distributed over the interval $(-\pi, \pi)$ & independent of ω calculate the PSD.

A.
$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,e^{j\omega\tau}\,d\omega \quad \rightarrow (1)$$

for a real RP

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\cos\omega\tau\,d\omega \quad \rightarrow (2)$$

$$R_{xx}(\tau) = \frac{A^2}{2}E\left[\cos\omega\tau\right] = \frac{A^2}{2}\int_{-\infty}^{\infty} f(\omega)\cos\omega\tau\,d\omega \quad \rightarrow (3)$$

$\therefore$ (2) = (3) $\quad \dfrac{S_{xx}(\omega)}{2\pi} = \dfrac{A^2}{2}f(\omega)$

$\therefore$ PSD $S_{xx}(\omega) = A^2\pi\, f(\omega)$

Q: Consider a r.p $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$ where A & B are two uncorrelated r.v's with zero mean & equal variance, ω_0 is real constant. Find the acf of $x(t)$ & hence its Pds.

$E(A) = E(B) = 0$

$E(A^2) = E(B^2) = \sigma^2$

$E(AB) = 0$

A. $R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

$$= E\left[\{A\cos\omega_0 t + B\sin\omega_0 t\}\{A\cos(\omega_0 t+\omega_0\tau) + B\sin(\omega_0 t+\omega_0\tau)\}\right]$$

$$= E[A^2]\cos\omega_0 t\cos(\omega_0 t+\omega_0\tau) + E(AB)\cos\omega_0 t\sin(\omega_0 t+\omega_0\tau)$$
$$+ E(AB)\sin\omega_0 t\cos(\omega_0 t+\omega_0\tau) + E(B^2)\sin\omega_0 t\sin(\omega_0 t+\omega_0\tau)$$

$$R_{xx}(\tau) = \sigma^2\left(\cos\omega_0 t\cos(\omega_0 t+\omega_0\tau) + \sin\omega_0 t\sin(\omega_0 t+\omega_0\tau)\right)$$

$$= \sigma^2\cos\omega_0\tau$$

$$S_{xx}(\omega) = F[R_{xx}(\tau)] = \sigma^2 F[\cos\omega_0\tau]$$

$$= \sigma^2\pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

Q: For a r.p the acf is $R(\tau) = a e^{-b|\tau|}$ find the Psd where a & b are constants.

A. $$S(\omega) = F[R(\tau)] = F[a e^{-b|\tau|}] = aF[e^{-b|\tau|}]$$

$$= a\left(\int_{-\infty}^{0} e^{b\tau} e^{-j\omega\tau}\,d\tau + \int_0^{\infty} e^{-b\tau} e^{-j\omega\tau}\,d\tau\right)$$

$$= a\left[\int_{-\infty}^{0} e^{(b-j\omega)\tau}\,d\tau + \int_0^{\infty} e^{-(b+j\omega)\tau}\,d\tau\right]$$

$$= a\left[\frac{1}{b-j\omega} + \frac{1}{b+j\omega}\right] = \frac{2ab}{b^2+\omega^2}$$

Q: The PSD of a r.p is given by $S_{xx}(\omega) = \pi \quad |\omega| \le 1$
$= 0$ else
find its a.c.f.

S: $$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1} \pi \cdot e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2}\, \frac{e^{j\omega\tau}}{j\tau}\Big|_{-1}^{1} = \frac{e^{j\tau} - e^{-j\tau}}{2j\tau} = \frac{\sin\tau}{\tau}$$

$$= \operatorname{Sinc} \tau$$

Q: Determine $R_{xx}(\tau)$ if $S_{xx}(\omega) = \frac{1}{(4+\omega^2)^2}$

S: $$S_{xx}(\omega) = \frac{1}{(4+\omega^2)^2} = \frac{1}{[(2+j\omega)(2-j\omega)]^2}$$

$$= \left[\frac{(2+j\omega) + (2-j\omega)}{4\,[(2+j\omega)(2-j\omega)]}\right]^2$$

$$= \frac{1}{16}\left[\frac{1}{(2-j\omega)^2} + \frac{1}{(2+j\omega)^2} + \frac{2}{4+\omega^2}\right]$$

$$R_{xx}(\tau) = \frac{1}{16}\, u(\tau)\, \tau e^{-2\tau} + \frac{1}{16}\, u(\tau)\tau e^{2\tau} + \frac{1}{32} \cdot e^{-2|\tau|}$$

Q: If $x(t)$ is real r.p s.t $R(0) - R(\tau) \geq \frac{1}{4}\left(R(0) - R(2\tau)\right)$

S: $$1 - \cos\theta = 2\sin^2\theta/2 \geq \frac{1}{4}\left(1 - \cos 2\theta\right)$$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) \cos\omega\tau \, d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) \cos\omega\tau \, d\omega$$

$$R(0)-R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)\,d\omega-\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)\cos\omega\tau\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)\left[1-\cos\omega\tau\right]d\omega$$

$$\geq\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)\cdot\frac{1}{4}\left(1-\cos 2\omega\tau\right]d\omega$$

$$\geq\frac{1}{4}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)\,d\omega-\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)\cos 2\omega\tau\,d\omega\right\}$$

$$\geq\frac{1}{4}\left[R(0)-R(2\tau)\right]$$

|||ly $R(0)-R(2\tau)\geq\frac{1}{4}\left[R(0)-R(4\tau)\right)$

$$\therefore\ R(0)-R(\tau)\geq\frac{1}{4}\left[\frac{1}{4}(R(0)-R(4\tau)\right]$$

$$\geq\frac{1}{4^2}\left(R(0)-R(2^2\tau)\right]$$

$$\geq\frac{1}{4^n}\left(R(0)-R(2^n\tau)\right]$$

Q: S.T if $y(t)=x(t+a)-x(t-a)$ then (1) $R_{yy}(\tau)=2R_{xx}(\tau)-R_{xx}(\tau+2a)-R_{xx}(\tau-2a)$ (2) $S_{yy}(\omega)=4S_{xx}(\omega)\sin^2 a\omega$

A: 1) $R_{yy}(\tau)=E\left[\{x(t+a)-x(t-a)\}\{x(t+\tau+a)-x(t+\tau-a)\}\right]$

$$=E\left[x(t+a)x(t+\tau+a)\right]-E\left[x(t+a)x(t+\tau-a)\right]$$
$$-E\left[x(t-a)x(t+\tau+a)\right]+E\left[x(t-a)x(t+\tau-a)\right]$$

$= R_{xx}(\tau) - E\left[x(t+a)\ldots\right] - E\left[x(t\ldots\right.$

$$- E\left[x(t-a)\,x(t+a+\tau+2a)\right] - R_{xx}(\tau)$$

$$= 2R_{xx}(\tau) - R_{xx}(\tau-2a) - R_{xx}(\tau+2a)$$

2) $$S_{yy}(\omega) = \int_{-\infty}^{\infty} R_{yy}(\tau)\, e^{-j\omega\tau}\, d\tau = F\left[R_{yy}(\tau)\right]$$

$$= 2F\left[R_{xx}(\tau)\right] - F\left[R_{xx}(\tau-2a)\right] - F\left[R_{xx}(\tau+2a)\right]$$

$$= 2S_{xx}(\omega) - S_{xx}(\omega)\, e^{-j2\omega a} - S_{xx}(\omega)\, e^{j2\omega a}$$

$$= 2S_{xx}(\omega) - S_{xx}(\omega)\left[e^{j2a\omega} + e^{-j2a\omega}\right]$$

$$= 2S_{xx}(\omega) - 2S_{xx}(\omega)\cos 2a\omega$$

$$= 2S_{xx}(\omega)\left[1 - \cos 2a\omega\right]$$

$$= 4S_{xx}(\omega)\sin^2 a\omega$$

Q: A stationary R.P with sample fn $x(t)$ has a psd of $S_{xx}(\omega) = \frac{16}{\omega^2+16}$ and an independent stationary R.P with sample fn $y(t)$ has a psd of $S_{yy}(\omega) = \frac{\omega^2}{\omega^2+4}$. $x(t)$ & $y(t)$ are zero mean. Find the psd of

1) $v(t) = x(t) + y(t)$ (2) $S_{xy}(\omega)$ & $S_{xv}(\omega)$

A: 1) $$E[x(t)] = 0 = E[y(t)]$$

$$R_{vv}(\tau) = E\left[v(t)\, v(t+\tau)\right]$$

$$= E\left[\{x(t)+y(t)\}\{x(t+\tau)+y(t+\tau)\}\right]$$

$$= R_{xx}(\tau) + \underset{\downarrow 0}{R_{xy}(\tau)} + \underset{\downarrow 0}{R_{yx}(\tau)} + R_{yy}(\tau)$$

$$= R_{xx}(\tau) + R_{yy}(\tau)$$

$$S_{UU}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) = \frac{16}{\omega^2+16} + \frac{\omega^2}{\omega^2+16}$$

$$= \frac{16+\omega^2}{\omega^2+16} = 1$$

2) $S_{XY}(\omega) = F\left(R_{XY}(\tau)\right)$

$$R_{XY}(\tau) = E\left(X(t)\,Y(t+\tau)\right) = 0$$

$$\therefore\ S_{XY}(\omega) = 0$$

3) $S_{XU}(\omega) = F\left[R_{XU}(\tau)\right]$

$$R_{XU}(\tau) = E\left[X(t)\,U(t+\tau)\right] = E\left[X(t)X(t+\tau)\right] + E\left(X(t)Y(t+\tau)\right)$$

$$= R_{XX}(\tau)$$

$$S_{XU}(\omega) = S_{XX}(\omega) = \frac{16}{\omega^2+16}$$

Q. If the acf of WSS process is $R(\tau) = ke^{-k|\tau|}$ S.T psd is $S(\omega) = \dfrac{2}{1+\left(\frac{\omega}{k}\right)^2}$

A. $S(\omega) = F\left[R(\tau)\right] = k\,F\left[e^{-k|\tau|}\right] = k.\dfrac{2k}{k^2+\omega^2}$

$$= \frac{2k^2}{k^2+\omega^2} = \frac{2}{1+\left(\frac{\omega}{k}\right)^2}$$

Q. Find the PSD of a RP $X(t)$ if $E[X(t)] = 1$ & $R_{XX}(\tau) = 1+e^{-\alpha|\tau|}$

A. $S_{XX}(\omega) = F\left[R_{XX}(\tau)\right] = F\left[1\right] + F\left[e^{-\alpha|\tau|}\right]$

$= 2\pi\,\delta(\omega) + \dfrac{2\alpha}{\ldots}$

Q: Determine which of the following fns are valid Psd fns.

a) $$S(\omega) = \frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$$

$$S(-\omega) = \frac{(-\omega)^2}{(-\omega)^6 + 3(-\omega)^2 + 3} = \frac{\omega^2}{\omega^6 + 3\omega^2 + 3} = S(\omega)$$

$\therefore S(\omega)$ is Valid Psd

b) $$S(\omega) = e^{-(\omega-1)^2}$$

$$S(-\omega) = e^{-(-\omega-1)^2} = e^{(\omega+1)^2} \neq S(\omega)$$

$\therefore$ $S(\omega)$ is not a valid Psd

3) $$S(\omega) = \frac{\omega^2}{\omega^4 + 1}$$

$$S(-\omega) = \frac{(-\omega)^2}{(-\omega)^4 + 1} = \frac{\omega^2}{\omega^4 + 1} = S(\omega)$$

