Math 228B Lec 1

Numerical solutions of PDEs

Examples of PDEs

hyperbolic, linear

wave equation: $u_{tt} = c^2 \Delta u$

$\Delta = \nabla^2$ = Laplacian $= \partial_x^2 + \partial_y^2 + \partial_z^2$

c = wave speed

subscripts are partial derivatives

$u_t = \dfrac{\partial u}{\partial t}$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, $\partial_x = \dfrac{\partial}{\partial x}$, etc.

transport equation: $u_t + a u_x = 0$

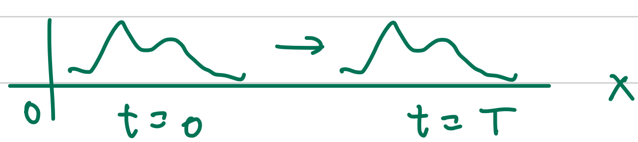
solution is $u(x,t) = u_0(x - at)$

$u_0(x)$ = given initial condition $(x \in \mathbb{R})$

check: $u_t = u_0'(x-at) \cdot (-a)$ chain rule

$u_x = u_0'(x-at) \cdot (1)$

$\Rightarrow u_t + a u_x = 0$ ✓

traveling wave 

initial condition is translated in time without changing its shape hyperbolic, nonlinear

inviscid Burgers' equation: $u_t + uu_x = 0$

the wave speed depends on the height (au_x replaced by uu_x)

solutions develop shocks (discontinuities)

parabolic, linear

heat equation: $u_t = \alpha \Delta u$

α = diffusion coefficient ($\frac{cm^2}{s}$)

flux: $\vec{q} = -\kappa \nabla u$

$\alpha = \frac{\kappa}{c\rho}$

c = specific heat

ρ = density

closely related

Schrödinger equation $i\hbar u_t = \left(-\frac{\hbar^2}{2m}\Delta + V(x,t)\right)u$

$\hbar$ = reduced Planck constant

m = particle mass, V = potential

Elliptic equations

Poisson equation: $-\Delta u = f$

linear elasticity (static): $\mu \Delta \vec{u} + (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) = f$

Stokes equations: $\begin{cases} -\mu \Delta \vec{u} + \nabla p = \vec{f} \\ \nabla\cdot\vec{u} = 0 \end{cases}$

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} \vec{u} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

hyperbolic, linear systems of PDE

elastic vibrations: $\rho u_{tt} = \mu \Delta u + (\lambda+\mu)\nabla(\nabla\cdot u)$

Maxwell equations:

$$\nabla\cdot E = 4\pi\rho$$
$$\nabla\cdot B = 0$$
$$\nabla\times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$
$$\nabla\times B = \frac{1}{c}\left(4\pi J + \frac{\partial E}{\partial t}\right)$$

nonlinear examples

Navier-Stokes: $\rho(u_t + u\cdot\nabla u) = -\nabla p + \mu\Delta u$

$\nabla\cdot u = 0$

μ = viscosity

solutions sometimes behave like those of hyperbolic equations and sometimes like those of elliptic equations.

Eikonal equation: $|\nabla u| = 1$

first arrival time of a signal

viscous Burgers' equation: $u_t + u u_x = \nu u_{xx}$

a 1D variant of Navier-Stokes

KdV: $u_t + u u_x = -\nu^2 u_{xxx}$

solutions form solitons

KdV is dispersive while viscous Burgers' is dissipative.

traffic equation: $\rho_t + (\rho U(\rho))_x = 0$

ρ = density of cars on a highway

$U(\rho)$ = driving speed

solutions form shocks

- Unlike ODEs, there's no general theory of PDEs
- Each type of equation has special features that must be understood and incorporated into a numerical method
- Boundary conditions are often a major challenge for PDEs in complex geometries

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Heat equation, background theory
(next time: finite difference methods)

nondimensional, 1D variant: $u_t = u_{xx}$

to non-dimensionalize: $\hat{u}, \hat{x}, \hat{t}, \alpha$ carry dimensions

$$\hat{u}_{\hat{t}} = \alpha \hat{u}_{\hat{x}\hat{x}}, \quad x = \frac{\pi}{L}\hat{x}, \quad t = \alpha \frac{\pi^2}{L^2}\hat{t}, \quad u = \frac{\hat{u}}{1^\circ C}$$

$$\frac{\partial}{\partial \hat{t}} = \frac{\partial t}{\partial \hat{t}} \frac{\partial}{\partial t} = \alpha \frac{\pi^2}{L^2} \frac{\partial}{\partial t}$$

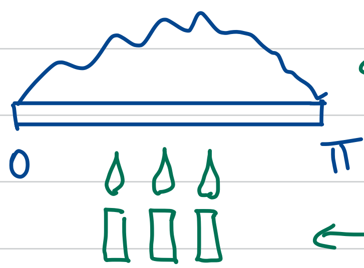
$$\frac{\partial^2}{\partial \hat{x}^2} = \left(\frac{\pi}{L}\frac{\partial}{\partial x}\right)\left(\frac{\pi}{L}\frac{\partial}{\partial x}\right) = \frac{\pi^2}{L^2}\frac{\partial^2}{\partial x^2}$$

setup. 2 options

1. rod of finite length $0 \le x \le L = \pi$

2. infinite domain $-\infty < x < \infty$

case 1.



