# Generalized functions
# Exercise sheet 5

Exercises marked with * are optional.
Fix $W \subseteq V$ and $E$ finite dimensional topological vector spaces over a local field $F$. Put $n = \dim_F V$.

**Exercise 1.** Show that $C^{-\infty}(V) \otimes E \simeq (C_c^\infty(V, E^* \otimes \mathrm{Haar}(V)))^*$.

**Exercise 2.** Given $1 \leq k \leq n$, find an isomorphism $\Lambda^k(V^*) \simeq \Lambda^k(V)^*$.

**Exercise 3.** Recall that

$$\Omega^{\mathrm{top}}(V) = \Lambda^n(V^*), \quad |V| = \{f : V \to \mathbb{R} : \forall \alpha \in F,\ f(\alpha v) = |\alpha|\, f(v)\},$$

and

$$\mathrm{Dens}(V) = \{f : V^n \to \mathbb{R} : f(Av_1, \ldots, Av_n) = |\det(A)|\, f(v_1, \ldots, v_n)\}.$$

Show that $\left|\Omega^{\mathrm{top}}(V)\right| \simeq \mathrm{Dens}(V)$.

**Exercise 4.**

(1) Show that $\mathrm{Haar}(W) \otimes \mathrm{Haar}(V/W) \simeq_{\mathrm{can}} \mathrm{Haar}(V)$.
(2) Show that $\mathrm{Haar}(V^*) \simeq_{\mathrm{can}} \mathrm{Haar}(V)^*$.
(3) Conclude that $\mathrm{Haar}(W) \otimes \mathrm{Haar}(V^*) \simeq_{\mathrm{can}} \mathrm{Haar}(W^\perp)$, where $W^\perp = (V/W)^*$.

**Exercise 5.** Assume $F = \mathbb{R}$. Find a distribution $\xi \in \mathrm{Dist}(V \setminus W)$ such that $\nexists \eta \in \mathrm{Dist}(V)$ with $\eta_{V \setminus W} = \xi$.

**Exercise 6.** * Let $F_m(W) = (C_c^\infty(V)/V_m(W))^* = \{\xi \in \mathrm{Dist}(V) : \langle \xi, f \rangle = 0 \text{ for any } f \in V_m(W)\}$, where $V_m(W)$ is as defined in class. Put $G_m(W) = F_m(W) \otimes \mathrm{Haar}(W)^*$, the corresponding quotient of the space of generalized functions. Show that

$$G_m(W)/G_m(W) \simeq_{\mathrm{can}} C^{-\infty}(W, \mathrm{Sym}^m(W^\perp) \otimes \mathrm{Haar}(W^\perp)).$$

## The tangent space of a manifold

This part of the exercise is not mandatory, but it highly recommended that you solve and submit these questions.

Let $M$ be a $C^\infty$-manifold over $\mathbb{R}$, and let $x \in M$. In this part of the exercise we will prove the equivalence of the definitions given in class for the tangent space $T_x(M)$ of $M$ at $x$.

We recall some notation:

- $\mathcal{C}_x(M)$ denotes the set of smooth curves $\gamma : (-1,1) \to M$ such that $\gamma(0) = x$. This set is endowed with a relation $\sim$ such that $\gamma_1 \sim \gamma_2$ if for any $x \in U \subseteq M$ and any diffeomorphism $\varphi : U \to \mathbb{R}^n$
$$\frac{d}{dt}\left(\varphi \circ \gamma_1 - \varphi \circ \gamma_2\right) |_{t=0} = 0.$$
- $\mathrm{Der}_x(C^\infty(M), \mathbb{R})$ denotes the vector space of derivations of $C^\infty(M)$ with values in $\mathbb{R}$ over $x$. That is- linear maps $d : C^\infty(M) \to \mathbb{R}$ which satisfy the Leibniz rule, i.e., for any $f, g \in C^\infty(M)$,
$$d(f \cdot g) = f(x)d(g) + g(x)d(f).$$
- Let $\mathfrak{m}_x := \{f \in C^\infty(M) : f(x) = 0\}$ and $\mathfrak{m}_x^2 = \left\{\sum_{i=1}^N f_i g_i : N \in \mathbb{N}, f_1, g_1, \ldots, f_N, g_N \in \mathfrak{m}_x\right\}$.