$\therefore$ $S(\omega)$ is a valid Psd

4) $$S(\omega) = \frac{\omega^4}{1 + \omega^2 + j\omega^6}$$

$$S(-\omega) = \frac{(-\omega)^4}{1 + (-\omega)^2 + j(-\omega)^6} = \frac{\omega^4}{1 + \omega^2 + j\omega^6} = S(\omega)$$

$\therefore$ $S(\omega)$ is a valid Psd

Q: find the acf of the following PSD's:

1) $S_{xx}(\omega) = \dfrac{157 + 12\omega^2}{(16+\omega^2)(9+\omega^2)}$

A: $S_{xx}(\omega) = \dfrac{5}{16+\omega^2} + \dfrac{7}{9+\omega^2}$

$= \dfrac{5}{4^2+\omega^2} + \dfrac{7}{3^2+\omega^2}$

$= \dfrac{5}{8} \cdot \dfrac{2 \cdot 4}{4^2+\omega^2} + \dfrac{7}{6} \dfrac{2 \cdot 3}{3^2+\omega^2}$

$R_{xx}(\tau) = \dfrac{5}{8} F^{-1}\left[\dfrac{2 \cdot 4}{4^2+\omega^2}\right] + \dfrac{7}{6} F^{-1}\left(\dfrac{2 \cdot 3}{3^2+\omega^2}\right)$

$= \dfrac{5}{8} e^{-4|\tau|} + \dfrac{7}{6} e^{-3|\tau|}$

2) $S_{xx}(\omega) = \dfrac{8}{(9+\omega^2)^2} = \dfrac{8}{(3^2+\omega^2)^2}$

$R_{xx}(\tau) = F^{-1}\left[\dfrac{8}{(3^2+\omega^2)^2}\right] = \dfrac{8}{6} F^{-1}\left[\dfrac{2 \cdot 3}{(3^2+\omega^2)^2}\right]$

$= \dfrac{8}{6} \tau e^{-3|\tau|}$

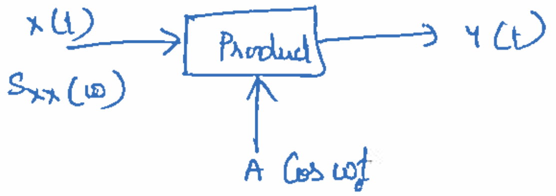
Q: A WSS noise process $N(t)$ has ACF $R_{NN}(\tau) = P e^{-3|\tau|}$ find PSD.

A: $S_{NN}(\omega) = F[R_{NN}(\tau)] = F[P \cdot e^{-3|\tau|}]$

$= P \cdot \dfrac{2 \cdot 3}{3^2+\omega^2} = \dfrac{6P}{9+\omega^2}$

Q: find $R_{YY}(\tau)$ & $S_{YY}(\omega)$ interms of $S_{XX}(\omega)$ for the product device shown in below fig if $X(t)$ is WSS.

A.



$= A\, X(t) \cos\omega_0 t$

$$R_{YY}(\tau) = E[y(t)\, y(t+\tau)]$$

$$= E[A\, x(t) \cos\omega_0 t \cdot A\, x(t+\tau) \cos(\omega_0 t + \omega_0 \tau)]$$

$$= \frac{A^2}{2} E[x(t)\, x(t+\tau)]\, E[\cos\omega_0\tau + \cos(2\omega t + \omega_0\tau)]$$

$$= \frac{A^2}{2} R_{XX}(\tau) \cos\omega_0\tau$$

$$S_{YY}(\omega) = F[R_{YY}(\tau)] = \frac{A^2}{2} F[R_{XX}(\tau) \cos\omega_0\tau]$$

$$= \frac{A^2}{2}\left(S_{XX}(\omega-\omega_0) + S_{XX}(\omega+\omega_0)\right)$$

Q. A RP has the pds $S_{XX}(\omega) = \frac{\omega^2}{1+\omega^2}$ find the average power in the process.

A: $$S_{XX}(\omega) = \frac{\omega^2}{1+\omega^2} = 1 - \frac{1}{1+\omega^2} = 1 - \frac{1}{2} \cdot \frac{2 \cdot 1}{1^2+\omega^2}$$

$$R_{XX}(\tau) = F^{-1}\left(1 - \frac{1}{2} \cdot \frac{2}{1^2+\omega^2}\right)$$

$$= \delta(\tau) - \frac{1}{2} e^{-|\tau|}$$

average power $= R_{XX}(0) = \delta(0) - \frac{1}{2} e^{-|0|} = 1 - \frac{1}{2} = \frac{1}{2}$

Q. The PSD of a stationary RP is given by

$S_{XX}(\omega) = A \quad -K < \omega < K$ find ACF.

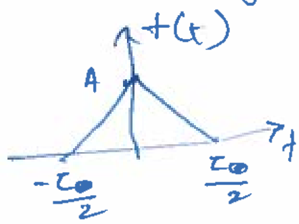
A. $$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-K}^{K} A \cdot e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} A \frac{\left[e^{j\omega\tau}\right]_{-k}^{k}}{j\tau} = \frac{A}{\pi\tau} \frac{\left[e^{jk\tau} - e^{-jk\tau}\right]}{2j}$$

$$= \frac{A}{\tau} \sin k\tau = \frac{Ak}{\pi} \frac{\sin k\tau}{k\tau}$$

$$= \frac{Ak}{\pi} Sa(k\tau)$$

Q: find the acf for the noise shown in below fig



A: $$f(t) = A + \left(\frac{2A}{\tau}\right)t \qquad -\frac{\tau_0}{2} < t < 0$$

$$A - \left(\frac{2A}{\tau}\right)t \qquad 0 < t < \frac{\tau_0}{2}$$

$$R(\tau) = \int_{-\tau_0/2}^{0} \left\{\left(A + \frac{2A}{\tau}\right)t\right\} \left\{ A + \frac{2A}{\tau}(t+\tau) \right\} dt$$

$$+ \int_{0}^{\tau/2} \left\{ A - \frac{2A}{\tau} t \right\} \left\{ A - \frac{2A}{\tau}(t+\tau) \right\} dt$$

$$= \int_{-\tau/2}^{0} A^2 + \frac{2A^2}{\tau}(t+\tau) + \frac{2A^2}{\tau} t + \left(\frac{2A}{\tau}\right)^2 \left[t^2 + t\tau\right] dt$$

$$+ \int_{0}^{\tau/2} A^2 - \frac{2A^2}{\tau}(t+\tau) - \frac{2A^2}{\tau} t + \left(\frac{2A}{\tau}\right)^2 \left[t^2 + t\tau\right] dt$$

$$= A^2 t + \frac{A^2}{\tau} t^2 + 2A^2 t + \frac{A^2}{\tau} t^2 + \frac{4A^2}{\tau^2} \frac{t^3}{3} + \frac{2A^2}{\tau} t^2 \Big|_{-\tau/2}^{0}$$

$$+ A^2 t - \frac{A^2}{\tau} t^2 - 2A^2 t - \frac{A^2}{\tau} t^2 + \frac{4A^2}{\tau^2} \frac{t^3}{3} + \frac{2A^2}{\tau} t^2 \Big|_{0}^{\tau/2}$$

$$= \frac{A^2 \tau}{3}$$

Q: An ergodic r.p is known to have an acf of the form $R_{xx}(\tau) = 1 - |\tau| \quad |\tau| < 1$ S.T the psd of the process is $S_{xx}(\omega) = \left[\frac{\sin \omega/2}{\omega/2}\right]^2$

A: $$S_{xx}(\omega) = \int_{-1}^{0} (1+\tau) e^{-j\omega\tau} d\tau + \int_{0}^{1} (1-\tau) e^{-j\omega\tau} d\tau$$

$$\int_{-1}^{0} e^{-j\omega\tau} d\tau = \frac{e^{-j\omega\tau}}{-j\omega}\Big|_{-1}^{0} = \frac{e^{j\omega} - 1}{j\omega}$$

$$\int_{0}^{1} e^{-j\omega\tau} d\tau = \frac{e^{-j\omega\tau}}{-j\omega}\Big|_{0}^{1} = \frac{1 - e^{-j\omega}}{j\omega}$$

$$\int_{-1}^{0} \tau \cdot e^{-j\omega\tau} d\tau = e^{-j\omega\tau}\left[\frac{\tau}{-j\omega} - \frac{1}{(-j\omega)^2}\right]\Big|_{-1}^{0}$$

$$= \frac{e^{j\omega}}{-j\omega} + \frac{1}{(j\omega)^2}\left[e^{j\omega} - 1\right]$$

$$\int_{0}^{1} \tau\, e^{-j\omega\tau} d\tau = e^{-j\omega\tau}\left[\frac{\tau}{-j\omega} - \frac{1}{(-j\omega)^2}\right]\Big|_{0}^{1}$$

$$= -\frac{e^{-j\omega}}{j\omega} + \frac{1}{(j\omega)^2}\left[1 - e^{-j\omega}\right]$$

$$S_{xx}(\omega) = \frac{e^{j\omega} - 1}{j\omega} - \frac{e^{j\omega}}{j\omega} + \frac{1}{(j\omega)^2}\left[e^{j\omega} - 1\right]$$

$$+ \frac{1 - e^{-j\omega}}{j\omega} + \frac{e^{-j\omega}}{j\omega} - \frac{1}{(j\omega)^2}\left[1 - e^{-j\omega}\right]$$

$$= \frac{2}{\omega^2} - \frac{(e^{j\omega} + e^{-j\omega})}{\omega^2} = \frac{2}{\omega^2} - \frac{2\cos\omega}{\omega^2}$$

$$= \frac{2}{\omega^2}\left[1 - \cos\omega\right] = \frac{4\sin^2 \omega/2}{\omega^2} = \left(\frac{\sin \omega/2}{\omega/2}\right)^2$$

Q: The acf of a wss rp is $R_{xx}(\tau) = a e^{-(\tau/b)^2}$ find the PSd & normalized power of a signal.

A: $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = a \int_{-\infty}^{\infty} e^{-(\frac{\tau}{b})^2} e^{-j\omega\tau} d\tau$

$$= a \int_{-\infty}^{\infty} e^{\left(-\frac{\tau^2}{b^2} - j\omega\tau\right)} d\tau$$

$$= a \int_{-\infty}^{\infty} e^{\left[-\frac{\tau^2}{b^2} - \frac{2j\omega\tau}{b} \cdot \frac{b}{2} + \left(\frac{b}{2}j\omega\right)^2 - \left(\frac{b}{2}j\omega\right)^2\right]} d\tau$$

$$= a \int_{-\infty}^{\infty} e^{-\left[\frac{\tau}{b} + \frac{j\omega b}{2}\right]^2} \cdot e^{\left(\frac{j\omega b}{2}\right)^2} d\tau$$

$$= a e^{-\frac{\omega^2 b^2}{4}} \int_{-\infty}^{\infty} e^{-\left[\frac{\tau}{b} + \frac{j\omega b}{2}\right]^2} d\tau$$

Let $\frac{\tau}{b} + \frac{j\omega b}{2} = \frac{z}{\sqrt{2}}$

$d\tau = \frac{b}{\sqrt{2}} dz$

$$= a e^{-\frac{\omega^2 b^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \frac{b}{\sqrt{2}} dz$$

$$= \frac{ab}{\sqrt{2}} e^{-\frac{\omega^2 b^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \frac{ab}{\cancel{\sqrt{2}}} e^{-\frac{\omega^2 b^2}{4}} \sqrt{\pi} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz\right]$$

$$= ab\sqrt{\pi}\, e^{-\frac{\omega^2 b^2}{4}}$$

average power $= R_{xx}(0) = a$ watts

Q: A Stationary rp $x(t)$ has acf $R_{xx}(\tau) = 10 + 5\cos 2\tau + 10 e^{-2|\tau|}$ find the dc & ac power of $x(t)$.

