Algebraic Geometry 2
Tutorial session 7

Lecturer: Rami Aizenbud
TA: Shai Shechter

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Recollection – gluing two schemes along open subsets

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1. $|X| = (|X_1| \sqcup |X_2|)/\sim$ where $\sim$ is the equivalence relation generated by $\{(x, \varphi(x)) : x \in U_1\}$. $V \subseteq |X|$ is open iff $V \cap |X_1|$ and $V \cap |X_2|$ are both open.

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2. Given $V \subseteq X$ open write $V_i = V \cap |X_i|$ $(i = 1, 2)$ and set

$$\mathcal{O}_X(V)$$
$$= \left\{(s_1, s_2) \in \mathcal{O}_{X_1}(V_1) \times \mathcal{O}_{X_2}(V_2) : s_1 \mid_{V_1 \cap U_1} = \varphi^\sharp(s_2 \mid_{V_2 \cap U_2})\right\}.$$

Remark

We slightly abused notation in the previous slide, by identifying $|X_i|$ with a subspace of $|X|$. This is possible, because both are naturally embedded in $|X|$.
In Hartshorne, Example 2.3.5, the author is more righteous, and uses explicit maps $i_j : X_j \to X$.

An example – $\mathbb{A}^2_k \setminus \{\text{origin}\}$

Exercise

Let k be a field, and, as usual $\mathbb{A}^2_k = \operatorname{Spec}(k[x, y])$. Show that $\mathbb{A}^2_k \setminus \{(x, y)\}$ is not affine.

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Here (x, y) is the maximal ideal corresponding to the closed point of the origin.

Solution.

Put $Z = \mathbb{A}^2_k \setminus \{(x, y)\}$.

1. $Z = D(x) \cup D(y)$, a union of principal open sets, with $D(x) \cap D(y) = D(xy)$.

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   - Thus $\Gamma(Z, \mathcal{O}_Z) = k[x, y]$ and, if Z is affine, then $Z = \mathbb{A}^2_k$.

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    - Thus $\Gamma(Z, \mathcal{O}_Z) = k[x, y]$ and, if Z is affine, then $Z = \mathbb{A}^2_k$.
3. However, (x) and (y) intersect non-trivially as elements of $\mathbb{A}^2_k$, and not of Z. Thus Z is not affine.

$\square$

An aside

Corollary

Any regular function on $\mathbb{A}_k^2 \setminus \{(x, y)\}$ can be extended uniquely to a regular function on $\mathbb{A}_k^2$.

This is, in fact, true without the assumption that k is a field.

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2. We can also take φ to be the isomorphism induced from $t \mapsto t^{-1}$ on U_1. The resulting space is denoted $\mathbb{P}^1_A$.

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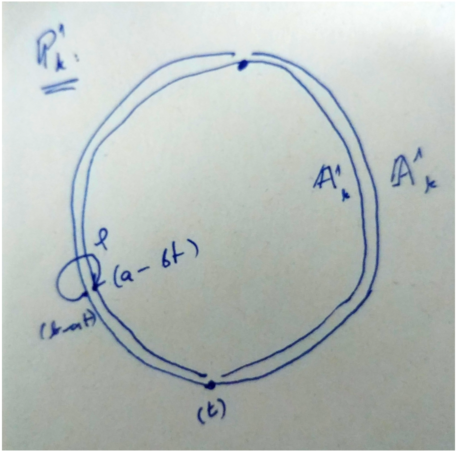
The space $\mathbb{P}^1_k$ can be thus identified with the space of pairs $[a : b]$ with $a, b \in k$ not both zero, upto multiplication of both a and b by a constant $c \in k^\times$. The identification is such that, if $a \neq 0$ then $[a : b]$ is mapped to the ideal $(bt - a)$ on one affine line, and if $b \neq 0$, it is mapped to the ideal $(at - b)$ on the second affine line.



