Introduction to Game Theory	Date: 30^{th} *April 2020*
Instructor: *Sujit Prakash Gujar*	Abhigyan , Issac , Shaily

Room-Allocation Rent Division

1 Introduction

There are a set of agents and a set of rooms, each agent has preference over rooms, now how should these agents divide the overall rent among themselves, and which room will they get. This room allocation rent division problem falls under fair division of indivisible goods and many solutions are proposed for the same. Goal of this problem is to fairly allocate rent and rooms to agents such that desirable properties like Envy-free, Efficient, Individual Eationality, having non negative Price,... are fulfilled. In this report, we first go through a naive implementation to this problem, further discussing two popular existing solutions, and evaluating few examples based on them

2 Preliminaries

There are N agents, $\{1, 2, \ldots, n\}$ and N rooms, $\{1, 2, \ldots, n\}$ in a house. Let N denote set of all rooms and A denote set of all agents. For simplicity, if a room is supposed to be shared between two agents, we consider that as 2 rooms. Rent of the overall house is fixed c. Each agents have its valuations for rooms, v_{ij} is the valuation of agent i for room j. Assume, sum of all the valuation of different rooms sums up to the total rent of house, $\forall i, \; \sum_{j=1}^{n} v_{ij} = c$ Let $p \in R^n$ be the price vector for, $p_j \geq 0$ will be the price of room j. We have a quasi linear setting, for any room j , agent i's utility $u_i(r, p) = v_{ij} - p_j$ A allocation $\mu \in \mathcal{R}^n$ is feasible, if each agents gets exactly one room, with non negative prices, μ_i represents the room allocated to agent i. By Envy free we mean, every agent should feel that their share is atleast as good as share of other agents, i.e. no one would want to switch to other agent's room and rent price. Prices should be non negative, you should not pay a agent to live in the house An allocation (μ_i, p) is envy-free if and only if $u_i(\mu_i, p_{\mu_i}) \geq u_i(j, p_j), \;\; \forall i \in A$ and $j \in N$

3 Naive Implementation

Each agent reports its valuation v_i for all the rooms, algorithm computes price of each room j buy averaging over all the valuation for room j , $p_j = \frac{1}{n} \sum_{i=1}^{n} v_{ij}$ We assign the room to the agent who had highest valuation for room j, and pays the price p_j.

For example, rent of the house is given 2000 units. For the following valuations, price $p = (1000, 800, 700)$ and allocation will be $\mu = (1, 3, 2)$. Agent 1 gets room1 at the price of 1000 units. Similarly Agent 2 gets the room2 at price of 800 units, and Agent 3 gets room3 at the price of 700 units.

	room1	room2	room3
agent1	1200	800	500
agent2	800	600	1100
agent3	1000	1000	500

However, this implementation lacks detailing, such as what happens when there's a tie of valuations for any room j, or what if highest bidder for two rooms are the same agents. Such an example scenario for house rent 1800 units can be,

	room1	room2	room3
agent1	700	500	600
agent2	1000	800	0
agent3	400	500	900

Moreover, this implementation is neither strategy-proof nor envy-free. In the first table, agent1 can lie its valuation about room1 to reduce the average but still get the room, and also that leads to increase in the prices of other rooms.

In the following table, house rent = 2500 units, an example that this mechanism is not envy free. For the following valuations, price $p = (1100, 600, 700)$ and allocation will be $\mu = (1, 3, 2)$. Agent 1 gets room1 at the price of 1100 units. Similarly Agent 2 gets the room2 at price of 600 units, and Agent 3 gets room3 at the price of 700 units. Utility of agent1, for room1 is $(1200-1000) = 100$, while if he gets room2 at the price at which agent3 got i.e. 600 units, its utility will be $(800 - 600) = 200$, so at the given price, agent1 will prefer room2 more than room1.

	room1	room2	room3
agent1	1200	800	500
agent2	1000	200	1300
agent3	1100	800	600

4 Splitting rent fairly with triangles

Suppose there are 3 agents and 3 rooms. We consider an equilateral triangle, If we represent each point of this triangle as distance from all its sides i.e. (d_1, d_2, d_3) where d_i is distance of that point from i^{th} side.

