Algebraic Geometry 2
Tutorial session 3

Lecturer: Rami Aizenbud
TA: Shai Shechter

May 4, 2020

The image presheaf

Example

Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\mathcal{F} = C^\infty$ be the sheaf of smooth real-valued functions on X.

The image presheaf

Example

Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\mathcal{F} = C^\infty$ be the sheaf of smooth real-valued functions on X. Put

$$\mathcal{G}(U) = \left\{ \xi = (\xi_1, \xi_2) : U \to \mathbb{R}^2 \text{ smooth} \mid \frac{\partial \xi_1}{\partial y} = \frac{\partial \xi_2}{\partial x} \right\}.$$

The image presheaf

Example

Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\mathcal{F} = C^\infty$ be the sheaf of smooth real-valued functions on X. Put

$$\mathcal{G}(U) = \left\{ \xi = (\xi_1, \xi_2) : U \to \mathbb{R}^2 \text{ smooth} \mid \frac{\partial \xi_1}{\partial y} = \frac{\partial \xi_2}{\partial x} \right\}.$$

Define $\varphi : \mathcal{F} \to \mathcal{G}$ by $\varphi_U(f) = \nabla f = (\frac{df}{dx}, \frac{df}{dy})$ for any $f \in \mathcal{F}(U)$.

The image presheaf - example cont

- Given $U \subseteq X$ *simply connected*, by a theorem from calculus, any $\xi \in \mathcal{G}(U)$ is conservative, and hence of the form $\varphi_U(f)$ for some $f \in \mathcal{F}(U)$. Therefore, φ_U is surjective for U simply connected.

The image presheaf - example cont

- Given $U \subseteq X$ *simply connected*, by a theorem from calculus, any $\xi \in \mathcal{G}(U)$ is conservative, and hence of the form $\varphi_U(f)$ for some $f \in \mathcal{F}(U)$. Therefore, φ_U is surjective for U simply connected.
- On the other hand, φ_X is *not* surjective. For example, the field $\xi_0(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ lies in $\mathcal{G}(X)$ and admits no preimage under φ_X.

The image presheaf - example cont

- Given $U \subseteq X$ *simply connected*, by a theorem from calculus, any $\xi \in \mathcal{G}(U)$ is conservative, and hence of the form $\varphi_U(f)$ for some $f \in \mathcal{F}(U)$. Therefore, φ_U is surjective for U simply connected.
- On the other hand, φ_X is *not* surjective. For example, the field $\xi_0(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ lies in $\mathcal{G}(X)$ and admits no preimage under φ_X.
- Let $X = \bigcup U_\alpha$ be an open cover by simply connected sets. For any α, $\xi_\alpha := \xi_0 \mid_{U_\alpha}$ lies in $\mathrm{Im}\varphi_{U_\alpha}$. Also, the ξ_α's agree on intersections.

The image presheaf - example cont

- Given $U \subseteq X$ *simply connected*, by a theorem from calculus, any $\xi \in \mathcal{G}(U)$ is conservative, and hence of the form $\varphi_U(f)$ for some $f \in \mathcal{F}(U)$. Therefore, φ_U is surjective for U simply connected.
- On the other hand, φ_X is *not* surjective. For example, the field $\xi_0(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ lies in $\mathcal{G}(X)$ and admits no preimage under φ_X.
- Let $X = \bigcup U_\alpha$ be an open cover by simply connected sets. For any α, $\xi_\alpha := \xi_0 \mid_{U_\alpha}$ lies in $\mathrm{Im}\varphi_{U_\alpha}$. Also, the ξ_α's agree on intersections. However, they glue uniquely (by identity) to $\xi_0 \in \mathcal{G}(X)$, which is not an element of $\mathrm{Im}\varphi_X$.

The image presheaf - example cont

- Given $U \subseteq X$ *simply connected*, by a theorem from calculus, any $\xi \in \mathcal{G}(U)$ is conservative, and hence of the form $\varphi_U(f)$ for some $f \in \mathcal{F}(U)$. Therefore, φ_U is surjective for U simply connected.
- On the other hand, φ_X is *not* surjective. For example, the field $\xi_0(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ lies in $\mathcal{G}(X)$ and admits no preimage under φ_X.
- Let $X = \bigcup U_\alpha$ be an open cover by simply connected sets. For any α, $\xi_\alpha := \xi_0 \mid_{U_\alpha}$ lies in $\mathrm{Im}\varphi_{U_\alpha}$. Also, the ξ_α's agree on intersections. However, they glue uniquely (by identity) to $\xi_0 \in \mathcal{G}(X)$, which is not an element of $\mathrm{Im}\varphi_X$.
- Therefore the assignment $U \mapsto \mathrm{Im}\varphi_U$ is not a sheaf.

This example shows that, usually, the obvious candidate for the image of a morphism of sheaves is *not* a sheaf.

This example shows that, usually, the obvious candidate for the image of a morphism of sheaves is *not* a sheaf. The true (in the categorical sense) image of a sheaf is defined by applying the sheafification operation to the image presheaf, which we now recall.

Grothendieck sheaves and Leray sheaves

Definition (Leray sheaf)

A Leray sheaf on a topological space X is a pair (E, p), where E is a topological space and $p : E \to X$ is a local homeomorphism.

