On Regular Representations of Groups

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Introduction

Jaikin's Example- $\zeta_{\operatorname{SL}_2(\mathfrak{o})}$

Norm-1 Subgroups of Local Division Algebras

Regular Representations

Motivation

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- ightharpoonup Γ is a topological group.
- ▶ $Irr(\Gamma)$ is the set of continuous irreducible complex **finite-dimensional** representations of Γ , upto equivalence.
- ▶ **Mission-** Understand $Irr(\Gamma)$.



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• Parametrize all irreducible representations of Γ according to some known parametrizing space

$$X \leftrightarrow \operatorname{Irr}(\Gamma)$$
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▶ For each class $[\rho] \in \operatorname{Irr}(\Gamma)$, present a complex finite dimensional vector space V_{ρ} , and the homomorphism $\rho : \Gamma \to \operatorname{Aut}(V_{\rho})$.

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- ▶ **Another Option** Describe all characters of Γ explicitly

$$\chi_{\rho}(\gamma) = \text{Tr}(\rho(\gamma)) \quad (\gamma \in \Gamma).$$



▶ **Asymptotics**- For any $n \in \mathbb{N}$, define

$$r_n(\Gamma) := \# \{ [\rho] \in \operatorname{Irr}(\Gamma) \mid \dim(\rho) = n \} \in \mathbb{N} \cup \{0, \infty\} .$$



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The group Γ is said to be **rigid** if $r_n(\Gamma) < \infty$ for all n.



Let Γ be an arithmetic group, e.g. $\Gamma = SL_3(\mathbb{Z})$. The group Γ is said to have **the congruence subgroup property** (CSP) if any finite-index normal subgroup of Γ contains a principal congruence subgroup.

For example, the principal congruence subgroups of $SL_3(\mathbb{Z})$ are all subgroups of the form $Ker(SL_3(\mathbb{Z}) \to SL_3(\mathbb{Z}/n\mathbb{Z}))$, for some $n \in \mathbb{N}$.

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The group $SL_2(\mathbb{Z})$, however, does not have CSP.



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The **representation zeta function** of Γ is

$$\zeta_{\Gamma}(s) = \sum_{n=1}^{\infty} r_n(\Gamma) n^{-s} = \sum_{[\rho] \in \operatorname{Irr}(\Gamma)} \dim(\rho)^{-s}, \quad (s \in \mathbb{C}).$$
 (*)

The representation zeta function of Γ admits an Euler product-

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- $\zeta_{SL_3(\mathbb{Z})}$ enumerates arbitrary representations of Γ .
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- ▶ The non-archimedean factors $\zeta_{SL_3(\mathbb{Z}_p)}$ enumerate continuous representations.



p-adic Groups

Theorem (Jaikin-Zapirain, '07)

Fix an odd prime $p \in \mathbb{Z}$. Let G be a rigid compact p-adic analytic group. There exist rational functions $f_1(t), \ldots, f_k(t) \in \mathbb{Q}(t)$ and numbers $n_1, \ldots, n_k \in \mathbb{N}$ such that

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Example (AKOV) Assume p > 2 and let $G \subseteq SL_2(\mathbb{Z}_p)$ be given by $SL_2(\mathbb{Z}_p) \cap (1 + pM_2(\mathbb{Z}_p))$. Then

$$\zeta_G(s) = p^3 \frac{1 - p^{-2-s}}{1 - p^{1-s}}.$$





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3. Model theoretic proof of the rationality of such integrals (Denef).



Jaikin's Example

Computing $\zeta_{\mathrm{SL}_2(\mathfrak{o})}$

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- ▶ Fix $G = \operatorname{SL}_2(\mathfrak{o})$. The subgroups $G^m = \operatorname{SL}_2(\mathfrak{o}) \cap (1 + \pi^m \operatorname{M}_2(\mathfrak{o}))$ are called the **principal congruence subgroups** of G. The sequence $\{G^m\}$ is a neighbourhood basis at 1.

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 .

