Generalised functions
Tutorial notes

Tutorial 2

2.1. Topologies on $C_c^\infty(\mathbb{R})$ and $(C_c^\infty(\mathbb{R}))^*$.

Definition 2.1 (Convergence in $C_c^\infty(\mathbb{R})$). A sequence $(f_n)_{n=1}^\infty$ of elements of $C_c^\infty(\mathbb{R})$ is said to converge to $f \in C_c^\infty(\mathbb{R})$ if:

(1) There exists a compact set $K \subseteq \mathbb{R}$ such that $\bigcup_{n=1}^\infty \operatorname{Supp}(f_n) \subseteq K$; and
(2) For every $k \in \mathbb{N}$ the derivatives $(f_n^{(k)})_n$ converge uniformly to the derivative $f^{(k)}$.

Definition 2.2 (Distributions). A linear functional $\xi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ is *continuous* if for every convergent sequence $(f_n)_n$ in $C_c^\infty(\mathbb{R})$, we have

$$\lim_{n\to\infty} \langle \xi, f_n \rangle = \langle \xi, \lim_{n\to\infty} f_n \rangle.$$

A continuous linear functional is also called a *distribution* of a *generalized function.*

Remark 2.3. One can indeed classify continuous functionals by their behaviour with respect to Cauchy sequences, according to the following exercise (to be proved in the future).

Exercise 2.4. *A linear operator between semi-normed spaces is continuous if and only if it maps Cauchy sequences to Cauchy sequences.*

Remark 2.5. As mentioned in the lecture, at the moment we make no distinction between the space generalized functions, which we denote $C^{-\infty}(\mathbb{R})$ and of distributions, as these spaces coincide over $\mathbb{R}$. We will discuss the difference between the two in later parts of the course, where they will be relevant.

Let $L^1_{\text{loc}}(\mathbb{R})$ denote the space of locally L^1-functions on $\mathbb{R}$, i.e. $f : \mathbb{R} \to \mathbb{R}$ such that $f \cdot \mathbf{1}_K \in L^1(\mathbb{R})$ for every compact set K, and recall that we have natural inclusions $C(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R}) \subseteq C^{-\infty}(\mathbb{R})$, where the final inclusion is via the map $f \mapsto \xi_f$, where $\xi_f(g) = \int_{\mathbb{R}} f(x)g(x)dx$.

Exercise 2.6. *Show that ξ_f is a well-defined distribution for all $f \in L^1_{\text{loc}}(\mathbb{R})$.*

Solution. The only non-obvious statement is continuity of ξ_f. Let $(g_n)_n$ be a convergent sequence in $C_c^\infty(\mathbb{R})$, with limit g and let $K \supseteq \operatorname{Supp}(g) \cup \bigcup_n \operatorname{Supp}(g_n)$ be compact, as in the definition. By uniform converges, there exists $n_0 \in \mathbb{N}$ such that $|g - g_n|_\infty < 1$ for all $n > n_0$. In particular, $|g_n| < |g| + I_K$ for all $n > n_0$, and $\int_{\mathbb{R}}(g + I_K)(x)dx = \int_{\mathbb{R}} g(x)dx + \operatorname{vol}(K) < \infty$.

Similarly, we have that $fg_n \leq fg + fI_K$, where the RHS is absolutely integrable, for all but finitely many n's, and, by Dominated Convergence, we have that

$$\lim_{n\to\infty} \xi_f(g_n) = \lim_{n\to\infty} \int_{\mathbb{R}} f(x)g_n(x)dx = \int_{\mathbb{R}} f(x)g(x)dx = \xi_f(g).$$

□

DEFINITION 2.7 (Weak convergence in L^1_{loc}). A sequence function $(f_n)_n$ in L^1_{loc} is said to converge weakly to f if for every $g \in C_c^\infty(\mathbb{R})$, $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)dx = \int_{\mathbb{R}} f(x)g(x)$. In other words, $\xi_{f_n}(g)$ tends to $\xi_f(g)$, for any $g \in C_c^\infty(\mathbb{R})$, as $n \to \infty$.

