CSE 394 Circuits and Signals

Discrete Time Fourier Transform

&

DFT

By

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Tutorials

Question: Find the DTFT of the following sequence

$$x[n] = \{1, 2, 1\}$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

Answer:

The DTFT of a signal $x[n]$ is given as

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Here the input x[n] is defined for n = 0, n = 1 and n = 2. So

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{2} x[n]e^{-j\omega n}$$

Tutorials

Question: Find the DTFT of the following sequence

$$x[n] = \{1, 2, 1\}$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

Answer:

$$X(e^{j\omega}) = \sum_{n=0}^{2} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = x[0] + x[1]e^{-j\omega} + x[2]e^{-j2\omega}$$

$$X(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$X(e^{j\omega}) = e^{-j\omega}\left[e^{j\omega} + 2 + e^{-j\omega}\right]$$

Using

$$e^{j\omega} + e^{-j\omega} = 2.\cos(\omega)$$

$$X(e^{j\omega}) = e^{-j\omega}[2 + 2.\cos(\omega)]$$

$$X(e^{j\omega}) = e^{-j\omega}2[1 + \cos(\omega)]$$

Tutorials

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$$x[n] = \{1, 2, 1\}$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

<u>Answer:</u>

$$X\left(e^{j\omega}\right) = e^{-j\omega} 2[1 + \cos(\omega)]$$

The magnitude response will be

$$\left|X\left(e^{j\omega}\right)\right| = \left|e^{-j\omega} 2[1 + \cos(\omega)]\right|$$

$$\left|X\left(e^{j\omega}\right)\right| = |2[1 + \cos(\omega)]|$$

The phase response will be

$$\phi = \angle\left|e^{-j\omega} 2[1 + \cos(\omega)]\right|$$

$$\phi = -\omega$$

Tutorials

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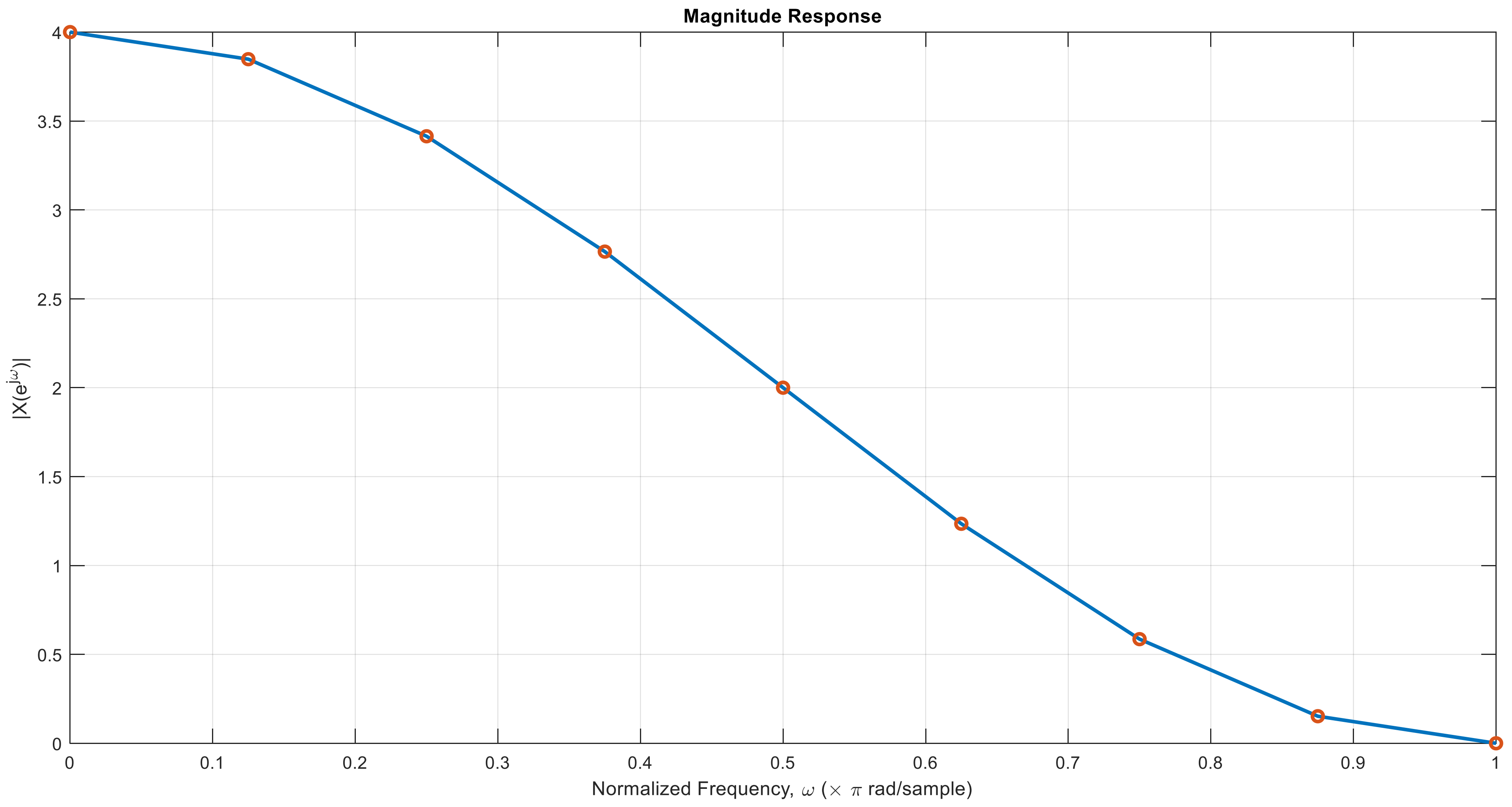
Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

