Approximating the Representation Zeta Function of Finite Groups of Lie-Type

Advanced School on Representations of Pro-p Groups
ICMAT, Madrid

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Local set-up

- Let $\mathfrak{o} \supseteq \mathbb{Z}_p$ be a complete dvr with maximal ideal $\mathfrak{p}$ and fraction field F.
- Let $\underline{\mathbf{G}} \subseteq \mathrm{GL}_N(F)$ be a semisimple $\mathfrak{o}$-defined algebraic group with Lie algebra $\underline{\mathfrak{g}}$ (e.g. $\underline{\mathbf{G}} = \mathrm{SL}_n(\mathbb{Q}_p)$ and $\underline{\mathfrak{g}} = \mathfrak{sl}_n(\mathbb{Q}_p)$).
- Put $G = \underline{\mathbf{G}}(\mathfrak{o}) = \underline{\mathbf{G}} \cap \mathrm{GL}_N(\mathfrak{o})$ and $G^k = \mathrm{Ker}(G \to \underline{\mathbf{G}}(\mathfrak{o}/\mathfrak{p}^k)) = G \cap 1 + \mathfrak{p}^k \, \mathrm{M}_n(\mathfrak{o})$.
  Analogously, write $\mathfrak{g} = \underline{\mathfrak{g}}(\mathfrak{o}) = \underline{\mathfrak{g}} \cap \mathrm{M}_N(\mathfrak{o})$ and $\mathfrak{g}^k = \mathfrak{p}^k \mathfrak{g}$.

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Assumption

We assume throughout that $\mathrm{char}(\mathfrak{o}/\mathfrak{p})$ is large enough so G^1 is amenable to the Kirillov Orbit Method. In particular, the series $\exp(x) = 1 + x + \frac{x^2}{2} + \cdots$ defines a bijection $\mathfrak{g}^1 \xrightarrow{1-1} G^1$.

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Motivation

"The anomaly of the zeroth principal congruence subgroup"

Observation (cf [AKOV 13, Proposition 3.7])

Write $q = |\mathfrak{o}/\mathfrak{p}|$ and $d = \dim \underline{\mathbf{G}}$. The sequence $\left(q^{-k\cdot d}\zeta_{G^k}\right)_{k\geq 1}$ is a constant sequence of functions, i.e.

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Proof

By the Kirillov orbit method and Lie-correspondence:

$$\zeta_{G^k}(s) = \sum_{\Omega \in \mathrm{Ad}(G^k)\backslash \widehat{\mathfrak{g}^k}} |\Omega|^{-s/2} = \sum_{\omega \in \widehat{\mathfrak{g}^k}} |\mathfrak{g}^k : \mathrm{Rad}_{\mathfrak{g}^k}(\omega)|^{-s/2-1} \qquad (*)$$

where $\mathrm{Rad}_{\mathfrak{g}^k}(\omega) = \left\{ x \in \mathfrak{g}^k : \omega([x, y]) = 1 \text{ for all } y \in \mathfrak{g}^k \right\}$.

The bijection

$$\varphi = (x \mapsto \pi^{k-1} x) : \mathfrak{g}^1 \to \mathfrak{g}^k$$

where $\mathfrak{p} = \pi \mathfrak{o}$, induces a surjective $q^{(k-1)d}$-to-one map

$$\varphi^* : \widehat{\mathfrak{g}^k} \to \widehat{\mathfrak{g}^1} \text{ given by } \varphi^*(\omega)(x) = \omega(\pi^{k-1} x)$$

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The observation **does not** extend to $k=0$. E.g.

$$\zeta_{\mathrm{SL}_2(\mathfrak{o})}(s) = \zeta_{\mathrm{SL}_2(\mathbb{F}_p)}(s) + \frac{4q\left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \frac{(q-1)^2}{2}(q^2+q)^{-s}}{1-q^{1-s}}$$

[Jaikin-Zapirain 06, Theorem 7.5] ,

$$q^{-3k}\zeta_{\mathrm{SL}_2^k(\mathfrak{o})}(s) = \frac{1-q^{-2-s}}{1-q^{1-s}} \qquad \text{for all } k \geq 1$$

