# Mathematical Building Blocks for AI and DS

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# EQUATION OF A LINE

slope → $y = mx + b$ ← constant

variables

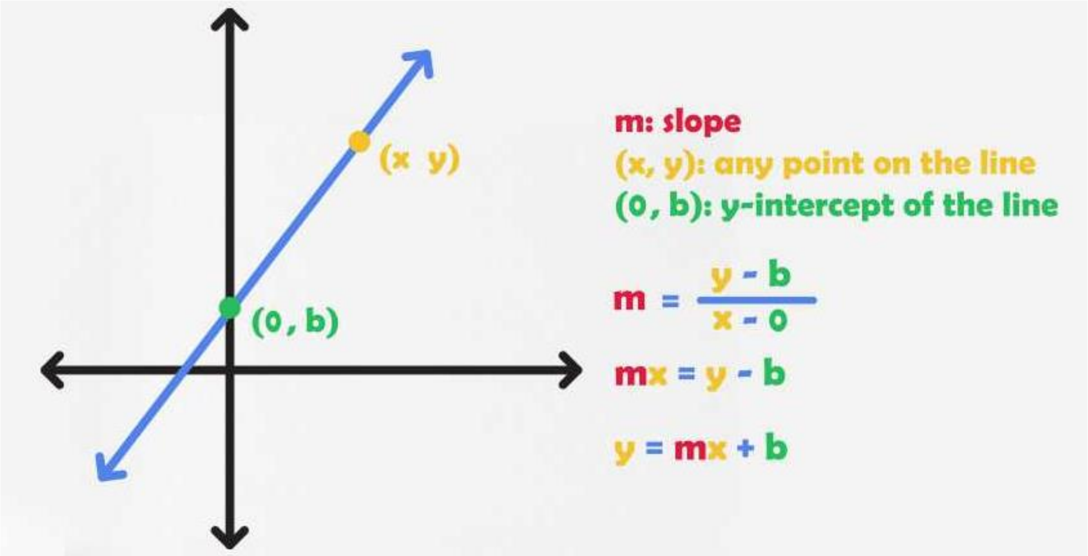
$Ax + By = C$,

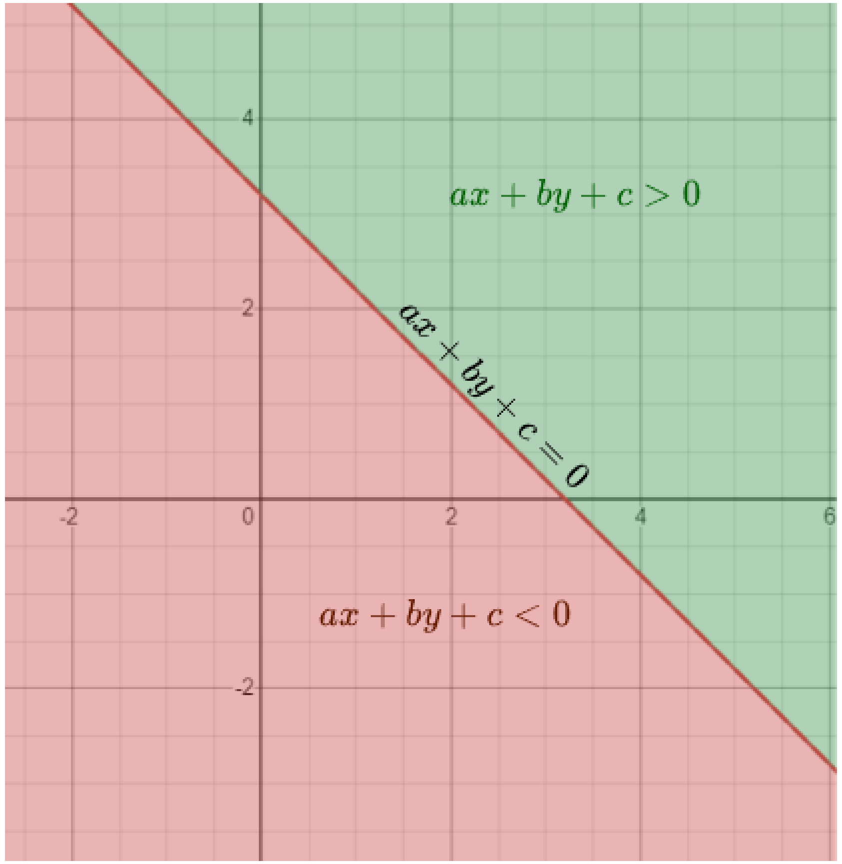
where:

$A, B, C$ = constants

(only integers!)

$x, y$ = variables

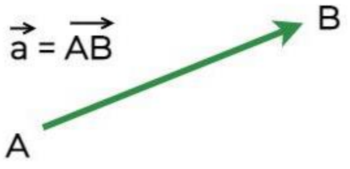


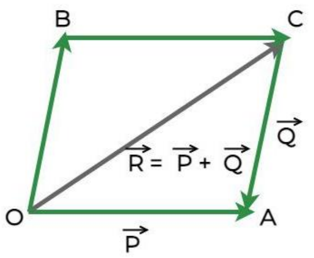
# VECTORS AND VECTORSPACE

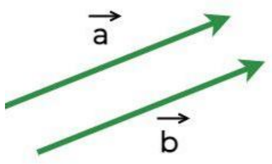
A vector is a quantity or phenomenon that has two independent properties: magnitude and direction.

