228B Lec 10

Plan:
- implicit methods
- backward Euler
- Crank-Nicolson
- truncation error of C-N

Implicit methods: the timestep restriction $\nu = \frac{k}{h^2} \le \frac{1}{2}$ makes the schemes we have studied so far impractical. Switching to an implicit method removes this stability constraint.

Simplest example: backward Euler (backward time, centered space BTCS)

$$D_t^+ u_j^n = D_x^+ D_x^- u^{n+1}$$

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2}$$

$$-\nu u_{j+1}^{n+1} + (1+2\nu) u_j^{n+1} - \nu u_{j-1}^{n+1} = u_j^n \qquad \left(\nu = \frac{k}{h^2}\right)$$

$$B u^{n+1} = u^n$$

z-transform

$$G(\xi)\hat{u}^{n+1}(\xi) = \hat{u}^n(\xi)$$

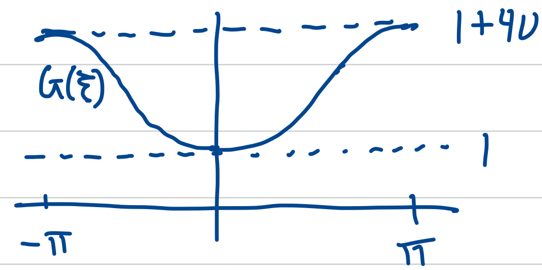
$$\hat{u}^{n+1}(\xi) = \frac{1}{G(\xi)}\hat{u}^n(\xi)$$

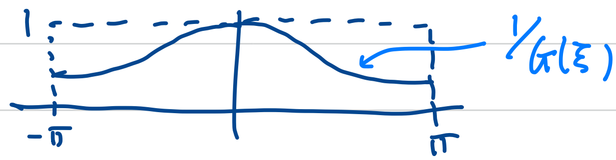
$$G(\xi) = -\nu e^{i\xi} + (1+2\nu) - \nu e^{-i\xi}$$

$$= 1 + 2\nu(1 - \cos\xi) = 1 + 4\nu \sin^2(\xi/2)$$

$G(\xi) \geq 1$, $0 < \dfrac{1}{G(\xi)} \leq 1$

$$\|B^{-1}\|_{\ell_h^2} = \left\| \frac{1}{G(\xi)} \right\|_\infty = 1$$





So the scheme $u^{n+1} = B^{-1} u^n$ is unconditionally stable. Even on the refinement path $h = k$, $\nu = \frac{1}{k}$ we still have $\|(B^{-1})^n\| \leq K = 1$ for $0 < nk \leq T$

the amplification factor of $\begin{Bmatrix} B = I \\ \alpha B_1 + \beta B_2 \\ B_1 B_2 \\ B^{-1} \end{Bmatrix}$ is $\begin{Bmatrix} G(\xi) = 1 \\ \alpha G_1(\xi) + \beta G_2(\xi) \\ G_1(\xi) G_2(\xi) \\ \dfrac{1}{G(\xi)} \end{Bmatrix}$

B is invertible on ℓ_h^2 iff $|G(\xi)| \neq 0$ for $\xi \in [-\pi, \pi]$

problem with BTCS : the truncation error is still $O(k + h^2)$. This was fine when k was νh^2 but now $O(k)$ is too big.

$\uparrow$ fixed constant

solution: use a more accurate timestepper

implicit trapezoidal rule $\rightarrow$ Crank-Nicolson scheme

$$D_t^+ u_j^n = \frac{1}{2}\left[D_x^+ D_x^- u_j^n + D_x^+ D_x^- u_j^{n+1} \right]$$

$$\frac{u^{n+1} - u^n}{k} = \frac{1}{2}\,\frac{Bu^n}{h^2} + \frac{1}{2}\,\frac{Bu^{n+1}}{h^2}$$

$$Bu_j = u_{j+1} - 2u_j + u_{j-1}$$

$$G(\xi) = e^{i\xi} - 2 + e^{-i\xi} = -2(1-\cos\xi) = -4\sin^2 \xi/2$$

$$\left(I - \tfrac{\nu}{2}B\right)u^{n+1} = \left(I + \tfrac{\nu}{2}B\right)u^n \qquad \nu = \frac{k}{h^2}$$

$$\left(1 - \tfrac{\nu}{2}G(\xi)\right)\hat{u}^{n+1}(\xi) = \left(1 + \tfrac{\nu}{2}G(\xi)\right)\hat{u}^n(\xi)$$

$$u^{n+1} = \left(I - \tfrac{\nu}{2}B\right)^{-1}\left(I + \tfrac{\nu}{2}B\right)u^n = B_1 u^n$$

$$\hat{u}^{n+1}(\xi) = \frac{1 + \frac{\nu}{2}G(\xi)}{1 - \frac{\nu}{2}G(\xi)} = \frac{1 - 2\nu\sin^2(\xi/2)}{1 + 2\nu\sin^2(\xi/2)} = G_1(\xi)$$

$$|1-a| \le 1 + |a| = 1 + a \qquad \left(a = 2\nu\sin^2(\xi/2) \ge 0\right)$$

$$\Rightarrow |G_1(\xi)| = \frac{|1-a|}{|1+a|} \le 1 \qquad (\text{and } G(0) = 1)$$