Theorem 1. *(Viviani's Theorem) The sum of distances from any point inside an equilateral triangle to its sides is constant*

So for any point p, $d_1 + d_2 + d_3$ is also going to be constant, and that's the height of the triangle. We can look at this as height of an equilateral triangle is the rent of the house, and each point inside it represents a possible division of rent c. So a point p $= (x, y, z)$ corresponds to room1 having x rent, room2 having y rent and room3 having z rent. For any vertex of this triangle, corresponds to one room having rent c while the other two are free of cost.

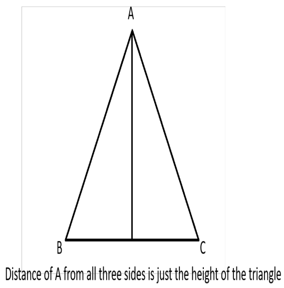

Figure 1

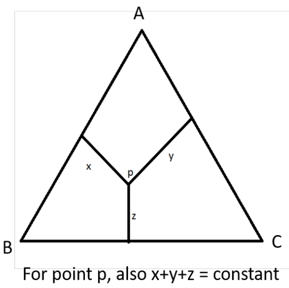

Figure 2

Now, out of infinitely many points, to find the optimal division of rent, we will use Sperner's Lemma

Lemma 1. *(Sperner, 1928) Any Sperner-labelled triangulation of a n-simplex S must contain an odd number of fully labelled elementary n-simplices. In particular, there is at least one.*

Definition 1. *A proper coloring of a simplicial subdivision is an assignment of n + 1 colors to the vertices of the subdivision, so that the vertices of S receive all different colors, and points on each face of S use only the colors of the vertices defining the respective face of S.*

2-D simplex will be a triangle, 3-D simplex will e a tetrahedron, and so on. Sperner lemma means every properly colored simplicial subdivision contains a cell whose vertices have all different colors

Lets take an example equilateral triangle to explain Sperner Lemma, Triangulate this equilateral triangle in any number of parts, color each vertex by different color. For the sides of this equilateral triangle, the nodes of this side can be colored by any of the color of its vertices. E.g. if a side has red and blue as vertices, inner nodes of this side can be either colored by red or blue. And for the nodes, inside this equilateral triangle, we can chose any color to color it. So Sperner Lemma says that there exists atleast one inner triangle, whose vertices have all different colors. Further, it also says that number of such fulled colored vertices triangles are odd.

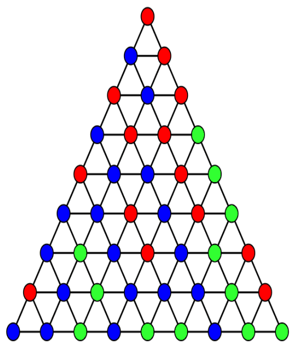

Figure 3: After coloring all the vertices (Ref:lesswrong)

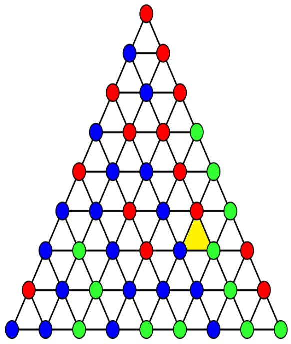

Figure 4: There exists atleast one triangle with all vertices having different color

An analogical proof to this is that, consider the equilateral triangle is house, and all the inner triangle are its room, now, we fix any two colors, say blue and green, any side that has this two colors will be the door of the room. A room can either have 0, 1 or 2 doors. We are trying to find the special room which has one door, i.e. its vertices have different colors. The doors are going to be along any of the side of equilateral triangle, and there will be odd numbers of doors.

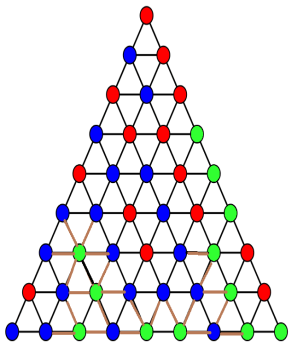

Figure 5: Doors are marked in light brown color, odds numbers of doors

Now, if we enter the house from a door, two things will happen either we have entered a room with one door or two doors. If its a room with one door we can stop, as we found our special room i.e. the triangle whose vertices have different colors. If its a room with two doors, we can exit the room from the other door, and this goes on till we find the room with one door. Now, say in a traversal, we didn't find any room with only one door, then we leave the house. and again enter with another door. If we leave the house, there always atleast one door remaining to enter, as number of doors in house is going to be odd. And eventually this will terminate leading us to room having only one door.

