Generalized functions
Tutorial notes

Part 1. Tutorial 1

1. Technical issues

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2. Basic definitions and properties

Definition 2.1. (1) Let $C^\infty(\mathbb{R})$ denote the space of smooth functions on $\mathbb{R}$, i.e. functions $f : \mathbb{R} \to \mathbb{R}$ which have derivatives of any order at any $x \in \mathbb{R}$.
(2) For a function $f : \mathbb{R} \to \mathbb{R}$ define the *support* of f to be

$$\mathrm{Supp}(f) = \mathrm{cl}\left(\{x \in \mathbb{R} : f(x) \neq 0\}\right).$$

(3) Let $C_c^\infty(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$ denote the space of smooth functions on $\mathbb{R}$ with *compact support.*

Exercise 2.2. *Prove $C_c^\infty(\mathbb{R}) \neq 0$.*

Solution. We show the following:

For any $a < b$ real numbers, there exists a function $f_{a,b} \neq 0$ such that $\mathrm{Supp}(f_{a,b}) = [a, b]$.

Fix $a \in \mathbb{R}$, and define

$$h_a(x) = e^{-1/(x-a)^2} \cdot \delta_{x>a}(x) = \begin{cases} e^{-1/(x-a)^2} & \text{if } x > a \\ 0 & \text{otherwise} \end{cases}.$$

The following are easy:

(1) $h_a(x) = 0$ if $x < a$ and $h_a(x) > 0$ if $x \leq a$;
(2) $h_a(x) > 0$ if $x > a$ and $\lim_{x\to a^+} h_a(x) = 0$; In particular, $h_a(x)$ is continuous;
(3) $\lim_{x\to+\infty} h_a(x) = 1$.

We show $h_a(x) \in C^\infty(\mathbb{R})$. Once, we have this, we may take $f_{a,b}(x) = h_a(x) \cdot h_{-b}(-x)$.

Since $h_a(x) = h_0(x - a)$, it suffices to verify that $h_0(x) = e^{-1/x^2} \in C^\infty(\mathbb{R})$. We show that the k-th derivative of h_0 is continuous on $\mathbb{R}$ for any $k \in \mathbb{N}$.

Claim. For any $k \in \mathbb{N}$, there exists a polynomial $p_k(x)$ of degree $< 3k$ such that in the domain $x \neq 0$

$$h_0^{(k)}(x) = \frac{p_k(x)}{x^{3k}} h_0(x).$$

Proof of claim. In the domain $x < 0$ h_0 is constant zero, so there is nothing to prove. For $x > 0$ we argue by induction on k, the case $k = 1$ being true since $h_0'(x) = \frac{2}{x^3} h_0(x)$ in this domain. The induction step is:

$$h_0^{(k+1)}(x) = \frac{p_k'(x)x^{3k} - 3kx^{3k-1}p_k(x)}{x^{6k}} h_0 - \frac{p_k(x)}{x^{3k}} h_0'(x)$$
$$= \left(\frac{p_k'(x)x^3 - 3kx^2 p_k(x) - 2p_k(x)}{x^{3k+3}} \right) e^{-1/x^2}$$

and the denominator on the LHS has degree $\leq \deg p_k + 2 < 3k + 3$.

The smoothness of h_0 follows from the following fact, left as a home-exercise.

Claim. $\lim_{x \to 0} \frac{1}{x^m} e^{-1/x^2} = 0$ for any $m \in \mathbb{N}$.

In particular, the two claims imply that

$$\lim_{x \to 0^-} h_0^{(k)}(x) = \lim_{x \to 0^+} h_0^{(k)}(x) = 0,$$

for any $k \in \mathbb{N}$. □

Definition 2.3. Given absolutely differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ one may define the convolution of f and g to be

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx = \int_{-\infty}^{\infty} f(x)g(t-x)dx.$$

Example 2.4. For example, if f is continuous and $g = I_{[0,1]}$, where $I_{[0,1]}$ is the indicator function of $[0, 1]$, then

$$(f * g)(t) = \int_t^{t-1} f(x)dx. \tag{2.1}$$

That is, $f * g$ is obtained from f by summing over a sliding window of width 1.

