Representation Zeta Function of Norm 1 Subgroups of Local Division Algebras

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Definition of ζ_G

- Let G be a topological group. For any $n \in \mathbb{N}$, let $r_n(G) \in \mathbb{N} \cup \{0, \infty\}$ denote the number of isomorphism classes of complex continuous irreducible representations of G.
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- The group G is said to have **polynomial representation growth** (PRG) if $r_n(G)$ is bounded above by a polynomial in n.
- In the case where G has PRG, one defines the **representation zeta function of** G to be the Dirichlet generating function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}, \quad s \in \mathbb{C}.$$

  This series is absolutely convergent in a half-plane $\{z \in \mathbb{C} \mid \Re(z) > \alpha\}$, for some $\alpha \in \mathbb{R}$.
  The infimum of such α's is called the **abscissa of convergence** of ζ_G.

Arithmetic Groups

Let $F/\mathbb{Q}$ be a number field with $\mathfrak{R} \subseteq F$ the ring of integers. Let G be a connected simply-connected algebraic group (e.g. $G = \mathrm{SL}_d$).

- The group $G(\mathfrak{R})$ is said to have the **congruence subgroup property** (CSP) if the map $\widehat{G(\mathfrak{R})} \to G(\widehat{\mathfrak{R}})$ is an isomorphism.
- It is said to have **weak CSP** (wCSP) if the kernel of the above map is finite.

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Example

The group $SL_d(\mathfrak{R})$ has PRG iff $d > 2$.

Euler Factorization

Theorem (Larsen, Lubotzky)

If $G(\mathfrak{R})$ has CSP then

$$\zeta_{G(\mathfrak{R})}(s) = \zeta_{G(\mathbb{C})}(s)^{|F:\mathbb{Q}|} \cdot \prod_{\substack{\mathfrak{p} \lhd \mathfrak{R} \\ \textit{prime}}} \zeta_{G(\mathfrak{R}_{\mathfrak{p}})}(s).$$

- The archemedian factor $\zeta_{G(\mathbb{C})}(s)$ enumerates irreducible finite-dimensional rational representations of $G(\mathbb{C})$, which are reflected in the root system of G.

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- The archemedian factor $\zeta_{G(\mathbb{C})}(s)$ enumerates irreducible finite-dimensional rational representations of $G(\mathbb{C})$, which are reflected in the root system of G.
- The factors $\zeta_{G(\mathfrak{R}_{\mathfrak{p}})}(s)$ enumerate continuous representations of the compact p-adic analytic groups $G(\mathfrak{R}_{\mathfrak{p}})$, where $\mathfrak{R}_{\mathfrak{p}}$ is the completion of $\mathfrak{R}$ at the prime $\mathfrak{p}$.

Compact p-adic Analytic Groups

Theorem (Jaikin-Zapirin)

Let G be a finitely generated compact p-adic analytic group. Assume $p > 2$ and that G is rigid. Then there exist rational functions $f_1(T), \ldots, f_k(T) \in \mathbb{Q}(T)$ and numbers $n_1, \ldots, n_k \in \mathbb{N}$ such that

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Remark

A similar result holds for compact 2-adic analytic groups, under some additional assumptions on G.

Some Notation

- From here on we fix K to be the completion of F at a fixed prime $\mathfrak{p}$, and let $\mathscr{O}$ denote the completion of $\mathfrak{R}$ and $\wp$ be its maximal ideal, generated by an element $\pi \in \mathscr{O}$.
- We put p to denote the characteristic of the residue field $\mathscr{O}/\wp$ and $q = p^\alpha$ to be its cardinality.

Forms of $\mathrm{SL}_\ell(\mathfrak{K})$

- Let ℓ be a prime number, distinct from p.
- Assume that the group $G(\mathfrak{K})$ is a form of $\mathrm{SL}_\ell(\mathfrak{K})$ for some $\ell \in \mathbb{N}$. For the case where ℓ is prime, for all but finitely many primes $\mathfrak{p}$, the group $G(\mathscr{O})$ is isomorphic to one of the following groups:
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- For the finitely many exceptional primes, the group $G(\mathscr{O})$ is isomorphic to $\mathrm{SL}_1(D)$, the group of norm-1 elements of a central division algebra D of degree ℓ over K.

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- For the finitely many exceptional primes, the group $G(\mathscr{O})$ is isomorphic to $\mathrm{SL}_1(D)$, the group of norm-1 elements of a central division algebra D of degree ℓ over K.
- In all 3 cases, the group $G(\mathscr{O})$ is a compact p-adic analytic group, and contains a maximal pro-p group $G^1(\mathscr{O})$, given explicitly as the kernel of the map $G(\mathscr{O}) \to G(\mathscr{O}/\wp)$.

