Generalized functions - Tirgul

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TA hours: 11:15-13:00, Thursday.

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Definition 1.1.

1. *Denote by $C^\infty(\mathbb{R})$ the space of smooth functions $f:\mathbb{R}\to\mathbb{R}$ (i.e. having derivatives of any order).*
2. *Define the support of a function $f:\mathbb{R}\to\mathbb{R}$ by $\operatorname{supp}(f)=\overline{\{x\in\mathbb{R}:f(x)\neq 0\}}$, the closure of the set in which it does not vanish.*
3. *Denote by $C_c^\infty(\mathbb{R})\subset C^\infty(\mathbb{R})$ the space of compactly supported function in $C^\infty(\mathbb{R})$.*

Exercise 1.2. *Prove that $C_c^\infty(\mathbb{R})\neq\{0\}$.*

Proof. We construct a smooth function with compact support. Assume $a<b$ and consider

$$\eta_{a,b}(x):=e^{-\frac{1}{(x-a)^2}}\cdot e^{-\frac{1}{(x-b)^2}}I_{(a,b)},$$

where $I_{(a,b)}$ is the indicator function of (a,b). Obviously $\operatorname{supp}(\eta_{a,b})=\overline{(a,b)}=[a,b]$ is compact. It is left to show that η is smooth at a and b (it is smooth at the other points as composition of smooth functions). This is true since $\lim\limits_{x\to a^+}\frac{e^{-\frac{1}{(x-a)^2}}-0}{x-a}=0=\lim\limits_{x\to a^-}\frac{0}{x-a}$. The result follows for arbitrary derivatives $\eta_{a,b}^{(k)}$ since e^{-x^2} decays faster than any polynomial. □

Definition 1.3. *Recall that for two functions the convolution is defined via $f*g=\int\limits_{-\infty}^{\infty}f(t)g(x-t)dt$.*

Remark 1.4. *The value of the convolution of two functions should be intuitively thought of as integrating a fixed function against a shifted sliding window.*

Exercise 1.5. *Show that given $f,g\in C_c^\infty(\mathbb{R})$ we have that $f*g\in C_c^\infty(\mathbb{R})$ in several steps.*

1. *Show that convolution is commutative.*
2. *Show that for every function $f\in C_c^\infty(\mathbb{R})$ and $g\in C_c(\mathbb{R})$ absolutely integrable the convolution $g*f$ is a smooth function.*
3. *Show that* $\operatorname{supp}(f*g)\subseteq\overline{\operatorname{supp}(f)+\operatorname{supp}(g)}$, *where* $\operatorname{supp}(f)+\operatorname{supp}(g)$ *is the* Minkowski sum *of* $\operatorname{supp}(f)$ *and* $\operatorname{supp}(g)$. *This is not an equality e.g. $f=1_{[-1,1]}\cdot x$ and $g=1_{[-2,2]}$ where the Minkowski sum is $[-3,3]$ but* $\operatorname{supp}(f*g)$ *does not contain $(-1,1)$.*

Proof.

1. The first item follows from a change of variables.
2. For the second item, given $x \in \mathbb{R}$,

$$(g*f)'(x) = \lim_{h \to 0} \frac{g*f(x+h) - g*f(x)}{h} = \lim_{h \to 0} \left(\frac{\int\limits_{-\infty}^{\infty} g(t)(f(-t+x+h) - f(-t+x))}{h} dt \right).$$

   Now since f is smooth, the integrand is bounded by $H(t) = M_x(t)g(t)$ for $|h|$ small enough, and since g is absolutely integrable, so is $H(t)$. By Lebesgue's dominated convergence theorem, thus we get

$$= \int\limits_{-\infty}^{\infty} \lim_{h \to 0} \left(\frac{g(t)(f(-t+x+h) - f(-t+x))}{h} \right) dt = \int\limits_{-\infty}^{\infty} g(t)f'(-t+x)dt = g*(f').$$

   Since f is smooth, we are done.

3. Set $S = \mathrm{supp}(f) + \mathrm{supp}(g)$. We have that,

$$g * f(x) = \int\limits_{-\infty}^{\infty} g(t)f(x-t)dt = \int_{\mathrm{supp}(g)\cap(x-\mathrm{supp}(f))} g(t)f(x-t)dt.$$

   Now, if $x \notin S$, then $\mathrm{supp}(g) \cap (x - \mathrm{supp}(f)) = \varnothing$ which then implies that $g(t)f(x-t) = 0$. In particular $g * f(x) = 0$, and recalling the definition of support we are done.

   Note that since the functions are of compact support we get that $\overline{\mathrm{supp}(f) + \mathrm{supp}(g)} = \mathrm{supp}(f) + \mathrm{supp}(g)$ by continuity of addition.

□

Remark 1.6. *This means that $C_c^\infty(\mathbb{R})$ forms an algebra under convolution. Does it have a unit?*

Remark 1.7. *Minkowski sum of a closed set and a compact set is closed, this is not true for the Minkowski sum of two closed set (take the hyperbola and a suitable line).*

Exercise 1.8. *Let $K \subset \mathbb{R}$ be a compact set, and let $K \subset U$ be an open neighborhood of K. Show there exists a function $f \in C_c^\infty(U)$ such that $f_{|K} \equiv 1$.*

Proof. Set $\epsilon = d(K, U^c)$, and let $K_\delta = \{x \in U : d(x, K) \leq \delta\}$. By Urysohn's lemma there exists a function g such that $g_{|K_{\frac{\epsilon}{4}}} \equiv 1$ and $g_{|\overline{K^c_{\frac{\epsilon}{2}}}} \equiv 0$. Note that g is compactly supported since its support is contained in the compact set $K_{\frac{\epsilon}{2}}$. Now, we can consider $f = g*\eta$ where η is a smooth positive bump function, non-vanishing at 0, such that $\mathrm{diam}(\mathrm{supp}(\eta)) < \frac{\epsilon}{4}$ and $\int_{\mathbb{R}} \eta(x)dx = 1$. The function f is smooth, and by the previous exercise $\mathrm{supp}(f) = \mathrm{supp}(g * \eta) \subset K_{\frac{3\epsilon}{4}} \subset U$ so we are done. □

Exercise 1.9. *Show that $C_c^\infty(\mathbb{R})$ is dense in $C_c(\mathbb{R})$ with the topology of uniform convergence.*

Proof. Consider the sequence of positive bump functions functions $f_n = \frac{\eta_{-\frac{1}{n},\frac{1}{n}}(x)}{I_n}$ where $I_n = \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx$, and $\mathrm{supp}(\eta_{-\frac{1}{n},\frac{1}{n}}) \subset [-\frac{1}{n},\frac{1}{n}]$. Set $G_n(y) = \max\{g(y-x) : x \in [-\frac{1}{n},\frac{1}{n}]\}$ and $g_n(y) = \min\{g(y-x) : x \in [-\frac{1}{n},\frac{1}{n}]\}$ and note that for an arbitrary $g(x) \in C_c^\infty(\mathbb{R})$:

$$g*f_n(y) = \int\limits_{-\infty}^{\infty} g(y-x)f_n(x)dx = \frac{1}{I_n}\int\limits_{-\infty}^{\infty} g(y-x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx = \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(y-x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx.$$

Now, note that

$$\frac{g_n(y)}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(y-x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le \frac{G_n(y)}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx.$$

This implies,

$$g_n(y) \le \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(y-x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le G_n(y),$$

yielding the required statement as $\lim\limits_{n\to\infty} G_n(y) = \lim\limits_{n\to\infty} g_n(y) = g(y)$ and all functions are contained in a suitable compact set. $\square$

Exercise 1.10 (Partition of unity)**.** *Let $f \in C_c^\infty(\mathbb{R})$, I an indexing set and $\mathbb{R} = \bigcup\limits_{i\in I} U_i$ a union of open sets. Then f can be written as a sum $f = \sum\limits_{i\in I} f_i$ where $f_i \in C_c^\infty(U_i)$.*

Proof. We can assume that U_i are open balls (otherwise, replace each U_i by a collection of balls covering it). Set $K := \mathrm{supp}(f)$, it is a compact set covered by $\bigcup\limits_{i\in I} U_i$, so there exists a finite sub-cover: $K \subseteq \bigcup\limits_{i=1}^{n} U_i = \bigcup\limits_{i=1}^{n} B(x_i, r_i)$. Since the cover is open and K is closed, there exists $\epsilon > 0$ such that $K \subseteq \bigcup\limits_{i=1}^{n} B(x_i, r_i - \epsilon)$. Let ρ_i be smooth non-negative bump functions satisfying $\rho_{i|B(x_i,r_i-\varepsilon)} \equiv 1$ and $\rho_i|_{B(x_i,r_i)^c} \equiv 0$ (we can use the last exercise). Since $\forall x \in K$ we have that $\sum\limits_{i=1}^{n} \rho_i(x) \ne 0$, we can define $f_i = \frac{\rho_i \cdot f}{\sum\limits_{j=1}^{n} \rho_j}$. This finishes the proof. $\square$

Remark 1.11. *A similar statement holds for a general topological space, where one can either choose the supports of f_i to be compact or to have* $\mathrm{supp}(f_i) \subseteq U_i$.

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Definition 2.1. *We say that $\{f_n\}$ converges to f if*

1. *There is a compact set $K \subset \mathbb{R}$ s.t.* $\mathrm{supp}(f) \bigcup\limits_{n\in\mathrm{N}} \mathrm{supp}(f_n) \subseteq K$.

2. *For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$ (recall that uniform convergence means that the δ chosen can be taken to be independent of x).*

We call the space of continuous functionals $\varphi : C_c^\infty(X) \to \mathbb{R}$, w.r.t. the convergence defined above *distributions*, denote them by $(C_c^\infty(\mathbb{R}))^*$ and write $\langle \varphi, f \rangle$ for $\varphi(f)$.

We can indeed define continuous functionals in this way because of the following exercise which we will not prove now.

Exercise 2.2. *A linear operator between semi-normed spaces is continuous if and only if it sends Cauchy sequences to Cauchy sequences.*

Remark 2.3. *For now we have no distinction between generalized functions which we denote by $C^{-\infty}(\mathbb{R})$ and distributions, as there is no difference for $\mathbb{R}$. We will discuss the difference in a later part of the course, when it will be relevant.*

Definition 2.4. *We say that a sequence of functions $\{f_n\}$ weakly converges to f if for every $g(x) \in C_c^\infty(\mathbb{R})$ we have that $\lim\limits_{n\to\infty} \int\limits_{-\infty}^{\infty} g(x)f_n(x)dx = \int\limits_{-\infty}^{\infty} g(x)f(x)dx$.*

Exercise 2.5. *Find a sequence of functions $\{f_n\}$ in $C_c^\infty(\mathbb{R})$ that converges weakly as distributions to the Dirac delta function at zero, δ_0.*

Proof. First note that by definition $\langle \delta_0, g(x) \rangle = \int\limits_{-\infty}^{\infty} g(x)\delta_0(x)dx = g(0)$. Now, consider the sequence of functions $f_n = \frac{\eta_{-\frac{1}{n},\frac{1}{n}}(x)}{I_n}$ where $I_n = \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx$, set $G_n = \max\{g(x) | x \in [-\frac{1}{n}, \frac{1}{n}]\}$ and $g_n = \min\{g(x) | x \in [-\frac{1}{n}, \frac{1}{n}]\}$ and note that for an arbitrary $g(x) \in C_c^\infty(\mathbb{R})$:

$$\int\limits_{-\infty}^{\infty} g(x)f_n(x)dx = \frac{1}{I_n}\int\limits_{-\infty}^{\infty} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx = \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx.$$

Now, note that

$$\frac{g_n}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le \frac{G_n}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx.$$

This implies,

$$g_n \le \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le G_n,$$

yielding the required statement as $\lim\limits_{n\to\infty} G_n = \lim\limits_{n\to\infty} g_n = g(0)$. □

Remark 2.6. *Note that if the sequence f_n converges weakly to f, it need not converge pointwise to f.*

Exercise 2.7. *Find a sequence of functions such that the remark above holds.*

Definition 2.8. *We say a function $f:\mathbb{R}\to\mathbb{R}$ is locally L^1, if $f_{|K}\in L^1(\mathbb{R})$ for every compact $K\subseteq\mathbb{R}$. We denote all such functions by $L^1_{\text{loc}}(\mathbb{R})$.*

Remark 2.9. *Note that every locally L^1 function can be naturally considered as a distribution via integration.*

Definition 2.10. *Define the derivative of a distribution η by $\langle \eta', f\rangle = -\langle \eta, f'\rangle$.*

Exercise 2.11. *Find a function $f\in L^1_{\text{loc}}(\mathbb{R})$ for which $f'=\delta_0$.*

Proof. Consider the Heaviside step function,

$$H(x)=\begin{cases}0, & \text{for } x<0,\\ 1, & \text{for } x\geq 0.\end{cases}$$

Every compact $K\subseteq\mathbb{R}$ is closed and bounded, so $H(x)$ is locally-L^1. Also, it is smooth and its derivative is 0 in $\mathbb{R}\backslash\{0\}$, but it is not continuous at 0, and thus its derivative is not a function on $\mathbb{R}$. We would like to interpret it as a distribution, indeed using integration by parts:

$$\langle H'(x), g(x)\rangle = \int\limits_{-\infty}^{\infty} g(x)H'(x)dx = g(x)H(x)\Big|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} g'(x)H(x)dx.$$

Recalling that g is compactly supported ($g(x)\in C_c^\infty(\mathbb{R})$), and using the fundamental theorem of calculus we see that,

$$g(x)H(x)\Big|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} g'(x)H(x)dx = 0-\int\limits_{0}^{\infty} g'(x)dx = 0-(-g(0)) = g(0) = \langle \delta_0, g(x)\rangle.$$

□

Definition 2.12. *For $\xi\in C^{-\infty}(\mathbb{R})$ we say that $\xi_{|U}\equiv 0$ if $\langle\xi,f\rangle=0$ for all $f\in C_c^\infty(U)$. Additionally we define* $\operatorname{supp}(\xi)=\bigcap\limits_{\xi_{|D_\beta^c}\equiv 0} D_\beta$*, where D_β are taken to be closed.*

Remark 2.13. *Note that* $\operatorname{supp}(\xi)$ *is always a closed set.*

Example 2.14. $\operatorname{supp}(\delta_x)=\{x\}$.

Exercise 2.15. *Show the identity axiom for $C^{-\infty}(\mathbb{R})$, i.e. for every $\xi\in(C_c^\infty(\mathbb{R}))^*$, if there exists a cover $\{U_i\}_{i\in I}$ of $\mathbb{R}$ such that $\xi_{|U_i}\equiv 0$ for all $i\in I$, then $\xi\equiv 0$.*

Proof. Let $g\in C_c^\infty(\mathbb{R})$. Since $\operatorname{supp}(g)$ is compact, we have a finite sub-cover $\{U_i\}_{i=1}^n$ of $\{U_i\}_{i\in I}$. Use partition of unity with respect to this cover and obtain compactly supported functions $f_i:\mathbb{R}\to[0,1]$ such that $\operatorname{supp}(f_i)\subseteq U_i$. Now,

$$\langle\xi,g\rangle = \langle\xi,\sum_{k=0}^m gf_k\rangle = \sum_{k=0}^m\langle\xi,gf_k\rangle = 0,$$

where the last equality is true since $\xi_{|U_i}\equiv 0$ for every $i\in I$, and since $\operatorname{supp}(gf_k)$ is compact and contained in U_k. □

Remark 2.16.

1. *Note that the argument for the previous exercise holds for any paracompact space, in particular for any real manifold.*
2. *Warning! It might not be the case that* $f_{|U} \in C_c^\infty(U)$ *even if* $f \in C_c^\infty(V)$ *and* $U \subset V$.

Exercise 2.17. *Prove that,*

$$\{\xi \in (C_c^\infty(\mathbb{R}))^* |\ \mathrm{supp}(\xi) = \{0\}\} = < \{\delta^{(k)}\}_{k=0}^\infty >_{\mathbb{R}} .$$

We prove two lemmas which will yield the desired result when combined.

Lemma 2.18. *Let* ξ *be a distribution supported on* $\{0\}$*, then there exists* $k \in \mathbb{N}$ *such that* $\xi x^k = 0$.

Proof. Take a bump function ψ such that $\psi \equiv 1$ in some neighborhood of 0 and $\mathrm{supp}(\psi) \subseteq (-1,1)$, and set $\psi_\epsilon(x) = \psi(\epsilon^{-1}x)$. For every $f \in C_c^\infty(\mathbb{R})$ and $\epsilon > 0$, since $0 \notin \mathrm{supp}(f - f\psi_\epsilon)$, we have that $\langle \xi, f - \psi_\epsilon f\rangle = 0$, implying $\langle \xi, f\rangle = \langle \xi, \psi_\epsilon f\rangle$. Since $\xi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ is a continuous linear map, for every compact $D \subseteq \mathbb{R}$ there exists $k_D \geq 0$ and $C_D > 0$ such that for all $f \in C_c^\infty(D)$ (you will prove this at home),

$$|\langle \xi, f\rangle| \leq C_D \big\|f\big\|_{C^{k_D}} = C_D \sup_{x\in D} \sum_{i=0}^{k_D} |f^{(i)}(x)|.$$

Thus for every $d \in \mathbb{N}$ with $k = k_{[-1,1]}$ and $C = C_{[-1,1]}$,

$$\big|\langle \xi x^d, f\rangle\big| = \big|\langle \xi, x^d f\rangle\big| = \big|\langle \xi, x^d f\psi_\epsilon\rangle\big| \leq C \sup_{x\in(-\epsilon,\epsilon)} \sum_{i=0}^{k} |(x^d f\psi_\epsilon)^{(i)}(x)|.$$

Now, note that since $(f\psi_\epsilon)$ is compactly supported and smooth, we can set $M = \max\limits_{x\in[-1,1], i,j\leq k} |\{f^{(i)}(x)\psi^{(j)}(\epsilon^{-1}x)\}|$. Inspect each summand for $x \in [-\epsilon, \epsilon]$,

$$\begin{aligned}
\Big|(x^d f\psi_\epsilon)^{(i)}(x)\Big| &= \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} \Big|(x^d)^{(i_1)} f^{(i_2)}(x)\psi^{(i_3)}(\epsilon^{-1}x)\epsilon^{-i_3}\Big| \\
&\leq \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} d\cdots(d-i_1+1)|x^{(d-i_1)}|M\epsilon^{-i_3} \\
&\leq \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} d\cdots(d-i_1+1)M\epsilon^{d-i_1-i_3}.
\end{aligned}$$

We can now evaluate the entire expression,

$$C \sup_{x\in(-\epsilon,\epsilon)} \sum_{i=0}^{k} |(x^d f\psi_\epsilon)^{(i)}(x)| \leq C \sum_{i=0}^{k} \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} d\cdots(d-i_1+1)M\epsilon^{d-i_1-i_3},$$

but since this holds for every $\epsilon > 0$, we can take $d > k+1$ and obtain that $\big|\langle \xi x^d, f\rangle\big| = 0$ for every $f \in C_c^\infty(\mathbb{R})$. $\square$

Lemma 2.19. *If $\xi x^k = 0$, that is $\langle \xi x^k, f\rangle = \langle \xi, x^k f\rangle = 0$ for every $f \in C_c^\infty(\mathbb{R})$ then $\xi = \sum\limits_{i=0}^{k-1} c_i \delta_0^{(i)}$ for some $c_i \in \mathbb{R}$.*

Proof. Our strategy will be to prove the claim for $k = 1$, and then to use induction. Assume $\xi x = 0$. First note that for every $f \in C_c^\infty(\mathbb{R})$ we can write,

$$f(x) = f(0) + x \int_0^1 f'(xt)dt. \tag{1}$$

Now, take a test function $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(0) = 1$, and use Formula 1 as above twice to obtain (once for $f(x)$ and once for $\psi(0)$):

$$f(x) = f(0)\psi(0) + x \int_0^1 f'(xt)dt = f(0)\psi(x) - x\Big(f(0)\int_0^1 \psi'(xt)dt - \int_0^1 f'(xt)dt\Big).$$

Note that each summand is a smooth compactly supported function as the two $f(0)$ constant terms cancel in the second summand. Now, we see that,

$$\begin{aligned}
\langle \xi, f\rangle &= \langle \xi, f(0)\psi(x)\rangle + \langle \xi, x\Big(- f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\Big)\rangle \\
&= f(0)\langle \xi, \psi(x)\rangle + \underbrace{\langle \xi x, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle}_{=0} \\
&= f(0)\langle \xi, \psi(x)\rangle.
\end{aligned}$$

Since $\langle \xi, \psi(x)\rangle$ is independent of ψ a long as $\psi(0) = 1$ (note we can use Formula 1 to write ψ as a combination of any other test function φ as long as $\varphi(0) = 1$), we can set $c_0 = \langle \xi, \psi\rangle$ and we are done.