Grothendieck sheaves and Leray sheaves

Definition (Leray sheaf)

A Leray sheaf on a topological space X is a pair (E, p), where E is a topological space and $p : E \to X$ is a local homeomorphism.

We'll usually write L-sheaf a Leray sheaf, and G-sheaf (or just sheaf) for Grothendieck sheaves, i.e. sheaves as defined in Hartshorne.

Grothendieck sheaves and Leray sheaves

Definition (Leray sheaf)

A Leray sheaf on a topological space X is a pair (E, p), where E is a topological space and $p : E \to X$ is a local homeomorphism.

We'll usually write L-sheaf a Leray sheaf, and G-sheaf (or just sheaf) for Grothendieck sheaves, i.e. sheaves as defined in Hartshorne.
The terminology G-Sheaves and L-Sheaves is uncommon; usually both are referred to as sheaves.

Grothendieck sheaves and Leray sheaves

Definition (Leray sheaf)

A Leray sheaf on a topological space X is a pair (E, p), where E is a topological space and $p : E \to X$ is a local homeomorphism.

We'll usually write L-sheaf a Leray sheaf, and G-sheaf (or just sheaf) for Grothendieck sheaves, i.e. sheaves as defined in Hartshorne.
The terminology G-Sheaves and L-Sheaves is uncommon; usually both are referred to as sheaves. Historically, Leray was the first to formalize the notion of a sheaf in trying to prove certain fixed-point theorems for PDEs.

From L-sheaves to G-sheaves

From L-sheaves to G-sheaves

Given an L-sheaf E over a top space X, we can construct a G-sheaf $G(E)$ by taking continuous sections. That is

$$G(E)(U) = \Gamma(U, E) := \{s : U \to E \text{ continuous} \mid p \circ s = \mathbf{1}_U\}$$

for all $U \subseteq X$ open.

From L-sheaves to G-sheaves

Given an L-sheaf E over a top space X, we can construct a G-sheaf $G(E)$ by taking continuous sections. That is

$$G(E)(U) = \Gamma(U, E) := \{s : U \to E \text{ continuous} \mid p \circ s = \mathbf{1}_U\}$$

for all $U \subseteq X$ open.

$G(E)$ is easily verified to be a sheaf.

From G-sheaves to L-sheaves

From G-sheaves to L-sheaves

Given a G-sheaf $\mathcal{F}$ over X, we construct a Leray sheaf whose underlying set is

$$L(\mathcal{F}) = \bigsqcup_{x \in X} \{x\} \times \mathcal{F}_x,$$

with $p : L(\mathcal{F}) \to X$ the projection onto the first coordinate.

From G-sheaves to L-sheaves

Given a G-sheaf $\mathcal{F}$ over X, we construct a Leray sheaf whose underlying set is

$$L(\mathcal{F}) = \bigsqcup_{x \in X} \{x\} \times \mathcal{F}_x,$$

with $p : L(\mathcal{F}) \to X$ the projection onto the first coordinate. The topology on $L(\mathcal{F})$ is generated by the sets

$$V_{\xi,U} = \{(x, \sigma_x) : x \in U \text{ and } \sigma_x = [\xi, U] \text{ in } \mathcal{F}_x\} \quad (U \text{ open}, \ \xi \in \mathcal{F}(U)).$$

Lemma (Home exercise)

$p \mid_{V_{\xi,U}}$ is a homeomorphism onto U.

Note that the construction of $L(\mathcal{F})$ can also be applied to presheaves.

Exercise

Let $\mathcal{F}$ be a presheaf over X. Find a natural map $\varphi : \mathcal{F} \to G(L(\mathcal{F}))$. Show that it is injective if $\mathcal{F}$ satisfies the identity axiom and that φ_U is for all U surjective if $\mathcal{F}$ satisfies gluing.

Exercise

Let $\mathcal{F}$ be a presheaf over X. Find a natural map $\varphi : \mathcal{F} \to G(L(\mathcal{F}))$. Show that it is injective if $\mathcal{F}$ satisfies the identity axiom and that φ_U is for all U surjective if $\mathcal{F}$ satisfies gluing.

Proof.

Given $U \subseteq X$ open, we define $\varphi_U : \mathcal{F}(U) \to \Gamma(U, L(\mathcal{F}))$ by $\varphi_U(\xi) = s_\xi$ where $s_\xi : U \to L(\mathcal{F})$ is defined by

$$s_\xi(x) = (x, [\xi, U]) \in \{x\} \times \mathcal{F}_x.$$

Exercise

Let $\mathcal{F}$ be a presheaf over X. Find a natural map $\varphi : \mathcal{F} \to G(L(\mathcal{F}))$. Show that it is injective if $\mathcal{F}$ satisfies the identity axiom and that φ_U is for all U surjective if $\mathcal{F}$ satisfies gluing.

Proof.

Given $U \subseteq X$ open, we define $\varphi_U : \mathcal{F}(U) \to \Gamma(U, L(\mathcal{F}))$ by $\varphi_U(\xi) = s_\xi$ where $s_\xi : U \to L(\mathcal{F})$ is defined by

$$s_\xi(x) = (x, [\xi, U]) \in \{x\} \times \mathcal{F}_x.$$

<u>Claim</u>: s_ξ *is continuous.*

Exercise

Let $\mathcal{F}$ be a presheaf over X. Find a natural map $\varphi : \mathcal{F} \to G(L(\mathcal{F}))$. Show that it is injective if $\mathcal{F}$ satisfies the identity axiom and that φ_U is for all U surjective if $\mathcal{F}$ satisfies gluing.

