PSET 03

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Q1. → E.V. → trace → determinant
→ E.$\vec{v}$ → Rank

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{vmatrix}$$

$$0 = (1-\lambda)\left[(5-\lambda)(9-\lambda) - 48\right] - 2\left[4(9-\lambda) - 42\right] + 3\left[32 - 7(5-\lambda)\right]$$

$$(1-\lambda)\left[\underbrace{45 - 14\lambda + \lambda^2 - 48}_{\lambda^2 - 14\lambda - 3}\right] - 2\left[\underbrace{36 - 4\lambda - 42}_{-4\lambda - 6}\right] + 3\left[\underbrace{32 - 35 + 7\lambda}_{7\lambda - 3}\right] = 0$$

$$\lambda^2 - 14\lambda - 3 - \lambda^3 + 14\lambda^2 + 3\lambda + 8\lambda + 12 + 21\lambda - 9 = 0$$

$$-\lambda^3 + 15\lambda^2 + 18\lambda = 0$$

$$-\lambda(\lambda^2 - 15\lambda - 18) = 0$$

$-\lambda = 0$, $\lambda^2 - 15\lambda - 18 = 0$

$$\lambda = \frac{15 \pm \sqrt{225 + 72}}{2}$$

$$= \frac{15 \pm 3\sqrt{33}}{2}$$

$$\lambda_1 = 0$$

$$\lambda_2 = \frac{15 + 3\sqrt{33}}{2}$$

$$\lambda_3 = \frac{15 - 3\sqrt{33}}{2}$$

$\Rightarrow \lambda = 0 \qquad Ax = 0 \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

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Solving $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 6 & 12 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 = free variable

$$x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$$

$x_2 = -2x_3$

$x_1 + 2x_2 + 3x_3 = 0$

$x_1 - 4x_3 + 3x_3 = 0$

$x_1 = x_3$

$$\boxed{V_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

$\Rightarrow \lambda = \dfrac{15 + 3\sqrt{33}}{2} \qquad (A - \lambda I)x = 0$

$$\begin{bmatrix} \dfrac{-3\sqrt{33} - 13}{2} & 4 & 6 \\ 8 & \dfrac{-3\sqrt{33} - 5}{2} & 12 \\ 14 & 16 & \dfrac{-3\sqrt{33} + 3}{2} \end{bmatrix}$$

Using cramer rule

$$\Rightarrow V_2 = \begin{bmatrix} \dfrac{3\sqrt{33} - 11}{22} \\ \dfrac{3\sqrt{33} + 11}{44} \\ 1 \end{bmatrix}$$

$\Downarrow$

$$\cancel{\begin{bmatrix} -3\sqrt{33} - 13 & 4 & 6 \\ 0 & -330 - 54\sqrt{33} & 204 + 36\sqrt{33} \\ 0 & 48\sqrt{33} + 264 & -174 + 30\sqrt{33} \end{bmatrix}}$$

$$\cancel{\begin{bmatrix} -3\sqrt{33} - 13 \\ 0 \\ 0 \end{bmatrix}}$$

$\Rightarrow \lambda = \dfrac{15 - 3\sqrt{33}}{2} \Rightarrow V_3 = \begin{bmatrix} \dfrac{-3\sqrt{33} - 11}{22} \\ \dfrac{-3\sqrt{33} - 11}{44} \\ 1 \end{bmatrix}$

Q.2. 1) x is $q \times 1$

y is $p \times 1$

$$y_{p\times 1} = A_{p\times q}\, x_{q\times 1}$$

dimension : $p \times q$

2)