Now, if $\xi x^{k+1} = 0$, then $(\xi x)x^k = 0$ and thus $\xi x = \sum\limits_{i=0}^{k-1} c_i \delta_0^{(i)}$ by induction. We take $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(0) = 1$ and $\psi^{(i)}(0) = 0$ for $1 \le i \le k$, using the same expansion as before,

$$\begin{aligned}
\langle \xi, f\rangle &= \langle \xi, f(0)\psi(x)\rangle + \langle \xi, x\Big(- f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\Big)\rangle \\
&= f(0)\langle \xi, \psi(x)\rangle + \langle \xi x, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle \\
&= f(0)\langle \xi, \psi(x)\rangle + \sum_{i=0}^{k-1} c_i \langle \delta_0^{(i)}, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle.
\end{aligned}$$

Now, observe that

$$-f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt = \begin{cases} f'(x), & \text{if } x = 0, \\ \frac{f(x) - f(0)\psi(x)}{x}, & \text{if } x \neq 0 \end{cases}$$

is smooth (note this follows from the choice of ψ such that sufficiently many derivatives vanish), so we have

$$\langle\delta_0^{(i)}, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle = (-1)^i f^{(i+1)}(0).$$

Since $\langle\xi,\psi\rangle$ is independent of ψ (considering functions ψ with desired conditions), we are done. $\square$

Definition 2.20. *Recall that for two functions the convolution is defined via $f * g = \int_{-\infty}^{\infty} f(t)g(x-t)dt$. We define the convolution of $f \in C_c^\infty(\mathbb{R})$ with a distribution ξ by $(\xi * f)(x) = \langle\xi, f(x-t)\rangle$.*

Exercise 2.21. *Show that for every distribution ξ and $f \in C_c^\infty(\mathbb{R})$ the convolution $\xi * f$ is a smooth function.*

Proof. Notice that since ξ is linear and continuous,

$$\begin{aligned}(\xi * f)'(x) &= \lim_{h\to 0}\frac{\xi * f(x+h) - \xi * f(x)}{h}\\ &= \lim_{h\to 0}\frac{\langle\xi, f(x-t+h)\rangle - \langle\xi, f(x-t)\rangle}{h}\\ &= \langle\xi, \lim_{h\to 0}\frac{f(x-t+h)-f(x-t)}{h}\rangle = \langle\xi, f'(x-t)\rangle = (\xi * f')(x).\end{aligned}$$

Thus we see that $(\xi * f)^{(k)} = \xi * f^{(k)}$, and since $f^{(k)} \in C_c^\infty(\mathbb{R})$ for all $k \in \mathbb{N}$ we are done. $\square$

Definition 2.22. *For two compactly supported distributions define $\langle\xi_1 * \xi_2, f\rangle = \langle\xi_1, (\xi_2 * f(-t))(-x)\rangle$. For the next exercise we also denote $\overline{f(x)} = f(-x)$ and $L_t(f)(x) = f(t+x)$. Note: this is very bad notation which should not be used elsewhere, if you have a better idea, you are free to tell me.*

Fact 2.23. *Convolution of functions is commutative and associative.*

Exercise 2.24. *Show the following identities for any compactly supported distributions ξ_1, ξ_2 and ξ_3.*

1. $\delta_0 * \xi_1 = \xi_1$.
2. $\delta_0' * \xi_1 = \xi_1'$.
3. $\xi_1 * \xi_2 = \xi_2 * \xi_1$.
4. $\xi_1 * (\xi_2 * \xi_3) = (\xi_1 * \xi_2) * \xi_3$.
5. $(\xi_1 * \xi_2)' = \xi_1 * \xi_2' = \xi_1' * \xi_2$.

Proof.

1. $\langle\delta_0 * \xi, f\rangle = \langle\delta_0, \overline{(\xi * \overline{f})}\rangle = (\xi * \overline{f})(0) = \langle\xi, f(t)\rangle = \langle\xi, f\rangle$.
2. $\langle\delta_0' * \xi, f\rangle = \langle\delta_0', \overline{(\xi * \overline{f})}\rangle = -(\xi * \overline{f})'(0) = -\langle\xi, f'\rangle = \langle\xi', f\rangle$.

3. Take an approximation of identity $\eta_n \in C_c^\infty(\mathbb{R})$, and see that,

$$\begin{aligned}\langle \xi_1 * \xi_2, f\rangle &= \langle (\xi_1 * \delta_0), \overline{\xi_2 * \overline{f}}\rangle = \langle \delta_0, \overline{(\xi_1 * \delta_0) * \xi_2 * \overline{f}}\rangle = \langle \delta_0, \overline{\langle \xi_1 * \delta_0, \overline{L_t(\xi_2 * \overline{f})}\rangle}\rangle \\ &= \langle \delta_0, \overline{\lim_{n\to\infty} \langle \xi_1 * \eta_n, \overline{L_t(\xi_2 * \overline{f})}\rangle}\rangle = \langle \delta_0, \overline{\lim_{n\to\infty} (\xi_1 * \eta_n) * (\xi_2 * \overline{f})}\rangle \\ &= \langle \delta_0, \overline{\lim_{n\to\infty} (\xi_2 * \overline{f}) * (\xi_1 * \eta_n)}\rangle = \langle \delta_0, \overline{\lim_{n\to\infty} \xi_2 * (\xi_1 * \eta_n * \overline{f})}\rangle \\ &= \langle \delta_0, \overline{(\xi_2 * \xi_1 * \delta_0) * \overline{f}}\rangle = \langle \xi_2 * \xi_1, f\rangle.\end{aligned}$$

4. Omitted.

5. Combine the above and see that, $(\xi_1 * \xi_2)' = \delta_0' * (\xi_1 * \xi_2) = (\delta_0' * \xi_1) * \xi_2 = \xi_1' * \xi_2$. For showing the last equality recall that $\langle \xi', f\rangle = -\langle \xi, f'\rangle$, and see that

$$\langle (\xi_1 * \xi_2)', f\rangle = -\langle (\xi_1, \overline{(\xi_2 * \overline{f'})}\rangle = -\langle (\xi_1, \overline{(\xi_2' * (-\overline{f}))}\rangle = \langle \xi_1 * \xi_2', f\rangle.$$

□

3 Tirgul 2

Definition 3.1. *A topological vector space V is a vector space over a field F such that addition $+ : V \times V \to V$ and multiplication by a scalar $\cdot : (F, V) \to V$ are continuous functions. Throughout these notes (and in the course) we will also assume V is Hausdorff.*

Definition 3.2. *Let V be a topological vector space over F.*

1. *We say that a set $A \subseteq V$ is convex if for every $a, b \in A$ the linear combination $ta + (1-t)b \in A$ where $t \in [0,1]$.*
2. *We say that V is locally convex if it has a basis of its topology which consists of convex sets.*
3. *We say that a set $W \subseteq V$ is balanced if $\lambda W \subseteq W$ for all $|\lambda| \le 1$ where $\lambda \in F$.*
4. *For every open convex balanced set $0 \in C$ in V we set ($x \in V$):*

$$N_C(x) = \inf\{\alpha \in \mathbb{R}_{\ge 0} : \frac{x}{\alpha} \in C\}.$$

Exercise 3.3. *Find a topological vector space which is not locally convex.*

Proof. For $0 < p < 1$ define $\|x\|_p = \sum_{i=0}^{\infty} |x_i|^p$ and consider the space

$$\ell^p(\mathbb{C}) = \{(x_n)_{n\in\mathbb{N}} : x_i \in \mathbb{C}, \|x\|^p < \infty\},$$

with the topology induced by the metric $d(x,y) = \|x - y\|_p$. We claim it is not locally convex. Indeed, if it was locally convex then in any open ball $B_r(0)$ around 0 with radius r we would have an open convex set C_B, which will in turn contain a smaller open δ-ball, denote it by $B_\delta(0)$. The convex hull of $B_\delta(0)$ is then contained in C_B, but taking the following convex combination we see this cannot be true:

$$\left|\frac{1}{n}(\delta,0,0,\ldots)+\frac{1}{n}(0,\delta,0,\ldots)+\ldots+\frac{1}{n}(0,\ldots,\underbrace{\delta}_{x_n},0,\ldots)\right|_p \leq \sum_{i=1}^{n}\left(\frac{\delta}{n}\right)^p = \frac{\delta^p}{n^{p-1}} < r,$$

as this inequality should hold for every n, and $\lim\limits_{n\to\infty}\frac{1}{n^{p-1}} = \infty$ for $0<p<1$. $\square$

Exercise 3.4. *Let $0 \in C$ be an open convex set in a topological vector space V.*

1. *Show that $N_C(x) < \infty$ for all $x \in V$.*
2. *Show that if furthermore C is balanced then $N_C(x)$ is a semi-norm.*

Proof.

1. Assume the contrary, there exists $v \in V$ such that $\frac{v}{\alpha} \notin C$ for all $\alpha \in \mathbb{R}$. Thus, $v_n = \frac{v}{n}$ is a sequence in the closed set C^c, but it converges to $0 \notin C^c$, this is a contradiction.
2. Easy to check.

$\square$

Exercise 3.5. *Let $0 \in C$ be an open convex set in a topological vector space V. Find a locally convex topological vector space V such that V has no continuous norm on it.*

Exercise 3.6. *Show that a linear operator between semi-normed spaces is continuous if and only if it sends Cauchy sequences to Cauchy sequences.*

Exercise 3.7. *Let V be a linear topological space. Prove that V is Hausdorff iff $\{0\}$ is a closed set.*

Proof. ($\Rightarrow$:) Assume V is Hausdorff, then for every $a \in V$, there exist open sets $a \in U_1$ and $0 \in U_2$ such that $U_1 \cap U_2 = \varnothing$. In particular $a \in U_1 \subseteq \{0\}^c$.
($\Leftarrow$:) Consider the inverse image of $\{0\}$ with respect to the minus map $- : V \times V \to V$, which is exactly the diagonal $\Delta := \{(x,x) : x \in V\} \subset V \times V$. Note it is a closed set since $\{0\}$ is. Given two points $a, b \in V$, since $(a,b) \in \Delta$, we have an open set $(a,b) \in V_1 \times V_2 \subset V \times V$ such that $V_1 \times V_2 \cap \Delta = \varnothing$. But this exactly implies V is Hausdorff since we have $a \in V_1$ and $b \in V_2$ such that $V_1 \cap V_2 = \varnothing$. $\square$

Exercise 3.8. *Let V be a topological vector space, show that for every neighborhood U of 0 there exists an open balanced set W such that $0 \in W \subseteq U$.*

Exercise 3.9. *Show that every finite dimensional vector space V which is Hausdorff is isomorphic to F^n.*

Proof. Take the standard basis $\{e_i\}_{i=1}^n$ of F^n and choose a basis $\{x_i\}_{i=1}^n$ of V. Set $\varphi : F^n \to V$ by $\varphi(e_i) = v_i$, it is an isomorphism of vector spaces, we would like to show it is also a homeomorphism. First, note that since addition and multiplication by scalar are continuous, so is φ, since we can view it as a composition of two functions $F^n \to (F \times V)^n \to V$, where the first is the injection $(\lambda_1, \ldots, \lambda_n) \mapsto ((\lambda_1, x_1), \ldots, (\lambda_n, x_n))$, and the second map is multiplication and then addition. To show that φ is open, we do the following. First consider the unit sphere

$$S_F^{n-1} = \{x \in F^n : \|x\|_{F^n} = 1\} \subseteq F^n.$$

It is compact, and since φ is continuous $\varphi(S_F^{n-1})$ is compact, and thus closed as V is Hausdorff. Now, the complement $\varphi(S_F^{n-1})^c$ is open, contains 0, and by an exercise it must contain a balanced neighborhood W of 0. Since φ^{-1} is linear, $\varphi^{-1}(W)$ is a balanced neighborhood of 0 in F^n, and by the construction if $x \in \varphi^{-1}(W)$, we have that $\|x\|_{F^n} < 1$ (why?). Now, for each $1 \leq i \leq n$ let $\ell_i : F^n \to F$ be the projection map to the i-th coordinate. Since $\ell_i \circ \varphi^{-1} : V \to F$ is a bounded linear functional on a neighborhood of 0 as $|\ell_i \circ \varphi^{-1}(x)|_F < 1$ for all $x \in W$, it is continuous (this is not hard, see [6, Theorem 6.21] for details), and thus $\varphi^{-1} = \sum_{i=1}^{n} \ell_i \circ \varphi^{-1} e_i$ is continuous. □

Remark 3.10. *If V is furthermore locally convex, we can finish off the argument by showing that in the image of each open ϵ-ball around 0 under φ there exists around each point an open convex set, since the open balls are a basis for the topology of F^n we are done.*

Exercise 3.11. *Let $W \subseteq V$ be locally convex topological vector spaces, and set V' and W' to be the entire duals of V and W respectively, and let $()^*$ denote the usual continuous dual.*

1. *Show that the restriction map $V' \to W'$ is onto.*
2. *Show that the restriction map $V^* \to W^*$ is onto.*

Theorem 3.12. *(Hahn-Banach) Let V be a normed vector space, $W \subseteq V$ a (not necessarily closed) linear subspace and let $f : W \to \mathbb{R}$ be a linear functional such that $\exists 0 < C \in \mathbb{R}$ such that $|f(x)| \leq C\|x\|$ for every $x \in W$. Then, there exists a linear functional $\tilde{f} : V \to \mathbb{R}$ extending f such that $\tilde{f}_{|W} = f$ and $|\tilde{f}(x)| \leq C\|x\|$ for every $x \in V$.*

Exercise 3.13. *Let V be a locally convex topological vector space, and let $f : W \to \mathbb{R}$ be a continuous linear functional, where $W \subseteq V$ is a closed linear subspace of V. Show that f can be extended to V.*

Proof. Recall that the topology of V is generated by open convex sets, each of which corresponds to a semi-norm. f is continuous, and thus it is bounded with respect to some semi-norm $N_C(x)$ where $0 \in C$ is convex and open, we thus have:

$$|f(x)| \leq MN_C(x).$$

Note that this follows since $0 \in f^{-1}(-\epsilon, \epsilon)$ contains an open convex set, and we can take an open convex inside of it. Now, consider the topological vector space $\tilde{V} = V/\ker N_C(x)$, and denote the projection by $p : V \to \tilde{V}$. Since we have a bound on f, if $N_C(w) = 0$ we get that $|f(w)| \leq MN_C(w) = 0$ and thus $\ker N_C \subseteq \ker(f)$, implying that f is defined on $\tilde{V}$, by setting $f(\bar{w}) = f \circ p^{-1}(\bar{w})$. On $\tilde{V}$ the semi-norm $N_C(x)$ is a norm, and thus we can use the Hahn-Banach theorem to extend $f \circ p^{-1}$ to $\tilde{f}$, with the same bound. To finish off the argument, define the extension of f to V to be $\tilde{f} \circ p$.

$\square$

Definition 3.14. *Let V be a topological vector space.*

1. *We say that a sequence $\{v_n\}_{i=1}^{\infty}$ is a Cauchy sequence if for every neighborhood U of 0 there is $n_0 \in \mathbb{N}$ such that $v_n - v_m \in U$ for all $m, n > n_0$.*
2. *We say that $\{v_n\}_{i=1}^{\infty}$ converges to ℓ if for every neighborhood U of ℓ there is $n_0 \in \mathbb{N}$ such that $v_n \in U$ for all $n > n_0$.*
3. *V is called sequentially complete if every Cauchy sequence $\{v_n\}_{i=1}^{\infty}$ converges to some $\ell \in V$.*
4. *V is called complete if for every map $\phi : V \to W$ which maps V homeomorphically into $\phi(V)$, the image $\phi(V)$ is closed in W.*

Exercise 3.15. *Find a topological space which is sequentially complete but is not complete.*

Proof. Taken from [4, Chapter 2 Example 3], this argument holds for any field K with its natural topology. Set $X_d = \mathbb{R}^d$ for $d > \aleph_0$, where $\mathbb{R}$ is equipped with its natural topology, and take $H \subseteq X_d$ to be the subspace of vectors with only countably many non-zero entries. Note that a basis for the topology of $\mathbb{R}^d$ is comprised of sets $\prod_{i \in d} U_i$ such that each U_i is open in $\mathbb{R}$ and only finitely many $U_i \neq \mathbb{R}$. Now, H is sequentially complete, since if $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then so is $v_n(\alpha)$ (where $\alpha \in d$), and it converges in $\mathbb{R}$. We can thus define v by $v(i) = \lim_{n\to\infty} v_n(i)$, for each $i \in d$. Since each v_n had only countable many non-zero entries, v can only countably many entries which are non-zero and thus $v \in H$. Take a basic open set $W = \prod_{i \in d} W_i$ around v, since only finitely many $W_i \neq \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that if $n > n_0$ all $v_n \in W$, and thus H is sequentially complete. H is dense in X_d since every basic open set of X_d intersects H as it only has finitely many elements of the product which are different than $\mathbb{R}$, and thus it is not a complete space. $\square$

Definition 3.16. *Let $\bar{V}$ be a complete space, and $i : V \to \bar{V}$ be an embedding. We say that $\bar{V}$ is a completion of V if one of the following equivalent definitions holds:*

1. *$i(V) \simeq V$ and $\overline{i(V)} \simeq \bar{V}$.*
2. *For every complete W and a map $f : V \to W$ there exists a unique map $\varphi_W : \bar{V} \to W$ such that $f = \varphi_W \circ i$.*

If $V \simeq \bar{V}$ we say that V is complete.

Exercise 3.17. *Show that these two definitions are indeed equivalent.*

Proof. (2) $\Rightarrow$ (1): We have $i : V \to \bar{V}$, set $V' = \overline{V} \subseteq \bar{V}$, it is closed in a complete space and thus complete (proof uses Cauchy filters, see Proposition 5.4 of [5]), we then have $i_{V'}: V \to V'$ by restricting the range of i to V'. We thus have a unique map $\varphi_{V'}: \bar{V} \to V'$ such that the following diagram is commutative,

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "i_{V'}"'] & & \bar{V} \arrow[dl, dashed, "\varphi_{V'}"] \\ & V' & \end{tikzcd} .$$

Since we also have the injection $j: V' \to \bar{V}$, we get two maps, $j \circ \varphi_{V'}: \bar{V} \to \bar{V}$ and $\varphi_{V'} \circ j: V' \to V'$, since by the universal property of (2) the only map from a complete space to itself is the identity map, and both V' and $\bar{V}$ are complete we get that $j \circ \varphi_{V'}$ and $\varphi_{V'} \circ j$ are the identity maps, and thus $\bar{V} \simeq V' = \overline{i(V)}$. Now, note that we can also assume that we have a completion w.r.t (1) (this always exists by the next exercise), and thus we have a map $i': V \to \hat{V}$ such that $\overline{i'(V)} = \hat{V}$ and V is homeomorphic to its image. By the universal property of (2) we then have a continuous $f: \bar{V} \to \hat{V}$ such that $i' = f \circ i$, but then if $U \subseteq V$ is open, so is $i'(U)$, implying that $f^{-1}(i'(U)) = f^{-1} \circ f \circ i(U)$ is open. Since $f^{-1}|_{i'(V)} = i \circ i'^{-1}$ where i'^{-1} is defined on $i'(V)$ (i' is injective there) and i is injective, $f^{-1} \circ f \circ i(U) = i(U)$ is open in $\bar{V}$. This finishes the proof.
(1) $\Rightarrow$ (2): Assume $\overline{i(V)} = \bar{V}$ and i maps V homeomorphically into its image in $\bar{V}$. Assume we have a map $f: V \to W$, then since V is dense in $\bar{V}$ it determines uniquely a map $\varphi_W: \bar{V} \to W$ such that $f = \varphi_W \circ i$. $\square$

Exercise 3.18. *Let V be a topological vector space, show it has a completion.*

Proof. Construction uses either Cauchy filters or Cauchy nets, see [5, Theorem 5.2] for details. $\square$

Definition 3.19. *A topological space (X, τ) is said to be metrizable if there exists a metric which induces the topology τ on X.*

Definition 3.20. *We say that a topological space is S_1 or first countable if every point has a countable basis of open sets.*

Exercise 3.21. *Show that in the category of first-countable vector spaces V is complete if and only if it is sequentially complete.*

Proof. This is essentially a statement about filters, see [5, Prop. 8.2]. $\square$

Definition 3.22. *A Fréchet space is a locally convex, complete and metrizable topological vector space.*

Example 3.23. *Let K be a compact set, and $k, n \in \mathbb{N}_0$. $C^\infty(\mathbb{R}^n)$ and $C_K^\infty(\mathbb{R}^n)$ are Fréchet. $C_K^k(\mathbb{R}^n)$ is a Banach space. $C_c^\infty(\mathbb{R}^n)$ is not Fréchet.*

Exercise 3.24. *Show that for a locally convex, complete topological vector space V the following three conditions are equivalent, thus each implying that V is a Fréchet space.*

1. *V is metrizable.*

2. *V is first countable.*

3. *There is a countable collection of semi-norms $\{n_i\}_{i\in\mathbb{N}}$ that defines the basis for the topology over V.*

Proof. Best to show that metrizable ⇒ first countable ⇒ semi-norms ⇒ metrizable, given as an exercise. □

Exercise 3.25. *Prove that $C^\infty(S^1) \simeq S(\mathbb{Z})$.*

Proof. Both of the spaces are Fréchet, they have the same topology with the homeomorphism given by the Fourier transform. Recall that $\mathcal{F} : C^\infty(S^1) \to S(\mathbb{Z})$ is given by $\mathcal{F}(f) = \{\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x)e^{-inx}dx\}_{n\in\mathbb{Z}}$, and in the other direction by $\mathcal{F}^{-1} : \{c_n\}_{n\in\mathbb{Z}} \mapsto \sum\limits_{n=-\infty}^{\infty} c_n e^{inx}$. Given an ϵ-ball with regards to the semi-norm $n_j(f) = \sup\limits_{x\in S_1} |f^{(j)}(x)|$, its image is given by,

$$\mathcal{F}(B_{n_j,\epsilon}) = \left\{ \{c_n\}_{n\in\mathbb{Z}} : \left| \sum_{n=-\infty}^{\infty} c_n n^j e^{inx} \right| < \epsilon \right\}.$$

Consider the norm m_j of $S(\mathbb{Z})$ given by $m_j(\{c_k\}_{k\in\mathbb{Z}}) = \sup\limits_{k\in\mathbb{Z}} |k^j c_k|$. Note that for $\mathcal{F}(f) = \{c_n\}_{n\in\mathbb{Z}}$, we have for any $k \in \mathbb{Z}$,

$$\begin{aligned}
|k^j c_k| = |\langle f, e^{-ikx}\rangle| &= \left| \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} c_n n^j e^{i(n-k)x} dx \right| \\
&\le \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} c_n n^j e^{i(n-k)x} \right| dx = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} |f^{(j)} e^{-ikx}| dx \\
&\le \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \sup_{x\in S^1} |f^{(j)}(x)| dx = \sup_{x\in S^1} |f^{(j)}(x)| = n_j(f).
\end{aligned}$$

This implies that $\sup\limits_{k\in\mathbb{Z}} |k^j c_k| = \|\mathcal{F}(f)\|_{m_j} \le \|f\|_{n_j}$, meaning that $\mathcal{F}$ is bounded and hence continuous. Alternatively one can view it as $\mathcal{F}(B_{n_i,\epsilon}) \subseteq B_{m_i,\epsilon}$. For the other side, take f in the image of $\mathcal{F}^{-1}$,

$$\begin{aligned}
n_j(f) = \sup_{x\in S^1} |f^{(j)}(x)| = \sup_{x\in S^1} \left| \sum_{n=-\infty}^{\infty} c_n (ni)^j e^{inx} \right| &\le \sum_{0\ne n=-\infty}^{\infty} \frac{1}{n^2} |c_n n^{j+2}| \\
\le \sum_{0\ne n=-\infty}^{\infty} \frac{1}{n^2} \sup_{k\in\mathbb{Z}} |k^{j+2} c_k| &= \frac{2\pi^2}{6} m_{j+2}(\{c_n\}_{n\in\mathbb{Z}}).
\end{aligned}$$

This implies that $\mathcal{F}^{-1}$ is bounded and thus continuous, which means that $\mathcal{F}$ is open and hence a homeomorphism. □

Exercise 3.26. *Let $A_{k_n,\epsilon_n} = \{f \in C^\infty(\mathbb{R}) : \mathrm{supp}(f) \subseteq [-n, n], |f^{(k_n)}| < \epsilon_n\}$. Show that the following two bases generate topologies on $C_c^\infty(\mathbb{R})$ which are equivalent,*

1. $U_{k_n,\epsilon_n} = \sum_{n\in\mathbb{N}} A_{k_n,\epsilon_n}$.

