Math 228B Lec 5

plan
- finish functional analysis review
- stability analysis of $D_t^+ u = D_x^+ D_x^- u$ in the 1-norm, 2-norm
- amplification factor (how it is used. Theory next time)

Linear operators $A : X \to Y$ $\qquad A(x+y) = Ax + Ay$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad A(\alpha x) = \alpha Ax$

Banach spaces

linear functionals $\rho : X \to \mathbb{C}$ $\qquad \rho(f+g) = \rho(f) + \rho(g)$

(or $\mathbb{R}$ if X is real) $\qquad \rho(\alpha f) = \alpha \rho(f)$

An operator is bounded if there is a constant C s.t.

$$\|Ax\| \le C\|x\| \qquad \forall x \in X \qquad (*)$$

The smallest constant C that works is the norm of the operator

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

sup means supremum, or least upper bound (it's the maximum value attained or approached, which is equivalent to being the smallest among all upper bounds)

note $\|A\| \ge \dfrac{\|Ax_0\|}{\|x_0\|}$ for any particular x_0

if $\|Ax\| \le C\|x\|$ for all x, then $\|A\| \le C$ $\left(C \text{ is an upper bound on } \left\{ \frac{\|Ax\|}{\|x\|} : x \neq 0 \right\} \right)$

To prove that $\|A\| = C$

① show $\|Ax\| \le C\|x\| \quad \forall x \in X$ (so $\|A\| \le C$ and $C \ge 0$)

and

- ②a $\exists\, x_0 \neq 0$ s.t. $\|Ax_0\| \ge C\|x_0\|$ $\left(\text{so } \|A\| \ge \frac{\|Ax_0\|}{\|x_0\|} \ge C\right)$
  (this covers the case of $C=0$)
- or
- ②b If $0 < K < C$ then $\exists\, x_0$ s.t. $\|Ax_0\| > K\|x_0\|$

$\left(\text{so } \|A\| \ge \frac{\|Ax_0\|}{\|x_0\|} > K \text{ for all } K<C, \text{ which implies } \|A\| \ge C\right)$

(note that $\|Ax_0\| > K\|x_0\| \ge 0 \Rightarrow x_0 \neq 0$)

②a is usually easier to prove if such an x_0 exists, but in ∞-dimensions, there may not be a maximizer of $\|Ax\|$ over $\|x\|=1$ (that's why we write sup instead of max)

The norm notation is used because the space of bounded operators $A: X \to Y$ is a Banach space with this norm $\left(\begin{array}{ll} A+B \text{ is the operator } & (A+B)x = Ax + Bx \\ \alpha A \quad " \quad " \quad " & (\alpha A)x = \alpha(Ax) \end{array} \right)$

exercise: show that

① $\|A+B\| \le \|A\| + \|B\|$ $\quad (A, B : X \to Y)$

② $\|BA\| \le \|B\| \cdot \|A\|$ $\quad (B : Y \to Z)$

③ $\|A^n\| \le \|A\|^n$ $\quad (Y = X)$

An $m \times n$ matrix is an operator from $\mathbb{R}^n \to \mathbb{R}^m$ (or $\mathbb{C}^n \to \mathbb{C}^m$)

1-norm: $\|A\|_1 = \max_j \sum_i |a_{ij}|$ — maximum absolute column sum (sum over the entries of each column)

∞-norm: $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ — max absolute row sum

2-norm: $\|A\|_2 = \sigma_1$ — largest singular value

$A = USV^H$, ← Hermitian transpose $V^H = \overline{V^T}$

$S = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ — $m \geq n$ case

U $m \times n$, V $n \times n$

U, V isometries ($U^H U = I$)

A.K.A. Conjugate transpose

Proof for the ∞-norm case.

Let i_0 be the row maximizing the absolute row sum

and let $C = \sum_j |a_{i_0 j}|$

step 1. for any $x \in \mathbb{R}^n$,

$$|(Ax)_i| = |a_{i1}x_1 + \cdots + a_{in}x_n|$$

$$\leq \sum_j |a_{ij}||x_j|$$

$$\leq \left(\sum_j |a_{ij}|\right) \max_k |x_k| \leq C\|x\|_\infty$$

So $\|Ax\| = \max_i |(Ax)_i| \leq C\|x\|_\infty$

Step 2. let $y_j = \operatorname{sgn}(a_{i_0 j}) = \begin{cases} 1 & a_{i_0,j} > 0 \\ 0 & " = 0 \\ -1 & " < 0 \end{cases}$ (writing y instead of x_0)

Then $\|y\|_\infty = 1$ and $(Ay)_{i_0} = \sum_j a_{i_0 j} \operatorname{sgn}(a_{i_0 j}) = C$

so $\|Ay\|_\infty \geq C\|y\|_\infty$. $\therefore \|A\| = C$.

