Experiment 14: Jumping Beans

1 Jumping Beans

A simple toy called a jumping bean can be constructed by putting a metal ball inside of a pill capsule. Placed on an inclined surface at a certain inclination, the jumping bean will tumble down in a rather surprising way, seemingly standing up-right, flipping end to end, instead of rolling. Investigate its motion. Find the dimensions of the fastest and slowest beans for a given inclination.

2 Introduction

We will need to explain the motion of the jumping bean's movement first. The haphazard motion it undergoes can be explained in 3 stages.

Initially, when the cylinder is rested against the slope, the ball within starts to roll down, under the influence of gravity, till it reaches the end of the cylinder. Due to the conservation of energy and friction, some of the ball's total kinetic energy is transferred to the cylinder, allowing the cylinder to rise. Once the cylinder fully rotates, it falls back down onto the slope due to gravity too and the process repeats.

The following objectives to solve this problem are as follow: to build a theoretical model of the process, find the dimensions necessary for the fastest and slowest beans and carry out experiments to ensure our theoretical predictions matches our experimental data.

3 The Theoretical Model

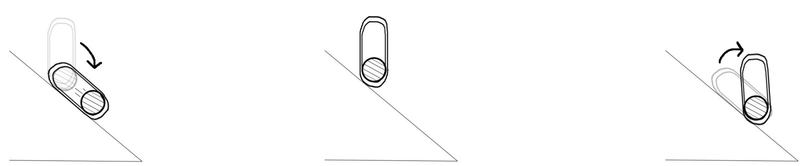

Figure 1: Stages of movement

The basis for our theoretical model is the conservation of energy.

As you can see in Fig.1, the toy moves in 3 stages. Starting from the stage in which the cylinder is rested against the slope, the ball within the toy begins to roll down due to gravity till it reaches the end of the cylinder.

The initial gravitational potential energy (mgh) of the ball is converted to the total kinetic energy of the ball and its final gravitational potential energy (mgh'). The total kinetic energy of the ball is the sum of its translational kinetic energy ($\frac{1}{2}mv^2$) of its center of mass and its rotational kinetic energy about its center of mass ($\frac{1}{2}Iw^2$).

Therefore,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}Iw^2 + mgh' \tag{1}$$

where

$$I = \frac{2}{5}mR^2 \tag{2}$$

and

$$w = \frac{v}{R} \tag{3}$$

Substituting (2) and (3) into (1):

$$mgh = \tfrac{1}{2}(mv^2 + \tfrac{2}{5}mR^2w^2) + mgh'$$

$$2g(h - h') = (1 + \tfrac{2}{5})v^2$$

$$v = \sqrt{\tfrac{2g(h-h')}{1+\frac{2}{5}}}$$

Since the change in height, (h-h') can be expressed as:

$$(h - h') = (d - 2r)sin(\alpha)$$

$$v = \sqrt{\frac{2g(d - 2r)sin(\alpha)}{1 + \frac{2}{5}}} \tag{4}$$

where α is the angle of inclination, d is the length of the cylinder and r is the radius of the ball.

Using kinematics, with initial speed, $u = 0$, acceleration, $a = gsin(\alpha)$, the time taken for the ball to travel down the cylinder is

$$v_f = u + at_1$$

$$t_1 = \tfrac{v_f}{a}$$

$$t_1 = \frac{\sqrt{\frac{2g(d-2r)sin(\alpha)}{1+\frac{2}{5}}}}{gsin(\alpha)} \tag{5}$$

Proceeding on, the next stage of the movement of the jumping bean is the rotation of the cylinder. Considering friction acting between the metal ball and the cylinder, rotating the cylinder causes the ball to slow down. Using the following three equations:

$$F = \mu N$$

$$\tau = I\alpha'$$

$$\omega = \omega_0 + \alpha' t \tag{6}$$

where F is the frictional force, $\mu = 0.6$ is the coefficient of friction, N is the normal contact force, τ is the torque, α' is the angular acceleration and ω is the angular velocity.

Therefore,

$$\begin{aligned}\alpha' &= \frac{\mu N r}{\frac{2}{5}mr^2} \\ &= \frac{\mu N}{\frac{2}{5}mr} \\ &= \frac{\mu mgcos(\alpha)}{\frac{2}{5}mr} \\ &= \frac{\mu gcos(\alpha)}{\frac{2}{5}r}\end{aligned}$$

Substituting the above into (6), we obtain

$$\omega = \frac{\mu gcos(\alpha)}{cr} t + \omega_0 \tag{7}$$

Integrating (7) by dt from θ_0 to θ will get us:

$$\theta - \theta_0 = \frac{1}{2}\frac{\mu gcos(\alpha)}{cr} t_2^2 + \omega_0 t_2$$

$$t_2 = \frac{cr\omega_0}{\mu gcos(\alpha)} + \sqrt{(\frac{cr\omega_0}{\mu gcos(\alpha)})^2 + \frac{2\pi cr}{\mu gcos(\alpha)}} \tag{8}$$

where we let $\beta = \frac{cr\omega_0}{\mu gcos(\alpha)}$.