Vector Notation

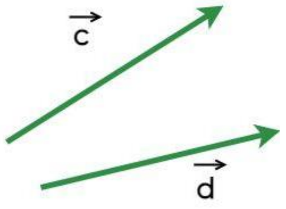



Parallelogram Law of Vector Addition



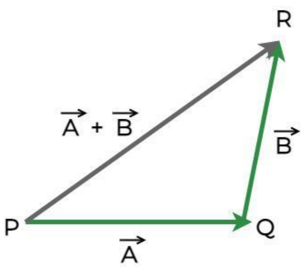



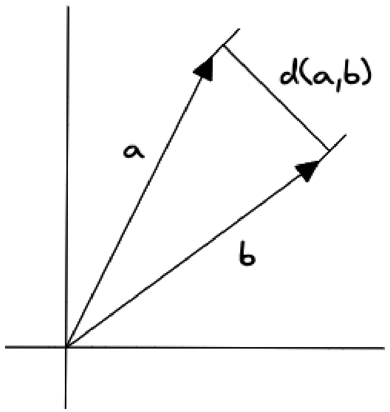

Equal Vectors



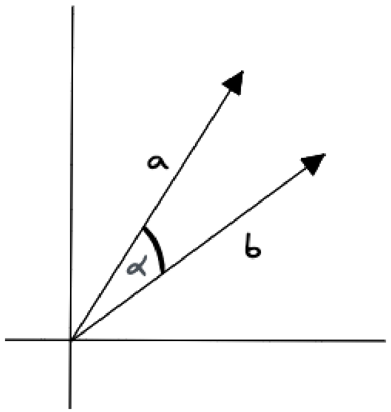

Unequal Vectors

Triangle Law of Vector Addition

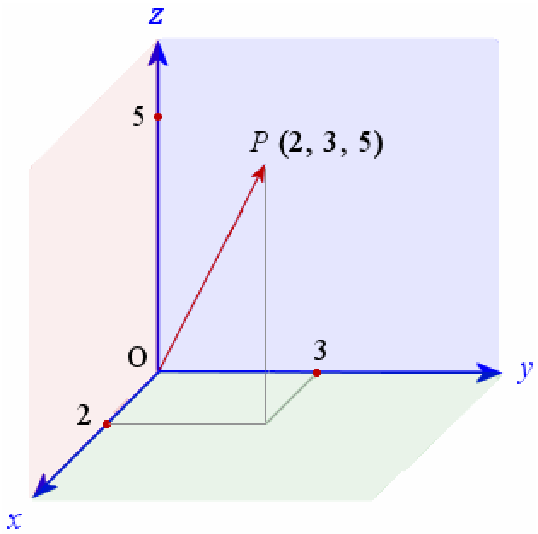



Euclidean Distance




Cosine Similarity

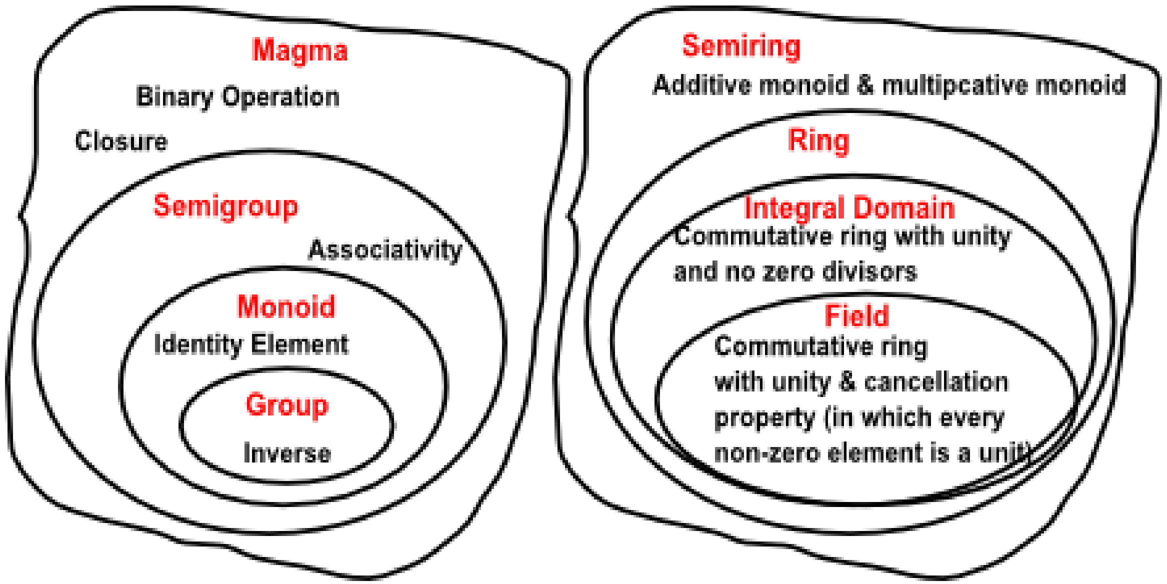


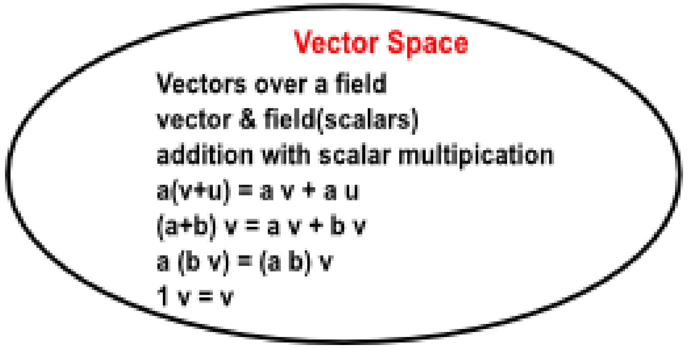

*n*-Dimensional Vectors and Points

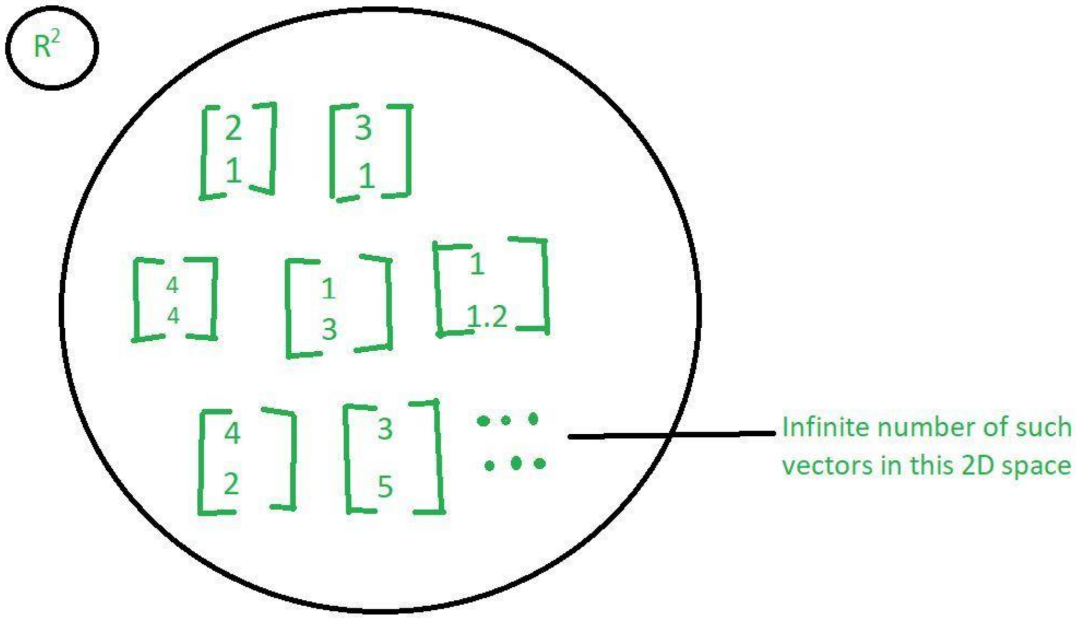
$$v = (v_1, v_2, \cdots, v_n)$$

$$v' = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

# VECTORS AND VECTORSPACE



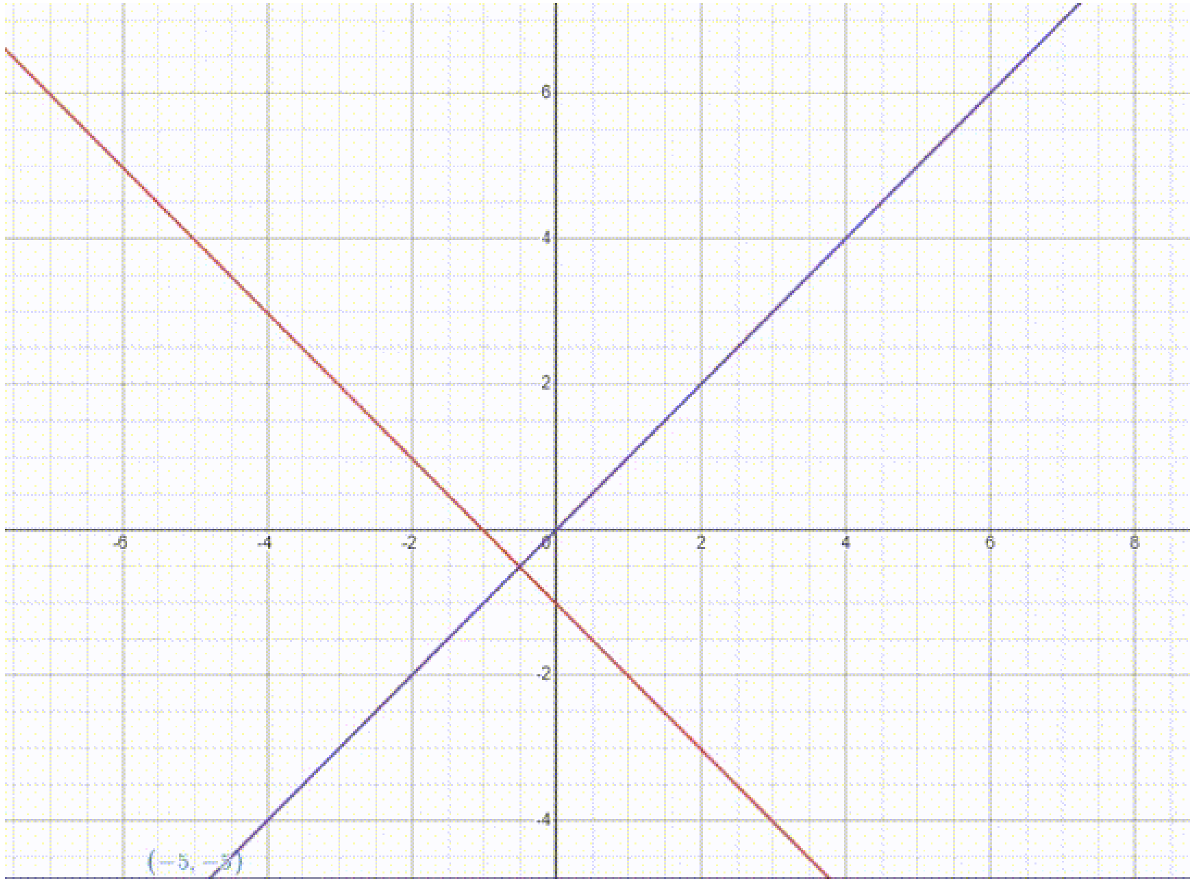




$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Basis of $R^2$

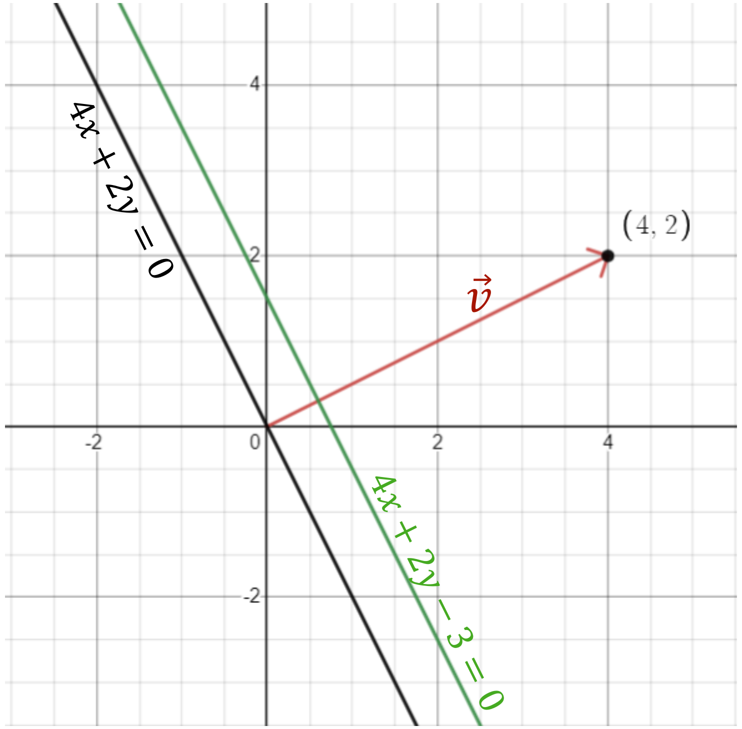
# LINE AND NORMAL




*The line* $ax + by + c = 0$

*and the vector* $(a, b)$

$\vec{v} = (a, b)$ *is always normal to* $ax + by + c = 0$

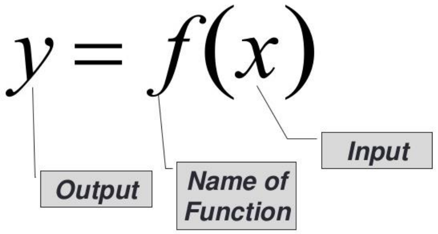


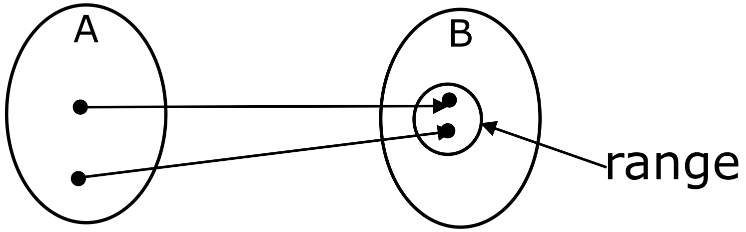
# FUNCTIONS

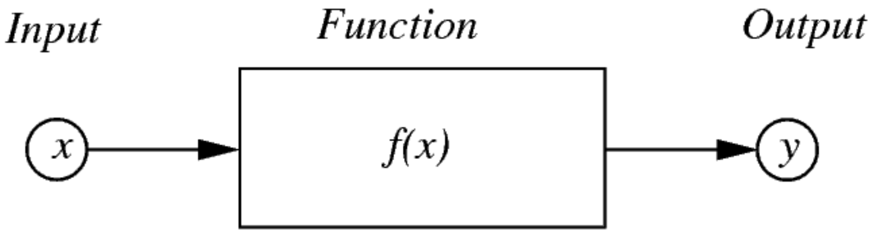
A function relates every element in a set to exactly one element in another set