Recall that we can think of our discrete evolution operator B $(u^{n+1} = Bu^n)$ as an infinite tridiagonal Toeplitz matrix:

$$B = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \nu & 1-2\nu & \nu & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \longleftarrow \text{row zero}$$

The max norm (operator norm of B acting on ℓ^∞) is the maximum absolute row sum

$$\|B\|_\infty = |\nu| + |1-2\nu| + |\nu| = \begin{cases} 1 & 0 \leq \nu \leq \frac{1}{2} \\ 4\nu - 1 & \nu > \frac{1}{2} \\ 1-4\nu & \nu < 0 \end{cases}$$

(all row sums are the same)

less important since $\nu = \frac{k}{h^2} > 0$

Similarly, $\|B\|_1 =$ max absolute column sum which has the same numerical value as $\|B\|_\infty$

$$|\nu| + |1-2\nu| + |\nu| \longleftarrow \begin{pmatrix} \ddots & & \\ \ddots & \nu & \\ \ddots & 1-2\nu & \ddots \\ & \nu & \ddots \\ & & \ddots \end{pmatrix}$$

One can repeat the Lax-Richtmyer analysis in the 1-norm.

First, this is reasonable as the heat equation does not lead to growth in the 1-norm

$$u_t = u_{xx} \qquad u(x,0) = g(x) \quad \longrightarrow \quad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} g(\xi)\, d\xi$$

so $$|u(x,t)| \leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} |g(\xi)|\, d\xi$$

equality if $g(x) \geq 0$

$$\therefore \int_{-\infty}^{\infty} |u(x,t)|\, dx \leq \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} |g(\xi)|\, d\xi\, dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-\xi)^2}{4t}} dx \right) |g(\xi)|\, d\xi$$

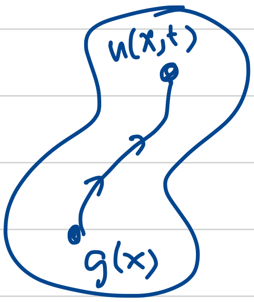
legal to change order of integration when the integrand is positive.

result: for all positive times,

$$\int_{-\infty}^{\infty} |u(x,t)|\, dx \leq \int_{-\infty}^{\infty} |g(x)|\, dx$$

and if $g(x) \geq 0$ this inequality is an equality.
(it becomes an inequality again on a finite interval with Dirichlet b.c.'s)

In our "evolution on a Banach space" framework,



$\mathcal{B}$

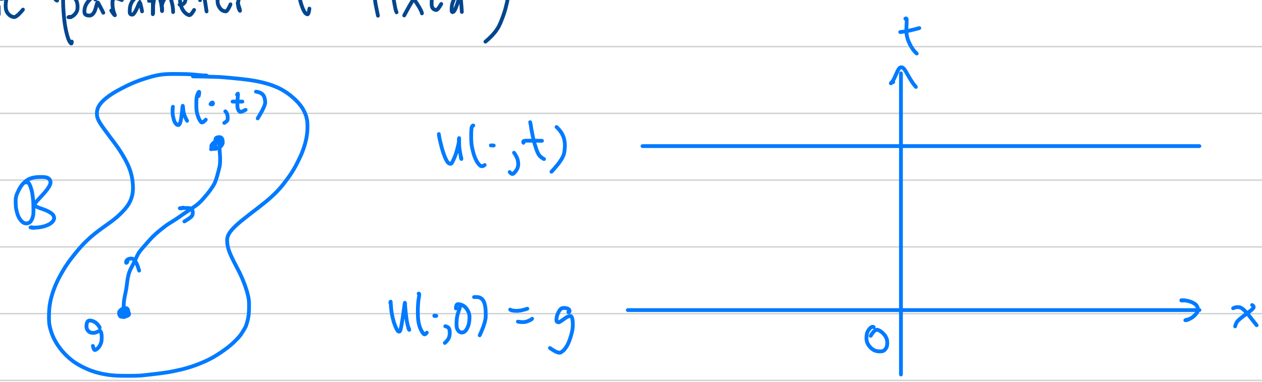
$\mathcal{B} = L^1(\mathbb{R}) =$ "integrable functions on $\mathbb{R}$"