A: average power of dc component is $P_{dc} = 10$ watts

" " " ac " is $P_{ac} = R_{xx}(0)$

$(10 + 5\cos 2\tau)$ $= 10 + 5 = 15\,W$

average power $= P_{xx} = R_{xx}(0) = 10 + 5 + 10 = 25\,W$

Q: A r.p has the psd $S_{xx}(\omega) = \dfrac{6\omega^2}{1+\omega^4}$ find the average power in the process.

A: $P_{xx} = E[x^2(t)] = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$

$\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{a^4 + x^4}\, dx = \frac{\pi}{2\sqrt{2}\,a}$

$= \dfrac{6}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^4}\, d\omega = \frac{6}{2\pi}\left(\frac{\pi}{2\sqrt{2}(1)}\right)$

$= 1.06$ watts

Q: The cpsd $S_{xy}(\omega) = \dfrac{1}{(a+j\omega)^2}$ $a > 0$, find the ccf

A. $R_{xy}(\tau) = F^{-1}[S_{xy}(\omega)] = F^{-1}\left(\dfrac{1}{(a+j\omega)^2}\right) = \tau e^{-a\tau} u(\tau)$

Q: The psd of $x(t)$ is given by

$S_{xx}(\omega) = 1 + \omega^2 \quad |\omega| < 1$

$\qquad\quad\; 0 \quad$ else

find the acf

A: $R_{xx}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-1}^{1} (1+\omega^2)\, e^{j\omega\tau}\, d\omega$

$= \dfrac{1}{2\pi}\left[\displaystyle\int_{-1}^{1} e^{j\omega\tau}\, d\omega + \int_{-1}^{1} \omega^2 e^{j\omega\tau}\, d\omega\right]$

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$$= \frac{1}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\Big|_{-1}^{1} + e^{j\omega\tau}\left[\frac{\omega^2}{j\tau} - \frac{2\omega}{(j\tau)^2} + \frac{2}{(j\tau)^3}\right]\Big|_{-1}^{1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\tau}-e^{-j\tau}}{j\tau} + \frac{e^{j\tau}-e^{-j\tau}}{j\tau} + \frac{2}{\tau^2}\left(e^{j\tau}+e^{-j\tau}\right) - \frac{2}{j\tau^3}\left(e^{j\tau}-e^{-j\tau}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\sin\tau}{\tau} + \frac{2\sin\tau}{\tau} + \frac{4\cos\tau}{\tau^2} - \frac{4}{\tau^3}\sin\tau\right]$$

$$= \frac{2}{\pi\tau^3}\left[\tau^2\sin\tau + \tau\cos\tau - \sin\tau\right]$$

Q: If the acf of a wss process is $R_{xx}(\tau) = k e^{-k|\tau|}$ s.t its psd is $S_{xx}(\omega) = \frac{2}{1+(\omega/k)^2}$

A: $S_{xx}(\omega) = F\left[R_{xx}(\tau)\right] = k\, F\left[e^{-k|\tau|}\right]$

$$= k\cdot\frac{2k}{k^2+\omega^2} = \frac{2k^2}{k^2+\omega^2} = \frac{2}{1+\left(\frac{\omega}{k}\right)^2}$$

Q. A r.p has the pds $S_{xx}(\omega) = \frac{6\omega^2}{1+\omega^4}$ find the average power in the process

A: $P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{6\omega^2}{1+\omega^4}\, d\omega = \frac{6}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2}{1+\omega^4}\, d\omega$$

$$= \frac{6}{\dots}\left(\frac{\pi}{2\sqrt{2}}\right) = 1.06 \text{ watts}$$

Questions

1) Explain the concept of Random process.
2) Explain how Random process are classified with neat sketches.
3) Explain the distribution and density fn in the Context of Stationary and independent random process.
4) Distinguish between stationary and non-stationary random process.
5) State the condition for a WSS R.P.
6) Explain briefly about time average and ergodicity.
7) Eplain briefly about mean ergodic & correlation ergodic process.
8) state & prove the properties of ACF
9) state & prove the properties of CCF.
10) Define PSD
11) Define CPSD
12) state & prove properties of PSD
13) state & prove properties of CPSD
14) state & prove the relationship b/w ACF & PSD
15) State & prove the relationship b/w CCF & CPSD

Additional Problems

1) A RP is given as $x(t) = At$ where A is an uniformly distributed RV on $(0, 2)$ find whether $x(t)$ is WSS or not

2) If $y_1(t) = x_1 \cos \omega t + x_2 \sin \omega t$
& $y_2(t) = x_1 \sin \omega t + x_2 \cos \omega t$ where x_1 & x_2 are zero mean independent RV's with unity variance, show that the RP $y_1(t)$ & $y_2(t)$ are individually WSS but not jointly WSS.

3) If $x(t)$ is a RP with mean 3 & acf of $9+4e^{-0.2|\tau|}$ find the mean & variance of the RV $x(t)$

4) Two RPs $u(t)$ & $v(t)$ are defined as $u(t) = x(t)+y(t)$ & $v(t) = 2x(t) + 3y(t)$ where $x(t)$ & $y(t)$ are two independent RP find $R_{uu}(\tau)$, $R_{vv}(\tau)$, $R_{uv}(\tau)$ & $R_{vu}(\tau)$ interms of $R_{xx}(\tau)$ & $R_{yy}(\tau)$

5) A Complex RP $z(t) = x(t) + j\,y(t)$ is defined by jointly stationary real process $x(t)$ & $y(t)$ S.T $E[|z(t)|^2] = R_{xx}(0) + R_{yy}(0)$

6) Let $x_1(t)$ & $x_2(t)$, $y_1(t)$ & $y_2(t)$ are real RP and define $z_1(t) = x_1(t) + j y_1(t)$, $z_2(t) = x_2(t) + j y_2(t)$ find the expressions for the cross correlation fn of $z_1(t)$ & $z_2(t)$ if

a) all the real processes are correlated

b) they are uncorrelated

c) they are uncorrelated with zero mean

7) A RP has the pds $S_{xx}(\omega) = \frac{6\omega^2}{1+\omega^2}$ find the average power in the process.

8) find the average power of WSS RP $x(t)$ which has psd $S_{xx}(\omega) = \frac{\omega^2 - 17}{(\omega^2+49)(\omega^2+16)}$

9) A WSS RP $x(t)$ has psd $S_{xx}(\omega) = \frac{\omega^2}{(\omega^4 + 10\omega^2 + 9)}$ find the acf and mean Square value of the process.

10) The spectral density of a WSS RP $x(t)$ is given by $S_{xx}(\omega) = \frac{\omega^2}{(\omega^4 + 13\omega^2 + 36)}$ find the acf & average power of the process.

11) S.T the pds of the acf $R_{xx}(\tau) = e^{-a|\tau|}(1 + a|\tau|)$ is equal to $\frac{4a^2}{(a^2+\omega^2)^2}$

Multiple Choice Questions

1) The collection of all the sample fns is reffered to as [ ]

a) ensemble b) assumble c) average d) set

2) If the future values of a sample fn cannot be predicted based on its past values the process is called as ( )

a) Deterministic b) non deterministic

c) Independent d) statistical process

3) If $R_{xy} = 0$ then x & y [ ]

a) Independent b) orthogonal

c) both a & b d) none

4) If $R_{xy} = \bar{x}\,\bar{y}$ then x & y are ( )

a) Independent b) orthogonal

c) both a & b d) none

5) If the future values of a sample fn can be predicted based on its past values then the process is called as ________ [ ]

a) Deterministic b) non deterministic

c) Independent d) statistical process.

6) For a WSS RP [ ]

a) mean is constant c) both a & b

b) ACF depends on time difference d) None

7) SSS RP is also [ ] MCQ-2

a) WSS (b) stationary to order 1

c) Stationary to order 2 (d) All a, b & c

8) For an ergodic RP the statistical or ensemble averages are ________ to the time averages [ ]

a) not equal (b) equal (c) Independent

d) none

9) Mean square of x(t) is ____ [ ]

a) $R_{xx}(\tau)$ (b) $R_{xx}(0)$ (c) $R_{xx}(\tau)$ (d) none.

10) $R_{xx}(-\tau) =$ ________ [ ]

a) $R_{xx}(\tau)$ (b) $-R_{xx}(\tau)$ (c) both a & b

d) none.

11) The acf has maximum magnitude at $\tau =$ ____ [ ]

a) Infinity (b) origin (c) finite value (d) none

12) If x(t) is ergodic zero mean & has no periodic component then [ ]

a) $\lim_{\tau \to \infty} R_{xx}(\tau) = \overline{X}^2$ (b) $\lim_{\tau \to 0} R_{xx}(0) = \overline{X}^2$

c) mean $\overline{X} = 0$ d) mean is constant

13) $R_{yx}(-\tau) =$ ________ ( )

a) $R_{yx}(\tau)$ (b) $R_{xy}(\tau)$ (c) $-R_{yx}(\tau)$ (d) $-R_{xy}(\tau)$

14) $|R_{xy}(\tau)| =$ ________ ( )

a) $\sqrt{R_{xx}(0) + R_{yy}(0)}$ (b) $\sqrt{R_{xx}(0)\, R_{yy}(0)}$

c) $\sqrt{R_{xx}(0) - R_{yy}(0)}$ (d) $\sqrt{R_{xx}(0) * R_{yy}(0)}$

15) $|R_{xy}(\tau)| =$ ________ ( )

a) $\frac{1}{2}[R_{xx}(0) \cdot R_{yy}(0)]$ (b) $\frac{1}{2}(R_{xx}(0) * R_{yy}(0))$

c) $\frac{1}{2}(R_{xx}(0) - R_{yy}(0)]$ (d) $\frac{1}{2}[R_{xx}(0) + R_{yy}(0))$

16) If $x(t)$ & $y(t)$ are independent then

$R_{xy}(\tau) = R_{yx}(\tau) =$ ________ ( )

a) $\bar{x} \cdot \bar{y}$ (b) $\bar{x} + \bar{y}$ (c) $\bar{x} * \bar{y}$ (d) $\bar{x} - \bar{y}$

17) For a WSS RP $x(t)$ & $y(t)$
$C_{xx}(\tau) =$ ________ ( )

a) $R_{xy}(\tau) + \bar{x}\,\bar{y}$ (b) $R_{xy}(\tau) - \bar{x}\,\bar{y}$

c) $R_{xy}(\tau) * \bar{x}\,\bar{y}$ d) $R_{xy}(\tau) \cdot \bar{x}\,\bar{y}$

18) For a poisson RP which is correct ( )

a) mean $= \lambda t$ (b) variance $= \lambda t$

19) $S_{XX}(\omega)$ = ________ [ ] MCQ-4

a) Fourier transform of $R_{XX}(\tau)$

b) z transform of $R_{XX}(\tau)$

c) Laplace transform of $R_{XX}(\tau)$

d) none

20) $S_{XX}(-\omega)$ = ________ [ ]

a) $-S_{XX}(\omega)$ b) $S_{XX}(\omega)$ c) both a & b

d) none.

