INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

MAXIMIN ALLOCATION

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INDIVIDUAL FAIRNESS

Two-Sided Online Platforms

Twitter Trends

Spliddit

New York Times - Rent calculator

CourseMatch

HOW TO ALLOCATE ITEMS FAIRLY

How do you define fair?

Can we achieve such a fairness?

What can be done?

PROPORTIONALITY

Get at least 1/n of total share of items

							total
Agent 1	1	6	1	2	1	1	12
Agent 2	4	1	2	1	1	3	12
Agent 3	1	1	3	3	3	1	12

Agent 2 **Agent 1** **Agent 3**

Any other proportional allocations?

							total
Agent 1	1	6	1	2	1	1	12
Agent 2	4	1	2	1	1	3	12
Agent 3	1	1	3	3	3	1	12

Does it exist?

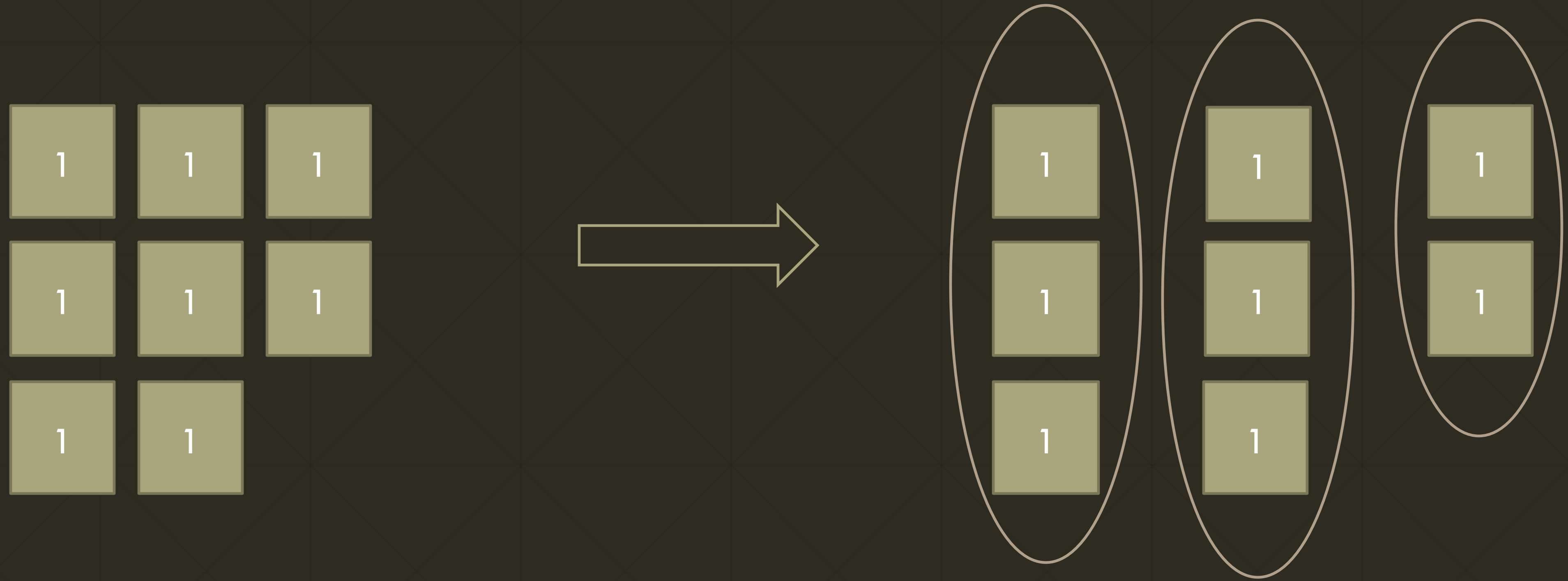
relaxation

??

CUT AND CHOOSE PROTOCOL?

You divide the items into bundles and chose last

DIVIDE INTO 3 BUNDLES



Question : can we find proportional allocation?

MAXIMIN SHARE

Maximin value we can guarantee, when we get to divide items into bundles, and then chose last, i.e., choose the minimum.

Question : Prop implies MMS Allocation?

TODAY'S SESSION

MMS Allocation?

Existence?

Complexity – MMS Share, and Allocation?

Approximation Algorithms?

How different it is for goods and chores?

