Algebraic Geometry 2
Tutorial session 9

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Morphisms of finite type; finite morphisms

Definition

A morphism $f : X \to Y$ is *locally of finite type* if there exists an open cover $Y = \bigcup_i V_i$ with $V_i = \mathrm{Spec}(B_i)$ affine, such that for any i, $f^{-1}(V_i) = \bigcup_j U_{i,j}$ with $U_{i,j} = \mathrm{Spec}(A_{i,j})$ where $A_{i,j}$ is f.g. over B_i.

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A morphism $f : X \to Y$ is *finite* if there exists an affine open cover $Y = \bigcup_i V_i$, with $V_i = \mathrm{Spec}(B_i)$ such that $f^{-1}(V_i) = \mathrm{Spec}(A_i)$ is affine, with A_i a *finite module* over B_i.

Exercise

1. Show that $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasicompact (i.e. $f^{-1}(V)$ is qc for all open affine $V \subseteq Y$).

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1. loc fin type + qc implies finite type directly from definition. Also, finite type implies loc fin type, so we only need to show finite type $\Rightarrow$ qc. Let $W \subseteq Y$ be open affine. *Exercise*: if $Y = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup_{j=1}^{n(i)} U_{i,j}$ are open affine covers as in the definition, there exists $i_1, \ldots, i_N$ such that

$$f^{-1}(V) = \bigcup_{\substack{k=1,\ldots,N \\ j=1,\ldots,n(i_k)}} f^{-1}(V) \cap U_{i_k,j}.$$

Proof.

Consequently, it suffices to prove

Lemma

Let $f : X = \mathrm{Spec}(A) \to Y = \mathrm{Spec}(B)$ with A a fg B-algebra, and $V \subseteq Y$ open (not necessarily affine). Then $f^{-1}(V)$ is qc.

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Also, by definition, A is quotient of $B[x_1, \ldots, x_n]$, for some n, and therefore X is a closed subscheme of $\mathbb{A}^n_B$. Since the lemma is completely topological, it is enough to verify the lemma for the case $X = \mathbb{A}^n_B$ and f is induced from the inclusion $B \subseteq B[x_1, \ldots, x_n]$.

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Proof.

2. We will use the following lemma, which will be proved (more generally) in the home exercise:

Affine communication lemma

Let X be a scheme over $S = \mathrm{Spec}(R)$. Assume there exists an affine open cover $X = \bigcup_i U_i$ where $U_i = \mathrm{Spec}(A_i)$ with A_i fg R-algebras. Then, for any $V = \mathrm{Spec}(B) \subseteq X$ open affine, B is a fg R-algebra.

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Let $f : X \to Y$ be of finite type and let $V \subseteq Y$ be open affine. $f^{-1}(V)$ is qc and covered by affine open schemes whose global sections are fg over $\Gamma(V, \mathcal{O}_Y)$. The result follows from the lemma.

$\square$

Properties of morphisms of finite type

Exercise

Let $f : X \to Y$ be a morphism of schemes. Show the following:

1. If f is a closed embedding then f is of finite type.
2. If f is a quasi-compact (i.e. $f^{-1}(V)$ is q.c. for all open affine $V \subseteq Y$) an open embedding then f is of finite type.
3. If f and finite type and $g : Y \to Z$ is also finite type, then $g \circ f$ is also finite type.

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3. The composition of morphisms locally of finite type is locally of finite type, by the ACL. The composition of qc morphisms is also qc.

Exercise

Show that a finite morphism has finite fibers.

Solution.

Let $f : X \to Y$ be a finite morphism and $y \in Y$. By definition, the exists $y \in V = \mathrm{Spec}(B) \subseteq Y$ open affine such that $f^{-1}(V) = \mathrm{Spec}(A)$ with A a finite module over B. Since we only care about y, we may assume $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ to begin with, and f is defined by a ring homomorphism $\varphi : B \to A$.

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Let $\mathfrak{p}$ be the prime ideal of B corresponding to y. We want to describe the set

$$f^{-1}(y) = \left\{\mathfrak{q} \in \mathrm{Spec}(A) : \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\right\}.$$

Solution- contd.

By moding out $\mathfrak{p}$ from the target and the domain, we may consider $\bar{\varphi} : B/\mathfrak{p} \to A/\varphi(\mathfrak{p})A$, and ask which primes in $A/\varphi(\mathfrak{p})A$ are pulled back to zero in $B/\mathfrak{p}$.