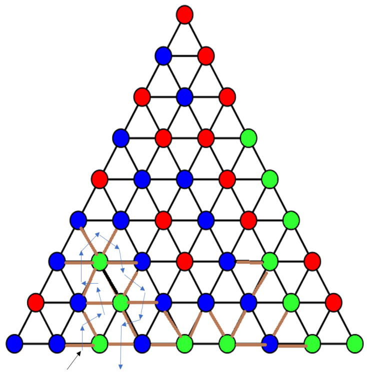

Figure 6: Entered house from a door, unsuccessful to find special room, left the house

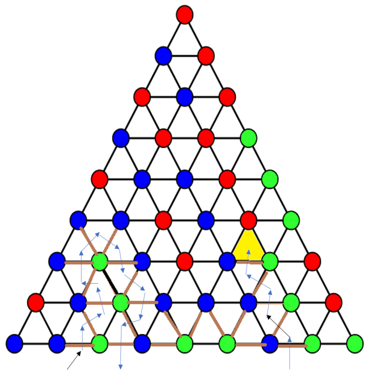

Figure 7: Re-entered the house and found the special room

Now, how does Sperner Lemma and Viviani Theorem solve our rent division problem is, to give an intuition, from Viviani theorem, we know that now each point in the equilateral triangle is a possible division of rent among three rooms. And from Sperner Lemma we find out which of this division will be optimal, i.e. on which inner triangle all the agents will agree upon. Viviani theorem will give us the division of rent and Sperner Lemma will give us allocation in rough sense.
We have 3 agents, lets call them 1,2 and 3. We take an equilateral triangle and triangulate it in as many layers we want. For each triangle now, we label the vertices as 1,2,3.

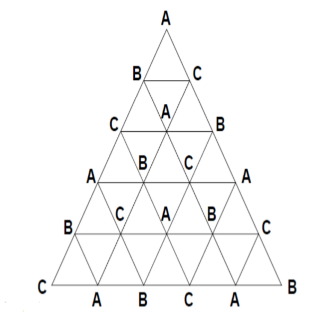

Figure 8: Labelling all the nodes (Ref:CMU)

As point in this triangle represent division of rent, all the points on the side of the triangle will always have one room free of cost. While the vertices of this equilateral triangle will always have two rooms free of cost. One of our assumption here is that anyone who has been offered a free room, will always prefer that compared to other options. Now we will go on coloring each node, our color here represent which room a node will get, i.e. which room the owner of that node will get.

So on the side excluding vertices of the equilateral triangle, always agents will prefer free room. The three color will be (Red, Green and Blue) corresponding to respective rooms. So we don't require preference elicitation in this step. Another assumption that we have is that, given two free rooms, each agent has different preference, then only we can apply sperner's lemma. To apply Sperner Lemma, we need to have that all our vertices of equilateral triangle are colored differently

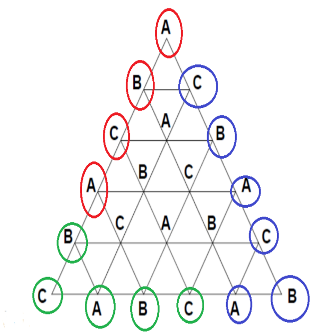

Figure 9: Coloring the side nodes (Assigning rooms)

In the next step, we need to do preference elicitation, we will transverse each node, and question the owner of the node, that based on this distribution of rent which room they will prefer and you assign color accordingly to that node. For example , house rent is 500 units. Now, say for the point (50,50,400), i.e. Room1 (Red)'s rent is 50 units, Room2 (Green)'s rent is 100 units, and Room3 (Blue)'s rent is 400 units and the owner of the node is A, then lets say A choose room2 then we will color that node green. We keep eliciting until all the nodes are colored.

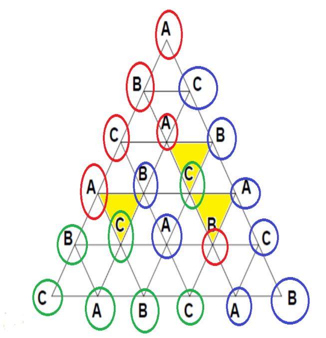

Figure 10: Coloring all the nodes

Once the coloring is done, we can transverse, and find the triangle with all three different colored vertices. In the above example, we found 3 such triangles, meaning all these distribution will be optimal Once we find the triangle, we know the allocation of room, but still rent distribution is not exactly clear. As on the vertices of this triangle have three different distribution of rent For example, the vertices of this triangle is (120,150,230), (120,160,220) and (130,150,220) for a house rent of 500 units Overall difference of rent happens to be 10 units about these vertices/ As we increase layers in our triangulation, this difference will keep on decreasing. However, we can always just take a mean over these vertices to get our rent division

Computationally, we will never do elicitation on each inner nodes. We will color the node as we traverse, and as seen, number of nodes that we actually traverse are quite low compared to total number of nodes.

