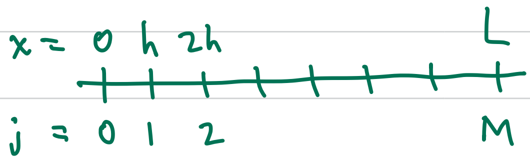
Math 228B Lec 7

plan:
- Boundary conditions in homework 1
- $\|B^n\|_{\ell^2_h} = \|G^n\|_{L^2(-\pi,\pi)}$
- 2-norm of a multiplication operator $\left(\|G^n\|_{L^2} = \|G\|_\infty^n \right)$
- z-transform of a sampled function
  (Fourier transform, Poisson summation formula)

Boundary conditions



$h = \frac{L}{M}$

(L=1 in homework)

For Dirichlet conditions

$u(0,t) = 0, \quad u(L,t) = 0$

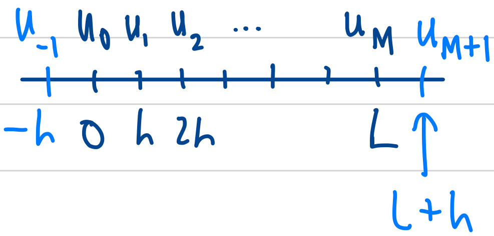
$D_t^+ u = \alpha D_x^+ D_x^- u$

update: $\begin{cases} u_j^{n+1} = \nu u_{j-1}^n + (1-2\nu) u_j^n + \nu u_{j+1}^n & 1 \le j \le M-1 \\ u_0^{n+1} = 0, \quad u_M^{n+1} = 0 \end{cases}$

(or don't store u_0^n, u_M^n and use a modified formula for $j=1$, $j=M-1$)

Neumann b.c.'s $\quad u_x(0,t) = 0, \quad u_x(L,t) = 0$

- include u_0, u_M among the unknowns
- use "ghost nodes" at $x_{-1} = -h$, $x_{M+1} = L+h$



$(\circledast)$ $u_{-1} = u_1$ (since $u_x(0,t) = 0$)

$u_{M+1} = u_{M-1}$ (since $u_x(L,t) = 0$)

Often the exact solution satisfies these reflective symmetries

$$u(-x,t) = u(x,t), \quad u(L+x,t) = u(L-x,t)$$

- eliminate u_{-1}, u_{M+1} from the update using $\circledast$

e.g. $$u_0^{n+1} = \nu u_{-1}^n + (1-2\nu)u_0^n + \nu u_1^n$$
$$= (1-2\nu)u_0^n + 2\nu u_1^n$$

final update for u_j^{n+1} with $j=0$

(similar formula for $j=M$)

(next: $\|B^n\|_{\ell_h^2} = \|\hat{g}^n\|_{L^2(-\pi,\pi)} = \|G\|_\infty^n$)

Observation: The norm of a unitary operator $Q: X \to Y$ is 1.

X, Y Hilbert spaces, $Q^{-1} = Q^*$ $\Rightarrow$ $Q^*Q = I$ so

$$\|Qx\| = \sqrt{\langle Qx, Qx\rangle} = \sqrt{\langle Q^*Qx, x\rangle} = \sqrt{\langle x, x\rangle} = \|x\| \quad \forall x \in X.$$

$$\|Q\| = \sup_{x\neq 0} \frac{\|Qx\|}{\|x\|} = \sup_{x\neq 0}\frac{\|x\|}{\|x\|} = \sup_{x\neq 0} 1 = 1$$

We saw last time that the Z transform is unitary
and diagonalizes any finite difference operator

(up to a constant factor)

$$\begin{tikzcd} \ell^2_h \arrow[r, "B"] \arrow[d, "Z"'] & \ell^2_h \arrow[d, "Z"] \\ L^2 \arrow[r, "\mathcal{G}"] & L^2 \end{tikzcd}$$

$$Bu_j = \sum_j c_m u_{j+m}$$

$$G(\xi) = \sum_m c_m e^{im\xi}$$

$$\mathcal{G}f(\xi) = G(\xi) f(\xi)$$

$B = Z^{-1}\mathcal{G}Z \leftarrow$ just like the eigen-decomposition $A = Q\Lambda Q^{-1}$ of a normal matrix

