Regular Characters of Classical Groups

Forth International Workshop on Zeta Functions in Algebra and Geometry

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This work is part of the authors Ph.D thesis, under the supervision of Uri Onn.

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Definition

The **level** of a continuous irreducible representation ρ of G is the minimal integer ℓ such that $\rho(G^{\ell+1}) = \{\mathbf{1}\}$. The level of a character χ is the level of the associated representation.

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Consider $G = \mathrm{GL}_N(\mathfrak{o})$.
The congruence subgroups of G are $G^m = 1 + \mathfrak{p}^m \mathrm{M}_N(\mathfrak{o})$.
Given a character $\chi \in \mathrm{Irr}(G)$, of level $\ell > 0$, identify χ with a character of $G/G^{\ell+1}$ and consider the its restriction to the group $G^\ell/G^{\ell+1} \simeq \mathrm{M}_n(\mathbb{F}_q)$.

- By Clifford theory, the constituents of $\chi_{|G^\ell/G^{\ell+1}}$ comprise a complete orbit Ω in the Pontryagin dual of $\mathrm{M}_N(\mathbb{F}_q)$ for the coadjoint action of $\mathrm{GL}_N(\mathbb{F}_q)$, i.e.

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- By non-degeneracy of

$$(X,Y)\mapsto \mathrm{Tr}(XY) : \mathrm{M}_N(\mathbb{F}_q)\times \mathrm{M}_N(\mathbb{F}_q)\to \mathbb{F}_q,$$

the orbit Θ is associated uniquely with a similarity class $\Theta\subseteq \mathrm{M}_N(\mathbb{F}_q)$.

$$\chi \in \mathrm{Irr}(G) \text{ of level } \ell > 0$$
$$\downarrow$$
$$\Omega \in \mathrm{Ad}^*(\mathrm{GL}_N(\mathbb{F}_q))\backslash\backslash\widehat{\mathrm{M}_N(\mathbb{F}_q)}$$
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Definition (Regular characters of $\mathrm{GL}_N(\mathfrak{o})$)

A character $\chi \in \mathrm{Irr}(\mathrm{GL}_N(\mathfrak{o}))$ is called **regular** if the associated similarity class Θ consists of regular (cyclic) matrices.

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A character $\chi \in \mathrm{Irr}(G)$ is said to be regular if the associated orbit Θ consists of regular elements of $\mathfrak{g}(\mathbb{F}_q)$.

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    1. Independent development. Proof of structural properties via implementation of algebraic geometry over local rings of finite length (viz. the *Greenberg functor*).
    2. Construction of several classes of regular characters of odd degree (*cuspidal* and *split-regular* characters).

Recent Results

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    1. Classification and enumeration of all regular characters of $\mathrm{GL}_N(\mathfrak{o})$ and $\mathrm{U}_N(\mathfrak{o})$, for $\mathfrak{o}$ of odd residual characteristic.
    2. Computation of regular representation zeta functions.
- Independently, Stasinski and Stevens [Stasinski and Stevens, 2016] completed the construction of all regular characters of $\mathrm{GL}_N(\mathfrak{o})$, without restrictions on the residual characteristic of $\mathfrak{o}$.

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- **Methods.**
  1. An extension of Hill's methods to the study of regular orbits in a classical Lie-algebra over $\mathfrak{o}$.
  2. Classification of regular conjugacy classes in classical Lie-algebras.
- **Results.**
  1. A uniform formula for the regular representation zeta function of $\mathbf{G}(\mathfrak{o})$.
  2. Computation of the regular zeta function of $\mathbf{G}(\mathfrak{o})$.

Regular Zeta Functions of Classical Groups

The regular representation zeta function of $G = \mathbf{G}(\mathfrak{o})$ is defined by the Dirichlet generating function

$$\zeta_G^{\text{reg.}}(s) = \sum_{\chi \in \mathrm{Irr}^{\text{reg.}}(G)} \chi(1)^{-s} \qquad (s \in \mathbb{C}).$$

Theorem

Assume $\mathrm{char}(\mathbb{F}_q) \neq 2$*, and let* $\mathbf{G}$ *be a classical group of type* B_n*,* C_n *or* D_n *over* $\mathfrak{o}$*, with* $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$.