PAPERS

An improved approximation algorithm for maximin shares

Algorithms for Max-Min Share Fair Allocation of Indivisible Chores

An Algorithmic Framework for Approximating Maximin Share Allocation of Chores

PRELIMINARIES

Additive valuations

N = [n] agents , M = [m] items

Agent i has a valuation function $v_i = 2^M \rightarrow R$

Utility $u_i\,(S) = v_i(S)$

Goods : positive valuations, Chores: negative

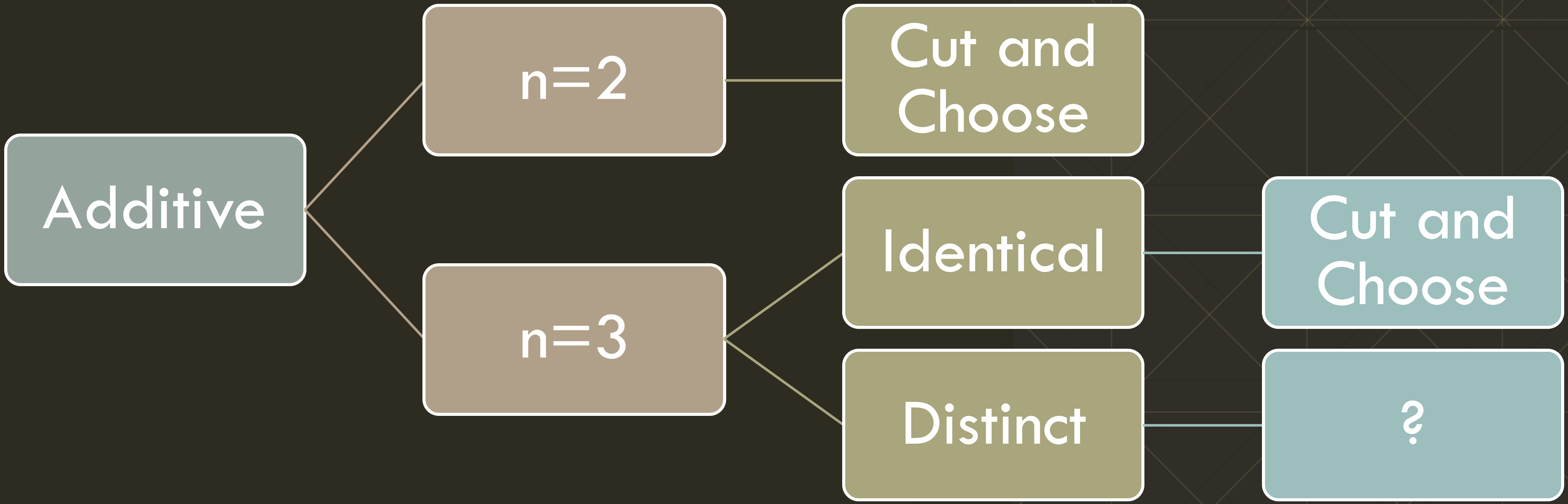
Complete allocation of items

MMS ALLOCATION

An Allocation A is said to be MMS allocation,$\forall i, u_i(A_i) \geq \mu_i$

$$\mu_i = \max_{(A_1,A_2,\dots,A_n)\in\prod_n(M)} \min_{j\in N} u\_i(A\_j)$$

EXISTENCE

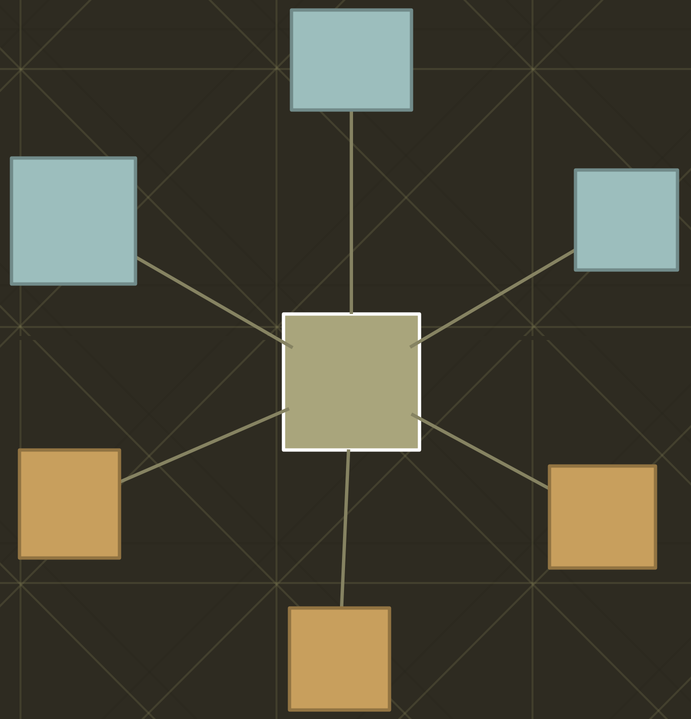


Additive
n=2
n=3
Cut and Choose
Identical
Distinct
Cut and Choose
?

3 AGENTS

Agent 1	μ_1	μ_1	μ_1
Agent 2	μ_2	μ_2	μ_2
Agent 3	μ_3	μ_3	μ_3

MMS Share Partition



All are Intersecting Partitions

Can we assign MMS allocation?

	1	2	3	4	5	6	7	8	9	10	11	12
Agent1	380	349	330	320	310	273	219	210	130	120	109	100
Agent2	380	349	330	320	310	273	220	209	130	119	110	100
Agent3	380	350	329	320	310	273	219	210	129	120	110	100

Total : 2850
Each : 950

Agent 1	2,6,7,11	1,4,9,10	3,5,8,12