Example 1: The case $\ell = 2$

- Jaikin-Zapirin introduced a computation of $\zeta_{\mathrm{SL}_2(\mathscr{O})}(s)$, based on a classification of the characters of $\mathrm{SL}_2(\mathscr{O})$ and on Clifford theory.

Theorem (Jaikin-Zapirin)

Suppose $p =\neq 2$. Then

$$\zeta_{\mathrm{SL}_2(\mathscr{O})}(s) = 1 + q^{-s} + \frac{q-3}{2}(q+1)^{-s} + 2\left(\frac{q+1}{2}\right)^{-s} + \frac{q-1}{2}(q-1)^{-s} + \\ + \frac{4q\left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \frac{(q-1)^2}{2}(q^2+q)^{-s}}{1-q^{-s+1}}.$$

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- An important feature of this computation is the fact that it is actually independent of the characteristic of K, and holds for positive characteristic as well.

Example 1

- Avni, Klopsch, Onn and Voll introduced a *p*-adic formalism for the computation of the representation zeta function of a class of pro-*p* groups. This formalism allows for the calculation of the representation zeta function with via certain Igusa-type zeta integrals.

Example 1

- Avni, Klopsch, Onn and Voll introduced a p-adic formalism for the computation of the representation zeta function of a class of pro-p groups. This formalism allows for the calculation of the representation zeta function with via certain Igusa-type zeta integrals.
- This development includes and application of the Kirillov orbit method and an analysis of the adjoint action in an associated Lie-ring defined over $\mathscr{O}$.
- As an outcome of this formalism, the authors managed to compute the representation zeta function of the maximal pro-p subgroup of $\mathrm{SL}_2(\mathscr{O})$ and of $\mathrm{SL}_1(D)$ for D a division algebra of degree 2 over K.

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Division Algebras of Degree 2

A division algebra D of degree 2 over K can be described by the following data:

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- The reduced norm of an element $\mathbf{x} = \alpha + \mathbf{i}\beta + \mathbf{j}\gamma + \mathbf{ij}\delta$ is given explicitly by the formula:

$$\mathrm{Nrd}_{D/K}(\mathbf{x}) = \alpha^2 - a\beta^2 + \pi(\gamma^2 - a\delta^2).$$

Remark

Compare to the case of $\mathbb{H}$, the Hamiltonian quaternions over $\mathbb{R}$.

Example 1

Under some mild assumptions on the ramification index of $K/\mathbb{Q}_p$, Avni, Klopsch, Onn and Voll computed the following.

Theorem (AKOV)

Suppose $p \neq 2$ and let D be a division algebra of degree 2 with $\mathbf{Z}(D) = K$. Then

$$\zeta_{\mathrm{SL}_1^1(D)}(s) = q^3 \frac{q - q^{-1-s}}{1 - q^{1-s}},$$

where $SL_1^1(D)$ is the maximal pro-p subgroup of $\mathrm{SL}_1(D)$.

Example 1

Applying tools from Clifford theory, the authors obtained the computation for $\mathrm{SL}_1(D)$ as well.

Theorem (AKOV)

Suppose $p \neq 2$ and let D be a division algebra of degree 2 with $\mathbf{Z}(D) = K$. Then

$$\zeta_{\mathrm{SL}_1(D)}(s) = \frac{(q+1)(1-q^{-s}) + 4(q-1)\left(\frac{q+1}{2}\right)^{-s}}{1-q^{1-s}}.$$

Example 2: $\ell = 3$

Theorem (AKOV)

Suppose $p \neq 3$, and let $G(\mathscr{O})$ be $\mathrm{SL}_3(\mathscr{O})$ or $\mathrm{SU}_3(\mathscr{O})$. Then

$$\zeta_{G(\mathscr{O})}(s) = q^8 \frac{1 + u(q)q^{-3-2s} + u(q^{-1})q^{-2-3s} + q^{-5-5s}}{(1 - q^{1-2s})(1 - q^{2-3s})}$$

where

$$u(T) = \begin{cases} T^3 + T^2 - T - 1 - T^{-1} & \text{if } G(\mathscr{O}) = \mathrm{SL}_3(\mathscr{O}) \\ -T^3 + T^2 - T + 1 - T^{-1} & \text{if } G(\mathscr{O}) = \mathrm{SU}_3(\mathscr{O}). \end{cases}$$

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Remark

The computation for $\mathrm{SL}_3(\mathscr{O})$ and $\mathrm{SU}_3(\mathscr{O})$ is completed as well.

