2019 MCM Problem A: A Game of Ecology

Math 42 Spring 2021 Final Project

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Abstract

In the Game of Ecology problem, we were tasked with analyzing the characteristics and ecological interactions of three different dragons. We developed a first-order multivariable discrete time model that tracked four state variables over time: the weight of the dragon W in kilograms, its daily calorie requirement C, its height H, as well as the population P of the prey consumed by the dragon. After the construction of this baseline model, we altered parameters to represent the states of three dragons which each lived in a different climate. This allowed us to explore the impact of environmental factors on the growth of dragons. Interestingly, we found that dragons in cold, arctic environments tend to weigh more than those in warmer, arid regions. Furthermore, in larger habitats dragons can reach a greater weight due to the presence of an increased prey population. We then extended our original model to include stochasticity by modelling weight and height as random normal variables, and also considered environmental catastrophes that follow a Bernoulli distribution which affect the model parameters in different ways. Ultimately, however, the inclusion of stochasticity did not change the inherent behavior of the model.

Contributions

- **Shail:** Abstract, Sections 1, 2, 3.1, 4.1, 4.2, 4.3, 4.4, 5.
- **Tara** Abstract, Sections 3.2, 4.1, 4.3, 6, editing and formatting.
- **Akshat** Abstract, Sections 3.3, 3.3.1, 4.1, 4.3.

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1 Introduction

We chose to work on the **2019 MCM Problem A: Game of Ecology** which is based on the television series *Game of Thrones*. The problem required us to answer a few questions about dragons and their interactions with the environment. To construct our response, we applied our knowledge of Data-Driven Modeling from MATH 42 to develop a model that tracks different characteristics of the dragons and their respective environments.

1.1 Problem Statement

In the Game Of Ecology problem, we were asked to analyze the characteristics of three different dragons, and how they interacted with their respective environments. We decided to first break down the environmental interactions into two separate categories: ecological requirements (e.g. food, surrounding area, size of surrounding community) and ecological impacts that arise from the predatory and environmentally destructive behavior of dragons.

These factors are intrinsically linked: for instance, the amount of food needed depends on the weight of the dragon, which in turn may depends on the prey population that relies on the area/size of the community the dragon is in. For this reason, we decided to approach this problem by developing a multivariable discrete time model in order to explore the interactions between these multiple variables. As dragons age over several years, we thought it would be more appropriate to treat time as a discrete rather than continuous quantity. We used the fixed time step n, which represents the number of years since birth (or the age of the dragon).

We took an iterative approach to the model by beginning with a simplified general deterministic model that aimed to represent the ecological interaction for a 'typical' dragon. The parameters of this model were then adjusted to reflect the states of the three different dragons. After this, we would increase the complexity of our baseline models by adding random variables that may better represent the real-world uncertainty. In this paper, we focus on two types of stochastic approaches: Environmental stochasticity and Stochasticity in Height and Weight. Adding random chance into our deterministic multivariable discrete time model allows us to model the unpredictability of the real world and also compare the best case and worst case predictions of our model.

2 Assumptions and Variables

2.1 Global Assumptions

The following assumptions will be true for all models unless specified otherwise:

1. *At birth each dragon weighs 10kg, is 50cm tall, and consumes 500 kcals a day.* This assumption for the initial weight is derived from the problem statement itself ("MCM Problem A 2019"). The assumption about initial calories required is based on the average calories consumed by human infants ("How Many Calories Does a Baby Need"). These values are used to specify the initial state variables for our general model.

2. *The initial prey population is 950.*
   This assumption is also used to specify the initial state.

3. *A dragon lives for at least a 100 years.*
   This assumption will be important when simulating our model 100 years into a dragon's life to observe how each state changes.

4. *All dragons are females.*
   This assumption is based upon the typical convention of population modeling, and will be used to specify the coefficients of the Harris-Benedict equation (described later), which is used to calculate the energy expenditure and calories required by a dragon.

5. *There is only one dragon in each habitat, and the amount of prey that are killed is proportional to the weight of this dragon.*
   To simplify our model world, we will assume that the only predator within the system is a single dragon which consumes a certain proportion of the prey population year after year.

6. *A dragon may kill its prey but not consume it.*
   Since dragons are depicted as extremely powerful and aggressive creatures in mythology, we will assume that a dragon may kill prey for purposes besides consumption (Rose). As a result of this assumption, we have different parameters for prey eaten and prey deaths later in the model.

7. *The daily energy expenditure of a dragon follows the Harris-Benedict equation.*
   The Harris-Benedict equation for total daily energy expenditure is the most popular model for energy expenditure and calories required by humans (Douglas et al.). We will assume that dragons' energy expenditure follows the same model.

8. *The environmental conditions will remain the same throughout time.*
   To simplify our model from the real world, we will assume that all environmental conditions remain the same over time. In reality there are several factors such as global warming that can contribute to a change in environmental conditions (MacMillan and Turrentine). However, for the purpose of simplicity, we will assume that in our model world these environmental conditions are held constant over time.

9. *The food eaten and resources available does not impact the height of the dragon.*
   We will assume that the height of the dragon over time is not impacted by any other factors apart from the age of the dragon (n years).

2.2 Defining Variables

Since we are using a multivariable dynamic matrix model, we need to define the different state variables that we will track over time.

2.2.1 State Variables

- W: **Predator's (Dragon) Weight in kg**
- C: **Predator's Calories Required in kcal**
- P: **Prey Population**
- H: **Dragon's Height in cm**

2.2.2 Input Variables

- Time (n) as a discrete variable, where n is the number of years since the birth of the dragon
- The state vector from the current time step (this is a first order recurrence relation).

2.2.3 Output Variables

- The state vector for the next time step

2.2.4 Parameters (Constant Variables)

- s: **Scaling constant between calories and kilograms**
  To convert the excess calories eaten (kcal) into kilograms of weight gained/losted, we will use a proportionality constant for mass conversion of 5/7700.

- α: **Represents the number of calories per prey eaten**
  We will use the assumption that dragons are carnivores who, like lions, consume horse meat. Based upon research 1 pound of horse meat contains 600 kcals ("Game meat, horse, raw nutrition facts"). Each horse weighs on average 1400 pounds (SmokinAlphaMare). If we assume that 30 percent of the horse's weight contains horse meat then we can calculate the number of calories per prey from the following formula:
  $$600 \times 1400 \times 0.3 = 252000$$

- β: **Represents the proportion of preys consumed by the predator in a year**
  We will initially assume that 5 percent of the prey population is consumed each year.

- d: **The activity level of a dragon**
  This is an activity constant for the Harris-Benedict formula (Mucha and Zajac). Since dragons tend to fly large distances, we will assume that their activity level coefficient is 1.9, which is used for 'extremely active' humans.

- b: **Intrinsic growth rate for prey population**
  We will assume that the birth rate for the prey population is approximately 15 percent when it has not reached its carrying capacity.

- v: **Carrying capacity of prey population**
  We will assume that the carrying capacity will initially be 1000 for the prey population.

- u: **Proportionality Constant between Prey Population and Weight of Dragon**
  We will assume that the proportionality constant between the prey population and the weight of the dragon will be 0.0005.

- r: **The intrinsic growth rate for Height**
  We need to specify the intrinsic growth rate for height H, which is essentially the rate at which we expect the height to increase over time before it reaches its stabilized height. Based upon the height growth rate in early childhood of humans, we will assume that the intrinsic growth rate is 50 percent (Brusie).

- k: **Stabilizing height in cm**
  We will initially assume that a dragon's height will stabilize at 200cm.

3 Models

3.1 A Simplified Deterministic Model

3.1.1 Equations

This is a first-order recurrence relation, meaning that we are assuming that the current value of each state only depends on the value of the states in the previous time step (or previous year). In our model, we will be tracking 4 states over time as specified in the last section.

- $W(n+1)$ refers to the weight of the dragon in kg at the time step of $n+1$ years.
- $C(n+1)$ refers to the calories required by the dragon in kcals at the time step of $n+1$ years.
- $P(n+1)$ refers to the prey population at the time step of $n+1$ years.
- $H(n+1)$ refers to the height of the dragon in cm at the time step of $n+1$ years.