Agent 2	2,6,8,10	4,5,7,12	1,3,9,11
Agent 3	3,6,7,9	1,2,10,12	4,5,8,11

Intersecting partition

MMS Allocation doesn't exist

Thoughts!

Can you create another example for MMS non-existence?

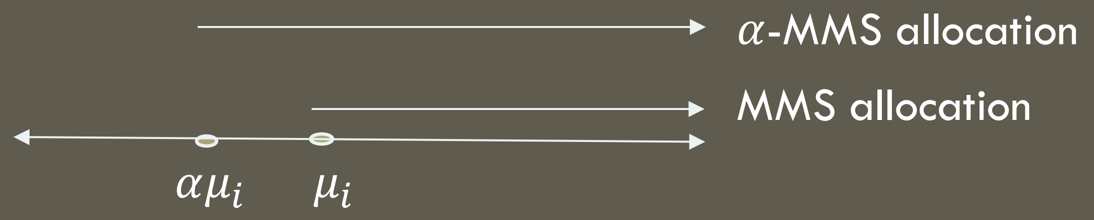
Is constructing such an example easy?

What can we say about likelihood, that an MMS allocation will exist?

How will it vary with n and m?

APPROXIMATION

α-MMS



$$u_i \geq \alpha\, \mu_i$$

Goods : $\alpha \in [0,1)$

Chores : $\alpha > 1$

GOODS : ½-MMS?

$\geq$ ½ MMS value

What if we directly assign them?

							mms
Agent 1	1	6	1	2	1	1	3
Agent 2	4	1	2	1	1	3	4
Agent 3	1	1	3	3	3	1	4

Agent 1

6 3 3

Agent 2

4 4 4

Agent 3

4 4 4

							mms
Agent 1	1	6	1	2	1	1	3
Agent 2	4	1	2	1	1	3	4
Agent 3	1	1	3	3	3	1	4

Lot of items >= ½ MMS

Agent 1

6 5

Re-Distribute

Agent 3

Agent 2

5 6

MMS guarantees increased

CAN WE ASSIGN IT?

Agent j gets good k

Remove Agent j and good g

Re-calculate MMS for rest

Redistribute items of A_l

A_l contains g item

Old : $A_1, A_2, \dots, A_n$

New : $A'_1, A'_2, \dots, A'_{n-1}$

$\forall k, A'_k \geq A_k$

$\forall i \; \mu'_i \geq \mu_i$

It doesn’t reduce MMS guarantees in the reduced instances.

We have enough goods left.