$y_1 = Ax_1$ distance of y_1 & y_2

$y_2 = Ax_2$ = distance of x_1 & x_2

$$\Rightarrow (y_1 - y_2)^T (y_1 - y_2) = (x_1 - x_2)^T (x_1 - x_2)$$

$$(Ax_1 - Ax_2)^T (Ax_1 - Ax_2)$$

$$(A(x_1 - x_2))^T \cdot A \cdot (x_1 - x_2)$$

$$(x_1 - x_2)^T A^T A (x_1 - x_2)$$

if $A^T A = I$ then distance will be same

3) a) $p=2 \quad q=2$

any orthogonal matrix will preserve distance

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

b) $q=2 \quad p=1$

$A_{1\times 2}$

$$\cancel{\begin{bmatrix}a & b\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = 1}$$

$$\cancel{a^2+b^2=1}$$

$$\cancel{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\end{bmatrix}}$$

$$\begin{bmatrix}a\\ b\end{bmatrix}_{2\times 1}\begin{bmatrix}a & b\end{bmatrix}_{1\times 2} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}a^2 & ab\\ ab & b^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$a^2=1 \qquad ab=0$$
$$b^2=1$$

No such A is possible

c) $q = 4 \quad p = 2$

$$A_{2\times4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

$$\underset{4\times2}{A^T} \underset{2\times4}{A} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cancel{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2} & \cancel{a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} + a_{14}a_{24}} \\ \cancel{a_{11}a_{21} + a_{22}a_{12} + a_{13}a_{22} + a_{24}a_{14}} & \cancel{a_{21}^2 + a_{22}^2 + a} \end{bmatrix}$$

$$\cancel{a_{11}^2 + a_{21}^2} \quad \cancel{a_{11}a_{12} + a_{21}a_{22}} \quad \cancel{a_{11}a_{13} + a_{21}a_{23}}$$

No such matrix exist that will keep distance same without putting any condition on X.

Q-3.

1. $l : w_1 x_1 + w_2 x_2 + w_3 = 0$

N points $D_1 = \{ x_1 = [x_1^i, x_2^i]^T \}$

$\mu = [\mu_1, \mu_2]^T$

- $A' = \begin{bmatrix} x_1^1 - \mu_1 \\ x_2^1 - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^1 - \mu_1 & x_2^1 - \mu_2 \end{bmatrix}$

$=$ column $\times$ row $\rightarrow$ always rank 1.

A' is rank 1.
Hence 1 non zero eigen values

- $A = \frac{1}{N} \sum_{i=1}^{N} \begin{bmatrix} x_1^i - \mu_1 \\ x_2^i - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^i - \mu_1 & x_2^i - \mu_2 \end{bmatrix}$

$\downarrow$ A is sum of rank 1 matrix.

Rank of A $\leq \underbrace{1 + 1 + \ldots + 1}_{N}$

$\leq N$

N=2 $\begin{bmatrix} x_1^1 - \mu_1 \\ x_2^1 - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^1 - \mu_1 & x_2^1 - \mu_2 \end{bmatrix}$

$+ \begin{bmatrix} x_1^2 - \mu_1 \\ x_2^2 - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^2 - \mu_1 & x_2^2 - \mu_2 \end{bmatrix}$

$$\mu_1 = \frac{x_1^1 + x_1^2}{2} \qquad \mu_2 = \frac{x_2^1 + x_2^2}{2}$$

$$x_1^1 - \mu_1 = \frac{x_1^1 - x_1^2}{2}$$

$$\Rightarrow \frac{1}{2} \times \frac{1}{4}\left(\begin{bmatrix} x_1^1 - x_1^2 \\ x_2^1 - x_2^2 \end{bmatrix} \begin{bmatrix} x_1^1 - x_1^2 & x_2^1 - x_2^2 \end{bmatrix} + \begin{bmatrix} x_1^2 - x_1^1 \\ x_2^2 - x_2^1 \end{bmatrix} \begin{bmatrix} x_1^2 - x_1^1 & x_2^2 - x_2^1 \end{bmatrix} \right)$$

$$\Rightarrow \frac{1}{2} \times \frac{1}{4}\left(\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} + \begin{bmatrix} -a \\ -b \end{bmatrix} \begin{bmatrix} -a & -b \end{bmatrix} \right)$$

$$= \frac{1}{2} \times \frac{1}{4}\left(2 \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \right) = \frac{1}{4}\left(\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \right)$$

column row

$\Downarrow$

rank 1.

for $N=2$, one non zero eigenvalue

$N=3$.