Remark 2.5. Note that $f * g$ is again absolutely integrable on $\mathbb{R}$, as

$$\int_{-\infty}^{\infty} |f * g(t)|\, dt \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t-x)g(x)|\, dxdt$$
$$= \int_{-\infty}^{\infty} |g(x)| \left(\int_{-\infty}^{\infty} |f(t-x)|\, dt \right) dx = \|g\|_1 \cdot \|f\|_1 ,$$

where the penultimate equality follow from Tonelli's Theorem.

In more compact form- $L^1(\mathbb{R}) * L^1(\mathbb{R}) \subseteq L^1(\mathbb{R})$. The next exercise proves two additional inclusions of this form.

Exercise 2.6. (1) *Show that convolution is commutative and associative.*

(2) *Given $f, g \in L^1(\mathbb{R})$ with f differentiable with compact support, show that $(f*g)' = f' * g$. Deduce $C_c^\infty * L^1(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$. Can we have $\subseteq C_c^\infty(\mathbb{R})$?*

(3) *Given $f, g \in C_c^\infty(\mathbb{R})$, show* $\mathrm{Supp}(f * g) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(g)$, *where $X + Y = \{x + y : x \in X, y \in Y\}$ is the* Minkowski sum. *Is this an equality? Deduce $C_c^\infty(R) * C_c^\infty(\mathbb{R}) \subseteq C_c^\infty(\mathbb{R})$.*

Solution. (1) Both assertions follow by change of variable; see (2.1).

(2) We have, for any $t \in \mathbb{R}$ and $\epsilon > 0$,

$$\frac{f * g(t+\epsilon) - f * g(t)}{\epsilon} = \int_{-\infty}^{\infty} \left(\frac{f(t+\epsilon-x) - f(t-x)}{\epsilon} \right) g(x)dx. \tag{2.2}$$

Taking lim as $\epsilon \to 0$, the claim would follow if we can justify replacing the limit and the integral.

Note that, as $\mathrm{Supp}(f)$ is compact and f is differentiable, we have that

$$M = \sup_{\substack{y \in \mathrm{Supp}(f) \\ \epsilon \in (0,1]}} \left| \frac{f(y+\epsilon) - f(y)}{\epsilon} \right| < \infty;$$

(home exercise; *hint*: $F(y,\epsilon) = \frac{f(y+\epsilon)-f(y)}{\epsilon}$ extends to a continuous function on a compact set in $\mathbb{R}^2$). In particular, if $0 < \epsilon \leq 1$ then the integrand in (2.2) is bounded above by $H(x) = M \cdot g(x)$, which is absolutely integrable, since g is. In particular, by the Dominated Convergence Theorem, we have that

$$\begin{aligned} (f * g)'(t) &= \lim_{\epsilon \to 0} \frac{f * g(t+\epsilon) - f * g(t)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \left(\frac{f(t+\epsilon-x) - f(t-x)}{\epsilon} \right) g(x)dx \\ &= \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \left(\frac{f(t+\epsilon-x) - f(t-x)}{\epsilon} \right) g(x)dx = f' * g(t) \end{aligned}$$

The assertion $C_c^\infty(\mathbb{R}) * L^1(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$ follows easily. The inclusion into $C_c^\infty(\mathbb{R})$ does not hold (home exercise).

(3) Let $t \notin \mathrm{Supp}(f) + \mathrm{Supp}(g)$, we want to show $f * g(t) = 0$. Indeed, if $f * g(t) \neq 0$ then

$$f * g(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx \neq 0,$$

which implies, in particular, that the function $f(t-x)g(x)$ is not everywhere zero. In particular, there exists $x \in \mathbb{R}$ such that both $f(t-x), g(x) \neq= 0$, i.e. $x \in \mathrm{Supp}(g)$ and $t - x \in \mathrm{Supp}(f)$. But then

$$t = (t-x) + x \in \mathrm{Supp}(f) + \mathrm{Supp}(g),$$

a contradiction.

The inclusion $\mathrm{Supp}(f * g) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(g)$ may be strict. As a non-example, we may take $f(x) = x \cdot I_{[-1,1]}$ and $g(x) = I_{[-2,2]}$, for which $\mathrm{Supp}(f) = [-1,1]$, $\mathrm{Supp}[g] = [-2,2]$ and $\mathrm{Supp}(f) + \mathrm{Supp}(g) = [-6,6]$, but $(-1,1) \cap \mathrm{Supp}(f * g) = \varnothing$. These are, of course, non-smooth compactly supported functions, but examples in $C_c^\infty(\mathbb{R})$ also exist (home exercise: find such!).