Example 2

Applying the p-adic formalism described by AKOV, and under some mild assumptions on the ramification index $e(K/\mathbb{Q}_p)$, we have the following

Theorem

Suppose $p \neq 3$ and assume that K contains a primitive cube root of unity. Let D be a division algebra of degree 3 with $\mathbf{Z}(D) = 3$. Then

$$\zeta_{\mathrm{SL}_1^1(D)}(s) = q^3 \frac{1 + q^{-s+1}(1 - q^{-3} - q^{-3s-3}}{1 - q^{-3s+2}},$$

and

$$\zeta_{\mathrm{SL}_1(D)}(s) = \frac{(1+q+q^2)(1-q^{-3s}) + 9\left(\frac{1+q+q^2}{3}\right)^{-s}(q^{-s+1}+1)(q-1)}{1-q^{-3s+2}}.$$

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- The case of $\mathrm{SL}_3(\mathscr{O})$ and $\mathrm{SU}_3(\mathscr{O})$ is significantly more involved and requires distinguishing between 8-10 types of conjugacy classes. The level of difficulty for $\ell > 3$ is still unclear.
- The case of $\mathrm{SL}_1(D)$ for division algebras of prime degree is, however, accessible.

Let ℓ be an arbitrary prime.
Assuming $p \neq \ell$ and some additional assumptions on $e(K/\mathbb{Q}_p)$, we have the following

Theorem

Suppose K contains a primitive ℓ-th root of unity, and let D be a division algebra of degree ℓ with $\mathbf{Z}(D) = K$. Then

$$\zeta_{\mathrm{SL}_1^1(D)}(s) = \frac{1 - q^{\binom{\ell}{2}s} + (q^\ell - 1) \cdot S_\ell(q^{1-\frac{\ell-1}{2}s})}{1 - q^{-\binom{\ell}{2}s+\ell-1}},$$

where $S_\ell(T) := T + \ldots + T^{\ell-1}$.

By an application of tools from Clifford theory, we also obtain the following.

Theorem

Let K be as above, with a primitive ℓ-th root of unity, and let D be a division algebra of degree ℓ with $\mathbf{Z}(D) = K$. Then

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Remark

It is possible that the assumption regarding the existence of primitive roots of unity can be omitted.

Questions ?

Kirillov Orbit Method

Prototype Case

- Consider the group $G = 1 + p\mathrm{Mat}_d(\mathbb{Z}_p)$ and the $\mathbb{Z}_p$-Lie algebra $\mathfrak{g} = \mathrm{Mat}_d(p\mathbb{Z}_p)$.

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- The exponential series $X \mapsto \sum_{j=0}^{\infty} \frac{1}{j!} X^j$ is a bijection of $\mathfrak{g}$ onto G. The map also induces a correspondence between sub-Lie algebras (resp. ideals) of $\mathfrak{g}$ and subgroups (resp. normal subgroups) of G.

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- The exponential series $X \mapsto \sum_{j=0}^{\infty} \frac{1}{j!} X^j$ is a bijection of $\mathfrak{g}$ onto G. The map also induces a correspondence between sub-Lie algebras (resp. ideals) of $\mathfrak{g}$ and subgroups (resp. normal subgroups) of G.
- The conjugation action of G on $\mathfrak{g}$ induces an action on the Pontryagin dual $\hat{\mathfrak{g}}$ of $\mathfrak{g}$.
- To any G-orbit $\Omega \subseteq \hat{\mathfrak{g}}$ we associate the class function

$$\chi_\Omega(\exp(X)) = \frac{1}{\sqrt{|\Omega|}} \sum f \in \Omega f(X).$$

- $\Omega \leftrightarrow \chi_\Omega$ is a bijection between the orbit space $\hat{\mathfrak{g}}/G$ and the set of finite-dimensional continuous irreducible characters of G.

Kirillov Orbit Method

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- In a more general setting, suppose G is a pro-p group and $\mathfrak{g}$ is a $\mathbb{Z}_p$-Lie algebra with a homeomorphism $\exp : \mathfrak{g} \to G$.
- Under assumptions on G guaranteeing the existence of a Lie-correspondence, as well as some other technical assumption, we have the following:

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Theorem (Howe)

The map $\Omega \mapsto \chi_\Omega$, where

$$\chi_\Omega((\exp(X)) = \frac{1}{\sqrt{|\Omega|}} \sum_{f \in \Omega} f(X),$$

is a bijection of $\hat{\mathfrak{g}}/G$ onto the set of irreducible continuous finite-dimensional characters of g.