21) If $x(t)$ is WSS RP & $\dot{x}(t)$ is the derivative of $x(t)$ then $S_{\dot{X}\dot{X}}(\omega)$ = ________ [ ]

a) $S_{XX}(\omega)$ b) $\omega^2 S_{XX}(\omega)$ c) $\frac{S_{XX}(\omega)}{\omega^2}$ d) none

22) $S_{XY}(\omega)$ = ________ [ ]

a) Fourier transform of $R_{XY}(\tau)$

b) Laplace transform of $R_{XY}(\tau)$

c) z transform of $R_{XY}(\tau)$

d) none

23) $S_{XY}(\omega)$ = ________ [ ]

a) $S_{YX}(-\omega)$ b) $S_{YX}^{*}(\omega)$ c) both a & b

d) none.

24) If $x(t)$ & $y(t)$ are orthogonal then [ ]

a) $S_{XY}(\omega) = S_{YX}(\omega) = 0$

b) $S_{XY}(\omega) = S_{YX}(\omega) = 2\pi\delta(\omega)$

c) $S_{XY}(\omega) = S_{YX}(\omega) = \bar{X}-\bar{Y}$

d) no...

MCQ-5

25) If $x(t)$ & $y(t)$ uncorrelated & Stationary then $S_{xy}(\omega) = S_{yx}(\omega) =$ ________ [ ]

a) $\bar{x} \cdot \bar{y}$ b) $2\pi \bar{x} \bar{y} \delta(\omega)$

c) ~~x~~ 0 d) ∞

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Unit - V

Random Signal Response of Linear System:

System Response:

Let $x(t)$ be a r.p applied as an i/p to LTI system with impulse response $h(t)$ & $y(t)$ be the r.p at the o/p of the system.

Let the i/p $x(t)$ be WSS

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\xi)\, h(t-\xi)\, d\xi$$

$$= \int_{-\infty}^{\infty} h(\xi)\, x(t-\xi)\, d\xi$$

Mean & Mean Squared Value of System Response

mean of o/p $y(t) = E[y(t)] = E\left[\int_{-\infty}^{\infty} h(\xi)\, x(t-\xi)\, d\xi\right]$

$$= \int_{-\infty}^{\infty} h(\xi)\, E[x(t-\xi)]\, d\xi$$

$$= \bar{X} \int_{-\infty}^{\infty} h(\xi)\, d\xi$$

W.K.T

$H(\omega) = \int_{-\infty}^{\infty} h(\xi)\, e^{-j\omega\xi}\, d\xi$

$H(0) = \int_{-\infty}^{\infty} h(\xi)\, d\xi$

$$\boxed{\bar{Y} = \bar{X} \cdot H(0)} = \text{Constant} \rightarrow (1)$$

The mean value of $y(t)$ equals the mean of $x(t)$ times the area under the impulse response if $x(t)$ is WSS

mean squared value of $y(t)$ is

$$E[y^2(t)] = E\left[\int_{-\infty}^{\infty} h(\xi_1)\, x(t-\xi_1)\, d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)\, x(t-\xi_2)\, d\xi_2\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\left[x(t-\xi_1)\, x(t-\xi_2)\right] d\xi_1\, d\xi_2\, h(\xi_1)\, h(\xi_2)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(\xi_2-\xi_1)\, h(\xi_1)\, h(\xi_2)\, d\xi_1\, d\xi_2$$

Auto correlation Function of Response:

The acf of the response $y(t)$ is

$$R_{yy}(t, t+\tau) = E\left[y(t)\, y(t+\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\xi_1)\, x(t-\xi_1)\, d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)\, x(t+\tau-\xi_2)\, d\xi_2\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\left[x(t-\xi_1)\, x(t+\tau-\xi_2)\right] h(\xi_1)\, h(\xi_2)\, d\xi_1\, d\xi_2$$

$$R_{yy}(t,t+\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(\tau+\xi_1-\xi_2)\, h(\xi_1)\, h(\xi_2)\, d\xi_1\, d\xi_2 \rightarrow (2)$$

from the above equ it is obvious that $y(t)$ is WSS if $x(t)$ is WSS because $R_{yy}(\tau)$ depends only on τ & not on time t.

$$R_{yy}(\tau) = R_{xx}(\tau) * h(-\tau) * h(\tau)$$

Cross correlation Functions of I/P & O/P:

The ccf of $x(t)$ & $y(t)$ is

$$R_{xy}(t, t+\tau) = E\left[x(t)\, y(t+\tau)\right] = E\left[x(t) \int_{-\infty}^{\infty} h(\xi)\, x(t+\tau-\xi)\, d\xi\right]$$

$$= \int_{-\infty}^{\infty} h(\xi)\, E\left[x(t)\, x(t+\tau-\xi)\right] d\xi$$

$$= \int_{-\infty}^{\infty} h(\xi)\, R_{xx}(\tau-\xi)\, d\xi = R_{xx}(\tau) * h(\tau) \rightarrow (3)$$

IIly $R_{yx}(\tau) = E[y(t)\, x(t+\tau)]$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau - \xi_1)\, h(-\xi_1)\, d\xi_1$$

$$= R_{xx}(\tau) * h(-\tau) \longrightarrow (4)$$

from equn (3) & (4) it is clear that the ccf depends only on τ & not on time t

$\therefore$ $x(t)$ & $y(t)$ are jointly WSS if $x(t)$ is WSS

~~Substituting~~ equn (3) $\Rightarrow$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau - \xi_2)\, h(\xi_2)\, d\xi_2$$

$$R_{xy}(\tau + \xi_1) = \int_{-\infty}^{\infty} R_{xx}(\tau + \xi_1 - \xi_2)\, h(\xi_2)\, d\xi_2 \rightarrow (5)$$

substituting (5) in (2) we get

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} R_{xy}(\tau + \xi_1)\, h(\xi_1)\, d\xi_1$$

$$\boxed{\therefore R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)} \longrightarrow (6)$$

(4) $\Rightarrow$ $R_{yx}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau - \xi_1)\, h(-\xi_1)\, d\xi_1$

$$R_{yx}(\tau - \xi_2) = \int_{-\infty}^{\infty} R_{xx}(\tau - \xi_2 - \xi_1)\, h(-\xi_1)\, d\xi_1$$

$\xi_1 = -\xi_1$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau + \xi_1 - \xi_2)\, h(\xi_1)\, d\xi_1 \rightarrow (7)$$

Substituting (7) in (2) we get

$$\boxed{R_{yy}(\tau) = \int_{-\infty}^{\infty} R_{yx}(\tau - \xi_2)\, h(\xi_2)\, d\xi_2 = R_{yx}(\tau) * h(\tau)} \rightarrow (8)$$

Spectral characteristics of System Response:

Power density Spectrum of Response:

The Pds of the Response $y(t)$ is

let $\xi = \tau + \xi_1 - \xi_2$

$d\xi = d\tau$

$$S_{yy}(\omega) = \int_{-\infty}^{\infty} R_{yy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1)\, h(\xi_2)\, R_{xx}(\tau+\xi_1-\xi_2)\, e^{-j\omega\tau}\, d\tau\, d\xi_1\, d\xi_2$$

$$= \int_{-\infty}^{\infty} h(\xi_1)\, d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)\, d\xi_2 \int_{-\infty}^{\infty} R_{xx}(\xi)\, e^{-j\omega(\xi-\xi_1+\xi_2)}\, d\xi$$

$$= \int_{-\infty}^{\infty} h(\xi_1)\, e^{j\omega\xi_1}\, d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)\, e^{-j\omega\xi_2}\, d\xi_2 \int_{-\infty}^{\infty} R_{xx}(\xi)\, e^{-j\omega\xi}\, d\xi$$

$$= H^*(\omega)\, H(\omega)\, S_{xx}(\omega)$$

$$= |H(\omega)|^2\, S_{xx}(\omega)$$

Cross power density Spectrum of i/p & o/p:

wkt (from equ ③) $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$

$$F[R_{xy}(\tau)] = F[R_{xx}(\tau) * h(\tau)]$$

$$\boxed{S_{xy}(\omega) = S_{xx}(\omega)\, H(\omega)}$$

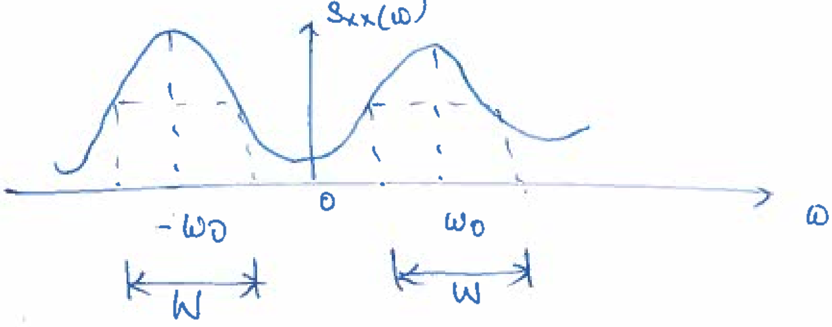
from equ ④ $R_{yx}(\tau) = R_{xx}(\tau) * h(-\tau)$

$$F[R_{yx}(\tau)] = F[R_{xx}(\tau) * h(-\tau)]$$

$$\boxed{S_{yx}(\omega) = S_{xx}(\omega)\, H(-\omega)}$$

Band Pass Process:

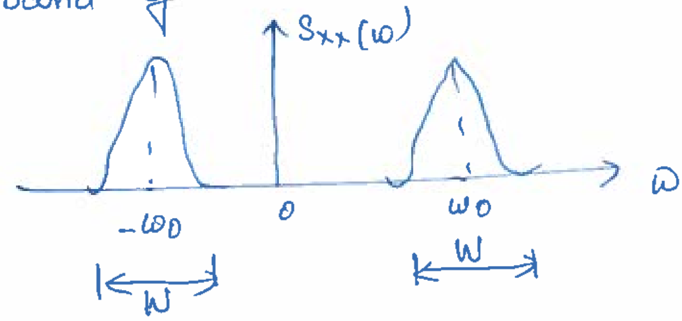
A r.p $x(t)$ is called a band pass process if its power spectral density $S_{xx}(\omega)$ has its significant components with in a band width W that does not include $\omega = 0$



The noise transmitting over a communication channel can be modelled as a band pass process.

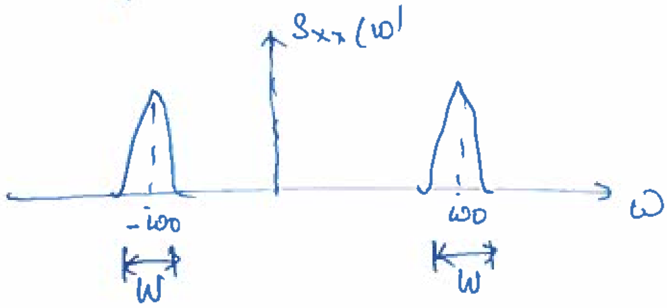
Band Limited process:

A bandpass r.p is said be band limited if its power spectrum characteristics are zero outside the frequency band of width W that does not include $\omega = 0$.