We encode these four in the state vector defined by the following recurrence relations:

$$\begin{pmatrix} W(n+1) \\ C(n+1) \\ P(n+1) \\ H(n+1) \end{pmatrix} = \begin{pmatrix} W(n) - s(C(n) - \frac{\alpha\beta P(n)}{365}) \\ d(10W(n) + 6.25H(n) - 5n - 161) \\ (1+b)(1 - \frac{P(n)}{v})P(n) + P(n) - (uP(n)W(n)) \\ H(n)(r(1 - \frac{H(n)}{k}) + 1) \end{pmatrix}$$

3.1.2 Theory Behind the Equations

As mentioned previously, these state variables interact with each other over time. We can see that a dragon's weight in the next state depends on the current weight, calories C and prey population P. The second part of this equation (given by $C(n) - \frac{\alpha\beta P(n)}{365}$) computes the difference between the daily calories required ($C(n)$) and the estimated daily calories eaten ($\frac{\alpha\beta P(n)}{365}$). Since β represents the proportion of the prey population eaten in a year, $\frac{\beta P(n)}{365}$ estimates the average amount of prey eaten in a day. We then multiply this value by α, which represents the total calories obtained by eating an individual prey in order to get the total daily calories eaten by the dragon on average. Once we determine whether or not the dragon is eating more than it is burning, we can calculate the amount of weight we expect it to lose/gain in a given year. We do this by multiplying this difference by a proportionality constant s, which also helps to convert calories to mass in kg.

For the other three states, we have used more conventional models to represent these real world processes. In our global assumptions section, we mentioned that one fundamental assumption we have made is that a dragon's energy expenditure can be modeled by the Harris-Benedict Basal Metabolic Rate (BMR) formula (Mucha and Zajac). We have also assumed that all dragons in our model are female, and therefore we will use the female version of this formula. By multiplying the BMR by an activity constant d we can estimate the daily calories required.

In the prey population recurrence relation we have used a modified logistic population model. The modification we have made is introducing the death of these preys. We assumed that the prey deaths due to the dragon is proportional to the number of prey in the population and the current weight of the dragon. We made this assumption because the more the dragon weighs, the more strength/power and desire for food it will have. This will likely lead to the dragon killing more of its prey (even if it does not consume it). This is why we have a different coefficient representing the proportion of the prey population eaten (β) and the proportionality constant for prey deaths (u).

Finally, we modeled the height of a dragon over time using a logistic model because we assumed that the growth cycle for a dragon's height mimics the growth cycle for a human's height. With humans we often see that height tends to grow rapidly during puberty and early childhood but it eventually stabilizes at a certain point (Yetman). The 'S'- shape graph of a logistic model mimics this qualitative behaviour.

3.1.3 Simulating the Model

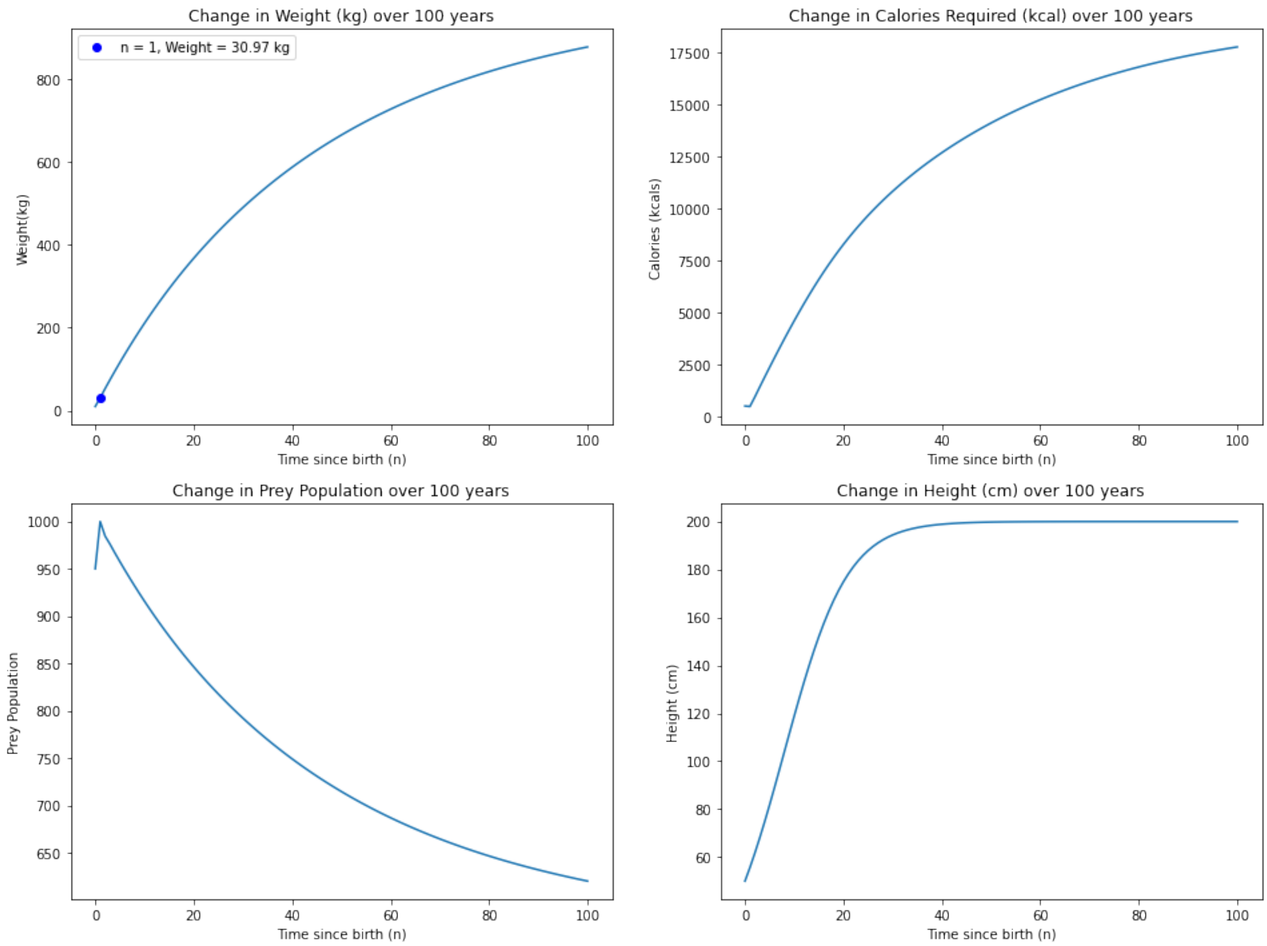

Figure 1

After running a simulation progressing 100 time steps (years) forward, we can observe the behaviour of all the four states in our general model in the Figure 1 below. We can see that the weight of the dragon initially grows extremely quickly, but over time it levels out to a more stable weight. We also can observe that the weight of the dragon starts to stabilize at around 870kg.

Another thing we can see from Figure 1 is that after one year from birth the dragon weighs approximately 31kg as indicated by the blue dot. This matches the quantitative behaviour specified in the problem, in which we expect the dragon to weigh between 30-40kg after one year. This suggests that our model may be a good representation of a dragon's weight.

Besides the realistic quantitative behaviour in our model, we can also see that the qualitative behaviour of the weight W and calories required C is extremely similar. This is likely because the weight of a dragon drives its BMR up, which in turn means that it requires more calories. In contrast, we can see that as the weight of the dragon increases, the prey population P steadily decreases. This is because in our model the amount of prey that die is proportional to the prey population and the dragon's weight. Therefore, when the weight of a dragon increases it leads to more prey dying, and thus the prey population declines.

We can see that toward the end of the 100 year simulation, both the weight and prey population are reaching their equilibrium at approximately 870kg and 640 prey respectively.

We can also observe the behaviour of the height variable over time. We see that the height steeply increases during the early life stages of a dragon but stabilizes at $k = 200cm$ after around 20-25 years into the dragon's life. Since we assumed that a dragon's height growth cycle mimics a humans, our model's qualitative behaviour is somewhat consistent with this assumption.

3.1.4 Varying parameters

As we are trying to analyze the behaviour of three different dragons (Dragons A, B and C) in this problem, we need to make additional assumptions about each dragon and observe how this impacts our model. We will make assumptions based upon the habitat of each dragon and will relate these to constant parameters in our model. We will focus on varying one parameter at a time while holding all others constant in order to draw insights from our model.

Additional assumptions about each dragon:

Dragon A:

- Lives in a hot arid climate
- Area of Habitat: Extremely Small

Dragon B:

- Lives in a temperate climate with moderate temperatures
- Area of Habitat: Extremely Large

Dragon C:

- Lives in a cold, arctic climate
- Area of Habitat: Medium-sized

(a) Varying Activity Level d

Let us assume that the temperature of a dragon's climate is positively correlated to its activity level coefficient (d). This assumption is made because we often see that a lot of animals who live in cooler climates tend to conserve energy through various methods like hibernation (Geiser). This would mean that their activity level coefficient would be lower than animals which live in warmer climates.