$\|g\| = \int_{-\infty}^{\infty} |g(x)|\, dx$ $\leftarrow$ norm in $\mathcal{B}$

the solution $u(x,t)$ of $\left\{ \begin{array}{l} u_t = u_{xx} \\ u(x,0) = g(x) \end{array} \right\}$ satisfies

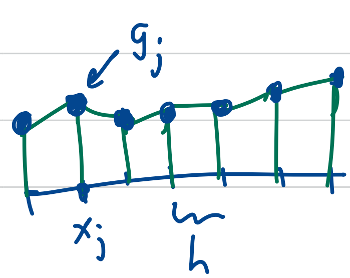
$$\|u(\cdot,t)\| \leq \|g\| \qquad (t \geq 0)$$

The dot indicates that we're thinking of u as a function of its first argument only (with the parameter t fixed)



$\mathcal{B}_h = \ell^1_h$ = "summable sequences"

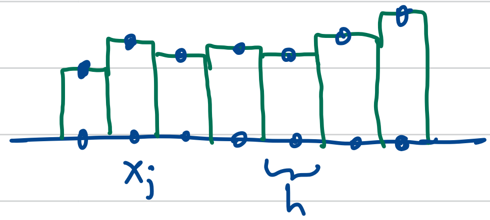
$$\|g\|_{1,h} = h \sum_{j=-\infty}^{\infty} |g_j|$$



This is the trapezoidal rule approximation

if you sample $g_j = g(x_j)$, $x_j = jh$

Same result but different picture:



(midpoint rule)

same quadrature rule

The h in the norm definition doesn't affect the "max absolute column sum" formula $\|B\|_1 = |\nu| + |1-2\nu| + |\nu|$

since $$\|B\|_{1,h} \quad \sup_{u\neq 0} \frac{\|Bu\|_{1,h}}{\|u\|_{1,h}} = \sup \frac{h\|Bu\|_1}{h\|u\|_1} = \|B\|_1$$

One can still prove that $$\|\tau^n\|_{1,h} \leq \begin{cases} \left(\frac{k}{2} + \frac{h^2}{12}\right)M & \nu \neq \frac{1}{6} \\ \left(\frac{k^2}{6} + \frac{h^4}{360}\right)M & \nu = \frac{1}{6} \end{cases} \quad (*)$$

but now (in a 1-norm analysis)

$$M = \max_{\substack{0\leq h\leq 1 \\ 0\leq x\leq h}} h\sum_j |g^{(\mu)}(x+jh)|$$ $\leftarrow$ worst (largest) discrete integral of $|g_{xxxx}(x)|$ or $|\partial_x^6 g(x)|$

$$\mu = \begin{cases} 4 & \nu \neq 1/6 \\ 6 & \nu = 1/6 \end{cases}$$

arbitrary upper bound on h

(See pages 38–40 of my 2007 notes for full details
posted on my Berkeley webpage
I use the Cauchy form of Taylor's theorem with remainder)

Lax-Richtmyer analysis

The error $e_j^n = u_j^n - u(jh, nk)$ satisfies $e^{n+1} = Be^n - k\tau^n$

$k \leq \varepsilon = \nu$

$$\Longrightarrow \quad \|e^n\| \leq K\, kn \max_{0 \leq \ell \leq n-1} \|\tau^\ell\|, \quad (nk \leq T)$$

need $\nu \leq \frac{1}{2}$ so $\|B^n\|_1 \leq K = 1$ and $h \leq 1$ to use ⊛ here

$\left(h = \sqrt{k/\nu} \leq 1 \iff k \leq \nu\right)$ so $\varepsilon = \nu$

result: if $\nu \leq \frac{1}{2}$ and $0 < k \leq \varepsilon = \nu$ then

$$\max_{0 \leq nk \leq T} h \sum_{j=-\infty}^{\infty} |e_j^n| \leq \begin{cases} \left(\frac{k}{2} + \frac{h^2}{12}\right) TM & \nu \neq \frac{1}{6},\ \nu \leq \frac{1}{2} \\ \left(\frac{k^2}{6} + \frac{h^4}{360}\right) TM & \nu = \frac{1}{6} \end{cases}$$

In the max norm analysis this was $\max_j |e_j^n|$ and M was $\max_x |g^{(\mu)}(x)|$