2. $V_{k_n,\epsilon_n} = \mathrm{conv}_{n\in\mathbb{N}}\{A_{k_n,\epsilon_n}\}$.

Note that above,

$$\mathrm{conv}_{n\in\mathbb{N}}\{A_{k_n,\epsilon_n}\} = \left\{\sum_{i\in\mathbb{N}} \lambda_i a_i : \sum_{i\in\mathbb{N}} \lambda_i = 1, \lambda_i \geq 0 \text{ and } a_i \neq 0 \text{ for finitely many } i\text{'s}\right\}.$$

Proof. First note that every two opens balls A_1 and A_2 in $C_c^\infty(\mathbb{R})$ are convex, and that we have that,

$$\frac{A_1 + A_2}{2} \subseteq \mathrm{conv}(A_1, A_2) \subseteq A_1 + A_2 \subseteq \mathrm{conv}(2A_1, 2A_2).$$

Now, for countably many open balls we use the same idea,

$$\sum_{i=1}^{\infty} \frac{A_i}{2^{i+1}} \subseteq \mathrm{conv}_{i\in\mathbb{N}}\{\frac{A_i}{2^i}\} \subseteq \sum_{i=1}^{\infty} \frac{A_i}{2^i} \subseteq \mathrm{conv}_{i\in\mathbb{N}}\{\frac{A_i}{2^{i-1}}\},$$

showing that we can find an open set from (1) in every set of (2) and vice-versa. Note that in both cases we only take finite sums and finite convex combinations, and that $\frac{A_{k_n,\epsilon_n}}{N} = A_{k_n,\frac{\epsilon_n}{N}}$. □

Exercise 3.27. *Show that $f_n \in C_c^\infty(\mathbb{R})$ converges to f with respect to the topology defined in the previous question if and only if it converges as was defined in Definition 2.1, i.e,*

1. *There is a compact set $K \subset \mathbb{R}$ s.t.* $\mathrm{supp}(f) \bigcup_{n\in\mathbb{N}} \mathrm{supp}(f_n) \subseteq K$.

2. *For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$.*

Proof. See homework. □

4 Tirgul 3

Definition 4.1. *Let V be a topological vector space, and set $U_{S,\epsilon} = \{\xi \in V^* : \forall f \in S, |\langle \xi, f\rangle| < \epsilon\}$, where V^* is the continuous dual of V.*

1. *We say a set $B \subseteq V$ is bounded if for every open $0 \in U \subseteq V$ there exists $\lambda \in \mathbb{R}$ such that $B \subseteq \lambda U$.*

2. *We define the weak(-*) topology on V^* by setting the basis of the topology at 0 to be $\mathcal{B}_w := \{U_{\epsilon,S} : \epsilon > 0, S \text{ finite}\}$.*

3. *We define the strong topology on V^* by setting the basis of the topology at 0 to be $\mathcal{B}_s := \{U_{\epsilon,S} : \epsilon > 0, S \text{ bounded}\}$.*

Remark 4.2. *When the topology on V is given by collection of semi-norms, a set is bounded in V if and only if it is bounded with respect to every semi-norm.*

Theorem 4.3. *(Banach-Steinhaus) Let X be a Fréchet space, Y be a normed vector space, and let $F = \{T_\alpha : X \to Y\}$ be a family of bounded linear operators. If for all $x \in X$ we have that $\sup_{T\in F} \|T(x)\|_Y < \infty$ then F is equicontinuous (i.e. for every open $0 \in W \subset Y$ there exists an open $0 \in V \subset X$ such that $T(V) \subset W$ for all $T \in F$).*

In particular, there exists a seminorm $\|\cdot\|_k$ on X such that

$$\sup_{T\in F, \|x\|_k=1} \|T(x)\|_Y < \infty.$$

Proof. This is a version of [6, Theorem 10.18]. □

Exercise 4.4. *The space of distributions $C_c^\infty(\mathbb{R})^*$ is sequentially complete with respect to the weak topology.*

Proof. Take a Cauchy sequence $\{\xi_n\}$ in $C_c^\infty(\mathbb{R})^*$. For every $f \in C_c^\infty(\mathbb{R})$, the limit operator $\xi = \lim_{n\to\infty} \xi_n$ is defined by $\langle \xi, f\rangle = \lim_{n\to\infty} \langle \xi_n, f\rangle$. It is indeed defined in each point since $\langle \xi_n, f\rangle$ is a Cauchy sequence of real numbers and thus converges, and it is also linear.

It is left to show that ξ is continuous. For this, it is enough to show that if $f_n \to f$ then $\langle \xi, f_n\rangle \to \langle \xi, f\rangle$. Let $\{f_n\}_{n=1}^\infty \in C_c^\infty(\mathbb{R})$ be a sequence of functions converging to f in $C_c^\infty(\mathbb{R})$, and set K compact such that $\bigcup_{n=1}^\infty \operatorname{supp}(f_n) \cup \operatorname{supp}(f) \subseteq K$. Since each ξ_n is continuous, it is bounded w.r.t. some semi-norm $\|\cdot\|_{K,n}$, and thus there exist $C_{K,n} > 0$ and $l_{K,n} > 0$ such that,

$$|\langle \xi_n, f\rangle| \le C_{K,n} \sup_{x\in K} \sum_{i=0}^{l_{K,n}} |f^{(i)}(x)|.$$

Now, since $\langle \xi_n, f\rangle$ is a Cauchy sequence of numbers, we get that $\sup_{n\in\mathbb{N}} \langle \xi_n, f\rangle$ is finite. Recalling that $C_K^\infty(\mathbb{R})$ is a Fréchet space, by the Banach-Steinhaus theorem for Fréchet spaces, there exists a bound for $\sup_{n\in\mathbb{N}, \|f\|_{K,n}=1} |\langle \xi_n, f\rangle|$, so there is uniform C_K which bound $|\langle \xi_n, f\rangle|$ for all n w.r.t $\|\cdot\|_K$. Now, for each f_k and all $n \in \mathbb{N}$ we have,

$$|\langle \xi, f - f_k\rangle| \le |\langle \xi - \xi_n, f - f_k\rangle| + |\langle \xi_n, f - f_k\rangle| \le |\langle \xi - \xi_n, f - f_k\rangle| + C_K\|f - f_k\|_K.$$

Since $\lim_{n\to\infty} |\langle \xi - \xi_n, f - f_k\rangle| = 0$, we get the required statement. □

Exercise 4.5. *Let $S \subseteq C_c^\infty(\mathbb{R})$ be a bounded set, then $\exists K$ compact such that $S \subseteq C_K^\infty(\mathbb{R})$.*

Exercise 4.6. *Consider the embedding $C_c^\infty(\mathbb{R}) \hookrightarrow C_c^\infty(\mathbb{R})^*$ defined by $f \mapsto \xi_f$. Show that the image of this map is dense in $C_c^\infty(\mathbb{R})^*$ w.r.t both weak and strong topologies.*

Proof. It is enough to show that for every basic open $U \in \mathcal{B}$ there exists $\xi_f \in U$. We start with showing this for compactly supported distributions. Assume that $\operatorname{supp}(\xi) = K$, we need to show that for each basic open set U of 0 we can find ξ_n such that $\xi - \xi_n \in U$. Take an approximation of identity η_n, and define $\xi_n = \xi * \eta_n$. ξ_n are compactly supported smooth functions (by previous exercises), and for each $f \in C_c^\infty(\mathbb{R})$ we have that $\langle \xi - \xi_n, f\rangle \to 0$, implying that for every $U \in \mathcal{B}_w$ we can take n large enough such that $\xi - \xi_n \in U$.

Showing this for the strong topology is trickier. By continuity of ξ we obtain for some $k \in \mathbb{N}$ (here $\|f\|_k = \sup |f^{(k)}(x)|$),

$$|\langle \xi - \xi_n, f\rangle| = |\langle \xi, f - \overline{\eta_n * \overline{f}}\rangle| \le C\|f - \overline{\eta_n * \overline{f}}\|_k.$$

Given $U_{S,\epsilon} \in \mathcal{B}_s$, since S is bounded it is contained in $\lambda B(0)_{\|\cdot\|_{k+1},1}$ for $\lambda \in \mathbb{R}$. We now show using Lagrange's mean value theorem that $\overline{\eta_n * \overline{f}} \to f$ uniformly with respect to the norm $\|\cdot\|_k$. Explicitly, we give a bound on $|f^{(k)}(x) - \overline{\eta_n * \overline{f^{(k)}}}(x)|$ which is independent on x and which converges to zero as n grows:

$$|f^{(k)}(x) - \overline{\eta_n * \overline{f^{(k)}}}(x)| = \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} (f^{(k)}(x) - f^{(k)}(x+t))\eta_n(t)dt \right|$$
$$\le \max_{t\in[-\frac{1}{n},\frac{1}{n}]} |f^{(k)}(x) - f^{(k)}(x+t)| \le \max_{t\in[-\frac{1}{n},\frac{1}{n}]} |f^{k+1}(c_t)t| \le \frac{\|f\|_{k+1}}{n}.$$

Now, for every $f \in S$ we have that $\|f\|_{k+1} < \lambda$, implying that for all $n \in \mathbb{N}$ we get,

$$\|f - \overline{\eta_n * \overline{f}}\|_k = \sup_{x\in\mathbb{R}} |f^{(k)}(x) - \overline{\eta_n * \overline{f^{(k)}(x)}}| < \frac{\lambda}{n}.$$

Since λ is independent of the function f, we can take $n_0 \in \mathbb{N}$ large enough such that $\xi - \xi_n \in U_{S,\epsilon}$ for $n > n_0 = \lceil \frac{C\lambda}{\epsilon} \rceil$, as required.

To finish off the proof, note that compactly supported distributions are dense in distributions by taking elements of the form $\xi_k = \xi I_{[-k,k]}$. We know that $C_c^\infty(\mathbb{R}) = \lim\limits_{\to K} C_K^\infty(\mathbb{R})$, and since for every compact K there exists $k \in \mathbb{N}$ large enough such that $K \subseteq [-k,k]$, combining with the fact that $S \subseteq C_K^\infty(\mathbb{R})$ for some K as it is bounded, we get that for n big enough $\xi_n = \xi$ on $S \subseteq C_K^\infty(\mathbb{R})$, and we are done. □

Definition 4.7. *Let $W \subseteq \mathbb{R}^n$ be a closed linear subspace, we define:*

$$V_m(C_c^\infty(\mathbb{R}^n), W) = \{f \in C_c^\infty(\mathbb{R}^n) : \frac{\partial^i f}{(\partial x)^i}_{|W} = 0, |i| \le m\},$$

and

$$F_m((C_c^\infty(\mathbb{R}^n))^*, W) = \{\xi \in (C_c^\infty(\mathbb{R}^n))^* : \xi_{|V_m} = 0\}.$$

Exercise 4.8. *Let $U = \mathbb{R}^n - \mathbb{R}^k$. Show that*

$$\overline{C_c^\infty(U)} = \bigcap_{m=0}^{\infty} V_m = \{f \in C_c^\infty(\mathbb{R}^n) : \frac{\partial^i f}{\partial x_i}_{|\mathbb{R}^k} = 0, |i| \in \mathbb{N}_0\}.$$

Proof. Given $f \in C_c^\infty(U)$, then $\langle \delta_{\vec{x}}^{\vec{i}}, f\rangle = 0$ if $\vec{x} \in \mathbb{R}^k$ since its support is by definition in U^c, thus $f \in \bigcap\limits_{m=0}^{\infty} V_m$. Since these are continuous operators, they are also zero on the closure, implying that $\overline{C_c^\infty(U)} \subseteq \bigcap\limits_{m=0}^{\infty} V_m$.

For the other direction, given $f \in \bigcap_{m=0}^{\infty} V_m$ use cutoff functions of I_U (which are roughly I_U convolved with approximations of identity, η_n) to construct a sequence of functions which are identically zero in an ϵ-neighborhood of $\mathbb{R}^k$, and thus in $C_c^\infty(U)$, converging to f. $\square$

5 Tirgul 4

5.1 Absolute values

We want to generalize the notion of completion of a field with respect to an absolute value.

Definition 5.1. *A function $|\cdot| : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is called an absolute value if for all $x, y \in \mathbb{Q}$:*

1. $|x+y| \leq |x| + |y|$ *(triangle inequality).*
2. $|xy| = |x||y|$.
3. $|x| = 0 \iff x = 0$.

If furthermore $|x+y| \leq \max\{|x|, |y|\}$, $|\cdot|$ is called a non-Archimedean absolute value (and Archimedean otherwise).

Definition 5.2. *We say that two absolute values $|\cdot|_1$ and $|\cdot|_2$ are equivalent if $\exists \alpha \in \mathbb{R}_{>0}$ such that $|\cdot|_1^\alpha = |\cdot|_2$.*

Theorem 5.3. *(Ostrowki's theorem) The only absolute values on $\mathbb{Q}$ up to equivalence are the following:*

1. *The real (Archimedean) absolute value,* $|x|_\infty = \begin{cases} -x, & \text{for } x < 0, \\ x, & \text{for } x \geq 0. \end{cases}$
2. *A p-adic (non-Archimedean) absolute value, if $x = p^n \frac{a}{b}$ and $a, b \in \mathbb{Z}$ are coprime to p, $n \in \mathbb{Z}$,* $|x|_p = \begin{cases} p^{-n}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$
3. *The discrete absolute value,* $|x|_{disc} = \begin{cases} 0, & \text{for } x = 0, \\ 1, & \text{for } x \neq 0. \end{cases}$

Example 5.4. *We have $|1|_7 = |10|_7 = |100|_7 = 1$, and $|49|_7 = |490|_7 = 5^{-2}$.*

Proof. (Ostrowski's theorem) Let $|\cdot|$ be an absolute value, we show it must be one of the above by cases. Assume $|\cdot|$ is non-Archimedean, i.e. $|x+y| \leq \max\{|x|, |y|\}$, and set $\mathfrak{a} = \{x \in \mathbb{Z} : |x| < 1\}$. This set is non empty as $|0| = 0$, and since we assume $|\cdot|$ is non-Archimedean it is an ideal of $\mathbb{Z}$ since,

$$|x + \ldots + x| \leq |x|, \tag{*}$$

and thus if $x \in \mathfrak{a}$, meaning that $|x| < 1$, then $xy = \underbrace{x + \ldots + x}_{y \text{ times}} \in \mathfrak{a}$. Consider a prime p. If $|p| = 1$ for every prime, we get that $|x| = 1$ for every $0 \neq x \in \mathbb{Q}$, as $|\frac{1}{p}| = |p|^{-1}$, and thus $|\cdot|$ is the discrete absolute value. Thus we can assume that there exists $p \in \mathfrak{a}$ (note that for every integer $|m| \leq 1$ by (*)), implying that $p\mathbb{Z} \subseteq \mathfrak{a} \subsetneq \mathbb{Z}$ and consequentially $p\mathbb{Z} = \mathfrak{a}$. Now, if we put $s = -\frac{\log |p|}{\log p}$, we see that $|p| = p^{-s}$. Taking $x = p^n \frac{a}{b}$ where $a, b, n \in \mathbb{Z}$ and a and b are coprime to p we get:

$$|x| = |p^n \frac{a}{b}| = |p^n| \underbrace{|\frac{a}{b}|}_{=1} = |p|^n = p^{-ns} = |x|_p^s,$$

showing $|\cdot|$ is equivalent to $|\cdot|_p$.

Now, assume $|\cdot|$ is an Archimedean absolute value. We must have that $|n| \geq 1$ for all non-zero integers $n \in \mathbb{Z}$. Otherwise, let n be the smallest positive number such that $|n| < 1$, and for every $n < x \in \mathbb{N}$ write it in base n:

$$x = a_0 + a_1 n + a_2 n^2 + \ldots + a_r n^r, \text{ for } 0 \leq a_i \leq n-1, \ n^r \leq x.$$

We have that $|a_i| \leq a_i \leq n$, thus

$$|x| \leq \sum_{i=0}^{r} |a_i n^i| \leq \sum_{i=0}^{r} n|n|^i = \frac{n(1-|n|^{r+1})}{1-|n|} \leq \frac{n}{1-|n|}.$$

Since $\frac{n}{1-|n|}$ is independent of r, it bounds every $x > n$, and thus we must have that $|x| \leq 1$ for every $x > n$ as otherwise $|x|^k > \frac{n}{1-|n|}$ for k large enough. We get that $|x| \leq 1$ for $x < n$ in the same way by considering $n < x^k$ for k large enough. But this means that $|x| \leq 1$ for all $x \in \mathbb{Z}$, so by the previous step it is equivalent to a p-adic absolute value, in contradiction to the fact that $|\cdot|$ is Archimedean.