- Dragon A lives in an arid habitat with the hottest temperature, we will classify it as an 'extra active' individual. as per the Harris-Benedict formula Thus, we let

$$d_A = 1.9$$

- Dragon B is classified as 'moderately active'.

$$d_B = 1.55$$

- Dragon C is 'sedentary'.

$$d_C = 1.25$$

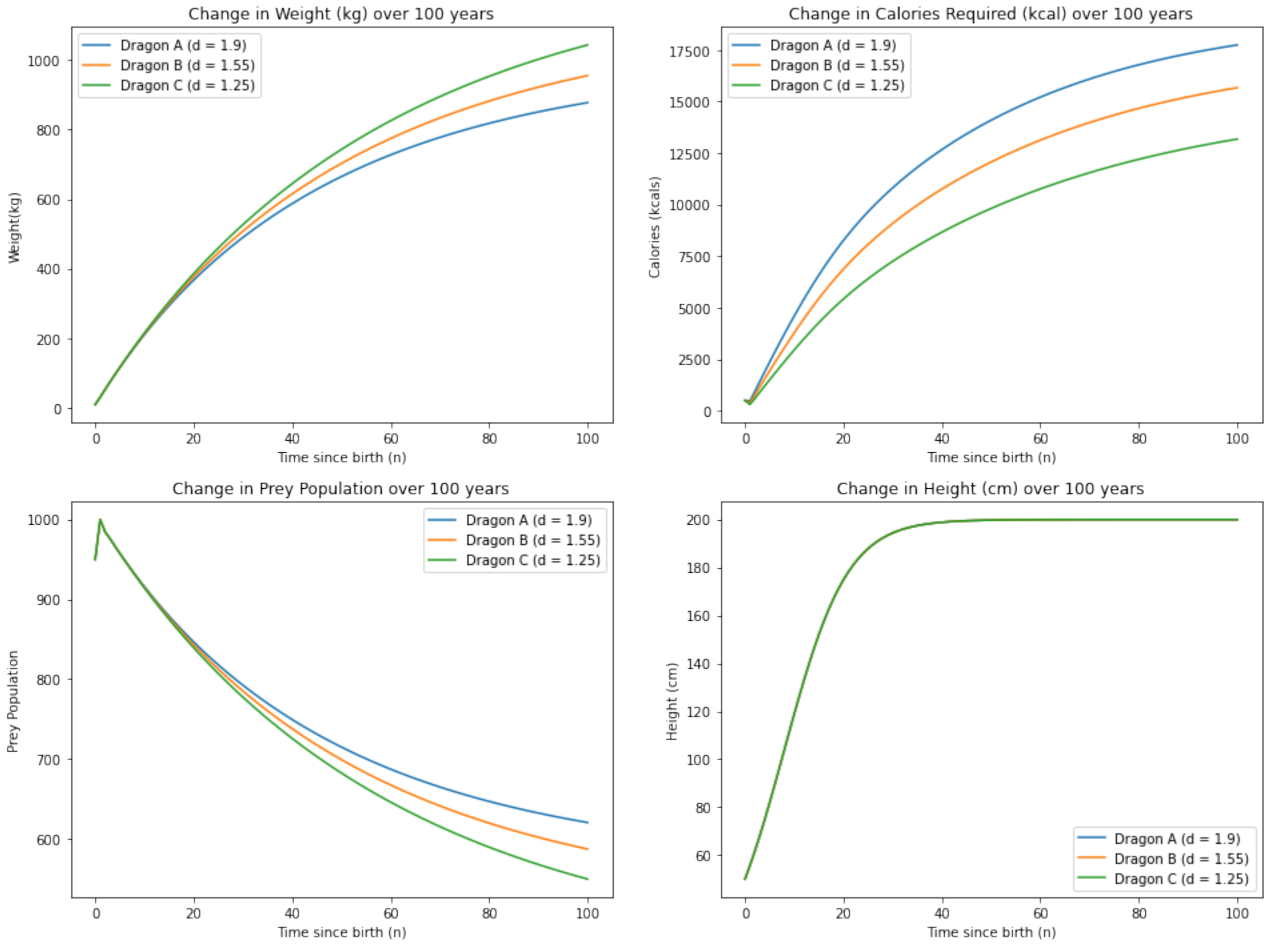

Figure 2

From Figure 2, we can see that changing the activity level parameter mainly impacts the weight, calories required and prey population states over time. We can see that by living a more 'sedentary' lifestyle, Dragon C gains more weight over time, and does so at a slightly faster rate. This may be because Dragon C is gaining extra weight to keep itself warm against its cold habitat; we often see this from other organisms that live in arctic climates (Leisher and Byington). Despite gaining the most weight over time, we can see that this dragon actually has the least calories required after 100 years. Since the proportion of preys eaten is the same amongst all the dragons in this simulation, Dragon C is likely over-consuming more calories, and therefore gaining more weight.

Finally, the prey population tends to decline the fastest when the dragon is the least active (dragon C), presumably because the amount of deaths for the prey population is impacted by the dragon's weight. This behaviour might be a potential limitation of our model as intuitively we may expect a more active dragon to be able to hunt down more of its preys than a sedentary dragon. Despite this, based upon our model and assumptions in the previous paragraph we can infer that dragons living in colder climates are likely to weigh more and require less calories over time. This inference is fundamentally tied to the assumption that the temperature of a climate influences the activity levels of the dragon.

(b) Varying the carrying capacity of the prey population v:

Next, we will move onto analyzing the prey population carrying capacity parameter v. We will assume that smaller environments are can only support low prey populations due to the scarcity of essential resources such as shelter. Furthermore since arid regions tend to have a lack of water, we will also assume that these regions only can support a low prey population.

- Dragon A lives in a small, arid region with a lower carrying capacity for the prey given by

$$v_A = 500.$$

  This means we must also start with a lower initial P (only for this model) and we thus set $P(0) = 500$ so that we do not exceed the carrying capacity.
- Dragon B lives in a large temperate climate with moderate temperatures, and therefore we will assume that a large amount of species can survive in such a climate. So we set

$$v_B = 2000.$$

- Dragon C lives in a cold arctic region, which can support larger populations than dry arid regions. Thus we assume

$$v_C = 1000.$$

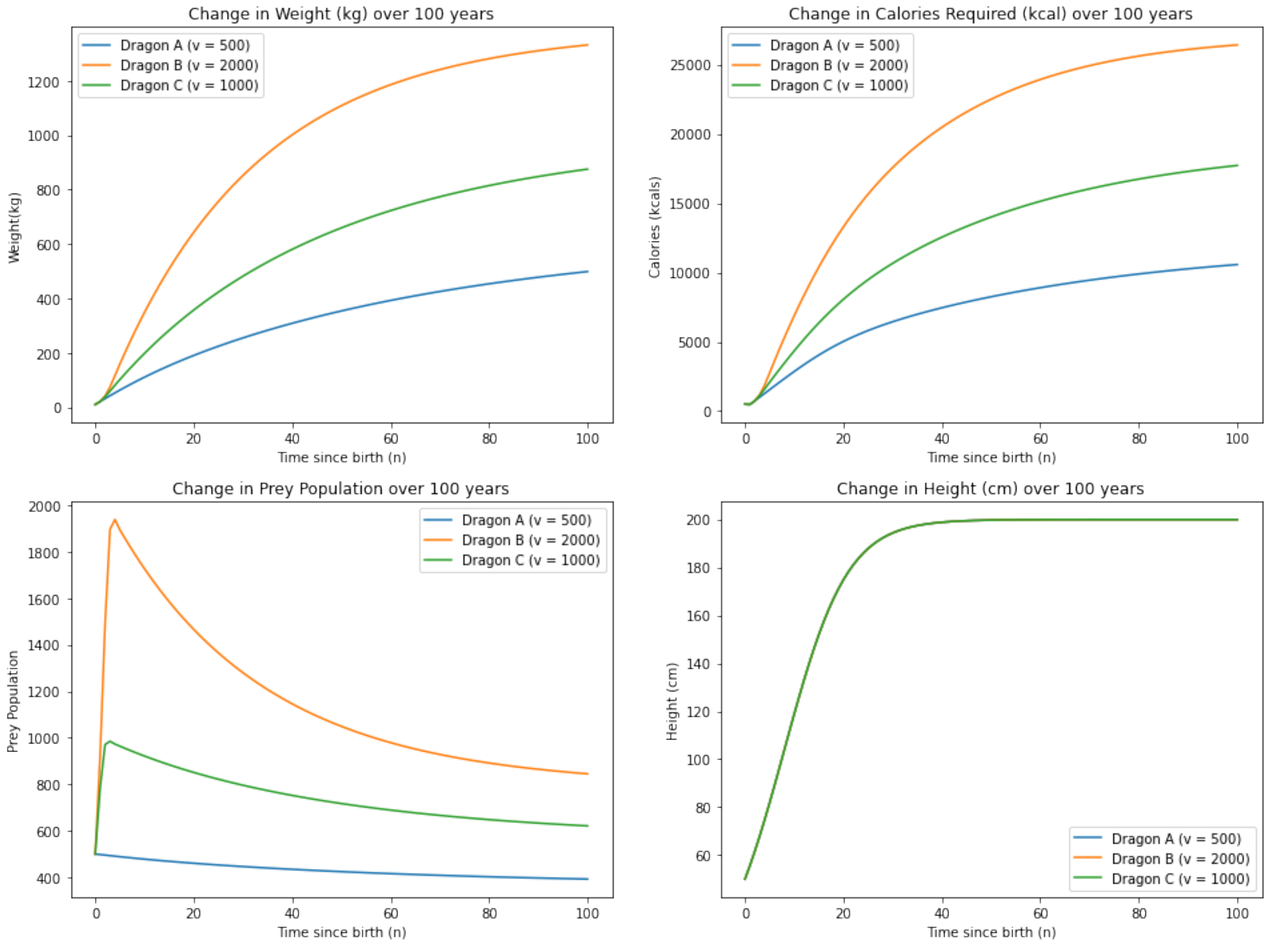

Figure 3

In Figure 3, we can see that the growth in the dragon's weight and calories required over time is heavily dependent on the carrying capacity of the prey population and therefore the size/type of environment the dragon lives in. The more prey the environment can support, the greater the weight of the dragon will be over time. This makes sense because a dragon's weight is heavily influenced by the food they are able to eat. If there is more prey within their environment then they would have more food to eat, and consequently would be able gain more weight over time.

