Math 228B Lec 11

plan:
- Crank-Nicolson on a finite domain
- efficient implementation of C-N
- 2D heat equation, C-N and ADI methods

Last time: Crank-Nicolson

$$u^{n+1} = B_1 u^n = \left(I - \tfrac{\nu}{2}B\right)^{-1}\left(I + \tfrac{\nu}{2}B\right)u^n$$

$$Bu_j = u_{j+1} - 2u_j + u_{j-1}$$

$$G_1(\xi) = \frac{1 + \frac{\nu}{2}G(\xi)}{1 - \frac{\nu}{2}G(\xi)} = \frac{1 - 2\nu\sin^2(\xi/2)}{1 + 2\nu\sin^2(\xi/2)} \in (-1, 1]$$

so $|G_1(\xi)| \le 1$

The finite domain cases from Lectures 8,9 work the same for implicit methods. E.g., for Neumann b.c.'s:

Building block operator

$$A = \begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}$$

$$\begin{tikzcd} u & \mathbb{R}_h^{M+1} \arrow[r, "A"] \arrow[d, "\Phi^*"] & \mathbb{R}_h^{M+1} & Au \\ \hat{u} = \Phi^* u & \mathbb{R}^{M+1} \arrow[r, "\Lambda"] & \mathbb{R}^{M+1} \arrow[u, "\Phi"] & \Lambda\hat{u} \end{tikzcd}$$

$$A = \Phi\Lambda\Phi^*, \quad \lambda_l = G\left(\frac{l\pi}{M}\right) \quad 0 \le l \le M, \quad G(\xi) = -4\sin^2\frac{\xi}{2}$$

$$\Phi = (\varphi_{j\ell})_{0\le j,\ell\le M}, \quad \varphi_{j\ell} = a_\ell \cos\frac{j\ell\pi}{M}, \quad a_\ell = \begin{cases} 1 & \ell = 0, M \\ \sqrt{2} & 1 \le \ell \le M-1 \end{cases}$$

$$\langle u, v\rangle_h = u^T \mathbb{M} v \qquad \Phi^* = \Phi^T \mathbb{M} = \Phi^{-1}$$

$$\mathbb{M} = \begin{pmatrix} \frac{h}{2} & & & \\ & h & & \\ & & \ddots & \\ & & & h & \\ & & & & h/2 \end{pmatrix} \qquad \Phi^T \mathbb{M} \Phi = I$$

[Φ^* plays the role of the Z-transform, but $\hat{u} = \Phi^* u$ lives in $\mathbb{R}^{M+1}$ instead of $L^2(-\pi,\pi)$] otherwise everything is the same

Crank-Nicolson: $$\left(I - \frac{\nu}{2}A\right)u^{n+1} = \left(I + \frac{\nu}{2}A\right)u^n$$

apply Φ^* $(\Phi^* A = \Lambda \Phi^*)$

$$\left(I - \frac{\nu}{2}\Lambda\right)\hat{u}^{n+1} = \left(I + \frac{\nu}{2}\Lambda\right)\hat{u}^n$$

$$\hat{u}^{n+1} = \Lambda_1 \hat{u}^n, \quad \Lambda_1 = \begin{pmatrix} \lambda_0^{(1)} & & \\ & \ddots & \\ & & \lambda_M^{(1)} \end{pmatrix}, \quad \lambda_\ell^{(1)} = \frac{1 + \frac{\nu}{2}\lambda_\ell}{1 - \frac{\nu}{2}\lambda_\ell} = \frac{1 - 2\nu\sin^2\left(\frac{\ell\pi}{2M}\right)}{1 + 2\nu\sin^2\left(\frac{\ell\pi}{2M}\right)}$$

$$\boxed{u^{n+1} = \underbrace{\left(I - \frac{\nu}{2}A\right)^{-1}\left(I + \frac{\nu}{2}A\right)}_{A_1} u^n, \quad A_1 = \Phi \Lambda_1 \Phi^*, \quad \lambda_\ell^{(1)} = G_1\left(\frac{\ell\pi}{M}\right)}$$

Since Φ is unitary, A_1 and Λ_1 have the same norm

$$\|A_1\|_{\mathbb{R}_h^{M+1}} = \|\Lambda_1\|_{\mathbb{R}^{M+1}} = \max_{0\le\ell\le M} |\lambda_\ell^{(1)}| \le \|G\|_\infty = 1$$

operator 2-norm of a diagonal matrix

unconditionally stable

And $I - \frac{\nu}{2}A$ is invertible since its eigenvalues are nonzero:

$$1 - \frac{\nu}{2}G\left(\frac{\ell\pi}{M}\right) = 1 + 2\nu\sin^2\left(\frac{\ell\pi}{2M}\right) \ge 1 \qquad 0 \le \ell \le M$$

Implementation of C-N

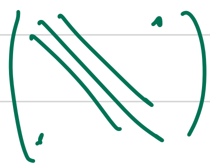
$$u^{n+\frac{1}{2}} = \left(I + \frac{\nu}{2}A\right)u^n \qquad \text{explicit half-step}$$

$$\left(I - \frac{\nu}{2}A\right)u^{n+1} = u^{n+\frac{1}{2}} \qquad \text{implicit half-step}$$

solve

in the Dirichlet & Neumann cases, $I - \frac{\nu}{2}A$ is tridiagonal, so can be solved in $O(M)$ time.