**Exercise 7.**

(1) Show that, given $\gamma_1, \gamma_2 \in \mathcal{C}_x(M)$, $\gamma_1 \sim \gamma_2$ if and only if there exists $x \in U \subseteq M$ open and a diffeomorphism $\varphi : U \to \mathbb{R}^n$ such that $\frac{d}{dt}(\varphi \circ \gamma_1 - \varphi \circ \gamma_2) |_{t=0} = 0$.

(2) Let $\gamma \in \mathcal{C}_x(M)$. Show that the map $d_\gamma : C^\infty(M) \to \mathbb{R}$, defined by $d_\gamma(f) = \frac{d}{dt}(f \circ \gamma) |_{t=0}$, is an element of $\mathrm{Der}_x(C^\infty(M), \mathbb{R})$.

(3) Show that $d_{\gamma_1} = d_{\gamma_2}$ if and only if[1] $\gamma_1 \sim \gamma_2$. Deduce that the map $[\gamma]_\sim \mapsto d_\gamma : (\mathcal{C}_x/\sim) \to \mathrm{Der}_x(C^\infty(M), \mathbb{R})$ is injective.

(4) Given $d \in \mathrm{Der}_x(C^\infty(M), \mathbb{R})$ and a diffeomorphism $\varphi = (\varphi_1, \ldots, \varphi_n) : U \to \mathbb{R}^n$ with $x \in U \subseteq M$ open, define $u_d = (d(\varphi_1), \ldots, d(\varphi_n))$ and put
$$\gamma_d(t) = \varphi^{-1}\left(\varphi(x) + t u_d\right).$$
Show that $\gamma_d$ is a smooth curve and that the association $d \mapsto \gamma_d$ is the inverse of the map defined in item (3).

**Exercise 8.**

(1) Let $f, g \in C^\infty(M)$ be such that $f |_U \equiv g |_U$ for some open neighbourhood $x \in U$. Show that $d(f) = d(g)$ for all $d \in \mathrm{Der}_x(C^\infty(M), \mathbb{R})$. (*Hint*: if $f$ vanishes in a neigbourhood of $x$ then there exist $f_1, f_2$ with $f_1(x) = f_2(x) = 0$ such that $f = f_1 f_2$.)

(2) Given $f \in C^\infty(M)$, $x \in U \subseteq M$ open and $\varphi : U \to \mathbb{R}^n$, a diffeomorphism with $\varphi(x) = 0$, show that $f \circ \varphi^{-1}(t_1, \ldots, t_n) = f(x) + \sum_{i=1}^n a_i t_i + h(t_1, \ldots, t_n)$ with $a_i = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(0)$, and $h \in C^\infty(\mathbb{R}^n)$ satisfying $\lim_{\mathbf{t} \to 0} \frac{h(\mathbf{t})}{\|\mathbf{t}\|^2} = 0$.

(3) Show that, for $d \in \mathrm{Der}_x(C^\infty(M), \mathbb{R})$, we have that $d \equiv 0$ if and only $d(\ell) = 0$ for any $\ell \in C^\infty(M)$ such that $\ell \circ \varphi^{-1}$ is linear.

(4) Show that for any $d \in \mathrm{Der}_x(C^\infty(M), \mathbb{R})$, the map $T_d : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{R}$, defined by $T_d(f + \mathfrak{m}_x^2) = d(f)$, is well-defined. Show that the map $d \mapsto T_d$ is injective.

(5) Given $T \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ define a map $d_T : C^\infty(M) \to \mathbb{R}$ by $d_T(f) = T((f - f(x)) + \mathfrak{m}_x^2)$. Show that $d_T \in \mathrm{Der}_x(C^\infty(M), \mathbb{R})$, and that the map $T \mapsto d_T$ is the inverse of the map defined in Item (3).

[1] For the *only if* implication, you may invoke the existence of smooth cut-off functions on $M$ without proof.