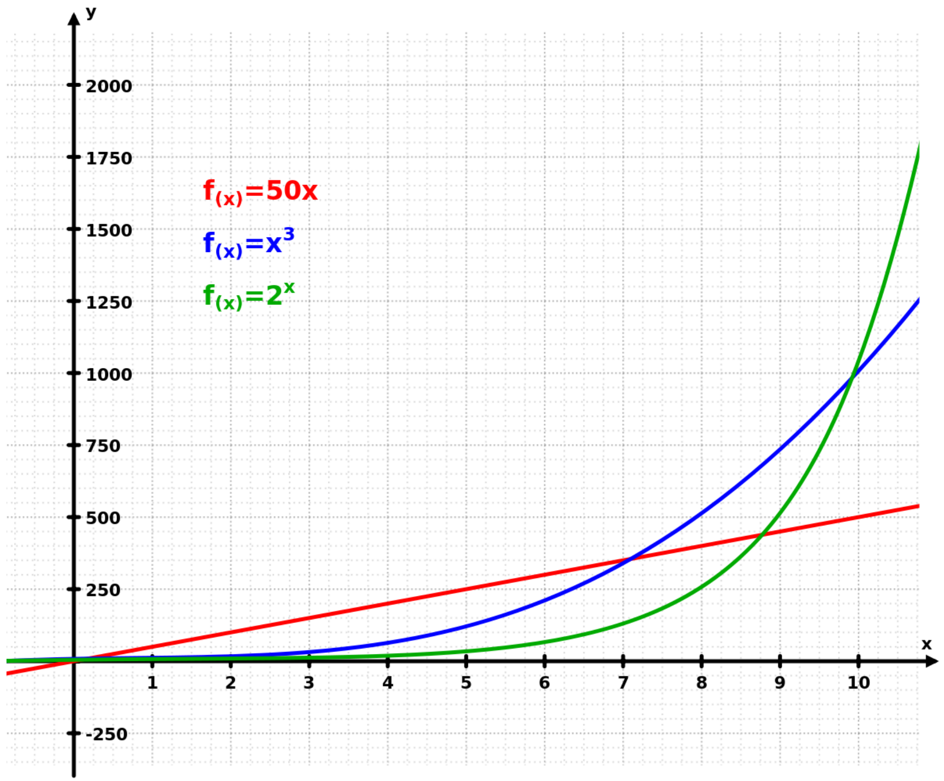
$$f : A \to B$$

function domain co-domain



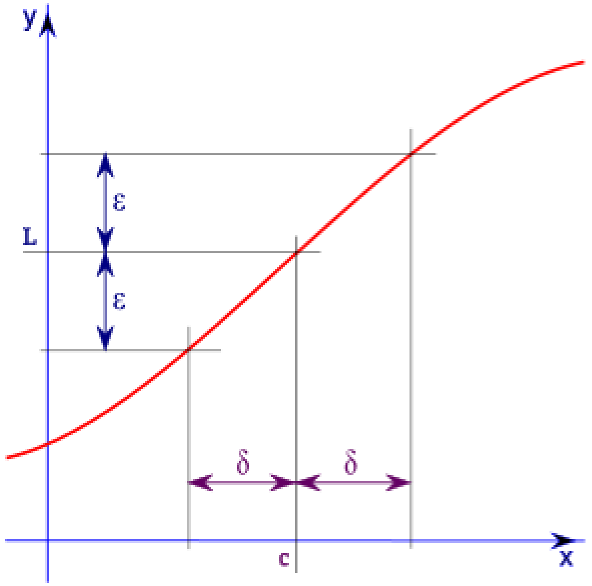




A function which has either  or one of its subsets as its range is called a **real valued function**. Further, if its domain is also either R or a subset of R, it is called a **real function**

# LIMIT OF A FUNCTION




$$\underbrace{\lim_{x\to a}}_{\text{“As you approach } a \text{ along the x-axis”}} \overbrace{f(x)}^{\text{function}} = \underbrace{L}_{\text{“What is the y-value getting closer to?”}}$$

The **limit** of a function at a point *a* in its domain (if it exists) is the value that the function approaches as its argument approaches *a*

$$\lim_{x\to c^-} f(x) = \lim_{h\to 0} f(x-h)$$

Left Hand Limit

$$\lim_{x\to c^+} f(x) = \lim_{h\to 0} f(x+h)$$

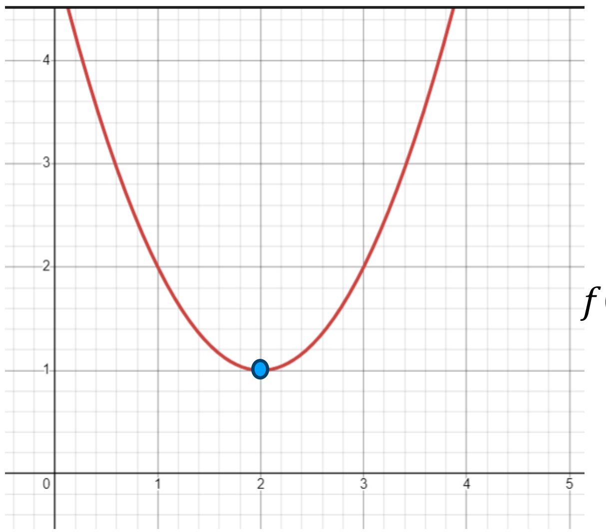
Right Hand Limit

The limit of a function exists if and only if the left-hand limit is equal to the right-hand limit.

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

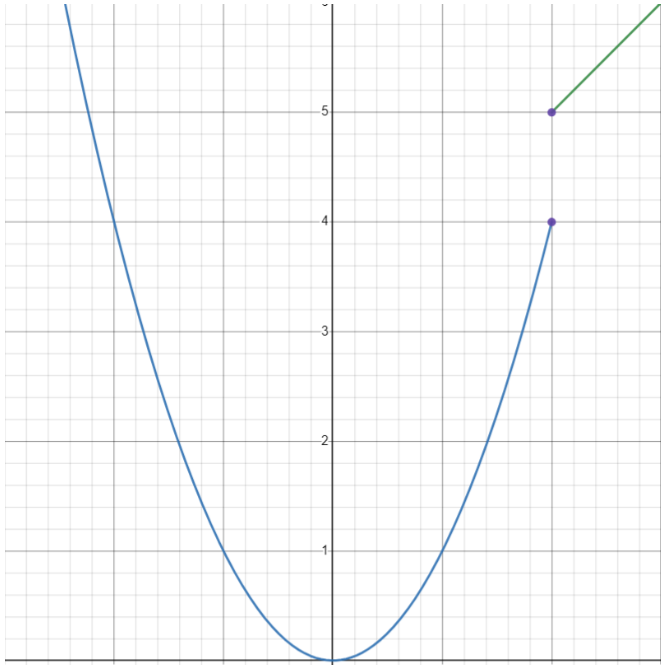
$$\lim_{x\to 5} f(x) = x + 4 = 9$$

# LIMIT OF A FUNCTION




$f(x) = (x-2)^2 + 1$

| $x^-$ | $f(x^-)$ | | $x^+$ | $f(x^+)$ |
|---|---|---|---|---|
| 1 | 2 | | 3 | 2 |
| 1.5 | 1.25 | | 2.5 | 1.25 |
| 1.9 | 1.01 | | 2.1 | 1.01 |
| 1.99 | 1.0001 | | 2.01 | 1.0001 |
| 1.999 | 1.000001 | | 2.001 | 1.000001 |
| 1.9999 | 1.00000001 | | 2.0001 | 1.00000001 |