$\mathcal{G}$ plays the role of the diagonal matrix Λ

$$\therefore \quad \|B^n\| \leq \|Z^{-1}\| \cdot \|\mathcal{G}^n\| \cdot \|Z\|$$

$$= \sqrt{\frac{h}{2\pi}}\, \|\mathcal{G}^n\| \cdot \sqrt{\frac{2\pi}{h}} = \|\mathcal{G}^n\|$$

$\left\|\sqrt{\frac{h}{2\pi}}\, Z\right\| = 1$

unitary

using $\mathcal{G}^n = Z B^n Z^{-1}$ we also have

$$\|\mathcal{G}^n\| \leq \underbrace{\|Z\|}_{\sqrt{\frac{2\pi}{h}}} \cdot \|B^n\| \cdot \underbrace{\|Z^{-1}\|}_{\sqrt{h/2\pi}} = \|B^n\|$$

so $\|B^n\|_{\ell^2_h} = \|\mathcal{G}^n\|_{L^2(-\pi,\pi)}$

__Note:__ $G(\xi) = \sum_m c_m e^{im\xi}$ is continuous (it's a finite sum)

Claim: $\|\mathcal{G}\|_{L^2} = \|G\|_\infty = \underbrace{\max_{-\pi \le \xi \le \pi} |G(\xi)|}_{C}$

proof: let C be the RHS:

step 1: show $\|\mathcal{G}f\|_{L^2} \le C\|f\|_{L^2}$ for all $f \in L^2(-\pi,\pi)$

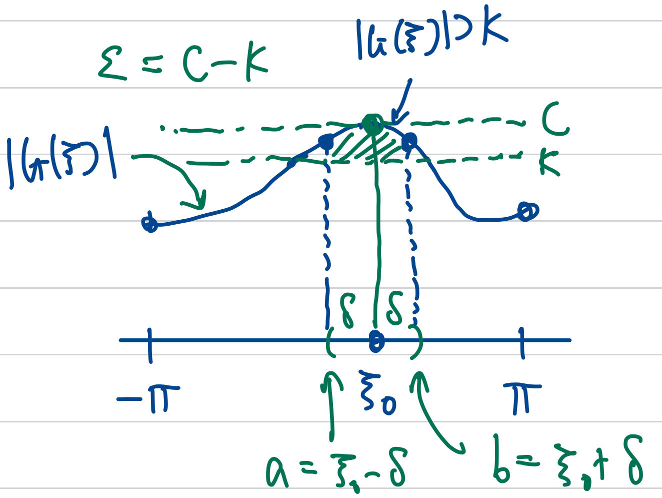
$$\|\mathcal{G}f\|^2 = \int_{-\pi}^{\pi} |\mathcal{G}f(\xi)|^2 d\xi = \int_{-\pi}^{\pi} |G(\xi)f(\xi)|^2 d\xi$$

$$\le C^2 \int_{-\pi}^{\pi} |f(\xi)|^2 d\xi = C^2\|f\|^2$$

↑ key step: $|G(\xi)|^2 \le C^2$ for every $\xi \in [-\pi,\pi]$.

step 2: show that if $K < C$ then $\exists f$ (exists) s.t. $\|\mathcal{G}f\| > K\|f\|$
(i.e. no smaller constant than C will work)

Since $|G(\xi)|$ is a continuous function, it achieves its maximum at some point $\xi_0 \in [-\pi,\pi]$, and there is a neighborhood (a,b) containing ξ_0 so that $|G(\xi)| > K$ for $a \le \xi \le b$.



Now define $f(\xi) = \begin{cases} 0 & \xi < a \\ 1 & a \le \xi \le b \\ 0 & \xi > b \end{cases}$

Then $\|\mathcal{G}f\|_{L^2}^2 = \int_{-\pi}^{\pi} |G(\xi) f(\xi)|^2 d\xi = \int_a^b |G(\xi)|^2 d\xi$

$> \int_a^b K^2 d\xi = K^2(b-a)$

and $\|f\|^2 = \int_{-\pi}^{\pi} |f(\xi)|^2 d\xi = \int_a^b 1 d\xi = b-a$

so $\|\mathcal{G}f\| > K\|f\|$

<u>Conclusion</u>: The 2-norm of a finite difference operator is the maximum absolute value of the amplification factor $G(\xi)$.

