Stable Matching with Proportionality Constraints

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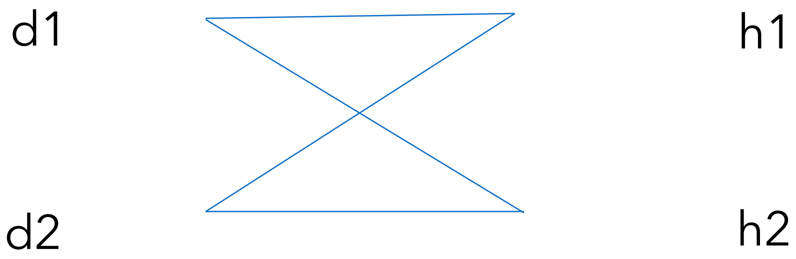
Bipartite Matching

- Single capacity
- Gale shapely algorithm
- Always exists
- Match $\mu := \{(h1,d1), (h2,d2)\}$
- Blocking pair - pair (h,d) that prefer each other

over assigned match μ

- Stable Matching - no blocking pair



Preferences:

d1: h1 > h2
h2: h1 > h2

h1: d1 > d2
h2: d1 > d2

Bipartite Matching

- $x(h,d)$ is 1 if h and d are assigned to each other
- Equations:
  - $x(h1,d1) + x(h2,d1) \leq 1$
  - $x(h1,d2) + x(h2,d2) \leq 1$
  - $x(h1,d1) + x(h1,d2) \leq 1$
  - $x(h2,d1) + x(h2,d2) \leq 1$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(h1,d1) \\ x(h1,d2) \\ x(h2,d1) \\ x(h2,d2) \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$Ax \leq b$

Scarf's Lemma

- $A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$
- $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$

Each row $i \in [m]$ of A has a strict order $\succ_i$ over the columns $\{j : a_{ij} > 0\}$

- A vector $x \in P$ **dominates** column r if there exists a row i such that.

1) $a_{ir} > 0, \sum_j a_{ij} x_j = b_i$ and
2) $k \succcurlyeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ AND $x_k > 0$

Scarf Lemma : P has an extreme point that dominates every column of A

Apply scarf's lemma on bipartite

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(h1,d1) \\ x(h1,d2) \\ x(h2,d1) \\ x(h2,d2) \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

col1 > col3

col2 > col4

col1 > col2

col3 > col4

Preferences:

d1: h1 > h2
h2: h1 > h2

h1: d1 > d2
h2: d1 > d2

$x=(1,0,0,1)^T$ is dominating

Column	Row	Cond - 1	Cond - 2	Cond - 3	Dominates
r=1	i=1	a[11] = 1	1+0+0+0=1	$k \in \{1\}$ $1 \succcurlyeq 1$	Y
r=2	i=1	a[12] = 0			
	i=2	a[22] = 0	0+1+0+0=1	$k \in \{2\}$ $2 \succcurlyeq 2$	Y
r=3	i=1	a[13]=1	1+0+0+0=1	$k \in \{1\}$ $1 \succcurlyeq 3$	Y
r=4	i=1	a[14]=0			
	i=2	a[24]=1	0+1+0+0=1	$k \in \{2\}$ $2 \succcurlyeq 4$	Y

$x=(0,1,1,0)^T$ is not dominating

Column	Row	Cond - 1	Cond - 2	Cond - 3	Dominates
r=1	i=1	a[11] = 1	0+0+1+0=1	$k \in \{3\}$ $1 \ngeq 3$	N
	i=2	a[21] = 0			N
	i=3	a[31] = 1	0+1+0+0=1	$k \in \{2\}$ $1 \ngeq 2$	N
	i=4	a[13]=0			N

Ordinal Matching

Finding a matching that is not dominated by any other

Gale Shapely (immediate rejection) for bipartite graph (single match)

Extending – matching many doctors to a hospital, immediate rejection method will not revisit doctors in round k+1 that got rejected in round k, need to define choice function accordingly

$Choice_h(.) : 2^D \to 2^D$, each hospital's h preferences over subsets of D respects $\succ_h$ as well as side constraints

μ is coalitional stable if for every set of doctors D^* who prefer h to their current match, $Choice_h(\mu(h) \cup D^*) = \mu(h)$

Scarf's lemma does not constrain us in this way

Matching with proportionality constraints

d1
Preference : h1>h2

d2
Preference: h1>h2

h1
Preference: d1>d2
$K_{h1}= 2$
Groups : {1,2}
$D^{h1}_{1} = \{d1\}$
$D^{h1}_{2} = \{d2\}$