VALID REDUCTION

Assigning High Valued Item

$S \subseteq M, \exists i \in N$

$v_i(S) \geq \alpha \cdot \mu_i^n(M)$

$\mu_{i'}^{n-1}(M \setminus S) \geq \mu_{i'}^n(M) \quad \forall i' \in N \setminus \{i\}$

½ MMS : HIGH VALUED ITEMS

Apply valid reduction, till no high value item is left

LOW VALUED ITEM

Remaining agents and items < ½ MMS

½ -MMS : LOW VALUED ITEMS

Moving Knife algorithm? – Proportional allocation for divisible goods

Cake

WHAT IF WE APPLY MOVING KNIFE

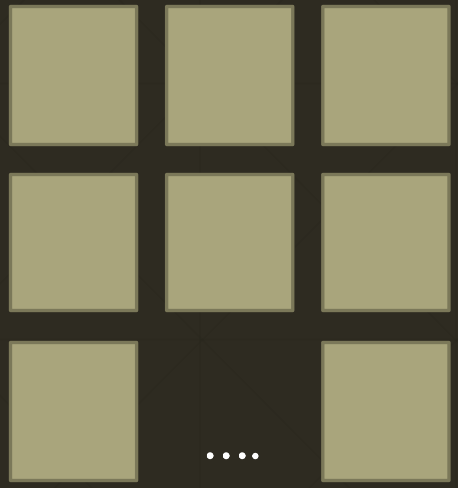
Take an empty bundle → Add Item1

Agent2 claims it >= ½ μ_i → Only one remaining agent

Assign the rest to it

						mms
Agent 1	1	6	1	2	1	3
Agent 2	4	1	2	1	3	4

WILL IT WORK ALWAYS?



$\mathrm{v_{ig}} \leq \frac{1}{2}\mu_i$

$\forall i, v_i(M) \geq n\ \mu_i$ (Why?)

Before someone claims it, adding last item g

$\frac{1}{2}\mu_i \leq v_{i(S)} < 1$

$\mathrm{v_{ig}} \leq \frac{1}{2}\mu_i$

Bundle S $< \frac{1}{2}\mu_i$

$\forall i, v_i(M \setminus S) > (n-1)\mu_i$ Enough goods are left!

½ MMS

$v_{ik} \geq \frac{1}{2}\mu_i$

- Valid reduction
- $\mu_i' \geq \mu_i$

$v_{ik} \leq \frac{1}{2}\mu_i$

- Bag Filling
- $\forall i, v_i(M \setminus S) > (n-1)\mu_i$

LOW VALUED ITEMS

Using Bag Filling, if we know that $v_{ik} \leq \delta$, and can we achieve α −MMS ?

δ= ??

LOW VALUED ITEMS

Assume $\forall i, \mu_i$ = 1 ; $\forall i, g, v_{ig} \leq \delta$

So again, we have bundle S, and just before adding last item g

$v_i(S) < \alpha \quad \Rightarrow \quad 1 > v_i(S) + v_{ig} \geq \alpha \quad \Rightarrow \quad 1 > v_i(S) + \delta \geq \alpha$

Now, $v_i(S) < \alpha$; $1 > v_i(S) + \delta \geq \alpha$,

what can be the maximum possible value of δ

$\delta = 1 - \alpha$

WHAT IF $\mu_i \neq 1$

Scale valuations

$$v'_{ik} = c_i v_{ik}; \mu'_i = c_i \mu_i;\ c_i > 0$$

$$v'_i(A_k) = c_i v_i(A_k) \geq c_i \alpha \mu_i = \alpha \mu'_i$$

Scale Invariance

BAG FILLING ALGORITHM

▶ **Proposition 5.** *Assume agents' valuations are normalized as defined in* (2)*, and that no agent values any item more than* $0 < \delta < 1/2$*:* $v_{ij} \leq \delta$ *for all* $j \in M$*, for all* $i \in N$*. Then, the bag filling algorithm gives a* $(1 - \delta)$ *MMS allocation.*

Image Credits - Approximating Maximin Share Allocations

QUICK REVIEW

- ½ MMS Allocation
- Valid Reduction
- Bag Filling
- Scaling Invariance

¾ MMS ALLOCATION

INTUITION

Bag Filling, $\forall v_{ig} \leq \frac{1}{4}$

Reduction, $\forall v_{ig} \geq 3/4$

Rest?

¾ MMS ALLOCATION

Create an ordered instance

Reduction for high value items

Modified Bag Filling

ORDERED INSTANCE?

Agent 1	1	6	1	2	1	1
Agent 2	4	1	2	1	1	3
Agent 3	1	1	3	3	3	1

Agent 1	6	2	1	1	1	1
Agent 2	4	3	2	1	1	1
Agent 3	3	3	3	1	1	1

Solve this now!

COUNTER-INTUITIVE?