Put
$$\operatorname{Irr}^k(G) := \{ \rho \in \operatorname{Irr}(G) \mid \rho \text{ of level } k \}$$
 . Then

$$\bigcup_{m\leq k} Irr^m(G) = Irr\left(SL_2(\mathfrak{o}/\mathfrak{p}^{k+1})\right).$$

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Computing $\zeta_{SL_2(\mathfrak{o})}$ - Strategy

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- ▶ In what follows, we will describe an explicit map between the set $Irr^k(G)$ that the space of orbits in a finite *G*-space.
- ▶ Using this map, we can compute the cardinality of $\operatorname{Irr}^k(G)$, as well as the dimensions of its elements, thereby allowing us to compute

$$\zeta_{\operatorname{SL}_2(\mathfrak o/\mathfrak p^{k+1})}(s) = \sum_{\substack{\rho \in \operatorname{Irr}^m(G) \\ m < k}} \dim(\rho)^{-s}.$$

Computing $\zeta_{SL_2(\mathfrak{o})}$ - Strategy (contd.)

Applying the process above for all $k \in \mathbb{N}$ we obtain-

Theorem (Jaikin-Zapirain, '07)

$$\begin{split} \zeta_{\mathrm{SL}_2(\mathfrak{o})}(s) = & \zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s) \\ &+ \frac{4q\left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \left(\frac{q-1}{2}\right)^2(q^2+q)^{-s}}{1-q^{1-s}}. \end{split}$$

General Notation

- \triangleright G^r superscript indices stand for congruence subgroups.
- ▶ G_m subscript indices stand for congruence quotients- i.e. quotients of G by a subgroup G^m .
- ▶ G_m^r both super- and subscript indices signify congruence subquotients, G^r/G^m .

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Similar notation will be used for Lie-algebras. From here on we fix $K \supseteq \mathfrak{o} \supseteq \mathfrak{p} = \pi \mathfrak{o}$.

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- ▶ Put $r = \lceil \frac{k}{2} \rceil$, and consider the restriction of ρ to the group G_k^r . The following hold
 - ▶ The group G_k^r is abelian.
 - ▶ The map $1 + \pi^r x \mapsto x$ is an isomorphism of abelian groups between G^r/G^k and the finite Lie-ring $\mathfrak{g}_{k-r} := \mathfrak{sl}_2(\mathfrak{o}/\mathfrak{p}^{k-r})$ of traceless 2×2 matrices over $\mathfrak{o}/\mathfrak{p}^{k-r}$.
 - ► There is a *G*-equivariant bijection of \mathfrak{g}_{k-r} with its Pontryagin dual $\widehat{\mathfrak{g}_{k-r}}$,

$$y \mapsto \phi_{y}(x) := \chi(\operatorname{Tr}(xy)).$$

Get a *G*-equivariant bijection

$$g_{k-r} \xrightarrow{1-1} \widehat{G_k^r} = \operatorname{Irr}(G_k^r)$$
$$y \mapsto \theta_y(1 + \pi^r x) = \phi_y(x).$$

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By Clifford Theory, there exists $y \in \mathfrak{g}_{k-r}$ which is unique up to G_k -conjugation and a fixed $e \in \mathbb{N}$, such that

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We obtain a mapping $\rho \mapsto Ad(G_k)y$.



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More generally- Suppose that there exists $\Theta \in Irr(S)$ whose restriction to G_k^r lies above θ_y and under ρ . Then the induced representation Θ^{G_k} is irreducible, and furthermore

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The question of computing the dimension of ρ is now converted to analysis of S and of the existence of extensions of θ_y to S.

Lucky Facts About \$\silon{\gamma}_2\$

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$$\mathbf{C}_{G_k}(y)\cdot (G_k^{k-r})\subseteq S. \tag{**}$$



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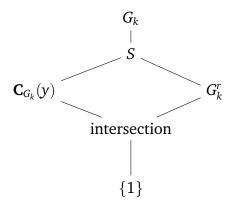
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- ▶ The following facts are true-
 - ▶ If *y* is non-zero modulo \mathfrak{p} , then $\mathbf{C}_{G_k}(y)$ is abelian.
 - ▶ In fact, $\mathbf{C}_{G_k}(y)$ belongs to one of 3 isomorphism types of abelian subgroups of G_k .
 - ▶ The inclusion (**) is an equality.



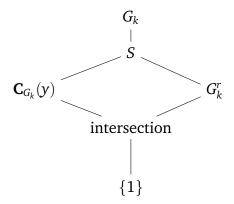
Extending θ_y - Even Level



Assume k = 2r.

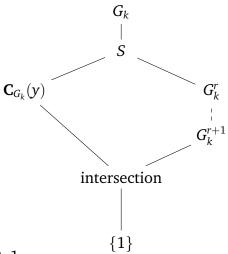


Extending θ_{v} - Even Level



Assume k = 2r. By gluing θ_v to any suitable representation of $\mathbf{C}_{G_{k}}(y)$, we obtain the desired extension $\Theta \in \operatorname{Irr}(S)$.

