HOMEWORK 2

DUE SATURDAY, FEB 21 AT 2 PM. (SUBMIT TO GRADESCOPE.)

Problem 1. (8 points) Recall from Lectures 7 and 8 that if $u_j = U(jh)$, the Z-transform $\hat{u}(\xi)$ is related to the Fourier transform $\widehat{U}(\kappa) = \int_{-\infty}^{\infty} U(x)e^{-ix\kappa}\,dx$ via

$$h\hat{u}(h\kappa) = \sum_{m=-\infty}^{\infty} \widehat{U}\left(\kappa - \frac{2\pi}{h}m\right).$$

Suppose that for $\kappa \in \mathbb{R}$, $\widehat{U}(\kappa)$ satisfies

$$\text{(a)} \quad \left|\widehat{U}(\kappa)\right| \le \frac{C_r}{1+|\kappa|^r}, \qquad \text{(b)} \quad \left|\widehat{U}(\kappa)\right| \le Ce^{-\rho|\kappa|},$$

where $r \ge 2$, $C_r > 0$, $C > 0$ and $\rho > 0$ are constants. Show that for $|\kappa| \le \pi/h$, $\left|h\hat{u}(h\kappa) - \widehat{U}(\kappa)\right|$ is bounded by

$$\text{(a)} \quad \frac{C_r h^r}{4\pi^{r-2}}, \qquad \text{(b)} \quad \frac{2Ce^{-\rho\pi/h}}{1-e^{-2\rho\pi/h}}. \qquad \text{Hints: } \left|\kappa \pm \frac{2\pi}{h}m\right| \ge \frac{2\pi m - \pi}{h}, \quad \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{\pi^2}{8}.$$

Problem 2. (6 points) A circulant matrix is constant along diagonals with entries that "wrap around"

$$A_{jk} = \left\{ \begin{array}{cc} r_{k-j} & k \ge j, \\ r_{N+k-j} & k < j \end{array} \right\}; \qquad N = 4 \text{ example: } \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \\ r_3 & r_0 & r_1 & r_2 \\ r_2 & r_3 & r_0 & r_1 \\ r_1 & r_2 & r_3 & r_0 \end{pmatrix}.$$

For convenience, we will index our matrices starting at zero. Show that $AU = U\Lambda$, with

$$U_{jk} = e^{2\pi ijk/N}, \qquad \begin{pmatrix} 0 \le j \le N-1 \\ 0 \le k \le N-1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{N-1} \end{pmatrix}$$

and $\lambda_k = \sum_{j=0}^{N-1} r_j e^{2\pi ijk/N}$.

Problem 3. (16 points) Use the Crank-Nicolson scheme to solve the equation

$$u_t = \alpha u_{xx} - \beta u_x, \qquad \alpha = \frac{1}{512}, \quad \beta = \frac{33}{32} \tag{$\star$}$$

on the interval $[0,1]$ with periodic boundary conditions and initial conditions

$$u(x,0) = g(x), \qquad g(x) = \big(\sin \pi x\big)^{100}.$$

For the spatial discretization, replace the right-hand side of $(\star)$ by $\alpha D_x^+ D_x^- u - \beta D_x^0 u$, and then use the trapezoidal rule in time.

(a) Plot $u(x,T)$ versus x, with $T \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, all on the same plot. (Use enough gridpoints in time and space so that these plots stop changing if you further refine the mesh.)

(b) Plot $u(0,t)$ for $0 \le t \le 1$. (Again use enough gridpoints and timesteps in your Crank-Nicolson scheme to get an accurate plot.)

(c) Compute (i.e., derive a simplified formula for) the amplification factor of the scheme and demonstrate that the scheme is unconditionally stable.

(d) Find a PDE satisfied by $v(x,t) = u(x+\beta t, t)$. Note that $v(x,0) = g(x)$.

(e) Derive a formula for the exact solution $v(x,t)$ expressed in terms of the Fourier coefficients c_k of $g(x) = \sum_{k=-50}^{50} c_k e^{2\pi i k x}$. Compute the c_k numerically by sampling $g(x)$ on a uniform grid with 128 gridpoints and applying the FFT. Also write down the formula for the exact solution $u(x,t)$.

(f) Evaluate $u(1/2, 1)$ with as much accuracy as you can get in double-precision using your formula in (e) for $u(x,t)$.

(g) Compute the error $e_h = u_h(1/2,1) - u(1/2,1)$, where $u_h(x,t)$ is the numerical solution from Crank-Nicolson with the refinement path $k = h$. Plot $\log_{10}|e_h|$ versus $\log_{10} h$ to show that the scheme converges at second order.