Narrow Band process:

A band limited r.p is said to be narrow band process if the band width W is very small compared to band centre freq i.e $W << \omega_0$



The narrow band process can be modelled as a cosine fn slowly varying in amplitude & phase with freq ω_0. It can be expressed as

$$N(t) = A(t) \cos[\omega_0 t + \theta(t)]$$

where $A(t)$ is an amplitude R.P

$\theta(t)$ is phase R.P

for any arbitary WSS R.P $N(t)$ the quadrature form of a narrow band process can be represented as

$$N(t) = X(t) \cos \omega_0 t - Y(t) \sin \omega_0 t$$

where $x(t)$ and $y(t)$ are called as in-phase and quadrature phase components of $N(t)$.

$$X(t) = A(t) \cos \theta(t)$$

$$Y(t) = A(t) \sin \theta(t)$$

where $A(t) = \sqrt{x^2(t) + y^2(t)}$

& $\theta(t) = \tan^{-1}\left[\frac{y(t)}{x(t)}\right]$

Properties of Band Limited Random process:

Let $N(t)$ be any Band limited WSS R.P with zero mean & PSD $S_{NN}(\omega)$. If the R.P is represented by

$$N(t) = X(t) \cos \omega_0 t - Y(t) \sin \omega_0 t$$

then $x(t)$ & $y(t)$ have the following properties

(i) If $N(t)$ is WSS then $X(t)$ & $Y(t)$ are jointly WSS

2) If $N(t)$ has zero mean then

$$E[x(t)] = E[y(t)] = 0$$

3) The mean square values of the processes are equal

i.e $E[N^2(t)] = E[x^2(t)] = E[y^2(t)]$

4) Both processes $x(t)$ & $y(t)$ have the same auto correlation fns.

i.e $R_{xx}(\tau) = R_{yy}(\tau)$.

5) The cross correlation fn of $x(t)$ & $y(t)$ are given by $R_{yx}(\tau) = -R_{xy}(\tau)$

If the process are orthogonal then

$$R_{yx}(\tau) = R_{xy}(\tau) = 0$$

6) Both $x(t)$ & $y(t)$ have same PSDs

$$S_{yy}(\omega) = S_{xx}(\omega) = \begin{cases} S_N(\omega-\omega_0) + S_N(\omega+\omega_0) & |\omega| \leq \omega_0 \\ 0 & \text{else} \end{cases}$$

7) The cross power spectrum are

$$S_{xy}(\omega) = -S_{yx}(\omega).$$

8) If $N(t)$ is Gaussian R.P then $x(t)$ & $y(t)$ are jointly gaussian.

9) The relationship b/w autocorrelation & Power spectrum $S_{NN}(\omega)$ is

$$R_{xx}(\tau) = \frac{1}{\pi}\int^{\infty} S_{xx}(\omega)\cos(\omega-\omega_0)\tau \, d\omega$$

& $R_{YY}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)\sin(\omega-\omega_0)\tau \, d\omega$

10) If $N(t)$ is a zero mean gaussian and its PSD $S_N(\omega)$ is symmetric about $\pm\omega_0$, then $X(t)$ & $Y(t)$ are statistically Independent.

Noise Definitions:

Noise is an unintentional fluctuation that tends to disturb transmission and reproduction of transmitted signals. Noise signals may or may not be predictable.

Predictable noise is known and can be eliminate easily. unpredictable noise is random.

White Noise or white Gaussian Noise:

white noise is a thermal noise whose frequency components extend from zero to infinity

Consider a WSS thermal noise $N(t)$. It is called white noise if Psd of $N(t)$ is constant at all freq's. i.e the Psd of white noise is independent of freq. It can be expressed as

$$S_{NN}(\omega) = \frac{N_0}{2}$$ where N_0 is the noise power.

According to central limit theorem, the pdf of white noise can be assumed to be Gaussian Distribution with zero mean. Hence it is known as a white Gaussian noise. Thermal noise may be considered as a white Gaussian noise.

The band width of white noise is infinity. It also has infinite average power so it is not physically realisable.

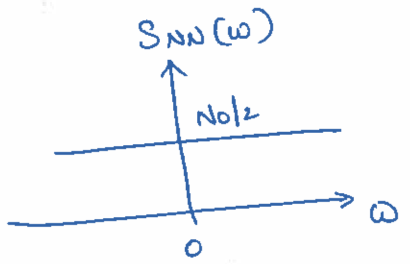
Psd of white noise is given by

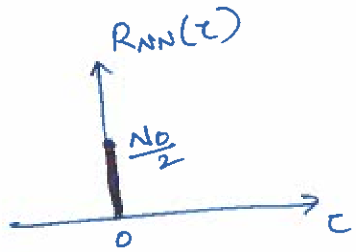
$$S_{NN}(\omega) = \frac{N_0}{2} \qquad -\infty < \omega < \infty$$

$$R_{NN}(\tau) = F^{-1}\left[S_{NN}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2} e^{j\omega\tau}\, d\tau$$

$$= \frac{N_0}{2} F^{-1}[1] \quad \cancel{= \frac{N_0}{2}\delta\tau}$$

$$= \frac{N_0}{2}\delta(\tau)$$





average power $= R_{NN}(0) = \frac{N_0}{2}$ watts

Coloured Noise:

If white noise exists within a freq range i.e it is band limited, then it is called coloured noise. Band limited noise can be realisable.

A white noise whose spectral components are finite over a band of freq's and zero at other freq's is called band limited white noise. It is also called filtered white noise.

For a band limited white noise, the Pds is

$$S_{NN}(\omega) = \frac{N_0}{2} \qquad -\omega_0 < \omega < \omega_0$$

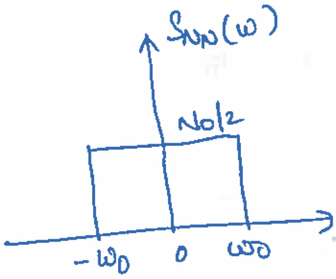
$$R_{NN}(\tau) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \frac{N_0}{2} e^{j\omega\tau}\, d\omega$$

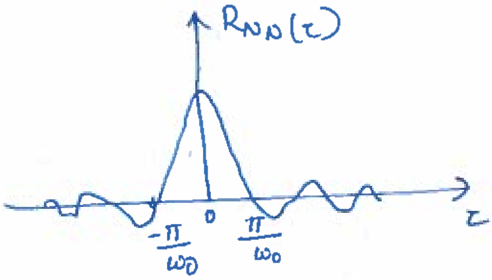
$$= \frac{1}{2\pi} \frac{N_0}{2} \frac{\left[e^{j\omega\tau}\right]_{-\omega_0}^{\omega_0}}{j\tau}$$

$$= \frac{N_0}{2\pi\tau} \frac{\left[e^{j\omega_0\tau} - e^{-j\omega_0\tau}\right]}{2j} = \frac{N_0}{2\pi\tau} \sin\omega_0\tau$$

$$= \frac{N_0 \omega_0}{2\pi} \left[\frac{\sin\omega_0\tau}{\omega_0\tau}\right] = \frac{N_0\omega_0}{2\pi} Sa(\omega\tau)$$

average power $= R_{NN}(0) = \frac{N_0\omega_0}{2\pi}$ watts





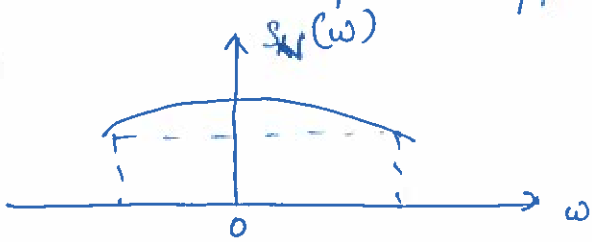
Thermal Noise & Resistor Noise:

In any conducting material, electrons move rando[m]. The noise produced is called thermal noise. It is also Thomson noise.

When temperature increases, random motion of electron increases and hence noise increases. At $0°K$, there is no motion of electrons and hence noise is zero. Thermal noise amplitude mainly depends on resistance. So it is also called resistor noise.

The thermal noise distribution may be approximate[d] to a Gaussian distribution with zero mean value

i.e $S_V(\omega) = 2KTR$



Q. A rp $x(t)$ is applied to a n/w with impulse response $h(t) = u(t)\, e^{-bt}$, $b>0$ the ccf of $x(t)$ with o/p $y(t)$ is known to be $R_{xy}(\tau) = u(\tau) e^{-b\tau}$ find the acf of $y(t)$ & average power in $y(t)$.

A:
$$h(t) = u(t)\, e^{-bt} \qquad R_{xy}(\tau) = u(\tau)\, e^{-b\tau}$$

$$H(\omega) = \frac{1}{b+j\omega} \qquad S_{xy}(\omega) = \frac{1}{(b+j\omega)}$$

wkt
$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

$$S_{xy}(\omega) = S_{xx}(\omega)\, H(\omega)$$

$$S_{xx}(\omega) = \frac{S_{xy}(\omega)}{H(\omega)} = 1$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = \left|\frac{1}{(b+j\omega)}\right|^2$$

$$= \frac{1}{b^2+\omega^2}$$

$$R_{yy}(\tau) = F^{-1}\left[S_{yy}(\omega)\right] = \frac{1}{2b} F^{-1}\left[\frac{2b}{b^2+\omega^2}\right]$$

$$= \frac{1}{2b} e^{-b|\tau|}$$

$$\text{avg power} = R_{yy}(0) = \frac{1}{2b} \text{ watts}$$

Q: The i/p voltage to an RLC series ckt is a stationary rp $x(t)$ with $E[x(t)] = 2$, $R_{xx}(\tau) = 4 + e^{-2|\tau|}$ let $y(t)$ be the voltage across the capacitor find $E[y(t)]$ & $S_{yy}(\omega)$

A:
$$H(\omega) = \frac{1/j\omega C}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{(1-\omega^2 LC) + j\omega RC}$$

$$|H(\omega)|^2 = \frac{1}{(1-\omega^2 LC)^2 + \omega^2 R^2C^2}$$

$R_{xx}(\tau) = 4 + e^{-2|\tau|}$

$S_{xx}(\omega) = 4(2\pi\,\delta(\omega)) + \frac{2\cdot 2}{2^2+\omega^2}$

$$E[y(t)] = H(0)\cdot\bar{X} = 1\,(2) = 2$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

$$= \frac{1}{(1-\omega^2 LC)^2 + (\omega RC)^2}\left[8\pi\,\delta(\omega) + \frac{4}{4+\omega^2}\right]$$

Q: White Noise $n(t)$ with psd $\frac{\eta}{2}$ is passed through RC LP n/w with 3db freq f_c. find
1) acf of the o/p noise of the n/w
2) $P(\tau) = R(\tau)/R(0)$ (3) find τ such that $P(\tau) \le 0$·

A: 1) $H(\omega) = \frac{1}{1+j\omega RC}$ $\qquad |H(\omega)|^2 = \frac{1}{1+(\omega RC)^2}$

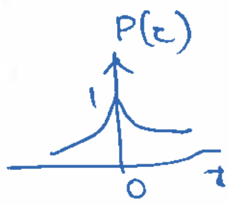
$$S_{xx}(\omega) = \frac{\eta}{2}$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = \frac{1}{1+(\omega RC)^2}\cdot\frac{\eta}{2}$$

$$R_{yy}(\tau) = F^{-1}\left[\frac{\eta}{2}\cdot\frac{1}{1+(\omega RC)^2}\right]$$

$$= \frac{\eta}{4RC}\,F^{-1}\left(\frac{2\cdot\frac{1}{RC}}{\omega^2+\left(\frac{1}{RC}\right)^2}\right)$$

$$R(\tau) = \frac{\eta}{4RC}\,e^{-\frac{|\tau|}{RC}}$$



2) $R(0) = \frac{\eta}{4RC}$ $\qquad P(\tau) = \frac{R(\tau)}{R(0)} = e^{-\frac{|\tau|}{RC}}$