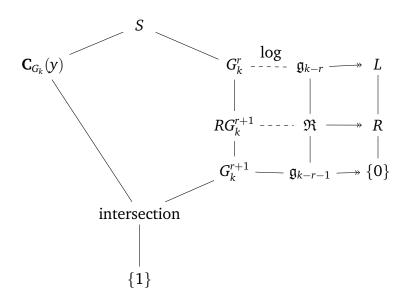
Extending θ_y - Odd Level



Assume k = 2r + 1.

Extending θ_y - Odd Level

In order to be able to extend θ_y to S, it is necessary to understand the set of representations of G^r/G^k lying above θ_y . This is done using the method of **Heisenberg lifts**.



The Norm-One Subgroups of Local Division Algebras

Division Algebras- General Properties

Let D be a division algebra, with $\mathbf{Z}(D) = K$ a local field of odd residual characteristic. Let $\ell = \deg D = (\dim_K D)^{1/2}$. Let L/K be an unramified field extension of degree ℓ .

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The algebra D embeds as a subalgebra of the matrix algebra $M_{\ell}(L)$, where an element $x=x_0+\nu x_1+\ldots+\nu^{\ell-1}x_{\ell-1}$ is mapped to the matrix

$$\Lambda_x = egin{pmatrix} x_0 & \pi\sigma(x_{\ell-1}) & \dots & \pi\sigma^{\ell-1}(x_1) \ x_1 & \sigma(x_0) & \dots & \pi\sigma^{\ell-1}(x_2) \ dots & \ddots & dots \ x_\ell & \sigma(x_{\ell-1}) & \dots & \sigma^{\ell-1}(x_0) \end{pmatrix},$$

where $\sigma: L \to L$ is a generates the Galois groups Gal(L/K).

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where $\sigma: L \to L$ is a generates the Galois groups $\operatorname{Gal}(L/K)$. Define $\operatorname{Nrd}(x) = \det \Lambda_x$ and $\operatorname{Trd}(x) = \operatorname{Tr}\Lambda_x$.

The Norm-One Subgroup of D^{\times}

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$$G = \operatorname{SL}_1(D) = \{x \in D \mid \operatorname{Nrd}(x) = 1\} \subseteq \mathcal{O}^{\times}.$$

The congruence subgroups of *G* are defined by

$$G^m := G \cap (1 +
u^m \mathbf{M}_{\ell}(\mathfrak{o}_L)),$$

where ν is the element of \mathcal{O} which is identified with the matrix

$$egin{pmatrix} 0 & & \dots & \pi \ 1 & 0 & & dots \ & \ddots & \ddots & \ & & 1 & 0 \end{pmatrix} \in \mathrm{M}_{\ell}(\mathfrak{o}_L).$$

The **level** of an irreducible representation $\rho \in Irr(G)$ is defined as in the case of $SL_2(\mathfrak{o})$, i.e. the minimal $m \in \mathbb{N}$ such that $G^{m+1} \subset Ker(\rho)$.

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Owing to the work of H. Koch, the centralizer in G_k of an element $y \in G_k^r$ can be effectively described. An explicit description of the centralizer was presented in the work of V. Singh. In particular- in the case where $\ell = \deg D$ is a prime number, all such centralizers are of the form

$$((L')^{\times}\cap G)\cdot G^{m'},$$

where $L' \supseteq K$ is a field extension, and $m' \in \mathbb{N}$ is dependent on y.

The case where ℓ is prime is amenable to a similar treatment as $SL_2(\mathfrak{o})$.

Results

Let D/K be a division algebra of prime degree ℓ , and put $\epsilon = \gcd(\ell, q-1) = \#\mu_{\ell}(K)$.

Theorem

1. There exist rational polynomials $a_m^{\epsilon}(t)$, $d_m^{\epsilon}(t) \in \mathbb{Q}(t)$ $(m \in \mathbb{N}_0)$, such that for any $m \in \mathbb{N}_0$, $\mathrm{SL}_1(D)$ has exactly $a_m^{\epsilon}(q)$ irreducible representations of level m. All such representations are of dimension $d_m^{\epsilon}(q)$.