We can thus assume $|n| \geq 1$ for $n \in \mathbb{N}$. Note that $r \leq \frac{\log x}{\log n}$, we now have:

$$|x| \leq \sum_{i=0}^{r} |a_i||n|^i \leq \left(1 + \frac{\log x}{\log n}\right) n|n|^{\frac{\log x}{\log n}}.$$

Using these bounds for x^k:

$$|x|^k \leq (1 + k\frac{\log x}{\log n}) n|n|^{\frac{k \log x}{\log n}},$$

implying

$$|x|^{\frac{1}{\log x}} \leq \sum_{i=0}^{r} |a_i||n|^i \leq \sqrt[k]{(1 + k\frac{\log x}{\log n}) n}|n|^{\frac{1}{\log n}}.$$

By taking $k \to \infty$, we get $|x|^{\frac{1}{\log x}} \leq |n|^{\frac{1}{\log n}}$. But by interchanging x and n we can get that $|n|^{\frac{1}{\log n}} \leq |x|^{\frac{1}{\log x}}$ (note that if $n < x$ we can repeat this process for x and n^k for k large enough). Thus $|x|^{\frac{1}{\log x}} = |n|^{\frac{1}{\log n}} = e^{\frac{\log |x|}{\log x}}$ is constant, implying that $s = \frac{\log |x|}{\log x} = \frac{\log |n|}{\log n}$ is constant. Now, note that $|x| = x^s$ for every x and get that,

$$|x| = x^s = |x|_\infty^s,$$

finishing the proof. $\square$

5.2 p-adic numbers

Definition 5.5. *We define the p-adic numbers $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$.*

Remark 5.6. *We get a space which is an uncountable field of characteristic 0, not algebraically closed, locally compact (every point has a compact neighborhood) and totally disconnected, i.e. every connected component is a point.*

Definition 5.7. *We define the p-adic integers $\mathbb{Z}_p$ to be the unit disc in $\mathbb{Q}_p$, explicitly $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.*

Exercise 5.8. *Show that $\sum\limits_{n=0}^{\infty} a_n$ converges $\iff$ $|a_n|_p \to 0$.*

Proof. One direction is the same as in usual real analysis; If the sum converges then the partial sums are a Cauchy sequence thus for every $\epsilon > 0$ there exists n_0 large enough such that for $n_0 < n_1 < n_2 \in \mathbb{N}$,

$$\left|\sum_{i=0}^{n_2} a_i - \sum_{i=0}^{n_1} a_i\right|_p = \left|\sum_{i=n_1+1}^{n_2} a_i\right|_p < \epsilon.$$

in particular take $n_0 < n-1, n \in \mathbb{N}$ and see that $\left|\sum\limits_{i=0}^{n} a_i - \sum\limits_{i=0}^{n-1} a_i\right| = |a_n| < \epsilon$. For the other direction, assume $|a_n|_p \xrightarrow{n\to\infty} 0$, thus for every $\epsilon > 0$ there exists n_0 large enough such that $|a_n|_p \leq p^{-m} < \epsilon$ for $n_0 < n$, but this exactly means that the sum converges as for $n_0 < n_1, n_2 \in \mathbb{N}$:

$$\left|\sum_{i=0}^{n_2} a_i - \sum_{i=0}^{n_1} a_i\right|_p = \left|\sum_{i=n_1+1}^{n_2} a_i\right|_p \leq \max_{n_1+1 \leq i \leq n_2} \{|a_i|_p\} \leq p^{-m} < \epsilon.$$

$\square$

Exercise 5.9. *Show that $\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.*

Proof. Recall that $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \{(x_0, x_1, x_2, \ldots) : x_i = x_j \bmod p^j, j < i\}$, with the topology being the weakest topology such that all the projections $p_n : \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ are continuous, and recall that $\mathbb{Z}/p^n\mathbb{Z}$ all have the discrete topology. We define a map $\varphi : \mathbb{Z}_p \to \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ by $\varphi(a) = (\sum\limits_{i=0}^{k} a_i p^i)_{k=0}^{\infty}$ (we saw that every element in $\mathbb{Q}_p$ can be written as an infinite sum, and since $|x| \leq 1$ for $x \in \mathbb{Z}_p$, their expansion starts on the 0-th term). This is indeed a map of rings, first note that $\varphi(0) = 0$ since 0 is divisible infinitely many times by p and thus $0 = 0 \mod p^n$ for every n, and furthermore $\varphi(1)$ is the constant sequence $(1)_{k=0}^{\infty}$, and it is evidently the unit element of the inverse limit (multiplication is done in each coordinate). To see that the addition and multiplication are sent to the correct elements note that each projection $q_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ by $q_n(x) = x \mod p^n$ is a map of rings, since reducing mod p^n commutes with addition and multiplication. Now, assume $\varphi(a) = 0$, thus $a = 0 \mod p^n$ for every $n \in \mathbb{N}_0$, since it means it is zero in each coordinate, but this must imply that $a = 0$ since this is the only element with $|a|_p = 0$. To see φ is onto, take some a in the right hand side, and construct an element $x \in \mathbb{Z}_p$ such that it the limit of the sequence (a_i). Because a was a compatible sequence in the inverse limit, for every $\epsilon > 0$ we can find n_0 such that for $n_0 < n < m$ we have

$$|a_m - a_n|_p = \left| \sum_{i=n+1}^{\infty} a_i p^i \right|_p \leq p^{-n_0} < \epsilon,$$

implying that (a_i) is a Cauchy sequence with respect to $|\cdot|_p$ in $\mathbb{Z}$, and thus a proper element in $\mathbb{Z}_p$. Now, φ is continuous, given a basic open set $U = p_n^{-1}(a_n) = \{(a_i) \in \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} : a_i = a_n \mod p^n\}$, we see that $\varphi^{-1}(U) = \{x \in \mathbb{Z}_p : x - a_n \in p^n\mathbb{Z}_p\} = a_n + p^n\mathbb{Z}_p$, which is open. Evidently so are inverse images of unions and intersections of such U's. To see φ is open, note that for $a \in \mathbb{Z}_p$:

$$\varphi(a + p^n\mathbb{Z}_p) = \left\{(x_i) \in \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} : x_i = a \mod p^n\right\} = p_n^{-1}(a).$$

Thus, φ is a homeomorphism of topological rings. □

Exercise 5.10. *Show that $\mathbb{Z}_p$ is compact.*

Proof. Using the previous exercise, we know that $\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, where the latter is a subspace of $\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$. Since each $\mathbb{Z}/p^n\mathbb{Z}$ is compact (it is finite), so is their product by Tychonoff's theorem, and since the inverse limit is a closed space of the product, it must be compact. To see it is closed note that $\mathbb{Z}_p = \bigcap_{i=1}^{\infty} C_n$ where $C_n = G_n \times \prod_{m \neq n, n+1} \mathbb{Z}/p^m\mathbb{Z}$ and

$$G_m = \{(x_{n+1} + p^{n+1}\mathbb{Z}, x_n + p^n\mathbb{Z}) : x_n = x_{n+1} \mod p^n\},$$

is the graph of the projection from $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. □

5.3 Properties of ℓ-spaces and analysis on ℓ-spaces

Definition 5.11. *An ℓ-space X is a locally compact, totally disconnected space which is Hausdorff. We furthermore say that X is countable at infinity or σ-compact if it is the countable union of compact sets.*

Definition 5.12. *Let X be an ℓ-space. We say that a function is smooth if it is locally constant, we denote all smooth (i.e. locally constant), compactly supported functions $f : X \to \mathbb{C}$ by $C_c^\infty(X)$ or $S(X)$.*

Exercise 5.13. *Let X be an ℓ-space, show it has a basis of clopen sets (i.e. it is zero-dimensional).*

Proof. Taken from [1, 3.1.7]. Assume we have a point $x \in \mathcal{W} \subseteq K$, with $\mathcal{W}$ open and $K = \overline{\mathcal{W}}$ compact and set $\mathcal{P}_x = \{U \subseteq K : U \text{ is clopen in } K \text{ and } x \in U\}$ and $P = \bigcap_{V \in \mathcal{P}_x} V$. Note that $K \in \mathcal{P}_x$, thus $x \in P \neq \varnothing$.

Now, we claim that for every closed subset F of K such that $F \cap P = \varnothing$ there exists some $W \in \mathcal{P}_x$ such that $W \cap F = \varnothing$. Otherwise, set $\eta = \{U \cap F : U \in \mathcal{P}_x\}$. By assumption, it is a family of non-empty closed subsets of F, and since F is compact if $\bigcap_{V \in \eta} V = \varnothing$, then there is a finite collection of V_i such that $\bigcap_{i=0}^{n} V_i = \bigcap_{i=0}^{n} U_i \cap F = \varnothing$ (note that this is an equivalent characterization of compactness via closed sets). Since $\mathcal{P}_x$ is closed under finite intersections, $\bigcap_{i=0}^{n} U_i \in \mathcal{P}_x$, but this is a contradiction since we assumed that every set in $\mathcal{P}_x$ intersects F non-trivially. Thus

$$\varnothing \neq \bigcap_{V \in \eta} V = \bigcap_{U \in \mathcal{P}_x} U \cap F = P \cap F,$$

contradicting the assumption that $P \cap F = \varnothing$, so we have a set $V \in \mathcal{P}_x$ such that $V \cap F = \varnothing$.

We now wish to show that $P = \{x\}$. Assume the contrary, i.e. $P \neq \{x\}$. P is disconnected since X is totally disconnected, so there exists non-empty closed $x \in A$ and B such that $A \cup B = P$ and $A \cap B = \varnothing$ which are open in K. Since K is regular, (Hausdorff + locally compact implies regular), there exist open disjoint sets $A \subseteq U$ and $B \subseteq V$ in K, where we have $F = K \backslash (U \cup V)$ closed in K and $P \cap F = \varnothing$. We showed that for such F we can find $W \in \mathcal{P}_x$ such that $F \cap W = \varnothing$. Now, observe that the open set $G = U \cap W$ is also closed in K as,

$$\overline{G} = \overline{U} \cap W \subseteq (K \backslash V) \cap (K \backslash F) = K \backslash (V \cup F) \subseteq U.$$

Therefore $\overline{G} \subseteq U \cap W = G$ (W was closed). Since $x \in G$, we have $G \in \mathcal{P}_x$, but as $G \cap B = \varnothing$, we get that $P = A \cup B$ is not contained in G, which is a contradiction, implying $P = \{x\}$.

Since for every open set $x \in O$ in K the set $K \backslash O$ is compact and $x \notin K \backslash O$, it follows from the above claim that O contains some $V \in \mathcal{P}_x$.

Now, given an open set $x \in \mathcal{A}$ in X, we have that $\mathcal{W} \cap \mathcal{A} \subset K \cap \mathcal{A}$, is open in K, and thus contains a clopen $\mathcal{U}$ in the topology of $K \cap \mathcal{A}$ from the above. Now, $\mathcal{U}$ is closed in K and thus closed in X, and open in $\overline{\mathcal{W} \cap \mathcal{A}}$ but contained in $\mathcal{W} \cap \mathcal{A}$ and thus open in X. This finishes the proof.

□

Remark 5.14. *Recall that a Radon measure is a positive (i.e, positive on positive functions) functional on the space of compactly supported continuous functions on X, or equivalently a locally finite, inner regular Borel measure.*

Theorem 5.15. *Let G be an ℓ-group. There exists up to a factor only one left-invariant distribution $\mu_G \in S^*(G)^G$, that is, a distribution such that:*

$$\langle g_0 \mu_G, f \rangle = \int_G f(g_0 g) d\mu_G(g) = \int_G f(g) d\mu_G(g) = \langle \mu_G, f \rangle$$

for all $f \in S(G)$ and $g_0 \in G$. Furthermore, we can take $\langle \mu_G, f \rangle > 0$ if $f \in S(G)$ is a non-zero non-negative function. This distribution is a measure which is called a (left-invariant) Haar measure on G.

Proof. We shall start by showing uniqueness. Set the right and left actions of G on itself by $\rho(h)(g) = gh^{-1}$ and $\lambda(h)(g) = hg$, and let $\{N_\alpha\}$ be a fundamental system of compact open subgroups of e_G where $\alpha \in I$. This kind of system exists for $\mathbb{Q}_p$ by taking $\{p\mathbb{Z}_p\}$, and by van-Dantzig's theorem for a general ℓ-group. We can suppose that there exists α_0 such that $N_\alpha \subseteq N_{\alpha_0}$ for all $\alpha \in I$, as if not, pick one N_β and consider the system of neighborhoods $\{N_\beta \cap N_\alpha\}_{\alpha\in I}$. Now, define

$$S_\alpha = \{f \in S(G) : \rho(g)f = f, \forall g \in N_\alpha\},$$

and note that if $N_\alpha \subseteq N_\beta$ we have that $S_\beta \subseteq S_\alpha$. Also, observe that $S(G) = \bigcup\limits_{\alpha\in I} S_\alpha$, since for every $f \in S(G)$ we can write $f = \sum\limits_{i=0}^{n} c_i I_{g_i N_i}$ with $I_{g_i N_i}$ being indicator functions of $g_i N_i$ (it is smooth), and thus f must be invariant w.r.t the right action of $N = \bigcap\limits_{i=0}^{n} N_i \neq \{e_G\}$ as $N_i k = N_i$ for all $k \in N_i$, and N must contain some N_α.

Each space S_α is preserved under the left action of G since if $\rho(h)I_{g_iN_i} = I_{g_iN_i}$ we also have that $\rho(h)\lambda(g)I_{g_iN_i} = \rho(h)I_{gg_iN_i} = I_{gg_iN_i}$ for all $g \in G$. Thus, if we have a directed system of compatible left G-invariant functionals $\mu_\alpha \in S_\alpha^*$, that is $S_\alpha \subseteq S_\beta$ implies $\mu_{\beta|S_\alpha} = \mu_\alpha$, we obtain a left G-invariant functional on $S(G) = \lim\limits_{\to\alpha} S_\alpha = \bigcup\limits_{\alpha\in I} S_\alpha$.

Now, note that as before each $f \in S_\alpha$ can be written as a finite sum of $I_{g_iN_\alpha}$ for some $g_i \in G$, as by definition $f(gN_\alpha) = f(g)$ for all $g \in G$ and f is compactly supported which means that g_iN_α are a cover for $\mathrm{supp}(f)$. This implies that S_α is generated by left translates of I_{N_α}, and thus μ_G is unique up to a scalar if it exists, since by determining its value on $I_{N_{\alpha_0}}$ we determine its value on every I_{N_α} by $I_{N_{\alpha_0}} = \sum\limits_{i=0}^{[N_{\alpha_0}:N_\alpha]} I_{g_iN_\alpha}$ and thus on every S_α.

Now, to construct such μ_G we can define for every $f \in S_\alpha$,

$$\langle \mu_\alpha, f\rangle = \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{g_i\in G/N_\alpha} f(g_i),$$

where this sum is finite since $\mathrm{supp}(f)$ is compact. Now, if $f \in S_\beta \subseteq S_\alpha$,

$$\begin{aligned}\langle \mu_\alpha, f\rangle &= \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{g_i\in G/N_\alpha} f(g_i) = \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{h_j\in G/N_\beta} \left(\sum_{g_i\in N_\beta/N_\alpha} f(h_jg_i)\right) \\ &= \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{h_j\in G/N_\beta} ([N_\beta : N_\alpha]f(h_j)) = \frac{1}{[N_{\alpha_0} : N_\beta]} \sum_{h_j\in G/N_\beta} f(h_j) = \langle \mu_\beta, f\rangle.\end{aligned}$$

This implies that the functionals μ_α are compatible. $\{\mu_\alpha\}$ are non-negative left G-invariant (the sum does not change after applying $\lambda(g)$), and thus establish the existence of a Haar measure on G.

Note that μ is indeed a Radon measure since it is inner regular, that is $\mu(U) = \sup\limits_{K\subset U \text{ cpt}} \mu(K)$ for every measurable set $U \subset G$, and locally finite, meaning that every point has a a neighborhood of finite volume. □

6 Tirgul 5

Definition 6.1. *For a topological vector space V we define* $\mathrm{Haar}(V) \subseteq \mu^\infty(V) \subseteq \mathrm{Dist}(V)$ *to be the one dimensional vector space of Haar measures on it, which exists by Haar's theorem.*

Exercise 6.2. *Let V be a finite dimensional linear space (either an ℓ-space or a real vector space). Show that every distribution on V which is translation invariant is a Haar measure, that is $(C_c^\infty(V)^*)^V = \mathrm{Haar}(V)$.*

Proof. (We show this again for an ℓ-space although we proved we have a Haar measure for an ℓ-group.) If V is an ℓ-space, every $f \in C_c^\infty(V)$ can be written as $f = \sum_{i=0}^{n} c_i I_{v_i+K_i}$ where K_i are compact open subgroups of V. Furthermore, the function f is invariant with respect to $K = \bigcap_{i=0}^{n} K_i$, and thus can be written as a finite sum of translations of K; $f = \sum_{i=0}^{m} c_i' I_{v_i+K}$. Now, if $\int_V f d\mu = 0$ for a choice of a Haar measure μ, we must have that $\sum_{i=0}^{m} c_i' = 0$ since $\mu(I_K) \neq 0$. Since $\xi \in (C_c^\infty(V)^*)^V$ is translation-invariant we get that,

$$\langle \xi, f\rangle = \langle \xi, \sum_{i=0}^{m} c_i' I_{v_i+K}\rangle = \langle \xi, \sum_{i=0}^{m} c_i' I_K\rangle = \sum_{i=0}^{m} c_i' \langle \xi, I_K\rangle = 0.$$

We now claim that for $\xi \in (C_c^\infty(V)^*)^V$, if $\langle \xi, f\rangle = 0$ where $\int_V f d\mu \neq 0$, then $\langle \xi, g\rangle = 0$ for all $g \in C_c^\infty(V)$. Indeed, if $\langle \xi, f\rangle = 0$, as we saw before this means that $\sum_{i=0}^{m} c_i' \langle \xi, I_K\rangle = 0$. Since the integral is non-zero, the sum of the coefficients is non-zero and thus $\langle \xi, I_K\rangle = 0$. We get that for every indicator function $I_{K'}$ where $K' \subset V$ is a compact open subgroup the index $[K : K \cap K']$ is finite, implying that,

$$\langle \xi, I_{K'}\rangle = \langle \xi, \sum_{j=0}^{[K':K\cap K']} I_{v_j+K\cap K'}\rangle = [K' : K\cap K']\langle \xi, I_{K\cap K'}\rangle = \frac{[K' : K\cap K']}{[K : K\cap K']}\langle \xi, I_K\rangle = 0.$$

Since this holds for every indicator of a compact open subgroup, this means that $\xi = 0$. To finish off this part of the proof pick some open compact open subgroup K, and note that the map $(C_c^\infty(V)^*)^V \to \mathbb{R}$ by $\xi \mapsto \langle \xi, I_K\rangle$ is an isomorphism since this is a map of vector spaces which is onto (there are non-zero distributions since the space of Haar measures is inside), and we just showed injectivity.

A different way to see this is that the space of functions whose integral vanishes, call it W, is of codimension one in $C_c^\infty(V)$, and thus we have that the space we are interested in is actually functionals on the one dimensional space $C_c^\infty(V)/W$.

Now, assume that $V = \mathbb{R}^n$ and ξ some invariant distribution. We follow the same method as in the first case. First note that using continuity of ξ, for every directional derivative,

$$\langle \xi, \frac{\partial f}{\partial \vec{v}}\rangle = \langle \xi, \lim_{h\to 0} \frac{f(x) - f(x - h\vec{v})}{h}\rangle = \lim_{h\to 0} \frac{\langle \xi, f\rangle - \langle L_{h\vec{v}}(\xi), f\rangle}{h},$$

where $L_{h\vec{v}}(\xi)$ denotes translation of ξ by $h\vec{v}$, and thus

$$\langle \xi, \frac{\partial f}{\partial \vec{v}}\rangle = \lim_{h\to 0} \frac{\langle \xi, f\rangle - \langle \xi, f\rangle}{h} = 0.$$

Now, assume that $\langle \xi, f \rangle = 0$ where $\int\limits_V f d\mu \neq 0$. Given any $g \in C_c^\infty(\mathbb{R}^n)$, assume that $\int\limits_V (f-g)d\mu = 0$ (otherwise normalize g). We claim that $f - g = \sum\limits_{i=0}^{n} \frac{\partial f_i}{\partial x_i}$ for some $f_i \in C_c^\infty(\mathbb{R}^n)$ which means that $\langle \xi, f-g \rangle = 0$ and thus $\langle \xi, g \rangle = \langle \xi, g + (f-g) \rangle = \langle \xi, f \rangle = 0$. Indeed, if $n = 1$, take $F(x) = \int\limits_{-\infty}^{x} (f-g)dx$, and this is the fundamental theorem of calculus (note that F is compactly supported since the integral of $f - g$ is zero).

For bigger n, set $h = f - g$ and continue by induction; Set $\alpha(x_n) = \int\limits_{\mathbb{R}^{n-1}} h dx_1 \dots dx_{n-1}$ and $H = h - \alpha(x_n)\Psi(x_1, \dots, x_{n-1})$, where Ψ is compactly supported and $\int\limits_{\mathbb{R}^{n-1}} \Psi dx_1 \dots dx_{n-1}$=1. The function $\alpha(x_n)$ is compactly supported since h is compactly supported.

Now, note that for every $x_n \in \mathbb{R}$ we have that, $\int\limits_{-\infty}^{\infty} H dx_1 \dots dx_{n-1} = 0$. Thus by the induction hypothesis, where x_n is fixed, $H = \sum\limits_{i=0}^{n-1} \frac{\partial H_i}{\partial x_i}$ for some $H_i \in C_c^\infty(\mathbb{R}^n)$. Note that H_i are indeed smooth in all variables, including x_n, since they vary smoothly when x_n varies (this can also be established by induction). Now, $h = H + \alpha\Psi$, but we see that $\int\limits_{-\infty}^{\infty} \alpha(x_n) = 0$, and thus $\alpha = \frac{\partial \beta}{\partial x_n}$ for some smooth, compactly supported β. Since Ψ is independent of x_n, this implies that $h = \sum\limits_{i=0}^{n-1} \frac{\partial H_i}{\partial x_i} + \frac{\partial \beta \Psi}{\partial x_n}$, and we're done by the same reasoning as in the first part. $\square$

6.1 The exterior algebra

Let V be a finite dimensional vector space, we define the exterior algebra as $\Lambda(V) = \bigoplus\limits_{i=0}^{\infty} \Lambda^i(V)$, where $\Lambda^k(V) = \left(\bigotimes\limits_{j=0}^{k} V \right) / J_k$ and J_k is the vector space spanned in $\bigotimes\limits_{j=0}^{k} V$ by the set $\{v_1 \otimes \ldots \otimes v_k : v_i = v_j \text{ for some } i \neq j\}$. Note that this implies that the elements of the exterior algebra are anti-symmetric, and that $\Lambda^k(V) = 0$ if $k > \dim V$, since after choosing a basis to V and decomposing an element in $\Lambda^k(V)$ to basic tensors, there must be a basis element which appears at least twice.