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Towards an approximate Kirillov formula

The 'anomaly' above is a consequence of the lack of a Kirillov formula for G. Namely,

$$\zeta_G(s) \neq \sum_{\omega \in \widehat{\mathfrak{g}}} |\mathfrak{g} : \mathrm{Rad}(\omega)|^{-s/2-1}$$

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Conjecture/Goal [Onn-Sh, ongoing]

There exists a relation $\approx$, independent of the residual characteristic of $\mathfrak{o}$, such that

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The main result

Some definitions

Let $0 \in C \subseteq \mathbb{Q}$ a finite set and $r \in \mathbb{N}$

1. Given $q \in \mathbb{N}$, let $\mathscr{D}_{C,r}(q)$ denote the set of Dirichlet polynomials with non-negative coefficients of the form

$$f(s) = \sum_{i=1}^{r} u_i(q) v_i(q)^{-s} \quad (s \in \mathbb{C})$$

with $u_1, \ldots, u_r, v_1, \ldots, v_r \in \mathbb{Q}[t]$ with coefficients in C

2. Given $q \in \mathbb{N}$ and f_1, f_2 Dirichlet polynomials with non-negative coefficients, write

$$f_1 \sim_{(C,q,r)} f_2$$

if

- $f_j(s) = \sum_{i=1}^{r} u_{i,j}(q) v_{i,j}(q)^{-s} \in \mathscr{D}_{C,r}(q)$, for $j = 1, 2$
- $\deg(u_{i,1}) = \deg(u_{i,2})$ and $\deg(v_{i,1}) = \deg(v_{i,2})$ for all $i = 1, \ldots, r$

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Recall

$$\zeta_{\mathrm{GL}_2(\mathbb{F}_q)}(s) = (q-1)\left(1 + q^{-s} + \frac{q-2}{2}(q+1)^{-s} + \frac{q-1}{2}(q-1)^{-s}\right).$$

Let $\mathcal{S}_2(\mathbb{F}_q)$ denote the set of similarity classes of 2×2 matrices over $\mathbb{F}_q$ and define $\epsilon_{\mathfrak{gl}_2(\mathbb{F}_q)}(s) = \sum_{[x] \in \mathcal{S}_2(\mathbb{F}_q)} |\mathfrak{gl}_2(\mathbb{F}_q) : \mathbf{C}_{\mathfrak{gl}_2(\mathbb{F}_q)}(x)|^{-s/2}$. Then:

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\epsilon_{\mathfrak{gl}_2(\mathbb{F}_q)} &= q(1 + (q-1)q^{-s}) \\
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Thus

$$\zeta_{\mathrm{GL}_2(\mathbb{F}_q)} \sim_{(C,q,4)} \epsilon_{\mathfrak{gl}_2(\mathbb{F}_q)} \text{ with } C = \frac{1}{2}\mathbb{Z} \cap [-3, 3].$$

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Given a connected reductive $\mathbb{F}_q$-defined group $\underline{\mathbf{G}}$ with Lie algebra $\underline{\mathfrak{g}}$, define the *adjoint class function* of $\underline{\mathfrak{g}}(\mathbb{F}_q)$

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Theorem (Sh)

Let $\mathscr{R} = (\Phi, X, \Phi^\vee, Y)$ be a root datum. There exist a prime $p_{\mathscr{R}}$, a finite set $C_{\mathscr{R}}$, and a natural number $r_{\mathscr{R}}$, such that the following holds for any finite field $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) > p_{\mathscr{R}}$. For any $\underline{\mathbf{G}}$ connected reductive $\mathbb{F}_q$-defined group with root datum $\mathscr{R}$ and Lie algebra $\underline{\mathfrak{g}}$

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Key component of proof

Jordan-type decompositions

Write $G = \underline{\mathbf{G}}(\mathbb{F}_q)$ and $\mathfrak{g} = \underline{\mathfrak{g}}(\mathbb{F}_q)$.