Theorem

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$$\zeta^{\mathrm{reg.}}_{\mathbf{G}(\mathfrak{o})}(s) = \frac{\mathcal{D}_{\mathfrak{g}(\mathfrak{o})}(s)}{1 - q^{n-\alpha s}}$$

where $\alpha = \frac{\dim \mathbf{G} - n}{2}$ *and* $n = \mathrm{rk}\mathbf{G}$*.*

Theorem (Cont.)

Furthermore, there exists polynomials $f_1, \ldots, f_r, g_1, \ldots, g_r \in \mathbb{Z}[t]$*, independent of* $\mathfrak{o}$*, such that*

$$\mathcal{D}_{\mathfrak{g}(\mathfrak{o})}(s) = \sum f_i(q) g_i(q)^{-s}.$$

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- Analysis of the locus of regular elements in $\mathfrak{g}(\mathfrak{o}/\mathfrak{p}^m)$, for $m \geq 1$.

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$\therefore$ **Theorem**

- Classification of regular adjoint orbits in $\mathfrak{g}(\mathbb{F}_q)$, and computation of the regular zeta function.

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$$\eta_{r,1}\left(\mathbf{C}_{\mathrm{M}_N(\mathfrak{o}/\mathfrak{p}^r)}(x_r)\right) = \mathbf{C}_{\mathrm{M}_N(\mathbb{F}_q)}(x_1)$$

In different terminology,

> *For x as above, the Lie algebra shadow of x coincides with the Lie-algebra of $\mathbb{F}_q$-points of the algebraic group obtained from $\mathbf{C}_{\mathrm{M}_N(\mathfrak{o})}(x)$ by reduction modulo $\mathfrak{p}$.*

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- For a general reductive group - ???

Examples

$$\zeta^{\mathrm{reg.}}_{\mathrm{SL}_2(\mathfrak{o})}(s) = \frac{4q\left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \left(\frac{q-1}{2}\right)^2 (q^2+q)^{-s}}{1-q^{1-s}}, \quad \text{(Jaikin)}.$$

$$\zeta^{\mathrm{reg.}}_{\mathrm{SL}_1(D)}(s) = \sum_{\lambda=0}^{\ell-2} \iota^2 (q-1) \frac{q^{\lambda} \cdot \left(q^{-\lambda\frac{(\ell-1)}{2}} \cdot \frac{q^{\ell}-1}{\iota(q-1)}\right)^{-s}}{1-q^{(\ell-1)-\binom{\ell}{2}s}}, \quad \text{(Sh)}$$

($\deg D$ is prime$\neq p$ and $\iota = \#\mu_\ell(\mathbb{F}_q)$).

Remark

In both cases above $\zeta_{\mathbf{G}(\mathfrak{o})}(s) = \zeta_{\mathbf{G}(\mathbb{F}_q)}(s) + \zeta^{\mathrm{reg.}}_{\mathbf{G}(\mathfrak{o})}(s)$.

For $\mathbf{G}_+ = \mathrm{GL}_n$ and $\mathbf{G}_- = \mathrm{U}_n$

$$\zeta^{\text{reg.}}_{\mathbf{G}_\epsilon(\mathfrak{o})}(s) = \frac{1}{1 - q^{n-\binom{n}{2}s}} \cdot \sum_{\tau \in \mathcal{A}_n} u^\tau_\epsilon(q) \prod_{i=1}^{n} \binom{\sum_j \tau_{i,j}}{\tau_{i,1}, \tau_{i,2}, \ldots, \tau_{i,n}} \binom{w_d(q)}{\sum_j \tau_{i,j}} \left(\frac{\left|\mathbf{G}_\epsilon(\mathbb{F}_q)\right|}{u^\tau_\epsilon(q)} \right)^{-s},$$

(KOS),

where

- $\mathcal{A}_n$ is the set of n-typical matrices,
- $w_d(q)$ is the number of irreducible polynomials of degree d over $\mathbb{F}_q$,
- $u^\tau_\epsilon(q)$ is the cardinality of the centralizer in $\mathbf{G}_\epsilon(\mathbb{F}_q)$ of a regular element associated to an n-typical matrix $\tau \in \mathcal{A}_n$.

