Math 228B, Lec 3

Last time: Truncation error for $D_t^+ u = D_x^+ D_x^- u$

$$\nu = k/h^2 \begin{cases} \text{if } \nu = \frac{1}{6}, & |\tau_j^n| \le M\left(\frac{k^2}{6} + \frac{h^4}{360}\right), \quad M = \sup_x |\partial_x^6 g(x)| \\ \text{otherwise} & |\tau_j^n| \le M\left(\frac{k}{2} + \frac{h^2}{12}\right), \quad M = \sup_x |\partial_x^4 g(x)| \end{cases}$$

Today:
- Consistency, stability, convergence
- Lax-Richtmyer equivalence theorem

recall that $h = \Delta x$, $k = \Delta t$

def: A scheme is consistent if $\tau_j^n \to 0$ as $k, h \to 0$

One can also specify the rate, e.g. $\tau_j^n = O(k + h^2)$

"first order in time, second order in space"

We'll be more precise about what $\tau_j^n \to 0$ means below.

We established uniform convergence above, with $|\tau_j^n| \le M\left(\frac{k}{2} + \frac{h^2}{12}\right)$

indep. of n, j

In words:

- A scheme is consistent if the exact solution of the PDE is an approximate solution of the scheme
- It is convergent if the exact solution of the scheme is an approximate solution of the PDE. (The solution of the scheme is close to the exact solution of the PDE)

Lax–Richtmyer equivalence theorem: A consistent finite difference scheme for a well-posed initial value problem is convergent iff it is stable.

main result here: consistency + stability $\Rightarrow$ convergence

setting of Lax–Richtmyer paper:

$$u_t = Au \quad (0 \leq t \leq T)$$
$$u(0) = g$$

linear, constant coefficient ODE in a Banach space $\mathcal{B}$

complete normed linear space

our case: $A = \dfrac{\partial^2}{\partial x^2}$

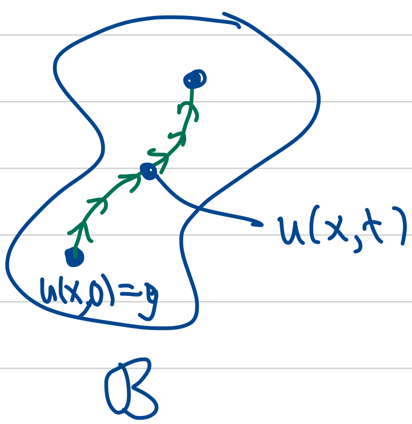
$\mathcal{B} = BC(\mathbb{R})$, bounded, continuous functions on $\mathbb{R}$

max norm: $\|g\| = \|g\|_\infty = \sup_{-\infty < x < \infty} |g(x)|$

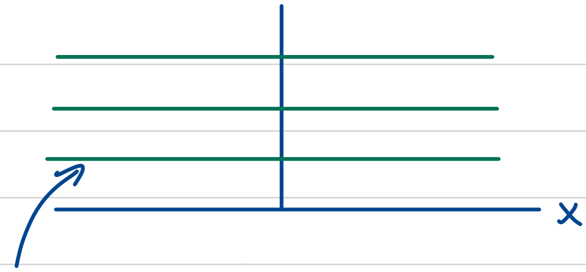
(other Banach spaces also work nicely for the heat equation)

e.g. $L^1(\mathbb{R})$, $L^2(\mathbb{R})$

abstract picture



each point in $\mathcal{B}$ is a function of x with t frozen



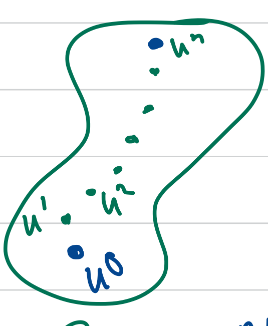
a time slice of the solution $u(x,t)$ is a function of x and can be thought of as a point on the solution curve in the space $\mathcal{B}$.

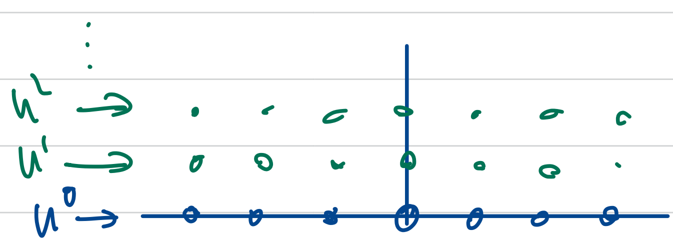
our case

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} g(\xi)\, d\xi$$

Next we set up a grid and define our finite difference scheme as an operator from one discrete time slice to the next

