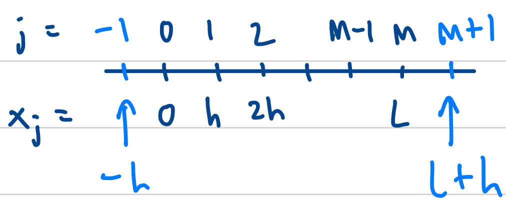
plan: finish stability analysis for finite domains (Neumann and periodic b.c.'s)

recall that for Neumann (insulating) boundary conditions, we use ghost nodes and reflective symmetry

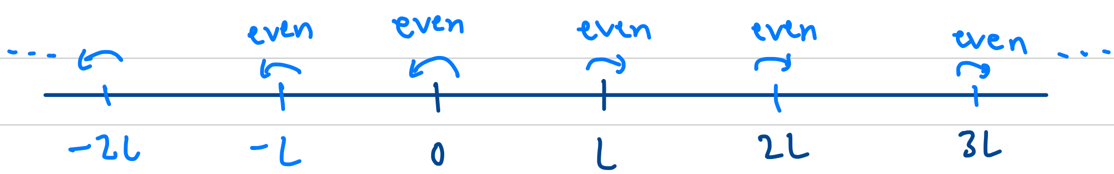


exact solution satisfies

$$u(-x,t) = u(x,t)$$
$$u(L+x,t) = u(L-x,t)$$

of a problem on all of $\mathbb{R}$ that satisfies $u_x(0,t)=0$, $u_x(L,t)=0$

reflective symmetries:



after eliminating the ghost node variables u^n_{-1}, u^n_{M+1}, the updates in u^n_j at $j=0,M$ are

$$u^{n+1}_0 = (1-2\nu)u^n_0 + 2\nu u^n_1$$

$$u^{n+1}_M = 2\nu u^n_{M-1} + (1-2\nu)u^n_M$$

or $u^{n+1} = Au^n$ with

$$A = \begin{pmatrix} \alpha & 2\beta & & & \\ \beta & \alpha & \beta & & \\ & \beta & \alpha & \ddots & \\ & & \ddots & \ddots & \\ & & \beta & \alpha & \beta \\ & & & 2\beta & \alpha \end{pmatrix}$$

$\alpha = 1-2\nu$

$\beta = \nu$

$(M+1)\times(M+1)$ matrix

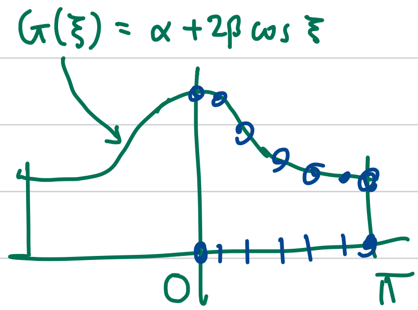
effect of ghost nodes

We will compute $\|A\|_2$ for arbitrary $\alpha, \beta \in \mathbb{R}$

Same construction as in the Dirichlet case (except $\mathrm{Im} \to \mathrm{Re}$)

$$w_j = e^{ij\xi}, \quad \xi = \frac{\ell\pi}{M} \quad (\ell \text{ fixed})$$

$$u_j = \mathrm{Re}\, w_j = \cos\frac{\ell j \pi}{M}, \quad 0 \le j \le M$$



$$\lambda_\ell = G\left(\frac{\ell\pi}{M}\right), \quad 0 \le \ell \le M$$

endpoints included

$$Au = \mathrm{Re}\left\{ \begin{pmatrix} \beta & \alpha & \beta & & & \\ & \beta & \alpha & \beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta & \alpha & \beta \end{pmatrix} \begin{pmatrix} w_{-1} \\ w_0 \\ w_1 \\ \vdots \\ w_{M-1} \\ w_M \\ w_{M+1} \end{pmatrix} \right\} = \mathrm{Re}\left\{ G(\xi) \begin{pmatrix} w_0 \\ \vdots \\ w_M \end{pmatrix} \right\}$$

$$= \lambda_\ell u$$

Here we used $\alpha, \beta, G(\xi)$ real, $Bw = G(\xi)w$, and $\mathrm{Re}\, w_{-1} = \mathrm{Re}\, w_1 = u_1$, $\mathrm{Re}\, w_{M+1} = \mathrm{Re}\, w_{M-1} = u_{M-1}$