3) $R(\tau) \leq 0.1$

$$\frac{\eta}{4RC} e^{-\frac{|\tau|}{RC}} \leq 0.1$$

$$\frac{\eta}{4RC} \leq 0.1\, e^{+\frac{|\tau|}{RC}}$$

$$e^{\frac{|\tau|}{RC}} \geq \frac{10\eta}{4RC} \Rightarrow |\tau| \geq RC \log\left(\frac{10\eta}{4RC}\right)$$

Q: The i/p voltage to RL LPF ckt is a stationary r.p $x(t)$ $E[x(t)] = 2$, $R_{xx}(\tau) = 4 + e^{-2|\tau|}$ let $y(t)$ be the voltage across the resistor find $E[y(t)]$, $S_{yy}(\omega)$

A $H(\omega) = \frac{R}{R + j\omega L} = \frac{1}{1 + \frac{j\omega L}{R}}$ $S_{xx}(\omega) = 8\pi\delta(\omega) + \frac{2.2}{2^2 + \omega^2}$

$$|H(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega L}{R}\right)^2} \qquad \bar{x} = 2$$

$H(0) = 1$, $E[y(t)] = H(0)\bar{x} = 1(2) = 2$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

$$= \left[\frac{1}{1 + \left(\frac{\omega L}{R}\right)^2}\right]\left(8\pi\delta(\omega) + \frac{4}{4 + \omega^2}\right)$$

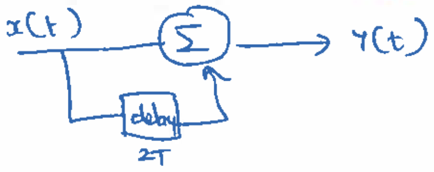
Q: A WSS r.p $x(t)$ is applied to the i/p of an LTI system whose impulse response is $5t\, e^{-2t}$ find the mean of the o/p of the system, if the mean of the i/p is 3.

$$h(t) = 5t\, e^{-2t}$$

A: $E[x(t)] = 3$ $H(\omega) = \frac{5}{(2 + j\omega)^2}$ $H(0) = \frac{5}{4}$

$E[y(t)] = H(0)\, E[x(t)] = \ldots 15/4$

Q: A WSS RP $x(t)$ with PSD $S_{xx}(\omega)$ is applied as the i/p of following system. find the PSD of $Y(t)$



A: $Y(t) = x(t) + x(t-2T)$

$$Y(\omega) = X(\omega) + X(\omega)\, e^{-j\omega 2T}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + e^{-j2\omega T} = 1 + \cos 2\omega T - j \sin 2\omega T$$

$$|H(\omega)|^2 = (1+\cos 2\omega T)^2 + (\sin 2\omega T)^2$$

$$= 2 + 2\cos 2\omega T = 2(1+\cos 2\omega T) = 4\cos^2 \omega T$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = 4\cos^2\omega T \cdot S_{xx}(\omega)$$

Q: A RP $x(t) = A \sin(\omega_0 t + \theta)$ where A & ω_0 are real +ve constants, θ is a RV uniformly distributed $(-\pi, \pi)$ is applied to the n/w. find the expression for the n/w response

$$x(t) \longrightarrow \boxed{h(t) = W u(t) e^{-Wt}} \longrightarrow y(t)$$

A: $$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} A \sin(\omega_0\tau + \theta) \cdot W u(t-\tau)\, e^{-W(t-\tau)}\, d\tau$$

$$= A W e^{-Wt} \int_{-\infty}^{t} e^{W\tau} \sin(\omega_0\tau + \theta)\, d\tau$$

Note: $$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$$

$$y(t) = A W e^{-Wt} \left[\frac{e^{W\tau}}{W^2+\omega_0^2}\left\{W \sin(\omega_0\tau+\theta) - \omega_0 \cos(\omega_0\tau+\theta)\right\}\right]_{-\infty}^{t}$$

$$= \frac{AW}{\omega^2+\omega_0^2}\left[\omega \sin(\omega_0 t+\theta) - \omega_0 \cos(\omega_0 t+\theta)\right]$$ 5.15

Q: A random noise x(t) having power spectrum $S_{xx}(\omega) = \frac{3}{49+\omega^2}$ is applied to a n/w for which $h(t) = u(t)\, t^2 e^{-7t}$ the n/w response is y(t) find 1) avg power ~~h(t)=u(t)t²e⁻⁷ᵗ~~ of x(t) (2) avg power of y(t) (3) psd of y(t).

A: $$S_{xx}(\omega) = \frac{3}{49+\omega^2} \qquad R_{xx}(\tau) = \frac{3}{14} e^{-7|\tau|}$$

1) average power of $x(t) = R_{xx}(0) = \frac{3}{14}$ watts

3) $$H(\omega) = \frac{2}{(7+j\omega)^2} \qquad S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

$$|H(\omega)|^2 = \frac{4}{(49+\omega^2)^2} \qquad = \frac{4}{(49+\omega^2)^2}\cdot\frac{3}{(7^2+\omega^2)}$$

$$= \frac{12}{(7^2+\omega^2)^3}$$

2) avg power of y(t) is

$$P_{yy} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)\, d\omega \qquad \omega = 7\tan\theta$$
$$d\omega = 7\sec^2\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{12}{(7^2+\omega^2)^3}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{12}{(7^2+7^2\tan^2\theta)^4}\, d\theta \cdot 7\sec^2\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{12}{7^8(1+\tan^2\theta)^4}\, d\theta \quad 7\sec^2\theta\, d\theta$$

$$= \frac{12}{2\pi 7^7}\int^{\pi/2} \frac{1}{\sec^8\theta}\cdot \sec^2\theta\, d\theta$$

$$= \frac{12}{2\pi (7)^7} \int_{-\pi/2}^{\pi/2} \cos^6\theta \, d\theta$$

$$\int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)} \frac{\pi}{2}$$

$$= \frac{\cancel{12}}{2\cancel{\pi}(7)^7} \cdot 2 \cdot \frac{5\times3\times1}{\cancel{6}\times4\times\cancel{2}} \cdot \frac{\cancel{\pi}}{2}$$

$$= \frac{15}{8(7)^7}$$

Q: A RP $n(t)$ has a psd $\frac{\eta}{2}$ $-\infty < f < \infty$ RP is applied through a LPF which has T.F $H(f) = 2$ for $-f_m \le f \le f_m$ find the PSd of at the o/p of the filter.

A: $$\text{o/p Psd} = |H(\omega)|^2 \cdot \text{i/p Psd} = |2|^2 \cdot \frac{\eta}{2} = 2\eta \qquad |f| \le f_m$$

Q find the o/p Psd, $R_{YY}(\tau)$ of an RC LPF which is subjected to a white noise psd of $\frac{N_0}{2}$.

A: $$S_{XX}(\omega) = \frac{N_0}{2} \qquad R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$H(\omega) = \frac{1/j\omega C}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

$$|H(\omega)|^2 = \frac{1}{1 + (\omega RC)^2}$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \frac{1}{1+(\omega RC)^2} \cdot \frac{N_0}{2}$$

$$R_{YY}(\tau) = \frac{N_0}{2} F^{-1}\left[\frac{1}{1+\omega^2 R^2C^2}\right] = \frac{N_0}{4RC} F^{-1}\left(\frac{2\cdot\frac{1}{RC}}{\omega^2 + \left(\frac{1}{RC}\right)^2}\right)$$

$$= \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}}$$

Q: $x(t)$ is a stationary RP with zero mean value & $R_{xx}(\tau) = e^{-2|\tau|}$ is applied to system of fn $H(\omega) = \frac{1}{2+j\omega}$ find mean & psd of its o/p.

A: $E[x(t)] = 0$, $H(\omega) = \frac{1}{2+j\omega}$, $H(0) = \frac{1}{2}$

$$E(y(t)) = H(0)\,E(x(t)) = \frac{1}{2}(0) = 0$$

$$R_{xx}(\tau) = e^{-2|\tau|} \qquad S_{xx}(\omega) = \frac{2 \cdot 2}{2^2+\omega^2} = \frac{4}{4+\omega^2}$$

$$|H(\omega)|^2 = \frac{1}{4+\omega^2}$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = \frac{4}{4+\omega^2} \cdot \frac{1}{4+\omega^2} = \frac{4}{(4+\omega^2)^2}$$

Q: A signal $x(t) = u(t)\,e^{-\alpha t}$ is applied to a n/w having an impulse response $h(t) = \omega\, u(t)\, e^{-\omega t}$ here α & ω are real +ve constants and $u(t)$ is an unit step fn find the o/p $y(t)$ & spectrum $Y(\omega)$

A: $H(\omega) = \frac{\omega}{\omega + j\omega}$ $\qquad X(\omega) = \frac{1}{\alpha + j\omega}$

$$Y(\omega) = X(\omega)\,H(\omega) = \frac{\omega}{\omega+j\omega} \cdot \frac{1}{\alpha+j\omega}$$

$$= \frac{\omega}{\omega-\alpha}\left[\frac{1}{\alpha+j\omega} - \frac{1}{\omega+j\omega}\right]$$

$$y(t) = \frac{\omega}{\omega-\alpha}\left[e^{-\alpha t}u(t) - e^{-\omega t}u(t)\right]$$

Q: A i/p $x(t)$ is applied to a n/w with impulse response $h(t) = e^{-bt}u(t)$ where $b>0$ is a constant. The ccf of $x(t)$ with o/p $y(t)$ is $R_{xy}(\tau) = u(\tau)\tau e^{-b\tau}$ find the acf of $y(t)$.

