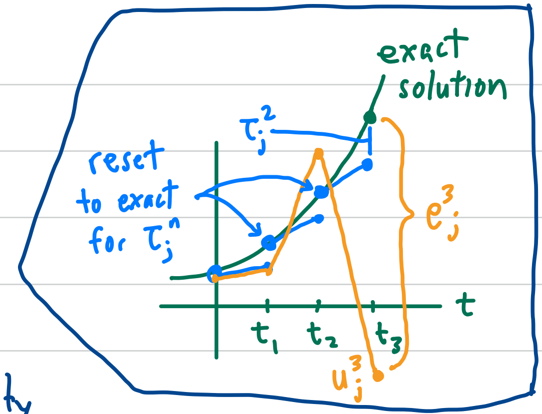
228B Lec 4



last time: Lax-Richtmyer theorem
(consistency + stability $\Rightarrow$ convergence)

<u>today</u>: sufficient condition for stability
stability analysis in the ∞-norm for $D_t^+ u = D_x^+ D_x^- u$
functional analysis review

<u>def</u>: A scheme is stable if $\forall T > 0$ $\exists K, \varepsilon > 0$ s.t.

$$\|B(k)^n\| \le K \quad \text{for } 0 < k < \varepsilon,\ 0 \le nk \le T$$

<u>Theorem</u>: A sufficient condition for stability is that

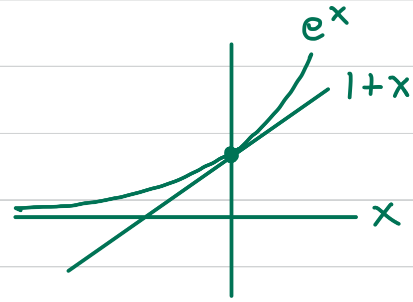
$$\exists\, C \ge 0 \text{ s.t. } \|B(k)\| \le 1 + Ck \quad \text{for } 0 < k < \varepsilon$$

<u>Proof</u>: $\|B(k)^n\| \le \|B(k)\|^n \le (1 + Ck)^n$ for $0 < k < \varepsilon$ and $0 \le nk \le T$

$$\le (e^{Ck})^n = e^{Ckn} \le \underbrace{e^{CT}}_{K}$$

$e^x = 1 + x + \frac{e^{\xi}}{2}x^2$
Taylor's theorem with remainder

$\left(\Rightarrow 1 + x \le e^x \text{ for } x \in \mathbb{R}\right)$



$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(\xi)x^2$, ξ between 0 and x

special case: $C = 0$, $\|B(k)\| \le 1$ for $0 < k < \varepsilon$

$\Rightarrow \|B(k)^n\| \le 1$ (so $K = 1$ works) stability of $B : \ell^\infty \to \ell^\infty$ from FTCS scheme for $u_t = u_{xx}$

$$Bu_j = \nu u_{j-1} + (1-2\nu)u_j + \nu u_{j+1}, \quad \nu = \frac{k}{h^2}$$

first let's just try computing $u^n = B^n u^0$ with $u^0_j = \begin{cases} 1 & j=0 \\ 0 & \text{o.w.} \end{cases}$

$k = 1/4$

$h = 1/4$

$\nu = 4$

$1-2\nu = -7$

solution becomes oscillatory and blows up

t										
$3k$	0	64	−336	780	−1015	780	−336	64	0	$\leftarrow u^3$
$2k$	0	0	16	−56	81	−56	16	0	0	$\leftarrow u^2$
k	0	0	0	4	−7	4	0	0	0	$\leftarrow u^1$
0	0	0	0	0	1	0	0	0	0	$\leftarrow u^0$
x	$-4h$	$-3h$	$-2h$	$-h$	0	h	$2h$	$3h$	$4h$	

$h = \frac{1}{1000}$

$k = 1/1000$

$\nu = 1000$

$1-2\nu = -1999$

hopelessly unstable

(remember, B propagates truncation errors)

try again

$k = \frac{1}{64}$

$h = \frac{1}{4}$

$\nu = \frac{1}{4}$

$1-2\nu = \frac{1}{2}$

solution decays and spreads out

t									
$3k$	0	1/64	6/64	15/64	20/64	15/64	6/64	1/64	0
$2k$	0	0	1/16	4/16	6/16	4/16	1/16	0	0
k	0	0	0	1/4	2/4	1/4	0	0	0
0	0	0	0	0	1	0	0	0	0
x	$-4h$	$-3h$	$-2h$	$-h$	0	h	$2h$	$3h$	$4h$

