Combinatrics

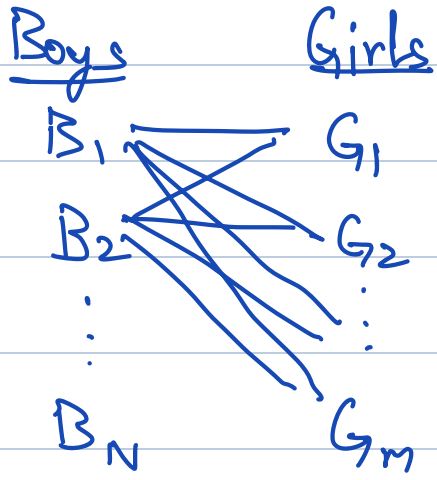
Today's Content:

- Addition and Multiplication Rule
- Permutations Basics
- Combination basics and properties.
- Pascal Triangle
- Find N-th column title.

Contest - 5th July 2024

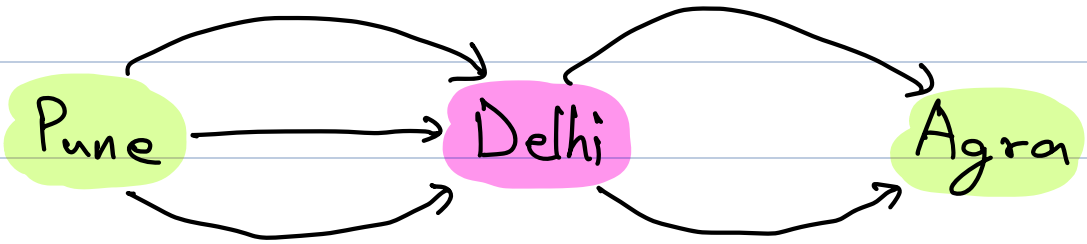
- Recursion, Backtracking
- Maths (MA & GCD, Combinatrics, Prime Numbers)
- OOPS

Question: Given 10 girls and 7 boys, how many different pairs can be formed?



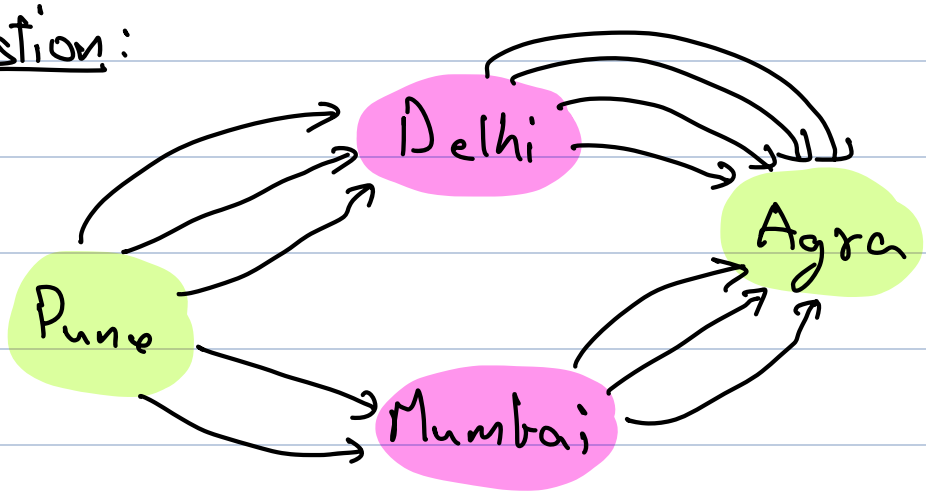
Ways of making 1 pair $= 7 * 10$
$= 70$

Question: Number of ways from Pune to Agra?



Ways to Delhi from Pune $= 3$
Ways to Agra from Delhi $= 2$
Ways to Agra from Delhi and Pune $= 3 * 2$
$= 6$.

Question:



AND → *
OR → +

to Agra from Delhi	to Agra from Mumbai
to Delhi = 3	to Mumbai = 2
to Agra = 4	to Agra = 3
Total = 3*4 = 12	Total = 2*3 = 6

Total = 12 + 6 = 18

Question: A[i][0] → Main course, A[i][1] → Dessert
A[i][2] → Beverage. Find which restaurant offers maximum choices to the customer

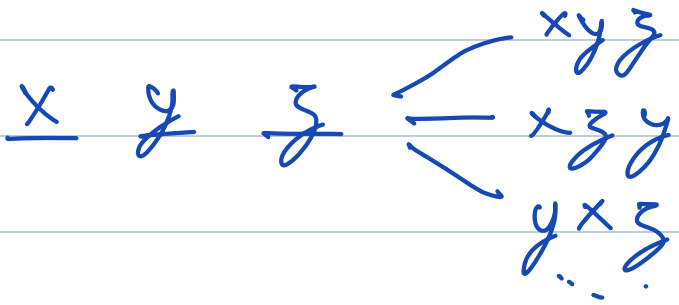
$$A = \begin{bmatrix} 3 & 2 & 2 \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

- R1 - 3*2*2 = 12
- R2 - 4*3*3 = 36
- R3 - 1*1*1 = 1

O/P = R2

Permutations - Ways of arrangements.