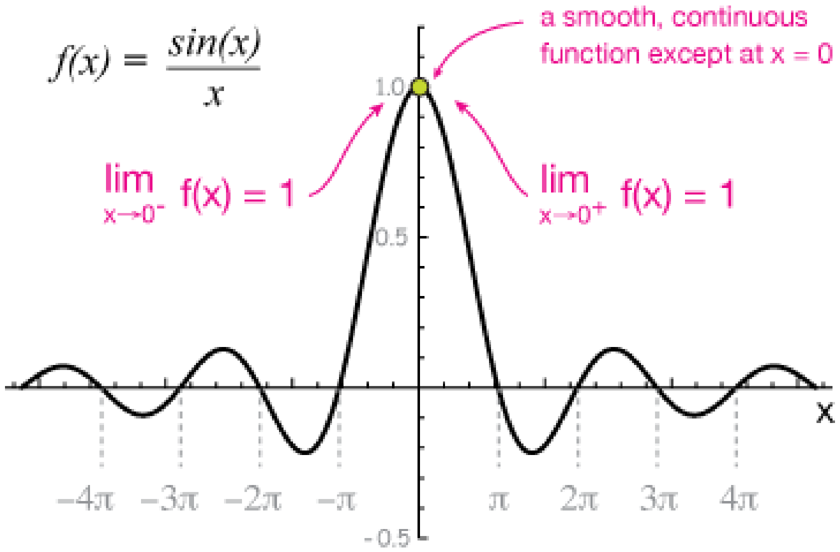



$$f(x) = \begin{cases} x^2 & if\ x < 2 \\ x+3 & otherwise \end{cases}$$

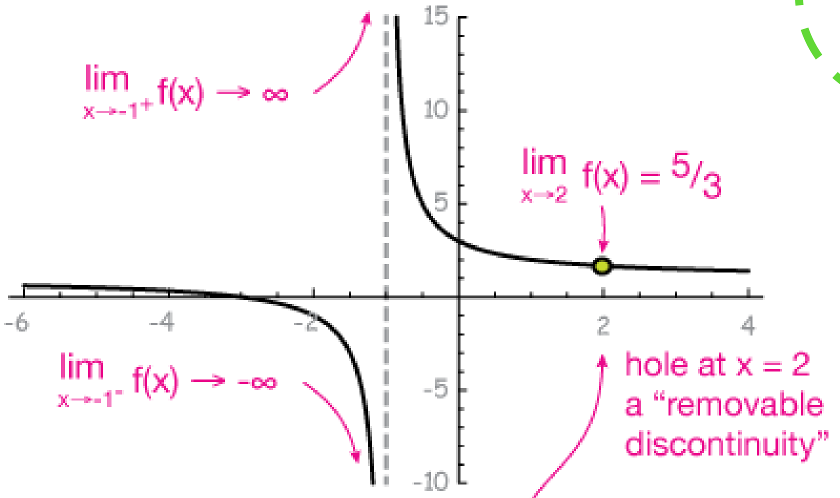
| $x^-$ | $f(x^-)$ | | $x^+$ | $f(x^+)$ |
|---|---|---|---|---|
| 1 | 1 | | 3 | 6 |
| 1.5 | 2.25 | | 2.5 | 5.5 |
| 1.9 | 3.61 | | 2.1 | 5.1 |
| 1.99 | 3.9601 | | 2.01 | 5.01 |
| 1.999 | 3.996001 | | 2.001 | 5.001 |
| 1.9999 | 3.99960001 | | 2.0001 | 5.0001 |

# LIMIT OF A FUNCTION

$$f(x) = \frac{\sin(x)}{x}$$



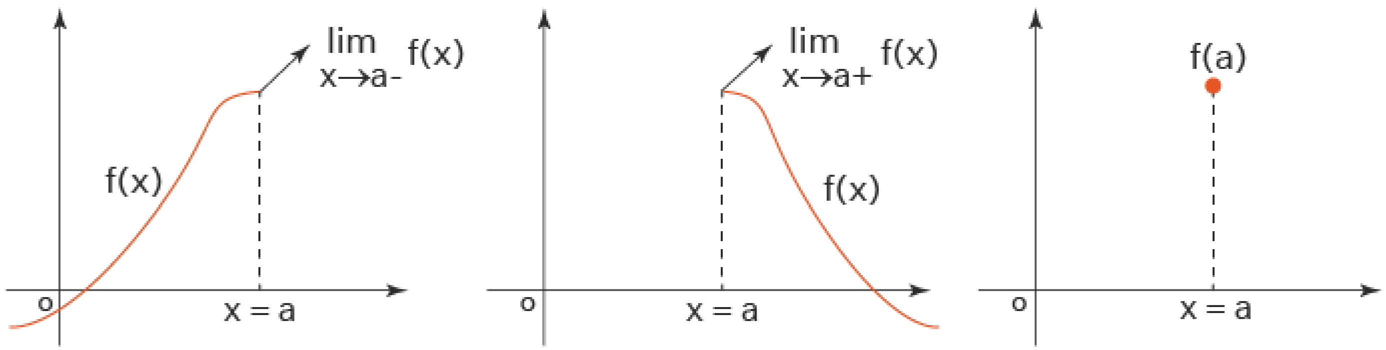

| x | sin(x)/x | x | sin(x)/x |
|---|---|---|---|
| π | 0.0 | -π | 0.0 |
| 1.0000 | 0.8414710 | -1.0000 | 0.8414710 |
| 0.1000 | 0.9983342 | -0.1000 | 0.9983342 |
| 0.0010 | 0.9999998 | -0.0010 | 0.9999998 |
| 0.0001 | 1.0000000 | -0.0001 | 1.0000000 |



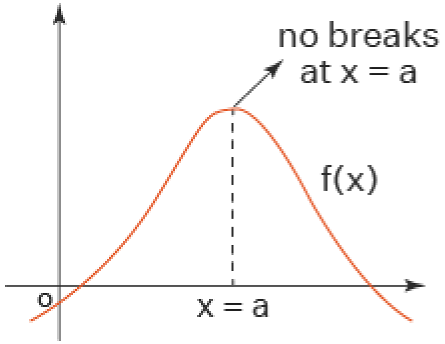

$$f(x) = \frac{x^2 + x - 6}{x^2 - x - 2} = \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}(x+1)}$$

| $x \to 2^-$ | f(x) | $x \to 2^+$ | f(x) |
|---|---|---|---|
| 1.50000 | 1.80000 | 2.50000 | 1.57143 |
| 1.75000 | 1.72727 | 2.25000 | 1.61538 |
| 1.90000 | 1.68966 | 2.10000 | 1.64516 |
| 1.95000 | 1.67797 | 2.05000 | 1.65574 |
| 1.99000 | 1.66890 | 2.01000 | 1.66445 |
| 1.99900 | 1.66689 | 2.00100 | 1.66644 |
| 1.99990 | 1.66669 | 2.00010 | 1.66664 |
| 1.99999 | 1.66667 | 2.00001 | 1.66666 |

# CONTINUITY




These three together will make the function f(x) continous at x = a




$\therefore \boxed{\lim_{x \to a} f(x) = f(a)} \Rightarrow$ f(x) is continuous at x = a

Check the continuity of the function *f* given by *f* (*x*) = 2*x* + 3 at *x* = 1.

$f(x)$ is continuous at $x = 1$

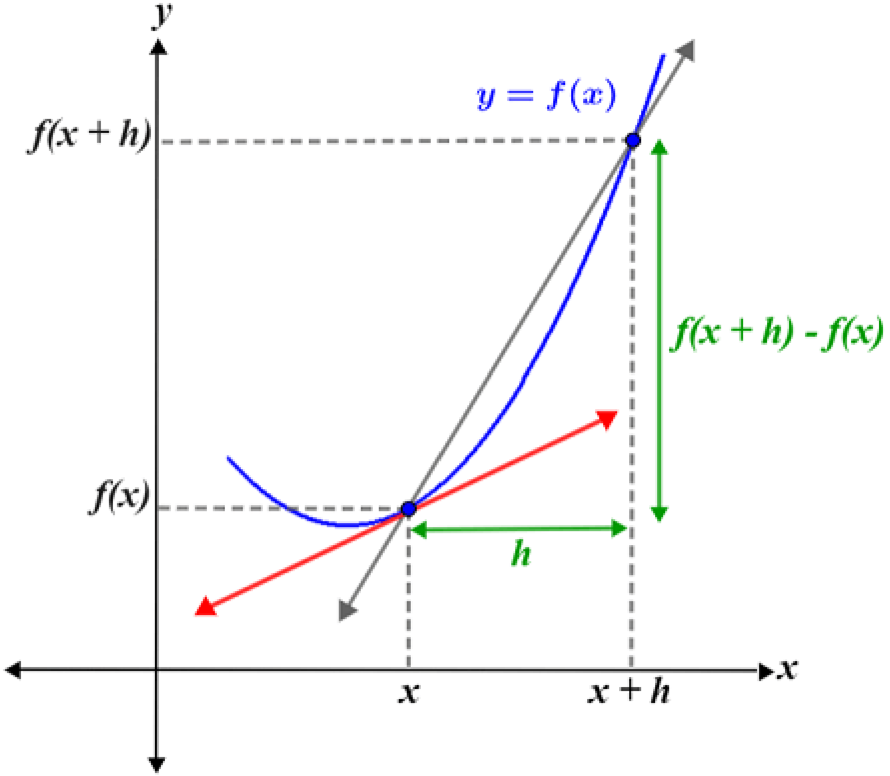
if $\lim_{x \to 1} f(x) = f(1)$

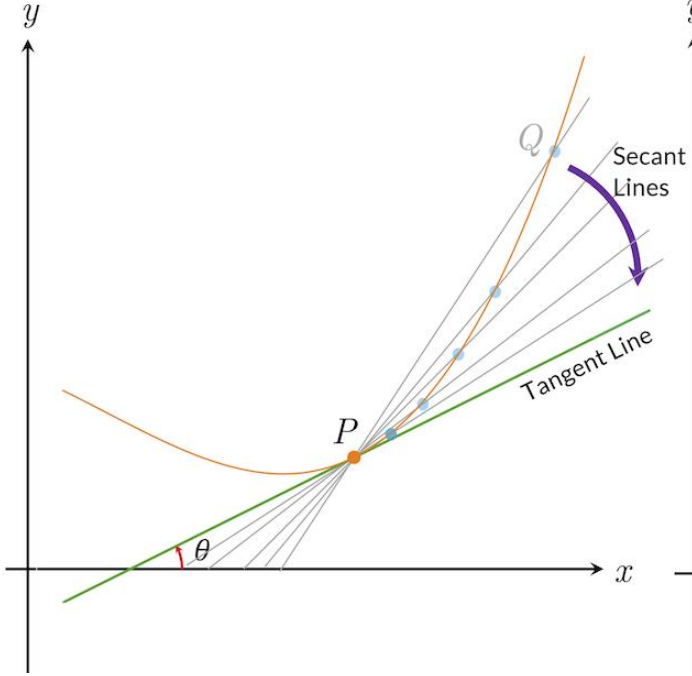
| L.H.S | R.H.S |
|---|---|
| $\lim_{x \to 1} f(x)$ | $f(1)$ |
| $= \lim_{x \to 1} (2x + 3)$ | $= 2 \times 1 + 3$ |
| $= 2 \times 1 + 3$ | $= 2 + 3$ |
| $= 2 + 3$ | **= 5** |
| **= 5** | |