EXERCISE 2.8. *Find a sequence of functions $(f_n)_n \in C_c^\infty(\mathbb{R})$ which converges weakly to δ_0, the Dirac delta function at zero.*

SOLUTION. Let $\psi \in C_c^\infty(\mathbb{R})$ be a non-negative, non-zero function with $\operatorname{Supp}(\psi) = [-1,1]$ and $\|\psi\|_1 = 1$, and define $f_n(x) = n\psi(nx)$, for any $n \in \mathbb{N}$. Then

$$\int_{\mathbb{R}} f_n(x)g(x)dx = \int_{-1/n}^{1/n} n\psi(nx)g(x)dx = \int_{-1}^{1} \psi(x)g(x/n)dx,$$

and the RHS is bounded above an below by $\sup_{|x|\leq 1/n} g(x)$ and $\inf_{|x|\leq 1/n} g(x)$, respectively. Since g is continuous, they both tend to $g(0)$ as $n \to \infty$. □

EXERCISE 2.9. *Find a sequence of functions $(f_n)_n$ converging weakly to f, which does not converge pointwise to f.*

SOLUTION. Let ψ be a bump function, supported on $[-1,1]$ and with $\psi(0) = 1$, put $f_n(x) = \psi(nx)$, for any $n \in \mathbb{N}$. In this setting f_n is easily verified to converge weakly to the zero function, as

$$\int_{\mathbb{R}} f_n(x)g(x) = \int_{-1/n}^{1/n} \psi(nx)g)(x)dx = \frac{1}{n}\int_{-1}^{1} \psi(x)g(x/n)dx,$$

which tends to 0 as n tends to infinity. However $f_n(0) = 1$ for all n, so $f_n \not\to 0$ pointwise.
□

DEFINITION 2.10. A sequence $(\xi_n)_n$ of generalized functions converges weakly to $\xi \in C^{-\infty}(\mathbb{R})$ if $\lim_{n\to\infty}\langle \xi_n, f\rangle = \langle \xi, f\rangle$ for any $f \in C_c^\infty(\mathbb{R})$.

Note that, by definition, the topology of weak convergence is generated by the semi-norms of the form $\xi \mapsto |\xi(f)| : (C_c(\mathbb{R}))^* \to \mathbb{R}$, where f ranges over all elements of$(C_c^\infty(\mathbb{R}))$. In particular, it has a neighbourhood base of sets of the form

$$U_{\boldsymbol{f},\boldsymbol{\epsilon}}(\xi) = \{\nu : |\nu(f_i) - \xi(f_i)| < \epsilon_i \text{ for all } i = 1,\ldots,r\}, \tag{2.1}$$

where $\xi \in C^{-\infty}(\mathbb{R})$, $r \in \mathbb{N}$, $\boldsymbol{f} = (f_1,\ldots,f_r)$ is an r-tuple of elements of $C_c^\infty(\mathbb{R})$, and $\boldsymbol{\epsilon} = (\epsilon_1,\ldots,\epsilon_r) \in (\mathbb{R}_{>0})^r$.

EXERCISE 2.11. *The map $f \mapsto \xi_f : C_c^\infty(\mathbb{R}) \to C^{-\infty}(\mathbb{R})$ is injective with dense image with respect to the weak topology.*

SOLUTION. Injectivity of $f \mapsto \xi_f$ is equivalent to the statement that $\xi_f \equiv 0$ if and only if $f \equiv 0$, which is easily verified by applying ξ_f to test functions with small support around points where f is non-vanishing.

To prove that the image is dense, we require the following lemma, which we'll prove in greater generality later on in the course.

LEMMA 2.12. *Put $V = C_c^\infty(\mathbb{R})$ and let $W \subseteq V^*$ be a subspace. Then W is dense in V^* with respect to weak topology if and only if $W^\perp = \{v \in V : \langle w,v\rangle = 0$ for all $w \in W\} = \{0\}$.*

Here $\langle \xi, f\rangle := \xi(f)$ is the previously defined pairing $C_c^\infty(\mathbb{R}) \times (C_c^\infty(\mathbb{R}))^* \to \mathbb{R}$. We will prove the *only if* implication below, as it is the only relevant one. Using the Lemma, we only need to verify that the perpendicular space to $W = \{\xi_f : f \in C_c^\infty(\mathbb{R})\}$ is zero. This holds, since $g \in W^\perp$ implies that $\int_{\mathbb{R}} f(x)g(x)dx = 0$ for all $f \in C_c^\infty(\mathbb{R})$, and, in particular, for test functions with arbitrarily small supports around any point $x \in \mathbb{R}$.