Theorem 2. *(Rental Harmony Theorem - Francis Edward Su) [1] Suppose n housemates in an n-bedroom house seek to decide who gets which room and for what part of the total rent. Also, suppose that the following conditions hold:*

1. *(Good House) In any partition of the rent, each person finds some room acceptable.*
2. *(Miserly Tenants) Each person always prefers a free room (one that costs no rent) to a non-free room.*
3. *(Closed Preference Sets) A person who prefers a room for a convergent sequence of prices prefers that room at the limiting price.*

Then there exists a partition of the rent so that each person prefers a different room.

Theorem 3. *(Alkan 1991) There exists no mechanism which is both envy-free and strategyproof*

The Rental Harmony Theorem establishes the existence of envy-free chore division and ϵ-approximate algorithm [1] Also here, agents never need to actually calculate their true valuations, even if they have their valuations, the algorithm will never ask for it, so there's also a notion of privacy. We discuss the scenario with 3 agents, however we can expand this concept for n agents. For n agents, we will have (n-1) dimension simplex. Many modifications have been made to this method, for example there's study which shows that if one of the agent want to keep their preference private or is not available for preference elicitation, we can still find optimal division of rent

5 A Market Approach

The idea behind this approach, is that the room market clears at price p if and only if for any group of agents the number of different rooms collectively demanded by the group is no less than the size of the group. [2] This is very similar to the Discrete Mathematics, Hall's Theorem.

Theorem 4. *(Hall's Condition) Given a set A, let $N(A)$ be the set of neighbors of A. Then the bipartite graph G with bi-partitions X and Y has a perfect matching iff $|N(A)| \geq |A|$ for all subsets A of X.*

Theorem 5. *A bipartite graph G consists of sets X and Y, $|X| \leq |Y|$, has a matching of size $|X|$ if and only if G satisfies Hall's Condition*

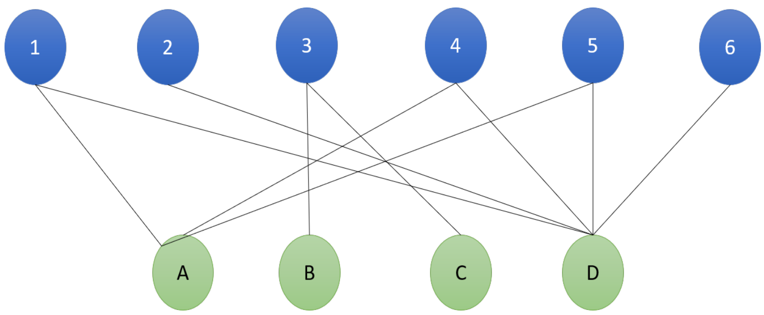

Figure 11: X={B,C} , N(X)={3} , $|N(x) = 1|$, $|X| = 2$, violates Halls Condition

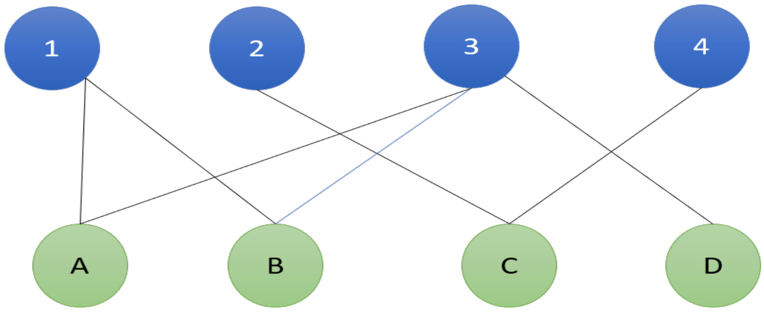

Figure 12: Halls Condition is satisfied, {C4, D3, A2,B1}

Now, coming to our room-agent allocation, we will have a bipartite graph with bi-partitions of agents and rooms. Given price p, an edge will only exists between an agent i and a room j , if the agents prefers the room j at price p_j If we have such a bipartite graph, and Hall's condition satisfies then we can be sure that there exists a match between rooms and agents. Basic idea is to keep changing the prices of rooms, till our bipartite graph reaches Hall's Condition, and once it satisfies Hall's Condition then we have our equilibrium price and allocation.