Proof.

Given $U \subseteq X$ open, we define $\varphi_U : \mathcal{F}(U) \to \Gamma(U, L(\mathcal{F}))$ by $\varphi_U(\xi) = s_\xi$ where $s_\xi : U \to L(\mathcal{F})$ is defined by

$$s_\xi(x) = (x, [\xi, U]) \in \{x\} \times \mathcal{F}_x.$$

<u>Claim</u>: *s_ξ is continuous.* Assume $s_\xi(x) \in V_{\nu,U'}$ for some $x \in U$ and $V_{\nu,U'} \subseteq L(\mathcal{F})$ basic open. By definition of $V_{\nu,U'}$, this means that $[\xi, U] = [\nu, U']$ as elements of $\mathcal{F}_x$.

Exercise

Let $\mathcal{F}$ be a presheaf over X. Find a natural map $\varphi : \mathcal{F} \to G(L(\mathcal{F}))$. Show that it is injective if $\mathcal{F}$ satisfies the identity axiom and that φ_U is for all U surjective if $\mathcal{F}$ satisfies gluing.

Proof.

Given $U \subseteq X$ open, we define $\varphi_U : \mathcal{F}(U) \to \Gamma(U, L(\mathcal{F}))$ by $\varphi_U(\xi) = s_\xi$ where $s_\xi : U \to L(\mathcal{F})$ is defined by

$$s_\xi(x) = (x, [\xi, U]) \in \{x\} \times \mathcal{F}_x.$$

Claim: s_ξ *is continuous.* Assume $s_\xi(x) \in V_{\nu,U'}$ for some $x \in U$ and $V_{\nu,U'} \subseteq L(\mathcal{F})$ basic open. By definition of $V_{\nu,U'}$, this means that $[\xi, U] = [\nu, U']$ as elements of $\mathcal{F}_x$. Thus $\exists W \subseteq U \cap U'$ such that $\xi\,|_W = \nu\,|_W$, and $s_\xi(W) \subseteq V_{\nu,U'}$.

Proof contd.

- Assume $\mathcal{F}$ has identity.

Proof contd.

- Assume $\mathcal{F}$ has identity. For $\xi \in \mathcal{F}(U)$ we have that

$$\varphi_U(\xi) = s_\xi = 0 \quad \Longleftrightarrow \quad \xi = 0 \text{ in } \mathcal{F}_x \text{ for all } x \in U$$

  which occurs iff any $x \in U$ has a neighbourhood V_x such that $\xi\,|_{V_x} \equiv 0$.

Proof contd.

- Assume $\mathcal{F}$ has identity. For $\xi \in \mathcal{F}(U)$ we have that

$$\varphi_U(\xi) = s_\xi = 0 \quad \Longleftrightarrow \quad \xi = 0 \text{ in } \mathcal{F}_x \text{ for all } x \in U$$

which occurs iff any $x \in U$ has a neighbourhood V_x such that $\xi\,|_{V_x} \equiv 0$. As $\mathcal{F}$ has the identity axiom, this implies $\xi = 0$, and therefore $\mathrm{Ker}(\varphi) = 0$.

Proof contd.

- Assume $\mathcal{F}$ has identity. For $\xi \in \mathcal{F}(U)$ we have that

$$\varphi_U(\xi) = s_\xi = 0 \quad \Longleftrightarrow \quad \xi = 0 \text{ in } \mathcal{F}_x \text{ for all } x \in U$$

  which occurs iff any $x \in U$ has a neighbourhood V_x such that $\xi\,|_{V_x} \equiv 0$. As $\mathcal{F}$ has the identity axiom, this implies $\xi = 0$, and therefore $\mathrm{Ker}(\varphi) = 0$.
- Assume $\mathcal{F}$ has gluing.

Proof contd.

- Assume $\mathcal{F}$ has identity. For $\xi \in \mathcal{F}(U)$ we have that

$$\varphi_U(\xi) = s_\xi = 0 \quad \iff \quad \xi = 0 \text{ in } \mathcal{F}_x \text{ for all } x \in U$$

which occurs iff any $x \in U$ has a neighbourhood V_x such that $\xi|_{V_x} \equiv 0$. As $\mathcal{F}$ has the identity axiom, this implies $\xi = 0$, and therefore $\mathrm{Ker}(\varphi) = 0$.

- Assume $\mathcal{F}$ has gluing. Let U be open and $s : U \to L(\mathcal{F})$ be a continuous section. Note that $s(x) = (x, s_2(x))$ for some $s_2(x) \in \mathcal{F}_x$.

Proof contd.

