Algebraic Geometry 2
Tutorial session 4

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May 17, 2020

The pullback and direct image - recollections

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We also defined the direct image sheaf $f^{-1}\mathcal{G}$ on X, by sheafification of the presheaf defined by

$$f^{-1}\mathcal{G}(U) = \lim_{V \supseteq f(U) \text{ open}} \mathcal{G}(V) \quad \text{for } U \subseteq X \text{ open.}$$

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Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y. Then there exists a natural bijection

$$\mathrm{Hom}_{\underline{\mathbf{Sh}}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\underline{\mathbf{Sh}}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

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Proof.

We construct natural maps going in both directions, and verify that their compositions are equivalent to identity (some details are left as exercises).

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$$f^{-1}\mathcal{G}(U) = \lim_{V' \supseteq f(U)} \mathcal{G}(V') = \mathcal{G}(V) \quad \text{and} \quad \mathcal{F}(U) = f_*\mathcal{F}(V).$$

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In particular, $\mathrm{Hom}(f^{-1}\mathcal{G}(U), \mathcal{F}(U)) = \mathrm{Hom}(\mathcal{G}(V), f_*\mathcal{F}(V))$, and we can define $F(\varphi)_V = \varphi_{f^{-1}(V)}$.

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Proof.

- $\underline{G : \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})}$: Given $\psi : \mathcal{G} \to f_*\mathcal{F}$, $U \subseteq X$ open and $f(U) \subseteq V \subseteq Y$ open, we have a map $g_{V,U} : \mathcal{G}(V) \to \mathcal{F}(U)$, given by the composition

$$\mathcal{G}(V) \xrightarrow{\psi_V} f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V) \xrightarrow{\mathrm{res}_{f^{-1}(V),U}} \mathcal{F}(U).$$

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- The equivalence $(GF\varphi)_U = \varphi_U$ for all $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ follows similarly, by unfolding the definitions ($\mathtt{Ex}$).

$\square$

Schemes

The spectrum of a ring

Let A be a commutative unital ring.

Definition

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3. $\mathrm{Spec}(\mathbb{Z}) = \{\langle 0 \rangle\} \sqcup \{\langle p \rangle : p \text{ prime}\}$.

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Given $I \lhd A$, define $V(I) := \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subseteq \mathfrak{p}\}$.

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The collection $\{V(I) : I \triangleleft R\}$ is the set of closed sets for a topology on $\mathrm{Spec}(R)$, which is known as the *Zariski Topolgy* of R.

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Let A be a ring.

1. Show that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, for all $\mathfrak{p} \in \mathrm{Spec}(A)$ and, in particular, that $\{\mathfrak{p}\}$ is closed iff $\mathfrak{p}$ is maximal.
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Solution.

1. By definition, and by the previous exercise:

$$\overline{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in F \text{ closed}} F = \bigcap_{\substack{I \triangleleft R \\ I \subseteq \mathfrak{p}}} V(I) = V\Big(\sum_{I \subseteq \mathfrak{p}} I\Big) = V(\mathfrak{p}).$$

In particular, $\{\mathfrak{p}\}$ is closed iff $\{\mathfrak{p}\} = V(\mathfrak{p})$ which occurs iff $\mathfrak{p}$ is maximal (o/w, take $\mathfrak{m} \supsetneq \mathfrak{p}$ maximal).

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   In particular, $\{\mathfrak{p}\}$ is closed iff $\{\mathfrak{p}\} = V(\mathfrak{p})$ which occurs iff $\mathfrak{p}$ is maximal (o/w, take $\mathfrak{m} \supsetneq \mathfrak{p}$ maximal).
2. Note: $(0) \in \operatorname{Spec}(R)$ iff R is a domain, in which case $V(0) = R$.

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Solution.

Assume A is domain, and $0 \neq I_1, I_2 \lhd A$ proper ideals such that $\mathrm{Spec}(A) = V(I_1) \cup V(I_2)$. Let $f_i \in I_i$ $(i = 1, 2)$. Then, since

$$V(I_1) \cup V(I_2) \subseteq V((f_1)) \cup V((f_2)) = V((f_1 f_2)),$$

we have that $f_1 f_2$ is necessarily nilpotent, and since A is a domain, either $f_1 = 0$ or $f_2 = 0$. By fixing $0 \neq f_1 \in I_1$ (wlog, assuming such exists) and letting $f_2 \in I_2$ vary we deduce that $I_2 = (0)$ and hence $V(I_2) = \mathrm{Spec}(A)$.

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we have that $f_1 f_2$ is necessarily nilpotent, and since A is a domain, either $f_1 = 0$ or $f_2 = 0$. By fixing $0 \neq f_1 \in I_1$ (wlog, assuming such exists) and letting $f_2 \in I_2$ vary we deduce that $I_2 = (0)$ and hence $V(I_2) = \mathrm{Spec}(A)$.
For the converse implication, assume $\mathrm{Spec}(A)$ is irreducible and let $f_1, f_2 \in A$ non-nilpotents. By assumption $V((f_1)) \cup V((f_2)) = V((f_1 f_2)) \subsetneq \mathrm{Spec}(A)$, and hence there exists $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $f_1 f_2 \notin \mathfrak{p}$. In particular, $f_1 f_2 \neq 0$. $\square$

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Corollary

If $\operatorname{Spec}(A)$ *is irreducible then there exists* $\xi \in \operatorname{Spec}(A)$ *such that* $\overline{\xi} = \operatorname{Spec}(A)$. *Such an element is called a* *generic point* *of* $\operatorname{Spec}(A)$.