$$\Rightarrow \begin{bmatrix} x_1^1 - \mu_1 \\ x_2^1 - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^1 - \mu_1 & x_2^1 - \mu_2 \end{bmatrix} + \begin{bmatrix} x_1^2 - \mu_1 \\ x_2^2 - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^2 - \mu_1 & x_2^2 - \mu_2 \end{bmatrix}$$

$$+ \begin{bmatrix} x_1^3 - \mu_1 \\ x_2^3 - \mu_2 \end{bmatrix} \begin{bmatrix} x_1^3 - \mu_1 & x_2^3 - \mu_2 \end{bmatrix}$$

$$x_1^1 - \mu_1 = \frac{3x_1^1 - x_1^1 - x_1^2 - x_1^3}{3} = \frac{2x_1^1 - x_1^2 - x_1^3}{3}$$

$$x_1^2 - \mu_1 = \frac{2x_1^2 - x_1^1 - x_1^3}{3}$$

here all terms are differe[nt]

Substituting. . .

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$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \begin{bmatrix} e & f \end{bmatrix}$$

$$\begin{bmatrix} aa & ba \\ ab & bb \end{bmatrix} + \begin{bmatrix} cc & dc \\ cd & dd \end{bmatrix} + \begin{bmatrix} ee & fe \\ ef & ff \end{bmatrix}$$

$$\begin{bmatrix} aa+cc+ee & ba+dc+fe \\ ab+cd+ef & bb+dd+ff \end{bmatrix}$$

↑ independent columns.

Hence rank = 2

$N = 1, 2 \Rightarrow$ rank = 1
1 non zero eigen vector

$N > 2 \Rightarrow$ rank = 2
2 non zero eigen vector

2). Line perpendicular to $l \Rightarrow$ is $l^{\perp}$ with m_2 slope

$l : w_1 x_1 + w_2 x_2 + w_3 = 0$

$$x_2 = \frac{-w_1}{w_2} x_1 - \frac{w_3}{w_2}$$

$$m_1 = -\frac{w_1}{w_2}$$

$m_1 m_2 = -1$ for $\perp^r$.

$$m_2 = \frac{w_2}{w_1}$$

~~and intercept can be anything until a point is given.~~

$$x_2 = m x_1 + c$$

$$x_2 = \frac{w_2 x_1}{w_1} + c$$

$$w_1 x_2 - w_2 x_1 = w_1 c$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$w_1 \mu_2 - w_2 \mu_1 = w_1 c$$

$$c = \frac{w_1 \mu_2 - w_2 \mu_1}{w_1}$$

line $\perp^r \Rightarrow \boxed{-w_2 x_1 + w_1 x_2 + (w_2 \mu_1 - w_1 \mu_2) = 0}$

and for $B' \Rightarrow$ column × row
Rank 1 ⇒ 1 non zero e.v.

$B \Rightarrow$ same explain as in previous.

3) The eigen values of the covariance matrix is the variability of data in orthogonal basis.

By finding e.v. and e.$\vec{v}$ of covariance matrix, we find that the eigenvectors with the largest eigenvalues correspond to the dimensions that have the strongest correlation in dataset.

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since covariance matrix is symmetric and positive definite, eigen vectors are orthogonal.

eigen value of covariance matrix = variance of data euler1.py 3.py

c: › Users › Shaily Mishra › Desktop › MS › Sem1 › SMAI › HWs › 3 › 3.py › ...

```python
import numpy
import matplotlib.pyplot as plt
import random
from numpy import linalg as LA
m, b = 5, 3
lower, upper = -50, 50
xstart = 0
xstop = 1000
x = numpy.linspace(-10,xstop,10)

samplepoints = 1000
x1 = [numpy.random.randint(xstart, xstop) for i in range(samplepoints)]
y1 = [numpy.random.randint(m*x+b+lower, m*x+b+upper) for x in x1]
meanx = numpy.mean(x1)
meany = numpy.mean(y1)
print('mean:', [meanx, meany])
X = numpy.stack((x1, y1), axis=0)
Covar = numpy.cov(X)
print('Covariance Matrix',Covar)
eigenvalue, eigenvector = LA.eig(Covar)
print('Eigen Values ', eigenvalue)
print('Eigen Vectors ', eigenvector)
plt.plot(x,m*x+b, label= "Line y =mx+c",linestyle='solid',color = "k")
plt.scatter(x1, y1, label="random points near line" , c='c')
plt.quiver(meanx,meany,eigenvector[0,0],eigenvector[0,1], color=['r'],scale=5, label="eigen vector[0]")
plt.quiver(meanx,meany,eigenvector[1,0],eigenvector[1,1], color=['g'],scale=5, label="eigen vector[1]")
plt.legend()
plt.show()
```