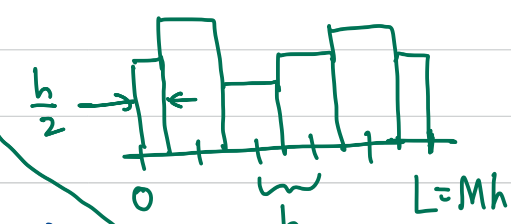
e.g. $\alpha u_0 + 2\beta u_1 = \mathrm{Re}\left(\beta w_{-1} + \alpha w_0 + \beta w_1\right)$

we needed ξ to be a multiple of $\frac{\pi}{M}$ for this to hold

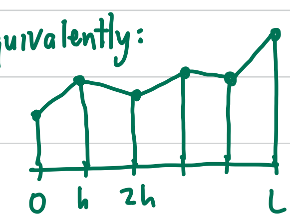
The "correct" discrete inner product is

$$\langle u, v\rangle_h = \frac{h}{2}\bar{u}_0 v_0 + h\sum_{j=1}^{M-1} \bar{u}_j v_j + \frac{h}{2}\bar{u}_M v_M$$

$$= \bar{u}^T \mathbb{M} v, \quad \mathbb{M} = \begin{pmatrix} h/2 & & & \\ & h & & \\ & & \ddots & \\ & & & h \\ & & & & h/2 \end{pmatrix}$$



equivalently:



In this inner product, A is self-adjoint:

$$\langle Au, v\rangle_h = \langle u, Av\rangle_h \qquad \forall\, u, v \in \mathbb{C}_h^{M+1}$$

and $\quad \Phi = \left(u^{(l=0)}, \sqrt{2}\, u^{(l=1)}, \ldots, \sqrt{2}\, u^{(l=M-1)}, u^{(l=M)} \right)$

is an orthogonal matrix $\quad \left(\Phi^* \Phi = I \right)$

a calculation

The adjoint of A satisfies $\quad \langle Au, v\rangle_h = \langle u, A^* v\rangle_h \quad \forall\, u, v$

$$\bar{u}^T \bar{A}^T \mathbb{M} v = \bar{u}^T \mathbb{M} A^* v \qquad \forall\, u, v$$

$$A^* = \mathbb{M}^{-1} \bar{A}^T \mathbb{M}$$

our case: $\quad A^* = \mathbb{M}^{-1} \bar{A}^T \mathbb{M} = A \quad$ and $\quad \Phi^* = \bar{\Phi}^T \mathbb{M} = \Phi^{-1}$

Φ actually maps $\mathbb{C}^{M+1}$ to $\mathbb{C}_h^{M+1}$

only the range has $\mathbb{M}$ in the inner product

$\bar{\varphi}_l^T \mathbb{M} \varphi_m = \langle \varphi_l, \varphi_m \rangle_h = \delta_{lm}$

$$\langle \Phi x, v\rangle_h = \langle x, \Phi^* v\rangle$$

$$\bar{x}^T \bar{\Phi}^T \mathbb{M} v = \bar{x}^T \Phi^* v$$

$$\underbrace{\begin{pmatrix} \frac{2}{h} & & & & \\ & \frac{1}{h} & & & \\ & & \ddots & & \\ & & & \frac{1}{h} & \\ & & & & \frac{2}{h} \end{pmatrix}}_{\mathbb{M}^{-1}} \underbrace{\begin{pmatrix} \alpha & \beta & & & \\ 2\beta & \alpha & \ddots & & \\ & \beta & \ddots & \beta & \\ & & \ddots & \ddots & 2\beta \\ & & & \beta & \alpha \end{pmatrix}}_{\bar{A}^T} \underbrace{\begin{pmatrix} \frac{h}{2} & & & & \\ & h & & & \\ & & \ddots & & \\ & & & h & \\ & & & & \frac{h}{2} \end{pmatrix}}_{\mathbb{M}} = \underbrace{\begin{pmatrix} \alpha & 2\beta & & & \\ \beta & \alpha & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & \beta & \alpha & \beta \\ & & & 2\beta & \alpha \end{pmatrix}}_{A} \quad \checkmark$$

We have shown that $A\Phi = \Phi\Lambda$, $\lambda_\ell = G(\frac{\ell\pi}{M})$. Next:

$\Phi^*\Phi = \Phi^T M \Phi = I$

<u>Discrete orthogonality, cosines</u>

Suppose $M \geq 0$ and $0 \leq \ell \leq m \leq M$.