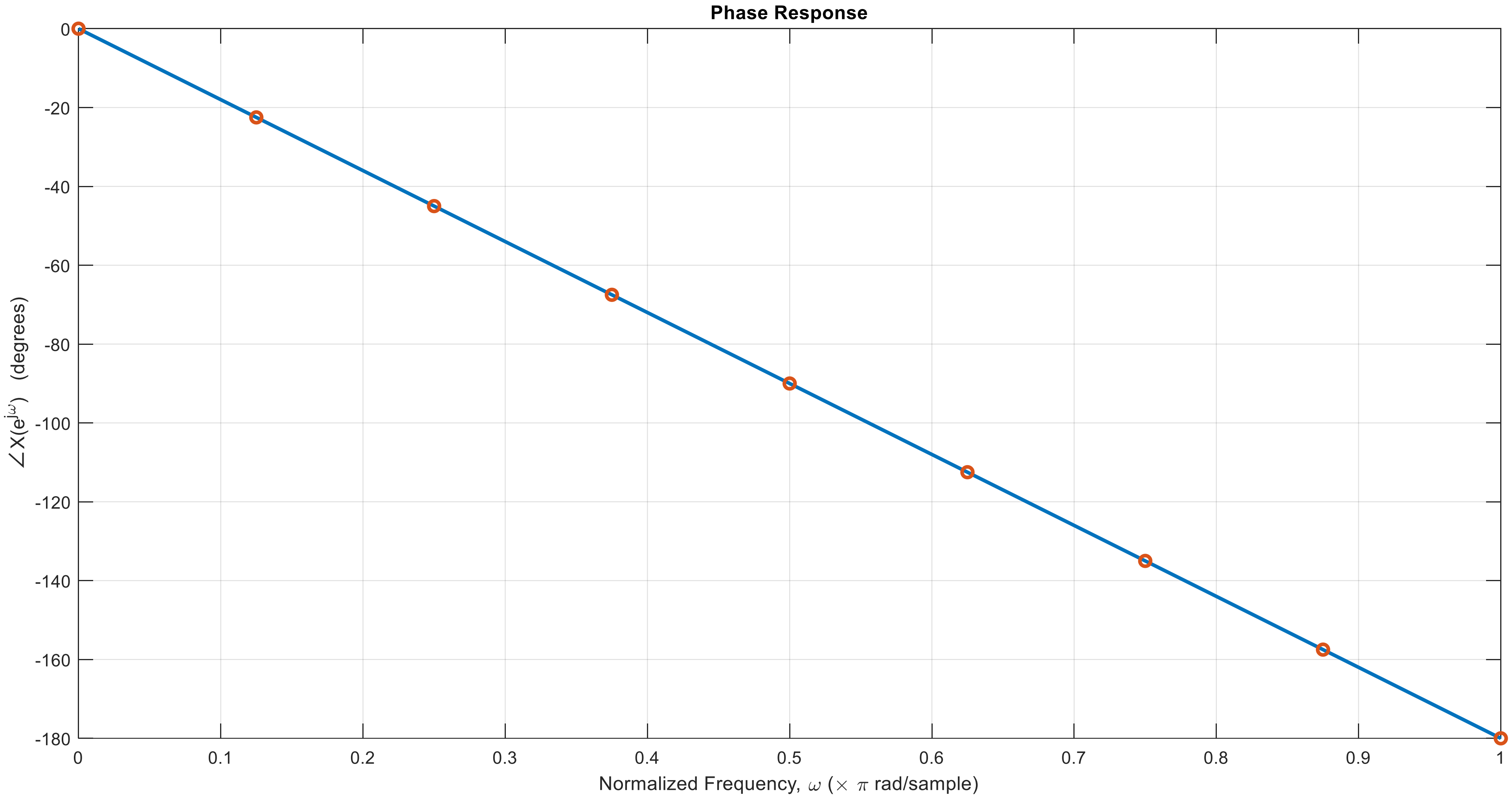
Answer:

$$\left|X\left(e^{j\omega}\right)\right| = |2[1 + \cos(\omega)]|$$

$$\phi = -\omega$$



Magnitude Response
$|X(e^{j\omega})|$
Normalized Frequency, ω ($\times$ π rad/sample)
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Phase Response
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
∠X(e^{jω}) (degrees)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency, ω (× π rad/sample)

Tutorials

Question: Find the DTFT of the following sequence

$$x[n] = 0.5^n u[n]$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

Answer:

The DTFT of a signal $x[n]$ is given as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Tutorials

Question: Find the DTFT of the following sequence

$$x[n] = 0.5^n u[n]$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

<u>Answer:</u>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (0.5)^n u[n] e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

Tutorials

Question: Find the DTFT of the following sequence

$$x[n] = 0.5^n u[n]$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

Answer:

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - 0.5\cos(\omega) + j0.5\sin(\omega)}$$

The magnitude response will be

$$|X(e^{j\omega})| = \left|\frac{1}{1 - 0.5\cos(\omega) + j0.5\sin(\omega)}\right|$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1 - 0.5\cos(\omega))^2 + (0.5\sin(\omega))^2}}$$

Tutorials

Question: Find the DTFT of the following sequence

$$x[n] = 0.5^n u[n]$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

Answer:

$$\left|X\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{(1 - 0.5\cos(\omega))^2 + (0.5\sin(\omega))^2}}$$

$$\left|X\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1 - \cos(\omega) + 0.25\cos^2(\omega) + 0.25\sin^2(\omega)}}$$

$$\left|X\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1.25 - \cos(\omega)}}$$

Tutorials

Question: Find the DTFT of the following sequence

$$x[n] = 0.5^n u[n]$$

Find the magnitude and phase response. Plot the magnitude and phase response taking points between 0 and π in steps of $\frac{\pi}{8}$