Also, $\mathcal{G}^n f(\xi) = G(\xi)^n f(\xi)$ and $\|G^n\|_\infty = \|G\|_\infty^n$

so $\|B^n\|_{\ell_h^2} = \|\mathcal{G}^n\|_{L^2} = \|G\|_\infty^n$

von Neumann analysis also handles implicit methods since

$B^{-1} = Z^{-1}\mathcal{G}^{-1}Z$, $\mathcal{G}^{-1}f(\xi) = \frac{1}{G(\xi)} f(\xi)$

and $\|B^{-1}\|_{\ell_h^2} = \|\mathcal{G}^{-1}\|_{L^2} = \|\frac{1}{G}\|_\infty = \max_{-\pi \le \xi \le \pi} \left|\frac{1}{G(\xi)}\right|$

So far we are using the l^2 version of the Z-transform, which is the easiest way to compute norms.

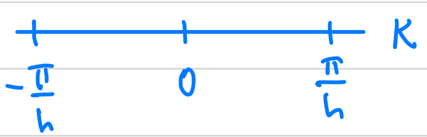
$$\hat{u}(\xi) = \sum_j u_j e^{-ij\xi} \quad \leftarrow$$ no change with different grid spacings

To interpret this physically, it is useful to introduce a variant that is tailored to the grid:

def: $$\tilde{u}(\kappa) = h \sum_j u_j e^{-ijh\kappa}$$

κ = wave number (units: $\frac{1}{\text{length}}$)

kappa



$\xi = h\kappa$ = dimensionless wave number

$\varphi_j(\kappa) = e^{-ijh\kappa}$ basis functions, $\tilde{u}(\kappa) = \sum_j h u_j \varphi_j(\kappa)$

orthogonality: $$\frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \overline{\varphi_j(\kappa)}\, \varphi_\ell(\kappa)\, d\kappa = \delta_{j\ell} = \begin{cases} 1 & j = \ell \\ 0 & j \neq \ell \end{cases}$$

$$\sum_j |h u_j|^2 = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} |\tilde{u}(\kappa)|^2\, d\kappa \quad \leftarrow$$ Parseval's identity

(just the Pythagorean theorem)

$$\therefore \quad h \sum_j |u_j|^2 = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} |\tilde{u}(\kappa)|^2\, d\kappa$$

<u>Observations</u>:

- $\tilde{u}(K) = h\,\hat{u}(hK)$
- $\hat{u}(\xi)$ and $\tilde{u}(K)$ are periodic with periods 2π and $\frac{2\pi}{h}$, respectively

Suppose we sample a continuous function $U(x)$, $x \in \mathbb{R}$:

$$u_j = U(jh), \quad -\infty < j < \infty$$

Physical z-transform of the sampled sequence u:

$$\tilde{u}(K) = h \sum_j u_j e^{-ijhK} = h \sum_j U(jh) e^{-i(jh)K} = h \sum_j V(jh)$$

$$V(x) = U(x) e^{-iKx} \quad \left(K \text{ fixed, } \begin{matrix} \text{different } K\text{'s give} \\ \text{different } V\text{'s} \end{matrix} \right)$$

Fourier transform of U, V: $\hat{U}(K) = \int_{-\infty}^{\infty} U(x) e^{-iKx}\, dx$

$$\hat{V}(r) = \int_{-\infty}^{\infty} V(x) e^{-irx} dx = \int_{-\infty}^{\infty} U(x) e^{-i(K+r)x} dx = \hat{U}(K+r)$$

Poisson summation formula

$$h \sum_{j=-\infty}^{\infty} V(jh) = \sum_{m=-\infty}^{\infty} \hat{V}\left(-\frac{2\pi}{h} m\right)$$

also true without the minus sign

$$\therefore \quad \tilde{u}(K) = \sum_m \hat{U}\left(K - \frac{2\pi}{h} m\right)$$

$\leftarrow$ significance will be explained next time...