then
$$\sum_{j=0}^{M} K_j \cos\frac{j\ell\pi}{M}\cos\frac{jm\pi}{M} = \begin{cases} M & \ell = m \in \{0, M\} \\ M/2 & \ell = m \in \{1,\dots,M-1\} \\ 0 & \ell \neq m \end{cases}$$

$$K_j = \begin{cases} 1/2 & j = 0, M \\ 1 & 1 \leq j \leq M-1 \end{cases}$$

<u>proof</u>:
$$\text{LHS} = \operatorname{Re}\left[\sum_{j=0}^{M} K_j \left(\frac{e^{ij\ell\pi/M} + e^{-ij\ell\pi/M}}{2}\right) e^{ijm\pi/M}\right]$$
$$= \frac{1}{2}\operatorname{Re}\sum_{j=0}^{M} K_j \left(e^{ij(m+\ell)\pi/M} + e^{ij(m-\ell)\pi/M}\right)$$

If $m+\ell$ and $m-\ell$ are both odd (in particular, $m \neq \ell$), let p be one of them and note that

$$\operatorname{Re}\sum_{j=0}^{M} K_j e^{ijp\pi/M} = \frac{1}{2}\operatorname{Re}\sum_{j=0}^{M} K_j \left(e^{ijp\pi/M} - \underbrace{e^{-ijp\pi/M}}_{e^{i(M-j)p\pi/M}}\right) = 0$$

($K_j = K_{M-j}$ for $0 \leq j \leq M$)

since the sum is purely imaginary

so $\text{LHS} = 0$

Otherwise, $m+l$ and $m-l$ are both even.

As a result, the $j=M$ term $e^{iM(m\pm l)\pi/M}$ is 1

so we can merge it with the $j=0$ term and drop the K_j factors:

$$\text{LHS} = \frac{1}{2}\,\mathrm{Re}\sum_{j=0}^{M-1}\left(e^{ij(m+l)\pi/M} + e^{ij(m-l)\pi/M}\right)$$

if $m=l\in\{0,M\}$, these terms are all ones and $\text{LHS}=M$

if $m=l\notin\{0,M\}$, then $2\le m+l\le 2M-2$

geometric series calculation, same as in sine case

and the first sum is zero while each term in the second sum is 1. $\therefore \text{LHS}=\frac{M}{2}$

If $m\neq l$, then both $m+l$ and $m-l$ are between 1 and $2M-1$, so the geometric series calculation gives $\text{LHS}=0$.

In all cases, we have shown that

$$\sum_{j=0}^{M} K_j \cos\frac{jl\pi}{M}\cos\frac{jm\pi}{M} = \begin{cases} M & l=m\in\{0,M\} \\ M/2 & l=m\in\{1,\dots,M-1\} \\ 0 & l\neq m \end{cases}$$

Case 3: periodic b.c.'s, $B = \sum_m c_m S^m$ arbitrary

(no need to assume B is real, symmetric or tridiagonal
$G(\xi)$ can be complex-valued)

These values of u_j represent the solution

$$\overbrace{\begin{array}{ccccccc} j = 0 & 1 & 2 & \cdots & & M \\ x_j = 0 & h & 2h & \cdots & L-h & L \end{array}}$$

$h = \dfrac{L}{M}$

$$A = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & & c_{-1} \\ c_{-1} & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & c_{m_2} \\ & & \ddots & \ddots & \ddots & \\ c_1 & c_2 & & \ddots & \ddots & \\ & & & & c_{m_1} & \end{pmatrix}$$

assume $m_1 \le 0$, $m_2 \ge 0$, $M \ge m_2 - m_1 + 1$

The ghost node values "wrap around", $u_{j \pm M} = u_j$

This causes the matrix entries of B to wrap around when constructing A

This time $u_j = w_j = e^{ij\xi}$ $\quad 0 \le j \le M-1$

need ξ so that $w_{j+M} = w_j$ for all j

$$e^{iM\xi} = 1, \qquad \xi = \frac{2\pi l}{M},$$

l ranges over any M consecutive integers
e.g. $-\lceil \frac{M}{2} \rceil + 1, \dots, \lfloor \frac{M}{2} \rfloor$

The spacing is double what it was in cases 1 and 2 above

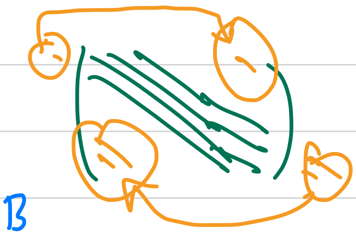
rows 0 through $M-1$ of B:

column index

$$\begin{array}{c|ccc|ccc|ccc|c} & -M & & -1 & 0 & & M-1 & M & & 2M-1 & \\ \hline \cdots\ 0 & 0 \cdots 0 & c_{m_1} \cdots & c_{-1} & c_0 & c_1 \cdots c_{m_2} \cdots & 0 & 0 & \cdots\ 0\ \cdots & 0 & 0 \cdots \\ & & 0\ \ddots & \vdots & c_{-1} & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots\ 0 & c_{m_1} & \vdots & \ddots & c_{m_2} & 0 & & \vdots & \vdots \\ & & & 0 & c_{m_1} & \ddots & \vdots & c_{m_2} & \ddots & \vdots & \\ & & & \vdots & 0 & \ddots & c_1 & \vdots & \ddots\ 0 & \vdots & \\ \cdots\ 0 & 0 \cdots 0 & 0 \cdots & 0 & 0 \cdots 0 & c_{m_1} \cdots c_{-1} & c_0 & c_1 & c_{m_2}\ 0 \cdots & 0 & 0 \cdots \\ & & A_{-1} & & & A_0 & & & A_1 & & \end{array}$$

A is the middle part of B with the "wings" mapped back inside

$$A = A_{-1} + A_0 + A_1$$



$\left\{ \begin{matrix} \text{cols} & -M:2M-1 \\ \text{rows} & 0:M-1 \end{matrix} \right\}$ of B

$$Au = A\begin{pmatrix} w_0 \\ \vdots \\ u_{M-1} \end{pmatrix} = \underbrace{\begin{pmatrix} A_{-1} & A_0 & A_1 \end{pmatrix}} \begin{pmatrix} w_{-M} \\ \vdots \\ w_{-1} \\ \hline w_0 \\ \vdots \\ w_{M-1} \\ \hline w_M \\ \vdots \\ w_{2M-1} \end{pmatrix} = G(\xi)\begin{pmatrix} w_0 \\ \vdots \\ w_{M-1} \end{pmatrix}$$

$$= \lambda_\ell u$$

(no Re or Im this time)

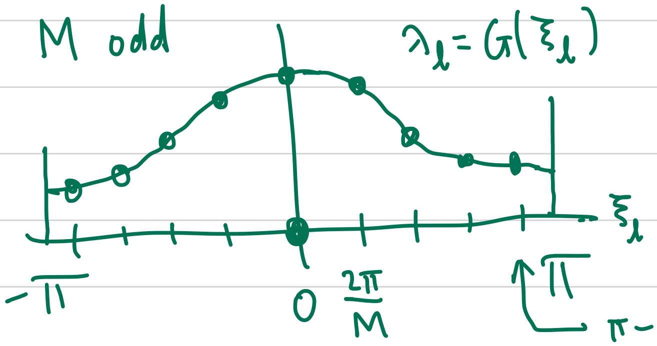
use $w_{j\pm M} = w_j$

(alternative direct derivation in homework)

result: (periodic B.C.'s)

M odd

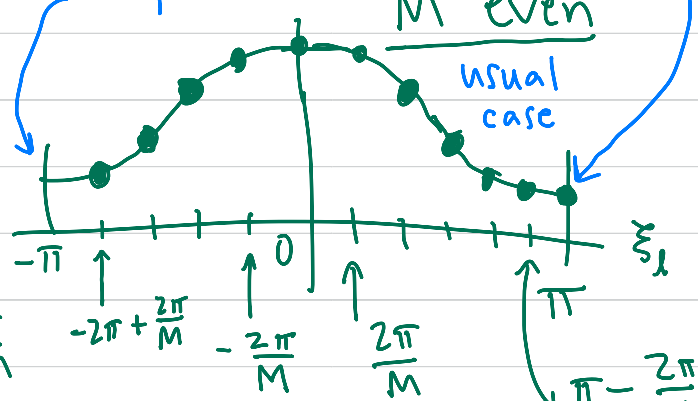
$\lambda_\ell = G(\xi_\ell)$



include endpoint only once

M even

usual case



$$-\left\lceil \frac{M}{2} \right\rceil + 1 \le \ell \le \left\lfloor \frac{M}{2} \right\rfloor$$

Discrete orthogonality for periodic b.c.'s:

$$\langle e^{ij\xi_\ell}, e^{ij\xi_m} \rangle = \frac{1}{M}\sum_{j=0}^{M-1} e^{ij\underbrace{(\xi_m - \xi_\ell)}_{\frac{2\pi}{M}(m-\ell)}} = \delta_{m\ell}$$

geometric series:

$$\sum_{j=0}^{M-1} \underbrace{e^{2\pi i j k/M}}_{a} = \begin{cases} M & k \in M\mathbb{Z} \\ \frac{1-a^M}{1-a} = 0 & k \notin M\mathbb{Z} \end{cases}$$

$a = 1$ in this case

$a^M = 1$ but $a \neq 1$ in this case

$|m - \ell| < M$