Since we have assumed that the carrying capacity of the prey population is influenced by the size and type of habitat the dragon lives in, we can say that the dragon's environment does influence its calories required and weight over time. For instance in arid, small regions (as in the case of Dragon A), a dragon is likely to be much lighter in weight after 100 years in comparison to a dragon that may be living in a large temperate region.

(c) Varying the predation coefficients β and u:

We can also analyze the impact of varying two coefficients that influence how many preys are eaten and killed: β and u. Another assumption we have to make is that in smaller areas, predators are more likely to encounter prey, and therefore the rate of predation may be larger. Thus, we adjust parameters as follows:

- For Dragon A, β and u increase:

$$\beta_A = 0.10 \text{ and } u_A = 0.0010$$

- For Dragon B, the parameters decrease:

$$\beta_B = 0.025 \text{ and } u_B = 0.00025$$

- For Dragon C, the parameters are held constant.

$$\beta_C = 0.05 \text{ and } u_C = 0.0005$$

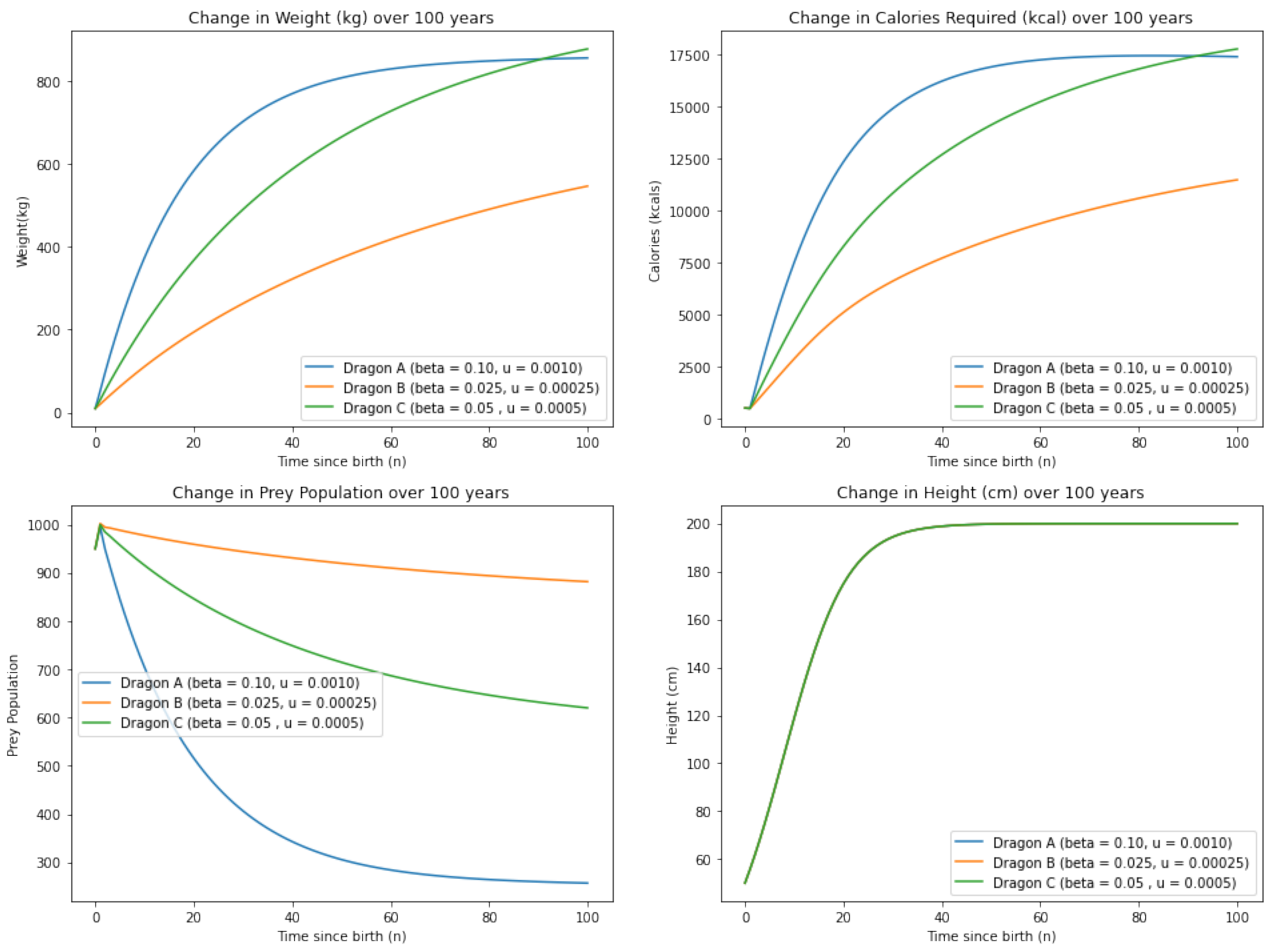

Figure 4

In Figure 4, we can see that in smaller habitats, where dragons kill more of their prey, there is a rapid weight gain and prey population decline particularly in the short run. We can also see that in the long run, the prey populations tend to be much lower as indicated by the blue curve for Dragon A.

One interesting thing to observe is that the weight of the dragon and calories required stabilizes much quicker when the predation coefficients are larger. We can actually see that the green curve overtakes the blue curve in the weight gained and calories required variables towards the end of the 100 year simulation. This might be because by eating/killing preys at a very quick rate, Dragon A has very little preys left within its population in the long term. This means that it is unable to gain as much weight due to scarcity of food. On the other hand, Dragon C paces its consumption of the prey population and therefore is able to grow to a larger weight over time. This suggests that a dragon should consume its prey at a moderate rate if it aims to maximize its weight gains in the long run.

(d) Varying the stabilizing height k:

All of our parameter variations so far have not influenced the height of the dragon over time because the recurrence relation for height depends only on the current/previous height variable. We will now focus on varying the stabilizing height parameter k and observe how changing the long term height influences the other state variables. We will assume that a dragon's height solely depends on their genetics. We thus vary the stabilizing height for each dragon k in cm as follows by setting

$$k_A = 150$$
$$k_B = 200$$
$$k_C = 250$$

for dragons A, B, and C respectively.

From Figure 5, we can clearly see that the height of a dragon only slightly impacts the state of the system over time. It seems that the height of a dragon has a minor influence on the difference in calories required between all three dragons. Consequently, this means that the weight and prey population for all three dragons are relatively similar over time. Thus according to our model, genetic factors that influence a dragon's height do not play a major role in the long term ecological impacts and requirements within the system.