Definition 2. *Demand of an agent i is defined as $D_i(p)$ at price p, $D_i(p) = \{r \in N : u_i(r, p_r) \geq u_i(s, p_s) \ \ \forall s \in N\}$*

$D_i(p)$ wil contain all the rooms for which agent i's valuation is lower or equal to set price of those room. For example, we have 3 agents and 3 rooms, and agent1 has valuation $v_1 = (100, 200, 300)$ current $p = (150, 100, 100)$, then $D_1(p) = \{2, 3\}$ i.e. Demand of agent 1 is room 2 and room 3 based on the current prize.

Theorem 6. *There exists a matching $\mu \in \mathcal{M}$ with $\mu_i \in D_i(p)$ for each $i \in A$ if and only if*

$$\forall J \subseteq A \ \ |\bigcup_{i \in J} D_i(p)| \ \geq \ |J|$$

In a bipartite graph with bipartitions of agents and rooms, there exists an edge between a room j and agent i if room $j \in D_i(p)$. If Hall's condition satisfies then we can find a matching between agents and rooms on price p If we set some prices randomly or just uniformly divide prices among all rooms, find the demand sets of agents, form a bipartite graph, and see whether it satisfies hall condition or not, chances are high that it won't as some rooms are overdemanded while others are not. We need an algorithm that keeps updating prices until we have a bipartite graph satisfies Hall's condition

Definition 3. *Set of rooms is said to be overdemanded at price p if the number of agents demanding only rooms in this set is greater than the number of the rooms in the set $S \subset N$ is overdemanded if*

$$|\{i \in A : D_i(p) \subseteq S\}| \geq |S|$$

For example, at some price p $S = \{2, 3\}$ i.e. room 2 and room 3. Now conisder $D_1(p) = \{2, 3\}$, $D_2(p) = \{3\}$ and $D_3(p) = \{2\}$, then $D_1(p) \subseteq S$, $D_2(p) \subseteq S$ and $D_3(p) \subseteq S$, i.e. the cardinality is 3 and $|S| = 2$, so clearly room2 and room3 are said to be overdemanded.
For example, $S = \{2\}$, and $D_1 = \{2\}$, $D_2 = \{2\}$, then again room 2 is said to be overdemanded. Now, in the same example if we have $D_3 = \{2, 3\}$, then for $S = \{2, 3\}$, then $D_1, D_2, D_3 \in S$, and set $\{2, 3\}$ is said to be overdemanded

Definition 4. *A set is said to be minimal overdemanded set, if it is overdemanded and none of its proper subsets is overdemanded*

So in the above example, $\{2, 3\}$ is not a minimal overdemaned set. Intuitively we can say that the algorithm should increase the prices of room in overdemanded sets, while decrease the price of

those which are not part of this set.
For, consider a scenario, $A = 1, 2, 3, 4$ and $N = 1, 2, 3, 4$, and $D_1(p) = D_2(p) = \{1\}$, $D_3(p) = \{2\}$ and $D_4(p) = \{1, 2\}$, so minimal overdemanded set will be $\{1\}$ but as we can see that b is also overdemanded.

Definition 5. *A full set of overdemanded rooms at price p , find minimal overdemanded set, further remove all the rooms which are in this set from the demand set of all agents, and again find minimal overdemanded set, keep repeating until we dont find any more overdemanded sets. Full set of overdemanded rooms will be union of all minimal overdemanded sets.*

In the above example, as $\{1\}$ is minimal overdemanded, then we removed it from all demand set of agents, and $D_1(p) = D_2(p) = \Phi$, $D_3(p) = \{2\}$ and $D_4(p) = \{2\}$, and we found that $\{2\}$ is also now minimal overdemanded set, and full set will be $\{1, 2\}$

Lemma 2. *Full set of overdemanded rooms $OD(p) = \Phi$ if and only if $|U_{i\in J} D_i(p)| \geq |J| \;\; \forall J \subseteq I$*