The condition for stability turns out to be $\nu \le \frac{1}{2}$:

$$(Bu)_j = Bu_j = \nu u_{j-1} + (1-2\nu)u_j + \nu u_{j-1}$$

$$|Bu_j| \le \underbrace{(\nu + |1-2\nu| + \nu)}_{1 \text{ if } \nu \le \frac{1}{2} \ (\nu > 0 \text{ by definition})} \max |u_j| = \|u\|_\infty$$

$$\|Bu\|_\infty \le \|u\|_\infty \quad \text{so} \quad \|B\| \le 1 \quad \text{and} \quad \|B^n\| \le 1$$

<u>note</u>: there's no superscript n on u. We're applying B to an arbitrary $u \in \ell^\infty$ here.

if $\nu > \frac{1}{2}$, let $u_j = (-1)^j$. (note $\|u\|_\infty = \max_j |u_j| = 1$)

$$Bu_j = \nu \underbrace{u_{j-1}}_{(-1)^{j-1}} + (1-2\nu)\underbrace{u_j}_{(-1)^j} + \nu \underbrace{u_{j+1}}_{(-1)^{j+1}} = (1-4\nu)(-1)^j$$

$$B^n u_j = (4\nu - 1)^n (-1)^{n+j}$$ (oscillates in sign and grows exponentially)

$$\|B^n u\|_\infty = (4\nu-1)^n \underbrace{\|u\|_\infty}_{1} \to \infty \text{ as } n \to \infty$$

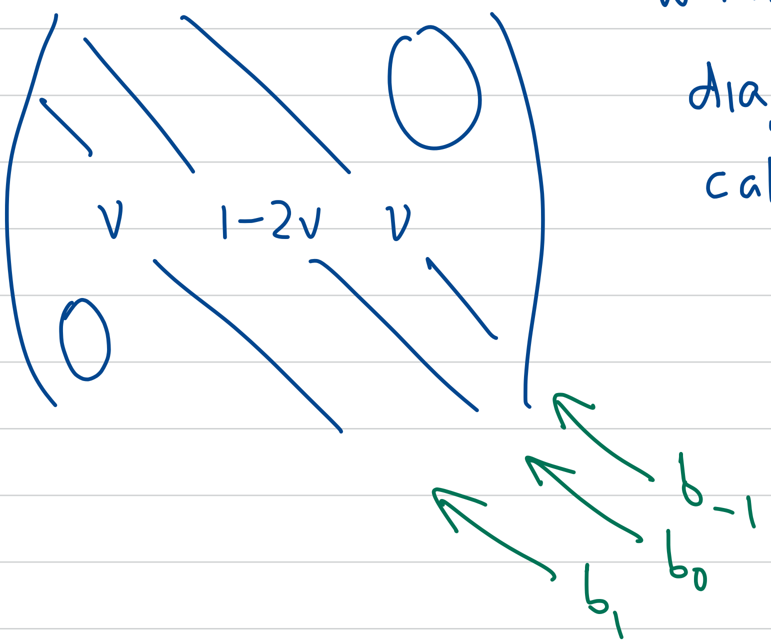
no matter how small k is.

so $\nexists\, K$ s.t. $\|B^n\| \le K$ for $0 \le nk \le T$

$$\|B^n\| \cdot \underbrace{\|u\|}_{1} \ge \|B^n u\| \to \infty$$

The finite difference operator $Bu_j = \nu u_{j-1} + (1-2\nu)u_j + \nu u_{j+1}$ can be thought of as an infinite tri-diagonal matrix with constants on the diagonals. (These are called Toeplitz matrices)

$$B_{ij} = b_{i-j}$$



(all other b_k's are zero)

As with finite matrices, the ∞-norm is the maximum absolute row sum. (We will prove this next time.)
But all the row sums are the same:

$$\|B\|_\infty = |\nu| + |1-2\nu| + |\nu| = \begin{cases} 1 & 0 \le \nu \le \frac{1}{2} \\ 4\nu - 1 & \nu > \frac{1}{2} \\ 1 - 4\nu & \nu < 0 \end{cases}$$

less important since $\nu = \frac{k}{h^2} > 0$

Functional analysis: study of linear algebra in infinite dimensions

A Banach space is a complete normed vector space.