Definition 6.3. *Let V be a finite dimensional vector space of dimension n over a local field F with absolute value $|\cdot|$.*

1. *We define the space of k-forms as $\Omega^k(V) = \Lambda^k(V^*)$.*
2. *For a 1-dimensional space V we define a real vector space $|V| = \{f : V^* \to \mathbb{R} : \forall \alpha \in F, f(\alpha v) = |\alpha| f(v)\}$.*
3. *We define the densities of V as $\mathrm{Dens}(V) = \{f : V^n \to \mathbb{R} : f(Av_1, \dots, Av_n) = |\det(A)| f(v_1, \dots, v_n)\}$.*

Exercise 6.4. *Show that*

$$\Omega^n(V) \simeq^{\text{can}} \{f : V^n \to F : f(Av_1, \ldots, Av_n) = \det(A) f(v_1, \ldots, v_n)\},$$

and that this space is one dimensional.

Proof. Set $T_n(V) = V \otimes \ldots \otimes V$ and

$$B = \{f : V^n \to F : f(Av_1, \ldots, Av_n) = \det(A) f(v_1, \ldots, v_n)\}.$$

Then we have that $T_n(V)^* \simeq^{\text{can}} T_n(V^*)$. Now, we can construct a map $\varphi : T_n(V)^* \to B$ via $\varphi(l)(v_1, \ldots, v_n) = \langle l, v_1 \otimes \ldots \otimes v_n \rangle$. Note that this is well defined since

$$\varphi(l)(Av_1, \ldots, Av_n) = \langle l, Av_1 \otimes \ldots \otimes Av_n \rangle = \det(A) \langle l, v_1 \otimes \ldots \otimes v_n \rangle.$$

Furthermore, note that $\ker(\varphi) = J_n$. Now, $\varphi(l) = 0 \iff$ there exists a basis $\{v_i\}$ such that $\langle \varphi(l), v_1, \ldots, v_n \rangle = 0$ (we can always find a matrix A transforming $\{v_i\}$ into every other basis). The claim follows since $\varphi(l)$ vanishes on some basis if and only if $l \in J_n$. □

Corollary 6.5. *We can thus also take as definition* $\text{Dens}(V) = |\Omega^{\text{top}}(V)|$.

Exercise 6.6. *Show that* $\text{Dens}(V) \simeq^{can} \text{Haar}(V)$.

Proof. A Haar measure can be viewed both as a functional on compactly supported, continuous functions and as a function on Borel sets. The absolute value of the determinant $|\det| : V^n \to \mathbb{R}$ is an element of the one dimensional space $\text{Dens}(V)$. We have a canonical isomorphism by bijecting between the element $\varphi_E \in \text{Dens}(V)$ such that $\varphi_E(e_1, \ldots, e_n) = 1$ for a basis $E = \{e_i\}_{i=1}^n$ of V, with the Haar measure μ_E normalized such that it has the value 1 on the parallelogram spanned by the vectors $\{e_i\}_{i=1}^n$.

This is independent of choice of basis since given a different basis both elements would be multiplied by the same factor of $|\det(M)|$, where M is the change of basis matrix between these two bases. Given a different basis $U = \{u_i\}$ of V with change of basis matrix M from E to U, we have that $\varphi_E = \lambda \varphi_B$ since $\text{Dens}(V)$ is a one dimensional space (for some scalar $\lambda \in \mathbb{R}$). Now, $\varphi_E(b_1, \ldots, b_n) = |\det(M)| \varphi_E(e_1, \ldots, e_n) = |\det(M)|$, and thus

$$|\det(M)| = \varphi_E(b_1, \ldots, b_n) = \lambda \varphi_B(b_1, \ldots, b_n) = \lambda.$$

We get the same situation for Haar measures as $\nu = \nu \mu_B(B) = \mu_E(B) = |\det(M)|$. □

Definition 6.7. *For a topological vector space V which is an ℓ-space we define the space of smooth measures as* $S(V, \text{Haar}(V)) \simeq S(V) \otimes \text{Haar}(V)$.

Remark 6.8. *Note that for such V as in the definition above we have that the smooth measures* $S(V, \text{Haar}(V))$ *are the compactly supported locally constant functions with values in* $\text{Haar}(V)$.

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Definition 7.1. *We say that a topological space X is paracompact if for every open cover $\{U_\alpha\}_{\alpha\in I}$ of X and point $x \in X$ there is a neighborhood $x \in V$ and a refinement $\{U_\beta\}_{\beta\in J}$ such that V intersects only finitely many sets of $\{U_\beta\}_{\beta\in J}$.*

Definition 7.2. *A topological manifold M is a topological space which is locally homeomorphic to $\mathbb{R}^n$, and is furthermore paracompact and Hausdorff.*

Exercise 7.3. *Find a space X which is locally homeomorphic to $\mathbb{R}^n$ at every point and is paracompact but is not Hausdorff.*

Proof. Consider the space obtained by gluing two copies of $\mathbb{R}$ along an open set, say $(0,\infty)$. At each point we can find a neighborhood small enough which is homeomorphic to $\mathbb{R}$. Note that this also works for the points 0_1 and 0_2 arising from the points zero in each copy of $\mathbb{R}$, since an open set around each can be taken only in one copy of $\mathbb{R}$ (recall that the quotient topology is defined to be the weakest such that the quotient map is continuous).

This space is also paracompact; take a small neighborhood with a compact closure, we can refine any open cover such that its interior meets only finitely many sets.

This space is not Hausdorff since the two points 0_1 and 0_2, cannot be separated by disjoint open sets, every two such sets must intersect in some interval as $(-\epsilon, 0_1) = (-\epsilon, 0_2)$ for every $\epsilon \in \mathbb{R}_{>0}$. $\square$

Definition 7.4. *An analytic F-manifold is a space M which is locally isomorphic to $\mathcal{O}_F^n$ together with a sheaf of functions*

$$An(U) = \{f : U \to F : \forall x \in U, \exists r > 0 \ s.t. \ f_{|B_r(x)}(y) = \sum_{\vec{k}\in\mathbb{N}^n} a_{\vec{k}}(x-y)^{\vec{k}}\},$$

where $B_r(x)$ is the ball of radius r around x, and $\vec{k}$ is a multi index, thus $(x-y)^{\vec{k}} = \prod_{i=0}^{n}(x_i - y_i)^{k_i}$.

Definition 7.5. *Let M be a smooth manifold or a p-adic analytic manifold. A real vector bundle over M is a tuple (E,p) where E is a topological space and $p : E \to M$ is a continuous surjection such that:*

1. *For every $x \in M$ we have that $p^{-1}(x) = V_x$ is a finite dimensional real vector space.*
2. *For every $x \in M$ there exists an open $x \in U$ and a trivialization $\varphi_U : V_x \times U \to p^{-1}(U)$ where φ_U is a homeomorphism and $p \circ \varphi_U(v,x) = x$ for all $v \in V_x$.*
3. *The maps $v \mapsto \varphi_U(v,x)$ are linear isomorphisms.*

If $E \simeq V \times M$ we say (E,p) is a trivial bundle over M.

Exercise 7.6. *Find two non-isomorphic bundles E and E', such that $E \oplus F \simeq E' \oplus F$ for a bundle F.*

Proof. Let $M = S^2$ be the 2-sphere and set $E = TS^2$ the tangent bundle of S^2, and $E' = S^2 \times \mathbb{R}^2$ the trivial two dimensional bundle over S^2. Furthermore, take $F = NS^2 \simeq S^2 \times \mathbb{R}$ to be the normal bundle of S^2.

Now, $E \not\simeq E'$, for example since E has no smooth non-vanishing section (this is the hairy ball theorem), and E' has such section (just by taking a constant section). It is not hard to see that $E' \oplus F = S^2 \times \mathbb{R}^3$ is the trivial 3-dimensional bundle over S^2. To see that $E \oplus F \simeq S^2 \times \mathbb{R}^3$, we use the following claim: an n-dimensional vector bundle $p : E \to M$ is isomorphic to the trivial bundle $\iff$ it has n sections $s_1, \ldots, s_n$ such that $s_1(m), \ldots, s_n(m)$ are linear independent for every $m \in M$ (Lemma 1.1 of Hatcher's Vector bundles and K-theory). To find these sections we take the 3 vector fields e_1, e_2 and e_3 on $\mathbb{R}^3$. Each is a section of $E \oplus F$, and they are linearly independent, so we are done (it is helpful to remark here that when we project them to the tangent bundle each vanishes at some point, but has a normal component so it does not vanish completely). $\square$

Exercise 7.7. *Given a manifold M, a vector bundle (E, p) with fiber of constant dimension m over it, and a functor $F : \mathrm{Vect}^m \to \mathrm{Vect}^n$, construct a vector bundle $(F(E), q)$ over M as discussed in class.*

Proof. (Sketch) First, take a cover $\{U_\alpha\}$ which is a local trivialization of E (that is, $p^{-1}(U_\alpha) \simeq V \times U_\alpha$). Define the total space $F(E)$ over each U_α by $F(V) \times U_\alpha$, where the surjection q will be projecting to M, and glue every two pieces $q^{-1}(U_\alpha)$ and $q^{-1}(U_\beta)$ by setting $(v, x) \sim (g_{\alpha,\beta}(v), x)$ for every $x \in U_\alpha \cap U_\beta$ and $v \in V$, where $g_{\alpha,\beta} = F(\varphi_{U_\beta}^{-1}\varphi_{U_\alpha})$ and $\varphi_{U_\alpha} : V \times U_\alpha \to E$ and φ_{U_β} is defined similarly.

Finally, note that for any two elements of the cover $g_{\alpha,\beta}^{-1} = g_{\beta,\alpha}$, and that since we started with a vector bundle this map is linear. In order for our construction to be well defined we need to show the cocycle condition , namely that $g_{\beta,\gamma}g_{\alpha,\beta} = g_{\alpha,\gamma}$ when restricted to triple overlaps. This holds since

$$g_{\beta,\gamma}g_{\alpha,\beta} = F(\varphi_{U_\gamma}^{-1}\varphi_{U_\beta})F(\varphi_{U_\beta}^{-1}\varphi_{U_\alpha}) = F(\varphi_{U_\gamma}^{-1}\varphi_{U_\alpha}) = g_{\alpha,\gamma}.$$

Note that if we want $F(E)$ to have a smooth structure we need to demand that F preserves smooth maps. $\square$

We can utilize Exercise 7.7 to define a density bundle over a real manifold:

Definition 7.8. *Let M be a smooth manifold, we can define its density bundle by $D_M = |\Omega^{top}(TM)|$, that is the density bundle of the tangent bundle. We will have a different construction for analytic manifolds.*

Exercise 7.9. *Let M be a smooth n-dimensional Riemannian manifold, that is a smooth real manifold with an inner product on tangent spaces*

$$<,>_p : T_pM \times T_pM \to \mathbb{R}$$

which varies smoothly. Construct explicitly a density over M, that is a smooth section of the density bundle over M. The density should respect coordinate changes, and be the standard density when M is a linear space with the standard inner product.

Proof. Consider the Gram matrix $G_{M,p}(e_1,\dots,e_n)_{i,j} =< e_i, e_j >_p= (E^T E)_{i,j}$ for $e_i \in T_pM$. This matrix is positive semidefinite, so we can define $vol_{M,p} : (T_pM)^n \to \mathbb{R}$ by

$$vol_{M,p}(e_1,\dots,e_n) = \sqrt{\det(G_{M,p}(\vec{e})}).$$

For every $p \in M$ we have that $vol_{M,p} \in |\Omega^{\text{top}}(T_pM)| \simeq \text{Dens}(T_pM)$ since given $A \in \text{GL}(T_pM)$,

$$vol_{M,p}(A\vec{e}) = \sqrt{\det(G_{M,p}(A\vec{e}))} = \sqrt{\det((AE)^T AE)} = |\det(A)| vol_{M,p}(\vec{e}).$$

This implies that vol_M is a section of D_M, and since we defined it using global constructions (i.e. the tangent bundle), it is independent of coordinates.

Now, to show that $vol_M \in C^\infty(M, D_M)$, for $p \in M$ take an open $p \in U \simeq \mathbb{R}^n$ such that U trivializes D_M. Since the inner product changes smoothly over M and $\det(G_{M,p}) \not\equiv 0$ for every $p \in M$, we have that $vol_{M|_U} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a smooth function as composition of smooth functions and we are done. $\square$

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8.1 Constant sheaves and locally constant sheaves

Definition 8.1. *Let V be a finite dimensional vector space and X a topological space.*

1. *We define the constant sheaf $\underline{V}_X$ to be the sheafification of the constant presheaf, which assigns to every open set in X the vector space V.*

2. *We say that a sheaf $\mathcal{F}$ over X is locally constant if for every $x \in X$ there exists an open $x \in U_x$ and a finite dimensional vector space V_x such that $\mathcal{F}_{|U_x} \simeq \underline{V_x}_{U_x}$.*

Example 8.2. *What is the sheafification of the following presheaves (V is a finite dimensional vector space):*

1. *The presheaf on two points $\{*, *\}$ assigning to each open subset the vector space V (with the restriction maps being identity).*

2. *The presheaf on two points $\{*, *\}$ assigning to each point the vector space V and to the whole space the vector space $V \oplus V \oplus V$ (with the restriction maps being projecting to the first and second coordinate).*

Exercise 8.3. *Let V be a finite dimensional vector space and X a topological space.*

1. *Show that $\underline{V}_X(U)$ is the space of locally constant functions from U to V.*

2. *Show that if X is a σ-compact ℓ-space then every locally constant sheaf $\mathcal{F}$ where $\mathcal{F}_x \simeq \mathcal{F}_y$ for all $x, y \in X$ is isomorphic to the constant sheaf.*

Proof.

1. First note that the space of locally constant functions from U to V, which here we denote by $\mathcal{C}_V(U)$, is a sheaf as we can glue locally constant functions, and they are determined by their values on a cover of U (it is a sheaf of functions).

   Now, note that there is a map of presheaves from the constant presheaf to $\mathcal{C}_V$ by for each U sending $v \in V$ to the constant function with value v on U, which is the identity map on the stalks. Finally, the universal property of the sheafification of $\underline{V}_X$ holds for $\mathcal{C}_V$ since given a sheaf $\mathcal{G}$ and map of presheaves $p : \underline{V}_X \to \mathcal{G}$ it factors through $\mathcal{C}_V$ by sending the constant function with value $v \in \mathcal{C}_V(U)$ to $p(v)$, thus $\mathcal{C}_V \simeq \underline{V}_X$.

2. Take a cover $\{U_\alpha\}$ of X such that $\mathcal{F}_{|U_\alpha} \simeq \underline{V}_{U_\alpha}$. Since X is σ-compact, we can write X as an ascending union of compacts, that is $X = \bigcup\limits_{n=1}^{\infty} K_n$ with $K_n \subset K_{n+1}$. Now, for each K_n we can refine the collection of sets of $\{U_\alpha\}$ which intersect K_n non-trivially to a finite cover $\{V_{n,\beta}\}$ of K_n whose elements are mutually disjoint. Continuing with this process we get $\{V_{n,\beta}\}_{n=0}^{\infty}$, a countable and mutually disjoint cover of X, where for each W in this collection $\mathcal{F}_{|W} \simeq \underline{V}_W$. Note that in this process each U_α is altered at most once. Gluing the sheaves back together, we must have that $\mathcal{F} \simeq \underline{V}_X$.

□

Remark 8.4. *Note that in the second part of the previous exercise we used the disjointness of the elements of the cover since otherwise we would need to ensure that the cocycle conditions hold on triple overlaps $V_i \cap V_j \cap V_k$. For a precise reference see [7, Exercise 2.7D].*

Remark 8.5. *To make the discussion less cumbersome, for the next two exercises we discuss sheaves of sets.*

Definition 8.6. *Recall that a Leray sheaf over a topological space X is a pair (E, π) of a topological space E and a local homeomorphism π (i.e. every point $x \in E$ has $x \in U$ such that $\pi : U \to \pi(U)$ is a homeomorphism and $\pi(U)$ is open).*

Definition 8.7. *Let X be an F-analytic manifold, we can define a density sheaf in two fashions. As a Leray sheaf by, $D_X = \{(x, v) : x \in X, v \in |\Omega^{top}(X)_x|\}$ (see lecture notes for the explicit basis for the topology), or as a sheaf in the Grothendieck style by setting (here μ is a Haar measure on U)*

$$D_x(U) = \{\nu \in meas(U) : \forall \varphi : \mathcal{O}_F^n \hookrightarrow U \exists f \in C^\infty(\mathcal{O}_F^n), \nu_{|\mathrm{im}\varphi} = \varphi_*(f\mu)\}.$$

Exercise 8.8. *Show that the definition of a Leray sheaf is equivalent to the Grothendieck definition of a sheaf.*

Proof. We will sketch an equivalence of categories between the Leray definition of a sheaf and the Grothendieck one. We start by defining functors in two directions.

Assume we are given a Leray sheaf, that is an espace etale with a projection (E, p) such that for every $e \in E$ there exists a neighborhood $e \in U_e \subseteq E$ such that $p_{|U_e}$ is a homeomorphism to its image. We define a presheaf by,

$$\mathcal{F}(U) = \{f : U \to p^{-1}(U) : f \text{ cts}, \, p \circ f = \mathrm{Id}_U\}$$

with the obvious restriction maps. This is actually a sheaf; because the sections are functions we can glue compatible sections, and the identity axiom holds.

For the other direction, given a Grothendieck sheaf, we form its espace etale by taking $E = \coprod_{x\in X} \{x\} \times \mathcal{F}_x$, with the projection being $p(x, v) = x$. We furthermore endow E with the topology generated by the basis $U_{s,V} = \{(x, (s)_x) : x \in V\}$ where $V \subseteq X$ is open and $s \in \mathcal{F}(V)$. For each $(x, v) = e \in E$ take a basic open set U_{s,V_x}, where $x \in V_x$ and $s \in \mathcal{F}(V_x)$ such that $(s)_x = v$, then p is a homeomorphism from U_{s,V_x} onto $V_x \subseteq X$. Note that p is also continuous since for an open $V \subseteq X$, we have that $p^{-1}(V) = \coprod_{x\in V} \{x\} \times \mathcal{F}_x = \bigcup_{W\subseteq V,\text{open}} \bigcup_{s\in\mathcal{F}(W)} U_{s,W}$, which is open.

We sketch the equivalence of categories. Starting with a Grothendieck sheaf $\mathcal{F}$, we construct a Leray sheaf $(E_{\mathcal{F}}, p)$ and obtain a Grothendieck sheaf $\mathcal{G}_{E_{\mathcal{F}}}$. For an open $U \subseteq X$ the sections are,

$$\mathcal{G}_{E_{\mathcal{F}}}(U) = \{f : U \to \coprod_{x\in U} \{x\} \times \mathcal{F}_x : f \text{ cts}, \; p \circ f = \mathrm{Id}_U\}.$$

Since the basis for the topology of $E_{\mathcal{F}}$ was sets $U_{s,V}$, where $V \subset X$ is open and $s \in \mathcal{F}(V)$, for each s we have a continuous function $f_s \in \mathcal{G}_{E_{\mathcal{F}}}(U)$ where $f_s(x) = (x, (s)_x)$, so $\mathcal{F}(U) \subseteq \mathcal{G}_{E_{\mathcal{F}}}(U)$. Conversely, if we have a continuous section $f : U \to \coprod_{x\in U} \{x\} \times \mathcal{F}_x$, for each $(x, (s)_x)$ in the image, we can consider its germ given by a representative (s, W_s), where $x \in W_s \subseteq U$ is open. By continuity of f, we must have an open $x \in V_s \subseteq f^{-1}(U_{s,W_s})$ with $f(V_s) = U_{s,V_s}$, that is $f(V_s) = \coprod_{x\in V_s} (x, (s)_x)$. Since we started with a sheaf $\mathcal{F}$, using the gluing axiom for $s_\alpha \in \mathcal{F}(V_{s_\alpha})$ there exists a section $s \in \mathcal{F}(U)$ corresponding to f (note that they agree on overlaps, since these are values of the function f). This shows that $\mathcal{G}_{E_{\mathcal{F}}} \simeq \mathcal{F}$.

For the other direction, assume we start with a Leray sheaf (E, p), and consider $E_{\mathcal{F}_E} = \coprod_{x\in X} \{x\} \times (\mathcal{F}_E)_x$, where here $(\mathcal{F}_E)_x = \{f : \{x\} \to p^{-1}(x)\}$ which can be identified with $p^{-1}(x)$. We construct a homeomorphism $\psi : E_{\mathcal{F}_E} \to E$ by $\psi(x, (f)_x) = f(x)$. This map is surjective since for every $e \in E$ we have that $\psi(p(e), f_e) = e$ where $f_e \in (\mathcal{F}_E)_{p(e)}$ and $f_e(p(e)) = e$. Injectivity is clear since if $f_1(x) = f_2(y)$, then $x = p(f_1(x)) = p(f_2(y)) = y$ and f_1, f_2 are functions from a singleton.

The map ψ is continuous since given an open $W \subseteq E$, consider $(x, (f)_x) \in \psi^{-1}(W) = \{(x, (f)_x) : f(x) \in W\}$. Since $(f)_x \in (\mathcal{F}_E)_x$, by considering its germ there exists an open subset $V \subseteq p(W)$ (note that $p(W)$ is open) such that $f \in \mathcal{F}_E(V)$, that is $f : V \to p^{-1}(V) \subseteq W$ and is continuous. By the definition of the topology on $E_{\mathcal{F}_E}$, we have that $(x, (f)_x) \in U_{f,V} \subseteq \psi^{-1}(W)$.