Are we assigning one item to more than one agent in this?

CONVERT BACK THE ALLOCATION

$A = (\emptyset, \ldots, \emptyset)$ and $R \leftarrow M$;

for $j = 1$ *to* m **do**

- $a \leftarrow i : j \in A'_i$ (pick the agent assigned item j in A') ;
- $g \leftarrow \arg\max_{k \in R} v_{ak}$;
- $A_i \leftarrow A_i \cup \{g\}$ and $R \leftarrow M \setminus \{g\}$;

Image Credits - Approximating Maximin Share Allocations

Allocation:
Agent1 : Item1
Agent2 : Item2
Agent3 : Item3,4,5,6

Agent 1	6	2	1	1	1	1
Agent 2	4	3	2	1	1	1
Agent 3	3	3	3	1	1	1

Allocation:
Item1 => Agent1
Assign highest valued item from the remaining items

Agent1 => Item2
Agent2 => Item1
Agent3 : Item3,4,5,6

Agent 1	1	6	1	2	1	1
Agent 2	4	1	2	1	1	3
Agent 3	1	1	3	3	3	1

WHY THIS WILL WORK

Each iteration => assign 1 item to 1 agent

A is $\alpha - MMS$ in ordered, it is also $\alpha - MMS$ in original

j^{th} iteration, agent i, $j \in A_j' \Rightarrow k_j \in A_j$

k_j is top j most valuable item for agent i

As before $\mathrm{k}-1$ items have been allocated

Now, $\forall \mathrm{j} \in A_i'\,; v_i\left(k_j\right) \geq v_i'(j)$

$v_i(A_i) = \sum_{j \in A_j'} v_i\left(k_j\right) \geq \sum_{j \in A_j'} v'_i\ (j) = v_i'(A_i') \geq \alpha$

Create an ordered instance

Reduction for high value items

Modified Bag Filling

HIGH VALUE ITEM BUNDLE

S1	S2	S3	S4
• 1st item (most valued)	• (n,n+1)	• (2n-1,2n,2n+1)	• (1,2n+1)

INITIAL ASSIGNMENT

For an agent, if $\{S1, S2, S3, S4\}$ is ¾μ_i, assign the lowest index bundle S_k

$S_k = \phi$; if those items are not present

This is a valid reduction

REMAINING ITEMS? – MODIFIED BAG FILLING

After initial assignment, for remaining n and m

Make n bags with item i^{th} and $(2n+1-i)^{th}$

If any agent likes any bag $\geq \frac{3}{4}\mu_i$, assign it

For remaining agents and items, keep adding item to bag, until someone claims

WALK THROUGH EXAMPLE

	1	2	3	4	5	6	7	8	9	10	11	12
Agent1	380	349	330	320	310	273	219	210	130	120	109	100
Agent2	380	349	330	320	310	273	220	209	130	119	110	100
Agent3	380	350	329	320	310	273	219	210	129	120	110	100

$\forall i;\ \mu_i = 950$; $\geq$¾ $\mu_i = 712.5$

Agent1	S1 = {1} 380	S2 = {3,4} 650	S3 = {5,6,7} 802	S4 = {1,7} 599
Agent2	S1 = {1} 380	S2 = {3,4} 650	S3 = {5,6,7} 803	S4 = {1,7} 600
Agent3	S1 = {1} 380	S2 = {3,4} 649	S3 = {5,6,7} 802	S4 = {1,7} 599

S3 qualifies, assign it arbitrarily – agent 1; reduce the instance $A_1 = \{5,6,7\}$

original		**1**	**2**	**3**	**4**	**8**	**9**	**10**	**11**	**12**
		1	2	3	4	5	6	7	8	9
Agent1	Agent2	380	349	330	320	209	130	119	110	100
Agent2	Agent3	380	350	329	320	210	129	120	110	100

Agent2	S1 = {1} 380	S2 = {3,4} 650	S3 = {5,6,7} 458	S4 = {1,7} 499
Agent3	S1 = {1} 380	S2 = {3,4} 649	S3 = {5,6,7} 459	S4 = {1,7} 500

Can't reduce any further! => Bag Filling

Bag1 =
{1,4}
720

Bag1 =
{2,3}
649

Both the agents like bag1; $\geq \frac{3}{4}\ \mu_i$

Assign to agent2

$$A_1 = \{5,6,7\} \Rightarrow 802$$
$$A_1 = \{1,4\ \} \Rightarrow 720$$
$$A_1 = \{2,3,8,9,10,11,12\} \Rightarrow 1348$$