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As we do not care about the non-zero ideals of $B/\mathfrak{p}$, we can localize this ring at 0. Write $K = \mathrm{Frac}(B/\mathfrak{p})$. The map $\bar{\varphi}$ defines a finite map $K \to ((A/\varphi(\mathfrak{p})A) \otimes_B K$ and there is a bijection between prime ideals of $A/\varphi(\mathfrak{p})$ and prime ideals of $(A/\varphi(\mathfrak{p})A) \otimes K$. That is

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Since the latter is a finite dimensional K-algebra, it has only finitely primes (a consequence of being artinian). $\square$

Dimension and codimension

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of closed irreducible subsets of X. For an arbitrary closed subset Y we define

$$\mathrm{codim}(Y, X) = \inf_{Z \subseteq Y \text{ irr}} \mathrm{codim}(Z, X)$$

Examples

1. $\dim \mathrm{Spec}(k) = 0$ and $\dim \mathbb{A}_k^n = n$ for any field k;
2. $\dim \mathrm{Spec}(\mathbb{Z}) = 1$;
3. More generally, if $X = \mathrm{Spec}(A)$ then $\dim(X) = \dim(A)$, where the RHS is the Krull dimesion, i.e the length of a maximal descending chain of prime ideals.
4. For a noetherian ring A, $\dim(\mathrm{Spec}(A[x_1, \ldots, x_n])) = \dim(A) + n$.

Exercise

Let X be an integral scheme of finite type over a field k.

1. For any closed point $x \in X$, $\dim X = \dim \mathcal{O}_{X,x}$
2. Given a closed subset $Y \subseteq X$, show that $\dim(Y) + \mathrm{codim}(Y, X) = \dim(X)$.
3. Let $U \subseteq X$ be a non-empty open subset. Show that $\dim(U) = \dim(X)$.

Solution.

1. In general, we have an bijective map $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$, with image within an affine open subset, which implies

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathrm{Spec}(\mathcal{O}_{X,x})) \leq \dim(X).$$

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Conversely, assume first that $X = \mathrm{Spec}(A)$ for A a f.g. domain over k. Then $x = \mathfrak{m}$ is a maximal ideal and, by Theorem 1.8A in Hartshorne

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In the more general case, we have that $X = \bigcup_{i=1}^{n} X_i$, a finite union of spectra of f.g. domains over k. We have that

$$\dim(X) = \max\{\dim X_i : i = 1, \dots, n\},$$

from which the claim follows.

Solution- contd.

2. Assume first that Y is irreducible and $X = \mathrm{Spec}(A)$ for A f.g. domain over k. Then $Y = V(\mathfrak{p})$ for a prime $\mathfrak{p}$ and, by definition $\mathrm{codim}(Y, X) = \mathrm{ht}(\mathfrak{p})$ and $\dim(Y) = \dim(A/\mathfrak{p})$. The result then follows, again, from Theorem 1.8A:

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3. To prove the last item, it suffices to see that any non-empty open subset contains a closed point. For the case $X = \mathrm{Spec}(A)$ and $U = D(f) \neq \varnothing$, this is equivalent to finding a maximal ideal not containing f. But if no such maximal exists, then f is in the Jacobson radical of A, which is zero.

□

Exercise

Let $R = \mathbb{C}[\![x]\!]$ and $X = \mathrm{Spec}(R[t])$. Show that all statements in the previous exercise fail for X.

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Solution.

Note that $\dim(X) = \dim(R) + 1 = 2$, since R is a dvr.
Consider $\mathfrak{p} = (xt - 1)$. Then $\mathfrak{p} \supseteq (x - 1, t - 1)$ is prime of height 1, hence

$$\dim(\mathcal{O}_{X,\mathfrak{p}}) = \dim(R[t]_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) = 1 < \dim X.$$

Moreover, for $Y = V(\mathfrak{p})$ we have that $\mathrm{codim}(Y, X) = 1$. However,

$$\dim(Y) = \dim \operatorname{Spec}(\mathbb{C}[\![x]\!][t]/(xt - 1)) = \dim \operatorname{Spec}(\mathbb{C}(\!(x)\!)) = 0,$$

since the latter is a field. So $\dim(Y) + \mathrm{codim}(Y, X) < \dim(X)$.

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since the latter is a field. So $\dim(Y) + \operatorname{codim}(Y, X) < \dim(X)$.
Finally, the localization of $R[t]$ by x is a polynomial ring over the field $\mathbb{C}(\!(x)\!)$, hence one-dimensional. So $\dim(D(x)) = 1 < \dim(X)$. □

The fiber product

Let X, Y be schemes over a scheme S. The fiber product of X and Y over S is a scheme $X \times_S Y$ with maps to X and Y, such that for any scheme W with maps $W \to X$, $W \to Y$ whose compositions with the morphism to S coincide, $\exists!$ morphism $W \to X \times_S Y$ such that the following diagram commutes:

$$\begin{tikzcd}
W \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dashed, "\exists! h"] & & \\
& X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

Theorem

The fiber product exists.

Fibers of morphisms

One application of the fiber product is to endow the fibers of a morphism with a natural structure of a scheme.
Given a morphism $f : X \to Y$ and a set-theoretic point $y \in Y$, recall y is determined by the inclusion $\mathrm{Spec}(k_y) \to Y$, where $k_y = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ is the residue field at $y \in Y$.

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Definition

The fiber X_y of X over y is the scheme given by the fiber product diagram:

$$\begin{tikzcd} X_y := \mathrm{Spec}(k_y) \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k_y) \arrow[r, "f"] & Y \end{tikzcd}$$

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A similar notion exists for schematic points and geometric points.