Thus,

$$t_2 = \beta w_0 + \sqrt{(\beta w_0)^2 + 2\pi\beta} \tag{9}$$

From here, we can obtain the total time taken, t_n for the "jumping bean" to travel down the length of the slope, l, from the general equation:

$$t_n = ceil(\tfrac{l}{d})t_1 + floor(\tfrac{l}{d})t_2$$

$$t_n = ceil(\frac{l}{d})(\frac{\sqrt{\frac{2g(d-2r)sin(\alpha)}{1+\frac{2}{5}}}}{gsin(\alpha)}) + floor(\frac{l}{d})(\frac{cr\omega_0}{\mu gcos(\alpha)} + \sqrt{(\frac{cr\omega_0}{\mu gcos(\alpha)})^2 + \frac{2\pi cr}{\mu gcos(\alpha)}}) \quad (10)$$

where the *ceiling* and *floor* functions represent the number of times the ball has to travel down the cylinder and the number of times the cylinder has to rotate respectively.

It should be mentioned that although ω increases for each subsequent stage in the movement based on the equation

$$w_{x+1} = \frac{1}{\beta}t_2 + \omega_x \quad (11)$$

we have chosen to keep ω constant so as to simplify the general equation (10). Additionally, our equation assumes the ball only begins to roll once the cylinder is resting against the slope.

4 Experimental setup to measure the time

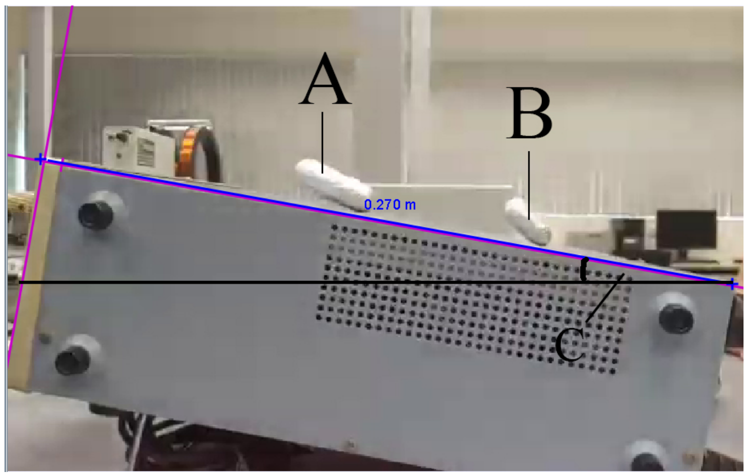

Figure 2: Experimental set-up example. A - Jumping bean of length A; B - Jumping bean of length B; C - Angle of inclination

Referring to Fig.2, the experimental set-up is as follows. Jumping beans, of various lengths, are made of an aluminium cylindrical casing and a metal ball of radius 1.35cm inside it. Individual beans are then placed in an upright position at the top-edge of a smooth slope of length 32.5cm and inclination angle of 0.24 radians. The beans are then released and allowed to travel down the slope.

All measurements of distance and length were obtained using a plastic ruler of resolution 0.05cm.

The motion of the beans were measured by a digital camera (60 FPS) and the time they took to travel down the slope was obtained from a video tracking software. The timing starts when the cylinder first falls and rests against the slope and ends when the jumping bean flips over the edge.

Throughout the experiment, the lengths of the beans were varied from 10.40cm to 20.60cm

The following data were gathered and then compared against the expected timing for each individual lengths.

5 Discussion of results

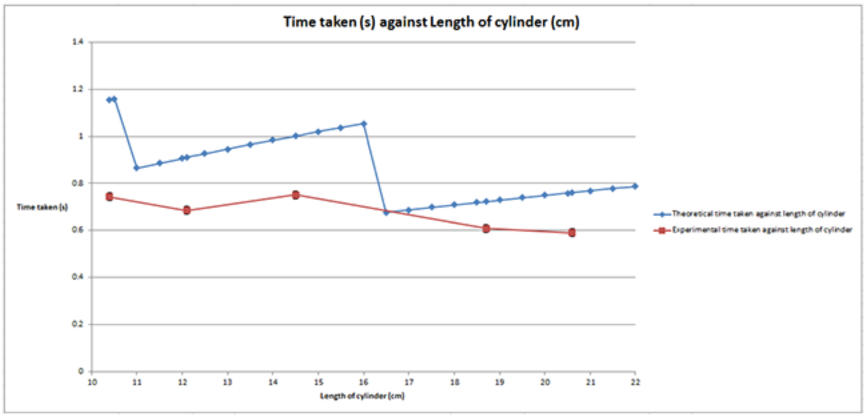

Figure 3: Graph showing the time taken for a jumping bean to travel down a slope and its length.

