228B Lec 8

plan:
- mesh-dependent Z-transform (finish discussion)
- finite domain (Dirichlet b.c.'s)

last time: $\hat{u}(\xi) = \sum_j u_j e^{-ij\xi}$ unscaled Z-transform

$\tilde{u}(k) = h \sum_j u_j e^{-ijhk}$ mesh dependent (physical) Z-transform

we showed that if $u_j = U(jh)$ ← sampled function

then
$$\tilde{u}(k) = \sum_{m=-\infty}^{\infty} \hat{U}\left(k - \frac{2\pi}{h} m\right)$$

Fourier transform of U: $\hat{U}(k) = \int_{-\infty}^{\infty} U(x) e^{-ikx} dx$

- sampling $u_j = U(jh)$ and computing $\tilde{u}(k)$ yields a $\frac{2\pi}{h}$-periodic function
- Summing $\hat{U}\left(k - \frac{2\pi}{h} m\right)$ periodizes $\hat{U}(k)$

It's amazing that they're exactly equal!

This is similar to the aliasing formula for the discrete Fourier transform (DFT or FFT). Ref: 228A Lec 12b

$f(x) = \sum_k c_k e^{ikx}$, $\quad$ samples $y_j = f(jh)$, $\quad h = \frac{2\pi}{N}$, $0 \le j < N$, $\quad \hat{y}_k = \frac{1}{N} \sum_{j=0}^{N-1} y_j e^{-2\pi i jk/N}$

$$\hat{y}_k = \sum_{m \in \mathbb{Z}} c_{k+mN}$$

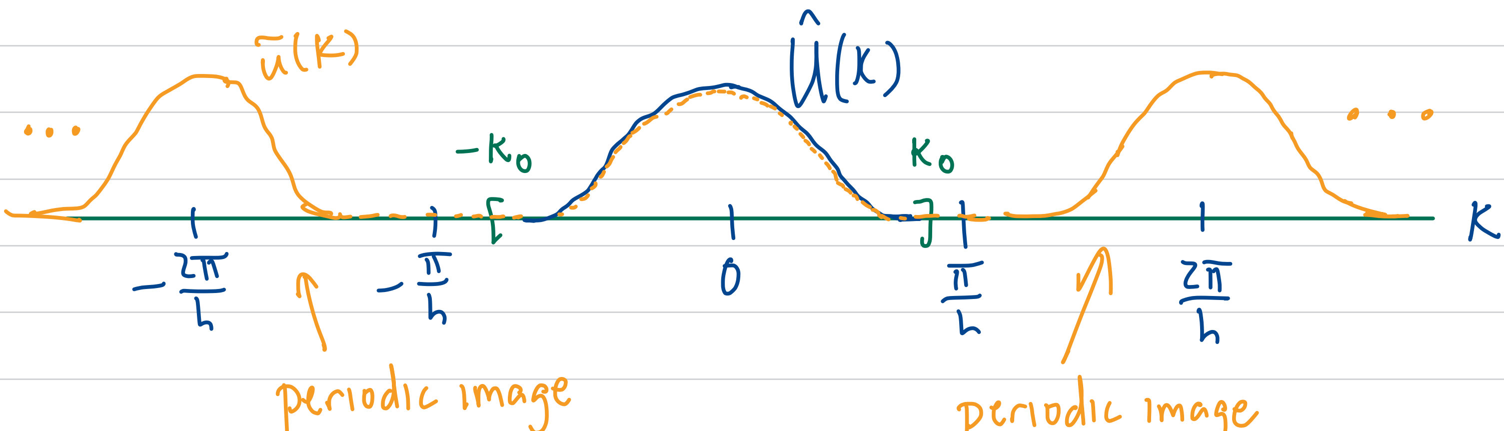
← exact formula!

the DFT of the sampled data is N-periodic

periodizes the c_k

back to the Z-transform:

Picture for the case where $\hat{u}(k) = 0$ for $|k| \ge k_0 > \frac{\pi}{h}$:



if $\hat{u}(k)$ decays reasonably fast, then

$$\tilde{u}(k) \to \hat{u}(k) \quad \text{as } h \to 0.$$

(The other terms in the sum $\sum_m \hat{u}\left(k - \frac{2\pi}{h} m\right)$ are small for $-\frac{\pi}{h} \le k \le \frac{\pi}{h}$ when h is small.)

And if $\hat{U}(K)$ is supported in $\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$, then

$$\hat{U}(k) = \tilde{u}(K)\, \chi_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]}(K) \leftarrow$$ no errors at all!