Regular Zeta Functions

Let $\mathcal{X}_n$ consist of triplets $(r,A,B) \in \mathbb{Z}_{\geq 0} \times \mathrm{M}_n(\mathbb{Z}_{\geq 0}) \times \mathrm{M}_n(\mathbb{Z}_{\geq 0})$ with $r + \sum_{d,e}(A_{d,e} + B_{d,e}) = n$, and $\mathcal{X}_n^0$ the subset of elements with $r = 0$.

$$\mathfrak{D}_{\mathfrak{sp}_{2n}(\mathfrak{o})} =$$

$$q^n \sum_{(0,A,B)\in\mathcal{X}_n^0} M_{(0,A,B)} \prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}}$$

$$\left(q^{2n^2} \frac{\prod_{i=1}^n (1-q^{-2i})}{\prod_{d,e}(1+q^{-d})^{A_{d,e}}(1-q^{-d})^{B_{d,e}}} \right)^{-s}$$

$$+ \; 4q^n \sum_{(r,A,B)\in\mathcal{X}_n \setminus \mathcal{X}_n^0} M_{(r,A,B)} \prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}}$$

$$\left(q^{2n^2} \frac{\prod_{i=1}^n (1-q^{-2i})}{2 \cdot \prod_{d,e}(1+q^{-d})^{A_{d,e}}(1-q^{-d})^{B_{d,e}}} \right)^{-s}.$$

$$\mathfrak{D}_{\mathfrak{so}_{2n+1}(\mathfrak{o})} = q^n \sum_{(r,A,B)\in\mathcal{X}_n} M_{(r,A,B)} \prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}} \left(q^{2n^2} \frac{\prod_{i=1}^{n}(1-q^{-2i})}{\prod_{d,e}(1+q^{-d})^{A_{d,e}}(1-q^{-d})^{B_{d,e}}} \right)^{-s}.$$

Let $\mathcal{X}_n^0$ denote the set of triplets $\mathbf{t} = (r, A, B) \in \mathcal{X}_n$ with $r = 0$, and let $\mathcal{X}_n^{0,+}$ denote the the subset of $\mathcal{X}_n^0$ consisting of element $(0, A, B)$ such that $\sum_{d,e} eA_{d,e}$ is even and $\mathcal{X}_n^{0,-} = \mathcal{X}_n^0 \setminus \mathcal{X}_n^{0,+}$.

$$\begin{aligned}
\mathfrak{D}_{\mathfrak{so}_{2n}^{+}(\mathfrak{o})} = & \\
& q^n \sum_{(0,A,B)\in\mathcal{X}_n^{0,+}} M_{(0,A,B)} \prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}} \\
& \quad \left(q^{2n^2-n} \frac{(1-q^{-n}) \prod_{i=1}^{n-1} (1-q^{-2i})}{\prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}}} \right)^{-s} \\
& + 2q^n \sum_{(r,A,B)\in\mathcal{X}_n \setminus \mathcal{X}_n^0} M_{(r,A,B)} \prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}} \\
& \qquad \left(q^{2n^2-n} \frac{(1-q^{-n}) \prod_{i=1}^{n-1} (1-q^{-2i})}{\prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}}} \right)^{-s},
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{D}_{\mathfrak{so}_{2n}^{-}(\mathfrak{o})} = \\
& q^{n} \sum_{(0,A,B)\in\mathcal{X}_n^{0,-}} M_{(0,A,B)} \prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}} \\
& \quad \left(q^{2n^2-n} \frac{(1+q^{-n})\prod_{i=1}^{n-1}(1-q^{-2i})}{\prod_{d,e}(1+q^{-d})^{A_{d,e}}(1-q^{-d})^{B_{d,e}}} \right)^{-s} \\
& + 2q^{n} \sum_{(r,A,B)\in\mathcal{X}_n\setminus\mathcal{X}_n^{0}} M_{(r,A,B)} \prod_{d,e} (1+q^{-d})^{A_{d,e}} (1-q^{-d})^{B_{d,e}} \\
& \quad \left(q^{2n^2-n} \frac{(1+q^{-n})\prod_{i=1}^{n-1}(1-q^{-2i})}{\prod_{d,e}(1+q^{-d})^{A_{d,e}}(1-q^{-d})^{B_{d,e}}} \right)^{-s} . \qquad (0.a)
\end{aligned}
$$

Questions?

Thank you!

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