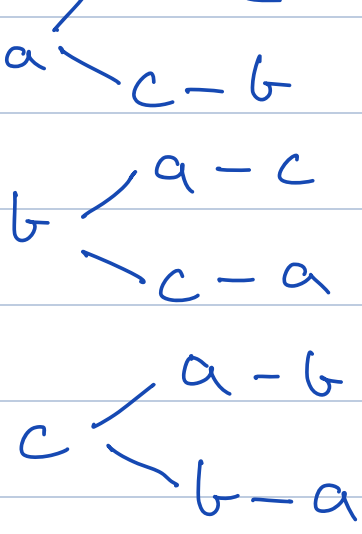
↳ Order matters



Question: Given N distinct characters, in how many ways can you arrange them?

s = a b c

3 * 2 * 1 = 6



s = a b c d

4 * 3 * 2 * 1 = 24

For N: N N-1 N-2 . . 2 1

$N*(N-1)*(N-2)*\cdots 2*1 = N!$

Question: How many ways can we arrange 2 out of the given characters?

$s = a\ b\ c\ d$

$4 * 3 = \boxed{12}$

$N = 4, r = 2$

${}^{N}P_r = {}^{4}P_2 = \frac{4!}{2!}$

$= \frac{24}{2} = \boxed{12}$

$s = a\ b\ c$

$3 * 2 = 6$

$N = 3, r = 2$

${}^{N}P_r = {}^{3}P_2$

$= \frac{3!}{1!} = \frac{6}{1} = 6$

$s = a\ b\ c\ d\ e$

$5 * 4 = 20$

$N = 5, r = 2$

${}^{N}P_r = {}^{5}P_2$

$= \frac{5!}{3!} = \frac{120}{6}$

$= 20$

Question: From N distinct characters, arrange r characters.

$\underbrace{N}_{1st} \quad \underbrace{N-1}_{2nd} \quad \underbrace{N-2}_{3rd} \quad \underbrace{N-3} \quad \ldots \quad \underbrace{(N-(r-1))}_{r\text{th}}$

$= N * (N-1) * (N-2) * \ldots\ldots (N-(r-1))$

$= \frac{N * (N-1) * (N-2) * \ldots\ldots (N-(r-1)) * (N-r) * (N-r-1) \cdots 2 * 1}{(N-r) * (N-r-1) \cdots 2 * 1}$

$= \frac{N!}{(N-r)!} = {}^{N}P_r =$ Ways of arranging r ele. out of N elements

Combination - Ways of selection

↳ Order does not matter.

$(i, j) == (j, i)$

Question: Given 4 players, count the number of ways of selecting 3 players

Available = P_1 P_2 P_3 P_4

Selected = P_1 P_2 P_3

P_1 P_2 P_4

P_1 P_3 P_4

P_2 P_3 P_4

} Combinations / Selections

$r!$	$r!$	$r!$	$r!$
P_1 P_2 P_3	P_1 P_2 P_4	P_1 P_3 P_4	P_2 P_3 P_4
P_1 P_3 P_2	P_1 P_4 P_2	P_1 P_4 P_3	P_2 P_4 P_3
P_2 P_1 P_3	P_2 P_1 P_4	P_3 P_1 P_4	P_3 P_2 P_4
P_2 P_3 P_1	P_2 P_4 P_1	P_3 P_4 P_1	P_3 P_4 P_2
P_3 P_1 P_2	P_4 P_1 P_2	P_4 P_1 P_3	P_4 P_2 P_3
P_3 P_2 P_1	P_4 P_2 P_1	P_4 P_3 P_1	P_4 P_3 P_2

$\{P_1, P_2, P_3\}$ $\{P_1, P_2, P_4\}$ $\{P_1, P_3, P_4\}$ $\{P_2, P_3, P_4\}$

Question: Given N elements, how many ways can you select r items?