- Assume $\mathcal{F}$ has identity. For $\xi \in \mathcal{F}(U)$ we have that

$$\varphi_U(\xi) = s_\xi = 0 \quad \Longleftrightarrow \quad \xi = 0 \text{ in } \mathcal{F}_x \text{ for all } x \in U$$

  which occurs iff any $x \in U$ has a neighbourhood V_x such that $\xi\, |_{V_x} \equiv 0$. As $\mathcal{F}$ has the identity axiom, this implies $\xi = 0$, and therefore $\mathrm{Ker}(\varphi) = 0$.
- Assume $\mathcal{F}$ has gluing. Let U be open and $s : U \to L(\mathcal{F})$ be a continuous section. Note that $s(x) = (x, s_2(x))$ for some $s_2(x) \in \mathcal{F}_x$. The continuity of s implies that for any $x \in U$ there exists an open set $x \in W_x \subseteq U$ and $\xi_x \in \mathcal{F}(W_x)$ such that $s_2(y) = [\xi_x, W_x] \in \mathcal{F}_y$ for all $y \in W_x$.

Proof contd.

- Assume $\mathcal{F}$ has identity. For $\xi \in \mathcal{F}(U)$ we have that

$$\varphi_U(\xi) = s_\xi = 0 \quad \iff \quad \xi = 0 \text{ in } \mathcal{F}_x \text{ for all } x \in U$$

  which occurs iff any $x \in U$ has a neighbourhood V_x such that $\xi\,|_{V_x} \equiv 0$. As $\mathcal{F}$ has the identity axiom, this implies $\xi = 0$, and therefore $\mathrm{Ker}(\varphi) = 0$.
- Assume $\mathcal{F}$ has gluing. Let U be open and $s : U \to L(\mathcal{F})$ be a continuous section. Note that $s(x) = (x, s_2(x))$ for some $s_2(x) \in \mathcal{F}_x$. The continuity of s implies that for any $x \in U$ there exists an open set $x \in W_x \subseteq U$ and $\xi_x \in \mathcal{F}(W_x)$ such that $s_2(y) = [\xi_x, W_x] \in \mathcal{F}_y$ for all $y \in W_x$. Refining the cover $\{W_x\}$ if necessary and applying the gluing axiom, the ξ_x's give rise to an element $\xi \in \mathcal{F}(U)$ such that $\xi\,|_{W_x} = \xi_x$. Then $s = \varphi_U(\xi)$.

$\square$

Question

Given an L-sheaf $E \xrightarrow{p} X$, can we construct a natural map

$$\begin{tikzcd} L(G(E)) \arrow[rr] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd} ?$$

Sheafification

Definition

The sheafification of a presheaf $\mathcal{F}$ over X is the sheaf $\mathcal{F}^+ := G(L(\mathcal{F}))$ described above.

Sheafification

Definition

The sheafification of a presheaf $\mathcal{F}$ over X is the sheaf $\mathcal{F}^+ := G(L(\mathcal{F}))$ described above.

It has the following properties:

Sheafification

Definition

The sheafification of a presheaf $\mathcal{F}$ over X is the sheaf $\mathcal{F}^+ := G(L(\mathcal{F}))$ described above.

It has the following properties:

1. It is unique up to unique isomorphism.

Sheafification

Definition

The sheafification of a presheaf $\mathcal{F}$ over X is the sheaf $\mathcal{F}^+ := G(L(\mathcal{F}))$ described above.

It has the following properties:

1. It is unique up to unique isomorphism.
2. If $\mathcal{F}$ is already a sheaf then $\mathcal{F}^+ \simeq \mathcal{F}$.

Sheafification

Definition

The sheafification of a presheaf $\mathcal{F}$ over X is the sheaf $\mathcal{F}^+ := G(L(\mathcal{F}))$ described above.

It has the following properties:

1. It is unique up to unique isomorphism.
2. If $\mathcal{F}$ is already a sheaf then $\mathcal{F}^+ \simeq \mathcal{F}$.
3. For any $x \in X$ we have $\mathcal{F}_x \simeq \mathcal{F}_x^+$.

Sheafification

Definition

The sheafification of a presheaf $\mathcal{F}$ over X is the sheaf $\mathcal{F}^+ := G(L(\mathcal{F}))$ described above.

It has the following properties:

1. It is unique up to unique isomorphism.
2. If $\mathcal{F}$ is already a sheaf then $\mathcal{F}^+ \simeq \mathcal{F}$.
3. For any $x \in X$ we have $\mathcal{F}_x \simeq \mathcal{F}_x^+$.
4. Given $\mathcal{G}$ a sheaf on X and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves, there exists a unique $\widetilde{\varphi} : \mathcal{F}^+ \to \mathcal{G}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, dashed, "\widetilde{\varphi}"'] & \end{tikzcd}$$

commutes.

Sheafification

Definition

The sheafification of a presheaf $\mathcal{F}$ over X is the sheaf $\mathcal{F}^+ := G(L(\mathcal{F}))$ described above.

It has the following properties:

1. It is unique up to unique isomorphism.
2. If $\mathcal{F}$ is already a sheaf then $\mathcal{F}^+ \simeq \mathcal{F}$.
3. For any $x \in X$ we have $\mathcal{F}_x \simeq \mathcal{F}_x^+$.
4. Given $\mathcal{G}$ a sheaf on X and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves, there exists a unique $\widetilde{\varphi} : \mathcal{F}^+ \to \mathcal{G}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, dashed, "\widetilde{\varphi}"'] & \end{tikzcd}$$

commutes. That is $\mathrm{Hom}_{\underline{\mathbf{PSh}}_X}(\mathcal{F}, \mathcal{G}^{\text{forget}}) = \mathrm{Hom}_{\underline{\mathbf{Sh}}_X}(\mathcal{F}^+, \mathcal{G})$.