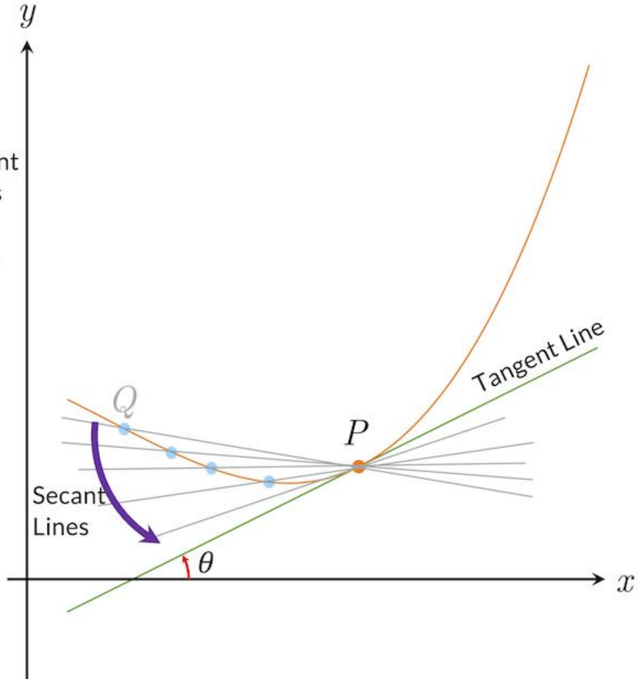
Since, L.H.S = R.H.S

∴ Function is **continuous.**

# DERIVATIVE OF A FUNCTION




$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

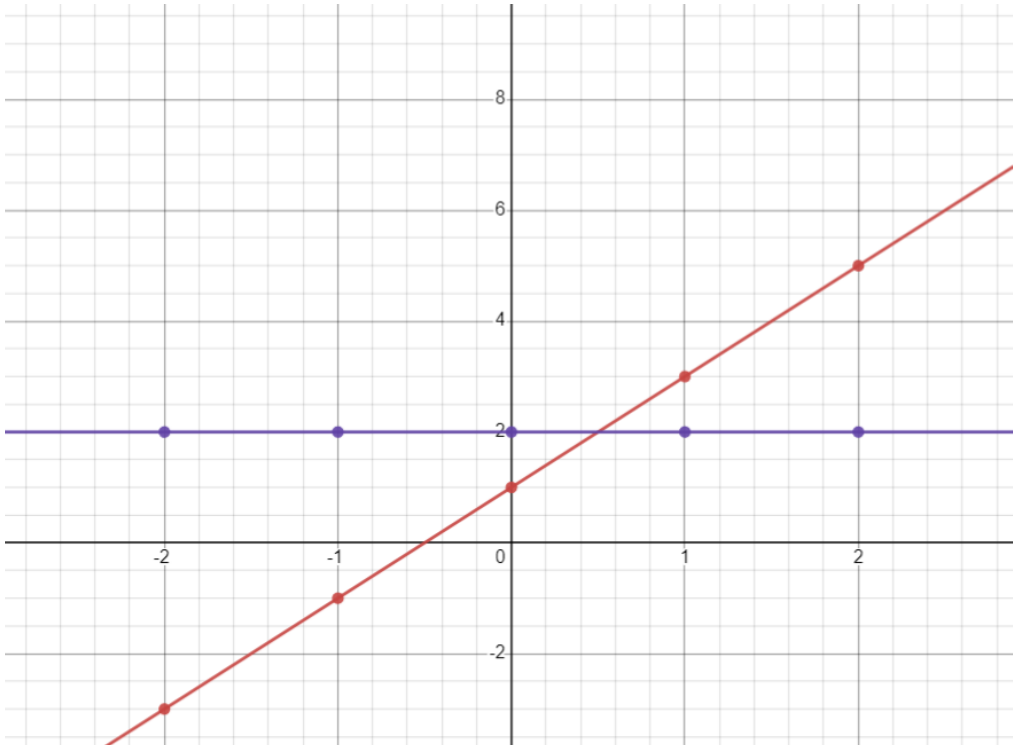
$$\text{Slope of Secant} = \frac{f(x+h) - f(x)}{h}$$

("Difference quotient")

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(if limit exists)

# DERIVATIVE OF A FUNCTION




$f(x) = 2x + 1$

$\frac{d}{dx} f(x) = 2$

| x | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| f(x) | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |

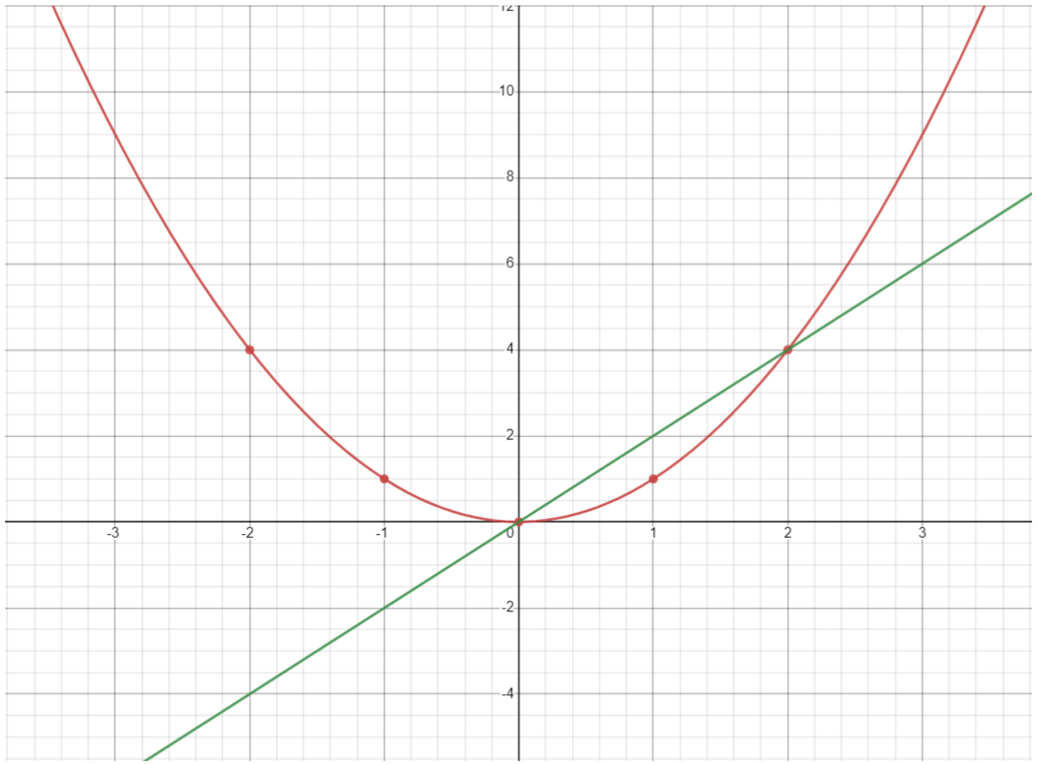
$$\frac{d}{dx} f(x) = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

$$\frac{d}{dx} f(x) = \frac{9-3}{4-1} = \frac{6}{3} = 2$$

$$\frac{d}{dx} f(x) = \frac{7-5}{3-2} = \frac{2}{1} = 2$$

$$\frac{d}{dx} f(x) = \frac{17-9}{8-4} = \frac{8}{4} = 2$$

# DERIVATIVE OF A FUNCTION




$f(x) = x^2$ $\qquad$ $\dfrac{d}{dx} f(x) = 2x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 1 | 4 | 9 | 16 | 25 | 36 | 49 |

1 $\quad$ 3 $\quad$ 5 $\quad$ 7 $\quad$ 9

$\frac{(1+3)}{2} = \mathbf{2}$ $\quad$ $\frac{(3+5)}{2} = \mathbf{4}$ $\quad$ $\frac{(5+7)}{2} = \mathbf{6}$ $\quad$ $\frac{(7+9)}{2} = \mathbf{8}$

$\dfrac{d}{dx} f(x)$ *at* $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4