Example (Fiber of a $\mathbb{C}$-point)

Let $X = \mathrm{Spec}(\mathbb{C})$, $Y = \mathrm{Spec}(\mathbb{R})$ and $\eta \in Y(\mathbb{C})$ the point corresponding to the inclusion $\mathbb{R} \to \mathbb{C}$. What is the fiber of η?

Example (Fiber of a $\mathbb{C}$-point)

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$$Y_\eta = \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}} (\mathbb{R})\mathrm{Spec}(\mathbb{C}) = \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Spec}(\mathbb{C} \times \mathbb{C})$$

In particular, the topological space underlying the fiber over η has two points.

Exercise

1. Let $f : X \to Y$ be a morphism of schemes. Show that $|X_y| \simeq f^{-1}(y)$ as a topological space, for any $y \in Y$.
2. Let $X = \mathrm{Spec}(k[s,t]/(s-t^2))$ and $Y = \mathrm{Spec}(k[s])$ be k-schemes with the morphism f associated to the map $s \mapsto s : k[s] \to k[s,t]$ and k a field. Compute the fibers of f.

Solution.

1. We may assume $Y = \mathrm{Spec}(B)$ is affine. Assume first that $X = \mathrm{Spec}(A)$ is affine as well and $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is given by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ for $\varphi : B \to A$.

Solution.

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For X non-affine, we need to show that the fiber of f over y is obtained by gluing of the fibers of $f\,|_{U_i}$, where $X = \bigcup_i \operatorname{Spec}(U_i)$ is an affine cover. This will be shown in the home exercise.

$\square$

Proof.

Now $f : X = \mathrm{Spec}(k[s,t]/(s-t^2)) \to Y = \mathrm{Spec}(k[s])$ is associated to the map $s \mapsto s$. The prime ideals of $k[s]$ are either $\mathfrak{p} = (s-\lambda)$ or $\mathfrak{p} = 0$ and the residue field of $k[s]$ of $\mathfrak{p}$ is $k[s]/(p(s))$ for $p \in k[s]$ irreducible in the first case, or $k(s)$ over the generic point.

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   $$X_y = \mathrm{Spec}(k[s,t](s-t^2) \otimes_{k[s]} k[s]/(p(s))) = \mathrm{Spec}(K[t]/(s^2-t))$$

   where K is the splitting field of p. If s has a square root in K then $X_y \simeq \mathrm{Spec}(K[t]/(t-\sqrt{s})) \bigsqcup \mathrm{Spec}(K[t]/(t+\sqrt{s}))$ and hes two points. Otherwise, X_y is the spectrum of a field and has one point.

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2. If $p(s) = s$ then X_y is non-reduced and has one point.

   $$X_y = \mathrm{Spec}(k[s,t]/(s-t^2) \otimes_{k[s]} k_{s=0}) = \mathrm{Spec}(k[t]/t^2).$$

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3. Finally, if $p = 0$, then

$$\begin{aligned} X_y &= \mathrm{Spec}(k[s,t]/(s-t^2) \otimes k(s)) \simeq \mathrm{Spec}(k[t] \otimes k(t^2)) \\ &= \mathrm{Spec}(k(t)). \end{aligned}$$

$\square$

Base change

Another useful application of fiber products is the ability to change the base of our scheme. Given a scheme X over a scheme S, and S' another S-scheme, we get a new scheme $X_{S'}$ over S' by setting $X_{S'} = X \times_S S'$. This defines a functor $\underline{\mathbf{Sch}}_S \to \underline{\mathbf{Sch}}_{S'}$.

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Exercise (Home exercise)

Show that the following properties are stable under base change:

1. $X \to S$ is a closed embedding
2. $X \to S$ is an open embedding
3. $X \to S$ is quasicompact
4. $X \to S$ is of finite type

Separated and proper morphisms

General topogical facts

- A topological space X is Hausdorff iff the diagonal embedding $X \xrightarrow{\Delta} X \times X$ is closed, iff $\Delta(X)$ is closed.
- A Hausdorff topological space X is compact iff for any Hausdorff topological space Y, the projection map $X \times Y \to Y$ is closed.

Definition

A morphism $X \to S$ of schemes is said to be separated if the diagonal embedding $X \to X \times_S X$ is closed.

Example

If X and S are both affine then $X \to S$ is separated.

Exercise

Let $X \to S$ be a separated morphism with S affine. Let U and V be open affine subsets of X. Show that $U \cap V$ is also affine.

Is this true if X is not separated?

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Proof.

Write $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$.

1. *Step 1.* $U \times_S V = \mathrm{Spec}(A \times_R B)$ is an open affine subscheme of $X \times_S X$, and $\Delta_S(X) \cap U \times_S V$ is a closed subscheme of it.

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This fails for non-separated X; for example- if X is the affine line with doubled origin, and U and V are the two copies of $\mathbb{A}^1$ within it, then $U \cap V = D(0)$, which is not affine. $\square$

Questions?