Proof of lemma. By our description of the weak convergence topology, it suffices to show that any set of the form $U_{\boldsymbol{f},\boldsymbol{\epsilon}}(\xi)$ contains an element of the form ξ_g for some $g \in C_c^\infty(\mathbb{R})$. We will show something stronger; namely, given $\xi \in C^{-\infty}(\mathbb{R})$ and $S = \{f_1, \dots, f_r\} \subseteq C_c^\infty(\mathbb{R})$ a finite set, there exists $g \in C_c^\infty(\mathbb{R})$ such that $\xi \mid_S = \xi_g \mid_S$. Wlog, we may assume S is linearly independent, and define $\rho = \rho_S : (C_c^\infty(\mathbb{R}))^* \to \mathbb{R}^r$ by $\rho(\eta) = (\langle \eta, f_1\rangle, \dots, \langle \eta, f_r\rangle)$. Then $\rho \mid_W$ is surjective, otherwise it is contained in a hyperplane (i.e. a one codimensional space) of the form $\{\sum_{i=1}^r c_i x_i = 0\}$ for some fixed $c_1, \dots, c_r$, not all zero. But then $(c_1, \dots, c_r) \cdot \rho(w) = \sum_{i=1}^r c_i \langle w, f_i\rangle = \langle w, \sum_{i=1}^r c_i f_i\rangle = 0$, for all $i = 1, \dots, r$, which implies $0 \neq \sum c_i f_i \in W^\perp$, a contradiction. In particular, as $\rho \mid_W$ is surjective, we there exists $g \in C_c^\infty(\mathbb{R})$ with $\xi_g \in W$ such that $\rho(\xi_g) = \rho(\xi)$, as required.

□

2.2. Distributional derivatives.

Definition 2.13. The derivative of a distribution ξ is defined via the rule $\langle \xi', f\rangle = -\langle \xi, f'\rangle$.

In the case where $\xi = \xi_f$ for $f \in C^\infty(\mathbb{R})$, we have

$$\langle \xi_f', g\rangle = \int_{\mathbb{R}} f'(x)g(x)dx = [f(x)g(x)]_{x=-\infty}^{\infty} - \int_{\mathbb{R}} f(x)g'(x)dx = -\langle \xi_{f'}, g\rangle.$$

Note that we do not require f to have compact support.

Exercise 2.14. *Find a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ whose distributional derivative is δ_0.*

Solution. Note that, for any $g \in C_c^\infty(\mathbb{R})$, we have that

$$\langle \delta_0, g\rangle = g(0) = g(0) - \lim_{x\to\infty} g(x) = -\int_0^\infty g'(x)dx,$$

by the fundamental theorem of calculus. Writing $H = \mathbf{1}_{\{x\geq 0\}}$ for the indicator function of the non-negative half axis (that is- the Heaviside function), we deduce that

$$\langle \delta_0, g\rangle = -\langle \xi_H, g'\rangle = \langle \xi_H', g\rangle,$$

as wanted. □

Exercise 2.15. *Compute the derivatives of* $|\sin(x)|$, $|x|\sin(x)$ *and* $\tanh(1/x)$*, considered as distributions.*

Solution. Let $g \in C_c^\infty(\mathbb{R})$ be arbitrary.

$$\begin{aligned}\langle \xi'_{|\sin|}, g\rangle &= -\int_{\mathbb{R}} |\sin(x)|\, g'(x)dx \\ &= \sum_{n\in\mathbb{Z}} \left(-\int_{2n\pi}^{(2n+1)\pi} \sin(x)g'(x)dx + \int_{(2n+1)\pi}^{(2n+2)\pi} \sin(x)g'(x)dx \right) \\ &= \int_{\mathbb{R}} \cos(x)\mathrm{sgn}(\sin(x))g(x)dx = \langle \xi_{\cos(x)\mathrm{sgn}(\sin(x))}, g\rangle\end{aligned}$$

where $\mathrm{sgn}(y) = \frac{y}{|y|}$ is the sign function. Thus $(|\sin|)' = \cos(x)\mathrm{sgn}(\sin(x))$.

