Algebraic Geometry 2
Tutorial session 6

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Functor of points

Definition

A functor $F : \underline{\mathbf{C}}^{\mathrm{op}} \to \underline{\mathbf{Sets}}$ is said to be *representable* if there exists $A \in \underline{\mathbf{C}}$ such that $F = h^A$; i.e. $F(B) \simeq \mathrm{Mor}_{\underline{\mathbf{C}}}(B, A)$ for all $B \in \underline{\mathbf{C}}$.

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There is also an analogous definition for covariant functors. There $F : \underline{\mathbf{C}} \to \underline{\mathbf{Set}}$ is said to be representable if

$$F = h_A = (B \mapsto \mathrm{Mor}_{\underline{\mathbf{C}}}(A, B)) \text{ for some } A \in \underline{\mathbf{C}}.$$

New definition of (affine) schemes

Given a scheme $(X, \mathcal{O}_X)$, we can define a functor $\underline{\textbf{Ring}} \to \underline{\textbf{Set}}$ by

$$F_X(R) = \mathrm{Hom}_{\underline{\textbf{Sch}}}(\mathrm{Spec}(R), X).$$

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This is called the *functor of points* of X. By an exercise, a scheme is affine iff F_X is representable; specifically, $F_X(\cdot) = \mathrm{Hom}(\mathcal{O}_X(X), \cdot)$.

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A general scheme, in this setting, would be a functor which is "locally representable".

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The associated functor of points, on the other hand, is very easy. Namely-

$$F_{\mathrm{Spec}(A)}(R) = \mathrm{Hom}_{\underline{\mathbf{Rings}}}(\mathbb{Z}[t, t^{-1}], R) = R^{\times},$$

since choosing a homomorphism $\mathbb{Z}[t, t^{-1}] \to R$ amounts to choosing the image of t, which is necessarily in $R^{\times}$.

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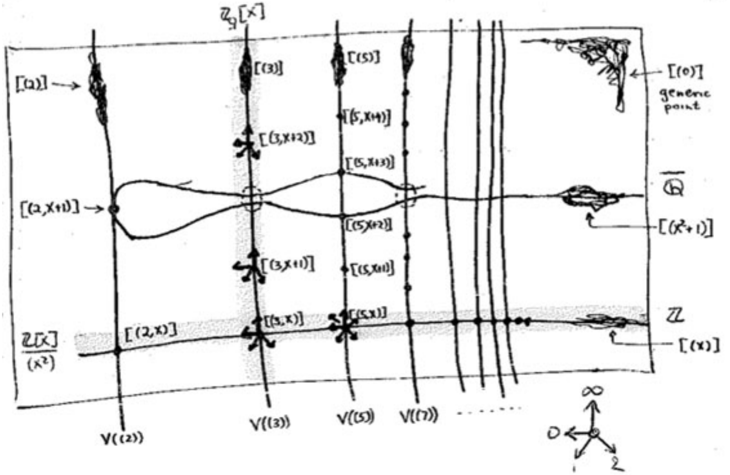
In fact, we can also encode the group laws of $R^{\times}$ in terms natural transformations of $F_{\mathrm{Spec}(A)}$, giving rise to $F_{\mathrm{Spec}(A)}$ as a *group scheme.* For example, the inversion in $R^{\times}$ is "encoded" in the map

$$t \mapsto t^{-1} : A \to A.$$

A very nice example is Mumford's doodle of $\mathrm{Spec}(\mathbb{Z}[t])$:



See `http://www.neverendingbooks.org/grothendiecks-functor-of-points` for more information.

Products and fibered products

Exercise

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1. Show that the map of *sets*, $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^2_k$, defined by $((x-a),(y-b)) \mapsto (x-a, y-b)$ is not surjective.

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Solution.

2. By Nullstellensatz, any proper ideal I of $k[x,y]$ admits a point $(a,b) \in k^2$ such that $f(a,b) = 0$ for all $f \in I$.

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1. For example, the ideal (x) is prime and not in the image of this map.

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On the other hand, taking the functor of points approach, we will have that, for example, $\mathbb{A}^1(R) \times \mathbb{A}^1(R) \simeq \mathbb{A}^2(R)$ for any ring R.

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$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists ! h"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

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\begin{tikzcd}
X \times Z \arrow[drr, bend left, "\pi_Z"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Y Z \arrow[ul, dashed, "\exists! \varphi"'] \arrow[r] \arrow[d] & Z \\
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1. The map $X \times_Y Z \xrightarrow{\varphi} X \times Z$ always exists, by UP of product.

The fibered product

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$$\begin{tikzcd}
X \times Z \arrow[drr, bend left, "\pi_Z"] \arrow[ddr, bend right, "\pi_X"'] \arrow[dr, dashed, shift right, "\exists!\psi"'] & & \\
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2. Since Y is terminal, we have that $f \circ \pi_X = g \circ \pi_Z$ and both equal the unique morphism $X \times Z \to Y$.