Theorem

2.

$$egin{aligned} \zeta_{\mathrm{SL}_1(D)}(s) &= \zeta_{\mathrm{SL}_1(\mathbb{F}_{q^\ell}|\mathbb{F}_q)}(s) + \sum_{m=1}^\infty a_m^\epsilon(q) d_m^\epsilon(q)^{-s} \ &= rac{q^\ell - 1}{q-1} \left(1 - q^{-\binom{\ell}{2}}s
ight) + \left(rac{q^\ell - 1}{\epsilon(q-1)}
ight)^{-s} \epsilon^2 \left(q-1
ight) \left(\sum_{\lambda = 0}^{\ell-2} q^{\lambda(1 - rac{\ell - 1}{2}s)}
ight)}{1 - q^{(\ell-1) - \binom{\ell}{2}s}} \end{aligned}$$

Theorem

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$$\begin{split} \zeta_{\operatorname{SL}_1(D)}(s) &= \zeta_{\operatorname{SL}_1(\mathbb{F}_{q^\ell}|\mathbb{F}_q)}(s) + \sum_{m=1}^\infty a_m^\epsilon(q) d_m^\epsilon(q)^{-s} \\ &= \frac{\frac{q^\ell - 1}{q - 1} \left(1 - q^{-\binom{\ell}{2}}s\right) + \left(\frac{q^\ell - 1}{\epsilon(q - 1)}\right)^{-s} \epsilon^2 \left(q - 1\right) \left(\sum_{\lambda = 0}^{\ell - 2} q^{\lambda(1 - \frac{\ell - 1}{2}s)}\right)}{1 - q^{(\ell - 1) - \binom{\ell}{2}s}} \end{split}$$

3. If $\ell > 2$ then all irreducible representations of $SL_1(D)$ are induced from one-dimensional representations of congruence subgroups.

Let $\mathbf{G} \subseteq \operatorname{GL}_n$ be a connected reductive algebraic group, which is defined over \mathfrak{o} . Put $G = \mathbf{G}(\mathfrak{o})$.



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The congruence subgroups of *G* are defined by

$$G^m := G \cap (1 + \pi^m \mathbf{M}_n(\mathfrak{o})), \quad (m \in \mathbb{N}),$$

and under some mild assumptions on p, we have that

$$G_m := G/G^m \cong \mathbf{G}(\mathfrak{o}/\mathfrak{p}^m).$$

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We retain the notation G_m^r for the subgroups G^r/G^m .

By virtue of **G** being reductive, it is possible to apply the process described above for $\mathbf{G} = \operatorname{SL}_2$ and to associate any representation $\rho \in \operatorname{Irr}(G)$ of level m with an orbit in a finite Lie-ring \mathfrak{g}_{m-r} , as above.

By virtue of **G** being reductive, it is possible to apply the process described above for $\mathbf{G} = \operatorname{SL}_2$ and to associate any representation $\rho \in \operatorname{Irr}(G)$ of level m with an orbit in a finite Lie-ring \mathfrak{g}_{m-r} , as above.

However, the description of stabilizers of the associated characters $\theta_y \in \operatorname{Irr}(G_m^r)$ is seldom as straightforward as in the previous cases.

Let \mathfrak{g} be the Lie algebra of $\mathbf{G}(\mathfrak{o})$, and put $\mathfrak{g}^m := \pi^m \mathfrak{g}$ and $\mathfrak{g}_m = \mathfrak{g}/\mathfrak{g}^m$.

Definition (Steinberg, Hill)

- ▶ An element $y \in \mathfrak{g}_1 = \mathfrak{g}(\mathbb{F}_q)$ is called regular if its centralizer $\mathbf{C}_{\mathbf{G}(\mathbb{F}_q)}(y)$ in G_1 is of minimal dimension.
- ▶ An element $y \in \mathfrak{g}_m$ is called regular if its image in \mathfrak{g}_1 is regular.
- ▶ A representation $\rho \in Irr(G)$ of level m is called regular, if the orbit associated to it is regular.

For example, in the case $\mathbf{G} = \mathrm{GL}_n$ it is true that an element $y \in \mathfrak{gl}_n(\mathfrak{o}/\mathfrak{p}^m)$ is regular if and only if its minimal polynomial is equal to its characteristic polynomial.