$$
\begin{aligned}
\langle \xi'_{|x|\sin(x)}, g\rangle = -\int_{\mathbb{R}} |x|\sin(x)g'(x)dx &= \int_{-\infty}^{0} x\sin(x)g'(x)dx - \int_{0}^{\infty} x\sin(x)g'(x)dx \\
= -\int_{-\infty}^{0}(\sin(x)+x\cos(x))g(x)dx &+ \int_{0}^{\infty}(\sin(x)+x\cos(x))g(x)dx \\
&= \int_{\mathbb{R}}(\sin|x| + |x|\cos(x))g(x)dx
\end{aligned}
$$

Thus $(|x|\sin(x))' = \sin|x| + |x|\cos(x)$. Note that the function in this case is C^1 on $\mathbb{R}$, so the computation above is in fact unnecessary.

$$
\begin{aligned}
\langle \xi'_{\tanh(1/x)}, g\rangle &= -\int_{\mathbb{R}} \tanh(1/x)g'(x)dx \\
&= -\left(\int_{0}^{\infty}\tanh(1/x)g'(x)dx + \int_{-\infty}^{0}\tanh(1/x)g'(x)dx\right) \\
&= -\left(-\lim_{x\to 0^+}\tanh(1/x)g(x) + \lim_{x\to 0^-}\tan(1/x)g(x) + \int_{\mathbb{R}}\frac{1}{x^2\cosh(1/x)}g(x)dx\right) \\
&= \langle 2\delta_0 - \xi_{(x^2\cosh(1/x)^2)^{-1}}, g\rangle
\end{aligned}
$$

and $(\tanh(1/x))' = 2\delta_0 + \frac{1}{x^2\cosh(1/x)^2}$. □

EXERCISE 2.16. *Let $\xi \in C^{-\infty}(\mathbb{R})$. Show that $\xi' = 0$ if and only if is of the form $\langle \xi, g\rangle = \int_{\mathbb{R}} ag(x)dx$ for $a \in \mathbb{R}$.*

SOLUTION.

$\Leftarrow$ Clear.

$$\langle \xi', g\rangle = -\int_{\mathbb{R}} ag'(x)dx = [ag(x)]_{x=-\infty}^{\infty} = 0.$$

$\Rightarrow$ Let $f \in C_c^\infty(\mathbb{R})$ and let $\psi \in C_c^\infty(\mathbb{R})$ be a test function with $\int_{\mathbb{R}}\psi(x)dx = 1$. Put $g = f - \psi \cdot \int_{\mathbb{R}} f(x)dx$. Then $g \in C_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} g(x)dx = 0$, and hence $G(x) = \int_{-\infty}^{x} g(t)dt$ is an anti-derivative of $g(x)$ in $C_c^\infty(\mathbb{R})$. Now, we have that

$$\langle \xi, f\rangle = \langle \xi, g\rangle + \int_{\mathbb{R}} f(x)dx \cdot \langle \xi, \psi\rangle = \xi(G') + \langle \xi_{\langle \xi,\psi\rangle}, f\rangle = \langle \xi_{\langle \xi,\psi\rangle}, f\rangle.$$

□

REMARK 2.17. Note that the proof of Exercise 2.16 extends verbatim to the case where $\xi' \mid_U \equiv 0$ for an open interval $U \subseteq \mathbb{R}$, implying that in this situation $\xi(g) = \int_{\mathbb{R}} ag(x)dx$ for all $g \in C_c^\infty(U)$.

2.3. Support of a generalized function.

DEFINITION 2.18 (Support of a generalized function). Given $\xi \in C^{-\infty}$ and $U \subseteq \mathbb{R}$ open, we say that $\xi \mid_U \equiv 0$ for all $f \in C_c^\infty(U)$. The *support* of ξ is defined to be $\mathrm{Supp}(\xi) = \bigcap_{\xi|_{D_\beta^c}\equiv 0} D_\beta$, where the D_β are taken to be closed.

Equivalently, $\mathrm{Supp}(\xi)$ is the complement of the largest open set on which ξ vanishes, and is, in particular, closed.