$\mathcal{B}_h = \ell^\infty(\mathbb{Z}) =$ space of bounded sequences $\{f_j\}_{j=-\infty}^{\infty}$





$\mathcal{B}_h$

$$u^{n+1} = \underbrace{B(k,h)}_{\text{bounded operator on } \mathcal{B}_h} u^n$$

our case:

$$B(k,h) = \nu u_{j+1} + (1-2\nu)u_j + \nu u_{j-1}\ , \quad \nu = \frac{k}{h^2}$$

next define a refinement path $h = H(k)$ relating k to h

our case: $\nu = \frac{k}{h^2} = \text{const}$, $\quad H(k) = \sqrt{k/\nu}$

Let $B = B(k)$ denote $B(k, H(k))$

<u>def:</u> A scheme is stable if $\forall T > 0\ \exists K, \varepsilon > 0$ s.t.

$$\|B(k)^n\| \leq K \quad \text{for} \quad \begin{array}{l} 0 < k < \varepsilon \\ 0 \leq nk \leq T \end{array}$$

operator norm: $\|A\| = \sup\limits_{\substack{u \neq 0 \\ u \in \mathcal{B}}} \dfrac{\|Au\|}{\|u\|}$ $\quad \left(\Rightarrow \|Au\| \leq \|A\| \cdot \|u\| \text{ for all } u \in \mathcal{B} \right)$

and $\|AB\| \leq \|A\| \cdot \|B\|$

$B(k)^n$ means $\underbrace{B(k) \cdot B(k) \cdots B(k)}_{n \text{ times}}$

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proof of Lax-Richtmyer:

$\left(\text{error } e_j^n = u_j^n - u(jh, nk) \right)$

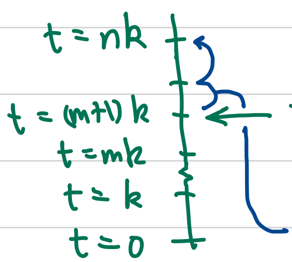
scheme: $\quad u_j^{n+1} = B(k) u_j^n$

exact: $\quad u(jh, (n+1)k) = B(k)\, u(jh, nk) + k\tau_j^n$

subtract: $\quad e_j^{n+1} = B(k) e_j^n - k\tau_j^n$

So $$e_j^1 = B(k)e_j^0 - k\tau_j^0$$
$$e_j^2 = B(k)^2 e_j^0 - B(k)k\tau_j^0 - k\tau_j^1$$
$$\vdots$$
$$e_j^n = B(k)^n e_j^0 - B(k)^{n-1}k\tau_j^0 - B(k)^{n-2}k\tau_j^1 - \cdots - B(k)k\tau_j^{n-2} - k\tau_j^{n-1}$$

picture:



τ^m enters here

$$\boxed{e_j^{m+1} = B(k)e_j^m - k\tau_j^m}$$

then gets propagated by $B(k)$ $\underbrace{n-(m+1)}_{\ell}$ times

$$B(k)^{\ell} k\tau_j^m$$

take norms, use triangle inequality and $\|B(k)^{\ell}\| \leq K$

$$\|e^n\| \leq K\underbrace{\|e^0\|}_{0} + Kk\left(\|\tau^0\| + \|\tau^1\| + \cdots + \|\tau^{n-1}\|\right)$$
$$\leq Knk \max_{1\leq \ell\leq n}\|\tau^{\ell}\| , \qquad nk \leq T$$

In our case, $\|\tau^{\ell}\| = \begin{cases} O(k+h^2) & \nu \neq \frac{1}{6} \\ O(k^2+h^4) & \nu = 1/6 \end{cases}$ and we will show that the scheme is stable if $\nu \leq \frac{1}{2}$ $\left(\begin{matrix} K=1, \varepsilon=\infty \\ \text{works} \end{matrix}\right)$

Conclusion :

$$\|e^n\| \leq \begin{cases} TM\left(\frac{h}{2} + \frac{h^2}{12}\right) & \nu \neq 1/6,\ 0 < \nu \leq \frac{1}{2} \\ TM\left(\frac{h^2}{6} + \frac{h^4}{360}\right) = \frac{TMh^4}{135} & \nu = \frac{1}{6} \end{cases}$$

true for all n satisfying $0 \leq nk \leq T$

$$\max_{0 \leq nk \leq T} \ \sup_{-\infty < j < \infty} |e_j^n| = \max_n \|e^n\| \leq RHS$$

$\therefore$ The maximum value of the error on the grid goes to zero as $k, h \to 0$ with $\nu = \frac{k}{h^2} \leq \frac{1}{2}$ held fixed