>=33.33% from D^{h1}_{1}
$\alpha^{h1}_{1} = 1/3$
$\alpha^{h1}_{2} = 0$

h2
Preference: d1>d2
$K_{h2}= 2$
Groups : {1,2}
$D^{h2}_{1} = \{d1,d2\}$

$\alpha^{h2}_{1} = 0$

Stable Match μ :
{(h1,d1), (h1,d2)}
$\mu(h1) = \{d1,d2\}$
$\mu(h1) = \{\emptyset\}$
$\mu(d1)=1$
$\mu(d2)=1$

Proportionality Constraint $$\alpha_t^h \cdot \sum_{d \in D} x(h,d) \leq \sum_{d \in D_t^h} x(h,d) \ \forall t = 1, \ldots, T_h, \ \forall h \in H$$

Bilateral Stability

Feasible matching - satisfies capacity and proportionality constraints

A feasible matching is bilaterally stable if

- Each hospital with a non empty waitlist is at its effective capacity

- If da is on the waitlist of h, dr ∈ μ (h) and da > dr, then dr is protected and da and dr are not of the same type

Wait listed Doctors - when d and h are mutually acceptable. If D^h_t does not contain any wait listed doctors, h cannot increase the #admitted doctors of type t as they have already matched to more preferred.

Effective Capacity $k_h^\mu := \min\{k_h, \min_{t \in T_0} \frac{1}{\alpha_t^h} |\mu(h) \cap D_t^h|\},\ \textit{and if } T_0 = \varnothing \textit{ or } \alpha_t^h = 0,\ \textit{then } k_h^\mu := k_h$

Protected Type of doctors $|\mu(h) \cap D_t^h| = \alpha_t^h \cdot k_h^\mu$

Bilateral Stability and Coalitional Stability

If μ is bilateral stable matching, then μ is also coalitional stable

(Maximal) Given a feasible matching that is coalitional stable, there is no other feasible matching that assigns more doctors to hospitals such that no doctor is worse off

A stable matching need not exist

- Doctors are divisible (fractional matching)
- Rounding the fractional solution

Fractional Stable Matching

- x(h1,d1) + x(h2,d1) ≤ 1
- x(h1,d2) + x(h2,d2) ≤ 1
- x(h1,d1) + x(h1,d2) ≤ 2
- x(h2,d1) + x(h2,d2) ≤ 2
- $\frac{1}{3}$[x(h1,d1) + x(h1,d2)] ≤ x(h1,d1)
- 0[x(h1,d1) + x(h1,d2)] ≤ x(h1,d2)
- 0[x(h2,d1) + x(h2,d2)] ≤ x(h2,d1) + x(h2,d2)

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(h1,d1) \\ x(h1,d2) \\ x(h2,d1) \\ x(h2,d2) \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Cannot apply scarf's lemma directly

$$\begin{bmatrix} 2/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(h1,d1) \\ x(h1,d2) \\ x(h2,d1) \\ x(h2,d2) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Mx ≥ 0
Polyhedral cone

To apply Scarf's Lemma

$$\{x \in \mathbb{R}^n_+ | \mathcal{M}x \geq 0\} \implies \{\mathcal{V}z | z \geq 0\}$$ Generator of Cone, non negative

$$\mathcal{Q} = \{z \geq 0 : \mathcal{A}\mathcal{V}z \leq b\} \qquad x^* = \mathcal{V}z^*$$

Generator of Cone

For each hospital h, we will have at most $T_h \prod_t |D_t^h|$ generators

For h1 , $T_{h1} = 2$, $|D_1^{h1}| = 1$, $|D_2^{h1}| = 1$, total 2 generators

For each hospital h, select each doctor from D_t^h and for each doctor, select a extreme point

(a) Choose an index $t^* \in \{1, \dots, T_h\}$ and and set $v(d_{t^*}, h) = 1 - \sum_{t \neq t^*} \alpha_t^h \geq \alpha_{t^*}^h$.

(b) For all $t \neq t^*$, set $v(d_t, h) = \alpha_t^h$.