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2. Since Y is terminal, we have that $f \circ \pi_X = g \circ \pi_Z$ and both equal the unique morphism $X \times Z \to Y$. Get $X \times_Y Z \xrightarrow{\psi} X \times Y$.

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3. $\varphi \circ \psi$ is the unique map $X \times Z$ which commutes with the projection maps; i.e. it is $\mathbf{1}_{X \times Z}$. Similarly, $\psi \circ \varphi = \mathbf{1}_{X \times_Y Z}$.

Fibered products

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In the case where X, Y and Z are all affine, the existence of fibered product is easy to prove. Essentially, it follows from the UP of the tensor product. In particular

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(C) \simeq \mathrm{Spec}(A \otimes_B C),$$

for A and C algebras over B.

Example

$\mathbb{A}^1_k(R) \times \mathbb{A}^1_k(R) \simeq \mathbb{A}^2_k(R)$ for any ring k and k-algebra R.

Indeed, we have that

$$\begin{aligned}\mathbb{A}^2(R) = \mathrm{Hom}_k(k[x,y], R) &\simeq \mathrm{Hom}_k(k[x] \otimes_k k[y], R) \\ &\simeq \mathrm{Hom}(k[x], R) \times \mathrm{Hom}(k[y], R) = \mathbb{A}^1_k(R) \times \mathbb{A}^1_k(R)\end{aligned}$$

where the second-to-last isomorphism is the UP of $\otimes$.

Some facts about tensor products

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1. Given an ideal $I \lhd A$, we have a natural isomorphism $(A/I) \otimes_A B \simeq B/(\varphi(I))$.
2. Given a multiplicatively closed subset $S \subseteq A$, we have a natural isomorphism $(S^{-1}A) \otimes_A B \simeq \varphi(S)^{-1}B$.

Proof of item (1)

Key tool

Tensor products are right-exact.

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Proof.

Given a ses of modules $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ and another module M consider

$$A \otimes M \xrightarrow{\alpha \otimes \mathbf{1}_M} B \otimes M \xrightarrow{\beta \otimes \mathbf{1}_M} C \otimes M \to 0.$$

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Showing $\beta \otimes \mathbf{1}_M$ is surjective and $\mathrm{Im}(\alpha \otimes \mathbf{1}_M) \subseteq \mathrm{Ker}(\beta \otimes \mathbf{1}_M)$ is easy. Setting $I = \mathrm{Im}(\alpha \otimes \mathbf{1}_M)$, we get a map

$$B \otimes M / I \to C \otimes M.$$

Check that $(c, m) \mapsto \beta^{-1}(c) \otimes m + I$ is a well-defined inverse. $\square$

We can also prove exactness using the more general fact that $\cdot \otimes M$ has a left adjoint (namely- $hom(M, \cdot)$), hence is right exact. This type of proof is neater, but beyond the scope of this tutorial.

Proof of item (1) - contd

We can apply this to the short exact sequence (of A-modules)

$$I \to A \to A/I \to 0$$

and with the functor $\cdot \otimes_A B$, we get

$$\begin{tikzcd}
I \otimes B \arrow[r] \arrow[d, equal, "\wr"] & A \otimes B \arrow[r] \arrow[d, equal, "\wr"] & (A/I) \otimes B \arrow[r] \arrow[d, dotted, "\wr"] & 0 \\
\varphi(I)B \arrow[r] & B \arrow[r] & B/\varphi(I)B \arrow[r] & 0
\end{tikzcd}$$

where the first two columns are the multiplication map.

Proof of item (2)

We have a natural map

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given on generators by

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Surjectivity is clear, because the image contains all fractions of the form $\frac{b}{\varphi(s)}$ for $b \in B$ and $s \in S$.

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The general case is similar- if $\sum_i \frac{a_i}{s_i} \otimes b_i$ is mapped to zero, then

$$\sum_i \frac{\varphi(a_i)b_i}{\varphi(s_i)} = \frac{\sum_i \overbrace{\varphi(a_i) \cdot (\prod_{j \neq i} \varphi(s_j))}^{=\varphi(a_i')} \cdot b_i}{\prod_i \varphi(s_i)} = 0.$$

Put $a_i' = a_i \sum_{j \neq i} s_j$. Then, for some $t' \in S$, we have

$$t'(\sum_i a_i' \otimes b_i) \in \mathrm{Ker}(A \otimes B \to B)$$

Corollary

All examples of morphisms of affine schemes discussed in the previous tutorial are preserved under base change.

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All examples of morphisms of affine schemes discussed in the previous tutorial are preserved under base change. More generally- closed embeddings, open embeddings and localizations of affine schemes are preserved under base change.

Models of schemes

Let X be a $\mathbb{Q}$-scheme, i.e, a scheme with a morphism $X \to \mathrm{Spec}(\mathbb{Q})$.

Definition

A $\mathbb{Z}$-model for X is a scheme X' (over $\mathbb{Z}$) such that $X' \times_{\mathbb{Z}} \mathbb{Q} \simeq X$.