Division Algebras

Let D be a division algebra of degree ℓ, with $K = \mathbf{Z}(D)$. Let L denote the unramified extension of K of degree L.

- It is known that L embeds into D as a maximal subfield of D. In particular L splits D, that is-

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$$D \otimes_K L \cong \mathrm{Mat}_\ell(L).$$

- The map $x \mapsto x \otimes 1$ embeds D as a sub-K-algebra of $\mathrm{Mat}_\ell(L)$ endowing D with a reduced norm $\mathrm{Nrd}_{D/K}(\cdot)$ and trace $\mathrm{Trd}_{D/K}(\cdot)$.

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- The absolute value given on K extends uniquely to D and satisfies

$$|\mathbf{x}|_\wp = \left|\mathrm{Nrd}_{D/K}(\mathbf{x})\right|_\wp^{1/\ell}, \quad \forall \mathbf{x} \in D.$$

- Put $\mathcal{O} = \left\{\mathbf{x} \mid |\mathbf{x}|_\wp \leq 1\right\}$ and $\mathcal{P} = \left\{\mathbf{x} \mid |\mathbf{x}|_\wp < 1\right\}$.

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- The group $\mathrm{SL}_1^1(D)$ consists of elements $\mathbf{x} \in \mathrm{SL}_1(D)$ such that $x - 1 \in \mathcal{P}$, and is a maximal pro-p sub group of $\mathrm{SL}_1(D)$.
  Under some mild assumptions on $e(K/\mathbb{Q}_p)$, the standard exponential map is convergent on the $\mathscr{O}$-Lie algebra $\mathfrak{sl}_1^1(D)$ of traceless elements in $\mathcal{P}$, mapping it bijectively onto $\mathrm{SL}_1^1(D)$ and making $\mathrm{SL}_1^1(D)$ amenable to the Kirrilov orbit method.

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- The quotient $\mathrm{SL}_1(D)/\mathrm{SL}_1^1(D)$ is embeddable into a subgroup of the cyclic group $\mathbb{F}_{q^\ell}$.

Denote $H = \mathrm{SL}_1^1(D)$ and $\mathfrak{h} = \mathfrak{sl}_1^1(D)$.

$$\zeta_{\mathrm{SL}_1^1(D)}(s) = \sum_{\Omega \in \widehat{\mathfrak{h}}/H} \chi_\Omega(1)^{-s} = \sum_{\Omega \in \widehat{\mathfrak{h}/H}} |\Omega|^{-s/2}$$

$$= \sum_{\omega \in \widehat{\mathfrak{h}}} |H.\omega|^{-\frac{s}{2}-1} = \sum_{\omega \in \widehat{\mathfrak{h}}} |H : \mathbf{C}_H(\omega)|^{-\frac{s}{2}-1}$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{\omega \in \widehat{\mathfrak{h}/\pi^n \mathfrak{h}}} |H : \mathbf{C}_H(\omega)|^{-\frac{s}{2}-1}$$

- **Fact 1**: For any $n \in \mathbb{N}$, there exists a surjection of the set $\pi^{-1}\mathfrak{g} \setminus \mathfrak{g}$ onto $\widehat{\mathfrak{h}/\pi^n\mathfrak{h}}$, whose fibres are cosets of $\pi^{n-1}\mathfrak{g}$ (where $\mathfrak{g}$ is the Lie-algebra of traceless elements in $\mathcal{O}$).

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- **Fact 2** The stabilizer of an element $\omega \in \widehat{\mathfrak{h}/\pi^n\mathfrak{h}}$ equals the set

$$\mathbf{St}^n(\mathbf{x}_\omega) := \left\{ h \in H \mid \mathbf{x}_\omega - h^{-1}\mathbf{x}_\omega h \in \pi^{n-1}\mathfrak{g} \right\}$$

  where $\mathbf{x}_\omega$ is any element which is mapped onto ω by the above bijection.

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Theorem

Let $\mathbf{x} \in \mathfrak{h}$ and $n \in \mathbb{N}$. Then

$$\left|\mathrm{SL}_1^1(D) : \mathbf{St}^n(\mathbf{x})\right| = \begin{cases} q^{(\ell-1)(\ell n - \mu(\mathbf{x}))} & \text{if } \mu(\mathbf{x}) \in \ell\mathbb{Z}, \\ q^{(\ell-1)(\ell n - \mu(\mathbf{x}) - 1)} & \text{if } \mu(\mathbf{x}) \notin \ell\mathbb{Z}, \end{cases}$$

- From this, the computation of the representation zeta function of $\mathrm{SL}_1^1(D)$ follows.

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Thank You