*so the derivative of* $\boldsymbol{f(x) = x^2}$ $\qquad$ $\boldsymbol{\dfrac{d}{dx} f(x) = 2x}$

$$f(x) = x^2 \qquad \frac{d}{dx} f(x) = 2x$$

| | $x$ | f(x) | $\frac{df}{dx} = \frac{f(x_0) - f(x)}{x_0 - x}$ | $x$ | f(x) | $\frac{df}{dx} = \frac{f(x_0) - f(x)}{x_0 - x}$ | $x$ | f(x) | $\frac{df}{dx} = \frac{f(x_0) - f(x)}{x_0 - x}$ |
|---|---|---|---|---|---|---|---|---|---|
| $x_0^-$ | **0** | **0** | **1** | **1** | **1** | **3** | **2** | **4** | **5** |
| | 0.1 | 0.01 | 1.1 | 1.1 | 1.21 | 3.1 | 2.1 | 4.41 | 5.1 |
| | 0.5 | 0.25 | 1.5 | 1.5 | 2.25 | 3.5 | 2.5 | 6.25 | 5.5 |
| | 0.9 | 0.81 | 1.9 | 1.9 | 3.61 | 3.9 | 2.9 | 8.41 | 5.9 |
| | 0.99 | 0.9801 | 1.99 | 1.99 | 3.9601 | 3.99 | 2.99 | 8.9401 | 5.99 |
| | 0.999 | 0.998001 | 1.999 | 1.999 | 3.996001 | 3.999 | 2.999 | 8.994001 | 5.999 |
| $x_0$ | **1** | **1** | **2** | **2** | **4** | **4** | **3** | **9** | **6** |
| | 1.001 | 1.002001 | 2.001 | 2.001 | 4.004001 | 4.001 | 3.001 | 9.006001 | 6.001 |
| | 1.01 | 1.0201 | 2.01 | 2.01 | 4.0401 | 4.01 | 3.01 | 9.0601 | 6.01 |
| | 1.1 | 1.21 | 2.1 | 2.1 | 4.41 | 4.1 | 3.1 | 9.61 | 6.1 |
| | 1.5 | 2.25 | 2.5 | 2.5 | 6.25 | 4.5 | 3.5 | 12.25 | 6.5 |
| | 1.9 | 3.61 | 2.9 | 2.9 | 8.41 | 4.9 | 3.9 | 15.21 | 6.9 |
| $x_0^+$ | **2** | **4** | **3** | **3** | **9** | **5** | **4** | **16** | **7** |

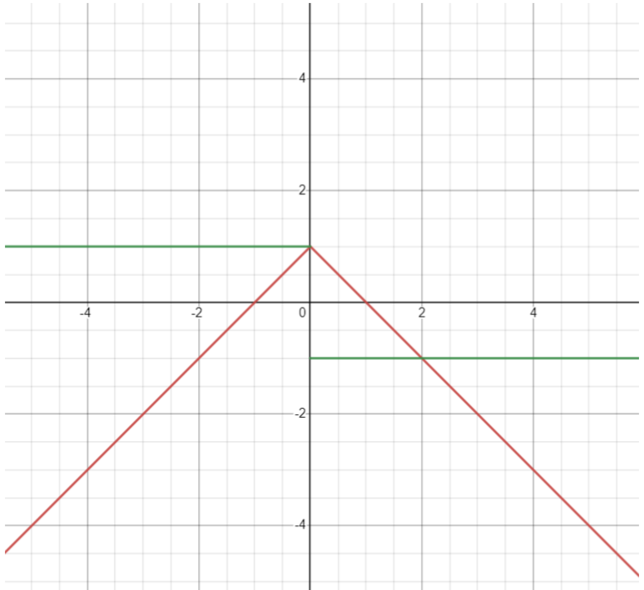
# WHY DERIVATIVE EXSIST AND NOT EXSIST ?

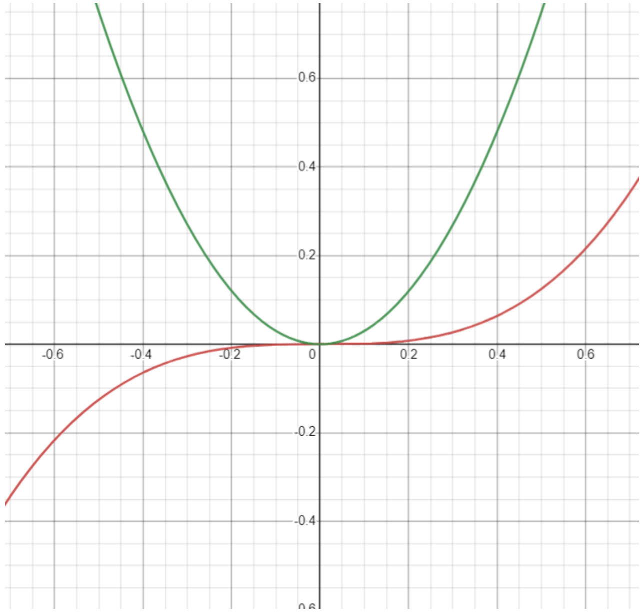
$f(x) = 1 - |x|$

$$\frac{d}{dx} f(x) = \begin{cases} -1 & if\ x > 0 \\ 1 & if\ x < 0 \end{cases}$$

$f(x) = x^3$

$$\frac{d}{dx} f(x) = 2x^2$$




| h | $\frac{f(x-h)-f(x)}{-h}$ | $\frac{f(x+h)-f(x)}{h}$ |
|---|---|---|
| 1 | 1 | -1 |
| 0.1 | 1 | -1 |
| 0.01 | 1 | -1 |
| 0.001 | 1 | -1 |
| 0.0001 | 1 | -1 |
| 0.00001 | 1 | -1 |
| 0.000001 | 1 | -1 |

| h | $\frac{f(x-h)-f(x)}{-h}$ | $\frac{f(x+h)-f(x)}{h}$ |
|---|---|---|
| 1 | 1 | 1 |
| 0.1 | 0.1 | 0.1 |
| 0.01 | 0.01 | 0.01 |
| 0.001 | 0.001 | 0.001 |
| 0.0001 | 0.0001 | 0.0001 |
| 0.00001 | 0.00001 | 0.00001 |
| 0.000001 | 0.000001 | 0.000001 |

# FUNCTION OF SEVERAL REAL VARIABLES

A **real-valued function of *n* real variables** is a [function](#) that takes as input *n* [real numbers](#), commonly represented by the [variables](#) $x_1, x_2, \ldots, xn$ for producing another real number, the *value* of the function, commonly denoted $f(x_1, x_2, \ldots, xn)$.

$$y = f(x_1, x_2)$$

$$y = f(x_1, x_2, x_3)$$

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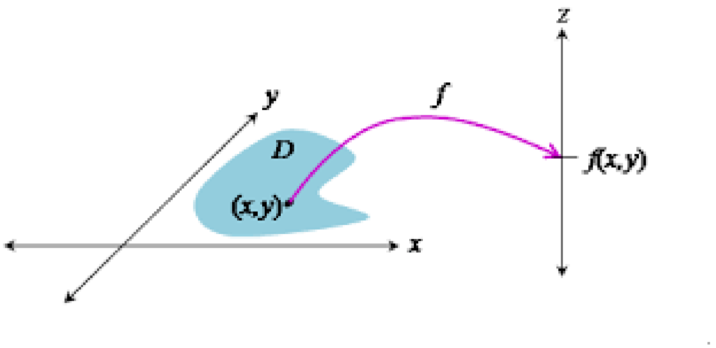
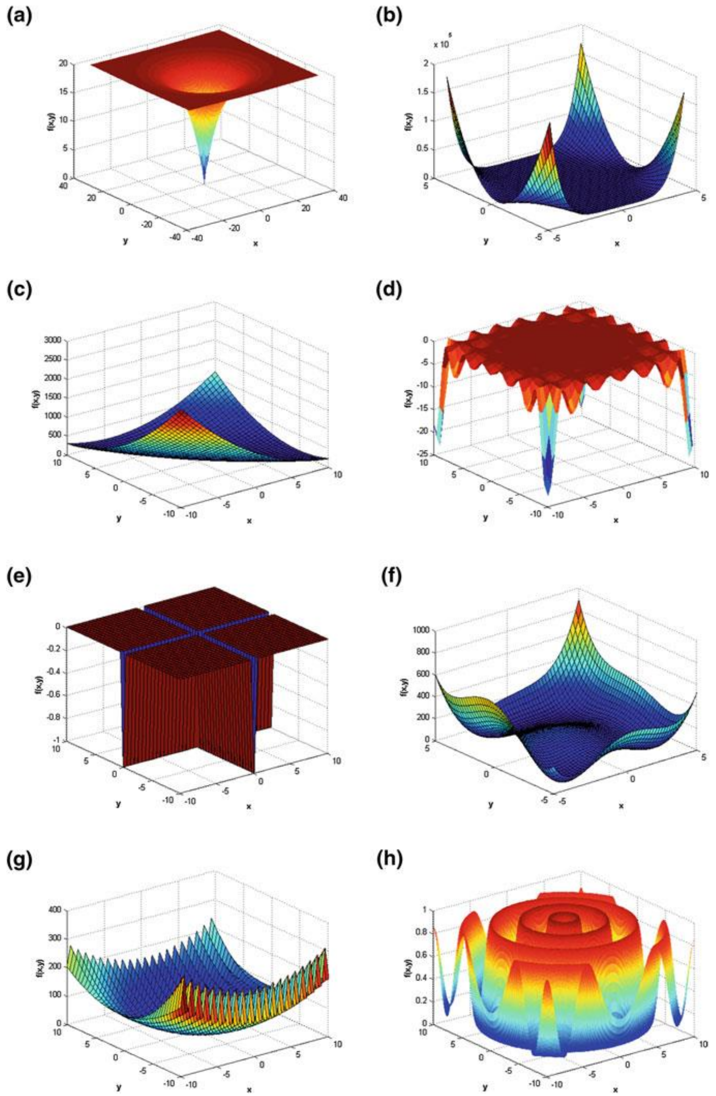
$$y = f(x_1, x_2, x_3, \ldots \ldots, x_n)$$