Sheafification - contd

1. The assertion $\mathcal{F}_x \simeq \mathcal{F}_x^+$ for any $x \in X$ may be verified directly. It follows from the more general fact that, for any Leray sheaf $E \xrightarrow{p} X$, the stalk of $G(E)$ over a point x is canonically isomorphic to the fiber $p^{-1}(x)$. (this is a consequence of p being a local homeomorphism.)

Sheafification - contd

1. The assertion $\mathcal{F}_x \simeq \mathcal{F}_x^+$ for any $x \in X$ may be verified directly. It follows from the more general fact that, for any Leray sheaf $E \xrightarrow{p} X$, the stalk of $G(E)$ over a point x is canonically isomorphic to the fiber $p^{-1}(x)$. (this is a consequence of p being a local homeomorphism.)
2. One possible way to define the map $\widetilde{\varphi} : \mathcal{F}^+ \to \mathcal{G}$ is by defining a map $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$, by setting

$$\widetilde{\varphi}_U(s)(x) = (x, \varphi_x(s(x))) \quad (s \in \mathcal{F}(U),\ U \text{ open}),$$

where φ_x denotes the induced homomorphism $\mathcal{F}_x \to \mathcal{G}_x$. To get $\widetilde{\varphi}$, compose φ^+ with the isomorphism $\mathcal{G}^+ \xrightarrow{\sim} \mathcal{G}$.

Another construction – a remark

There is another construction of the sheafification which is independent of choosing points. Let $\mathcal{F}$ be a presheaf over a top space X.

1. Given U open and an open cover $U = \bigcup V_\alpha$, we can define

$$\mathcal{F}^{\oplus}_{U=\bigcup V_\alpha}(U) := \left\{ (\xi_\alpha)_\alpha \mid \xi_\alpha \in \mathcal{F}(V_\alpha) \text{ and } \xi_\alpha \mid_{V_\alpha \cap V_\beta} = \xi_\beta \mid_{V_\alpha \cap V_\beta} \right\}.$$

Another construction – a remark

There is another construction of the sheafification which is independent of choosing points. Let $\mathcal{F}$ be a presheaf over a top space X.

1. Given U open and an open cover $U = \bigcup V_\alpha$, we can define
$$\mathcal{F}^{\oplus}_{U=\bigcup V_\alpha}(U) := \left\{ (\xi_\alpha)_\alpha \mid \xi_\alpha \in \mathcal{F}(V_\alpha) \text{ and } \xi_\alpha \mid_{V_\alpha \cap V_\beta} = \xi_\beta \mid_{V_\alpha \cap V_\beta} \right\}.$$
2. We then take a direct limit and define
$$\mathcal{F}^{\oplus}(U) := \lim_{U=\bigcup V_\alpha} \mathcal{F}^{\oplus}_{U=\bigcup V_\alpha}(U).$$

Another construction – a remark

There is another construction of the sheafification which is independent of choosing points. Let $\mathcal{F}$ be a presheaf over a top space X.

1. Given U open and an open cover $U = \bigcup V_\alpha$, we can define
$$\mathcal{F}^{\oplus}_{U=\bigcup V_\alpha}(U) := \left\{(\xi_\alpha)_\alpha \mid \xi_\alpha \in \mathcal{F}(V_\alpha) \text{ and } \xi_\alpha|_{V_\alpha \cap V_\beta} = \xi_\beta|_{V_\alpha \cap V_\beta}\right\}.$$
2. We then take a direct limit and define
$$\mathcal{F}^{\oplus}(U) := \lim_{U=\bigcup V_\alpha} \mathcal{F}^{\oplus}_{U=\bigcup V_\alpha}(U).$$
3. If $\mathcal{F}$ has the identity axiom then $\mathcal{F}^{\oplus} \simeq \mathcal{F}^{+}$ is a sheaf. Otherwise $\mathcal{F}^{\oplus}$ has identity, and then $\mathcal{F}^{\oplus\oplus}$ is a sheaf, isomorphic to $\mathcal{F}^{+}$.

Operations on sheaves

Let $f : X \to Y$ a continuous map of topological spaces.

Definition (Direct image sheaf)

Given a sheaf $\mathcal{F}$ on X, we can define a new sheaf on Y by setting

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)),$$

for any $V \subseteq Y$ open.

Exercise

Let $f : X \to Y$ be a finite covering map and $y \in Y$. What is $(f_*\mathcal{F})_y$?

Exercise

Let $f : X \to Y$ be a finite covering map and $y \in Y$. What is $(f_*\mathcal{F})_y$?

Solution.

Since f is a finite covering map, there exists $V \ni y$ open such that $f^{-1}V = \bigsqcup_{i=1}^n U_i$ with $f|_{U_i}$ a homeomorphism onto V.

Exercise

Let $f : X \to Y$ be a finite covering map and $y \in Y$. What is $(f_*\mathcal{F})_y$?

Solution.