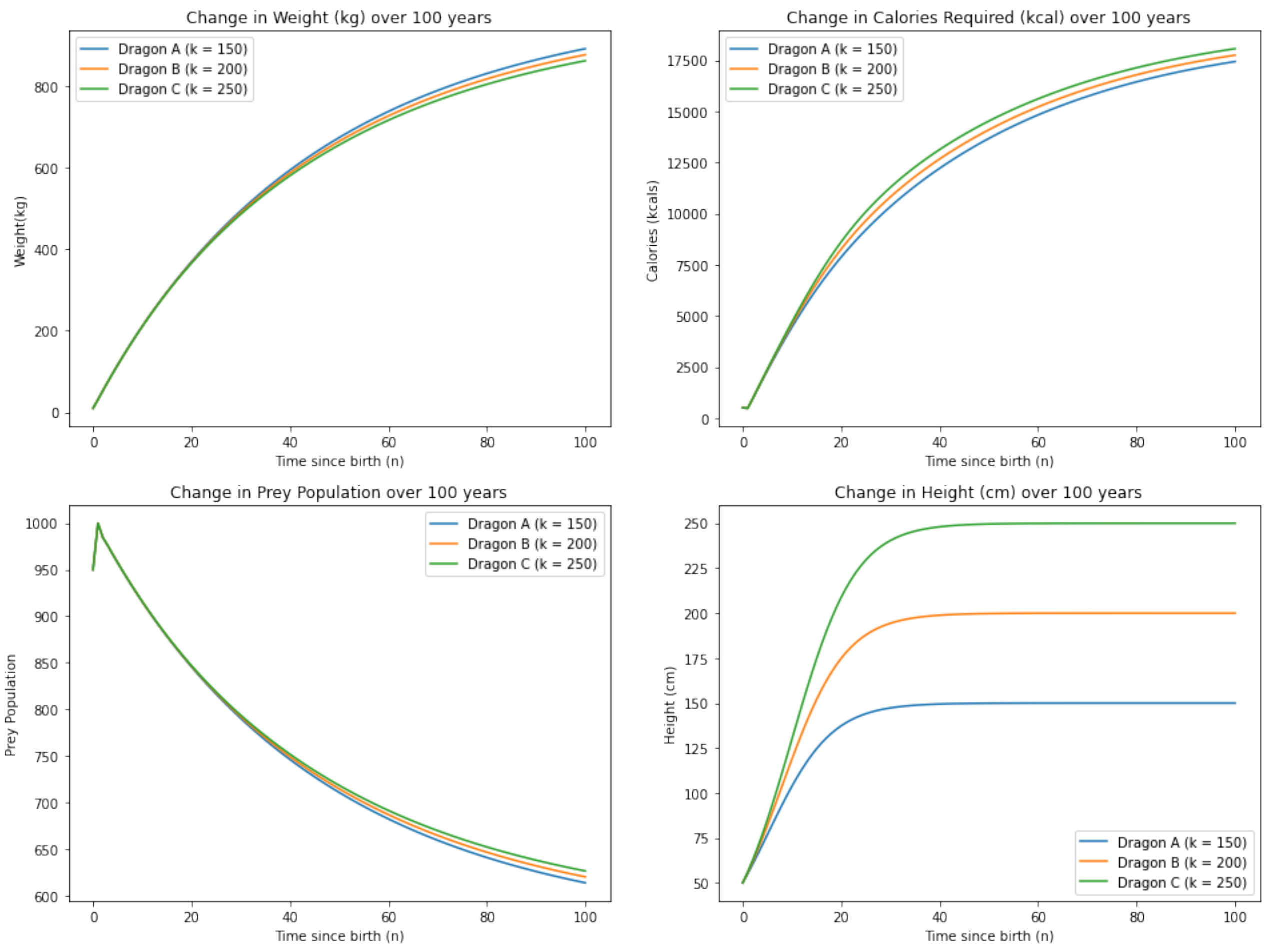

Figure 5

3.2 Introducing Stochasticity in Weight and Height

In order to model the random fluctuations in parameters which occur in real life, we relied on the inherent stochasticity of two of the parameters, namely, weight W and height H. According to numerous studies, the weights and heights of humans follow a normal distribution(Trussell and Bloom), and we thus extended our original model to reflect this empirical reality.

In order to build our model quantitatively, we had to make the following assumptions:

1. The weights W_{normal} and heights H_{normal} of dragons at each time step can be represented using a random variable that follows a normal distribution with mean equal to the weight and height values respectively generated by the simple deterministic model at that time step.

2. The standard deviation of these distributions will be proportional to the time step as the accumulation of environmental factors with time will result in further variation in these values.

Putting together the first two assumptions yields

$$W_{\text{normal}}(n) \sim N(W(n), 0.2n) \text{ and}$$
$$H_{\text{normal}}(n) \sim N(H(n), 0.1n).$$

We use $W_{\text{normal}}(n+1)$ and $H_{\text{normal}}(n+1)$ in place of $W(n+1)$ and $H(n+1)$ respectively in our state vector.

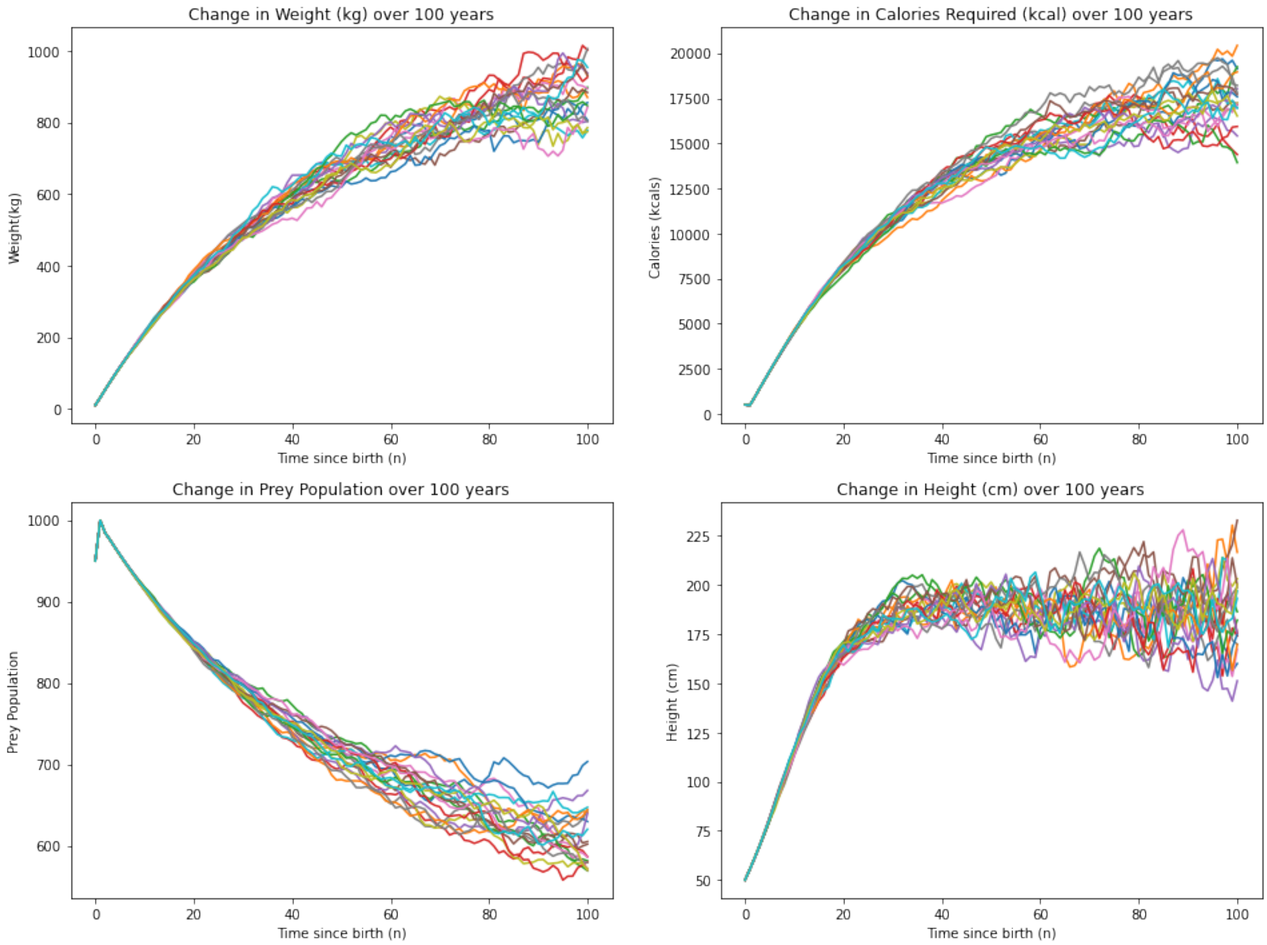

Figure 6

We thus generated the four graphs in Figure 6, whose overall behavior resembles that of the graphs generated by the simple deterministic model with added variation. This variation is beneficial as it allows us to visualize a range for each state variable at each point in time. As we described earlier, these variations increase with time, which is realistic due to the accumulation of environmental factors. In the case of humans, the weights and heights of babies are quite similar; however, the weights and heights of adults exhibit a far larger standard deviation. These observations can also be extended to dragons. Furthermore, since we constructed our model such that the change in prey population and the change in calories required depend on the weight of the dragon, these variables also exhibit resultant stochasticity.

A drawback of the model, however, is that fluctuation of the height variable results in decreases in height between each time step. While it is possible for organism height to reduce rather than grow as they age, it is a relatively rare occurrence and the decrease in height occurs over a long period of time. Whereas, our model at times shows a sharp decrease in height in between two time steps. We may correct this by treating the parameter r, which is the intrinsic growth rate of height, as a random variable rather than H itself. We could potentially also vary the parameter k, which is the stabilizing growth rate.

3.3 Introducing Environmental Stochasticity

In order to account for environmental stochasticity, we consider environmental events that are characterized by randomness. In this course, having focused on catastrophes, we assume that notable environmental occurrences that affect the state of the system mainly take the form of a catastrophe. To fit this in the context of the T.V. show, the *Game of Thrones*, we can consider the eruptions of **Fourteen Flames**, a chain of volcanoes extending across the Valyrian Peninsula (A Wiki Of Ice and Fire).

