Generalized functions
Exercise sheet 3

Exercise 1. Given a compact subset $K \subseteq \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$, put

$$B^K_{\epsilon,k} := \left\{ f \in C_c^\infty(\mathbb{R}) : \mathrm{Supp}(f) \subseteq K, \sup_{x\in\mathbb{R}} \left| f^{(k)}(x) \right| < \epsilon \right\}.$$

(1) Let I denote the set of sequences $(\epsilon_n, k_n)_{n=1}^\infty$, with $\epsilon_n > 0$ and $k_n \in \mathbb{Z}_{\geq 0}$. Show that the following collections generate the same topology on $C_c^\infty(\mathbb{R})$:
- $\mathfrak{T}_1 = \left\{ U_{(\epsilon_n,k_n)} \right\}_{(\epsilon_n,k_n)\in I}$ where $U_{(\epsilon_n,k_n)} := \sum_{n\in\mathbb{N}} B^{[-n,n]}_{\epsilon_n,k_n}$; and
- $\mathfrak{T}_2 := \left\{ V_{(\epsilon_n,k_n)} \right\}_{(\epsilon_n,k_n)\in I}$ where $V_{(\epsilon_n,k_n)} := \mathrm{conv}\left(\bigcup_{n\in\mathbb{N}} B^{[-n,n]}_{\epsilon_n,k_n} \right)$.

Recall that $\sum_{n\in\mathbb{N}} X_n := \left\{ \sum_{n\in\mathbb{N}} x_n : x_n \in X_n \text{ and } x_n = 0 \text{ for } n \gg 0 \right\}$, for $X_1, X_2, \ldots$ subsets of a vector space.

(2) Show that a sequence $(f_n)_{n=1}^\infty$ in $C_c^\infty(\mathbb{R})$ converges to f in the sense defined in the first lecture if and only if it converges to f with respect to the topology generated by the collections defined in the previous exercise.

Exercise 2. Recall that, given a topological vector space V, we write V^* and $V^\sharp$ to denote the continuous and full dual spaces of V, respectively.

(1) Let V be a Frèchet space. Show that given a finite linearly independent set $\{v_1, \ldots, v_n\}$ and values $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, there exists $\xi \in V^*$ such that $\langle \xi, v_i \rangle = \lambda_i$ for all $i = 1, \ldots, n$.

(2) Show that $(C_c^\infty(\mathbb{R}))^*$ is dense in $(C_c^\infty(\mathbb{R}))^\sharp$ with respect to the weak topology.

Remark. The weak topology on $V^\sharp$ is defined similarly to V^*; describe a neighbourhood system for this topology at 0 as part of the exercise.

(3) Conclude that $(C_c^\infty(\mathbb{R}))^*$ is not complete with respect to weak topology.

(4) * Show that $(C_c^\infty(\mathbb{R}))^\sharp$ is complete with respect to weak topology. Conclude that it is the weak completion of $(C_c^\infty(\mathbb{R}))^*$

Exercise 3. Recall that given a subspace W of $\mathbb{R}^n$ and $m \in \mathbb{Z}_{\geq 0}$ we defined

$$V_m(C_c^\infty(\mathbb{R}^n), W) = \left\{ f \in C_c^\infty(\mathbb{R}^n) : \frac{\partial^\alpha}{(\partial x)^\alpha} f \mid_W \equiv 0 \text{ for any } \alpha \text{ with } |\alpha| \leq m \right\},$$

and

$$F_m((C_c^\infty(\mathbb{R}^n))^*, W) = \{ \xi \in (C_c^\infty(\mathbb{R}^n))^* : \xi \mid_{V_m} \equiv 0 \}.$$

(1) Show that, $\overline{C_c^\infty(\mathbb{R}^n \setminus W)} = \bigcap_{m=0}^\infty V_m$.

(2) Compute $\overline{C_c^\infty(\mathbb{R}^n \setminus \{0\})}$.

(3) Show that $\bigcup_{m=0}^\infty F_m \neq (C_W^\infty(\mathbb{R}^n))^*$.

(4) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that $\varphi(W) \subseteq W$. Prove that F_m is invariant under the map $\xi \mapsto \varphi^*(\xi)$, where $\langle \varphi^*(\xi), f \rangle = \langle \xi, f \circ \varphi \rangle$.

(5) Define $G_m = \bigoplus_{|\alpha|\leq m} \frac{\partial^\alpha}{(\partial x)^\alpha} C^{-\infty}(W)$. Show that $F_m = F_{m-1} \oplus G_m$, and that G_m is *not* invariant with respect to smooth coordinate change.

Exercise 4.

(1) Prove that the map $\varphi : C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n \times \mathbb{R}^k)$, defined by

$$\varphi(f, g)(x, y) = f(x)g(y)$$

is bilinear and continuous.

(2) Prove that the map $\bar{\varphi} : C_c^\infty(\mathbb{R}^n) \otimes C_c^\infty(\mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n \times \mathbb{R}^k)$, determined on generators by φ, is injective and has dense image.

(3) * Is $\bar{\varphi}$ surjective? Prove this or find a counterexample.