Math 228B Lec 6

plan: von Neumann stability analysis (theory)
Z-transform, diagonalizing finite difference operators

goal: general method to compute $\|B\|_{2,h}$, $\|B^n\|_{2,h}$
for a finite difference operator $B: \ell^2_h \to \ell^2_h$

main tool: Z transform

$$\ell^2(\mathbb{Z}) \underset{Z^{-1}}{\overset{Z}{\rightleftarrows}} L^2(-\pi,\pi)$$

$$u \longmapsto \hat{u}(\xi) = \sum_{j=-\infty}^{\infty} u_j e^{-ij\xi}$$

$$f_j^\vee = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi) e^{ij\xi}\, d\xi \longleftarrow f$$

The $f_j^\vee$ are Fourier series coefficients of $f(\xi)$
(up to the sign convention using $e^{ij\xi}$ here instead of $e^{-ij\xi}$)

Parseval's theorem: $\displaystyle\sum_j |u_j|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{u}(\xi)|^2\, d\xi$

$$\|u\|_{\ell^2} = \frac{1}{\sqrt{2\pi}} \|\hat{u}\|_{L^2}$$

$(\mathcal{B}_h = \ell^2_h)$

$$\|u\|_{\ell^2_h} = \sqrt{h\sum_j |u_j|^2} = \sqrt{h}\,\|u\|_{\ell^2} = \sqrt{\frac{h}{2\pi}}\,\|\hat{u}\|_{L^2}$$

$$\|u\|_{\ell^2_h} = \left\| \sqrt{\frac{h}{2\pi}}\, Z u \right\|_{L^2} \qquad \text{for all } u \in \ell^2_h$$

so $\sqrt{\frac{h}{2\pi}}\, Z$ is unitary from ℓ^2_h to $L^2(-\pi,\pi)$

(unitary means it is onto and preserves lengths.
Its adjoint is its inverse: $U^{-1} = U^*$)

Now consider the shift operator $Su_j = u_{j+1}$ on ℓ^2_h

matrix rep. of S:
$$\begin{array}{cc} & j \rightarrow \\ i\downarrow & \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ & 0 & & 0 & 1 \\ & & & & \ddots \end{pmatrix} \end{array} \qquad S_{ij} = \begin{cases} 1 & j = i+1 \\ 0 & \text{o.w.} \end{cases}$$

first superdiagonal $(j = i+1)$
main diagonal $(i = j)$

$$S \underbrace{\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}}_{Su} \qquad \downarrow j$$

each component shifted up (or left)

The inverse of S shifts down (i.e., right)

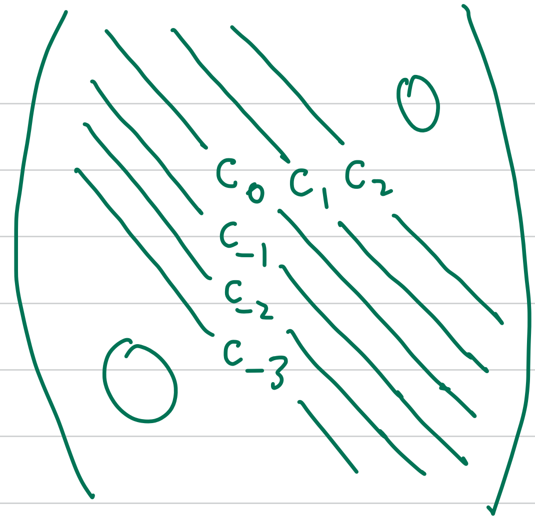
$$S^{-1}u_j = (S^{-1}u)_j = u_{j-1}$$

A general finite difference operator (also called a banded Toeplitz matrix) has the form

$$B = \sum_{m=m_1}^{m_2} c_m S^m = \sum_{m=-\infty}^{\infty} c_m S^m \qquad B_{ij} = c_{j-i}$$

$c_m = 0$ if $m < m_1$ or $m > m_2$

picture for $m_1 = -3$, $m_2 = 2$:



examples:

$$D_x^+ u = \frac{S - I}{h} \qquad (S^0 = I) \quad \left(c_1 = \frac{1}{h},\ c_0 = -\frac{1}{h}\right)$$

and our FTCS scheme for the heat equation is

$$u^{n+1} = Bu^n, \qquad B = \nu S^{-1} + (1-2\nu) S^0 + \nu S^1$$

Note: In numerical linear algebra, the index convention for Toeplitz matrices is often reversed. $B_{ij} = c_{i-j}$

$$c_m \leftrightarrow c_{-m}, \qquad Bu_j = \sum_m c_{-m} u_{j+m} = \sum_\ell u_{j-\ell} c_\ell$$

This has the advantage of giving the discrete convolution of u and c, $Bu = c * u = u * c$.