To factor in the environmental stochasticity in our model using quantitatively, we must make the following simplifying assumptions:

1. Only one of the fourteen volcanoes can erupt at once, i.e. two or more volcanoes do not erupt simultaneously. Letting X_i for $i \in \{1, 2, \ldots, 14\}$ be the event that the ith volcano erupts, we assume that $P(X_i \cap X_j) = 0$ for $i \neq j$.
2. We assume that there is at most one eruption per year.
3. We assume that the event that there is an eruption follows a Bernoulli Distribution with probability $p = 0.04$, i.e. 4 percent. Thus,
$$X_i \sim \text{Bernoulli}(p = 0.04)$$

To arrive at the value for p = 0.04, we considered any possible literature on active volcanic eruptions. We tried to draw parallels between the volcanoes in the Game of Thrones and and active, real-life volcanoes. Choosing the Yellowstone Volcano as a benchmark for active volcanoes (National Park Service), we found that the probability of a caldera forming eruption can be given as 0.00014 percent (USGS). However, not all volcanic eruptions are caldera forming eruptions and since scientists believe that it is highly unlikely that there will be another catastrophic eruption at Yellowstone, the probability of the volcanoes erupting in our model should be much higher. We chose to follow the textbook's p-value of 0.04 used in the *Bobcats* project (Mooney and Swift).

When running the simulation, at each time step there will be a possibility that there is a catastrophe. Hence, at each time step we can 'sample' a catastrophe from a Bernoulli distribution. If there is a catastrophe we should see an impact on the parameters in our model that help define the state.

- Should a catastrophe occur, we should expect to see a decrease in the birth rate of the prey - a notion that requires little explanation. We expect about a 5 percent decrease in birth rate for the prey population in the event of a catastrophe. Thus, setting the initial value of $b_{\text{env}} = b$, when there is a volcanic eruption,
$$b_{\text{env}} -= 0.05$$
- Moreover, in a catastrophic event we would expect that dragons would be less active in seeking prey to safeguard themselves from harm, so we witness a 5 percent decrease in the activity constant. Thus, in this case, after setting initial value equal to the baseline value, when there is a catastrophe,
$$d_{\text{env}} *= 0.95$$
- Due to fatalities caused by the catastrophe we infer that fewer prey will be left. Hence, if the dragons eat the same amount of prey from a slightly smaller population, they will consume a larger proportion of

that population. Therefore, we witness an increase in β, which is the proportion of the prey population consumed. Thus,

$$\beta_{\text{env}} = 0.07,$$

increasing from baseline value 0.05.

- A catastrophe would also destroy a lot of resources which would significantly reduce the carrying capacity v of the prey population at that time step.

Upon running a simulation for 100 years, we produce the graphs below, which we compare against the graphs generated by the simple deterministic model.

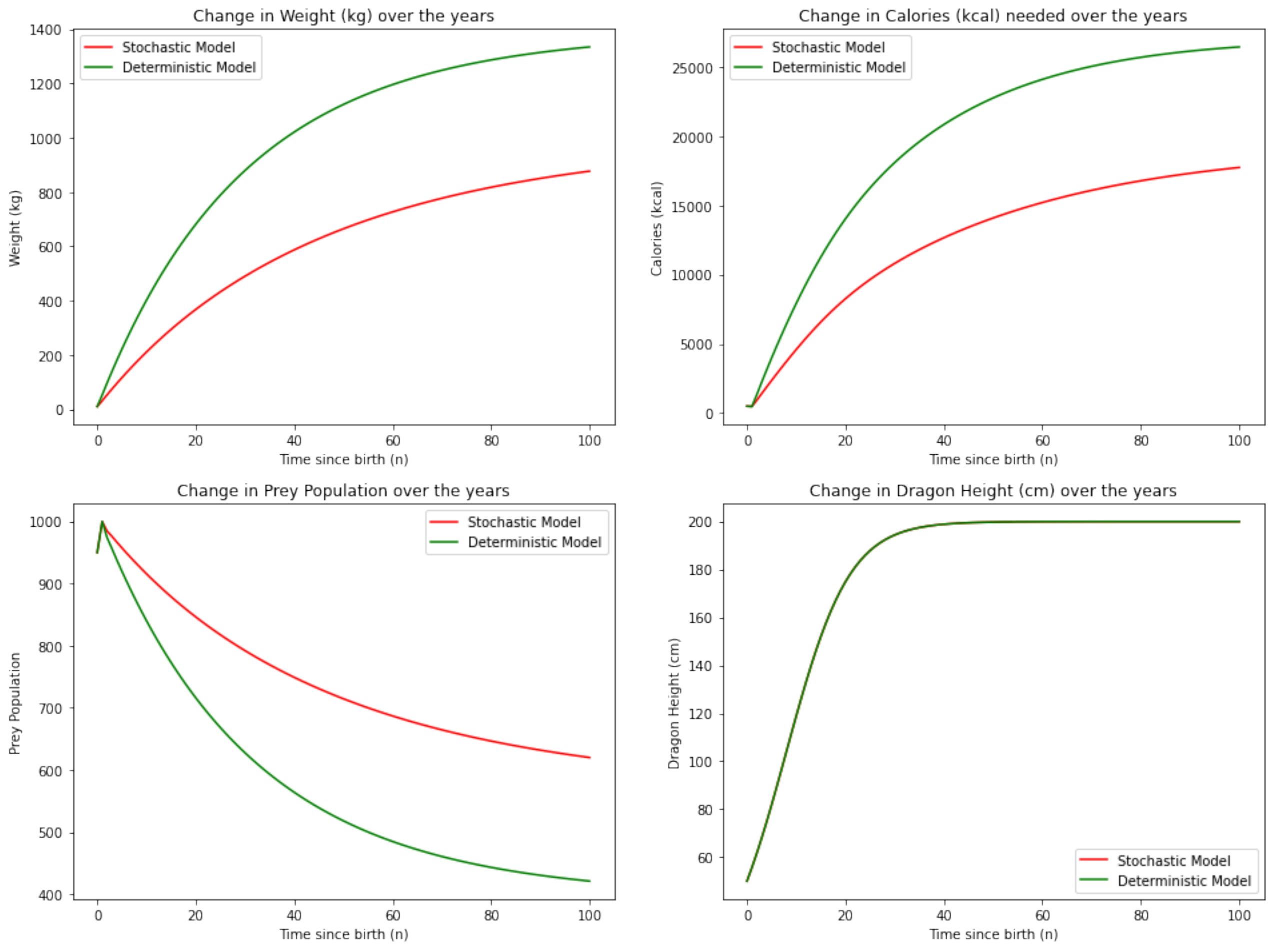

Figure 7

Notice that according to the stochastic model, the weight of the dragon ends up much lower than the weight from the deterministic model. Similarly, we can see that the calories required by the dragon is much lower for the stochastic model as compared to that for the deterministic model. However, notice that the population of prey ends up stabilizing at a higher value in the stochastic model. The height of the dragon is not affected by an environmental catastrophe. These differences, or lack of, can be explained by keeping the equations in mind.

A catastrophe would mean a lower prey population than that from the deterministic model. This would mean a lower weight in the next time step than that of the deterministic model. However, given the interaction term between weight and prey population, a lower weight of the dragon would mean fewer prey get killed. Also if there is no disaster in the next time step both parameters b and v go back to their baseline values, so more prey end up surviving. Hence in the long run, the weight increases more slowly in the stochastic model, resulting in a lower eventual weight. However a lower weight means that the dragon's eating less prey which means more prey end up surviving. A similar explanation can be applied to the calories needed.

3.3.1 Considering Three Dragons in Different Regions

We then decided to run the environmental stochastic model to compare the three dragons over time. We varied the parameters for Dragon A, B and C based upon our assumptions in section 3.1.3. We ran this simulation by altering all the model parameters simultaneously rather than focusing on one and holding the rest constant. In this simulation we started with an initial prey population of 500 in order to ensure we do not start above the carrying capacity for prey population in Dragon A's habitat.