Length of cylinder, d (cm)	Experimental time taken, t (s)	Theoretical time taken (s)	Percentage discrepancy (%)
10.40	0.743	1.154	35.61547541
12.10	0.686	0.911	24.67386566
14.50	0.752	1.002	24.92703341
18.70	0.609	0.723	15.80579746
20.60	0.591	0.761	22.33878357

Figure 4: Table showing the percentage discrepancy between the experimental and theoretical timings.

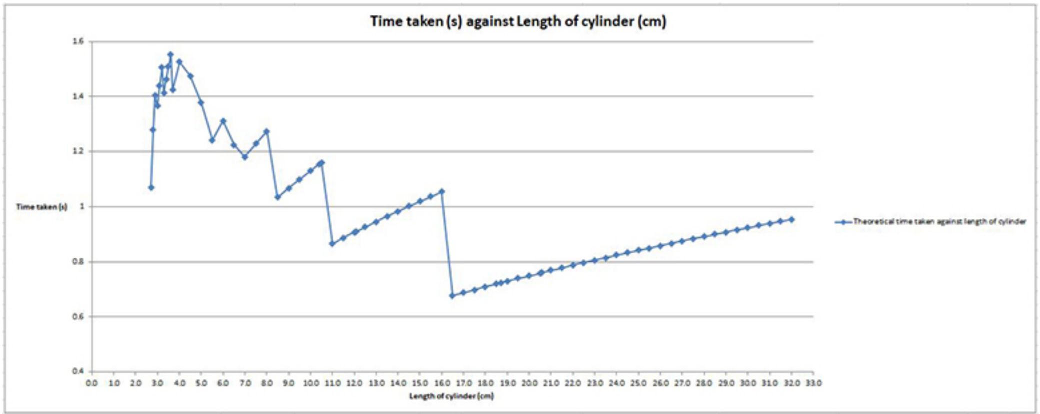

Figure 5: Graph showing the theoretical time taken for a jumping bean to travel down a slope and its length.

As seen from above in Fig.3 and Fig.4, the experimental timing we have obtained is faster than we theorized. However, this was expected due to the fact that we have kept ω constant to keep equation (10) simple. Due to the slower ω, this causes the theoretical time to be larger than it should be.

Furthermore, during the experiments, the metal ball within may have already started rolling before the cylinder has rested against the slope. This causes t_1 to be shorter than expected.

Thus, these two causes are the main contributors to the discrepancies obtained.

Despite so, our experimental data still follows a general trend similar to that of the theoretical one.

Note that the standard uncertainty for t_n is 0.017s

We have also observed that curiously, the jumping beans are independent of the mass of the metal balls used inside of them. We were able to run a small experiment involving two similar jumping beans with metal balls of the same radius but different masses and were able to confirm it as so.

In theory, as seen from Fig.5, the fastest jumping bean occurs when the length of the cylinder is exactly half of the distance it needs to travel. Meanwhile, the slowest jumping occurs when the length of the cylinder is about $\frac{1}{9}$ that of the total distance it needs to travel. Naturally, the larger the metal ball used, the faster the jumping beans are, as per our equation.

6 Conclusion

To summarize, in this study, we were able to prove the relationship between the length of the cylinder of a jumping bean and the time it takes to travel down a slope. Initially, as the length of cylinder is close to the radius of the ball, the jumping bean can be compared similarly to that of a ball rolling down a slope. The time required to travel continues to increase sharply till the length of the cylinder reaches $\frac{1}{9}th$ that of the slope. At this point, the time required is the highest at 1.552s. After that, the time required decreases generally, with a cylinder length half of that of the slope giving us the fastest jumping bean. At this point, it gives us a time of about 0.677s.

7 References

1. Charles Kittel (1973), Mechanics (Berkley Physics Course Volume 1). ISBN-13: 978-0070048805

2. Hugh D. Young, Roger A. Freedman, (2015) University Physics with Modern Physics (14th Edition). ISBN-13: 978-0133977981

3. Coefficient of friction between iron and aluminium. `http://www-eng.lbl.gov/~ajdemell/coefficients_of_friction.html`

8 Links on Youtube

1. https://www.youtube.com/watch?v=Azb9bDktZVcfeature=youtu.be

2. `https://www.youtube.com/watch?v=_395DY8V7PI&t=327s`

3. https://www.youtube.com/watch?v=06mpikpl0$_I$