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It is also true in this case that the centralizer $\mathbf{C}_{G_m}(y)$ is an abelian group, and in fact equals to the group

$$(\mathfrak{o}/\mathfrak{p}^m)[y] \cap \mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}^m).$$

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In addition, if y is taken to be in the orbit associated to a regular representation $\rho \in \operatorname{Irr}(\operatorname{GL}_n(\mathfrak{o}))$ of level m, and $\theta_y \in \operatorname{Irr}(G_m^r)$ is the associated character, then

$$Stab^{G_m}(\theta_y) = \mathbf{C}_{G_m}(y) \cdot \mathbf{G}_m^{m-r},$$

as in the case of SL₂.



Using this stabilizer structure, G. Hill managed to produce an explicit description of all regular characters of $GL_n(\mathfrak{o})$.

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Recent works (KOS, Stasinsky, Takase, S.) have applied the construction of regular representations to the study of representation growth, and attempted to generalize the construction to other families of reductive groups.

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A possible application to the study of regular representation is to the analysis of the regular part of the representation zeta function of G, in view of the decomposition-

$$\zeta_G(s) = \zeta_{\mathbf{G}(\mathbb{F}_q)}(s) + \zeta_G^{\text{reg.}}(s) + \zeta_G^{\text{irreg.}}(s).$$

Regular Zeta Functions of Groups of Type A_{n-1}

Let $\mathbf{G}_{+1} = \mathrm{GL}_n$ and $\mathbf{G}_{-1} = \mathrm{U}_n$ the unitary group attached to an unramified quadratic extension L/K. Let $\epsilon \in \{\pm 1\}$.



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As was recently shown by Krokovski, Onn and Singla, there exist

- $ightharpoonup \mathcal{A}_n$ a 'nice' set,
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As was recently shown by Krokovski, Onn and Singla, there exist

- A_n a 'nice' set,
- an explicit map from the regular part of $\operatorname{Irr}^m(\mathbf{G}_{\epsilon}(\mathfrak{o}))$ to \mathcal{A}_n ,
- ▶ polynomials $u_{\epsilon}^{\tau}(t)$, for all $\tau \in A_n$

such that, if $\rho \in \operatorname{Irr}^m(\mathbf{G}_{\epsilon}(\mathfrak{o}))$ is mapped to $\tau \in \mathcal{A}_n$ then

$$\dim(
ho) = q^{m\binom{n}{2}} \left(rac{|\mathbf{G}_{\epsilon}(\mathbb{F}_q)|}{u^{ au}_{\epsilon}(q)}
ight).$$



The number of regular representations of a given type $\tau \in \mathcal{A}_n$ can also be described, using the polynomial $w_d(q)$, which gives the number of irreducible polynomials of degree d over \mathbb{F}_q .

The number of regular representations of a given type $\tau \in \mathcal{A}_n$ can also be described, using the polynomial $w_d(q)$, which gives the number of irreducible polynomials of degree d over \mathbb{F}_q . From this, one obtains the regular zeta function of $\mathbf{G}_{\epsilon}(\mathfrak{o})$.

Theorem (Krakovski, Onn, Singla, '16)

$$egin{aligned} \zeta_{\mathbf{G}_{\epsilon}(\mathfrak{o})}^{\mathrm{reg.}}(s) = & rac{1}{1 - q^{n - {n \choose 2}s}} \ & \cdot \sum_{ au \in \mathcal{A}_n} u_{\epsilon}^{ au}(q) \prod_{i=1}^n inom{\sum_j au_{i,j}}{ au_{i,1}, au_{i,2}, \dots, au_{i,n}} inom{w_d(q)}{\sum_j au_{i,j}} \left(rac{|\mathbf{G}_{\epsilon}(\mathbb{F}_q)|}{u_{\epsilon}^{ au}(q)}
ight)^{-s}. \end{aligned}$$

Regular Zeta Function for Classical Groups

A similar analysis has be thus far accomplished for classical groups defined over o, i.e. linear algebraic groups of the form

$$\left\{ \mathbf{X} = (x_{i,j}) \in \mathbf{M}_n \mid \mathbf{X}^T \mathbf{J} \mathbf{X} = \mathbf{J} \right\},$$

where $J \in GL_n(\mathfrak{o})$ is a fixed symmetric or anti-symmetric matrix.

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where $\mathbf{J} \in \mathrm{GL}_n(\mathfrak{o})$ is a fixed symmetric or anti-symmetric matrix. In the specific cases of the symplectic group Sp_{2n} and the odd-orthogonal group SO_{2n+1} , the regular representation zeta function has been computed.

Questions?

Thank You.