The paper proposes the following algorithm,

1. Initially set the price $p = \left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right)$, i.e. equally divide rent among all rooms
2. Find $OP(p)$, and if $OP(p) = \Phi$, then by Lemma1 and halls theorem we know there exists a mapping, and we can terminate
3. Otherwise, continuously increase prices of all the rooms in $OP(p)$ equally by $dx \to 0$, and decrease prices of all the rooms not in $OP(p)$ equally by $dy \to 0$ such that $|OP(p)|dx = (n - |OP(p)|)dy$
4. Update the price vector and go to step 2

When discrete setting, instead of increasing $dx \to 0$, the paper gives a formula to increase prices of rooms in OP(p)

$$x(p) = \left\{ \begin{array}{ll} \min\limits_{j\in J(p)} \left(\max\limits_{r\in \mathcal{N}} u_i(r, p_r) - \max\limits_{s\in N/OD(p)} u_j(s, ps) \right) & \text{if } OD(p) \neq \Phi \\ 0 & \text{otherwise} \end{array} \right\}$$

For all $r \in N$, price updated is defined as

$$f_r(p) = \left\{ \begin{array}{ll} p_r - \frac{|OD(p)|}{n} x(p) & \text{if } r \notin OD(p) \\ p_r + \frac{|OD(p)|}{n} x(p) & \text{if } r \in OD(p) \end{array} \right\}$$

Theorem 7. *Let p^t be the price sequence in the discrete-price auction. There exists finite T such that $OP(p^T) = \Phi$*

In the continuous-price auction the only instances that are crucial are those instances where some agent's demand changes [2] So we can always create a discrete equivalent of continuous-price auction.

Proposition 1. *An allocation (μ, p) is envy-free if and only if $\mu_i \in D_i(p)$ for each agent i*

Proposition 2. *(Svensson 1983) Let (μ, p) be an envy-free allocation, then (μ, p) is efficient.*

This auction mechanism is efficient, envy-free, individually-rational and it yields a non-negative price to each room whenever that is possible with envy-freeness. [**?**]

6 Computing Examples

NYTimes has implemented the Francis Su mechanism (Sperner Lemma) and Spliddit has implemented Atila's mechanism So we have evaluated 5 examples using these platforms. In all the 5 examples, the allocation assigned by both were same, there were slight differences in prices.

	room1	room2	room3
agent1	700	500	600
agent2	1000	8000	0
agent3	400	500	900

Allocation	Price paid (Francis Su)	Price paid (Atila)
room2	393.75	400
room1	525	600
room3	881.25	800

Table 1: Budget of house : 1800 units

	room1	room2	room3
agent1	500	300	200
agent2	200	200	600
agent3	400	400	200

Allocation Assigned	Price paid (Francis Su)	Price paid (Atila)
room3	395	333.33
room1	333.33	433.33
room2	270.33	233.33

Table 2: Budget of house : 1000 units

	room1	room2	room3
agent1	300	300	600
agent2	500	200	500
agent3	400	400	400

Allocation Assigned	Price paid (Francis Su)	Price paid (Atila)
room3	475	500
room1	400	400
room2	325	300

Table 3: Budget of house : 1200 units

	room1	room2	room3
agent1	500	500	500
agent2	800	700	0
agent3	100	700	700

Allocation Assigned	Price paid (Francis Su)	Price paid (Atila)
room3	484.37	466.67
room1	531.25	566.67
room2	484.37	466.67

Table 4: Budget of house : 1500 units

	room1	room2	room3
agent1	558	306	336
agent2	100	906	194
agent3	266	114	820

Allocation Assigned	Price paid (Francis Su)	Price paid (Atila)
room1	50	196.67
room2	575	544.67
room3	575	458.67

Table 5: Budget of house : 1200 units

7 Conclusion

In this report, we first started with a naive implementation which was neither strategyproof nor envy-free. Further we studied two mechanism - Using Sperner Lemma to find allocation by Francis Su and Using Hall's theorem to find allocation by Atila Abdulkadirog. Both being envy-free but not strategyproof. There are many mechanism designed in this family of class each focus on improving one of the properties like efficiency, strategyproof, fairness, etc.

References

[1] Francis Edward Su. Rental harmony: Sperner's lemma in fair division. *The American Mathematical Monthly*, 106(10):930–942, 1999.

[2] Atila Abdulkadiroglu, Tayfun Sönmez, and M. Utku Ünver. Room assignment-rent division: A market approach. *Social Choice and Welfare*, 22(3):515–538, 2004.