(Better not to form $\left(I - \frac{\nu}{2}A\right)^{-1}$ as that is a dense matrix, so just multiplying $\left(I-\frac{\nu}{2}A\right)^{-1}u^{n+\frac{1}{2}}$ requires $O(M^2)$ flops)

for periodic b.c.'s, $A =$  is not tridiagonal,

but you can declare $\left(I - \frac{\nu}{2}A\right)$ as a sparse matrix to avoid fill-in when computing $\left(I-\frac{\nu}{2}A\right)\backslash u^{n+\frac{1}{2}}$

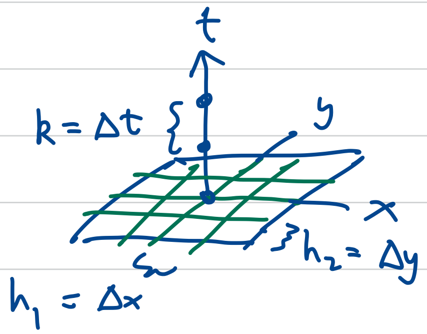
<u>higher dimensions</u>: $u_t = \Delta u \qquad (\Delta u = \nabla^2 u)$

<u>in 2D</u>: $u_t = u_{xx} + u_{yy}, \qquad u(x,y,0) = g(x,y)$
initial condition

<u>exact solution</u>:

$$u(x,y,t) = \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}}\, g(\xi,\eta)\, d\xi\, d\eta$$

numerics:



2d discrete grids at each time $t_n = nk$

$$u^n_{j\ell} \approx u(jh_1, \ell h_2, nk)$$

discrete 2-norm

$$\|u\|^2_{2,h} = h_1 h_2 \sum_{j,\ell=-\infty}^{\infty} |u_{j\ell}|^2$$

FTCS: $\quad D_t^+ u^n_{j\ell} = D_x^+ D_x^- u^n_{j\ell} + D_y^+ D_y^- u^n_{j\ell}$

$$B_1 u = u_{j+1,\ell} - 2u_{j\ell} + u_{j-1,\ell}$$

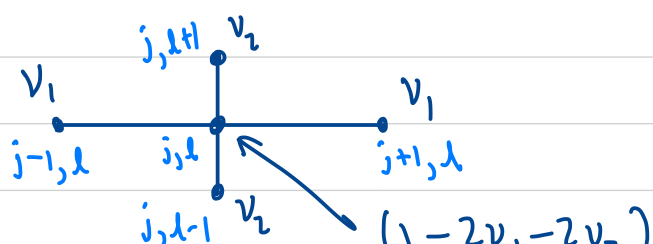
$$B_2 u = u_{j,\ell+1} - 2u_{j\ell} + u_{j,\ell-1}$$

$$\nu_1 = \frac{k}{h_1^2}$$

$$u^{n+1}_{j\ell} = \underbrace{(I + \nu_1 B_1 + \nu_2 B_2)}_{B} u^n_{j\ell}$$

$$\nu_2 = \frac{k}{h_2^2}$$

stencil of B:

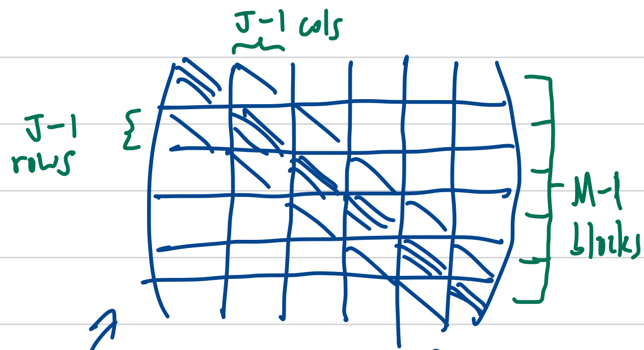


C-N in 2D:

$$\left(I - \frac{\nu_1}{2}B_1 - \frac{\nu_2}{2}B_2\right)u^{n+1} = \left(I + \frac{\nu_1}{2}B_1 + \frac{\nu_2}{2}B_2\right)u^n$$

problem with C-N in 2d:

the matrix A is not tightly banded, so it's expensive to solve using Gaussian elimination



sparsity pattern for A

flattening a 2d array into one dimension $\longrightarrow$ $u_{jl} \Leftrightarrow u((l-1)(J-1)+j)$

$1 \le j \le J-1, \; 1 \le l \le M-1$