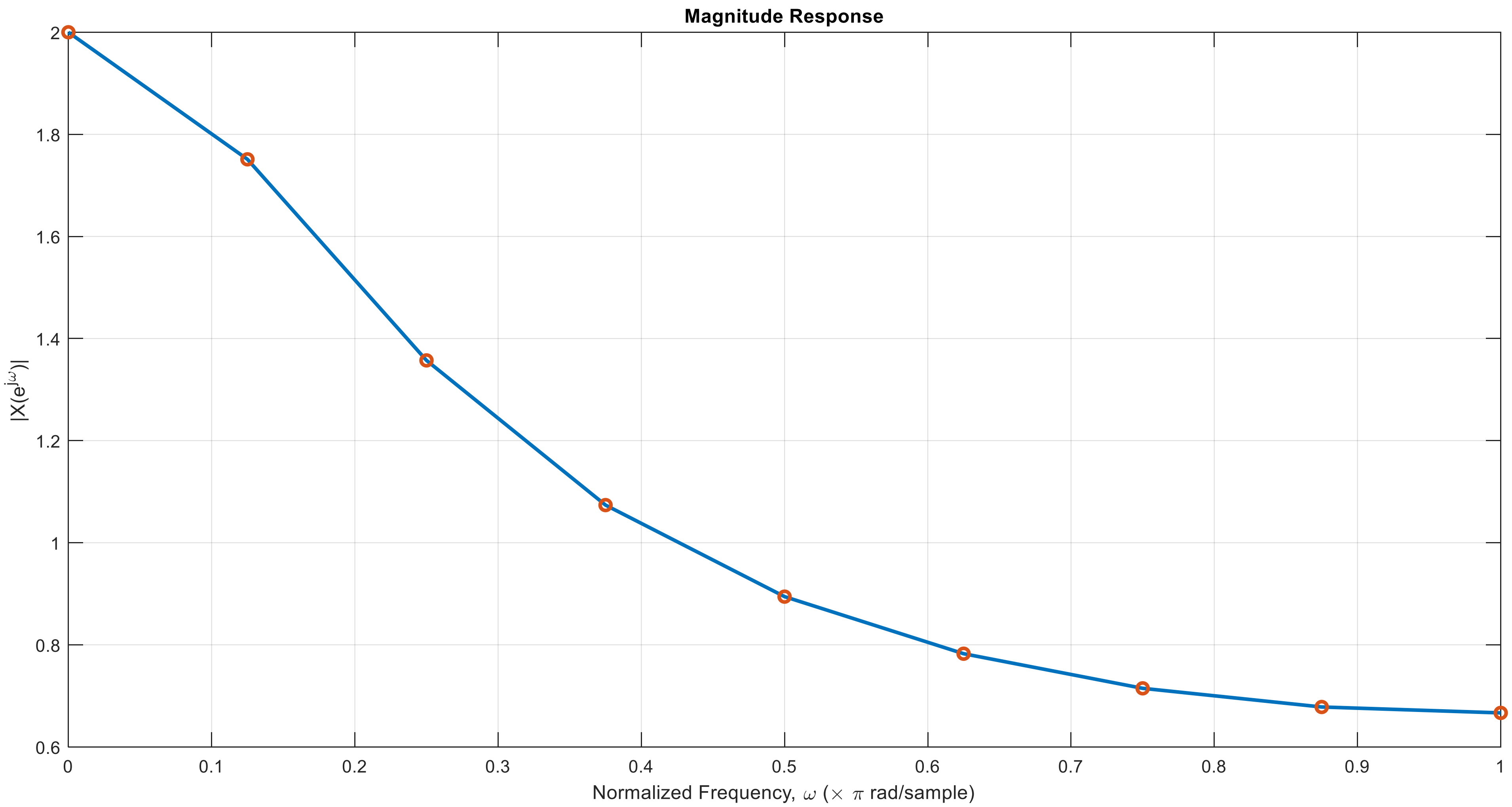
Answer:

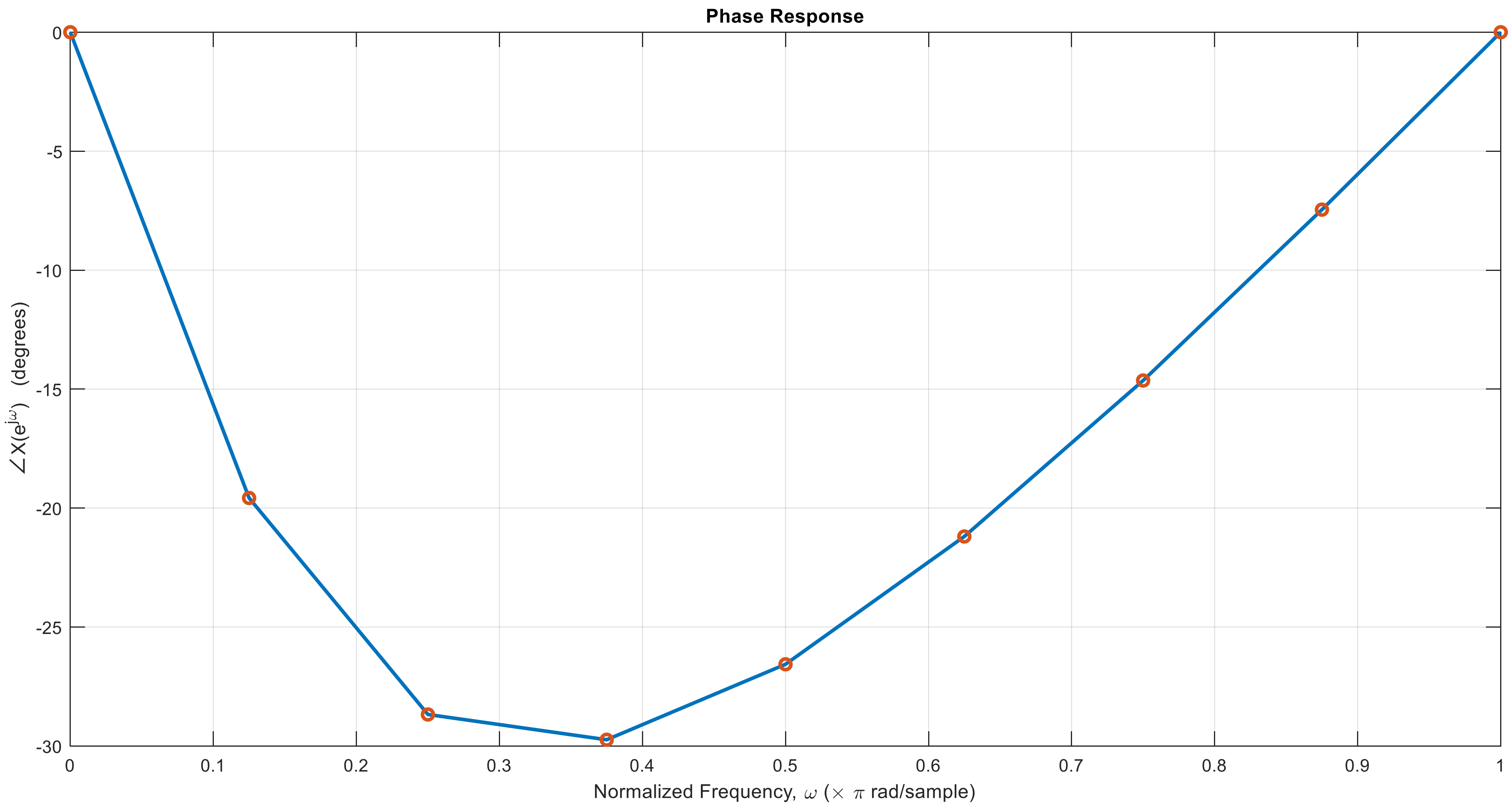
$$X(e^{j\omega}) = \frac{1}{1 - 0.5\cos(\omega) + j0.5\sin(\omega)}$$

The phase response will be

$$\phi = \angle \frac{1}{1 - 0.5\cos(\omega) + j0.5\sin(\omega)}$$

$$\phi = -\tan^{-1}\left(\frac{0.5\sin(\omega)}{1 - 0.5\cos(\omega)}\right)$$


Magnitude Response
$|X(e^{j\omega})|$
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency, ω ($\times$ π rad/sample)


Phase Response
0
-5
-10
-15
-20
-25
-30
∠X(e^{jω}) (degrees)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency, ω (× π rad/sample)