The inverse Fourier transform of this equation is the Shannon sampling theorem (Math 118):

$$\chi_A(k) = \begin{cases} 1 & k \in A \\ 0 & k \notin A \end{cases}$$

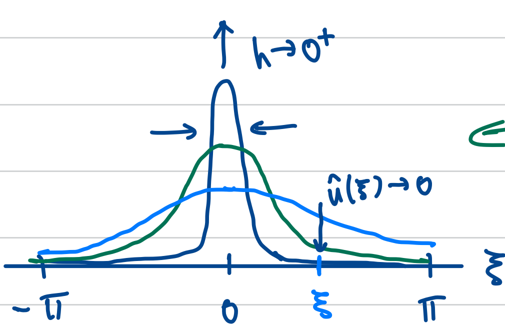
characteristic function

$$U(x) = \sum_j U(jh)\, \mathrm{sinc}\left(\frac{\pi}{h}(x - jh)\right), \quad \mathrm{sinc}\, x = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x \neq 0 \end{cases}$$

$h = \frac{\pi}{\Omega}$

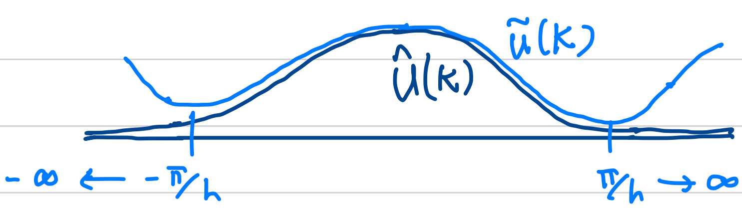
exact reconstruction of a band-limited function from sampled values.

I prefer to work with $\hat{u}(\xi)$ for stability analysis. But it's nice to understand how to rescale it for the $h \to 0^+$ limit to be meaningful.



plots of $\hat{u}(\xi)$ as $h \to 0^+$ for sampled values $u_j = U(jh)$.

For fixed $\xi \neq 0$, $\hat{u}(\xi) \to 0$.



But for fixed K,

$$\tilde{u}(K) = h\, \hat{u}(hK) \longrightarrow \hat{U}(K)$$

as $h \to 0^+$

<u>Finite domains</u>

the amplification factor allows us to compute the 2-norm of a finite difference operator on the whole real line (on ℓ_h^2)

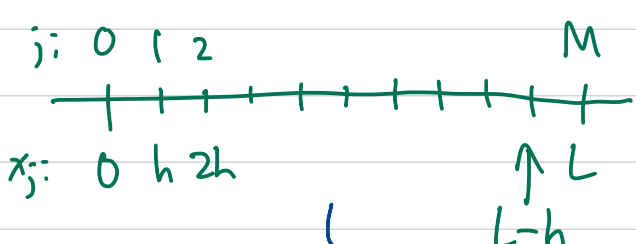
$$Bu_j = \sum_m c_m u_{j+m}, \qquad G(\xi) = \sum_m c_m e^{im\xi}$$

$$\|B\|_{\ell_h^2} = \|G\|_{L^\infty[-\pi,\pi]}$$

What does it tell us about finite domain problems?

Answer: in some cases the eigenvalues of the finite version of B (call it A) are sampled values of $G(\xi)$.

<u>case 1:</u> Dirichlet B.C.'s and B is real, symmetric, tri-diagonal

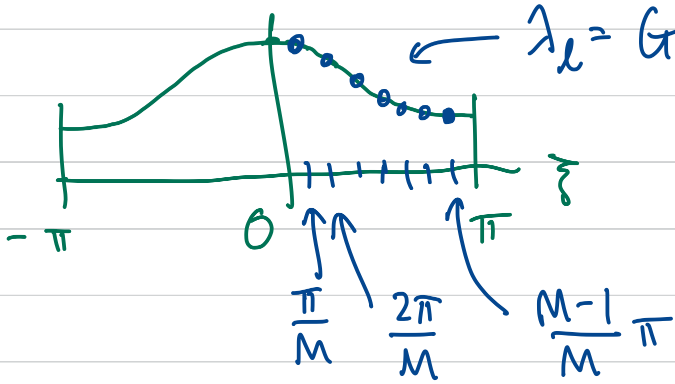


$$h = \frac{L}{M}$$

$$A = \underbrace{\begin{pmatrix} \alpha & \beta & & & \\ \beta & \alpha & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & & \alpha & \beta \\ & & & \beta & \alpha \end{pmatrix}}_{(M-1)\times(M-1)}$$

<u>Claim:</u> the eigenvalues of A are the values of $G(\xi)$ sampled at equal intervals $\xi_\ell = \frac{\ell\pi}{M}$, $\quad 1 \le \ell \le M-1$

$$\left(\text{so} \quad \|A\| = \max_{1 \le \ell \le M-1} |\lambda_\ell| \le \|G\|_\infty = \|B\| \right)$$



$$\lambda_\ell = G(\xi_\ell) = G\left(\frac{\ell\pi}{M}\right)$$

$$1 \le \ell \le M-1$$

<u>proof</u>: first note that $G(\xi) = \beta e^{i\xi} + \alpha e^0 + \beta e^{-i\xi}$ is real-valued.

$$= \alpha + 2\beta \cos \xi$$

We know $w \in \ell^\infty$ given by

$$w_j = e^{ij\xi} \qquad -\infty < j < \infty \qquad (\xi \text{ fixed})$$

satisfies $\quad Bw_j = G(\xi)\, w_j$.