A: $h(t) = e^{-bt}u(t)$ $\qquad R_{xy}(\tau) = u(\tau)\tau e^{-b\tau}$

$H(\omega) = \dfrac{1}{(b+j\omega)}$ $\qquad S_{xy}(\omega) = \dfrac{1}{(b+j\omega)^2}$

$|H(\omega)|^2 = \dfrac{1}{b^2+\omega^2}$ $\qquad R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$

$S_{yy}(\omega) = S_{xy}(\omega)\, H^*(\omega)$

$$S_{yy}(\omega) = \frac{1}{(b+j\omega)^2}\cdot\frac{1}{(b-j\omega)} = \frac{1}{(b^2+\omega^2)(b+j\omega)}$$

$$= \frac{1}{4b^2}\left[\frac{1}{b+j\omega} + \frac{1}{b-j\omega}\right] + \frac{1}{2b}\left[\frac{1}{(b+j\omega)^2}\right]$$

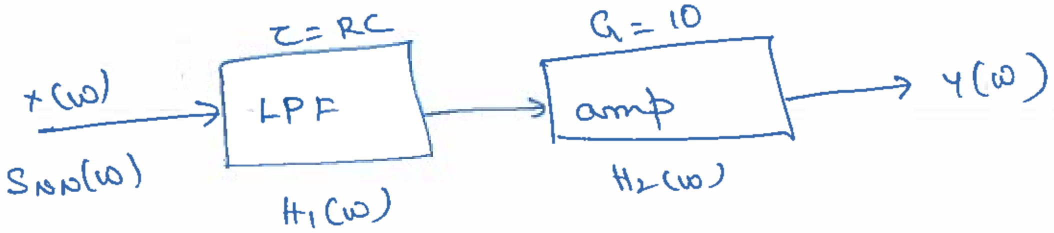
$$R_{yy}(\tau) = \frac{1}{4b^2}\left[e^{-b\tau}u(\tau) + e^{b\tau}u(\tau)\right] + \frac{1}{2b}\tau e^{-b\tau}u(\tau)$$

Q: white noise with two sided psd $\frac{\eta}{2}$ is passed through a LP RC n/w with time constant $\tau = RC$ and there after through an ideal amplifier with a voltage gain of 10.

a) write an expression for the acf of the white noise

b) " " " " " psd of the noise at the o/p of the amplifier.

c) write an expression for the acf of the o/p noise in (b)



$$S_{NN}(\omega) = S_{xx}(\omega) = \frac{\eta}{2}$$

$$H_1(\omega) = \frac{1}{1+j\omega RC} \qquad H_2(\omega) = 10$$

$$H(\omega) = H_1(\omega)\, H_2(\omega) = \frac{10}{1+j\omega RC}$$

a) $$R_{NN}(\tau) = F^{-1}\left(S_{NN}(\omega)\right) = F^{-1}\left(\frac{\eta}{2}\right) = \frac{\eta}{2}\,\delta(\tau)$$

b) $$S_{YY}(\omega) = |H(\omega)|^2 S_{NN}(\omega) = \frac{100}{1+(\omega RC)^2} \cdot \frac{\eta}{2}$$

$$= 50\eta \cdot \frac{1}{(RC)^2} \cdot \frac{1}{2 \cdot \frac{1}{RC}} \cdot \frac{2 \cdot \frac{1}{RC}}{\omega^2 + \left(\frac{1}{RC}\right)^2}$$

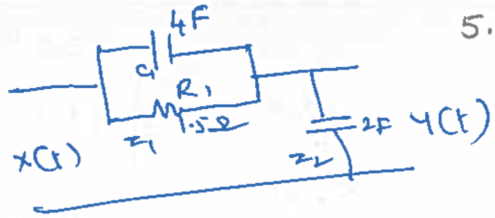
c) $$R_{YY}(\tau) = F^{-1}\left[S_{YY}(\omega)\right]$$

$$= \frac{25\eta}{RC}\, F^{-1}\left[\frac{2 \cdot \frac{1}{RC}}{\omega^2 + \left(\frac{1}{RC}\right)^2}\right]$$

$$= \frac{25\eta}{RC}\, e^{-\frac{|\tau|}{RC}}$$

Q: A stationary r.p $x(t)$ having an acf $R_{xx}(\tau) = 2e^{-4|\tau|}$ is applied to the n/w shown below find

1) $S_{xx}(\omega)$ 2) $|H(\omega)|^2$ 3) $S_{YY}(\omega)$



1: $Z_1 = \dfrac{R \cdot \frac{1}{j\omega C_1}}{R + \frac{1}{j\omega C_1}} = \dfrac{R}{1 + j\omega C_1 R}$

$$H(\omega) = \frac{Z_2}{Z_1 + Z_2} = \frac{1/j\omega C_2}{\frac{1}{j\omega C_2} + \frac{R}{1+j\omega C_1 R}} = \frac{1 + j\omega C_1 R}{1 + j\omega R (C_1 + C_2)}$$

$R = 1.5\Omega$
$C_1 = 4F$
$C_2 = 2F$

$$= \frac{1 + j\omega 6}{1 + j\omega 9}$$

$$S_{xx}(\omega) = 2 \cdot \frac{2 \cdot 4}{4^2 + \omega^2} = \frac{16}{16 + \omega^2}$$

$$|H(\omega)|^2 = \frac{1 + 36\omega^2}{1 + 81\omega^2}$$

3) $$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = \frac{(1 + 36\omega^2)}{(1 + 81\omega^2)} \cdot \frac{16}{(\omega^2 + 16)}$$

Q: white noise with Psd $\frac{N_o}{2}$ is applied to a n/w with impulse response $h(t) = u(t)\, Wt\, e^{-Wt}$ where $W > 0$ is a constant find the ccf of i/p & o/p.

1: $S_{xx}(\omega) = \dfrac{N_o}{2}$ $\qquad h(t) = Wt\, e^{-Wt}$

$$H(\omega) = \frac{W}{(W + j\omega)^2}$$

$$S_{xy}(\omega) = S_{xx}(\omega) H(\omega) = \frac{W}{(W + j\omega)^2} \cdot \frac{N_o}{2}$$

$$R_{xy}(\tau) = \frac{N_o}{2} F^{-1}\left[\frac{W}{(W + j\omega)^2}\right] = \frac{N_o}{2} W\tau e^{-W\tau} u(\tau)$$

Q: A i/p $x(t)$ has a Psd $G(f) = 10^{-4}$ $\; -\alpha < f \le \alpha$ the i/p is passed through an LPF whose T.F is $H(f) = 100$ $\; -f_M \le f \le f_M$ find the Psd of the waveform at the o/p of the filter.

A: $G(f) = 10^{-4} \quad -\alpha \le f \le \alpha$

$H(f) = 100 \quad -f_m \le f \le f_m$

$$\text{o/p PSD} = |H(f)|^2 G(f) = (100)^2 \times 10^{-4}$$

$$= 1 \quad -f_m \le f \le f_m$$

Q. Consider a linear system as shown below where $x(t)$ is the i/p & $y(t)$ is the o/p of the system. The acf of $x(t)$ is $R_{xx}(\tau) = 5\,\delta(\tau)$. Find the PSD, acf, mean square value of the o/p $y(t)$.

$$x(t) \longrightarrow \boxed{\frac{1}{6+j\omega}} \longrightarrow y(t)$$

A: $H(\omega) = \dfrac{1}{6+j\omega}$ $\qquad R_{xx}(\tau) = 5\,\delta(\tau)$

$\qquad\qquad\qquad\qquad\qquad S_{xx}(\omega) = 5$

$$|H(\omega)|^2 = \frac{1}{36+\omega^2}$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = \frac{5}{36+\omega^2}$$

$$R_{yy}(\tau) = F^{-1}\left(S_{yy}(\omega)\right) = F^{-1}\left(\frac{5}{6^2+\omega^2}\right) = \frac{5}{12} F^{-1}\left(\frac{2 \cdot 6}{6^2+\omega^2}\right)$$

$$= \frac{5}{12} e^{-6|\tau|}$$

mean square value of $y(t) = R_{yy}(0) = \dfrac{5}{12}$ watts

Questions

1) S.T if the i/p to LTI system is WSS the o/p is also WSS.

2) S.T o/p PSD $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ for a LTI system whose i/p $x(t)$ is WSS.

3) Define white noise.

4) Define colour noise

5) Find the acf of white noise

6) Define Band pass process, Band limited process & Narrow Band process.

7) State the properties of Band limited process.

Additional Problems

1) Consider a linear time invariant system as shown below

$$x(t) \longrightarrow \boxed{\frac{1}{6+j\omega}} \longrightarrow y(t)$$

where $x(t)$ is the i/p & $y(t)$ is the o/p of the system. The acf of $x(t)$ is $R_{xx}(\tau) = 5\,\delta(\tau)$ find the PSD, acf & mean square value of o/p $y(t)$.

2) White noise with power density $\frac{N_0}{2}$ is applied to a n/w with impulse response $h(t) = u(t)\,\omega t\, e^{-\omega t}$, $\omega > 0$ find the cross correlation of i/p & o/p.

3) The o/p of a filter is given by

$y(t) = x(t+T) - x(t-T)$ where $x(t)$ is WSS process with power spectrum $S_{xx}(\omega)$ and T, is a constant find the power spectrum of $y(t)$.

4) A WSS RP $x(t)$ is applied to the i/p of an LTI system whose impulse response is $5t\,e^{-2t}u(t)$. The mean of $x(t)$ is 3. find the mean of the o/p of the system.

5) A WSS process $x(t)$ with $R_{xx}(\tau) = A\,e^{-a|\tau|}$, where A & a are real positive constants is applied to the i/p of an LTI system with $h(t) = e^{-bt}u(t)$ where b is a real positive constant. find the PSD of the o/p of the system.

Multiple choice Questions

1) The System response y(t) of LTI System with i/p x(t) & impulse response h(t) is ________ [ ]

a) $y(t) = x(t) \cdot h(t)$ b) $y(t) = x(t) * h(t)$

c) $y(t) = x(t) + h(t)$ d) None

2) If the i/p to LTI is WSS then the mean value of the o/p is ________ [ ]

a) $\bar{y} = \bar{x} \cdot H(0)$ b) $\bar{y}$ = Constant

c) both a & b d) None

3) The auto correlation fn of response y(t) of an LTI System is ________ if the i/p x(t) is WSS [ ]

a) $R_{yy}(\tau) = R_{xx}(\tau) * h(-\tau) * h(\tau)$

b) $R_{yy}(\tau) = R_{xx}(\tau)\, h(-\tau)\, h(\tau)$

c) $R_{yy}(\tau) = R_{xx}(\tau)$

d) none.

4) If the i/p to LTI System is WSS then o/p is ~~also WSS~~ ________ [ ]

a) Not WSS b) WSS c) 1st order Stationary

d) none.

5) The o/p PSD of LTI System is ________ [ ]

a) $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

b) $S_{YY}(\omega) = H(\omega) \cdot H(-\omega) \cdot S_{XX}(\omega)$

c) both a & b

d) None

6) $S_{XY}(\omega) =$ ______________ [ ]

a) $S_{XX}(\omega)\, H(\omega)$ b) $S_{XX}(\omega)\, H(-\omega)$

c) $S_{XX}(\omega) * H(\omega)$ d) None

7) $S_{YX}(\omega) =$ ______________ [ ]

a) $S_{XX}(\omega)\, H(\omega)$ b) $S_{XX}(\omega)\, H(-\omega)$

c) $S_{XX}(\omega) * H(-\omega)$ d) None

8) For a white noise PSD is ______________ [ ]

a) $S_{NN}(\omega) = \frac{N_0}{2}$ b) $S_{NN}(\omega) = \frac{N_0 \omega}{2}$

c) $S_{NN}(\omega) = \frac{N_0 \omega^2}{2}$ d) None

9) For a white Noise ACF is ______________ [ ]

a) $R_{NN}(\tau) = \frac{N_0}{2}\delta(\tau)$ b) $R_{NN}(\tau) = \tau \frac{N_0}{2}\delta(\tau)$

c) $R_{NN}(\tau) = \tau^2 \frac{N_0}{2}\delta(\tau)$ d) None

10) The ccf b/w $x(t)$ & $y(t)$ is $R_{XY}(\tau) =$ ______ [ ]

a) $h(\tau) * R_{XX}(\tau)$ b) $h(-\tau) * R_{XX}(\tau)$ c) $h(-\tau) * R_{XY}(\tau)$