To show ψ is an open map, take a basic open $U_{f,W} \subseteq E_{\mathcal{F}_E}$, where $W \subseteq X$ is open and $f : W \to p^{-1}(W)$ is a continuous section. If $e \in \psi(U_{f,W}) = f(W)$, then we have an open $e \in U_e \subseteq E$ such that $p_{|U_e}$ is a homeomorphism. In particular, $p(U_e) \cap W$ is open and since $f_{|p(U_e)\cap W} : p(U_e) \cap W \to p^{-1}(p(U_e)) \cap p^{-1}(W)$ is a continuous section, it must also be a homeomorphism, thus $e \in f(p(U_e) \cap W) \subseteq f(W)$ is open. This finishes the proof. $\square$

Remark 8.9. *Note that if we apply the procedure depicted above on a presheaf we end up with a sheaf. This is exactly the sheafification functor from presheaves* *to sheaves.*

Exercise 8.10. *Show that covering spaces correspond to locally constant sheaves, and that a covering space is trivial exactly when it corresponds to a constant sheaf. Give an example for a locally constant sheaf arising from a covering space which is not constant.*

Proof. Assume we are given a covering space (E, p), view it as an espace etale and reconstruct the corresponding Grothendieck sheaf $\mathcal{F}_E$. If X is our base space, we have a cover $\{U_\alpha\}$ of X such that $p^{-1}(U_\alpha) \simeq U_\alpha \times D$ and D is discrete. Thus $\mathcal{F}_{E|U_\alpha}(V) = \{f : V \to V \times D : f \text{ cts}, p \circ f = \mathrm{Id}_U\}$, which are exactly the locally constant functions since D is discrete (meaning that such f is constant on each open $f^{-1}(V, d)$), implying that $\mathcal{F}_{E|U_\alpha}$ are constant sheaves.

Conversely, given a locally constant sheaf with stalk D (if it has several stalks, redo this process), assemble its espace etale $(E_\mathcal{F}, p)$. The espace etale of the constant sheaf $\underline{D}_U$ corresponds to $U \times D$ with the product topology, since the stalk at every point is D, and the open sets are $U_{s,V} = V \times \{s\}$. This implies that given a cover $\{U_\alpha\}$ of X such that $\mathcal{F}_{|U_\alpha}$ is isomorphic to the constant sheaf, we will have that $p^{-1}(U_\alpha) \simeq U \times D$, as this is the espace etale of the constant sheaf $\underline{D}_{U_\alpha}$, showing that $(E_\mathcal{F}, p)$ is a covering space.

It is useful to note that while for a locally constant sheaf the stalks at all points can be isomorphic to one another, if the sheaf is not isomorphic to a constant sheaf the topology of the espace etale will be different than the product topology, as we will see in the example.

Finally, note that we get that a non-trivial covering space cannot give rise to a constant sheaf by using the equivalence of categories from Exercise 8.8. Otherwise, apply the functors forth and back, and get that the covering space you started with was isomorphic to a product space, i.e. trivial.

As an example to a locally constant sheaf arising from a covering space which is not constant, consider the double cover of the circle by itself (draw the picture, see where it fails!). Locally, the sheaf obtained is the constant sheaf, but there are no global sections; Given a section $f : S^1 \to S^1$, note that $f(S^1)$ is not open in S^1, and thus f cannot be continuous. To make this precise, take $f^{-1}(U)$ where U is an open neighborhood of the boundary point of $f(S^1)$, and see it is not open. Thus it cannot be the constant sheaf. $\square$

Remark 8.11. *Note that Exercise 8.10 shows that the picture for a σ-compact ℓ-space is very different from the case of a locally connected space. For the first every locally constant sheaf (with isomorphic stalks) was constant by Exercise 8.3, and for the latter every non-trivial covering space gives rise to a locally constant sheaf which is not constant. In particular, ℓ-spaces have no non-trivial covering spaces.*

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Exercise 9.1. *Let X be a smooth or an F-analytic manifold. Show that $\overline{C_c^\infty(X)}^w = C^{-\infty}(X)$.*

Proof. Recall that $C^{-\infty}(X) = \mu_c^\infty(X)^*$. Given a topological vector space V, for $W \subseteq V^*$ the space W is dense w.r.t the weak topology if and only if $W^\perp = \{v \in V : \langle w, v\rangle = 0\ \forall w \in W\} = \{0\}$. To see the relevant direction, if $W^\perp = \{0\}$, we will show that for every $\xi \in V^*$, finite set $S \subset V$ and $\epsilon < 0$ we can find $w \in W$ such that $\xi_{|S} = w_{|S}$. Given such $\xi \in V^*$, $S = \{v_1, \dots, v_n\}$ and $\epsilon > 0$, assume S is a linearly independent set, and consider $\rho : V^* \to \mathbb{R}^n$ by $\rho(\eta) = (\langle \eta, v_1 \rangle, \dots, \langle \eta, v_n \rangle)$. The map $\rho_{|W}$ is onto, since otherwise there exists some $c_i \in \mathbb{R}$ such that $\sum_{i=1}^{n} c_i \langle w, v_i \rangle = 0$ for all $w \in W$ (it must lie in some hyperplane, and all hyperplanes are of this form), but this means that $\langle w, \sum_{i=1}^{n} c_i v_i \rangle = 0$ implying $\sum_{i=1}^{n} c_i v_i \in W^{\perp} = \{0\}$. The surjectivity of $\rho_{|W}$ allows us to find the desired $w \in W$. Thus it is enough to show that given $\eta \in \mu_c^{\infty}(X)$, if $\langle f, \eta \rangle = 0$ for all $f \in C_c^{\infty}(X)$ then $\eta = 0$.

Assume X is a smooth manifold. Given a non-zero measure η, there exists some $\mathbb{R}^n \simeq U \subset X$ such that $\eta_{|U} \neq 0$, to see this either use the fact that distributions form a sheaf, or view it as a positive function on Borel sets. Now, since $U \simeq \mathbb{R}^n$ we must have that $\eta_{|U} = g \cdot \mu_{Haar}$ where $g \in C^{\infty}(\mathbb{R}^n)$. Taking some cutoff function $\psi \in C_c^{\infty}(\mathbb{R}^n)$ such that $\psi_{|B_1(0)} \equiv 1$ and $\psi \geq 0$ implies the desired result as $\langle g\psi, \eta \rangle = \langle g\psi, g \cdot \mu_{Haar} \rangle = \langle g^2 \psi, \mu_{Haar} \rangle > 0$ as this is an integral of a positive function.

For an F-analytic manifold we do the same procedure only this time ψ is the indicator function of the open unit ball in F^n. $\square$

Definition 9.2. *Let M a smooth manifold and E a smooth real bundle over it. We define the topology on $C_c^{\infty}(M, E)$ by taking a cover $\{U_\alpha\}$ which locally trivializes both M and E, and announcing that a set is open if its preimage in $\bigoplus C_c^{\infty}(U_\alpha, E_{|U_\alpha})$ is open.*

Exercise 9.3. *Assuming the topology on $C_c^{\infty}(\mathbb{R}^n, \mathbb{R}^k)$ was already determined (recall it is the product topology on $C_c^{\infty}(\mathbb{R}^n)^k$), show that the topology on $C_c^{\infty}(M, E)$ is well defined.*

Proof. We need to show that given a different cover $\{V_\beta\}$ of M which locally trivializes M and E, we get the same topology.

Consider the cover $\{W_{\alpha,\beta}\}$ for $W_{\alpha,\beta} = U_\alpha \cap V_\beta$ which refines both covers. We need to show that for the addition map,

$$\bigoplus_{\alpha \in I} \bigoplus_{\beta \in J} C_c^{\infty}(W_{\alpha,\beta}, E_{|W_{\alpha,\beta}}) \xrightarrow{+} \bigoplus_{\alpha \in I} C_c^{\infty}(U_\alpha, E_{|U_\alpha})$$

a set in the range is open if and only if its preimage is open, where $W_{\alpha,\beta} \subseteq U_\alpha \simeq \mathbb{R}^n$ and $E_{|W_{\alpha,\beta}} \simeq E_{|U_\alpha} \simeq \mathbb{R}^k$. In order to show the above, it is enough to handle each case $\bigoplus_{\beta \in J} C_c^{\infty}(W_{\alpha,\beta}, \mathbb{R}^k) \xrightarrow{+} C_c^{\infty}(U_\alpha, \mathbb{R}^k) \simeq C_c^{\infty}(\mathbb{R}^n, \mathbb{R}^k)$ separately, since in the direct sum topology a set is open if all the injections $D_i \hookrightarrow \bigoplus D_i$ are continuous (for us - a convex set). Given a basic open set $U_{(L_m, \epsilon_m, B_m)} \subseteq C_c^{\infty}(U_\alpha, \mathbb{R}^k)$ where L_m are mixed differentiations, $\epsilon_m \in \mathbb{R}_{>0}$ and B_m are compact sets such that $\bigcup_{m=1}^{\infty} B_m = \mathbb{R}^n$, it is of the form $U_{(L_m, \epsilon_m, B_m)} = \sum_{m \in \mathbb{N}} V_{L_m, \epsilon_m, B_m}$, where

$$V_{L_m, \epsilon_m, B_m} = \left\{ f \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^k) : \operatorname{supp}(f) \subseteq B_m, \sup_{x \in \mathbb{R}^n} ||L_m(f)|| < \epsilon_m \right\}.$$

Now, take a finite sum $\sum f_\beta \in +^{-1}(U_{\{L_m,\epsilon_m,B_m\}})$ for $\sum f_\beta = f = \sum_{i=0}^{l} f_{m_i}$ and $f_{m_i} \in V_{L_{m_i},\epsilon_{m_i},B_{m_i}}$. Set by $f_\beta = pr_\beta(f)$ the projection of f into $C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k)$, and define $N = \#\{\beta : pr_\beta(f) \neq 0\}$ and $\epsilon'_{m_i} = \frac{\epsilon_{m_i} - \sup ||L_{m_i}(f_{m_i})||}{N}$ and set $\epsilon'_m = \frac{\epsilon_m}{N}$ if $m \neq m_i$ for all $0 \leq i \leq l$. For $B'_{m,\beta} \subseteq W_{\alpha,\beta}$, compact sets which exhaust $W_{\alpha,\beta}$ and such that $B'_{m,\beta} \subseteq B_m$, the sets $U_{(L_m,\epsilon'_m,B'_{m,\beta})}$ are basic open sets in each $C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k)$, and their direct sum is open in the direct sum. Now, we claim that,

$$f \in \bigoplus_{\beta : f_\beta \neq 0} f_\beta + U_{(L_m,\epsilon'_m,B'_{m,\beta})} \subseteq +^{-1}(U_{(L_m,\epsilon_m,B_m)}).$$

Given $g = \sum_{\beta : f_\beta \neq 0} g_\beta$ where $g_\beta \in U_{(L_m,\epsilon'_m,B'_{m,\beta})}$, then $g_\beta = \sum_{i_\beta=1}^{l_\beta} g_{\beta,i_\beta}$ where $g_{\beta,i_\beta} \in V_{L_{n_{i_\beta}},\epsilon'_{n_{i_\beta}},B'_{n_{i_\beta},\beta}}$.

Thus if $n_{i_\beta} = m_i$ for some i, we have $\sup_{x \in B_{m_i}} ||L_{m_i}(g_{\beta,m_i})|| < \epsilon'_{m_i} = \frac{\epsilon_{m_i} - \sup ||L_{m_i}(f)||}{N}$ implying that,

$$\begin{aligned}
\sup_{x \in B_{m_i}} \Big|\Big| \sum_{\beta : f_\beta \neq 0} L_{m_i}(f_{m_i,\beta} + g_{\beta,m_i}) \Big|\Big| &\leq \sup_{x \in B_{m_i}} \Big|\Big| \sum_{\beta : f_\beta \neq 0} L_{m_i}(f_{m_i,\beta}) \Big|\Big| + \sum_{\beta : f_\beta \neq 0} \sup_{x \in B_{m_i}} ||L_{m_i}(g_{\beta,m_i})|| \\
&< \sup_{x \in B_{m_i}} \Big|\Big| L_{m_i}(f_{m_i}) \Big|\Big| + \sum_{\beta : f_\beta \neq 0} \left(\frac{\epsilon_{m_i} - \sup ||L_{m_i}(f_{m_i})||}{N} \right) \\
&= \epsilon_{m_i}.
\end{aligned}$$

Otherwise, if $n_{i_\beta} \neq m_i$ for all i, set $n' = n_{i_\beta}$, and using the requirement $\sup_{x \in B_{n'}} ||L_{n'}(g_{\beta,n'})|| < \frac{\epsilon_{n'}}{N}$ we note that:

$$\sup_{x \in B_{n'}} \Big|\Big| \sum_{\beta : f_\beta \neq 0} L_{n'}(g_{\beta,n'}) \Big|\Big| \leq \sum_{\beta : f_\beta \neq 0} \sup_{x \in B_{n'}} ||L_{n'}(g_{\beta,n'})|| < N \frac{\epsilon_{n'}}{N} = \epsilon_{n'}.$$

This allows us to conclude that $f + g = \sum_{\beta : f_\beta \neq 0} \sum_{i=1}^{l} f_{m_i,\beta} + \sum_{\beta : f_\beta \neq 0} \sum_{i_\beta=1}^{l_\beta} g_{\beta,i_\beta}$ lie in $U_{(L_m,\epsilon_m,B_m)} = \sum_{m \in \mathbb{N}} V_{L_m,\epsilon_m,B_m}$ for all such functions g, implying that the addition is continuous. For a less cumbersome approach, note that the embeddings $\bigoplus_{\beta \in J} C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k) \to \bigoplus_{\beta \in J} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$ are continuous (a cookie for the person who finds a quick proof for this), so it is enough to show that the addition map $\bigoplus_{\beta \in J} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k) \xrightarrow{+} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k) \simeq \bigoplus_{i=k}^{k} C_c^\infty(\mathbb{R}^n)$ is continuous. Since the domain has the direct sum topology, it is enough to check this for a finite direct sum, which follows by the continuity of addition in a topological vector space.

To show the map is open, it is enough to consider $\bigoplus_{\beta \in J} C_{K,c}^\infty(W_{\alpha,\beta},\mathbb{R}^k) \xrightarrow{+} C_K^\infty(\mathbb{R}^n,\mathbb{R}^k) \simeq \bigoplus_{j=1}^{k} C_K^\infty(\mathbb{R}^n)$, for every compact K, and since the domain has the direct sum topology and the basic open sets are finite sums of open sets in each coordinate, it is enough to show it for a finite direct sum $\bigoplus_{i=1}^{m} C^\infty_{K,c}(W_i, \mathbb{R}^k) \xrightarrow{+} \bigoplus_{j=1}^{k} C^\infty_K(\mathbb{R}^n)$ where $K \subset \bigcup W_i$. Now, use partition of unity f_i, with $C_i = \operatorname{supp}(f_i) \subset W_i$ where $\sum_{i=1}^{m} f_{i|_K} \equiv 1$ to get an onto map via the composition,

$$\bigoplus_{i=1}^{m} C^\infty_{K\cap C_i}(W_i, \mathbb{R}^k) \hookrightarrow \bigoplus_{i=1}^{m} C^\infty_{K,c}(W_i, \mathbb{R}^k) \xrightarrow{+} C^\infty_K(\mathbb{R}^n).$$

Since this is a continuous surjective map of Fréchet spaces, it must be open, implying that the addition is open since the embedding is continuous. $\square$

9.1 Operations on distributions

Definition 9.4. *Let $\varphi : X \to Y$ be map of manifolds (either smooth or F-analytic) and set,*

$$\operatorname{Dist}(X)_{\mathrm{prop},\varphi} = \{\xi \in \operatorname{Dist}(X) : \varphi_{|_{\mathrm{supp}(\xi)}} \text{ is proper}\}.$$

1. *If φ is proper, define the pushforward $\varphi_*(\xi) \in \operatorname{Dist}(X)$ for $f \in C^\infty_c(Y)$ by*
$$\langle \varphi_*(\xi), f\rangle = \langle \xi, \varphi^*(f)\rangle.$$
2. *For $\xi \in \operatorname{Dist}(X)_{\mathrm{prop},\varphi}$ or $\xi \in \operatorname{Dist}_c(X)$ and $f \in C^\infty_c(Y)$ we define the pushforward by*
$$\langle \varphi_*(\xi), f\rangle = \langle \xi, \rho_f \cdot \varphi^*(f)\rangle$$
where $\rho_f \in C^\infty_c(X)$ is a cut off function such that $\rho_{f|_U} \equiv 1$ and U is an open set such that $\varphi^{-1}(\operatorname{supp}(f)) \cap \operatorname{supp}(\xi) \subset U$.

Exercise 9.5. *Show that the definition above for pushing forward distributions in $Dist(X)_{\mathrm{prop},\varphi}$ is well defined.*

Proof. First, note that $\rho_f \varphi^*(f)$ is compactly supported in X since ρ_f is (and $\operatorname{supp}(\varphi^*(f))$ is closed).

The definition is independent of ρ_f, since given a different cutoff function ψ_f, then $(\rho_f - \psi_f)_{|V} \equiv 0$, where $\operatorname{supp}(\xi\varphi^*(f)) \subset V$ is an open set, implying $(\rho_f - \psi_f) \in C^\infty_c(X\backslash \operatorname{supp}(\xi\varphi^*(f)))$ and thus $\langle \xi\varphi^*(f), \rho_f - \psi_f\rangle = 0 \Rightarrow \langle \xi, (\rho_f)\varphi^*(f)\rangle = \langle \xi, (\psi_f)\varphi^*(f)\rangle$.

Finally, note that we can indeed find such ρ_f and U. Since $\varphi_{|_{\mathrm{supp}(\xi)}}$ is proper, we have that $\operatorname{supp}(\xi) \cap \varphi^{-1}(\operatorname{supp}(f))$ is compact, and thus by regularity we can find U whose closure is compact which contains it. Alternatively, we can just take a smooth compactly supported partition of unity with respect to some cover of $\operatorname{supp}(\xi) \cap \varphi^{-1}(\operatorname{supp}(f))$. The proof for the F-analytic case is analogous. $\square$

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10.1 Pontryagin duality and Fourier transform

Let G be an abelian locally compact group. Define its Pontryagin dual by,

$$G^\vee = \{\chi : G \to U_1(\mathbb{C}) = S^1 \subset \mathbb{C} : \chi(g_1g_2) = \chi(g_1)\chi(g_2),\ \chi \text{ is cts}\}.$$

The topology on $G^\vee$ is the compact open topology, i.e. a sub-basis for the topology is comprised of sets $M(K,V) = \{\chi \in G^\vee : \chi(K) \subseteq V\}$ where $K \subseteq G$ is compact and $V \subseteq S^1$ is open.

Remark 10.1. *The category of LCA groups is not abelian but quasi-abelian. Every LCA subgroup of an LCA is group is closed.*

Theorem 10.2. *For a locally compact abelian group G, we have that $G^{\vee\vee} \simeq G$.*

Proof. This is Theorem 4.31 of A Course in Abstract Harmonic Analysis, Second Edition, Folland. □

Exercise 10.3. *Let G be a locally compact, Hausdorff abelian group, then $G^\vee$ is a locally compact Hausdorff abelian group.*

Proof. (See Mathstackexchange Q. 1502405) We see that characters form an abelian group. Also, the compact open topology on $G^\vee$ is equivalent to the topology of uniform convergence on compact sets; functions in $M(K,V)$ converge uniformly on K to 1 when taking $1 \in V$ to be smaller and smaller, and the other way follows since if $f_n \to 1$ uniformly on K then $f_n \in M(K,V_n)$ where $\mu(V_n) \to 0$ and $1 \in V_n$. This also follows since S^1 is a topological group and thus admits a uniform structure. Thus, in order to show that the multiplication and inverse are continuous it is enough to show that if $f_n \to f$ and $g_n \to g$ uniformly on compact sets then $f_n \cdot g_n^{-1} \to f \cdot g^{-1}$ uniformly on compact sets. Now, if $K \subset G$ is compact, note that this follows from the following bound $(\forall x \in K)$:

$$|f_n g_n^{-1} - f g^{-1}| \leq |f_n(g_n^{-1} - g^{-1})| + |(f_n - f)g^{-1}| = |g_n - g| + |f_n - f|.$$

Now to show it is locally compact, consider the space $(S^1)^G$ of all functions $f : G \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ with the product topology (i.e. a basis is given by open sets in only finitely many components). It is a compact space by Tychonoff's theorem, and it has the space

$$\tilde{G} = \bigcap_{g_1,g_2 \in G} \{\chi : G \to S^1 : \chi(g_1 g_2) = \chi(g_1)(g_2)\},$$

as a closed subspace, implying that $\tilde{G}$ is compact. Furthermore, for every $S \subseteq G$ and $\epsilon > 0$ the set $A(S,\epsilon) = \{\chi \in (S^1)^G : \chi(S) \subseteq [-\epsilon,\epsilon]\}$ is also closed and compact in $(S^1)^G$ as the complement is a union of sets of the form $\{\chi : G \to S^1 : \chi(s) \in [-\epsilon,\epsilon]^c\}$ for some $s \in S$, which are open.