Exercise 2.19. *Prove the identity axiom of $C^{-\infty}(\mathbb{R})$, i.e. for every $\xi \in C^{-\infty}(\mathbb{R})$, if there exists an open cover $\{U_i\}_{i\in I}$ of $\mathbb{R}$ such that $\xi|_{U_i} \equiv 0$ for all i, then $\xi = 0$.*

Solution. Let $f \in C_c^\infty(\mathbb{R})$. By an exercise from the previous tutorial (smooth partition of unity), we may choose $i_1, \dots, i_r \in I$ such that $\mathrm{Supp}(f)$ is covered by $\bigcup_{j=1}^r U_{i_j}$ and find functions $f_1, \dots, f_r \in C_c^\infty(\mathbb{R})$ with $\mathrm{Supp}(f_j) \subseteq U_{i_j}$ such that $f = \sum_{j=1}^r f_j$. Since this is a finite sum, we have

$$\langle \xi, f \rangle = \sum_{j=1}^r \langle \xi, f_j \rangle = 0$$

□

Exercise 2.20. *Show that*

$$\mathrm{Supp}(\xi) \setminus \mathrm{Supp}(\xi)^\circ \subseteq \mathrm{Supp}(\xi') \subseteq \mathrm{Supp}(\xi),$$

for any $\xi \in C^{-\infty}(\mathbb{R})$.

Solution. The second inclusion is obvious: given $U \subseteq \mathrm{Supp}(\xi)^c$, and $f \in C_c^\infty(U)$, we have that $f' \in C_c^\infty(U)$ as well, and $\xi'(f) = -\xi(f') = 0$, implying that $\xi'|_U \equiv 0$.

For the first inclusion, let $U \subseteq \mathrm{Supp}(\xi')^c$ be an open *interval*. By Exercise 2.16, as $\xi'|_U \equiv 0$, there exists $a \in \mathbb{R}$ such that $\xi'|_U \equiv \int_{\mathbb{R}} a g(x)dx$, for all $g \in C_c^\infty(U)$. If $a = 0$ then $U \subseteq \mathrm{Supp}(\xi)^c$. Otherwise, if $a \neq 0$, by evaluating ξ on a positive test function, we have that $U \subseteq \mathrm{Supp}(\xi)^\circ$ (since U is open). Thus $\mathrm{Supp}(\xi')^c \subseteq \mathrm{Supp}(\xi)^c \cup \mathrm{Supp}(\xi)^\circ$, as required. □

Proposition 2.21. *The space of generalized functions with support $\{0\}$ is spanned by the set of derivatives of δ_0.*

The proposition follows from the following two lemmas, the first of which is proved in the home-exercises.

Lemma 2.22. *Let ξ be a generalized function with support $\{0\}$. Then there exists $k \in \mathbb{N}$ such that $\xi x^k = 0$.*

Proof. Home exercise. □

Lemma 2.23. *Assume $\xi x^k = 0$ for some $k \in \mathbb{N}$. Then $\xi = \sum_{i=0}^{k-1} c_i \delta_0^{(i)}$ for some $c_i \in \mathbb{R}$.*

Proof. We argue by induction on k. The case $k = 0$ is trivial, but it is instructive to consider the case $k = 1$ before proceeding.

Note that, for any $f \in C_c^\infty(\mathbb{R})$, we have the following:

$$f(x) - f(0) = \int_0^x f'(t)dt = x \cdot \int_0^1 f'(xt)dt. \tag{2.2}$$

If the function $x \mapsto \int_0^1 f(xt)dt$ were of compact support the lemma would easily follow. However, this is rarely the case. Let $\psi \in C_c^\infty(\mathbb{R})$ have $\psi(0) = 1$. Applying (2.2) twice, we have:

$$f(x) = f(0)\psi(0) + \int_0^1 f'(xt)dt = f(0)\psi(x) - x \underbrace{\left(f(0) \int_0^1 \psi'(xt)dt - \int_0^1 f'(xt) \right)}_{\star}.$$

Note that the expression $(\star)$ is smooth and compactly supported, as it may be rewritten as

$$(\star) = f(0) \cdot \frac{\psi(x) - \psi(0)}{x} - \frac{f(x) - f(0)}{x} = \frac{1}{x}\left(f(0)\psi(x) - f(x)\right).$$