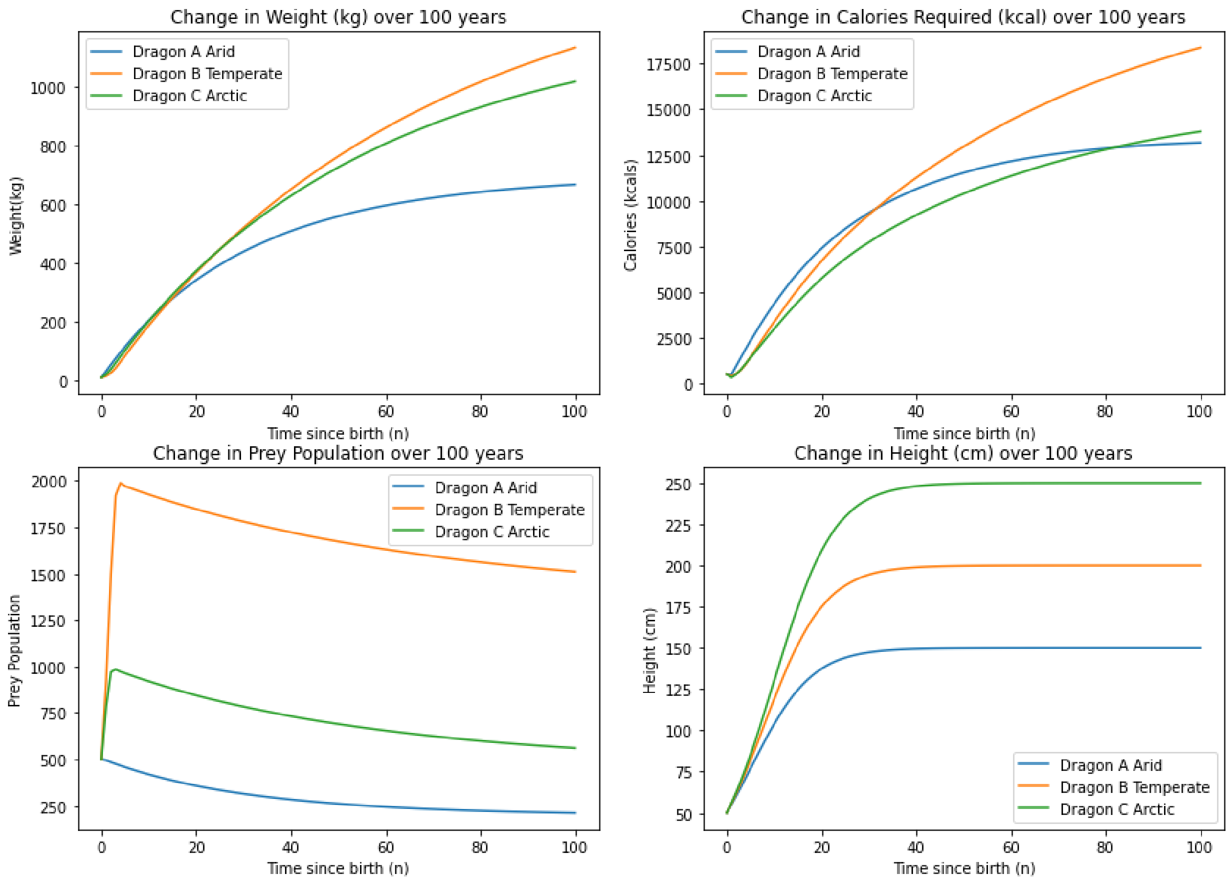

Figure 8

We can see that in all three regions most of the general trends from our original model still hold. We can clearly see that dragons in arctic conditions tend to be heavier over time than dragons in arid conditions. Furthermore, we can observe that the prey population for the temperate conditions is the largest, which suggests that this habitat has the most favourable living conditions. On the other hand, arid conditions tend to consist of low prey populations. This makes sense given this region's lack of water and other resources, which makes it have unfavourable living conditions.

4 Conclusion

4.1 Our Conclusion

We were thus able to analyze the characteristics of three different dragons to gain insight into how certain environmental factors impact the long term ecological effects/requirements of a dragon. First, by performing analysis on our simplified deterministic model we found that in colder environments dragons tend to gain more weight over time than dragons in warmer conditions, even though they require less food (calories) for survival. Next, we found that a dragon living in a smaller area, in which it hunts more prey, is likely to reach its stabilizing weight quicker than dragons in larger areas. We also found that variations in these parameters do not result in significant changes in the model behavior.

After generating these insights from our simplified deterministic model, we decided to add stochastic variables in order to better represent the unpredictability that often occurs in real-world ecosystems. We did this by using the inherent stochasticity of height and weight, as well as environmental stochasticity. While the introduction of stochasticity in height and weight helped us generate a range of values for the state variables at any given time step, they did not truly alter model behavior. Thus, by the principle of Occam's razor, the inclusion of this type of stochasticity is not a must when building a model regarding dragon behavior. If given accurate parameter values, it would be helpful in predicting a range for the heights and weights of dragons over time, but in the absence of concrete data, it does not reveal sharply different qualitative behaviour.

When we took into account environmental stochasticity, we found that the state variables W (weight), C (calories), and P (Prey Population) differ significantly for the stochastic model as compared to the deterministic model. This is because a catastrophe would negatively affect the birth rate b and carrying capacity v of the prey population. There was no difference, however, in the height H, which is not dependent on other quantities in the model.

4.2 Model Strengths

Our model has a lot of strengths, particularly when it comes to modeling the qualitative behaviour of real-world phenomena:

- In our model, we have seen that there is an inverse relationship between a dragon's weight and prey population. This intuitively makes sense because it suggests that the dragon is gaining its weight by eating the prey.
- The model reaches an equilibrium point where the dragon cannot gain more weight because the prey population will get too low. In the real-world, we often see animals reach a stable weight and stay close to it over time. Furthermore, in our baseline model we saw that this weight was around 870kg. This is similar to other fully grown carnivores such as lions, who weigh around 920kg when fully grown (Millburn). We also expect dragons to be much heavier than humans. All of this suggests our model has realistic quantitative and qualitative behaviour for the weight state.
- Our baseline model predicts that the dragon would weigh 30.97kg after one year, which matches the information given within the problem that states "after a year [the dragon will] grow to roughly 30-40 kg" ("MCM Problem A 2019").
- By relating the assumptions of a dragon's habitat to the parameters in model, we were able to use our model to understand and derive insights about the interactions between a dragon and the size, and type of habitat it lives in.

4.3 Model Limitations

Although our model quantitatively and qualitatively matched the behaviour from the problem statement, there were few limitations that we must take note of:

- By using a first order recurrence relation, one fundamental assumption we made in the model was that the weight and calories of a dragon in a given year is only influenced by the previous years calories. Research into metabolism and nutrition shows that there are several other factors such as the types of food eaten and amount of muscle mass that may influence these state variables ("10 Factors that Affect your Metabolism"). Furthermore, as animals eat more food they may actually burn more calories because of the thermal effect of food (Schutz). Therefore, our model might be too simplistic to account for the multi-faceted complexities of these processes.
- The model does not account for the dragon's activity level when estimating the number of prey deaths. In reality if a dragon lives a more 'sedentary' lifestyle, it may not be able to hunt/kill as many preys because of its low levels of movement.
- In the model, the weight and prey population states only stabilize towards the end of the 100 year simulation. In reality, the process of reaching a stable prey population and weight does not take so long for other living things. For instance, humans usually reach a relatively stable weight and height after puberty.
- This model assumes that the activity level, predation rate, and other environmental factors remain constant throughout the life cycle of a dragon. In reality, this may not be the case because things are constantly changing in the real-world. For instance, due to global warming, the temperatures of certain habitats may increase over time, which can impact the dragon's activity level and in turn influence the state of the system (MacMillan and Turrentine).

- This model assumes that there is only one prey species and the calories obtained by consuming it is constant. In reality, carnivores tend to consume several different prey species, which all have different nutritional content and calories.
- As previously mentioned, in the height/weight stochasticity model, the fluctuations of the height variable are not realistic as it is unlikely that the height of an organism will decrease sharply in just one year.
- When introducing environmental stochasticity one major drawback in this model is the treatment of parameters after the occurrence of a disaster. Once a disaster occurs, we modify certain parameters. However, in the next time step when there is no disaster, we set the parameters back to the baseline values immediately. This is not necessarily representative of how these parameters might change. To further understand what we mean, suppose that at time step at n there occurs a disaster and all the parameters are modified. For instance, the carrying capacity of the prey population is reduced. Suppose that at time step $n+1$ there is no disaster. This does not mean that the population's carrying capacity will instantly bounce back to baseline values but might gradually catch up to its baseline values. Our model does not account for this and instead considers that the population's parameters instantly bounce back to baseline values.

4.4 Future Work

There are several adjustments that could be made to improve our model and address its limitations.

For instance, we could enhance the complexity of our model in order to account for the complexity of the real world. Some ways we could do this is by introducing parameters and state variables to account for real-world processes such as global warming. If we tracked the temperature over time and used this state variable to determine the activity level of a dragon, then we may have been able to account for such real-world processes. Apart from tracking temperature, another potential improvement would be to add the activity level parameter into the prey population and weight equations in order to ensure that the amount a dragon moves impacts its ability to find food and kill its prey. In addition to improving the model's complexity, we could have also combined the different types of stochasticity into one model. This may have helped to better model the uncertainty present within the real world. Overall, these suggestions could help improve our model's representation of a dragon's interaction with its environment and potentially provide us with new insights.