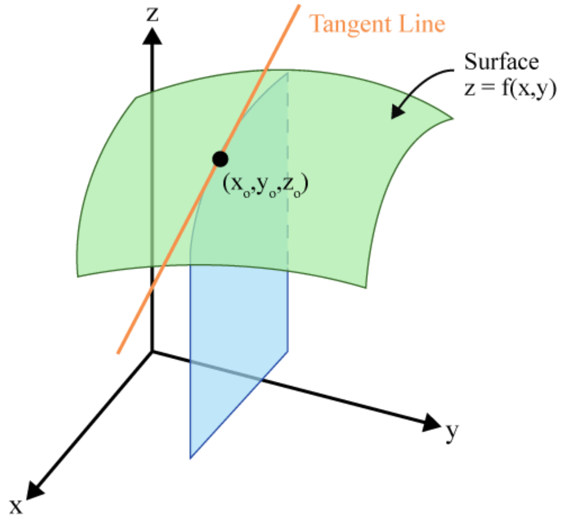



# PARTIAL DERIVATIVES

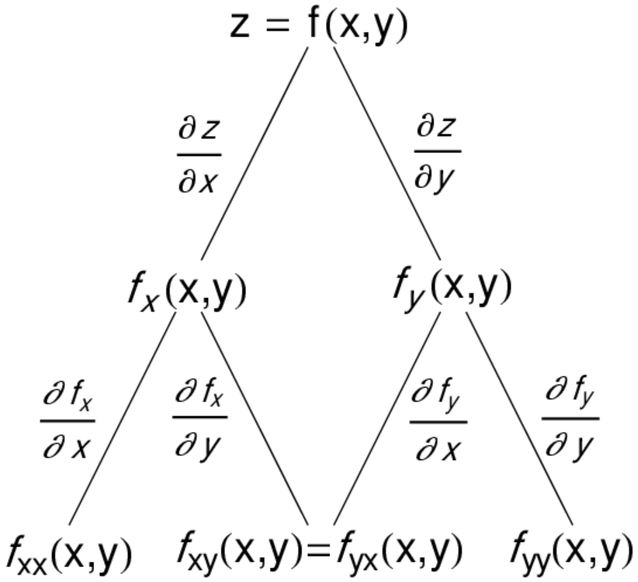
Partial derivatives of a function of two variables states that if $z = f(x, y)$, then the first order partial derivatives of $f$ with respect to $x$ and $y$, provided the limits exist and are finite, are:

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$



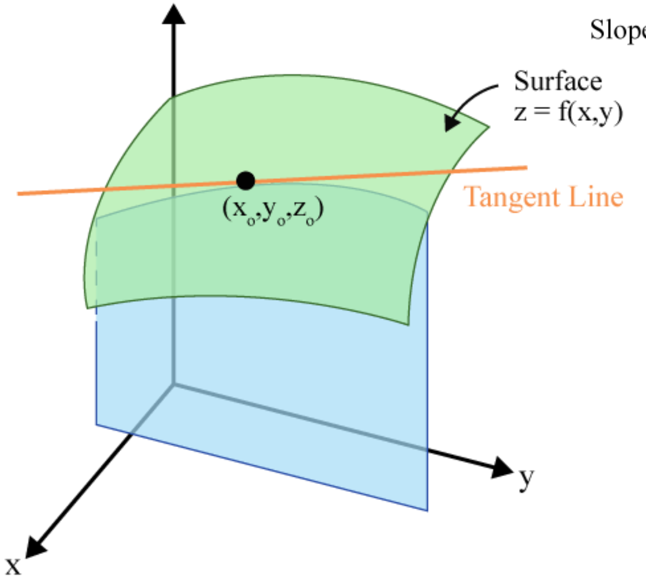

Slope of the surface in the x-direction

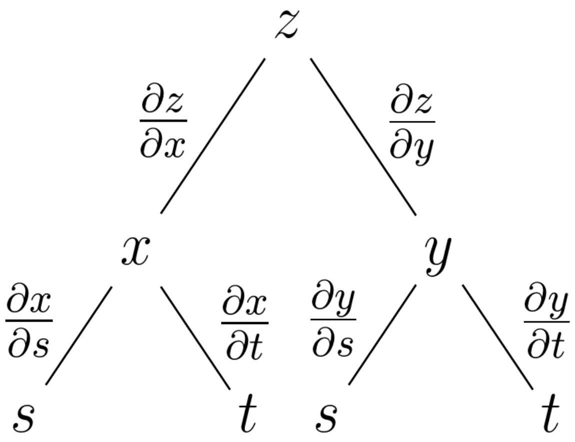



$$z = xy^2 - y\sin(x)$$

$$\frac{\partial z}{\partial x} = y^2 - y\cos(x)$$

$$\frac{\partial y}{\partial x} = 2xy - \sin(x)$$




$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

# PARTIAL DERIVATIVES ! WHY ?

**Isolating Independent Effects:** When a function depends on multiple variables, partial derivatives allow us to isolate the effect of changing one variable while holding the others constant. This is crucial for understanding how different factors interact and contribute to the overall behavior of the function

**Local Sensitivity Analysis:** Partial derivatives measure the instantaneous rate of change of a function with respect to each variable at a specific point. This provides insights into how sensitive the function is to small changes in each variable locally.

$$z = f(x,y) = 2x + 3y$$

$$\frac{\partial z}{\partial x} = 2 \quad \frac{\partial z}{\partial y} = 3$$

$$f(1,0) = 2 \rightarrow f(3,4) = 18$$

$$\Delta f = 18 - 2 = 16$$

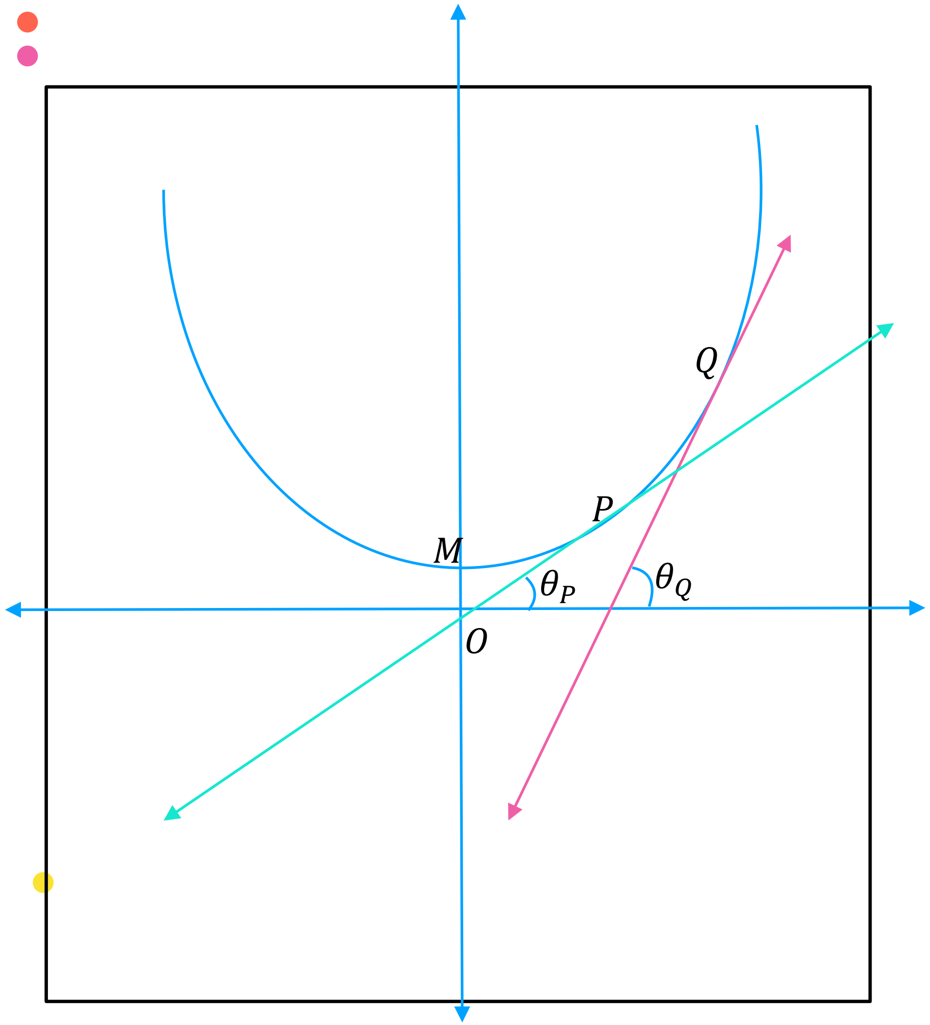
$$\Delta x = 2 \quad \Delta y = 4$$

$$\Delta f = \Delta x \,.\, 2 + \Delta y \,.\, 3 = 16$$

$$\Delta f = \Delta x \frac{\partial z}{\partial x} + \Delta y \frac{\partial z}{\partial y} = 16$$

| f | | y | | | | |
|---|---|---|---|---|---|---|
| | | **0** | **1** | **2** | **3** | **4** |
| x | **0** | 0 | 3 | 6 | 9 | 12 |
| | **1** | 2 | 5 | 8 | 11 | 14 |
| | **2** | 4 | 7 | 10 | 13 | 16 |
| | **3** | 6 | 9 | 12 | 15 | 18 |
| | **4** | 8 | 11 | 14 | 17 | 20 |

