Generalized functions
Tutorial notes

Tutorial 3

3.1. Complements from previous tutorial. At the beginning of the tutorial, we discussed two subjects which we did not manage to cover in the previous tutorial. These subjects were:

- distributions on $\mathbb{R}$ with support $\{0\}$; and
- convolution of distributions.

For the full discussion, see Tutorial 2 notes.

3.2. Topological vector spaces.

Definition 3.1. A topological vector space V is a vector space over a topological field F (i.e. a field with a topology under which the field operations are continuous) such that $+ : V \times V \to V$ and $\cdot : F \times V \to V$ are continuous.

Exercise 3.2. *Prove that a topological vector space is Hausdorff iff $\{0\}$ is a closed set.*

Solution.

$\Leftarrow$ Clear; in a Hausdorff space all points are closed.

$\Rightarrow$ Consider the set $\Delta = \{(x, x) : x \in V\}$. Then $\Delta = f^{-1}(\{0\})$, for $f(u, v) = u - v$, which is closed whenever $\{0\}$ is closed, since f is continuous. Let $u, v \in V$ be two distinct points. Then $(u, v) \notin \Delta$ and hence there exists $(u, v) \in W \subseteq V \times V$ open which is disjoint from Δ. Since the topology on $V \times V$ is generated by boxes, i.e. sets of the form $U_1 \times U_2$ For $U_1, U_2 \subseteq V$ open, we have that $(u, v) \in U_1 \times U_2 \subseteq W$. Hausdorffness follows, since $U_1 \times U_2 \cap \Delta = \varnothing$ is equivalent to $U_1 \cap U_2 = \varnothing$.

□

Definition 3.3 (Local convexity etc). Let V be a topological vector space over $F = \mathbb{R}$ or $\mathbb{C}$.

(1) A set $A \subseteq V$ is *convex* is $\lambda A + (1 - \lambda)A \subseteq A$ for all $\lambda \in [0, 1]$.
(2) V is said to be *locally convex* if its topology can be generated by open convex sets.
(3) A set $W \subseteq V$ is said to be *balanced* if $\lambda W \subseteq W$ whenever $|\lambda| \leq 1$ and $\lambda \in F$.
(4) Given a (balanced open convex) set $C \ni 0$, one defines

$$N_C(x) := \inf \{\alpha \in \mathbb{R}_{\geq 0} : x \in \alpha C\} \quad (x \in V).$$

(5) A set C is said to be *absorbent* if $N_C(x) < \infty$ for all $x \in V$.

Exercise 3.4. *Find a topological vector space V which is not locally convex.*

Solution. Note that if V is normed, or, more generally equipped with a translation invariant metric d such that $d(\lambda x, \lambda y) \leq |\lambda|\, d(x,y)$ for all $x, y \in V$ and $|\lambda| < 1$, then any open ball around 0 in V is convex and so V is locally convex.

Let $V = \ell^{1/2}(\mathbb{R}) = \left\{ (x_i)_{i=1}^{\infty} : x_i \in \mathbb{R},\ \sum \sqrt{|x_i|} < \infty \right\}$, equipped with the topology induced from the metric $d((x_i),(y_i)) = \sum_i \sqrt{|x_i - y_i|}$. Consider the open ball $B_1(0)$ of radius 1 around 0. Note that $B_1(0)$ is *not* convex, e.g. for $x = (1/2, 0, 0, \ldots), y = (0, 1/2, 0, 0, \ldots)$ we have that $d(x,0) = d(y,0) = 1/\sqrt{2} < 1$, but $d(\frac{1}{2}x + \frac{1}{2}y, 0) = \sqrt{1/4} + \sqrt{1/4} = 1$.

Assume towards a contradiction that V *is* is locally convex. Since $B_1(0)$ is open, it contains a convex open subset $0 \in C \subseteq B_1(0)$, which, in turn, contains a smaller open ball $B_\epsilon(0)$ around zero. Since C is convex, it follows that any convex combination of elements of $B_\epsilon(0)$ must be included in C. On the other hand, if we take $x_n = (x_n^i)_{i=1}^{\infty}$, defined by $x_n^i = \epsilon^2$ if $i = n$ and 0 otherwise, then, for any $n \in \mathbb{N}$,

$$d(\sum_{i=1}^{n} \frac{1}{n} x_n^i, 0) = \sum_{i=1}^{n} \frac{\epsilon}{\sqrt{n}} = \sqrt{n}\epsilon,$$

which tends to infinity as n grows, and , in particular, eventually escapes the ball $B_1(0)$. $\square$

Definition 3.5 (Seminorm). A *seminorm* on a topological vector space is a function $\eta : V \to \mathbb{R}$ which satisfyies the triangle inequality, homogeneity and non-negativity axioms, but such that $\eta(v) = 0$ may be possible for $v \neq 0$.

Exercise 3.6. *Let C be an open convex neighborhood of 0 in a tvs V over $\mathbb{R}$.*

(1) *Show that C is absorbent.*
(2) *Show that if C is further assumed to be balanced than N_C is a seminorm.*

Solution.

(1) Let $v \in V$ be arbitrary. Consider the set $\widetilde{C} = \{(\lambda, u) : \mathbb{R} \times V : \lambda u \in C\}$. This is just the preimage of C under scalar multiplication, and hence is open in $F \times V$. Also, it clearly contains $(0, v)$ (and, more generally, $(0, u)$ for all $u \in V$). In particular, there exist $0 \in U_1 \subseteq \mathbb{R}$ and $v \in U_2 \subseteq V$ open such that $(0, v) \in U_1 \times U_2 \subseteq \widetilde{C}$. Since U_1 is open in $\mathbb{R}$ it contains non-zero elements, and there exists $\lambda \neq 0$ such that $(\lambda, v) \in \widetilde{C}$, and so $\lambda v \in C$ and $v \in (\lambda^{-1})C$, as wanted.
(2) Home exercise.