Lastly, the inclusion $C_c^\infty(\mathbb{R}) * C_c^\infty(\mathbb{R}) \subseteq C_c^\infty(\mathbb{R})$, follows from the continuity of addition, as a map $\mathbb{R}^2 \to \mathbb{R}$ (maps the compact set $\mathrm{Supp}(f) \times \mathrm{Supp}(g)$ onto a compact set). Note that, in general, if X and Y are closed (but not compact) then $X+Y$ may not be closed (e.g. $X = \{(x,y) : xy = 1\}, Y = \{(x,y) : x = 0\} \subseteq \mathbb{R}^2$). □

EXERCISE 2.7. *Let $K \subseteq \mathbb{R}$ be a compact set and $U \supseteq K$ an open set. Show that there exists $f \in C_c^\infty(U) := \{g \in C_c^\infty(\mathbb{R}) : \mathrm{Supp}(g) \subseteq U\}$ such that $f\mid_K \equiv 1$.*

REMARK 2.8. This exercise holds in much greater generality.

Solution. Set $\epsilon = d(K, U^c) = \inf\{d(x,y) : x \in K,\ y \in U^c\}$, and, for any $\delta > 0$, put $K_\delta = \{x \in U : d(x,K) \le \delta\}$, a closed set in U. By Urysohn's Lemma, there exists $\varphi : U \to [0,1]$ such that $\varphi\mid_{K_{\epsilon/4}} \equiv 1$ and $\varphi\mid_{\mathrm{cl}\left(K^c_{3\epsilon/4}\right)} \equiv 0$. Let $\psi \in C_c^\infty(\mathbb{R})$ have $\mathrm{Supp}(\psi) = [-\epsilon/4, \epsilon/4]$, $\psi(0) \neq 0$ and $\int_{\mathbb{R}} \psi(x)dx = 1$ (normalise the function taken in Exercise 2.2). By Exercise 2.6, $f = \varphi * \psi$ is smooth and has support $\mathrm{Supp}(f) = \mathrm{Supp}(\psi) + \mathrm{Supp}(\varphi) \subseteq K_{3\epsilon/4} + [-\epsilon/4, \epsilon/4] \subseteq U$. Furthermore, for any $x \in K$,

$$f(x) = \int_{\mathbb{R}} \varphi(x-t)\psi(t)dt = \int_{-\epsilon/4}^{\epsilon/4} \underbrace{\varphi(x-t)}_{x-t\in K_{\epsilon/4}} \psi(t)dt = \int_{-\epsilon/4}^{\epsilon/4} \psi(t)dt = 1.$$

□

REMARK 2.9. Note that the function f we obtained in Exercise 2.7 is, in addition, non-negative.

EXERCISE 2.10 (Partition of unity). *Let $f \in C_c^\infty(\mathbb{R})$, I an indexing set and $\mathbb{R} = \bigcup_{i\in I} U_i$ an open cover. Then there exist $f_i \in C_c^\infty(U_i)$, for any $i \in I$, such that $f = \sum_{i\in I} f_i$.*

Solution. Put $K = \mathrm{Supp}(f)$. Since K is compact, there exist $i_1, \dots, i_r \in I$ such that $K \subseteq \bigcup_{j=1}^r U_{i_j}$.

CLAIM. There exists open sets $V_1, \dots, V_r \subseteq \mathbb{R}$ such that $K \subseteq \bigcup_{j=1}^r V_j$ and such that $\mathrm{cl}\,(V_j) \subseteq U_{i_j}$, for any $j = 1, \dots, r$.