So $\|B_1\|_{\ell_h^2} = \|G_1\|_\infty = 1$

C.-N. is unconditionally stable

truncation error (Lax-Richtmyer framework)

$$u^{n+1} = B_1 u^n \qquad \left[B_1 = \left(I - \tfrac{\nu}{2}B\right)^{-1}\left(I + \tfrac{\nu}{2}B\right) \right]$$

$$u^{n+1}_{exact} = B_1 u^n_{exact} + k\tau^n$$

easier to compute

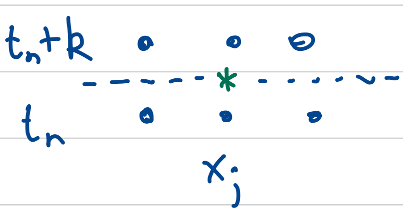
$$D_t^+ u^n_{exact} = \frac{1}{2}\left[D_x^+ D_x^- u^n_{exact} + D_x^+ D_x^- u^{n+1}_{exact} \right] + \tilde{\tau}^n$$

relationship between the two: $\tau^n = \left(I - \frac{\nu}{2}B\right)^{-1}\tilde{\tau}^n$

$$\|\tau^n\|_{2,h} \le \|\tilde{\tau}^n\|_{2,h} \quad \text{since} \quad \left| \frac{1}{1 - \frac{\nu}{2}G(\xi)} \right| \le 1$$

$1 + 2\nu \sin^2(\xi/2)$

stencil:



$\leftarrow$ do Taylor expansion at $(x_j, t_n + \frac{1}{2}k)$

(*) $$\left(D_t^+ u^n_{exact}\right)_j = \frac{u(jh, nk+k) - u(jh, nk)}{k}$$

$$= \frac{\left[u + \frac{k}{2}u_t + \frac{1}{2}\left(\frac{k}{2}\right)^2 u_{tt} + \frac{1}{6}\left(\frac{k}{2}\right)^3 u_{ttt} + \cdots\right] - \left[u - \frac{k}{2}u_t + \cdots\right]}{k}$$

$$= u_t + \frac{k^2}{24}u_{ttt} + O(k^4) \leftarrow \text{at } (jh, nk + \tfrac{k}{2})$$

(u_{tt} term cancelled by symmetry of midpoint) lecture 2

$$\left(D_x^+ D_x^- u^n_{exact}\right)_j = u_{xx}(jh, nk) + \frac{h^2}{12} u_{xxxx}(jh, nk) + O(h^4)$$

$$\left(D_x^+ D_x^- u^{n+1}_{exact}\right)_j = u_{xx}(jh, nk+k) + \frac{h^2}{12} u_{xxxx}(jh, nk+k) + O(h^4)$$

now we expand u_{xx}, u_{xxxx} at $(jh, nk + \frac{k}{2})$ in the time direction

e.g. $u_{xx}(jh, nk+k) = u_{xx} + \frac{k}{2} u_{xxt} + \frac{(k/2)^2}{2} u_{xxtt} + \frac{(k/2)^3}{6} u_{xxttt} + \dots$

$$\frac{1}{2}(\cdot + \cdot) = u_{xx} + \frac{k^2}{8} u_{xxtt} + O(k^4) \quad \leftarrow \text{at } (jh, nk + \tfrac{k}{2}) \qquad (**)$$

$$+ \frac{h^2}{12}\left(u_{xxxx} + \frac{k^2}{8} u_{xxxxtt} + O(k^4)\right) + O(h^4)$$

so $\tilde{\tau}_j^n = (*) - (**)$

$$= \underbrace{(u_t - u_{xx})}_{0} + \frac{k^2}{24} u_{ttt} - \frac{k^2}{8} u_{xxtt} - \frac{h^2}{12} u_{xxxx} + O(k^4 + h^4)$$

$$= O(k^2 + h^2)$$

$0 \leq (k^2 - h^2)^2$

$0 \leq k^4 - 2k^2h^2 + h^4$

$2k^2h^2 \leq k^4 + h^4$