For h1:

- Choosing d1 from D_1^{h1} and d2 from D_2^{h1}
- $t^* \in \{1,2\}$
- $t^* = 1$, $v(d2,h1) = \alpha^{h1}{}_2=0$, $v(d1,h1) = 1-0 = 1$
- $t^* = 2$, $v(d1,h1) = \alpha^{h1}{}_1 = 1/3$, $v(d1,h1) = 1-1/3 = 2/3$
- $V_{h1} = \{(1,0), (1/3, 2/3)\} = \{v1 , v2\}$
- v1 means with probability 1, assigns d1 to h1

For h2:

- Choosing d1 from D_1^{h2}
- $t^* \in \{1\}$
- $t^* = 1$, $v(d1,h1) = 1$
- Replicate the same for all doctors in this type
- $V_{h2} = \{(1,0), (0, 1)\} = \{v3 , v4\}$
- v3 means with probability 1, assigns d1 to h2

V (generator matrix) is union of all generators

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$V=\begin{bmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$AV=\begin{bmatrix} 1 & 1/3 & 1 & 0 \\ 0 & 2/3 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathcal{Q} = \{z \geq 0 : \mathcal{A}\mathcal{V}z \leq b\} \qquad x^* = \mathcal{V}z^*$$

Rules for strict ordering

For hospital h, v and v′ ∈ V_h, if d1 is lowest ranked doctor in v, and d1′ is of v′ (positive probability)

- If d > d′ for h , v ≻ v′

-If d = d′, compare v(d1,h) and v′(d1,h)

    -If v(d1,h) > v′(d1,h) , v ≻ v′

    -If v(d1,h) = v′(d1,h) , move to next lowest ranked doctors

For h1, d1 is lowest in both v1 with 1 prob and v2, d2 is lowest with 1/3 prob,

As d1 > d2 , v1 > v2

For h2, similar argument, v3>v4 as d1>d2

Rules for strict ordering

For each $d, v \in V$,

- $v \succ v'$, if d is assigned to higher ranked hospital in v than in v'
- If $v, v' \in V_h$ (assigned to same hospital)
  - $v(d,h) > v'(d',h)$, $v' \succ v$ (lower prob better)
  - $v(d,h) = v'(d',h)$, order in the same way h would have

For d1, $v2 \succ v1 \succ v3$

For d2, $v2 \succ v4$

Apply scarf's lemma

$$\begin{bmatrix} 1 & 1/3 & 1 & 0 \\ 0 & 2/3 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} z1 \\ z2 \\ z3 \\ z4 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

col2 > col1>col3

col2 > col4

col1 > col2

col3 > col4

$z=(1/2,3/2,0,0)^T$
$x = Vz$
$X = (1,1,0,0)^T$

Column	Row	Cond - 1	Cond - 2	Cond - 3	Dominates
r=1	i=1	a[11] = 1	1/2+1/2+0+0=1	$k \in \{1,2\}$ $2 \succcurlyeq 1, 1 \succcurlyeq 1$	Y
r=2	i=1	a[12] = 1/3	1/2+1/2+0+0=1	$k \in \{1,2\}$ $2 \succcurlyeq 1$	N
	i=2	a[22] = 2/3	0+1+0+0=1	$k \in \{2\}$ $2 \succcurlyeq 2$	Y
r=3	i=1	a[13]=1	1/2+1/2+0+0=1	$k \in \{1,2\}$ $1 \succcurlyeq 3, 2 \succcurlyeq 3$	Y
r=4	i=1	a[14]=0			N
	i=2	a[24]=1	0+1+0+0=1	$k \in \{2\}$ $2 \succcurlyeq 4$	Y

$z=(1,0,0,1)^T$
$x = Vz$
$X = (1,0,0,1)^T$
Not stable

Column	Row	Cond - 1	Cond - 2	Cond - 3	Dominates
r=1	i=1	a[11] = 1	1+0+0+0=1	$k \in \{1\}$ $1 \succcurlyeq 1$	Y
r=2	i=1	a[12] = 1/3	1+0+0+0=1	$k \in \{1\}$ $2 \succcurlyeq 1$	N
	i=2	a[22] = 2/3	0+0+0+1=1	$k \in \{4\}$ $2 \succcurlyeq 4$	N
	i=3	a[13]=1	1+0+0+0 = 2		N
	i=4	a[14]=0			N

Rounding fractional solution

- Capacities at the hospitals are not violated
- number of doctors for each type is rounded either up
  or down to the closest integral number
- And modify alphas

Summary

- Proportionality constraints : only lower bounds, both lower and upper bounds
- Stable matchings need not exists
- Fractional matchings always exists and is stable
- Violation of proportionality constraints at school h : O($\frac{1}{\# \, accepted \; students}$)
- Stable matching is maximal