Besides improving the model for this specific problem, we could potentially generalize it outside of the realm of fictional dragons. One situation in which our model could be potentially altered and generalized is to the context of a dominant carnivore such as a lion. To do so, however, we would need to change the model specifically for the weight and prey population state variables since lions tend to come in groups (called prides) and therefore it would be unreasonable to assume that there is only one lion doing the killing of its prey population. We might also need to modify the calories required and height model parameters so that our model fits the real-world growth and energy expenditure of a lion. Apart from lions, we may be able to adjust the model and generalize it to humans since most of the equations are based upon modified models used in the human context. Specifically, the height, weight and calories required equations use models that were originally developed for humans. One thing we would need to change in this context is the definition of a 'prey' population since humans eat many different foods that all contain different amounts of calories. Hence, our existing model will not account for this. Overall, there are several potential applications of our model to generate insights about other ecosystems outside the realm of fictional dragons.

5 Letter

Dear Mr. George R.R. Martin,

We are writing to you today in order to provide insight about the interactions between dragons and their surrounding environments. We derived these insights by developing a model that aims to track the weight, height, calories required and prey population for a given dragon over the first 100 years of its lifespan. The objective of this letter is to provide you with more information about this interaction so that you can maintain the realistic ecological underpinnings of your fascintating creation.

To make sure our model was consistent with the facts of your book, we assumed that the weight of a dragon at birth was 10kg. We modeled the weight of the dragon using basic thermodynamics, which involved finding the difference between calories required and the amount of calories eaten from the prey population. Essentially, by computing the difference between calories in and calories out, we were able to estimate the amount of weight a dragon would gain each year.

For the other three variables (height, calories required, prey population), we modified popular mathematical models that are used to model these processes in humans. One important thing to note is that when we ran a 100-year simulation of our model, we found that after one year from birth the predicted the weight of a dragon would be between 30-40kg, which is consistent with the narrative of your book.

Using our model and its key parameters we were able to find out how environmental factors could potentially impact the weight, height, calories and the prey population for a given dragon in the long run. In order to mimic the story in *A Song of Ice and Fire*, we assumed that there were three dragons called Dragon A, Dragon B and Dragon C (unfortunately, we lack your creativity!) that all lived in different environments. Dragon A lived in a small, hot, arid climate, Dragon B lived in a large temperate climate with moderate temperatures, and Dragon C lived in a medium-sized, arctic, cold climate.

We first wanted to see how the temperature of the environment impacted the each dragon's weight. We assumed that the cooler the temperature of a dragon's habitat, the less active it would be in order to conserve energy. We often see this with other animals through methods like hibernation (Geiser). By using this assumption to tweak the activity level parameter in our model for each dragon, we found that Dragon C, who lived in the coldest environment, gained the most weight over the long run. This intuitively makes sense since animals in colder climates may put on more weight to keep themselves warm. Apart from this, we found that in colder temperatures, the dragon actually required less calories, meaning that it was able to survive off less food. Therefore, one suggestion we had was to portray the dragons in colder climates as bigger in overall size. Furthermore, if a dragon moves from a cold climate to a more moderate or hot one, it should gradually lose weight and require more calories. These suggestions will help to maintain realistic ecological underpinning of the story.

Apart from the temperature of the climate, we also wanted to explore how the size of a dragon's habitat impacts the overall ecological state. We linked the size of a dragon's habitat to its behaviour by assuming that in smaller areas predation is more likely to occur. We made this assumption because in smaller areas, organisms are more likely to encounter each other, and therefore it will take less time for a dragon to find its prey. Ultimately, this means that the rate of predation in smaller environments is likely to be larger than in bigger environments. Using this assumption, we found that Dragon A tended to have the most rapid weight gain initially. However, by rapidly consuming so many of its prey so quickly, Dragon A was unable to continue gaining weight towards the end of the 100 years. Instead Dragon C, who consumed its preys at a moderate rate, actually overtook Dragon A in its weight towards the end of the 100 year simulation. We also found that at the end of the simulation the difference between the prey populations for Dragon A (highest predation rates) and Dragon C (average predation rates) was more than double the difference between the prey populations for Dragon B (lowest predation rates) and Dragon C (average predation rates). Based upon this finding, we recommend that smaller areas (with higher predation rates) in the book are described to have much fewer prey than larger areas (with lower predation rates).

Overall, as you can see, using our model we were able explore how various environmental factors such as the temperature, size, and type of habitat play a vital role in determining the growth, actions and interactions between a dragon and its ecosystem. We hope that our model's findings can help inform the narration of your book and guide your story to maintain a realistic ecological foundation.

Yours sincerely,

Akshat Srivastav, Shail Mirpuri, and Tara Jaigopal

6 References

"10 Factors That Affect Your Metabolism: Inspired Chiropractic." *Welcome to Inspired Chiropractic*, 12 June 2015, www.inspiredchiropractic.com/10-factors-that-affect-your-metabolism/.

Brusie, Chaunie. "What You Need to Know About Growth in Boys." *Healthline*, Healthline Media, 8 Mar. 2019, www.healthline.com/health/when-do-boys-stop-growing#median-height.

Douglas, Crystal C., et al. "Ability of the Harris-Benedict Formula to Predict Energy Requirements Differs with Weight History and Ethnicity." *Nutrition Research*, vol. 27, no. 4, 2007, pp. 194–199., doi:10.1016/j.nutres.2007.01.016.

Geiser, Fritz. Not Just Sleep: All about Hibernation. 19 Sept. 2019, www.science.org.au/curious/ hibernation#: :text=Hibernation%20is%20a%20way%20animals,in%20medical%20benefits%20for% 20people.

"How Many Calories Does a Baby Need?: SneakPeek." *SneakPeek*, SneakPeek, 3 May 2021, sneakpeektest.com/blog/how-many-calories-does-a-baby-need/#: :text=1%2D3 %20months%20 %E2%80 %93%20438%2D,717%2D768%20calories%20per%20day.

Leisher, Craig, and Cara Cannon Byington. "Winter Weight Gain and Tropical Diversity." *Cool Green Science*, 29 Aug. 2015, blog.nature.org/science/2014/01/22/winter-weight-gain-and-why-there-are-more-plants-and-animals-in-the-tropics/.

MacMillan, Amanda, and Jeff Turrentine. "Global Warming 101." NRDC, 10 May 2021, www.nrdc.org/ stories/global-warming-101.

MCM. "MCM Problem A 2019." *Game of Ecology*, 2019, www.mathmodels.org/Problems/2019/MCM-A/index.html.

Millburn, Naomi. "The Weight & Height of Lions." *Pets on Mom.com*, 19 Nov. 2020, animals.mom.com/ weight-height-lions-2378.html.

Mirpuri, Shail, et al. "Math 42 Final Code." *GitHub*, 2021, github.com/shailm99/math42final.

Mucha, Mateusz, and Aleksandra Zajac. "Harris-Benedict Calculator (Total Daily Energy Expenditure)." *Omni Calculator*, Omni Calculator, 19 Mar. 2021, www.omnicalculator.com/health/bmr-harris-benedict-equation#: :text=To%20determine%20your%20total%20daily%20calorie%20needs %2C%20multiply%20your %20BMR,Calorie%2DCalculation%20%3D%20BMR%20x%201.375.

National Park Service. "Volcano." *National Park Service*, https://www.nps.gov/yell/learn/nature/ volcano.htm#: :text=Is%20Yellowstone's%20volcano%20still%20active,also%20reveal%20activity %20below% 20ground. Accessed 10 July 2021.

NutritionData. "Game Meat, Horse, Raw Nutrition Facts." *NutritionData*, NutritionData, nutritiondata.self.com/facts/lamb-veal-and-game-products/4639/2.

Rose, Carol. *Giants, Monsters, and Dragons: an Encyclopedia of Folklore, Legend, and Myth.* Norton, 2001.

Schutz, Yves. "Physical Activity and Thermic Effect of Food." *Physical Activity and Obesity,* 2015, doi:10.5040/9781492595731.ch-033.

SmokinAlphaMare. *The Grim Facts – Horsemeat and What It's Worth*, 16 May 2013, theethicsofequinerescue.blogspot.com/2013/05/the-grim-facts-horsemeat-and-what-its.html#: :text= A%20%231%20meat%20horse%20is,they%20are%20in%20good%20flesh.

Trussell, James, and David E. Bloom. "A Model Distribution of Height or Weight at a given Age." *Human Biology*, vol. 51, no. 4, Dec. 1979, pp. 523–536.

USGS. "Yellowstone." *USGS*, https://www.usgs.gov/volcanoes/yellowstone/questions-about-supervolcanoes.

A Wiki Of Ice and Fire. "Fourteen Flames." *A Wiki Of Ice and Fire*, https://awoiaf.westeros.org/index.php/ Fourteen_Flames.