$\leftarrow$ $u(x,t)$ = temperature

$\leftarrow$ heat source

$$u_t - u_{xx} = f$$

(assume $f = 0$ today)

initial conditions: $u(x,0) = g(x)$

boundary conditions: $u(0) = u(\pi) = 0$

$g(x)$ is the initial temperature distribution

goal: find $u(x,t)$ for $t > 0$, $0 < x < \pi$

separation of variables

represent solution as a superposition of simple solutions of the form $u(x,t) = X(x)T(t)$

$$XT_t = X_{xx}T \qquad \frac{T_t}{T} = \frac{X_{xx}}{X} = \lambda$$

$$X'' = \lambda X \quad , \quad X(0) = X(\pi) = 0$$

$$X(x) = \sin kx \quad , \quad \lambda_k = -k^2 \quad , \quad k = 1, 2, 3, \ldots$$

$$T(t) = e^{-k^2 t}$$

superposition: use a Fourier series to represent the initial condition

$$g(x) = \sum_{k=1}^{\infty} c_k \sin kx \quad , \quad c_k = \frac{2}{\pi} \int_0^{\pi} g(x) \sin kx \, dx$$

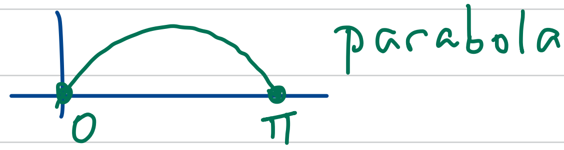
now evolve each component of $g(x)$ independently:

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-k^2 t} \sin kx \qquad \text{(exact solution)}$$

$g(x)$ is recovered as $t \to 0^+$ since

$$e^{-k^2 t} \to 1 \quad \text{as} \quad t \to 0$$

If you do the same thing for the <u>backward</u> heat equation $u_t = -u_{xx}$ the Fourier modes grow exponentially in time (rather than decaying) example: $g(x) = x(\pi - x)$



$$c_k = \frac{2}{\pi}\int_0^{\pi} g(x)\sin kx\,dx = \begin{cases} 0 & k \text{ even} \\ \frac{8}{\pi k^3} & k \text{ odd} \end{cases}$$

$$u_t = -u_{xx} \quad \Rightarrow \quad u(x,t) = \sum_{k \text{ odd}} \frac{8}{\pi k^3} e^{k^2 t} \sin kt$$

this formula for $u(x,t)$ diverges for any $t>0$

$\frac{8}{\pi k^3} e^{k^2 t} \to \infty$ as $k \to \infty$ no matter how small t is. ($t>0$ is fixed.)

$\therefore$ There's no solution with this seemingly nice initial condition!

The backward heat equation is ill-posed

well-posed means solutions exist, are unique, and depend continuously on the "data", i.e., on the initial condition $g(x)$.

case 2. $u_t = u_{xx}$, $x \in \mathbb{R}$ (real line)

the boundary conditions $u(0)=0$, $u(\pi)=0$ are replaced with the requirement that $u(x,t)$ is bounded as $x \to \infty$ or $x \to -\infty$ for each $t>0$. ($|u(x,t)| \le M$ for $t \ge 0$, $x \in \mathbb{R}$)

This problem can be solved using the Fourier transform

exact solution: $$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} g(\xi)\, d\xi$$

see Fritz John, Partial Differential Equations
or Walter Strauss, Partial Differential Equations

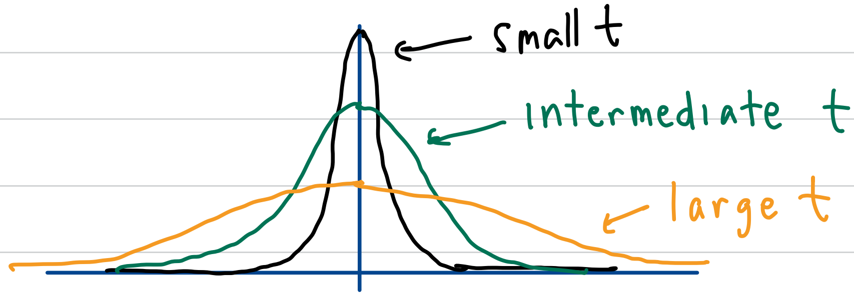
initial condition: $\lim_{t \to 0^+} u(x,t) = g(x)$

requirements on g:

$g(x)$ should be continuous and bounded

($\exists M < \infty$ s.t. $|g(x)| \le M$ for $x \in \mathbb{R}$)

$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is a Gaussian at each fixed time $t>0$.

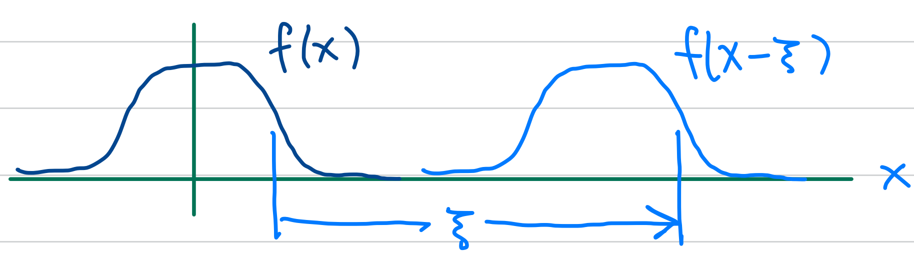


For all $t>0$, $\int_{-\infty}^{\infty} G(x,t)\,dx = 1$

Convolution: $f*g(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$

It gives a superposition of translations of $f(x)$ with weight $g(\xi)$

($f(x-\xi)$ translates $f(x)$ to the right by ξ)



Observations:

1. The exact solution is a smoothed out version of the initial condition. (Smoothed out by convolution with a smooth function)

$$u(\cdot, t) = g * G(\cdot, t) \qquad t > 0 \text{ fixed}$$

(g and $G(\cdot, t)$ are functions of $x \in \mathbb{R}$)

2. The value of $u(x,t)$ at x depends on all of $g(\xi)$ for every $t > 0$. (Information travels infinitely fast, though the contribution from $g(\xi)$ decays super-exponentially as $|x - \xi|$ grows.)

Next time: numerics.