Theorem: The Z-transform diagonalizes any finite difference operator $B = \sum_m c_m S^m$.

proof: We will show that $ZB = GZ$ where G is a multiplication operator, $Gf(\xi) = (Gf)(\xi) = G(\xi) f(\xi)$

So $B = Z^{-1} G Z$ (analogous to $A = Q \Lambda Q^{-1}$)

orthogonal matrix

diagonal matrix

$$\begin{tikzcd} \ell^2_h \arrow[r, "B"] \arrow[d, "Z"'] & \ell^2_h \arrow[d, "Z"] \\ L^2(-\pi,\pi) \arrow[r, "G"'] & L^2(-\pi,\pi) \end{tikzcd}$$

(commutative diagram)

$$(ZBu)(\xi) = \widehat{Bu}(\xi) = \sum_j Bu_j e^{-ij\xi}$$

$$= \sum_j \left(\sum_m c_m u_{j+m} \right) e^{-ij\xi} \qquad \begin{matrix} -\infty < j < \infty \\ m_1 \le m \le m_2 \end{matrix}$$

$$= \sum_m \sum_j c_m u_{j+m} e^{-ij\xi} \qquad \begin{matrix} l = j+m \\ j = l-m \end{matrix}$$

$$= \sum_m \sum_l c_m u_l e^{-i(l-m)\xi}$$

$$= \left(\sum_m c_m e^{im\xi} \right) \sum_l u_l e^{-il\xi} = G(\xi)\,\hat{u}(\xi)$$

$$= (GZu)(\xi)$$

so $ZB = GZ$ as claimed.

$G(\xi)$ is known as the amplification factor of B:

$$Bu_j = \sum_m c_m u_{j+m} \longrightarrow G(\xi) = \sum_m c_m e^{im\xi}$$

Fix $\xi \in (-\pi, \pi]$ and let w be the sequence with components $w_j = e^{ij\xi}$

Then Bw is the sequence

$$Bw_j = \sum_m c_m w_{j+m} = \sum_m c_m e^{i(j+m)\xi} = \underbrace{\left(\sum_m c_m e^{im\xi} \right)}_{G(\xi)} \underbrace{e^{ij\xi}}_{w_j}$$

so $Bw = \underbrace{G(\xi)}_{\text{eigenvalue}} \, w$ eigenvector

This is true in ℓ^∞ ($w \in \ell^\infty$ is an eigenvector of B)

but not in ℓ_h^2 since w is not normalizable

(it's not square summable : $h \sum_j |w_j|^2 = \infty$, so $w \notin \ell_h^2$)

The operator B acting on ℓ_h^2 doesn't have any eigenvalues since none of the candidate eigenvectors are in the space. Instead, B has a <u>continuous</u> spectrum

$$\underbrace{\sigma(B)}_{\substack{\text{spectrum} \\ \text{of } B}} = \underbrace{G([-\pi, \pi])}_{\text{range of } G(\xi)}$$

<u>def:</u> $\sigma(B) = \{ \lambda \in \mathbb{C} : B - \lambda I \text{ is not invertible} \}$

The discrete spectrum consists of eigenvalues ($B - \lambda I$ is not injective.) The continuous spectrum consists of $\lambda \in \mathbb{C}$

for which $B - \lambda I$ is injective and has dense range, but is not surjective. $(B-\lambda I)^{-1}$ is an unbounded operator in this case.

What matters to us in practice is that

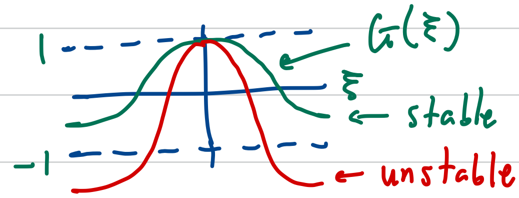
$$B = Z^{-1} G Z \quad , \quad B^2 = Z^{-1} G \underbrace{Z Z^{-1}}_{I} G Z = Z^{-1} G^2 Z$$

$$B^n = Z^{-1} G^n Z$$

We will see next time that $\quad \|B^n\|_{\ell_h^2} = \|G^n\|_{L^2(-\pi,\pi)}$

and $\quad \|G^n\|_{L^2(-\pi,\pi)} = \|G^n\|_\infty = \max\limits_{-\pi \le \xi \le \pi} |G(\xi)|^n$

Lax-Richtmyer: $\quad \|B(k)^n u\| \le K \quad$ for $\quad \begin{array}{l} 0 < k < \varepsilon \\ 0 \le kn \le T \end{array}$



unstable if $G(\xi) > 0$ or $G(\xi) < -1$ ⟶
for any $\xi \in [-\pi, \pi]$