¾ MMS ALLOCATION

Create an ordered instance

Reduction for high value items

Modified Bag Filling

GOODS : α —MMS

Valid Reduction

Scale Invariance

Ordered Instance

Bag Filling

¾ MMS Allocation

Proof : Do we have enough goods left

MMS Share – NP Hard

MMS Allocation – NP Hard

α –MMS Allocation – NP Hard (if)

PTAS exists for all

¾+1/12n – MMS

Polynomial time ¾-MMS

For different n, different bound are also proved

CHORES

INTUITION

How different it is from chores?

Reduction?

Bag Filling?

$\alpha - MMS?$ No chores is left unallocated

ROUND ROBIN

EF1

$\forall i, j\ v_i(A_i) \geq v_i(A_j) + v_i(k^{min})\ ;\ v_i(k^{min}) = \min_{k \in M} v\_i(k)$

Suppose total there are L rounds.

i goes first. Till L-1 rounds, $v_i(A_i^{L-1}) \geq v_i(A_j^{L-1})$

In the last round, say only agent i got an item;

It's a chore; so that item value $\geq v_i(k^{min})$

ROUND ROBIN

j goes first.

$v_i(k_j^1)$	$v_i(k_i^2) \geq v_i(k_j^1)$	$v_i(k_i^3) \geq v_i(k_j^2)$	$v_i(k_i^{L1}) \geq v_i(k_j^{L-1})$	$v_i(k_i^L)$

$$v_i(A_i) - v_i(A_j) = v_i(k_i^L) - v_i(k_j^L) + \cdots + v_i(k_i^2) - v_i(k_j^2) + v_i(k_i^1) - v_i(k_j^1)$$

$$\geq v_i(k_i^L) - v_i(k_j^1)$$

$$\geq v_i(k_i^L)$$ (chore; also L round, i may or may not get any chore)

$$\geq v_i(k^{min})$$

ROUND ROBIN

$\forall i, j, v_i(A_i) \geq v_j(A_j) + v_i\,(k^{min})$

Summing both side for $j = 1\ to\ n$

$v_i(A_i) \geq \frac{1}{n} v_j\,(M) + \frac{1}{n} v_i(k^{min})$

$v_i(k^{min}) \geq \mu_i\,;\ \frac{1}{n} v_j\,(M) \geq \mu_i$

$v_i(A_i) \geq 2\,\mu_i$

Gives 2-MMS

WHY RR WON'T WORK FOR GOODS

$$v_i\left(k^{min}\right) \geq \mu_i \; ; \; \frac{1}{n} v_j\,(M) \geq \mu_i$$

MODIFIED BAG FILLING?

Create an ordered instance

Keep filling chores from largest to lowest, until $\exists i, v_{i(S)} \geq \alpha\mu_i$

11/9 MMS

This modified bag filling gives 11/9-MMS

	1	2	3	4	5	6	7	8	9	10	11	12
Agent1	-380	-349	-330	-320	-310	-273	-219	-210	-130	-120	-109	-100
Agent2	-380	-349	-330	-320	-310	-273	-220	-209	-130	-119	-110	-100
Agent3	-380	-350	-329	-320	-310	-273	-219	-210	-129	-120	-110	-100

$\forall i;\ \mu_i = -950\ ; \geq \frac{11}{9}\,\mu_i = -1161.12$

Bag = {1,2,3} => -1059; doesn't violate any agent threshold
Bag = {1,2,3,4} => -1379; violates all
Only chore that can we added is chore 12
Bag = {1,2,3,12} => -1159; assign arbitrary; $A_1 = \{1,2,3,12\}$
Repeat the process
$A_2 = \{4,5,6,7\} \Rightarrow -1123\ ; A_3 = \{8,9,10,11\} \Rightarrow -569$

SUMMARY OF CHORES

Round Robin

Modified Bag Filling

Proof : No Chore is left unallocated

Best : 11/9 – MMS

Polynomial time 5/4-MMS

TO SUMMARIZE

Proportionality, MMS, $\alpha - MMS$

Goods : ½ MMS and ¾ MMS

Chores : 2 MMS and 11/9 MMS

Complexity

THANK YOU