11) If the PdS $S_{NN}(\omega)$ of a RP has its significant component clustered in a bandwidth B_N Hz that does not include $\omega = 0$. Then the process is ______ [ ]

a) band limited   b) band pass

c) narrow band   d) Stationary

12) If the power spectrum of a band pass RP is zero outside some freq bandwidth W that does not include $\omega = 0$, the process is called ______ [ ]

a) band limited   b) band pass

c) narrow band   d) Stationary

13) A process is said to be narrow band if the frequency bandwidth W is ______ freq near band centre [ ]

a) equal to   b) much greater than

c) much lesser than   d) twice the

14) If $R_{xx}(\tau) = 3\delta(\tau)$ & $H(\omega) = \frac{1}{6+j\omega}$ Then the mean square value of o/p is ______ ( )

a) 3   b) 4   c) $\frac{1}{3}$ d) $\frac{1}{4}$

15) $R_{yy}(\tau) =$ ______ ( )

a) $R_{xy}(\tau) * h(-\tau)$ b) $R_{xy}(\tau) * h(\tau)$

c) $R_{xy}(\tau) \cdot h(\tau)$ d) $R_{xy}(\tau)\, h(-\tau)$

MCQ-4

16) $R_{yy}(\tau) =$ ________ [ ]

a) $R_{yx}(\tau) * h(\tau)$

b) $R_{yx}(\tau) * h(-\tau)$

c) $R_{yx}(\tau) \cdot h(\tau)$

d) $R_{yx}(\tau) \cdot h(-\tau)$

.08	.09
.5319	.5359
.5714	.5753
.6103	.6141
.6480	.6517
.6844	.6879
.7190	.7224
.7517	.7549
.7823	.7852
.8106	.8133
.8365	.8389
.8599	.8621
.8810	.8830
.8997	.9015
.9162	.9177
.9306	.9319
.9429	.9441
.9535	.9545
.9625	.9633
.9699	.9706
.9761	.9767
.9812	.9817
.9854	.9857
.9887	.9890
.9913	.9916
.9934	.9936
.9951	.9952
.9963	.9964
.9973	.9974
.9980	.9981
.9986	.9986
.9990	.9990
.9993	.9993
.9995	.9995
.9996	.9997
.9998	.9998
.9998	.9998
.9999	.9999
.9999	.9999
.0000	1.0000

ent approx-

(B-8)

solute rela-
any $x \geq 0$.
approxima-

APPENDIX C

Useful Mathematical Quantities

TRIGONOMETRIC IDENTITIES

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \qquad \text{(C-1)}$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \qquad \text{(C-2)}$$

$$\cos\left(x \pm \frac{\pi}{2}\right) = \mp\sin(x) \qquad \text{(C-3)}$$

$$\sin\left(x \pm \frac{\pi}{2}\right) = \pm\cos(x) \qquad \text{(C-4)}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \text{(C-5)}$$

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \text{(C-6)}$$

$$2\cos(x) = c^{jx} + e^{-jx} \qquad \text{(C-7)}$$

$$2j\sin(x) = e^{jx} - e^{-jx} \qquad \text{(C-8)}$$

$$2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y) \qquad \text{(C-9)}$$

$$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y) \qquad \text{(C-10)}$$

$$2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y) \qquad \text{(C-11)}$$

$$2\cos^2(x) = 1 + \cos(2x) \qquad \text{(C-12)}$$

$$2\sin^2(x) = 1 - \cos(2x) \qquad \text{(C-13)}$$

$$4\cos^3(x) = 3\cos(x) + \cos(3x) \qquad \text{(C-14)}$$

$$4\sin^3(x) = 3\sin(x) - \sin(3x) \qquad \text{(C-15)}$$

$$8\cos^4(x) = 3 + 4\cos(2x) + \cos(4x) \qquad \text{(C-16)}$$

$$8\sin^4(x) = 3 - 4\cos(2x) + \cos(4x) \tag{C-17}$$

$$A\cos(x) - B\sin(x) = R\cos(x+\theta) \tag{C-18}$$

where

$$R = \sqrt{A^2 + B^2} \tag{C-19a}$$

$$\theta = \tan^{-1}(B/A) \tag{C-19b}$$

$$A = R\cos(\theta) \tag{C-19c}$$

$$B = R\sin(\theta) \tag{C-19d}$$

INDEFINITE INTEGRALS

Rational Algebraic Functions

$$\int (a+bx)^n\,dx = \frac{(a+bx)^{n+1}}{b(n+1)} \qquad 0 < n \tag{C-20}$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a+bx| \tag{C-21}$$

$$\int \frac{dx}{(a+bx)^n} = \frac{-1}{(n-1)b(a+bx)^{n-1}} \qquad 1 < n \tag{C-22}$$

$$\begin{aligned}\int \frac{dx}{c+bx+ax^2} &= \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) && b^2 < 4ac \\ &= \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| && b^2 > 4ac \\ &= \frac{-2}{2ax+b} && b^2 = 4ac \end{aligned} \tag{C-23}$$

$$\int \frac{x\,dx}{c+bx+ax^2} = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{2a}\int \frac{dx}{c+bx+ax^2} \tag{C-24}$$

$$\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) \tag{C-25}$$

$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(a^2+x^2) \tag{C-26}$$

$$\int \frac{x^2\,dx}{a^2+x^2} = x - a\tan^{-1}\left(\frac{x}{a}\right) \tag{C-27}$$

$$\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) \tag{C-28}$$

$$\int \frac{x\,dx}{(a^2+x^2)^2} = \frac{-1}{2(a^2+x^2)} \qquad \text{(C-29)}$$

$$\int \frac{x^2\,dx}{(a^2+x^2)^2} = \frac{-x}{2(a^2+x^2)} + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \text{(C-30)}$$

$$\int \frac{dx}{(a^2+x^2)^3} = \frac{x}{4a^2(a^2+x^2)^2} + \frac{3x}{8a^4(a^2+x^2)} + \frac{3}{8a^5}\tan^{-1}\left(\frac{x}{a}\right) \qquad \text{(C-31)}$$

$$\int \frac{x^2\,dx}{(a^2+x^2)^3} = \frac{-x}{4(a^2+x^2)^2} + \frac{x}{8a^2(a^2+x^2)} + \frac{1}{8a^3}\tan^{-1}\left(\frac{x}{a}\right) \qquad \text{(C-32)}$$

$$\int \frac{x^4\,dx}{(a^2+x^2)^3} = \frac{a^2x}{4(a^2+x^2)^2} - \frac{5x}{8(a^2+x^2)} + \frac{3}{8a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \text{(C-33)}$$

$$\int \frac{dx}{(a^2+x^2)^4} = \frac{x}{6a^2(a^2+x^2)^3} + \frac{5x}{24a^4(a^2+x^2)^2} + \frac{5x}{16a^6(a^2+x^2)} + \frac{5}{16a^7}\tan^{-1}\left(\frac{x}{a}\right) \qquad \text{(C-34)}$$

$$\int \frac{x^2\,dx}{(a^2+x^2)^4} = \frac{-x}{6(a^2+x^2)^3} + \frac{x}{24a^2(a^2+x^2)^2} + \frac{x}{16a^4(a^2+x^2)} + \frac{1}{16a^5}\tan^{-1}\left(\frac{x}{a}\right) \qquad \text{(C-35)}$$

$$\int \frac{x^4\,dx}{(a^2+x^2)^4} = \frac{a^2x}{6(a^2+x^2)^3} - \frac{7x}{24(a^2+x^2)^2} + \frac{x}{16a^2(a^2+x^2)} + \frac{1}{16a^3}\tan^{-1}\left(\frac{x}{a}\right) \qquad \text{(C-36)}$$

$$\int \frac{dx}{a^4+x^4} = \frac{1}{4a^3\sqrt{2}}\ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) + \frac{1}{2a^3\sqrt{2}}\tan^{-1}\left(\frac{ax\sqrt{2}}{a^2-x^2}\right) \qquad \text{(C-37)}$$

$$\int \frac{x^2\,dx}{a^4+x^4} = -\frac{1}{4a\sqrt{2}}\ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) + \frac{1}{2a\sqrt{2}}\tan^{-1}\left(\frac{ax\sqrt{2}}{a^2-x^2}\right) \qquad \text{(C-38)}$$

Trigonometric Functions

$$\int \cos(x)\,dx = \sin(x) \qquad \text{(C-39)}$$

$$\int x\cos(x)\,dx = \cos(x) + x\sin(x) \qquad \text{(C-40)}$$

$$\int x^2\cos(x)\,dx = 2x\cos(x) + (x^2-2)\sin(x) \qquad \text{(C-41)}$$

$$\int \sin(x)\,dx = -\cos(x) \tag{C-42}$$

$$\int x\sin(x)\,dx = \sin(x) - x\cos(x) \tag{C-43}$$

$$\int x^2\sin(x)\,dx = 2x\sin(x) - (x^2-2)\cos(x) \tag{C-44}$$

Exponential Functions

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} \qquad a \text{ real or complex} \tag{C-45}$$

$$\int xe^{ax}\,dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right] \qquad a \text{ real or complex} \tag{C-46}$$

$$\int x^2e^{ax}\,dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right] \qquad a \text{ real or complex} \tag{C-47}$$

$$\int x^3e^{ax}\,dx = e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right] \qquad a \text{ real or complex} \tag{C-48}$$

$$\int e^{ax}\sin(x)\,dx = \frac{e^{ax}}{a^2+1}[a\sin(x) - \cos(x)] \tag{C-49}$$

$$\int e^{ax}\cos(x)\,dx = \frac{e^{ax}}{a^2+1}[a\cos(x) + \sin(x)] \tag{C-50}$$

DEFINITE INTEGRALS

$$\int_{-\infty}^{\infty} e^{-a^2x^2+bx}\,dx = \frac{\sqrt{\pi}}{a}e^{b^2/(4a^2)} \qquad a > 0 \tag{C-51}$$

$$\int_0^{\infty} x^2e^{-x^2}\,dx = \sqrt{\pi}/4 \tag{C-52}$$

$$\int_0^{\infty} \mathrm{Sa}(x)\,dx = \int_0^{\infty}\frac{\sin(x)}{x}\,dx = \frac{\pi}{2} \tag{C-53}$$

$$\int_0^{\infty} \mathrm{Sa}^2(x)\,dx = \pi/2 \tag{C-54}$$

FINITE

$\sum_{n=1}^{N} n =$

$\sum_{n=1}^{N} n^2 =$

$\sum_{n=1}^{N} n^3 =$

$\sum_{n=0}^{N} x^n =$

$\sum_{n=0}^{N} \frac{N}{n!(N-}$

$\sum_{n=0}^{N} e^{j(\theta+n\phi}$

$\sum_{n=0}^{N} \binom{N}{n} =$

$\sum_{n=N_1}^{N_2} w^n =$

INFINITE

$e^x = 1 + x$

(C-42)
(C-43)
(C-44)
(C-45)
(C-46)
(C-47)
(C-48)
(C-49)
(C-50)
(C-51)
(C-52)
(C-53)
(C-54)

FINITE SERIES

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \tag{C-55}$$

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6} \tag{C-56}$$

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4} \tag{C-57}$$

$$\sum_{n=0}^{N} x^n = \frac{x^{N+1}-1}{x-1} \tag{C-58}$$

$$\sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^n y^{N-n} = (x+y)^N \tag{C-59}$$

$$\sum_{n=0}^{N} e^{j(\theta+n\phi)} = \frac{\sin[(N+1)\phi/2]}{\sin(\phi/2)} e^{j[\theta+(N\phi/2)]} \tag{C-60}$$

$$\sum_{n=0}^{N} \binom{N}{n} = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} = 2^N \tag{C-61}$$

$$\sum_{n=N_1}^{N_2} w^n = \frac{w^{N_1} + w^{N_2+1}}{1-w} \qquad \begin{cases} N_2 > N_1 \text{ and } w \\ \text{real or complex} \end{cases} \tag{C-62}$$

INFINITE SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{C-63}$$