In particular, taking an open neighborhood $e \in U \subset G$ the sets $V(U,\epsilon) = A(U,\epsilon) \cap \tilde{G}$ are closed and compact in $(S^1)^G$. Take $0 < \epsilon < \frac{1}{2}$, we show that we have that $V(U,\epsilon) \subseteq G^\vee$ are the compact neighborhoods we desire. Start with an open $e \in U_0 = U \subset G$, and choose a sequence of neighborhoods (U_n) containing e such that $U_{n+1} \cdot U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$ and set $\epsilon_n = \frac{\epsilon}{2^n}$. Taking $\chi \in V(U_n,\epsilon_n)$, we see that since for $x \in U_{n+1}$ we have that $\chi(x) \in [-\epsilon_n,\epsilon_n]$ and $x^2 \in U_n$ we get $\chi(x^2) = \chi(x)^2 \in [-\epsilon_n^2,\epsilon_n^2] \subseteq [-\frac{\epsilon_n}{2},\frac{\epsilon_n}{2}]$, implying that $V(U_n,\epsilon_n) \subseteq V(U_{n+1},\epsilon_{n+1})$.

Now, take $\chi \in V(U,\epsilon)$ and a basic open set $(-\delta,\delta) \subset S^1$ for $\delta > 0$. We have that $[-\epsilon_n,\epsilon_n] \subseteq (-\delta,\delta)$ for n big enough, implying that $e \in U_n \subseteq \chi^{-1}((-\delta,\delta))$ which means that χ is continuous at e. Since χ is a homomorphism, we can show it is continuous everywhere; if $\chi(g) \in W \subset S^1$ and W is open, we have that $(-\delta, \delta) \subseteq \chi(g^{-1})W$ for some $\delta > 0$ and that,

$$\begin{aligned}\chi^{-1}(\chi(g^{-1})W) &= \{y \in G : \chi(y) \in \chi^{-1}(g)W\} = \{y \in G : \chi(gy) \in W\} \\ &= g^{-1}\{gy \in G : \chi(gy) \in W\} = g^{-1}\chi^{-1}(W).\end{aligned}$$

Now, for some $m \in \mathbb{N}_0$ big enough, the following implies that $g \in gU_m \subseteq \chi^{-1}(W)$:

$$U_m \subset \chi^{-1}(\chi(g^{-1})W) = g^{-1}\chi^{-1}(W).$$

We know that $V(U, \epsilon)$ is compact in the product topology, and want to show it is compact with respect to the compact open topology. For this, it is enough to show that any net in $V(U, \epsilon)$ has a converging subnet in the compact open topology. Assume we are given some net $(x_\alpha) \in V(U, \epsilon)$, then it has a subnet $(f_\beta) \to f$ converging in the product topology with $f_\beta, f \in V(U, \epsilon)$. Now, note that $V(U, \epsilon)$ is uniformly equicontinuous, that is if $g_1, g_2 \in G$ and $g_1 g_2^{-1} \in U_n$ then for any $\chi \in V(U, \epsilon)$,

$$|\chi(g_1) - \chi(g_2)| = |\chi(g_1)\chi^{-1}(g_2) - 1| = |\chi(g_1 g_2^{-1}) - 1| \leq \epsilon_n.$$

Given a basic open neighborhood of the identity character $1_G \in M(K, B_{\epsilon'}(0))$, where K is compact, for every $g \in K$ we have that $g \in U_n g$. Now, taking any $g' \in U_n g$, we get that $gg'^{-1} \in U_n$ implying that for some big enough β we have that $|f(g) - f_\beta(g)| < \epsilon_n$ and that,

$$|f(g') - f_\beta(g')| \leq |f(g') - f(g)| + |f_\beta(g') - f_\beta(g)| + |f(g) - f_\beta(g)| < 3\epsilon_n.$$

Taking $n > n_g$ such that $\epsilon_n < \frac{\epsilon'}{3}$, we see that $f_\beta \to f$ uniformly on $U_n g$, but since K is compact we can cover it with finitely many sets of the form $U_{n_g} g$, and take $n = \max\limits_{0 \leq i \leq k}\{n_{g_i}\}$ and appropriate β.

To finish off the argument, note that by local compactness every $g \in G$ has a neighborhood $g \in U$ with compact closure $U \subset K$, and we have that $M(K, B_0(\epsilon)) \subseteq V(U, \epsilon)$ for an appropriate $\frac{1}{2} > \epsilon > 0$. $\square$

Exercise 10.4. *Let G be a locally compact, Hausdorff abelian group. Show that if G is compact then $G^\vee$ is discrete, and that if G is discrete then $G^\vee$ is compact.*

Proof. If G is compact, take $P = \{\theta : -\frac{1}{4} < \theta < \frac{1}{4}\} \subset \mathbb{R}/\mathbb{Z}$, and consider the open set $1_G \in M(G, P)$. Since the only subgroup in $P \subset S^1$ is $\{1\}$, we have that $M(G, P) = \{1_G\}$ is open, implying that every $\chi \in G^\vee$ is open and hence $G^\vee$ discrete.

If G is discrete, every character is continuous and since $G^\vee \subseteq \tilde{G}$, where $\tilde{G}$ is the space of all homomorphisms we have that $G^\vee = \tilde{G}$. Since $G^\vee = \tilde{G}$ is a closed space of the compact space $(S^1)^G$ it is compact in the product topology. Finally, the map $\mathrm{Id} : G^\vee_{prod} \to G^\vee_{c.o}$ is continuous since given a compact $K \subset G$, it must be a finite number of points implying that $M(K, V) = \bigcap_{i=0}^n M(\{s_i\}, V)$, which is open in the product topology, implying that its image is a compact set (in other words both topologies coincide). $\square$

Exercise 10.5. *Let G be a locally compact, Hausdorff abelian group, and $H \leq G$ a closed subgroup.*

1. *Show that Pontryagin duality is a contravariant endofunctor in the category of locally compact abelian groups.*

2. *Show that $H^\vee \simeq G^\vee/H^\perp$ where $H^\perp = \{\chi \in G^\vee : \chi(h) = 1\ \forall h \in H\}$.*

Proof. For the first item, we know by Exercise 10.3 that $G^\vee$ is a locally compact abelian group. Given a continuous homomorphism $\varphi : G \to G'$, it gives rise to a homomorphism $\varphi^\vee : G'^\vee \to G^\vee$ via precomposition, i.e. $\varphi^\vee(\chi)(g) = \chi \circ \varphi(g)$. Since $\mathrm{id_G}^\vee = \mathrm{id_{G^\vee}}$, and $(\varphi_1 \circ \varphi_2)^\vee = \varphi_2^\vee \circ \varphi_1^\vee$ for appropriate homomorphisms φ_1 and φ_2, it is left to show that it is continuous. This follows since for compact $K \subset G$ and open $U \subset S^1$ we have that,

$$\varphi^{\vee^{-1}}(M_G(K,U)) = \{\chi \in G'^\vee : \chi \circ \varphi(K) \subset U\} = M_{G'}(\varphi(K), U),$$

and since φ is continuous $\varphi(K)$ is compact and $M_{G'}(\varphi(K), U)$ is a basic open set in $G'^\vee$.

For the second item note that we have the inclusion $i : H \hookrightarrow G$ which gives rise to the surjective projection $p : G^\vee \to H^\vee$: Pontryagin duality is an anti-equivalence of the quasi-abelian category of LCA groups and sends exact sequences to exact sequences. To see this set K' to be the cokernel of the map p, i.e. $K' = H^\vee/p(G^\vee)$. If it is not zero, then $\varphi : (K')^\vee \to H$ is not the zero map, but $\varphi \circ i$ is the zero map. Since i was assumed to be injective φ must be trivial so p is surjective. Since $\ker(p) = \{\chi \in G^\vee : \chi(h) = 1\ \forall h \in H\} = H^\perp$, by Noether's isomorphism theorem we have that $H^\vee \simeq G^\vee/H^\perp$ as groups. To see this is indeed a homeomorphism, it is enough to show that p is an open map (the quotient map $q : G^\vee \to G^\vee/H^\perp$ is continuous and is also open as $q^{-1}(q(M_G(K,U))) = M_G(K,U)H^\perp$ - we need just continuity), but this holds since for a basic open set $M_G(K,U)$ we have that $p(M_G(K,U)) = (M_H(K \cap H, U))$, and $H \cap K$ is compact since H is closed. Note that for surjectivity of p we used knowledge about the functor $(\cdot)^\vee$, namely that it sends exact sequences to exact sequences. This gives rise to a version of the Hahn-Banach theorem for locally compact abelian groups. For a different way to prove this item (using Gelfand-Raikov theorem), see A Course in Abstract Harmonic Analysis, Second Edition, Folland, Thm 4.40 in the sources folder. □

Remark 10.6. *Note that we can identify $\mathbb{R}^n$ with $(\mathbb{R}^\vee)^n$ by $x(\xi) = e^{-i\xi\cdot x}$ where $\xi \in \mathbb{R}^n$ and $\xi \cdot x$ is the dot product. This identification is not canonical (it depends on the choice of an inner product).*

Definition 10.7. *Let V be a topological vector space over a local field. We define $\mathcal{G}(V) := S^*(V, \mathrm{Haar}(V))$.*

Definition 10.8. *We define the Fourier transform in several steps.*

1. *Firstly, for a vector space V (either Archimedean or non-Archimedean) define $\mathcal{F} : \mu_c(V) \to C(V^\vee)$ by $\mathcal{F}(\mu)(\chi) = \int \chi d\mu$ (Exercise - $\mathcal{F}(\mu)$ is continuous).*

2. *We note that for the subspace $\mu_c^\infty(V) \subset \mu_c(V)$ we have that $\mathcal{F}(\mu_c^\infty(V)) \subset S(V^\vee)$.*

3. *Since we also have that $\mu_c^\infty(V) \subset S(V, \mathrm{Haar}(V))$, and it is dense in $S(V, \mathrm{Haar}(V))$, we would like to define the Fourier transform on $S(V, \mathrm{Haar}(V))$ via continuity.*

4. *Finally, we define the Fourier transform $\mathcal{F} : S^*(V^\vee) \to \mathcal{G}(V) = S^*(V, \mathrm{Haar}(V))$ via duality.*

For the second and third steps we solve the following exercise.

Exercise 10.9. *Show that the Fourier transform $\mathcal{F} : S(V, \mathrm{Haar}(V)) \to S(V^\vee)$ is continuous for an Archimedean V and is indeed contained in $S(V^\vee)$.*

Proof. Assume V is a real vector space of dimension n, and recall that the topology on $S(V)$ is determined by the semi-norms $\|f\|_{\alpha,\beta} = \sup_{x\in V} |x^\alpha \frac{\partial^\beta f(x)}{\partial x^\beta}|$ where $\alpha, \beta \in \mathbb{N}_0^n$ and $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$. It is enough to show that for every semi-norm $\|\cdot\|_{\alpha,\beta}$ there exists a semi-norm $\|\cdot\|'$ and a constant C such that $\|\mathcal{F}(f)\|_{\alpha,\beta} \leq C\|f\|'$ for all $f \in C_c^\infty(V, \mathrm{Haar}(V))$. Now, recall that,

$$\frac{i\partial \mathcal{F}(f)}{\partial \xi_j} = \int_{\mathbb{R}^n} \frac{i\partial}{\partial \xi_j}(e^{-i\xi\cdot x} f(x))dx = \mathcal{F}(x_j f),$$

where one can differentiate directly using the definition to verify the above procedure. The other side of the coin is given by integration by parts,

$$\xi_j \mathcal{F}(f) = \int_{\mathbb{R}^n} \xi_j e^{-i\xi\cdot x} f(x)dx = \left[-e^{-i\xi\cdot x} f(x)\right]_{-\infty}^{\infty} - \int_{\mathbb{R}^n} \frac{\xi_j}{-i\xi_j} e^{-i\xi\cdot x} \frac{\partial f(x)}{\partial x_j} dx = \mathcal{F}(\frac{-i\partial(f)}{\partial x_j}).$$

Note that since the functions $e^{-i\xi\cdot x}$ converge weakly to zero as distributions as $|\xi| \to \infty$ the above shows that smooth, compactly supported measures are mapped into $S(V^\vee)$. We can now bound $\mathcal{F}(f)$ properly using the above relations:

$$\begin{aligned}
\|\mathcal{F}(f)\|_{\alpha,\beta} &= \sup_{x^\vee \in V^\vee} \left|(x^\vee)^\alpha \frac{(-i\partial)^\beta \mathcal{F}(f)(x^\vee)}{\partial (x^\vee)^\beta}\right| = \sup_{x^\vee \in V^\vee} \left| \int_V x^\vee \frac{(-i\partial)^\alpha((-x)^\beta f(x))}{\partial x^\alpha} d\mu(x) \right| \\
&\leq \sup_{x^\vee \in V^\vee} \int_V \left| x^\vee \frac{\partial^\alpha((-x)^\beta f(x))}{\partial x^\alpha}\right| d\mu(x) \leq C \sup_{x\in V} \left((1+|x|)^{n+1} \left| \frac{\partial^\alpha(|x|^\beta f)}{\partial x^\alpha} \right| \right).
\end{aligned}$$

where $C = \int_V \frac{1}{(1+|x|)^{n+1}} d\mu(x)$. Since the last expression is a linear combination of norms of the form $\|f\|_{\alpha',\beta'}$ for $|\alpha'| \leq |\alpha| + n + 1$ and $|\beta'| \leq |\beta|$, this implies that $\mathcal{F}$ is continuous. Note that we can also use this to show that $\mathcal{F}(f)$ is Schwartz, since if all the norms $\|\cdot\|_{\alpha,\beta}$ are bounded then the value of $|x^\alpha \frac{\partial^\beta f(x)}{\partial x^\beta}|$ decays to 0 as $|x| \to \infty$ for every α and β. $\square$

Exercise 10.10. *Let $L \subset V$ be a linear subspace of a finite dimensional space V. Show that $\mathcal{F}(\delta_L) = \delta_{L^\perp}$.*

Proof. Let $f \in C_c^\infty(V^\vee)$ and recall that $L^\perp = \{\chi \in V^\vee : \chi(l) = 1\ \forall l \in L\}$.

$$\begin{aligned}
\langle \mathcal{F}(\delta_L), f\rangle = \langle \delta_L, \mathcal{F}(f)\rangle &= \int_L \int_{V^\vee} f(\chi)\chi(l) d\chi dl \\
&= \int_L \int_{L^\vee} \int_{L^\perp} f(\lambda_1\lambda_2)(\lambda_1\lambda_2)(l) d\lambda_1 d\lambda_2 dl.
\end{aligned}$$

Since $\chi(l) = 1$ for every $\chi \in L^{\perp}$ and $l \in L$, setting $I_f(\lambda_2) = \int_{L^{\perp}} f(\lambda_1\lambda_2)d\lambda_1$ as a function on $L^{\vee}$ (and using $L \simeq L^{\vee\vee}$) we get the following:

$$\begin{aligned} &= \int_L \int_{L^{\vee}} l(\lambda_2) \int_{L^{\perp}} f(\lambda_1\lambda_2)d\lambda_1 d\lambda_2 dl = \int_L \int_{L^{\vee}} l(\lambda_2) I_f(\lambda_2) d\lambda_2 dl \\ &= \int_L \mathcal{F}(I_f)(l)dl = \mathcal{F} \circ \mathcal{F}(I_f)(0) = I_f(0), \end{aligned}$$

where $\mathcal{F} \circ \mathcal{F}(f)(x) = f(-x)$ by the Fourier inversion formula. Since $I_f(0) = \int_{L^{\perp}} f(\lambda_2)d\lambda_2 = \langle \delta_{L^{\perp}}, f\rangle$, we are done.

Note that under the identification of $V^{\vee}$ with V (through V^*), we can identify $L^{\perp}$ with the orthogonal complement of L in V. $\square$

Remark 10.11. *Note the above proof works for every locally compact abelian group G and closed subgroup L such that we can define Schwartz functions (and thus Schwartz distributions) on G.*

Remark 10.12. *Pushing forward a Schwartz measure along a submersion φ yields a Schwartz measure (otherwise we might get something non-smooth). If φ is a linear projection, this follows from Fubini's theorem.*

Exercise 10.13. *(Functoriality of Fourier transform) Let $W \subset V$ be vector spaces over a local field, denote the inclusion of W in V by i, and set $p : V^{\vee} \to W^{\vee}$ for its dual map, then the following diagrams commute:*

$$\begin{tikzcd} S(V) \arrow[r, "i^*"] & S(W) \\ S(V^{\vee}, \mathrm{Haar}(V^{\vee})) \arrow[u, "\mathcal{F}"] \arrow[r, "p_*"] & S(W^{\vee}, \mathrm{Haar}(W^{\vee})) \arrow[u, "\mathcal{F}"'] \end{tikzcd} \qquad \begin{tikzcd} S^*(V) \arrow[d, "\mathcal{F}"'] & S^*(W) \arrow[l, "i_*"'] \arrow[d, "\mathcal{F}"'] \\ \mathcal{G}(V^{\vee}) & \mathcal{G}(W^{\vee}). \arrow[l, "p^*"'] \end{tikzcd}$$

Note that this is possible since p is a linear submersion (surjective) so pushing Schwartz measures along it yields Schwartz measures.

Proof. We start by showing the right hand side diagram commutes. Since i_*, the Fourier transform and p^* are continuous with respect to the weak topology, it is enough to prove commutativity for a dense set in $S^*(W)$.

First take the delta function $\delta_0 \in S^*(W)$, it is a compactly supported measure, and it holds that $i_*(\delta_0) = \delta_0$. Furthermore, since $\mathcal{F} : S^*(V) \to \mathcal{G}(V^{\vee})$ is defined via duality we have that $\mathcal{F}(\delta_0) = 1$:

$$\langle \mathcal{F}(\delta_0), f\mu\rangle = \langle \delta_0, \mathcal{F}(f\mu)\rangle = \mathcal{F}(f\mu)(0_{V^{\vee\vee}}) = \int_{V^{\vee}} f d\mu = \langle 1, f\mu\rangle,$$

where the third equality is sensible since $\mathcal{F}(f\mu) \in S(V^{\vee\vee})$ and $0_{V^{\vee\vee}}(\chi) = 1$ for all $\chi \in V^{\vee}$. We can also show that $p^*(1) = 1$. Consider $\mathcal{G}(W^{\vee})$ as a subspace of $C^{-\infty}(W^{\vee})$, there the generalized Schwartz function 1 is a smooth function, and note that the following diagram, where the horizontal arrows are the inclusions is commutative:

$$\begin{tikzcd}
\mathcal{G}(W^\vee) \arrow[r, hook] \arrow[d, "p^*"'] & C^{-\infty}(W^\vee) \arrow[d, "p^*"'] & C^\infty(W^\vee) \arrow[l, hook'] \arrow[d, "p^*"'] \\
\mathcal{G}(V^\vee) \arrow[r, hook] & C^{-\infty}(V^\vee) & C^\infty(V^\vee). \arrow[l, hook']
\end{tikzcd}$$

Now, note that every measure $f\mu \in \mu_c^\infty(V^\vee)$ can be treated either as a functional on smooth functions (since it has compact support as a distribution), or as the parameter a generalized function takes values on. This is utilized in the third equality below to yield the required result:

$$\langle p_{\mathcal{G}}^*(1), f\mu\rangle = \langle p_{C^{-\infty}}^*(1), f\mu\rangle = \langle 1, p_*(f\mu)\rangle = \langle p_*(f\mu), 1\rangle = \langle f\mu, p_{C^\infty}^*(1)\rangle = \langle 1, f\mu\rangle.$$

Note that since p_* is a submersion pushing forward a compactly supported smooth measure along it yields a smooth compactly supported measure.

Since δ_w for any other $w \in W$ is just a translation of δ_0 by w, its Fourier transform is $\mathcal{F}(\delta_w)(\chi) = \chi(w)$, and i_* and p^* are invariant to translations, the diagram is commutative for delta distributions. The space of Delta distributions $\mathrm{span}_{\mathbb{R}}\{\delta_w\}_{w\in W}$ is dense w.r.t the weak topology since for every function f with $f(x_0) \neq 0$ we can take suitable $c \in \mathbb{R}$ such that $|\langle \xi - c\delta_x, f\rangle|$ is small as desired.

To see this implies the commutativity of the left diagram, it is enough to show that if $A^* = 0$ for $A^* : V_2^* \to V_1^*$ where A^* is the dual map to the linear map $A : V_1 \to V_2$, then $A = 0$, and use this for $\mathcal{F}i_* - p^*\mathcal{F}$. If $A^* = 0$, we have for every $\xi_2 \in V_2^*$ and $v_1 \in V_1$ that $0 = \langle A^*\xi_2, v_1\rangle = \langle \xi_2, Av_1\rangle$. If there exists $v_1 \in V_1$ such that $Av_1 \neq 0$, then we can define a non-zero linear functional $\xi : \mathrm{span}_{\mathbb{R}}\{Av_1\} \to \mathbb{R}$ via $\langle \xi, Av_1\rangle = 1$, and extend it to a non-zero continuous functional $\xi_2 \in V_2^*$ by the Hahn-Banach theorem. This yields a contradiction as

$$1 = \langle \xi_2, Av_1\rangle = \langle A^*\xi_2, v_1\rangle = \langle 0, v_1\rangle = 0.$$

□

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11.1 Wave front set

Motivated by the definition of Fourier transform for Schwartz distributions and hence for compactly supported distributions, we move to study smoothness of distributions. Roughly speaking, our intuition will be the following (following Hormander's philosophy in [2, Chapter VIII, Page 251]). A compactly supported distribution ξ is said to be smooth if it is a smooth function. Morally speaking the Fourier transform interchanges between smoothness and rapid decay, and we can show that a compactly supported distribution is a smooth function if its Fourier transform is a rapidly decaying function. If the Fourier transform of ξ is not a rapidly decreasing function, we can use the directions in $V^\vee \simeq V^* \otimes F^\vee \simeq V^*$ (choose a character $\varphi : F \to S^1$) in which it does not rapidly decay to get information on the lack of smoothness of ξ. The data about the lack of smoothness of ξ will be encoded in the wave front set of ξ.