Tutorials

Question: Find the DTFT of the following sequence

$$y[n] = (0.5)^n u[n] * (0.1)^n u[n]$$

Answer:

$$Y\left(e^{j\omega}\right) = DTFT\ \{(0.5)^n u[n] * (0.1)^n u[n]\}$$

Using convolution property

$$Y\left(e^{j\omega}\right) = DTFT\ \{(0.5)^n u[n]\} DTFT\ \{(0.1)^n u[n]\}$$

$$Y\left(e^{j\omega}\right) = \frac{1}{(1 - 0.5e^{-j\omega})(1 - 0.1e^{-j\omega})}$$

Discrete Fourier Transform

Discrete Fourier Transform is the sampled version of DTFT.

What is the period of DTFT?

It is 2π

So we will take N samples in the one period and we get the N point DFT

The DTFT of x[n] is given as

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

If we sample we get

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, 1, 2, 3, \ldots \ldots ..(N-1)$$

Discrete Fourier Transform

So the N point DFT is given as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, 1, 2, 3, \dots \dots .. (N-1)$$

And N point IDFT is given as

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}, \qquad n = 0, 1, 2, 3, \dots \dots .. (N-1)$$

Discrete Fourier Transform

So the N point DFT is given as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, 1, 2, 3, \ldots \ldots . . (N-1)$$

And N point IDFT is given as

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad n = 0, 1, 2, 3, \ldots \ldots . . (N-1)$$

Where

$$W_N = e^{-j\frac{2\pi}{N}}$$

Is called the twiddle factor

Discrete Fourier Transform

Find the 2 point DFT of

$$x[n] = \{1, 2\}$$

Answer

$$N = 2$$

The N point DFT is given as

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \qquad k = 0, 1, 2, 3, \dots \dots ..(N-1)$$

The 2 point DFT will then be

$$X[k] = \sum_{n=0}^{1} x[n]W_N^{kn} \qquad k = 0, 1$$

Twiddle Factor

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_2 = e^{-j\frac{2\pi}{2}} = e^{-j\pi} = \cos(\pi) - j.\sin(\pi) = -1$$

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

$$X[0] = \sum_{n=0}^{2-1} x[n]W_N^{0n}$$

$$X[0] = \sum_{n=0}^{1} x[n] = x[0] + x[1] = 1 + 2 = 3$$

$$X[1] = \sum_{n=0}^{2-1} x[n]W_2^{1n}$$

$$X[1] = x[0] + x[1]W_2 = 1 - 2 = -1$$

$$X[k] = \{3, -1\}$$

Matrix Form of 4 point DFT

The N point DFT is given as

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \qquad k = 0, 1, 2, 3, \ldots\ldots..(N-1)$$

For 4 point DFT, N = 4. So k = 0, 1, 2, 3

$$X[k] = \sum_{n=0}^{4-1} x[n]W_4^{kn}$$

$$X[k] = \sum_{n=0}^{3} x[n]W_4^{kn}$$

$$X[0] = \sum_{n=0}^{3} x[n] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = \sum_{n=0}^{3} x[n]W_4^{n} = x[0] + x[1]W_4^1 + x[2]W_4^2 + x[3]W_4^3$$

$$X[2] = \sum_{n=0}^{3} x[n]W_4^{2n} = x[0] + x[1]W_4^2 + x[2]W_4^4 + x[3]W_4^6$$

$$X[3] = \sum_{n=0}^{3} x[n]W_4^{3n} = x[0] + x[1]W_4^3 + x[2]W_4^6 + x[3]W_4^9$$