$\square$

Theorem 3.7 (Hahn-Banach). *Let V be a* normed *vector space and $W \subseteq V$ a subspace with $f : W \to \mathbb{R}$ a bounded linear functional (i.e. such that $\|f\| = \sup_{x \in W, \|x\|=1} \|f(x)\| < \infty$ for some $C > 0$ and for all $x \in W$). Then there exists a linear functional $\widetilde{f} : V \to \mathbb{R}$ such that $\widetilde{f}\,|_W = f$ and $\left\|\widetilde{f}\right\| = \|f\|$.*

Exercise 3.8. *Let V be a locally convex topological vector space and W a closed linear functional. Show that any continuous linear functional $f : W \to \mathbb{R}$ can be extended to V.*

Solution. Let $f : W \to \mathbb{R}$ be a continuous linear functional. By continuity and the definition of the induced topology, $f^{-1}(-1,1) = A \cap W$ for some open set A. By local convexity of V, we have that A contains an open convex set $C \ni 0$, which we can further assume to be balanced, by the home exercise. One easily verifies that $|f(x)| \leq N_C(x)$ for all $x \in W$; indeed $\frac{x}{N_C(x)+\epsilon} \in C \subseteq f^{-1}(-1,1)$, for all $\epsilon > 0$. Note that $U := \mathrm{Ker}(N_C) = \{x \in V : N_C(x) = 0\}$ is a closed[1] linear subspace of V; indeed, for any $x, y \in U$, $\lambda \in F$, $N_C(\lambda x) = |\lambda| N_C(x) = 0$ and $0 \leq N_C(x+y) \leq N_C(x) + N_c(y) = 0$. Furthermore, $f(U) = 0$, since $|f(x)| \leq N_C(x) = 0$ for all $x \in U$. In particular, f reduces to a linear functional on $\overline{W} = W/U \subseteq V/U = \overline{V}$. Finally, we note that N_C reduces to a *norm* on $\overline{V}$, and hence we can apply Hahn-Banach to extend the map induced from f on $\overline{W}$ to $\overline{V}$, and then pull back to an extension $\widetilde{f}$ of f to V. $\square$

COROLLARY 3.9. *Given a locally convex vector space V with a closed subspace W, the restriction maps $V^\sharp \to W^\sharp$ and $V^* \to W^*$ are surjective (here $V^\sharp$ is the abstract dual* $\mathrm{Hom}(V, \mathbb{R})$*, consisting of* all *linear maps $V \to \mathbb{R}$).*

3.3. Complete and sequentially complete topological vector spaces.

DEFINITION 3.10. Let V be a topological vector space.

1. A sequence $\{v_n\}_{n=1}^\infty$ in V is Cauchy if for every neighbourhood U of 0, there exists $n_0 \in \mathbb{N}$ such that $v_m - v_n \in U$ for any $m, n \in \mathbb{N}$.
2. A sequence $\{v_n\}_{n=1}^\infty$ is said converge to $v \in V$ if for every $0 \in U$ open, there exists $n_0 \in \mathbb{N}$ such that $v_n - v \in U$ for all $n > n_0$.
3. V is called *sequentially complete* if all Cauchy sequences converge to some limit in V;
4. V is said to be complete if for every $\phi : V \to W$ which maps V homeomorphically onto $\phi(V)$, the set $\phi(V)$ is closed in W.

EXERCISE 3.11. *Find a topological vector space which is complete sequentially complete but not complete.*

SOLUTION. Let $V = \mathbb{R}^\mathbb{R} = \{f : \mathbb{R} \to \mathbb{R}\}$ have the product topology ($\mathbb{R}$ is endowed with the standard topology), and let $U = \{f : \mathbb{R} \to \mathbb{R} \mid |\{x : f(x) \neq 0\}| \leq \aleph_0\}$. Given a Cauchy sequence $(f_n) \in U$, since coordinate projections are continuous in the product topology, the sequence $f_n(x)$ is Cauchy in $\mathbb{R}$ for all $x \in \mathbb{R}$. We may define $f(x) = \lim_{\to\infty} f_n(x)$, and this is clearly an element of U, since it can have at most countably many non-zero values. Also, using the definition of the product topology, one easily verifies that f_n converges to f in V. Thus U is sequentially complete.

Also, from the definition of the product topology, one has that U is dense in V, and doe not equal it, so it is not complete. $\square$

REMARK 3.12. Another important example of a sequentially complete but not complete space is the image of $C_c^\infty(\mathbb{R})$ in $C^{-\infty}(\mathbb{R})$, under the map $f \mapsto \xi_f$ (where $\langle \xi_f, g \rangle = \int_\mathbb{R} f(x)g(x)dx$), with respect to discrete topology on $C^{-\infty}(\mathbb{R})$. The proof of this fact, which relies on the Banach-Steinhaus Theorem, will appear in the next tutorial.

We also have the following universal description of the completion of V:

EXERCISE 3.13. *Let V be a topological vector space and $\iota : V \to \bar{V}$ be an embedding into another tvs. Prove that the following are equivalent:*

1. *$\iota(V) \simeq V$ and $\mathrm{cl}\,(\iota(V)) = \bar{V}$; and*
2. *For every complete space W and $f : V \to W$ there exists a unique map $\varphi_W : \bar{V} \to W$ such that $f = \varphi_W \circ \iota$.*

[1] Verify that you see why U is closed.

Finally, using Cauchy filters or Cauchy nets, one can construct the completion of a topological vector space explicitly, and prove:

EXERCISE 3.14. *The completion of a tvs V exists and is unique up to unique isomorphism.*