Suppose $\xi = \xi_\ell = \frac{\ell\pi}{M}$, $\quad 1 \le \ell \le M-1$, $\quad w_j = e^{ij\ell\pi/M}$

We claim $u \in \mathbb{R}^{M-1}$ given by $u_j = \operatorname{Im}(w_j) = \sin \frac{j\ell\pi}{M}$

is an eigenvector of A with eigenvalue $G(\xi_\ell)$. $\quad 1 \le j \le M-1$

Use the same formulas for $j = 0, M$

$$Au = \mathrm{Im}\left\{ \left(\begin{array}{c|ccccc|c} \beta & \alpha & \beta & & & & \\ & \beta & \alpha & \beta & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & \beta & \alpha & \beta \end{array}\right) \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \\ w_M \end{pmatrix} \right\} = \mathrm{Im}\left\{ G(\xi) \begin{pmatrix} w_1 \\ \vdots \\ w_{M-1}\end{pmatrix}\right\} = \underbrace{G(\xi)}_{\lambda_\ell}\, u$$

$\begin{pmatrix} \text{rows } 1 .. M-1 \\ \text{cols } 0 .. M \end{pmatrix}$ of B

here I use α, β are real, and $\mathrm{Im}\, w_0 = \mathrm{Im}\, w_M = 0$ and $G(\xi)$ is real

discrete orthogonality lemma (proved below):

$$\Phi = \sqrt{\frac{2}{M}} \left(u^{(\ell=1)}, \ldots, u^{(\ell=M-1)} \right) \quad \text{satisfies} \quad \Phi^T \Phi = I$$

so the columns of Φ are linearly independent and

$$A = \Phi \Lambda \Phi^{-1}, \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{M-1} \end{pmatrix}, \quad \lambda_\ell = G\left(\frac{\ell\pi}{M}\right)$$

discrete orthogonality lemma (for sines)

1. Suppose $M \geq 2$ and $1 \leq \ell \leq m \leq M-1$

then $$\sum_{j=1}^{M-1} \sin\frac{j\ell\pi}{M} \sin\frac{jm\pi}{M} = \frac{M}{2}\delta_{\ell m}$$

<u>proof</u>:

$$LHS = \operatorname{Im} \sum_{j=1}^{M-1} \left(\frac{e^{ij\ell\pi/M} - e^{-ij\ell\pi/M}}{2i} e^{ijm\pi/M} \right)$$

$$\circledast \quad = -\frac{1}{2} \operatorname{Re} \sum_{j=1}^{M-1} \left(e^{ij(m+\ell)\pi/M} - e^{ij(m-\ell)\pi/M} \right)$$

$(m+\ell) + (m-\ell) = 2m$ is even

so $m+\ell$ and $m-\ell$ are both even or both odd

<u>both odd</u>: This rules out $m=\ell$, so $1 \leq \ell < m \leq M-1$

Let $p = m+\ell$ or $p = m-\ell$. Then p is odd and $1 \leq p \leq 2M-3$

for $j \in \{1, .., M-1\}$, $e^{i(M-j)p\pi/M} = \underbrace{e^{ip\pi}}_{-1} e^{-ijp\pi/M}$

$$\text{so} \sum_{j=1}^{M-1} e^{ijp\pi/M} = \frac{1}{2} \sum_{j=1}^{M-1} \left(e^{ijp\pi/M} + e^{i(M-j)p\pi/M} \right)$$

$$= \frac{1}{2} \sum_{j=1}^{M-1} \left(e^{ijp\pi/M} - \overline{e^{ijp\pi/M}} \right) \text{ is purely imaginary}$$

$(z - \bar{z} = (x+iy) - (x-iy) = 2iy)$

Taking the real part in $\circledast$ gives $-\frac{1}{2}(0-0) = 0$, as claimed for $m \neq \ell$.

<u>both even:</u> we can extend the sum in ⊛ to $\sum_{j=0}^{M-1}$

since the $j=0$ term is $(e^0 - e^0) = (1-1) = 0$

If $1 \leq p \leq 2M-1$ and p is even, then

$$\sum_{j=0}^{M-1} e^{ijp\pi/M} = \sum_{j=0}^{M-1} a^j , \quad a = e^{ip\pi/M} \neq 1$$

$$= \frac{1-a^M}{1-a} = \frac{1-e^{ip\pi}}{1-a} = \frac{1-1}{1-a} = 0$$

In ⊛, $m+l$ is even and $2 \leq m+l \leq 2M-2$

so the first term in the sum gives 0.

$m-l$ is also even and $0 \leq m-l \leq M \leq 2M-1$.

(we assumed $m \geq l$).

If $m \neq l$, then $1 \leq m-l \leq 2M-1$ and the second term in the sum is also 0.

If $m=l$, we instead have

$$⊛ = -\frac{1}{2}\text{Re}\left\{0 - \sum_{j=0}^{M-1} e^0\right\} = \frac{M}{2}$$

In all cases, $\displaystyle\sum_{j=1}^{M-1} \sin\frac{jl\pi}{M} \sin\frac{jm\pi}{M} = \frac{M}{2}\delta_{lm}$