CSE 394 Circuits and Signals

Discrete Time Fourier Transform

By

Tony Aby Varkey M

Assistant Professor

Dept. of ECE

Presidency University

Properties of Discrete Time Fourier Transform

Time Reversal Property

Statement

If

$$x[n] \leftrightarrow X(e^{j\omega})$$

Then

$$y[n] = x[-n] \leftrightarrow Y(e^{j\omega}) = X(e^{-j\omega})$$

Proof

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}$$

Put

$$p = -n$$

$$X(e^{j\omega}) = \sum_{p=\infty}^{-\infty} x[p]e^{j\omega p}$$

$$Y(e^{j\omega}) = \sum_{p=-\infty}^{\infty} x[p]e^{j\omega n}$$

$$Y(e^{j\omega}) = X(e^{-j\omega})$$

Properties of Discrete Time Fourier Transform

Convolution Property

Statement

If

$$x[n] \leftrightarrow X(e^{j\omega}) \quad \text{and} \quad h[n] \leftrightarrow H(e^{j\omega})$$

Then

$$y[n] = x[n] * h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Proof

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] * h[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k]\, e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]\, e^{-j\omega n}$$

Substitute

$$p = n - k$$

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] \sum_{p=-\infty}^{\infty} h[p]\, e^{-j\omega(p+k)}$$

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] \sum_{p=-\infty}^{\infty} h[p]\, e^{-j\omega p} e^{-j\omega k}$$

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \sum_{p=-\infty}^{\infty} h[p]\, e^{-j\omega p}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Properties of Discrete Time Fourier Transform

Linearity Property

If

$$x_1[n] \leftrightarrow X_1(e^{j\omega}) \quad \text{and} \quad x_2[n] \leftrightarrow X_2(e^{j\omega})$$

Then

$$c_1x_1[n] + c_2x_2[n] \leftrightarrow c_1X_1(e^{j\omega}) + c_2X_2(e^{j\omega})$$

Time Shifting Property

If

$$x[n] \leftrightarrow X(e^{j\omega})$$

Then

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

Frequency Shifting Property

If

$$x[n] \leftrightarrow X(e^{j\omega})$$

Then

$$e^{j\beta n}x[n] \leftrightarrow X(e^{j(\omega-\beta)})$$

Time Reversal Property

If

$$x[n] \leftrightarrow X(e^{j\omega})$$

Then

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

Convolution Property

If

$$x[n] \leftrightarrow X(e^{j\omega}) \quad \text{and} \quad h[n] \leftrightarrow H(e^{j\omega})$$

Then

$$x[n] * h[n] \leftrightarrow X(e^{j\omega})H(e^{j\omega})$$

Tutorials

Question: Using the properties of DTFT find the DTFT of

$$x_1[n] = e^{-j3n}(0.25)^{n+2}u[n-2]$$

Answer:

The signal can be represented as

$$x_1[n] = e^{-j3n}(0.25)^{n-2}(0.25)^4u[n-2]$$

$$x_1[n] = (0.25)^4e^{-j3n}(0.25)^{n-2}u[n-2]$$

This can be written as

$$x_1[n] = (0.25)^4x_2[n]$$

Where

$$x_2[n] = e^{-j3n}(0.25)^{n-2}u[n-2]$$

$$x_2[n] = e^{-j3n}x_3[n]$$

Where

$$x_3[n] = (0.25)^{n-2}u[n-2]$$

$$x_3[n] = x_4[n-2]$$

Where

$$x_4[n] = (0.25)^nu[n]$$

The DTFT of $x_4[n]$

$$X_4(e^{j\omega}) = DTFT\{(0.25)^nu[n]\}$$

$$X_4(e^{j\omega}) = \frac{1}{1-0.25e^{-j\omega}}$$

The DTFT of $x_3[n]$

$$X_3(e^{j\omega}) = DTFT\{x_4[n-2]\}$$

$$X_3(e^{j\omega}) = e^{-j2\omega}X_4(e^{j\omega})$$

$$X_3(e^{j\omega}) = \frac{e^{-j2\omega}}{1-0.25e^{-j\omega}}$$

The DTFT of $x_2[n]$

$$X_2(e^{j\omega}) = DTFT\left\{e^{-j3n}x_3[n]\right\}$$

$$X_2(e^{j\omega}) = X_3(e^{j(\omega-3)})$$

$$X_2(e^{j\omega}) = \frac{e^{-j2(\omega-3)}}{1-0.25e^{-j(\omega-3)}}$$

The DTFT of $x_1[n]$

$$X_1(e^{j\omega}) = DTFT\{(0.25)^4x_2[n]\}$$

$$X_1(e^{j\omega}) = (0.25)^4X_2(e^{j\omega})$$

$$X_1(e^{j\omega}) = \frac{(0.25)^4e^{-j2(\omega-3)}}{1-0.25e^{-j(\omega-3)}}$$

Tutorials

Question: Show that DTFT is periodic with period 2π

Answer:

The DTFT is given as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

What is $X\left(e^{j(\omega+2\pi)}\right)$

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n}$$

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\, e^{-j2\pi n}$$

What is $e^{-j2\pi n}$

$$e^{-j2\pi n} = \cos(2\pi n) - j.\sin(2\pi n)$$

$$e^{-j2\pi n} = 1$$

So

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X\left(e^{j(\omega+2\pi)}\right) = X(e^{j\omega})$$

Tutorials

Question: Using the properties of DTFT find the DTFT of

$$x[n] = 2.e^{-j\pi n}(0.5)^{n-3}u[n+4]$$

Answer:

$$x[n] = 2.e^{-j\pi n}(0.5)^{n+4}(0.5)^{-7}u[n+4]$$

$$x[n] = 2.(0.5)^{-7}e^{-j\pi n}(0.5)^{n+4}u[n+4]$$

$$x[n] = 2.(0.5)^{-7}x_1[n]$$

Where

$$x_1[n] = e^{-j\pi n}(0.5)^{n+4}u[n+4]$$

$$x_1[n] = e^{-j\pi n}x_2[n]$$

Where

$$x_2[n] = (0.5)^{n+4}u[n+4]$$

$$x_2[n] = x_3[n+4]$$

Where

$$x_3[n] = (0.5)^n u[n]$$

$$X_3(e^{j\omega}) = DTFT\{x_3[n]\}$$

$$X_3(e^{j\omega}) = DTFT\{(0.5)^n u[n]\}$$

$$DTFT\{(\alpha)^n u[n]\} = \frac{1}{1-\alpha e^{-j\omega}}$$

$$X_3(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}}$$

$$X_2(e^{j\omega}) = DTFT\{x_2[n]\}$$

$$X_2(e^{j\omega}) = DTFT\{x_3[n+4]\}$$

$$DTFT\{x[n-n_0]\} = e^{-j\omega n_0}X(e^{j\omega})$$

$$X_2(e^{j\omega}) = e^{j\omega 4}X_3(e^{j\omega}) = \frac{e^{j\omega 4}}{1-0.5e^{-j\omega}}$$

$$X_1(e^{j\omega}) = DTFT\{x_1[n]\}$$

$$X_1(e^{j\omega}) = DTFT\{e^{-j\pi n}x_2[n]\}$$

$$DTFT\{e^{j\beta n}x[n]\} = X\left(e^{j(\omega-\beta)}\right)$$

$$X_1(e^{j\omega}) = X_2\left(e^{j(\omega+\pi)}\right) = \frac{e^{j(\omega+\pi)4}}{1-0.5e^{-j(\omega+\pi)}}$$

$$X(e^{j\omega}) = DTFT\{x[n]\}$$

$$X(e^{j\omega}) = DTFT\{2.(0.5)^{-7}x_1[n]\}$$

$$X(e^{j\omega}) = 2.(0.5)^{-7}DTFT\{x_1[n]\}$$

$$X(e^{j\omega}) = 2.(0.5)^{-7}X_1(e^{j\omega})$$

$$X\left(e^{j\omega}\right) = 2.(0.5)^{-7}\left(\frac{e^{j(\omega+\pi)4}}{1-0.5e^{-j(\omega+\pi)}}\right)$$

Thank You