A vector space over a field ($\mathbb{R}$ or $\mathbb{C}$) is a collection of objects that you can add together and multiply by scalars

$$f, g \in V \Rightarrow \underbrace{\alpha f + \beta g}_{\text{scalars (in } \mathbb{R} \text{ or } \mathbb{C})} \in V$$

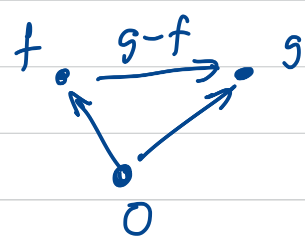
A norm is a mapping $\|\cdot\| : V \to \mathbb{R}$ that satisfies

$\|f+g\| \leq \|f\| + \|g\|$ triangle inequality

$\|\alpha f\| = |\alpha| \, \|f\|$ homogeneity

$\left.\begin{array}{l} \|f\| \geq 0 \\ \|f\| = 0 \text{ iff } u = 0 \end{array}\right\}$ positive definiteness

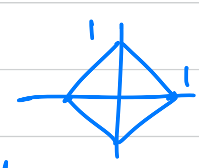
the distance from f to g is simply $\|g-f\|$
(translation invariant)



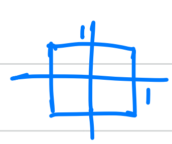
norms on $\mathbb{R}^n$ and $\mathbb{C}^n$:

shape of unit ball

one norm (Manhattan) $\quad \|x\|_1 = \sum_1^n |x_j|$



∞-norm (max) $\quad \|x\|_\infty = \max_j |x_j|$



2-norm (Euclidean) $\quad \|x\|_2 = \sqrt{\sum_1^n |x_j|^2}$

We need norms to talk about errors in our numerical solutions.

Convergence: $\underbrace{f_n \to f}_{\lim_{n\to\infty} f_n = f}$ in V, if $\|f_n - f\| \to 0$ in $\mathbb{R}$.

$$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } \underbrace{\text{if } n \geq N \text{ then } \|f_n - f\| < \varepsilon}_{\text{eventually all the remaining terms in the sequence are within that tolerance.}}$$

↑ tolerance given to you by the customer

A Cauchy sequence $f_1, f_2, \ldots$ is a sequence in which the terms eventually stay arbitrarily close to each other:

$$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } \forall n, m \geq N, \quad \|f_n - f_m\| < \varepsilon$$

Easy to show: every convergent sequence in a normed space is Cauchy.

A space is complete if the converse is also true (every Cauchy sequence converges to an element of the space)

A Hilbert space is a Banach space where the norm comes from an inner product $\|f\| = \sqrt{\langle f, f \rangle}$

In other words, a Hilbert space is a complete inner product space.

examples: $\begin{cases} \mathbb{C}^n, & \langle x, y\rangle = \bar{x}^T y \\ L^2(0,1), & \langle f, g\rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx \end{cases}$

inner product (physics convention):

1. $\langle f, f\rangle > 0$ if $f \neq 0$ (real and positive definite)
2. $\langle g, f\rangle = \overline{\langle f, g\rangle}$ (conjugate symmetry)

3a. $\langle f, \alpha g\rangle = \alpha \langle f, g\rangle$
3b. $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$
(bilinearity)
(or sesquilinearity)

3a implies $\langle f, 0\rangle = 0$, $\|0\| = 0$

2 & 3a imply $\langle \alpha f, g\rangle = \bar{\alpha} \langle f, g\rangle$

2 & 3b imply $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$

Cauchy-Schwarz: $|\langle f, g\rangle| \leq \|f\| \cdot \|g\|$

(needed e.g. to prove that $\|f\| = \sqrt{\langle f, f\rangle}$ is a norm)

interpretation (over $\mathbb{R}$): $\langle f, g\rangle = \|f\| \cdot \|g\| \cos\theta$

over $\mathbb{C}$, it's important to conjugate the first slot ($\begin{pmatrix}1\\ i\end{pmatrix}$ is not orthogonal to itself)

matlab's prime is the conjugate transpose ($x' = \bar{x}^T = x^H$)
also called the Hermitian transpose