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Exercise

Assume X is affine. Show that, by composing $X \to \mathrm{Spec}(\mathbb{Q})$ with the map $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$, we get a $\mathbb{Z}$-model for X.

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Exercise

Assume X is affine. Show that, by composing $X \to \mathrm{Spec}(\mathbb{Q})$ with the map $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$, we get a $\mathbb{Z}$-model for X. Does this generalize to any morphism of rings $R \to S$? Any inclusion?

Models for schemes

Solution.

Let $A = \Gamma(X, \mathcal{O}_X)$. The claim that $X \to \mathrm{Spec}(\mathbb{Z})$ is a $\mathbb{Z}$-model is equivalent to the following:

Lemma

Let A be a $\mathbb{Q}$-algebra, and let A' be its underlying ring (i.e. A, considered as a $\mathbb{Z}$-algebra). Then $A' \otimes_{\mathbb{Z}} \mathbb{Q} \simeq A$.

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Lemma

Let A be a $\mathbb{Q}$-algebra, and let A' be its underlying ring (i.e. A, considered as a $\mathbb{Z}$-algebra). Then $A' \otimes_{\mathbb{Z}} \mathbb{Q} \simeq A$.

The lemma holds because $\mathbb{Q} = S^{-1}\mathbb{Z}$, for $S = \mathbb{Z} \setminus \{0\}$, and by a previous exercise:

$$A' \otimes_{\mathbb{Z}} \mathbb{Q} = A' \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z} = S^{-1}A' = A,$$

since A' is already closed under multiplication by $\mathbb{Q}$. $\square$

Models for schemes

Solution– contd.

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1. S is a localization of R by some multiplicatively closed set, and the homomorphism is the natural map $R \to S$; or
2. $S = R/I$ for some ideal $I \triangleleft R$.

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Solution– contd.

Regarding generalizations of this statement for a ring homomorphism $R \to S$, and the case where we have an S-scheme X and we seek an R-model, the same argument will hold in the cases where:

1. S is a localization of R by some multiplicatively closed set, and the homomorphism is the natural map $R \to S$; or
2. $S = R/I$ for some ideal $I \lhd R$.

The argument **does not** hold for arbitrary inclusions. $\square$

Example

Let $X = \mathrm{Spec}(\mathbb{C})$, considered as a scheme over $\mathbb{C}$. Then X is not an $\mathbb{R}$-model of itself.

Example

Let $X = \mathrm{Spec}(\mathbb{C})$, considered as a scheme over $\mathbb{C}$. Then X is not an $\mathbb{R}$-model of itself.

Indeed,

$$X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) = \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Spec}(\mathbb{C} \times \mathbb{C}),$$

where the last equality holds since

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\simeq \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[t]/(t^2+1)) \\ &= \mathbb{C}[t]/(t^2+1) \\ &\simeq (\mathbb{C}[t]/(t-i)) \times (\mathbb{C}[t](t+i)) = \mathbb{C} \times \mathbb{C}. \end{aligned}$$

Thus $X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ has two points, while X has only one.

Definition (The fiber of a morphism)

Let $X \to Y$ be a morphism of schemes, and $y \in Y$ a point. The fiber of f at y is defined by the fibered product

$$\begin{tikzcd} X \times_Y \mathrm{Spec}(k(y)) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k(y)) \arrow[r] & Y \end{tikzcd}$$

where $\mathrm{Spec}(k(y)) \to Y$ is the canonical map with image $\{y\}$.

Exercise

Compute the fibers of $\mathrm{Spec}(\mathbb{Q}[x, y]/(y^2 - x)) \to \mathbb{A}^1_{\mathbb{Q}}$ at the maximal ideals $(x - 1), (x)$ and $(x + 1)$. Also at (0).

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1. $\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1) \simeq \mathbb{Q}[y]/(y^2-1) \simeq \mathbb{Q} \times \mathbb{Q}$, and the spectrum has two points.

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2. $\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x) \simeq \mathbb{Q}[y]/(y^2)$, and the spectrum has one *double* point.

Exercise

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3. $\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x+1) \simeq \mathbb{Q}[y]/(y^2+1) \simeq \mathbb{Q}[i]$, and the specturm has one point, but with a larger field.

Exercise

Compute the fibers of $\mathrm{Spec}(\mathbb{Q}[x,y]/(y^2-x)) \to \mathbb{A}^1_{\mathbb{Q}}$ at the maximal ideals $(x-1), (x)$ and $(x+1)$. Also at (0).

1. $\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1) \simeq \mathbb{Q}[y]/(y^2-1) \simeq \mathbb{Q} \times \mathbb{Q}$, and the spectrum has two points.
2. $\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x) \simeq \mathbb{Q}[y]/(y^2)$, and the spectrum has one *double* point.
3. $\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x+1) \simeq \mathbb{Q}[y]/(y^2+1) \simeq \mathbb{Q}[i]$, and the specturm has one point, but with a larger field.
4. $\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}(x) \simeq \mathbb{Q}[y] \otimes \mathbb{Q}(y^2) \simeq \mathbb{Q}(y)$

Questions?