$(\mu = 4$ if $\nu \neq \frac{1}{6}$, $\mu = 6$ if $\nu = \frac{1}{6})$

Final variant: 2-norm

$$\mathcal{B} = L^2(\mathbb{R}), \quad \|g\| = \sqrt{\int_{-\infty}^{\infty} |g(x)|^2 dx}$$

$$\mathcal{B}_h = \ell_h^2, \quad \|g\|_{2,h} = \sqrt{h \sum_j |g_j|^2}$$

To determine the stability of B acting on ℓ_h^2, one computes the amplification factor of the scheme finite difference operator $\quad B u_j = \nu u_{j+1} + (1-2\nu) u_j + \nu u_{j-1}$

amplification factor $\quad G(\xi) = \nu e^{i\xi} + (1-2\nu) e^{0} + \nu e^{-i\xi}$

$$= 1 + \nu \left(e^{i\xi/2} - e^{-i\xi/2} \right)^2$$

$$= 1 - 4\nu \sin^2 \frac{\xi}{2}$$

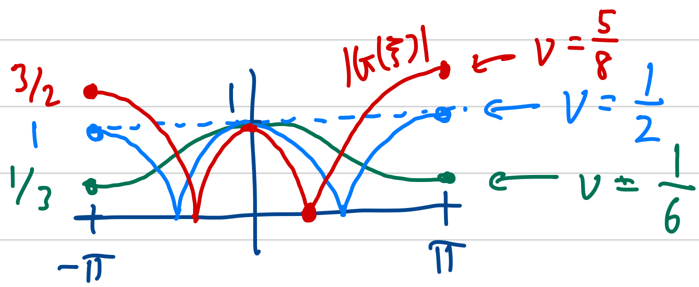
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sin^2 x = \frac{(\quad)^2}{-4}$$

We will see that $\|B\|_{\ell_h^2} = \|G\|_\infty$

max norm on $[-\pi, \pi]$

$$|G(\xi)| = \left| 1 - 4\nu \sin^2 \frac{\xi}{2} \right|$$



$$\left(\begin{array}{l} \text{if } \nu > \frac{1}{2} \text{ then} \\ -4\nu \sin^2 \frac{\pi}{2} < -2 \\ \text{so} \quad 1 - 4\nu \sin^2 \frac{\pi}{2} < -1 \\ \quad |1 - 4\nu \cdots| > 1 \end{array} \right)$$

As expected, the transition from $\|B\| \le 1$ to $\|B\| > 1$ occurs at $\nu = \frac{1}{2}$

The rest of the Lax-Richtmyer convergence proof in the 2-norm is very similar to the 1-norm analysis.

first show that $\|\tau^n\|_{2,h} \le \begin{cases} M\sqrt{\frac{2}{3}k^2 + \frac{1}{63}h^4} & \nu \neq \frac{1}{6} \\ M\sqrt{\frac{k^4}{10} + \frac{h^8}{39600}} & \nu = \frac{1}{6} \end{cases}$

(see scratch work below)

where $M = \max_{\substack{0 \le h \le 1 \\ 0 \le x \le h}} \sqrt{h \sum_j |g^{(\mu)}(x+jh)|^2}$

$\mu = 4$ if $\nu \neq \frac{1}{6}$

$= 6$ if $\nu = 1/6$

Conclusion from Lax-Richtmyer:

if $\nu \le \frac{1}{2}$ and $0 < k \le \nu$ then $\|B(k)^n\| \le 1$ and

$$\max_{0 \le nk \le T} \sqrt{h \sum_j |e_j^n|^2} \le \begin{cases} MT\sqrt{\frac{2}{3}k^2 + \frac{1}{63}h^4} & \nu \neq \frac{1}{6}, \nu \le \frac{1}{2} \\ MT\sqrt{\frac{k^4}{10} + \frac{h^8}{39600}} & \nu = \frac{1}{6} \end{cases}$$

The 2-norm analysis (von-Neumann stability analysis) is the version that generalizes most easily to implicit methods. This is the topic of the next lecture.

scratch work for $\nu \neq \frac{1}{6}$ case of ℓ_h^2 truncation error analysis

$$|\tau_j^n| \leq \begin{cases} k\int_0^1 |u_{xxxx}(x_j, t_n+\theta k)|(1-\theta)\,d\theta \\ + \frac{h^2}{6}\int_0^1 |u_{xxxx}(x_j+\theta h, t_n)|(1-\theta)^3 d\theta \\ + \frac{h^2}{6}\int_0^1 |u_{xxxx}(x_j-\theta h, t_n)|(1-\theta)^3 d\theta \end{cases}$$