Total No. of selections * No. of arrangements of each selection = Total No. of Arrangements

Total No. of selections $* r! = {}^{N}P_{r}$

$$= \frac{N!}{(N-r)!}$$

$$\text{Combinations} = \frac{N!}{(N-r)! * r!} = {}^{N}C_{r}$$

Properties of Combinations

1) $^nC_0 = \frac{n!}{n! * 0!} = 1$

2) $^nC_n = \frac{n!}{0! * n!} = 1$

3) $^nC_{n-r} = \frac{n!}{(n-(n-r))! * (n-r)!} = \frac{n!}{(n-r)!\, r!} = {}^nC_r$

$$\boxed{^nC_r = {}^nC_{n-r}}$$

4) $^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

_ _ _ _ _ _ _ _ → r spots / N players

1 selected → _ _ _ _ _ → r-1 spots / (N-1) players.

Not selecting → _ _ _ _ _ → r spots / (N-1) players

1 player

Question: Generate Pascal's triangle for given value of N.

N = 4 O/P

$^0C_0$

$^1C_0$ $^1C_1$

$^2C_0$ $^2C_1$ $^2C_2$

$^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$

$^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

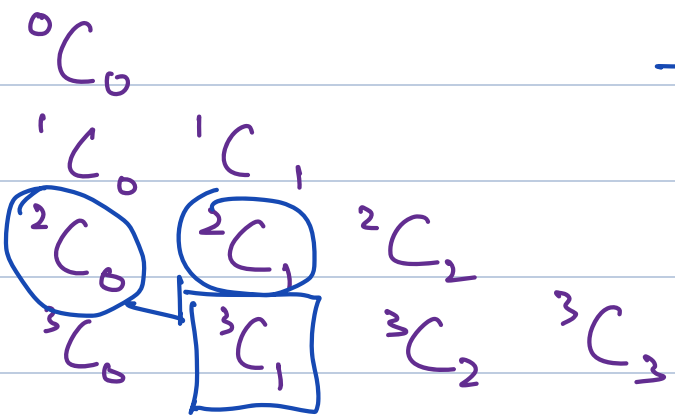
Brute Force Approach

```
ans = []
for (i=0; i<=N; i++){
    ans[i] = []
    for (j=0 ; j <= i ; j++){
        ans[i].add ( iCj ); → O(N)
    }
}
return ans;
```

$$\frac{i*(i-1)*(i-2)\cdots}{(i-j)*(i-j-1)*\cdots*j*(j-1)*\cdots}$$

T.C = $O(N^3)$ S.C = $O(1)$

Optimised



→

1
1 1
1 2 1
1 3 3 1

$$^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

$$^3C_1 = {}^2C_0 + {}^2C_1$$

```
func PascalsTriangle ( int n ) {
    ans [n+1][n+1];
    ans [0][0] = 1;
    for ( i = 1 ; i <= n ; i++ ) {
        ans[i][0] = 1 ;   ans[i][i] = 1 ;
        for ( j = 1 ; j < i ; j++ ) {
            ans[i][j] = ans[i-1][j] + ans[i-1][j-1];
        }
}
return ans; }
```

T.C = $O(N^2)$

S.C = $O(1)$

Question: Find the nth column title

1	2	3	4	...	26	27	...	52	53	...
A	B	C	D	...	Z	AA	AB	...	AZ	BA	...

N = 30 → O/P = AD

N = 50 → O/P = AX

(1 - 26) → (A - Z)
(27 - 52) → (AA - AZ)
(53 - 78) → (BA - BZ)
:

100 → CV
104 → CZ
103 → CY
102 → CX
101 → CW
100 → CV

Base 2	Base 8	Base 26
0	0	A
1	1	B
10	2	:
11	3	Z
100	4	AA
101	5	AB
.	6	:
.	7	AZ
.	10	BA
	11	:
	:	

A → 1 B → 2 - - - Z → 26.

100 →

26	100
	3
	0

22 → V ↑

3 → C

78 →

26	78
	3
	0

0 →

3 → C

Way 1

78 →

26	78
26	2
	0

26 → 2 ↑

2 → B

Way 2 → Convert to 0 index and then do conversion at each step

78th (1 index) → 77th column (0 index).

A → 0, B → 1 ... Z → 25

26	78 - 1	25 → Z
26	2 - 1	1 → B
	0	

```
func nthCol(int n){
    n = n - 1                          N = 1 → B
    char[] ans                                 ↓
    while(n > 0){                              66
        int quotient = n/26 - 1
        int remainder = n%26
        char cur = (char)(remainder + 65)
        n = quotient
        ans = cur + ans
    }
```

```
    return ans;
}
```

$TC = O(\log_{26} N)$ $\quad S.C = O(1)$