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Since both f_1 and f_2 are polynomials, this is only possible if both are constant, thus $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \simeq k$. If $\mathbb{P}^1_k$ were affine, it would have had only one point, which is false. $\square$

Gluing contd

Lemma (The gluing lemma, HS Ex 2.12)

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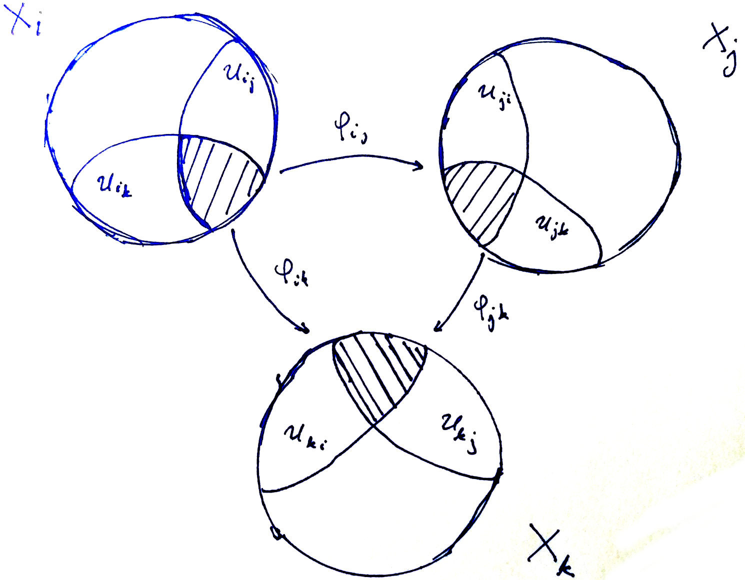
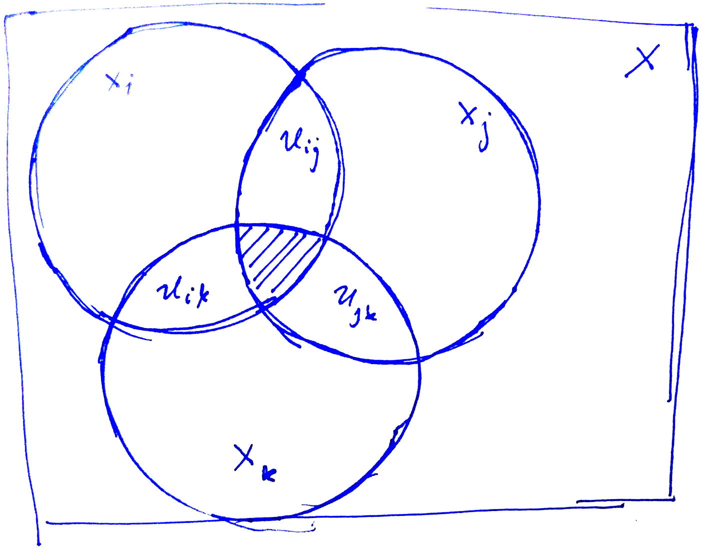
Gluing input



Gluing output



Remark

The data $(X_i, U_{i,j}, \varphi_{i,j})$ is often referred to as the **descent data** of the gluing.

Proof of the gluing lemma

1. **Topological space**. Let $\widetilde{X} = \bigsqcup |X_i|$, and let $\sim$ be the relation defined on $\widetilde{X}$ by

$$x \sim y \iff x = y \text{ or } \exists i, j \text{ st } x \in U_{i,j} \subseteq X_i \text{ and } y = \varphi_{i,j}(x).$$

The assumptions imply that $\sim$ is an equivalence relation. Let $|X| = \widetilde{X}/\sim$ with the quotient topology. Let $\psi_i : |X_i| \to |X|$ be the natural inclusions.

2. **Structure sheaf** Given $V \subseteq |X|$ open, put $V_i = V \cap |X_i|$ and define

$$\mathcal{O}_X(V) = \left\{ (s_i)_i \in \prod_i \mathcal{O}_{X_i}(V_i) : s_i \,|_{(V_i \cap V_j) \cap U_{i,j}} = \varphi_{i,j}^\sharp (s_j \,|_{(V_i \cap V_j) \cap U_{j,i}}) \right\}.$$

The restriction maps are defined coordinate-wise, i.e. $\mathrm{res}_{V,W} = (\mathrm{res}_{V_i,W_i})_i$ where $W_i = W \cap |X_i|$.