Since f is a finite covering map, there exists $V \ni y$ open such that $f^{-1}V = \bigsqcup_{i=1}^{n} U_i$ with $f\,|_{U_i}$ a homeomorphism onto V. Let $f^{-1}(y) = \{x_1, \ldots, x_n\}$ with $x_i \in U_i$.

Exercise

Let $f : X \to Y$ be a finite covering map and $y \in Y$. What is $(f_*\mathcal{F})_y$?

Solution.

Since f is a finite covering map, there exists $V \ni y$ open such that $f^{-1}V = \bigsqcup_{i=1}^n U_i$ with $f\,|_{U_i}$ a homeomorphism onto V. Let $f^{-1}(y) = \{x_1, \ldots, x_n\}$ with $x_i \in U_i$. Then

$$\begin{aligned}(f_*\mathcal{F})_y &= \lim_{y \in W \text{ open}} (f_*\mathcal{F})(W) = \lim_{y \in W \subseteq V} \mathcal{F}(f^{-1}(W)) \\ &= \lim_{y \in W \subseteq V} \mathcal{F}\Big(\bigsqcup_i (f^{-1}(W) \cap U_i)\Big) = \lim_{y \in W \subseteq V} \prod_i \mathcal{F}((f\,|_{U_i})^{-1}(W)) \\ &= \prod_{i=1}^n \lim_{x_i \in U' \subseteq U} \mathcal{F}(U') = \prod_{x \in f^{-1}(y)} \mathcal{F}_x.\end{aligned}$$

□

Operations on sheaves – Pullback

Now let $\mathcal{F}$ be a sheaf on Y and $f : X \to Y$ a continuous map. We want to obtain a sheaf $f^{-1}(\mathcal{F})$ on X. We describe it in the context of L-sheaves.

Recall that, given another continuous map $p : E \to Y$, the fiber product $X \times_Y E$ is defined by $\{(x, e) \mid f(x) = p(e)\} \subseteq X \times E$, with the subspace topology.

Lemma

Let $E \xrightarrow{p} Y$ be an L-sheaf over Y. Then $X \times_Y E \to X$ is an L-sheaf on X, wrt projection onto the first coordinate.

Pullback in L-sheaves

Proof of lemma.

Let π_1 denote the projection $X \times_Y E \to X$ and $(x, e) \in X \times_Y E$.

Pullback in L-sheaves

Proof of lemma.

Let π_1 denote the projection $X \times_Y E \to X$ and $(x, e) \in X \times_Y E$.

Pullback in L-sheaves

Proof of lemma.

Let π_1 denote the projection $X \times_Y E \to X$ and $(x, e) \in X \times_Y E$. Assume $e \in U \subseteq E$ is open such that $p\,|_U$ is a homeomorphism. Put $W = f^{-1}(p(U))$. Then $x \in W$, since $f(x) = p(e) \in p(U)$. Let $V := (W \times U) \cap (X \times_Y E)$.

Claim.$\pi_1\,|_V$ *is a homeomorphism*

Pullback in L-sheaves

Proof of lemma.

Let π_1 denote the projection $X \times_Y E \to X$ and $(x, e) \in X \times_Y E$. Assume $e \in U \subseteq E$ is open such that $p\mid_U$ is a homeomorphism. Put $W = f^{-1}(p(U))$. Then $x \in W$, since $f(x) = p(e) \in p(U)$. Let $V := (W \times U) \cap (X \times_Y E)$.

Claim.$\pi_1\mid_V$ *is a homeomorphism* It is continuous and open, as the restriction of a continuous open map.

Pullback in L-sheaves

Proof of lemma.

Let π_1 denote the projection $X \times_Y E \to X$ and $(x, e) \in X \times_Y E$. Assume $e \in U \subseteq E$ is open such that $p\,|_U$ is a homeomorphism. Put $W = f^{-1}(p(U))$. Then $x \in W$, since $f(x) = p(e) \in p(U)$. Let $V := (W \times U) \cap (X \times_Y E)$.

Claim.$\pi_1\,|_V$ *is a homeomorphism* It is continuous and open, as the restriction of a continuous open map. to verify that $\pi_1\,|_V$ is injective, note that

$$\pi_1(x_1, e_1) = \pi_1(x_2, e_2) \quad \Rightarrow \quad x_1 = x_2 \quad \Rightarrow \quad f(x_1) = f(x_2),$$

which, for $(x_i, e_i) \in X \times_Y E$ implies $p(e_1) = p(e_2)$. Since $p\,|_U$ is bijective, this implies $e_1 = e_2$ as well.

Pullback in L-sheaves

Proof of lemma.

Let π_1 denote the projection $X \times_Y E \to X$ and $(x, e) \in X \times_Y E$. Assume $e \in U \subseteq E$ is open such that $p\,|_U$ is a homeomorphism. Put $W = f^{-1}(p(U))$. Then $x \in W$, since $f(x) = p(e) \in p(U)$. Let $V := (W \times U) \cap (X \times_Y E)$.

Claim.$\pi_1\,|_V$ *is a homeomorphism* It is continuous and open, as the restriction of a continuous open map. to verify that $\pi_1\,|_V$ is injective, note that

$$\pi_1(x_1, e_1) = \pi_1(x_2, e_2) \quad \Rightarrow \quad x_1 = x_2 \quad \Rightarrow \quad f(x_1) = f(x_2),$$

which, for $(x_i, e_i) \in X \times_Y E$ implies $p(e_1) = p(e_2)$. Since $p\,|_U$ is bijective, this implies $e_1 = e_2$ as well. Therefore π_1 is a local homeomorphism.