# DERIVATIVES AND OPTIMIZATION



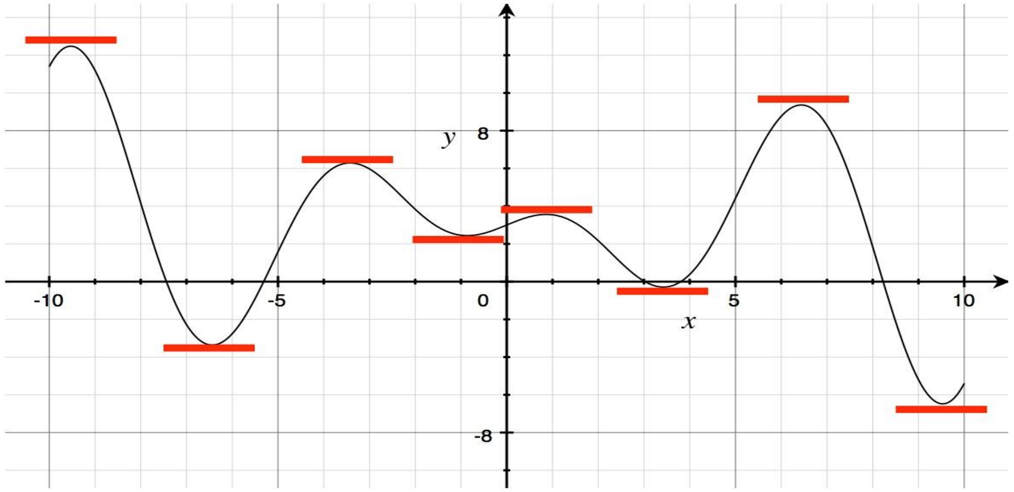

Here $\theta_p < \theta_Q \Leftrightarrow tan(\theta_P) < tan(\theta_Q)$ ; we know,

$$\frac{dy}{dx} \text{ at } P = tan(\theta_P) \ ; \ \frac{dy}{dx} \text{ at } Q = tan(\theta_Q)$$

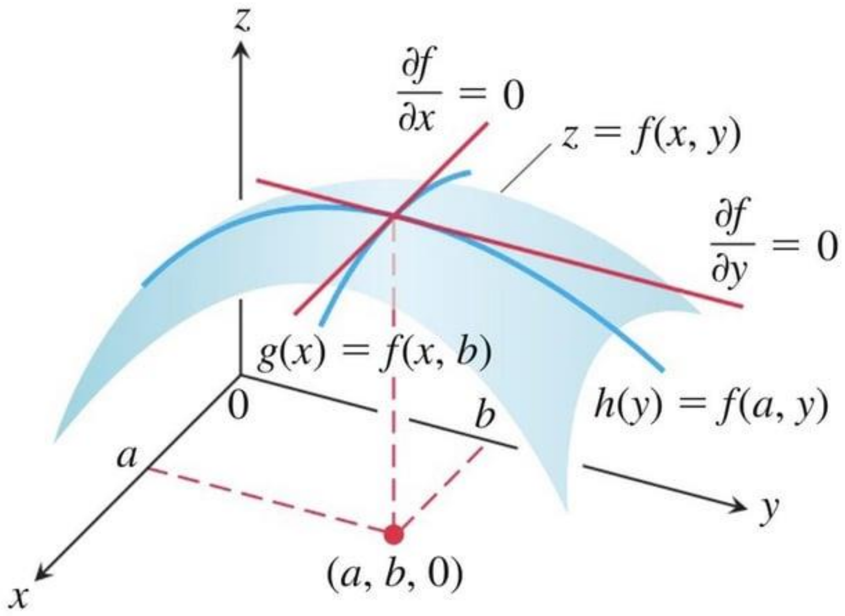
What is the angle of tangent at M ?

$$\theta_m = 0 \Rightarrow \tan(\theta_m) = 0 \Rightarrow \frac{dy}{dx} = 0 \text{ at } M$$

*Derivative at a point $x_c$ is 0 $\Rightarrow$ $x_c$ is an extremum point which is a maximum or minimum, then it can be used for* ***optimization***



# DERIVATIVES AND OPTIMIZATION




*Gradient of a function*

$$\nabla f(x_1, x_2, x_3, \ldots, x_n) = \begin{bmatrix} \frac{\delta}{\delta x_1}(f) \\ \frac{\delta}{\delta x_2}(f) \\ \frac{\delta}{\delta x_3}(f) \\ \vdots \\ \frac{\delta}{\delta x_n}(f) \end{bmatrix}$$

Find the extrema of $f(x, y) = x^2 + 2y^2 - 4(x + y)$

1. Gradient:

$$\nabla f = (2x - 4, 4y - 4)$$

2. Critical point: $(2, 1)$

*To find optimal points of a function* $f(x_0, x_1, x_2, \ldots\ldots x_n)$:

Calculate $\nabla f$

Find $x_0, x_1, \ldots\ldots x_n$ where $\nabla f(x_0, x_1, x_2, \ldots\ldots x_n) = 0$

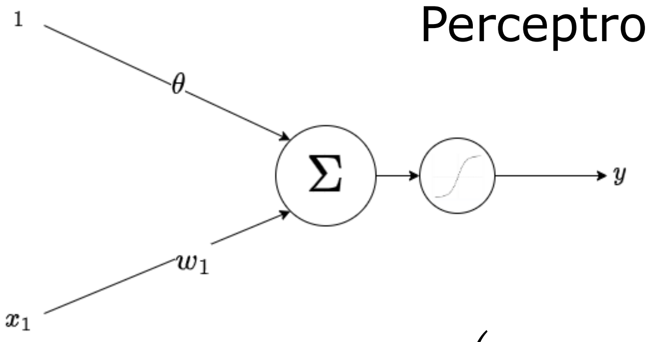
# ERROR FUNCTION AND SURFACE

## Linear Regression

$$\hat{y} = mx + b$$

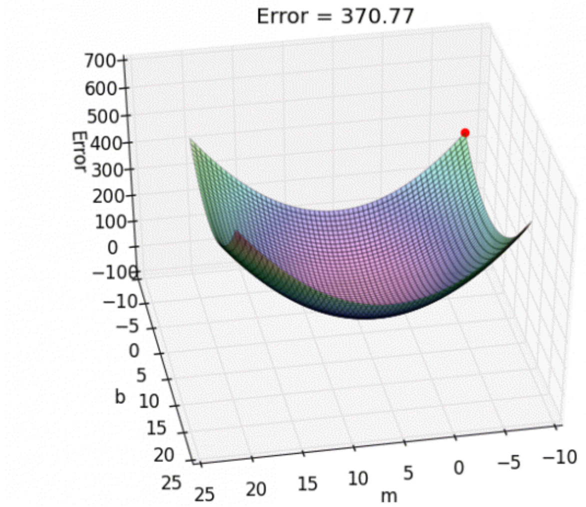
$$J = \sum (y - \hat{y})^2 = \sum (y - mx - b)^2 = J(m, b)$$

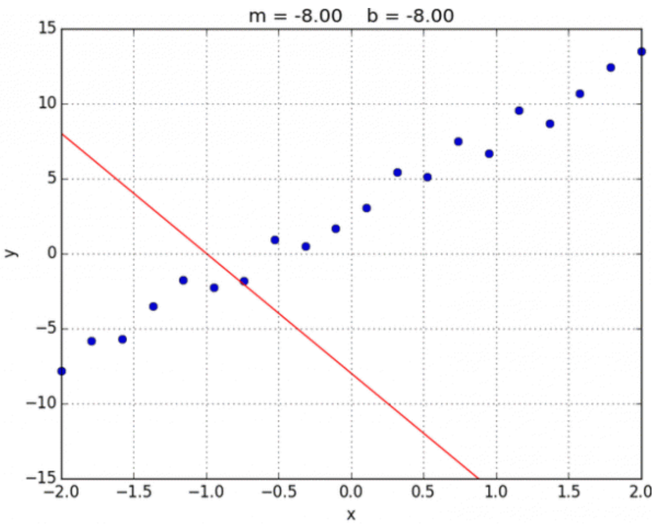
## Perceptron

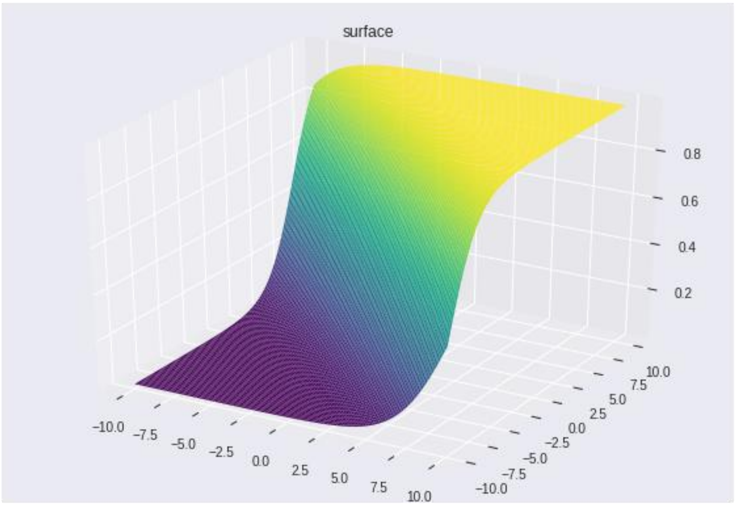



$$\hat{y} = \frac{1}{1 + e^{(-w_1 x_1 - \theta)}}$$

$$J = \sum (y - \hat{y})^2 = \sum \left( y - \frac{1}{1 + e^{(-w_1 x_1 - \theta)}} \right)^2 = J(w_1, \theta)$$

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QUESTIONS