Proof of Claim. Put $\mathcal{A} = \left\{V \subseteq \mathbb{R} \text{ open} : \mathrm{cl}\,(V) \subseteq U_{i_j} \text{ for some } j = 1, \dots, r\right\}$. Note that, as K is a regular topological space (T_3, i.e. separates points from closed sets), for any $x \in K$, there exists $V \in \mathcal{A}$ such that $x \in V$. In particular $\mathcal{A}$ is a cover of K, and there exists a finite subcover $\mathcal{B} \subseteq \mathcal{A}$. We can choose a function $f : \mathcal{B} \to \{1, \dots, r\}$, mapping each $V \in \mathcal{B}$ to $j \in \{1, \dots, r\}$ such that $\mathrm{cl}\,(V) \subseteq U_{i_j}$. Define, for any $j = 1, \dots, u_j$,

$$V_j := \bigcup f^{-1}(U_{i_j}).$$

As V_j is the union of finitely many open sets whose closures are contained in U_{i_j}, so does $\mathrm{cl}\,(V_j) \subseteq U_{i_j}$.

Using the claim and Exercise 2.7, for any $j = 1, \dots, r$ we may choose ρ_j such that $\rho_j\mid_{K\cap \mathrm{cl}(V_j)} \equiv 1$ and $\rho_j\mid_{U^c_{i_j}} \equiv 0$. In particular, since the ρ_j's can be taken to be non-negative, we have that $\sum_{j=1}^r \rho_j(x) \neq 0$ for any $x \in K$, and we may define

$$f_i = \begin{cases} \frac{\rho_j \cdot f}{\rho_1 + \cdots + \rho_r} & \text{if } i = i_j \in \{i_1, \dots, i_j\} \\ 0 & \text{otherwise} \end{cases}$$

□

EXERCISE 2.11. *Show that $C_c^\infty(\mathbb{R})$ is dense in $C_c(\mathbb{R})$ with respect to uniform convergence.*

REMARK 2.12. More generally, it is true that $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, for any $1 \le p < \infty$ (but not for $p = \infty$). This will be proved in the home exercise.

PROOF. Let $g \in C_c(\mathbb{R})$. We construct a sequence $\{g_n\}_{n\in\mathbb{N}}$ of elements of $C_c^\infty(\mathbb{R})$ such that $g_n \xrightarrow{n\to\infty} g$.

For any $n \in \mathbb{N}$, let $\chi_n \in C_c^\infty(\mathbb{R})$ be a non-negative bump function with $\mathrm{Supp}(\chi_n) = \left[-\frac{1}{n}, \frac{1}{n}\right]$ and $\|\chi_n\|_1 := \int_{\mathbb{R}} \chi_n = 1$ (e.g., take $\chi_n = \frac{f_{-\frac{1}{n},\frac{1}{n}}}{\left\|f_{-\frac{1}{n},\frac{1}{n}}\right\|_1}$, where $f_{a,b}$ is as in Exercise 2.2), and put

$$g_n = g * \chi_n \quad \text{for all} \quad n \in \mathbb{N}.$$

By Exercise 2.6, since g and χ_n have compact support and χ_n are smooth, we have that $g_n \in C_c^\infty(\mathbb{R})$ for all n. Moreover, again by Exercise 2.6.(3),

$$\mathrm{Supp}(g_n) = \mathrm{Supp}(g) + \left[-\frac{1}{n}, \frac{1}{n}\right] \subseteq \mathrm{Supp}(g) + [-1, 1],$$

and so, as this is a compact set, it suffices to show point-wise convergence in this domain. This holds since, for any $x \in \mathbb{R}$,

$$g_n(x) = \int_{\mathbb{R}} g(x-t)\chi_n(t)dt = \int_{-\frac{1}{n}}^{\frac{1}{n}} g(x-t)\chi_n(t)dt,$$

and

$$\begin{aligned}
\min_{|t'|\leq\frac{1}{n}} g(x-t') = \min_{|t'|\leq\frac{1}{n}} g(x-t') \cdot \int_{-\frac{1}{n}}^{\frac{1}{n}} \chi_n(t)dt \\
\leq \int_{-\frac{1}{n}}^{\frac{1}{n}} g(x-t)\chi_n(t)dt \\
\max_{|t'|\leq\frac{1}{n}} g(x-t') \cdot \int_{-\frac{1}{n}}^{\frac{1}{n}} \chi_n(t)dt \leq \max_{|t'|\leq\frac{1}{n}} g(x-t').
\end{aligned}$$

Since g is continuous on $\mathbb{R}$, both right-hand left-hand side of this inequality converge to $g(x)$ as $n \to \infty$, implying the claim. $\square$