Thus, we have that

$$\langle \xi, f \rangle = f(0)\langle \xi, \psi \rangle + \underbrace{\langle \xi, x \cdot (\star) \rangle}_{=0 \text{ by assumption}} .$$

Finally, note that $\langle \xi, \psi_1 \rangle = \langle \xi, \psi_2 \rangle$ for any two test functions with $\psi_1(0) = \psi_2(0) = 1$. Indeed, using (2.2),

$$\langle \xi, \psi_1 - \psi_2 \rangle = \langle \xi, x \int_0^1 (\psi_1'(xt) - \psi_2(xt) dt \rangle = 0,$$

using the same argument that $\int_0^1 (\psi_1'(xt) - \psi_2'(xt))dt \in C_c^\infty(\mathbb{R})$. Thus we may take $c_0 = \langle \xi, \psi \rangle$

Now for the induction step. By assumption $\xi x^{k+1} = (\xi x)x^k = 0$, and, using the induction hypothesis, $\xi x = \sum_{i=0}^{k-1} c_i \delta^{(i)}$. Using the same formula as above, for $f \in C_c^\infty(\mathbb{R})$, we have that

$$\langle \xi, f \rangle = f(0)\langle \xi, \psi \rangle + \langle \xi x, (\star) \rangle = f(0)\langle \xi, \psi \rangle + \sum_{i=0}^{k} c_i \langle \delta^{(i)}, (\star) \rangle$$

Using our above expansion of $(\star)$, and the explicit description of $\delta^{(i)}$ as $\langle \delta^{(i)}, g \rangle = (-1)^i g^{(i)}(0)$, by taking ψ have at least k zero derivatives at 0, we easily verify that

$$\langle \delta^{(i)}, (\star) \rangle = (-1)^i f^{(i+1)}(0) = -\langle \delta^{(i+1)}, f \rangle.$$

Using a similar argument to the induction step, noting that $\langle \xi, \psi \rangle$ is independent of the choice of step function with sufficiently many vanishing derivatives, the lemma follows. $\square$

2.4. Convolution and product of generalized functions.

Notation 2.24. Given a function $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, we write $\bar{f}$ for the function $\overline{f}(x) = f(-x)$ and $L_t f$ for the function $L_t f(x) = f(x + t)$.

Definition 2.25. Given $f \in C_c^\infty(\mathbb{R})$ and $\xi \in C^{-\infty}(\mathbb{R})$, we define the convolution to be the *function* $\xi * f(t) = \langle \xi, \overline{L_{-t} f} \rangle$.

Note that, for $\xi = \xi_g$, this coincides with the ordinary definition

$$\xi_g * f(t) = \int_{\mathbb{R}} g(x)\overline{L_{-t} f}(x) dx = \int_{\mathbb{R}} g(x) f(t - x) dx = g * f(t).$$

Exercise 2.26. *Given $f \in C_c^\infty(\mathbb{R})$ and $\xi \in C^{-\infty}(\mathbb{R})$, show that $\xi * f$ is a smooth function.*

Solution. Note that, for any $g \in C_c^\infty(\mathbb{R})$, the limit

$$\lim_{\epsilon \to 0} \frac{L_\epsilon g - g}{\epsilon} = g'$$

is with respect to the topology of C_c^∞. Indeed, we may restrict to $0 < |\epsilon| \leq 1$, and have a common compact set supporting all functions in this net. In particular, for any $t \in \mathbb{R}$, we have that

$$(\xi * f)'(t) = \lim_{\epsilon \to 0} \frac{\xi * f(t+\epsilon) - \xi * f(t)}{\epsilon}$$
$$= \lim_{\epsilon \to 0} \langle \xi, \frac{L_\epsilon \overline{L_{-t} f} - \overline{L_{-t} f}}{\epsilon} \rangle$$
$$= \langle \xi, (\overline{L_{-t} f})' \rangle = \langle \xi, \overline{L_{-t}(f')} \rangle = \xi * f'(t).$$

In particular, if f is k-times differentiable at t, then so is $\xi * f$. □

DEFINITION 2.27 (Convolution of distributions). Given two *compactly supported* distributions ξ_1, ξ_2, define $\langle \xi_1 * \xi_2, f \rangle = \langle \xi_1, \overline{\xi_2 * \overline{f}} \rangle$.

