Algebraic Geometry 2
Exercise sheet 2

Throughout, X and Y are topological spaces.

1. Leray and Grothendieck sheaves

Recall the following definition.

Definition 1 (Leray sheaf). A *Leray sheaf* (or L-sheaf, for short) is a pair (E, p), where E is a topological space and $p : E \to X$ is locally a homeomorphism.

To avoid confusion with L-sheaves, we sometimes call sheaves, as defined in the first lecture, Grothendieck sheaves, or G-sheaves for short.

Given an L-sheaf (E, p) on a topological space X, we defined a G-sheaf of sets $G(E)$ by

$$G(E)(U) := \Gamma(U, E) = \{s : U \to E \text{ continuous} : s \circ p = \mathrm{Id}_U\}$$

for $U \subseteq X$ open. Also, given a G-sheaf $\mathcal{F}$ we defined an L-sheaf $L(\mathcal{F})$ by

$$L(\mathcal{F}) := \dot{\bigcup_{x\in X}} \{x\} \times \mathcal{F}_x,$$

where $\mathcal{F}_x$ is the stalk at x, and $p : L(\mathcal{F}) \to X$ given by projection onto the first coordinate. The toplogy on $L(\mathcal{F})$ is generated by the open sets $V_{\xi,U} = \{[\xi, U]_{\mathcal{F}_x} : x \in U\}$ for $U \subseteq X$ open and $\xi \in \mathcal{F}(U)$, where $[\xi, U]_{\mathcal{F}_x}$ denotes the class of (ξ, U) in the stalk $\mathcal{F}_x$.

Exercise 1. Let $E \xrightarrow{p} X$ be an L-sheaf. Assume that the following conditions hold:

(i) For any $x \in X$, the fiber $p^{-1}(x)$ is endowed with the structure of an abelian group.
(ii) For any $x \in X$, there exists an open neighbourhood $x \in U \subseteq X$, a discrete abelian group A_U and a homeomorphism $\psi : U \times A_U \to p^{-1}(U)$ such that
  (a) $p \circ \psi(y, a) = y$ for $y \in U$, $a \in A_U$, and
  (b) the map $a \mapsto \psi(y, a) : A_U \to p^{-1}(y)$ is an isomorphism of abelian groups, for all $y \in U$.

Show that $G(E)$ is a sheaf of abelian groups over X with stalk $(G(E))_x \simeq p^{-1}(x)$ for all $x \in X$.

Conclude that, given a presheaf $\mathcal{F}$ of abelian groups over X with sheafification $\mathcal{F}^+$, we have $\mathcal{F}_x \simeq \mathcal{F}^+_x$ for all $x \in X$.

Exercise 2. Pullback of sheaves. Given a continuous map $f : X \to Y$ and an L-sheaf $E \xrightarrow{p} Y$, recall that the *pullback* E is given by the pair $(X \times_Y E, \pi_1)$, where $X \times_Y E$ is fiber product $\{(x, e) : f(x) = p(e)\} \subseteq X \times E$, with induced topology, and π_1 is projection onto the first coordinate.

(1) Show that π_1 is a local homeomorpism, and thus $(X \times_Y E, \pi_1)$ is an L-sheaf.
(2) Let $U \subseteq X$ and $f(U) \subseteq V \subseteq U$ be open. Given $s \in \Gamma(V, E)$, show that the map $\widetilde{s}(x) = (x, s \circ f(x))$ define a continuous section of π_1 over U.
(3) Conclude that there exists a natural injective map $\lim_{f(U)\subseteq V \text{ open}} \Gamma(V, E) \hookrightarrow \Gamma(U, X \times_Y E)$.
(4) Let $U \subseteq X$ be open. Assume there exists $W \subseteq E$ open such that $p\,|_W : W \to Y$ is a homeomorphism onto an open set and $f(U) \subseteq p(W)$. Show that $\Gamma(U, X \times_Y E) \simeq \lim_{f(U)\subseteq V \text{ open}} \Gamma(V, E)$, under the map of item (3).
(5) Conclude that, given a sheaf $\mathcal{F}$ on Y, the pullback sheaf $f^{-1}\mathcal{F}$ on X is isomorphic to the sheaf obtained by sheafifying the presheaf $U \mapsto \lim_{f(U)\subseteq V \text{ open}} \mathcal{F}(V)$.

Exercise 3. (Hartshorne, Ex 1.18). Let $f : X \to Y$ be a continuous map of topological spaces.

(1) Show that given a sheaf $\mathcal{F}$ on X, there is a natural morphism $f^{-1}f_*\mathcal{F} \to \mathcal{F}$.
(2) Show that given a sheaf $\mathcal{G}$ on Y, there is a morphism $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.
(3) Use the maps above to prove the existence of a natural bijection

$$\mathrm{Hom}_{\underline{\mathsf{Sh}}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\underline{\mathsf{Sh}}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

2. Spectra and Schemes

Exercise 3. Let A be a commutative unital ring. Recall that, for $I \lhd A$, we defined:

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) : I \subseteq \mathfrak{p}\}.$$

(1) Show that $\{V(I) : I \lhd A\}$ is a topology on $\operatorname{Spec}(A)$.
(2) Show that, for $I, J \lhd A$, we have $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.
(3) Show that the Zariski topology on $\operatorname{Spec}(A)$ is T_0, i.e. for any two distinct elements of $\operatorname{Spec}(A)$, there exists an open set which contains precisely one of them. $(*)$ Can this topology satisfy any stronger separation axioms (i.e. T_i for $i > 0$)?

Exercise 4. Let A be a ring and $(X, \mathcal{O}_X)$ a scheme. Given a morphism $f : X \to \operatorname{Spec}(A)$ with associated map of sheaves $f^\sharp : \mathcal{O}_{\operatorname{Spec}(A)} \to f_*\mathcal{O}_X$, by taking global sections we get a ring homomorphism

$$\alpha(f) := f^\sharp_{\operatorname{Spec}(A)} : \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) \to \Gamma(X, \mathcal{O}_X).$$

(1) Show that this defines a bijection

$$\alpha : \operatorname{Hom}_{\underline{\mathsf{Schemes}}}(X, \operatorname{Spec}(A)) \to \operatorname{Hom}_{\underline{\mathsf{Rings}}}(A, \Gamma(X, \mathcal{O}_X)).$$

Remark. You may use Proposition 2.3 of Hartshorne without proof in this exercise.
(2) Show that $\operatorname{Spec}\mathbb{Z}$ is a terminal object in the category of schemes, i.e. that each scheme X admits a *unique* morphism to $\operatorname{Spec}(\mathbb{Z})$.