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We need to verify

1. $\mathcal{O}_X$ is a sheaf;
2. $(X, \mathcal{O}_X)$ is a locally ringed space; and
3. $(X, \mathcal{O}_X)$ is covered by affine schemes.

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2. It is clear that $\mathcal{O}_X$ is a sheaf of rings. To verify that all stalks are local rings, we first consider the stalk of $\prod_i \mathcal{O}_{X_i}$ at a point $x \in X$. This stalk is given by

$$\prod_i A_i \quad \text{where} \quad A_i = \begin{cases} \mathcal{O}_{X_i,x} & \text{if } x \in X_i \\ 0 & \text{otherwise.} \end{cases}$$

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3. X is covered by the X_i, and these are covered by affine schemes.

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For each $i = 0, \ldots, n$, we let X_i be given by

$$\mathrm{Spec}(k[x_0^i, \ldots, x_n^i]/(x_i^i - 1)) \simeq \mathbb{A}_k^n.$$

Note that the variable x_i^i is a "dummy", and we can think of X_i has having coordinates $x_0^i, \ldots, x_{i-1}^i, x_{i+1}^i, \ldots, x_n^i$.

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Given $j = 0, \ldots, n$, we put $U_{i,j} = D_{X_i}(x_j^i)$. We wish to glue the X_i's together along the $U_{i,j}$'s. To do so, we need to specify the isomorphisms $\varphi_{i,j} : U_{i,j} \to U_{j,i}$.

Construction of $\varphi_{i,j}$

For $j \neq i$, we have $\mathcal{O}_{X_i}(U_{i,j}) = \mathrm{Spec}(k[x_0^i, \ldots, x_n^i, \frac{1}{x_j^i}]/(x_i^i - 1))$.

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Lemma

The data $(X_i, U_{i,j}, \varphi_{i,j})$ comprises a descent data, and therefore may be glues to a scheme $X = \mathbb{P}_k^n$.

Proof of lemma

1. The schemes U_i, U_j are clearly isomorphic (by change of variables), and that the map $\varphi_{i,j}$ defined above is invertible, with $\varphi_{i,j}^{-1} = \varphi_{j,i}$.

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$$x_k^j \xrightarrow{\varphi_{i,j}^\sharp} \frac{x_k^i}{x_j^i} \xrightarrow{\varphi_{j,i}^\sharp} \frac{x_k^j / x_i^j}{x_j^j / x_i^j} = x_k^j / x_j^j = x_k^j$$

as $x_j^j = 1$ in X_j.

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2. To show agreement on triple intersection, we compute the action of $\varphi_{k,j} \circ \varphi_{i,j}$ on x_r^j.

$$\begin{tikzcd}
x_r^k \arrow[r, mapsto, "\varphi_{j,k}^\sharp"] \arrow[rrd, mapsto, bend right, "\varphi_{i,k}^\sharp"'] & \frac{x_r^j}{x_k^j} \arrow[r, mapsto, "\varphi_{i,j}^\sharp"] & \frac{x_r^i / x_j^i}{x_k^i / x_j^i} \arrow[d, equal] \\
 & & \frac{x_r^i}{x_k^i}
\end{tikzcd}$$

Another construction which generalizes the notion of a projective variety appears in Hartshorne, p 76, using the Proj construction.

Another construction which generalizes the notion of a projective variety appears in Hartshorne, p 76, using the Proj construction. Using this construction, we have that

$$\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, \ldots, x_n]).$$

The corresponding cover by affine spaces is given by $X_i = D_+(x_i)$, with $U_{i,j} = D_+(x_i x_j)$. We will recall the definitions and prove this in the next tutorial.

Questions?