EXERCISE 2.28. *Prove the following identities for ξ, ξ_1, ξ_2, ξ_3 compactly supported distributions.*

(1) $\delta_0 * \xi = \xi * \delta_0 = \xi$;
(2) $\delta_0' * \xi = \xi'$;
(3) $\xi_1 * \xi_2 = \xi_2 * \xi_1$;
(4) $\xi_1 * (\xi_2 * \xi_3) = (\xi_1 * \xi_2) * \xi_3$;
(5) $(\xi_1 * \xi_2)' = \xi_1 * \xi_2' = \xi_1' * \xi_2$.

SOLUTION.

(1) $\langle \delta_0 * \xi, f \rangle = \langle \delta_0, \overline{\xi * \overline{f}} \rangle = \overline{\xi * \overline{f}}(0) = \xi * \overline{f}(0) = \langle \xi, L_0 f \rangle = \langle \xi, f \rangle$.
For the second equality, we note that
$$\overline{\delta_0 * \overline{f}}(t) = \langle \delta_0, \overline{L_t \overline{f}} \rangle = \overline{L_t \overline{f}}(0) = \overline{f}(-t) = f(t).$$
Thus $\langle \xi * \delta_0, f \rangle = \langle \xi, \widehat{\delta_0 * \overline{f}} \rangle = \langle \xi, f \rangle$.

(2) $\langle \delta_0' * \xi, f \rangle = -(\overline{\xi * \overline{f}})'(0) = -\xi * (\overline{f})'(0) = -\langle \xi, f' \rangle = \langle \xi', f \rangle$.

(3) Let $(\eta_n)_n$ be an approximation of identity.

$$\langle \xi_1 * \xi_2, f \rangle = \langle \delta_0 * (\xi_1 * \delta_0), \overline{\xi_2 * \overline{f}} \rangle = \langle \delta_0, \overline{(\xi * \delta_0) * (\xi_2 * \overline{f})} \rangle$$
$$= \langle \delta_0, \overline{\lim_{n\to\infty} (\xi_1 * \eta_n) * (\xi_2 * \overline{f})} \rangle = \langle \delta_0, \overline{\lim_{n\to\infty} (\xi_2 * \overline{f}) * (\xi_1 * \eta_n)} \rangle$$
$$= \langle \delta_0, \overline{\lim_{n\to\infty} \xi_2 * (\overline{f} * (\xi_1 * \eta_n))} \rangle = \langle \delta_0, \overline{\lim_{n\to\infty} \xi_2 * ((\xi_1 * \eta_n) * \overline{f})} \rangle$$
$$= \langle \delta_0, \overline{\xi_2 * ((\xi_1 * \delta_0) * \overline{f})} \rangle = \langle \delta_0, \overline{(\xi_2 * \xi_1) * \overline{f}} \rangle = \langle \xi_2 * \xi_1, f \rangle.$$

(4) We first note that $(\xi_1 * \xi_2) * f = \xi_1 * (\xi_2 * f)$. This may be verified explicitly:
$$((\xi_1 * \xi_2) * f)(t) = \langle \xi_1 * \xi_2, \overline{L_t f} \rangle = \langle \xi_1, \overline{\xi_2 * L_t f} \rangle,$$
and $\xi_1 * (\xi_2 * f)(t) = \langle \xi_1, \overline{L_t(\xi_2 * f)} \rangle$. The equality follows since
$$(\xi_2 * L_t f)(s) = \langle \xi_2, \overline{L_s L_t f} \rangle = \langle \xi_2, \overline{L_{s+t} f} \rangle = (\xi_2 * f))(s+t) = L_t(\xi_2 * f)(s).$$
Associativity follows from

$$\langle \xi_1 * (\xi_2 * \xi_3), f \rangle = \langle \xi_1, \overline{(\xi_2 * \xi_3) * \overline{f}} \rangle = \langle \xi_1, \overline{\xi_2 * (\xi_3 * \overline{f})} \rangle$$
$$= \langle \xi_1 * \xi_2, \overline{\xi_3 * \overline{f}} \rangle = \langle (\xi_1 * \xi_2) * \xi_3, f \rangle.$$

(5) Follows from (2),(3) and (4).

□