Definition 11.1. *Let X be an F-analytic manifold, W an F-vector space, F a non-Archimedean local field and $pr_X : X \times W \to X$ the standard projection. We set,*

$$S^W(X \times W) = \{f \in C^\infty(X \times W) : pr_{X|_{\operatorname{supp} f}} : \operatorname{supp}(f) \to X \text{ is proper}\}.$$

Definition 11.2. *Let X be a smooth manifold, W an F-topological vector space and F an Archimedean local field. We set $w^m = \prod_{i=1}^{\dim W} w_i^{m_i}$ and define $S^W(X \times W)$ to be:*

$$\{f \in C^\infty(X \times W) : \forall K \subset X, \forall m, n \in \mathbb{N}^{\dim W}, D \in \operatorname{Diff}(X), \|Df\|_{m,n,K} < \infty\},$$

where $\|Df(x,w)\|_{m,n,K} = \sup_{(x,w)\in K\times W} \left| D\frac{\partial^n f(x,w)}{\partial w_n} w^m \right|$.

Definition 11.3. *Let $f \in C^\infty(V)$ and $v \in V$ where V is a topological vector space over a local field F. We say that f vanishes asymptotically along v if $\exists U \ni v$ open neighborhood and $\rho \in C_c^\infty(U)$ where $\rho(v) \neq 0$ such that $p^*(\rho)a^*f \in S(U \times F)$ where $a : U \times F \to V$ via $(u, \lambda) \mapsto \lambda u$ and $p : U \times F \to U$ is the projection. One can interpret this as f being Schwartz in (a conical neighborhood of a) direction v.*

Example 11.4. *The function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = e^{-x^2}$ vanishes asymptotically for every $v \in \mathbb{R}^2$ not on the line $\{x = 0\}$.*

Proof. Take any $v = (a, b) \in \mathbb{R}^2$ such that $x \neq 0$. We can find an open ball or radius r, denoted B around v, small enough such that it doesn't intersect $\{x = 0\}$, we show that $p^*(\rho)a^*f \in S(B \times \mathbb{R})$ where $\rho \in C_c^\infty(B)$ and $\rho(v) \neq 0$.

Now, we have that $F(x,y,\lambda) = p^*(\rho)(x,y,\lambda)a^*f(x,y,\lambda) = \rho(x,y)e^{-\lambda^2x^2}$, we show it is indeed in $S(B \times \mathbb{R})$. It is enough to show that $F \in S(\mathbb{R}^3)$ since F is flat outside $B \times \mathbb{R}$ (i.e. its derivatives of any order are zero outside $B \times \mathbb{R}$). Now, set $K = \operatorname{supp}(\rho)$, for every α and β we have,

$$\|F(x,y,\lambda)\|_{\alpha,\beta} = \sup_{(x,y,\lambda)\in\mathbb{R}^3} \left|x^{\beta_1}y^{\beta_2}\lambda^{\beta_3}D_\alpha(F)\right| = \sup_{(x,y,\lambda)\in K\times\mathbb{R}} \left|x^{\beta_1}y^{\beta_2}\lambda^{\beta_3}D_\alpha(\rho(x,y)e^{-\lambda^2x^2})\right|,$$

where $D_\alpha = \frac{\partial^3}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial\lambda^{\alpha_3}}$ is a differential operator. Since this is bounded (tends to zero) when $\lambda \to \infty$, and K is compact, the norm is finite.

If $x = 0$, then for any neighborhood $v \in B$ and $K \subset B$ we can take $\alpha = 0$ and $\beta = (0,0,1)$ and get for every $y \in \mathbb{R}$ that $\lim_{\lambda\to\infty} |\lambda F(0,y,\lambda)| = |\lambda e^0| = \infty$, implying that f doesn't vanish asymptotically along $(0, y)$ for all $y \in \mathbb{R}$. □

Exercise 11.5. *Show that if $f \in C^\infty(V)$ vanishes asymptotically along 0 then $f \equiv 0$.*

Proof. If f vanishes asymptotically along 0 then there exists an open $0 \in U$ and $\rho \in C_c^\infty(U)$ such that $\rho(0) \neq 0$ and $F(v,\lambda) = p^*(\rho)a^*f \in S(U \times F)$.

Now, since F is Schwartz, if V is non-Archimedean it is compactly supported and thus $F(\frac{v}{\lambda}, \lambda) = f(v)\rho(\frac{v}{\lambda}) = 0$ for λ large enough, implying $f(v) = 0$ for all v ($\rho \neq 0$ in some neighborhood of 0 since it is locally constant). If V is Archimedean then $\rho(0)f(v) = \lim_{\lambda\to\infty} F(\frac{v}{\lambda}, \lambda) = 0$ and since $\rho(0) \neq 0$ we get $f(v) = 0$ as desired. □

Definition 11.6. *Let V be a vector space over a local field and $\xi \in C^{-\infty}(V)$. We say that ξ is smooth at $(x,w) \in V \times V^*$ if $\exists \rho \in C_c^\infty(V)$ with $\rho(x) \neq 0$ such that $\mathcal{F}(\rho\xi) \in C^\infty(V^\vee) \simeq C^\infty(V^*)$ vanishes asymptotically along w.*

Definition 11.7. *Let V be a vector space over a local field and $\xi \in C^{-\infty}(V)$, we define the wave front set of ξ by:*

$$WF(\xi) = \{(x,w) \in V \times V^* : \xi \text{ is not smooth at } (x,w)\}.$$

Remark 11.8. *For a manifold M one defines $WF(\xi) \subset T^*M$ analogously, where now a distribution is smooth at (x,w) if it is smooth locally, i.e. with respect to a chart $x \in U \subset M$.*

Example 11.9. *Compute the wave front set of the Dirac delta function $\delta \in Dist(\mathbb{R}^n)$.*

Proof. Since $\mathrm{supp}(\delta) = \{0\}$, we have that $WF(\delta) \subseteq \{0\} \times V^*$. This holds since for every $x \notin \mathrm{supp}(\delta)$, we can take a bump function $\rho \in C_c^\infty(\mathbb{R}^n)$ which is non-zero at x and zero at 0, and since $\rho\delta = 0$, its Fourier transform is smooth in every direction $w \in (\mathbb{R}^n)^\vee \simeq \mathbb{R}^n$.

For every $w \in V^*$ we get that $(0,w) \in WF(\delta)$ since if we take $\rho(0) = 1$ then $\mathcal{F}(\rho\delta) = \mathcal{F}(\delta) = 1$ which doesn't vanish asymptotically in any direction. □

Definition 11.10 (Pushforward and pullback of wavefront sets)**.** *Let $\varphi : M \to N$ be a suitable map between manifolds (i.e. smooth or F-analytic). It then induces a map $\varphi : TM \to TN$ on tangent bundles via*

$$d\varphi(m,v) = (\varphi(m), d_m\varphi(v)).$$

Since we cannot just get a map on cotangent bundles by duality (φ might not be a bijection), we define the following set,

$$\Delta_\varphi = \{(m,l),(n,t) \in T^*N \times T^*M : \varphi(m) = n \text{ and } d\varphi^*(l) = l \circ d_m\varphi = t\}.$$

We then define the pullback of $X \subseteq T^*M$ to be

$$\varphi^*(X) = \{(n,l) \in T^*N : ((n,l)(m,t)) \in \Delta_\varphi \text{ for some } (m,t) \in X\} \subseteq T^*N$$

and the pushforward of $Y \subseteq T^*N$ to be

$$\varphi_*(Y) = \{(m,t) \in T^*M : ((n,l)(m,t)) \in \Delta_\varphi \text{ for some } (n,l) \in Y\} \subseteq T^*M.$$

Corollary 11.11. *Let $\varphi : M \to N$ be a map between manifolds, $\xi_M \in C^{-\infty}(M)$ and $\xi_N \in C^{-\infty}(N)$, then the following holds:*

1. *If φ is a submersion, then $WF(\varphi^*(\xi_N)) = \varphi^*(WF(\xi_N))$.*
2. *If $\xi_M \in C^{-\infty}_{\mathrm{prop},\varphi}(M)$, then $WF(\varphi_*(\xi_M)) \subseteq \varphi_*(WF(\xi_M))$.*

Exercise 11.12. *Show that if $\xi \in C_c^{-\infty}(V)$ is smooth at (v,l) for a given $l \in V^*$ and all $v \in \mathrm{supp}(\xi)$ then $l_*(\xi) \in C^{-\infty}(\mathbb{R})$ is a smooth function with compact support (or in words - if a compactly supported generalized function is smooth at all vectors for a given codirection then it is a smooth function in this codirection).*

Proof. Note that since ξ is compactly supported, it is l-proper, $l_*(\xi)$ is compactly supported and we can use (2) of the previous corollary. Explicitly, we have that $WF(l_*(\xi)) \subseteq l_*(WF(\xi))$, we show that $l_*(WF(\xi)) = l(\text{supp}(\xi)) \times \{0\}$.

$l_*(WF(\xi))$ consists of all elements $(x, y) \in \mathbb{R} \times \mathbb{R}^*$ such that there exists $(v, w) \in WF(\xi)$ where $l(v) = x$ and $(d_v l)^*(y) = w$ (draw the picture). Since l is a linear functional, $(d_v l)^*(y) = y \circ l \in V^*$, but since $y \in \mathbb{R}^*$, the functional $y \circ l$ is just given by λl for some $\lambda \in \mathbb{R}$, so $(d_v l)^*(y) = w = \lambda l$. Since $WF(\xi)$ is conical, and ξ is smooth at (v, l), we get $(v, l) \notin WF(\xi)$ and hence $(v, \lambda l) \notin WF(\xi)$ for all $\lambda \in \mathbb{R}$.

This implies that there does not exist $(v, w) \in WF(\xi)$ such that $(d_v l)^*(w) = y$ for all $y \in \mathbb{R}^*$ and in particular $l_*(WF(\xi)) = l(\text{supp}(\xi)) \times \{0\}$. Since smoothness is a local property, there should be an analogous statement for manifolds (think about the generalization). □

Exercise 11.13. *Let $L \subseteq V$ be vector spaces over a local field F and $N \subset M$ F-manifolds (either smooth if F is Archimedean or F-analytic if F is non-Archimedean).*

1. *Compute $WF(i_*(\mu))$ where $\mu \in \text{Haar}(L)$ is a Haar measure on $i : L \hookrightarrow V$.*
2. *Compute $WF(i_*(\eta))$ where η is a smooth section of the density bundle of N and $i : N \hookrightarrow M$ is the embedding of N into M.*

Proof. 1. Consider the submersive map $p : V \to V/L$. We have that $p^* WF(\delta_0) = WF(p^*\delta_0) = WF(\delta_L)$. Since $WF(\delta_0) = \{0\} \times (V/L)^* = \{0\} \times L^\perp$, and $p^*(\{0\} \times L^\perp) = L \times L^\perp$, we get that $WF(i_*(\mu)) = L \times L^\perp$.

Different, longer approach: Since i is linear, we can use Corollary 11.11(2), we thus know that $WF(i_*(\mu)) \subseteq i_*(WF(\mu))$. Since $WF(\mu)$ is smooth on L, we have that $i_*(WF(\mu)) = i_*(L \times \{0\})$, which is exactly all those (v, w) such that $v \in \text{Im}(i) = L$ and $(di)_v^*(w) = w \circ i = 0$, meaning that

$$i_*(WF(\mu)) = \{(v, w) \in V \oplus V^* : v \in L, \langle w, x \rangle = 0 \; \forall x \in L\} = \text{CN}_L^V.$$

We claim that $WF(i_*(\mu)) = \text{CN}_N^M$, this amounts to showing that $i_*(\mu)$ is not smooth at (x, w) for all $(x, w) \in L \times L^\perp$. Take $(x, w) \in L \times L^\perp$ and $\rho \in C_c^\infty(V)$ such that $\rho(x) \neq 0$, using Exercise 10.13 (p is a submersion) we see that

$$\mathcal{F}(\rho i_*(\mu)) = \mathcal{F}(i_*(\rho_{|L}\mu)) = p^*\mathcal{F}(\rho_{|L}\mu) = p^*(\mathcal{F}(\rho_{|L}) * \mathcal{F}(\mu)) = p^*(\mathcal{F}(\rho_{|L}) * \delta_0) = p^*(\mathcal{F}(\rho_{|L})).$$

Since $\rho_{|L}$ is smooth and compactly supported, its Fourier transform is a Schwartz function, and $p^*(\mathcal{F}(\rho_{|L})) = \mathcal{F}(\rho_{|L}) \circ p$, which is constant on the $L^\perp$ axis in $V \simeq L \times L^\perp$. In particular, $a^*\mathcal{F}(\rho_{|L}) \circ p \notin S^F(U \times F)$ for all neighborhoods U of w.

2. Assume M and N are smooth manifolds which are already embedded into some $\mathbb{R}^N$ (this can be done by Whitney's theorem). Let $x \in N$ and take a neighborhood $x \in U_x \simeq \mathbb{R}^m$ in M and a diffeomorphism $\varphi : U_x \to \mathbb{R}^m$ such that $\varphi(U_x \cap N) \simeq \mathbb{R}^n \subset \mathbb{R}^m$ (possibly taking a smaller neighborhood U_x). Here, we arrive at the same situation as in 1. as a smooth section of the density bundle is locally just a smooth function multiplied by a Haar measure, and we know that $WF(\varphi_* i_*(\eta)) = \varphi(U_x) \cap \text{supp}(\varphi_* i_*(\eta)) \times (\mathbb{R}^n)^\perp \subseteq \text{CN}_{\mathbb{R}^n}^{\mathbb{R}^m}$. Now, by Hormander's theorem the wave front set is invariant to diffeomorphisms, i.e $\varphi_* WF(i_*(\eta)) = WF(\varphi_* i_*(\eta))$, and since smoothness is a local property, we get that $WF(i_*(\eta)) = \{(x, l) \in \text{CN}_N^M : x \in \text{supp}(i_*(\eta))\}$. □

Theorem 11.14 (Proof taken from [5, Theorem 25.4]). *Every tempered distribution on $\mathbb{R}^n$ is of the form*

$$T = \sum_{i=1}^{n} \frac{\partial^{k_i}}{\partial x^{k_i}}(f_i)$$

where f_i is a continuous function of polynomial growth (i.e. grows at most polynomially at infinity) and k_i are multi-indices.

Proof. A distribution T is tempered if and only if it is continuous as a functional on $C_c^\infty(\mathbb{R}^n)$ with respect to the topology induced from $\mathcal{S}(\mathbb{R}^n)$ on $C_c^\infty(\mathbb{R}^n)$, as the latter space is dense in the former. Since every T of the form as above is bounded with respect to some norm $\|\cdot\|_{\alpha,\beta}$, it is a tempered distribution.

Given a tempered distribution, since it is continuous there exist integers $m, h \geq 0$ and $C \in \mathbb{R}_{>0}$ such that for every test function $f \in C_c^\infty(\mathbb{R}^n)$ we have,

$$|\langle T, f\rangle| \leq C \sup_{|p|\leq m} \sup_{x\in\mathbb{R}^n} \left|(1+|x|^2)^h \frac{\partial^p f}{\partial x^p}(x)\right|,$$

where p is a multi-index and $|x|$ is the Euclidean norm on $\mathbb{R}^n$. Set $f_h(x) = (1+|x|^2)^h f(x)$. This is a test function, and the map $f \mapsto f_h$ is linear and bijective from $C_c^\infty(\mathbb{R}^n)$ to itself. Also, we can show that for every $h \in \mathbb{N}$ we have that,

$$\left|\frac{\partial^p f}{\partial x^p}(x)\right| \leq C_{p,h}(1+|x|^2)^{-h} \sum_{q\leq p} \left|\frac{\partial^q f_h}{\partial x^q}(x)\right|,$$

where $C_{p,h}$ depends only on p and h, and $q \leq p$ if $q_i \leq p_i$ for all i. This is evident for $h = 0$. For general $k+1$, by induction,

$$\begin{aligned}
\left|\frac{\partial^p f_{-k-1}}{\partial x^p}(x)\right| (1+|x|^2)^{k+1} &\leq (1+|x|^2) C_{p,k} \sum_{q\leq p} \left|\frac{\partial^q[(1+|x|^2)^{-1} f]}{\partial x^q}(x)\right| \\
&\leq C_{p,k+1} \sum_{q\leq p} \left|\frac{\partial^q f}{\partial x^q}(x)\right|,
\end{aligned}$$

where the last statement follows by induction on p (for $p = 1$):

$$\left|\frac{\partial f_{-1}}{\partial x_i}\right|(1+|x|^2) = \left|f_i' - \frac{(1+|x|^2)_i' f}{1+|x|^2}\right| \leq \left|\frac{\partial f}{\partial x_i}(x)\right| + |f(x)| \left|\frac{(1+|x|^2)_i'}{1+|x|^2}\right| \leq C\left(\left|\frac{\partial f}{\partial x_i}(x)\right| + |f(x)|\right).$$

We get that

$$\langle T, f\rangle \leq C' \sup_{|p|\leq m} \sup_{x\in\mathbb{R}^n} \left|\frac{\partial^p f_h}{\partial x^p}(x)\right|.$$

Now, note that since we have $f(y) = \int_{-\infty}^{y_n} \dots \int_{-\infty}^{y_1} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} f dx_1 \dots dx_n$, we deduce,

$$\sup_{y\in\mathbb{R}^n} |f(y)| \leq \sup_{y\in\mathbb{R}^n} \int_{-\infty}^{y_n} \dots \int_{-\infty}^{y_1} \left|\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} f\right| dx_1 \dots dx_n \leq \left\|\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} f\right\|_{L^1}.$$

We therefore obtain,

$$|\langle T, f\rangle| \leq C'' \sup_{|p|\leq m+n} \left\| \frac{\partial^p f_h}{\partial x^p} \right\|_{L^1}.$$

Let N be the number of n-tuples p such that $|p| \leq m+n$, and consider the space $(L^1(\mathbb{R}^n))^N$ with the injection $J(f) = (\frac{\partial^p f}{\partial x^p})_{|p|\leq m+n}$ of $C_c^\infty(\mathbb{R}^n)$ into $(L^1(\mathbb{R}^n))^N$. Note that the product $(L^1(\mathbb{R}^n))^N$ is a Banach space with respect to the norm $\sup\limits_{|p|\leq m+n} \left\| \frac{\partial^p f_h}{\partial x^p} \right\|_{L^1}$ as above (every such functional is absolutely continuous w.r.t the Haar measure, and thus is given by a measurable function which can then be shown to be bounded). The previous estimate shows that the functional

$$\langle \xi, J(f_h)\rangle = \langle T, f\rangle$$

is continuous on $J(C_c^\infty(\mathbb{R}^n))$ with respect to the topology induced from $(L^1(\mathbb{R}^n))^N$. By the Hahn-Banach theorem, we can thus extend ξ to a functional on $(L^1(\mathbb{R}^n))^N$. The dual of $(L^1(\mathbb{R}^n))^N$ is $(L^\infty(\mathbb{R}^n))^N$ (i.e. bounded functions), so there are N bounded functions $\{h_p\}_{|p|\leq m+n}$ such that

$$\langle T, f\rangle = \sum_{|p|\leq m+n} \langle h_p, \frac{\partial^p f_h}{\partial x^p}\rangle.$$

This means that $T = \sum\limits_{|p|\leq m+n} (1+|x|^2)^h (-1)^{|p|} \frac{\partial^p h_p}{\partial x^p}$. For each p set

$$g_p(x) = \int_0^{x_1} \dots \int_0^{x_n} h_p(t_1, \dots, t_n) dt_1 \dots dt_n.$$

Since each h_p is bounded, we get that each g_p is continuous, and that

$$|g_p(x)| \leq \int_0^{x_1} \dots \int_0^{x_n} \|h_p(t_1, \dots, t_n)\|_{L^\infty} dt_1 \dots dt_n \leq |x_1| \dots |x_n| \|h_p\|_{L^\infty}$$

(with the norm being the essential supremum norm). Now,

$$h_p = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} g_p,$$

and we get $T = \sum\limits_{|p|\leq m+n} (1+|x|^2)^h \frac{\partial^p}{\partial x^p} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} (\tilde{g}_p)$ where $\tilde{g}_p = (-1)^{|p|} g_p$. Since

$$(1+|x|^2)^h \frac{\partial^p}{\partial x^p} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} (\tilde{g}_p) = \sum_{q\leq p} \frac{\partial^q}{\partial x^q} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} [Q_{p,q,h} \tilde{g}_p]$$

where $Q_{p,q,h}$ are polynomials which depend only on p, h and q, we are done. $\square$

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