HOMEWORK 1
DUE SATURDAY, FEB 7, AT 2 PM. (SUBMIT TO GRADESCOPE)

Problem 1.

Let $q(x) = \dfrac{\sin(\pi x)}{2 - \cos(2\pi x)}$. Consider the heat equation on $[0, 1]$,

$$u_t = \alpha u_{xx} + f, \qquad u(x, 0) = g(x), \qquad u(0) = 0, \quad u(1) = 0$$

with diffusion constant $\alpha = \frac{1}{100}$ and the following two sets of data:

$$(i) \quad f(x, t) = 0, \quad g(x) = q(x); \qquad (ii) \quad f(x, t) = q(x)\sin(\pi t), \quad g(x) = 0.$$

(a) (5 points) write a code to evaluate the exact solution at $t = 1$ numerically. Make a plot of your solution $u(x, 1)$ as a function of x and report the value of $u(1/4, 1)$ to 12 correct digits. Do this for both cases (i) and (ii) above. The exact solution is:

$$(i) \quad u(x,t) = \sum_{k=1}^{\infty} c_k e^{-\alpha k^2 \pi^2 t} \sin(k\pi x), \qquad c_k = 2\int_0^1 q(x)\sin(k\pi x)\,dx.$$

$$(ii) \quad u(x,t) = \sum_{k=1}^{\infty} \left(\int_0^t e^{-\alpha k^2 \pi^2 (t-s)} \sin(\pi s)\,ds \right) c_k \sin(k\pi x).$$

The integral in parentheses can be done analytically. The integral for c_k must be done numerically. Use the trapezoidal rule, experimenting with the number of quadrature points needed for the integral to converge to machine precision. Compute enough c_k's that additional terms do not affect your result for $u(1/4, 1)$.

(b) (4 points) Code up the scheme $D_t^+ u = \alpha D_x^+ D_x^- u$ for problem (i) above (with $f = 0$). What values of $\nu = \alpha k/h^2$ lead to a stable scheme? What magic value of ν leads to a higher convergence rate? Verify that your scheme converges at the appropriate order for the two cases by comparing the numerical solution to the exact solution. For the error, use

$$\text{err} = \sqrt{h \sum_{j=0}^{M-1} \Big(u_{\text{numerical}}(x_j, T) - u_{\text{exact}}(x_j, T) \Big)^2}, \qquad h = 1/M, \;\; x_j = jh, \;\; T = 1.$$

Make a log-log plot of err versus h, e.g. with $h = 1/M$, $M \in \{30, 60, 90, 120, 180, 240, 360, 480\}$.

(c) (4 points) Code up the scheme $D_t^+ u = \alpha D_x^+ D_x^- u + f$ for problem (ii) and check the order of convergence for the magic value of ν for problem (i). Now try the scheme $D_t^+ u = \alpha D_x^+ D_x^- u + (1/3) f_j^n + (1/2) f_j^{n+1} + (1/12) f_{j+1}^n + (1/12) f_{j-1}^n$. Compute the truncation error of the more complicated scheme to explain your numerical results.

Problem 2.

Consider the PDE $u_t = \alpha u_{xx} + u$ with $\alpha = 1/20$ on the interval $0 \le x \le 1$ with insulating boundary conditions, $u_x(0,t) = 0$, $u_x(1,t) = 0$ and initial conditions $u(x,0) = g(x) = \sin^2(\pi x)$.

(a) (5 points) what is the exact solution?

(b) (4 points) compute the amplification factor of the scheme $D_t^+ u = \alpha D_x^+ D_x^- u + u$. Use it to show that the scheme is stable for $\nu = \alpha k/h^2 \le 1/2$.

(c) (4 points) devise a new scheme with a truncation error that is second order in time, fourth order in space. Show your work, i.e. that the $O(k + h^2)$ terms cancel in the formula for τ_j^n. Code it up and present a convergence study to show that your scheme works (as in problem 1).

Problem 3.

(4 points) Consider the linear functional $\rho(f) = \int_0^4 f(x)\,dx$ in two contexts, first acting on $L^1[0,4]$, and then on $L^2[0,4]$. Compute the norm of ρ in each of these cases, $\|\rho\|_1$ and $\|\rho\|_2$. (The norm of a linear functional is $\|\rho\| = \sup_{f \ne 0} |\rho(f)|/\|f\|$.)