$\square$

Pullback in G-sheaves

Question

What is the analog of pullback on G-sheaves?

Pullback in G-sheaves

Question

What is the analog of pullback on G-sheaves?

Note that, given $W \subseteq X$ open, $f(W) \subseteq V \subseteq Y$ open and a section $s : V \to E$ of p, we can cook-up a section $\widetilde{s} : W \to X \times_Y E$ by the formula $\widetilde{s}(w) = (w, s \circ f(w))$.

$$\begin{tikzcd} X \times_Y E \arrow[r] \arrow[d, "\pi_1"] & E \arrow[d, "p"'] \\ X \arrow[r, "f"] \arrow[u, bend left, dashed, "\widetilde{s}"] & Y \arrow[u, bend right, "s"'] \end{tikzcd} .$$

Pullback in G-sheaves

Question

What is the analog of pullback on G-sheaves?

Note that, given $W \subseteq X$ open, $f(W) \subseteq V \subseteq Y$ open and a section $s : V \to E$ of p, we can cook-up a section $\widetilde{s} : W \to X \times_Y E$ by the formula $\widetilde{s}(w) = (w, s \circ f(w))$.

$$\begin{tikzcd} X \times_Y E \arrow[r] \arrow[d, "\pi_1"] & E \arrow[d, "p"'] \\ X \arrow[r, "f"] \arrow[u, bend left, dashed, "\widetilde{s}"] & Y \arrow[u, bend right, "s"'] \end{tikzcd} \quad .$$

We get an inclusion $\lim_{f(W)\subseteq V} G(E)(V) \hookrightarrow G(X \times_Y E)(W)$.

Question

Are these all sections on $G(X \times_Y E)(W)$?

Question

Are these all sections on $G(X \times_Y E)(W)$?

Answer

If $W \subseteq f^{-1}(p(U))$ for $U \subseteq E$ open such that $p \mid U$ is a homeomorphism, then YES (home exercise).

Question

Are these all sections on $G(X \times_Y E)(W)$?

Answer

If $W \subseteq f^{-1}(p(U))$ for $U \subseteq E$ open such that $p \mid U$ is a homeomorphism, then YES (home exercise). Moreover, such W's form a basis for the topology of X. Thus- we obtain a description of $G(X \times_Y E)$ by sheafification.

Question

Are these all sections on $G(X \times_Y E)(W)$?

Answer

If $W \subseteq f^{-1}(p(U))$ for $U \subseteq E$ open such that $p \mid U$ is a homeomorphism, then YES (home exercise). Moreover, such W's form a basis for the topology of X. Thus- we obtain a description of $G(X \times_Y E)$ by sheafification.

Definition (Pullback of a sheaf)

Given $f : X \to Y$ a continuous map and $\mathcal{F}$ a sheaf on Y, the *pullback sheaf* $f^{-1}(\mathcal{F})$ is the sheaf obtained by sheafifying the presheaf

$$U \mapsto \lim_{f(U) \subseteq V \text{ open}} \mathcal{F}(V).$$

Adjointness of pullback and direct image

Exercise

Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y. Then there exists a natural bijection

$$\mathrm{Hom}_{\underline{\mathbf{Sh}}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\underline{\mathbf{Sh}}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Adjointness of pullback and direct image

Exercise

Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y. Then there exists a natural bijection

$$\mathrm{Hom}_{\underline{\mathbf{Sh}}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\underline{\mathbf{Sh}}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Proof.

We construct natural maps going in both directions, and verify that their compositions are equivalent to identity (some details are left as exercises).

Proof.

- $\underline{F : \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F})}$:

Proof.

- $\underline{F : \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F})}$: Let $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ and let $V \subseteq Y$ be open. Put $U = f^{-1}(V)$. Since $f(U)$ is open, we have that $f^{-1}\mathcal{G}(U) = \lim_{V' \supseteq f(U)} \mathcal{G}(V') = \mathcal{G}(V)$ (Ex).

Proof.

- $\underline{F : \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F})}$: Let $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ and let $V \subseteq Y$ be open. Put $U = f^{-1}(V)$. Since $f(U)$ is open, we have that $f^{-1}\mathcal{G}(U) = \lim_{V' \supseteq f(U)} \mathcal{G}(V') = \mathcal{G}(V)$ (Ex). Furthermore, by definition, $\mathcal{F}(U) = f^*\mathcal{F}(V)$.

Proof.

- $\underline{F : \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F})}$: Let $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ and let $V \subseteq Y$ be open. Put $U = f^{-1}(V)$. Since $f(U)$ is open, we have that $f^{-1}\mathcal{G}(U) = \lim_{V' \supseteq f(U)} \mathcal{G}(V') = \mathcal{G}(V)$ (Ex). Furthermore, by definition, $\mathcal{F}(U) = f^*\mathcal{F}(V)$. Therefore, we may define $F(\varphi)$ by setting

$$F(\varphi)_V = \varphi_{f^{-1}(V)} \in \mathrm{Hom}(\mathcal{G}(V), f^*\mathcal{F}(V)).$$

The definition $F(\varphi)$ is compatible with restrictions, and therefore $F(\varphi)$ is a morphism of sheaves (Ex).