$\tau = a + b + c$, each positive

$$\tau^2 \leq a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

$$\leq 2(a^2+b^2+c^2)$$

$$a^2 = k^2\left(\int_0^1 |\tilde{u}(\theta)|(1-\theta)\,d\theta\right)^2 \leq k^2\int_0^1 |\tilde{u}(\theta)|^2 d\theta \int_0^1 (1-\theta)^2 d\theta$$

$$= \frac{k^2}{3}\int_0^1 |\tilde{u}(\theta)|^2 d\theta$$

$$b^2 = \frac{h^4}{36}\left(\int_0^1 |\tilde{u}(\theta)|(1-\theta)^3 d\theta\right)^2 \leq \frac{h^4}{36}\int_0^1 |\tilde{u}(\theta)|^2 d\theta \int_0^1 (1-\theta)^6 d\theta$$

c^2 //

$$= \frac{h^4}{36\cdot 7}\int_0^1 |\tilde{u}(\theta)|^2 d\theta$$

$$h\sum_j 2(a^2+b^2+c^2) \leq \left(\frac{2}{3}k^2 + \frac{h^4}{63}\right)\max_x h\sum_j |\tilde{u}_j(\theta)|^2$$

$$|u(x)|^2 \leq \left(\frac{1}{\sqrt{4\pi t}}\int e^{-\frac{(x-\xi)^2}{4t}}|g(\xi)|\,d\xi\right)^2 \leq \frac{1}{\sqrt{4\pi t}}\int e^{\cdots}|g(\xi)|^2 d\xi$$

$$h\sum_j |u(x+jh)|^2 \leq \frac{1}{\sqrt{4\pi t}}\int e^{\cdots}\, h\sum |\tilde{g}(\xi+jh)|^2\,d\xi$$

$$\leq \max_x h\sum_j |\tilde{g}(\xi+jh)|^2$$

$$\sqrt{h\sum_j |\tau_j^n|^2} \leq \sqrt{\left(\frac{2}{3}k^2 + \frac{1}{63}h^4\right)}\sqrt{\max_{\substack{0\leq h\leq 1\\ 0\leq x\leq h}} h\sum_j |g_{xxxx}(x+jh)|^2}$$

$$\left(\sqrt{\frac{2}{3}}k + \sqrt{\frac{1}{63}}h^2\right)$$

scratch work for $\nu = \frac{1}{6}$ in l_h^2 analysis

$\nu = \frac{1}{6}$

$$\underbrace{\overset{u_{ttt}}{\frac{k^2}{12}\int_0^1 |\tilde{u}|(1-\theta)^2\,d\theta}}_{a} + \underbrace{\overset{u_{xxxxxx}}{\frac{h^4}{5!}\int_0^1 |\tilde{u}|(1-\theta)^5\,d\theta}}_{b,\ c}$$

$\tau = a + b + c$

$\tau^2 \leq 2(a^2 + b^2 + c^2)$

$$a^2 \leq \frac{k^4}{4}\int_0^1 |\tilde{u}|^2\,d\theta \int_0^1 (1-\theta)^4\,d\theta = \frac{k^4}{20}\int_0^1 |\tilde{u}|^2\,d\theta$$

$$b^2, c^2 \leq \frac{h^8}{(5!)^2}\int |\tilde{u}|^2\,d\theta \int_0^1 (1-\theta)^{10}\,d\theta = \frac{h^8}{(5!)^2 \cdot 11}\int_0^1 |\tilde{u}|^2\,d\theta$$

$$h\sum_j |\tau_j^n|^2 \leq \sum_j 2(a^2+b^2+c^2) \leq \left(\frac{k^4}{10} + \frac{4h^8}{(5!)^2 11}\right)\int_0^1 \underbrace{h\sum |\tilde{u}_j|^2}_{\text{max}}\,d\theta$$

max (get rid of integral)

$$\sqrt{h\sum_j |\tau_j^n|^2} \leq \sqrt{\frac{h^4}{10} + \frac{h^8}{39600}}\ M$$

$$M = \max_{\substack{0 \leq h \leq 1 \\ 0 \leq x \leq h}} \sqrt{h\sum_j |g^{(m)}(x+jh)|^2} \qquad \nu = \frac{k}{h^2} = \frac{1}{6} \qquad k = \frac{1}{6}h^2$$

$\sqrt{a^2+b^2} \leq |a| + |b|$ $\qquad \left\| \binom{a}{0} + \binom{0}{b} \right\|$ $\qquad \frac{h^2}{\sqrt{10}} + \frac{h^4}{60\sqrt{11}}$