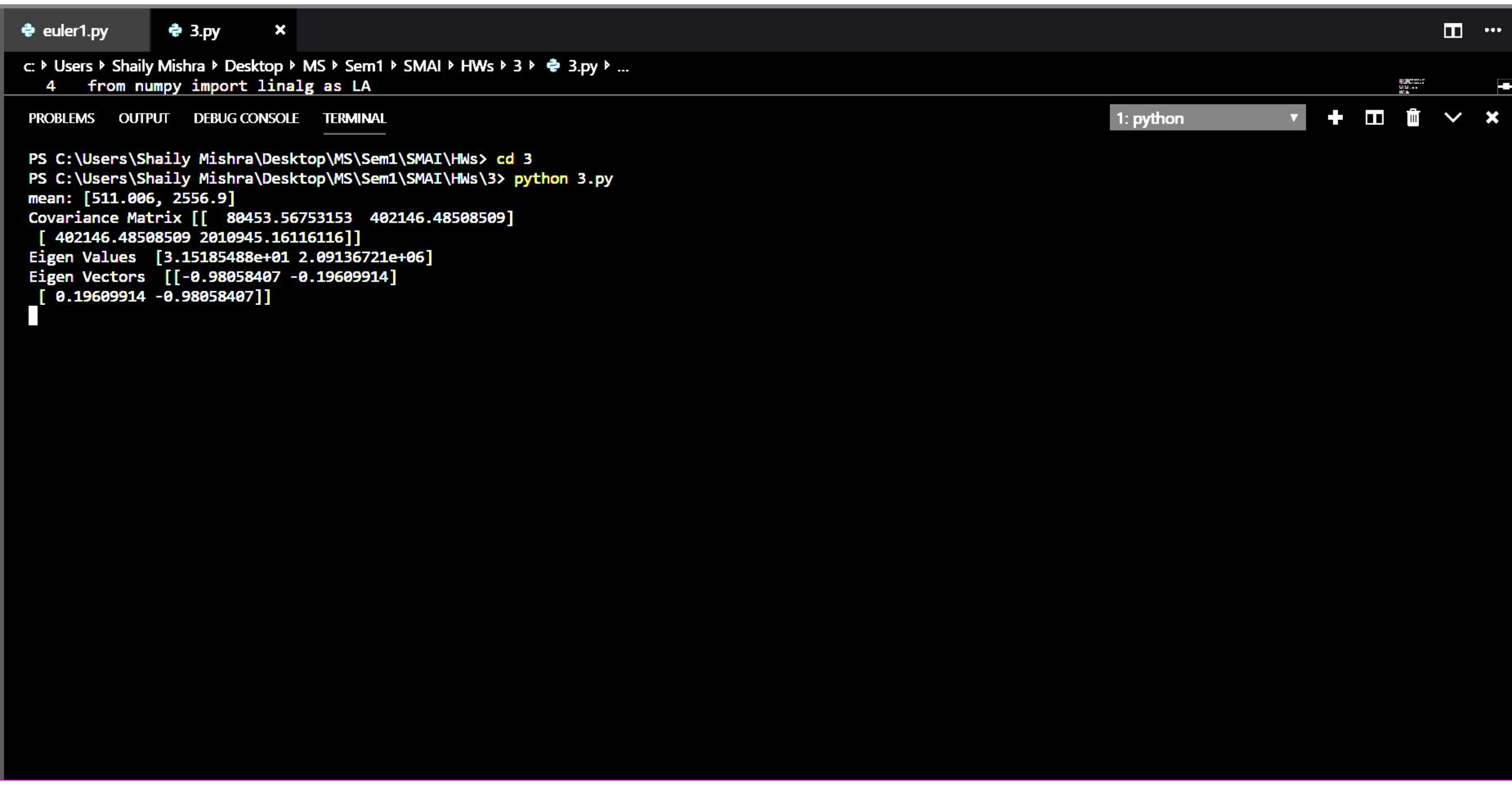
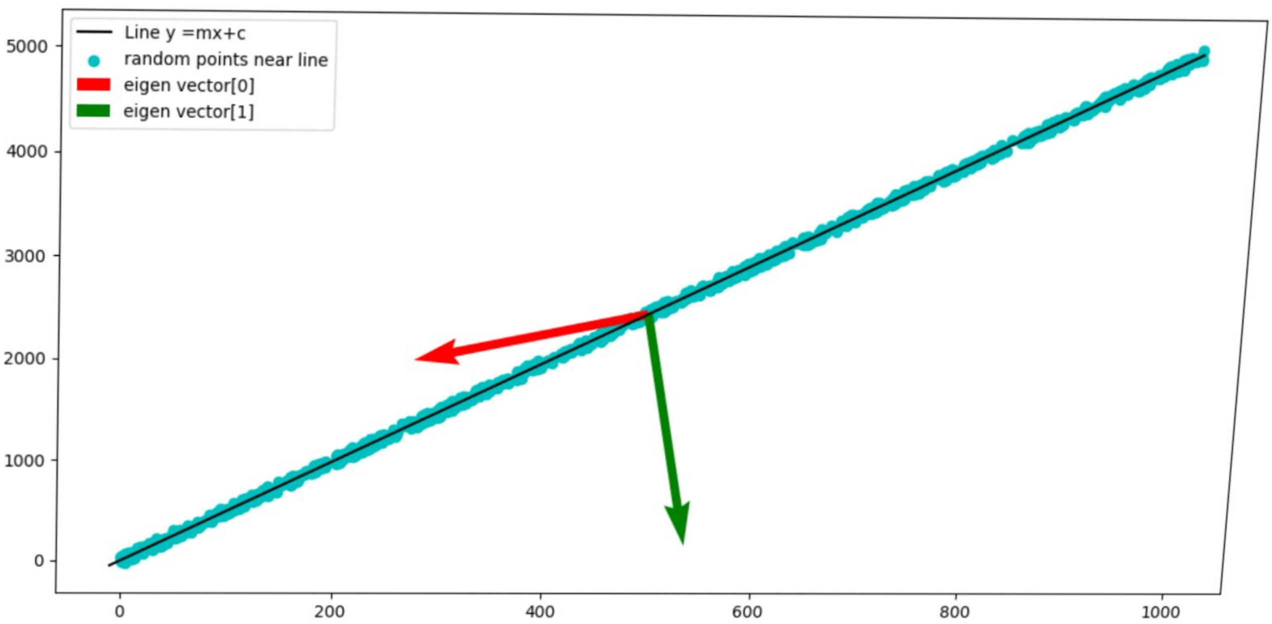

euler1.py
3.py
c: Users Shaily Mishra Desktop MS Sem1 SMAI HWs 3 3.py ...
4 from numpy import linalg as LA
PROBLEMS
OUTPUT
DEBUG CONSOLE
TERMINAL
1: python
PS C:\Users\Shaily Mishra\Desktop\MS\Sem1\SMAI\HWs> cd 3
PS C:\Users\Shaily Mishra\Desktop\MS\Sem1\SMAI\HWs\3> python 3.py
mean: [511.006, 2556.9]
Covariance Matrix [[80453.56753153 402146.48508509]
 [402146.48508509 2010945.16116116]]
Eigen Values [3.15185488e+01 2.09136721e+06]
Eigen Vectors [[-0.98058407 -0.19609914]
 [0.19609914 -0.98058407]]

Figure 1



Line y =mx+c
random points near line
eigen vector[0]
eigen vector[1]
5000
4000
3000
2000
1000
0
0
200
400
600
800
1000