□

Proof.

- $\underline{G : \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})}$: Given $\psi : \mathcal{G} \to f^*\mathcal{F}$, $U \subseteq X$ open and $f(U) \subseteq V \subseteq Y$ open, we have a map $g_{V,U} : \mathcal{G}(V) \to \mathcal{F}(U)$, given by the composition

$$\mathcal{G}(V) \xrightarrow{\psi_V} f^*\mathcal{F}(V) = \mathcal{F}(f^{-1}V) \xrightarrow{\mathrm{res}_{f^{-1}(V),U}} \mathcal{F}(U).$$

Proof.

- $\underline{G : \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})}$: Given $\psi : \mathcal{G} \to f^*\mathcal{F}$, $U \subseteq X$ open and $f(U) \subseteq V \subseteq Y$ open, we have a map $g_{V,U} : \mathcal{G}(V) \to \mathcal{F}(U)$, given by the composition

$$\mathcal{G}(V) \xrightarrow{\psi_V} f^*\mathcal{F}(V) = \mathcal{F}(f^{-1}V) \xrightarrow{\mathrm{res}_{f^{-1}(V),U}} \mathcal{F}(U).$$

The universal property of direct limit then gives a map

$$g_U = \lim_{V \supseteq f(U)} g_{V,U} : \lim_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U),$$

which makes a suitable diagram commutative.

Proof.

- $\underline{G : \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})}$: Given $\psi : \mathcal{G} \to f^*\mathcal{F}$, $U \subseteq X$ open and $f(U) \subseteq V \subseteq Y$ open, we have a map $g_{V,U} : \mathcal{G}(V) \to \mathcal{F}(U)$, given by the composition

$$\mathcal{G}(V) \xrightarrow{\psi_V} f^*\mathcal{F}(V) = \mathcal{F}(f^{-1}V) \xrightarrow{\mathrm{res}_{f^{-1}(V),U}} \mathcal{F}(U).$$

  The universal property of direct limit then gives a map

$$g_U = \lim_{V \supseteq f(U)} g_{V,U} : \lim_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U),$$

  which makes a suitable diagram commutative. The maps (g_U) comprise a morphisms of presheaves form $(U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V))$ to $\mathcal{F}$ ($\mathtt{Ex}$).

Proof.

- $\underline{G : \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})}$: Given $\psi : \mathcal{G} \to f^*\mathcal{F}$, $U \subseteq X$ open and $f(U) \subseteq V \subseteq Y$ open, we have a map $g_{V,U} : \mathcal{G}(V) \to \mathcal{F}(U)$, given by the composition

$$\mathcal{G}(V) \xrightarrow{\psi_V} f^*\mathcal{F}(V) = \mathcal{F}(f^{-1}V) \xrightarrow{\mathrm{res}_{f^{-1}(V),U}} \mathcal{F}(U).$$

The universal property of direct limit then gives a map

$$g_U = \lim_{V \supseteq f(U)} g_{V,U} : \lim_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U),$$

which makes a suitable diagram commutative. The maps (g_U) comprise a morphisms of presheaves form $(U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V))$ to $\mathcal{F}$ (Ex). Sheafifying, we set $G(\psi) = \widetilde{g} : f^{-1}\mathcal{G} \to \mathcal{F}$.

□

Proof.

- Given $\psi : \mathcal{G} \to f^*\mathcal{F}$ and $V \subseteq Y$ open, we have $(FG\psi)_V = (G\psi)_{f^{-1}(V)}$, where the RHS is given by a direct limit over open sets containing $f(f^{-1}(V)) = V$.

Proof.

- Given $\psi : \mathcal{G} \to f^*\mathcal{F}$ and $V \subseteq Y$ open, we have $(FG\psi)_V = (G\psi)_{f^{-1}(V)}$, where the RHS is given by a direct limit over open sets containing $f(f^{-1}(V)) = V$. Hence it is over a filtered limit with a terminal object, and equals to its value at V.

Proof.

- Given $\psi : \mathcal{G} \to f^*\mathcal{F}$ and $V \subseteq Y$ open, we have $(FG\psi)_V = (G\psi)_{f^{-1}(V)}$, where the RHS is given by a direct limit over open sets containing $f(f^{-1}(V)) = V$. Hence it is over a filtered limit with a terminal object, and equals to its value at V. It follows that $(FG\psi)_V = (G\psi)_{f^{-1}(V)} = \psi_V$.

$\square$

Proof.

- Given $\psi : \mathcal{G} \to f^*\mathcal{F}$ and $V \subseteq Y$ open, we have $(FG\psi)_V = (G\psi)_{f^{-1}(V)}$, where the RHS is given by a direct limit over open sets containing $f(f^{-1}(V)) = V$. Hence it is over a filtered limit with a terminal object, and equals to its value at V. It follows that $(FG\psi)_V = (G\psi)_{f^{-1}(V)} = \psi_V$.
- The equivalence $(GF\varphi)_U = \varphi_U$ for all $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ follows similarly, by unfolding the definitions (Ex).

$\square$