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4. A morphism of locally ringed spaces $(f, f^\sharp)$ is a morphism of ringed spaces with the **added requirement** that $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to f_*\mathcal{O}_{X,x}$ is a local homomorphism (i.e. preimage of the maximal ideal is maximal) for any $x \in X$.

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Explicit construction

The sheaf $\mathcal{O}$ is defined by sheafifying the presheaf on $\mathrm{Spec}(A)$ whose stalks are given by $A_\mathfrak{p} = \lim_{\mathfrak{p} \in D(f),\, f \in A} A_f$.

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Thus $(\mathrm{Spec}(A), \mathcal{O})$ is a locally ring space, with $\mathcal{O}(\mathrm{Spec}(A)) \simeq A$.

Explicit construction

The sheaf $\mathcal{O}$ is defined by sheafifying the presheaf on $\mathrm{Spec}(A)$ whose stalks are given by $A_\mathfrak{p} = \lim_{\mathfrak{p} \in D(f),\, f \in A} A_f$. Explicitly, $\mathcal{O}(U)$ consists of functions $s : U \to \bigsqcup_{\mathfrak{p} \in U} A_\mathfrak{p}$ for which there exists an open cover $U = \bigcup_\alpha V_\alpha$ such that, given α, $\exists a_\alpha, b_\alpha \in A$ with $V_\alpha \subseteq D(b_\alpha)$, such that $s(\mathfrak{q}) =$ “ the image of a_α / b_α in $A_\mathfrak{q}$”, for all $\mathfrak{q} \in V_\alpha$.

Proposition (2.3 in Hartshorne)

1. *Let* $\varphi : A \to B$ *be a homomorphism of rings. Then* φ *induces a natural morphism of locally ringed spaces*

$$(f, f^\sharp) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}).$$

2. *Conversely, any morphism of locally ring spaces* $(f, f^\sharp)$ *as above is induced from a ring homomorphism* $\varphi : A \to B$*.*

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Example

Example - contd On the other hand, if $\mathrm{Im}(f) = \{t_0\}$, and $(f, f^\sharp)$ is a morphism of ringed spaces, then $f^\sharp$ induces a homomorphism on stalks

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Assuming $R/\mathfrak{m} \not\cong K$, this is **not** a local homomorphism. Therefore, in this case, $(f, f^\sharp)$ is not induced from any morphism $R \to K$.

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   $\underline{f \text{ is continuous}}$. Given $I \lhd A$, we have

   $$\mathfrak{p} \in f^{-1}(V(I)) \iff f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) \supseteq I \iff \mathfrak{p} \supseteq \varphi(I).$$

   Thus $f^{-1}(V(I)) = V((\varphi(I)))$.

$\square$

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Construction of $f^\sharp$. For any $\mathfrak{p} \in \mathrm{Spec} B$, we have a localized map $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$, defined by $\varphi_\mathfrak{p}(a/b) = \varphi(a)/\varphi(b)$.

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Then $f^\sharp$ is a ring homomorphism. Furthermore, if $U \subseteq V$ is such that there exist $a, b \in A$ with $U \subseteq D(b)$ such that $s(\mathfrak{q}) = a/b \in A_\mathfrak{q}$ for all $\mathfrak{q} \in U$, then $f^{-1}(U) \subseteq D(\varphi(b))$ and $f^\sharp(s)(\mathfrak{q}') = \varphi(a)/\varphi(b) \in B_{\mathfrak{q}'}$ for all $\mathfrak{q}' \in f^{-1}(U)$. It follows that
$f^\sharp(s) \in \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(V)) = f_*\mathcal{O}_{\mathrm{Spec}(B)}(V)$.

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$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "l_A"'] & B \arrow[d, "l_B"] \\ A_{f(\mathfrak{p})} \arrow[r, "f^\sharp"', "\mathfrak{p}"] & B_{\mathfrak{p}} \end{tikzcd}.$$

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In particular, since the bottom row consists of local rings and a local homorphism, it must be that

$$\varphi^{-1}(\mathfrak{p}) = \varphi^{-1} \circ l_B^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = l_A^{-1} \circ f^\sharp(\mathfrak{p}B_{\mathfrak{p}}) = f(\mathfrak{p}).$$

Thus $f = \varphi^{-1}$.

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Let $R = \overline{\mathbb{F}_q}[[t]]$, and $\varphi : R \to R$ defined by $\varphi(\sum a_i t^i) = \sum a_i^q t^i$.

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- The associated map $f : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is the identity map.
- Given $V \subseteq \mathrm{Spec}(R)$ and $s \in \mathcal{O}_{\mathrm{Spec}(R)}(V)$, we have

$$f^\sharp(s)(\cdot) = (s(\cdot))^q.$$

Corollary (of the proposition)

Let A, B be rings. Then A and B are isomorphic if and only if $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are isomorphic as locally ringed spaces.

Definition

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Here $\mathcal{O}_X \mid_{U_\alpha}$ denotes the restricted sheaf $\mathcal{O}_X \mid_{U_\alpha} (V) = \mathcal{O}_X(V)$ for $V \subseteq U_\alpha$ open.