Matrix Form of 4 point DFT

$$X[0] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = x[0] + x[1]W_4^1 + x[2]W_4^2 + x[3]W_4^3$$

$$X[2] = x[0] + x[1]W_4^2 + x[2]W_4^4 + x[3]W_4^6$$

$$X[3] = x[0] + x[1]W_4^3 + x[2]W_4^6 + x[3]W_4^9$$

This can be written in matrix form as

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Matrix Form of 4 point IDFT

$$x[n] = \frac{1}{4}\sum_{k=0}^{4} X[k]W_N^{-kn}, \qquad n = 0, 1, 2, 3$$

$$x[0] = \frac{1}{4}\left[X[0] + X[1] + X[2] + X[3]\right]$$

$$x[1] = \frac{1}{4}\left[X[0] + X[1]W_4^1 + X[2]W_4^2 + X[3]W_4^3\right]$$

$$x[2] = \frac{1}{4}\left[X[0] + X[1]W_4^2 + X[2]W_4^4 + X[3]W_4^6\right]$$

$$x[3] = \frac{1}{4}\left[X[0] + X[1]W_4^3 + X[2]W_4^6 + X[3]W_4^9\right]$$

This can be written in matrix form as

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ 1 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ 1 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

Discrete Fourier Transform

Find the 4 point DFT of

$$x[n] = \{1, 2, 3, 5\}$$

Answer

$$N = 4$$

The N point DFT is given as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, 2, 3, \ldots \ldots . . (N-1)$$

The 4 point DFT will then be

$$X[k] = \sum_{n=0}^{3} x[n] W_N^{kn} \qquad k = 0, 1,2,3$$

Twiddle Factor

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-\frac{j\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j. \sin\left(\frac{\pi}{2}\right) = -j$$

$$W_4^2 = (-j)^2 = -1$$

$$W_4^3 = j$$

$$W_4^4 = 1$$

$$W_4^6 = -1$$

$$W_4^9 = -j$$

Discrete Fourier Transform

Find the 4 point DFT of

$$x[n] = \{1, 2, 3, 5\}$$

Answer

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$$

$$X[0] = 1 + 2 + 3 + 5 = 11$$

$$X[1] = 1 - j2 - 3 + j5 = -2 + j3$$

$$X[2] = 1 - 2 + 3 - 5 = -3$$

$$X[3] = 1 + j2 - 3 - j5 = -2 - j3$$

So the 4 point DFT is

$$X[k] = \{11, -2 + j3, -3, -2 - j3\}$$

Discrete Fourier Transform

Find the 4 point IDFT of

$$X[k] = \{11, -2 + j3, -3, -2 - j3\}$$

Answer

$$N = 4$$

The N point IDFT is given as

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad n = 0, 1, 2, 3, \ldots \ldots . . (N-1)$$

The 4 point IDFT will then be

$$x[0] = \frac{1}{4} \sum_{k=0}^{3} x[n] W_N^{kn} \quad n = 0, 1,2,3$$

Twiddle Factor

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_4^{-1} = e^{j\frac{2\pi}{4}} = e^{\frac{j\pi}{2}} = j$$

$$W_4^{-2} = (j)^2 = -1$$

$$W_4^{-3} = -j$$

$$W_4^{-4} = 1$$

$$W_4^{-6} = -1$$

$$W_4^{-9} = j$$

Discrete Fourier Transform

Find the 4 point IDFT of

$$X[k] = \{11, -2 + j3, -3, -2 - j3\}$$

Answer

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 11 \\ -2+j3 \\ -3 \\ -2-j3 \end{bmatrix}$$

$$x[0] = \frac{1}{4}[11 - 2 + j3 - 3 - 2 - j3] = 1$$

$$x[1] = \frac{1}{4}[11 + j(-2 + j3) + 3 - j(-2 - j3)]$$

$$x[1] = \frac{1}{4}[11 - j2 - 3 + 3 + j2 - 3] = \frac{1}{4}[8] = 2$$

$$x[2] = \frac{1}{4}[11 + 2 - j3 - 3 + 2 + j3] = 3$$

$$x[3] = \frac{1}{4}[11 - j(-2 + j3) + 3 + j(-2 - j3)]$$

$$x[3] = \frac{1}{4}[11 + j2 + 3 + 3 - j2 + 3] = \frac{1}{4}[20] = 5$$

So the 4 point IDFT is

$$x[n] = \{1, \quad 2, \quad 3, \quad 5\}$$

Thank You