- $\epsilon_{\mathfrak{g}}(s) = \sum_{\substack{[x_{ss}] \in \mathrm{Ad}(G) \backslash \mathfrak{g} \\ \text{semisimple}}} \left| \mathfrak{g} : \mathbf{C}_{\mathfrak{g}}(x_{ss}) \right|^{-s/2} \cdot \epsilon^{\mathrm{nil}}_{\mathbf{C}_{\mathfrak{g}}(x_{ss})}(s)$
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Example- unipotent zeta function of $\mathrm{GL}_n(\mathbb{F}_q)$

By definition, an irreducible character $\chi \in \mathrm{Irr}(G)$ is *unipotent* if there exists a $\mathbb{F}_q$-defined maximal torus $\underline{\mathbf{T}} \subseteq \underline{\mathbf{G}}$ such that χ occurs in $R^G_{\mathbf{T}(\mathbb{F}_q)}(\mathbf{1})$.

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For any finite field $\mathbb{F}_q$,

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Application of main result to arithmetic groups

Let K be a global number field with ring of integers $\mathcal{O}$ and let $\underline{\mathbf{G}}$ be an affine algebraic group scheme over $\mathcal{O}$ whose generic fiber $\underline{\mathbf{G}}_K$ is connected, simply connected and semisimple of absolute rank $r = \mathrm{rk}(\underline{\mathbf{G}})$ and absolute root system Φ.

The assumption of simply-connectedness implies that the constant coefficient of $\zeta_{\underline{\mathbf{G}}(\mathcal{O}/\mathfrak{p})}$ is 1 one for all but finitely many $\mathfrak{p}$'s. Therefore we may consider their infinite product.

Let $\mathrm{Irr}^{\mathrm{sqf}}(\underline{\mathbf{G}}(\mathcal{O}))$ denote the set of equivalence classes of irreducible representations of $\underline{\mathbf{G}}(\mathcal{O})$, which factor through a quotient of the form $\underline{\mathbf{G}}(\mathcal{O}/I)$ with I a square-free ideal.

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converges on the right half-plane $\left\{ \mathrm{Re}(s) > \frac{r+1}{|\Phi^+|} \right\}$.

Furthermore, there exists $\delta > 0$ such that $\zeta^{\mathrm{sqf}}_{\underline{\mathbf{G}}(\mathcal{O})}$ extends to a meromorphic function on $\left\{ \mathrm{Re}(s) > \frac{r+1}{|\Phi^+|} - \delta \right\}$ with a unique pole in this region. In particular, for suitable $\alpha, c > 0$ and $\beta \in \mathbb{N}$,

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Questions?

Thank you!

Unipotent representations and nilpotent classes

Given an irreducible root system Φ, let $\underline{\mathbf{H}}_\Phi$ be the split simple group of adjoint type with root system Φ. We assume $\Psi \leq \Phi \Rightarrow \underline{\mathbf{H}}_\Psi \subseteq \underline{\mathbf{H}}_\Phi$.

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Let $\Pi \subseteq \Phi$ be a basis and $\widetilde{\Pi} = \Pi \cup \{-\alpha_0\}$, where α_0 is the longest root. There exists a prime p_Φ, a finite set $1 \in C_\Phi \subseteq \mathbb{Q}$ and $r_\Phi \in \mathbb{N}$ such that for any finite field $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) = p$

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A non-example

The theorem *may fail* for non-reductive algebraic group.

Let $\underline{\mathbf{U}}_3$ denote the ($\mathbb{Z}$-defined) algebraic group of upper unitriangular matrices, with Lie algebra $\underline{\mathfrak{u}}_3$.

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Let $\underline{\mathbf{U}}_3$ denote the ($\mathbb{Z}$-defined) algebraic group of upper unitriangular matrices, with Lie algebra $\underline{\mathfrak{u}}_3$.

Then

$$\zeta_{\underline{\mathbf{U}}_3(\mathbb{F}_q)}(s) = q^2 + (q-1)q^{-s}$$

and

$$\epsilon_{\underline{\mathfrak{u}}_3(\mathbb{F}_q)}(s) = q + q^2 \cdot q^{-s/2}.$$

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