Generalized functions
Tutorial notes

Tutorial 6 - Absolute values, P-adics and tdlc spaces

6.1. Absolute values.

DEFINITION 6.1.1. A function $|\cdot|: \mathbb{Q} \to \mathbb{R}_+$ is called an absolute values, if for any $x, y \in \mathbb{Q}$

- $|x| = 0 \iff x = 0$
- $|xy| = |x||y|$
- $|x+y| \leq |x| + |y|$

EXAMPLE 6.1.1. Let $0 < \alpha \leq 1$ then $|\cdot|^\alpha := |x|^\alpha$ is an absolute value on $\mathbb{C}$.
(Hint: taking log changes sign).

EXAMPLE 6.1.2. A discrete absolute value $|x| = 1 \iff x \neq 0$

DEFINITION 6.1.2. A absolute value is said to be non-archimendian if $|x+y| \leq \max\{|x|, |y|\}$

EXAMPLE 6.1.3. Let p be a prime and $q \in \mathbb{Q}$ s.t. $q = p^n \frac{a}{b}$ with $(a,b) = 1,\ n \in \mathbb{Z}$ we set $|q|_p := p^{-n}$ (and 0 if $q = 0$).
(also the discrete one is non-archimendian).

LEMMA 6.1.1. TFAE

(1) $|\cdot|$ is non archimendian.
(2) $\exists n > 1$ s.t. $|n| \leq 1$.
(3) $\forall n \in \mathbb{Z}\ |n| \leq 1$

PROOF. (1)$\Rightarrow$ (2): Take $n = 2$ and observe that $|2| = |1+1| = \max\{|1|, |1|\} = 1$
(2)$\Rightarrow$ (3): Let $m > 1$ with $|m| \leq 1$, and take any $n \in \mathbb{N}$ writing in base m, set $r = \lfloor \log_m(n) \rfloor$ we get

$$|n| = |\sum_{i=0}^{r} a_i m^i| \leq_{t.e} \sum_{i=0}^{r} |a_i m^i| = \sum_{i=0}^{r} |a_i||m|^i \leq \sum_{i=0}^{r} |a_i| \leq (r+1)m$$

(since $|a_i| \leq a_i$ by t.e).
So we got that $|n| \leq (\log_m(n) + 1)m$ hence for n^k we have

$$|n|^k = |n^k| \leq (\log_m(n^k) + 1)m = (k \log_m(n) + 1)m$$

by taking limit on k to infinity we obtain

$$|n| = (|n|^k)^{1/k} \leq ((k \log_m(n) + 1)m)^{1/k} \to 1$$

(3)$\Rightarrow$ (1):

$$|x+y|^n = |(x+y)^n| = |\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}| \leq \sum_{i=0}^{n} |\binom{n}{i}|(\max\{|x|,|y|\})^n = (n+1)(\max\{|x|,|y|\})^n$$

hence taking the limit

$$|x+y| \leq (n+1)^{1/n} \max\{|x|,|y|\} \to \max\{|x|,|y|\}$$

$\square$

COROLLARY 6.1.1. The absolute value is archemedian if and only if $|2| > 1$.

DEFINITION 6.1.3. We say two absolute values $|\cdot|_1$, $|\cdot|_2$ are equivalent if there exist $\alpha > 0$ s.t. $|\cdot|_1 = |\cdot|_2^\alpha$

THEOREM 6.1.1 (Ostrowki). Up to equivalence, the only absolute values on $\mathbb{Q}$ are the usual one, the p-adics, the discrete.
You will prove this via a guided exercise, note that by the theorem there is just one archemedian absolute value on $\mathbb{Q}$, actually we show that this holds for $\mathbb{C}$.

REMARK 6.1.1. We may define a absolute value on a integral domain an we get a absolute value on the field of fractions.

LEMMA 6.1.2. In the case of non-arcemedian norm a_n is Cauchy if and only if for any ϵ there is some N s.t. for any $N < n$, $|a_n - a_{n+1}| \leq \epsilon$

PROOF. If Cauchy then of course the right hand side follows. And in the other direction for $n < m$ notice that

$$|a_n - a_m| = |a_n - a_{n+1} + a_{n+1} \pm ..a_{m-1} - a_m| \leq \max\{a_i - a_{i+1}\}_{i=n}^{m-1}$$

$\square$

6.2. p-adic field. We saw that for any p prime there is a non archemedian a.v. $|\cdot|_p$ on $\mathbb{Q}$.

LEMMA 6.2.1. The metric induced by $|\cdot|_p$ is not complete.

PROOF. Any complete field is uncountable By Baire category. $\square$

DEFINITION 6.2.1. The completion of $\mathbb{Q}$ wrt the $|\cdot|_p$ is denoted by $\mathbb{Q}_p$

LEMMA 6.2.2. In $\mathbb{Q}_p$ the sequence $f_n = \sum_{i=0}^{n} a_i$ converges if and only if $|a_n|_p \to 0$

PROOF. Prooof of the non trivial direction, assume $|a_n|_p \to 0$ then for any ϵ there is n_0 s.t. $|a_n|_p \leq p^{-m} < \epsilon$ for any $n_0 < n$ thus for any $n_0 < n$ we get

$$|f_n - f_{n+1}| = |a_{n+1}| < \epsilon$$

□

How to write down p-adic numbers:

THEOREM 6.2.1. Any $a \in \mathbb{Q}_p$ with $|a| \leq 1$ has unique representative C.S. $\{a_i\}$ such that for any i

(1) $0 \leq a_i \leq p^{i+1}$.
(2) $a_i \cong a_{i+1}$ $(mod\ p^{i+1})$

COROLLARY 6.2.1. Any $|a|_p \leq 1$ can be written as $a = \sum_{i=0}^{\infty} b_i p^i$.

COROLLARY 6.2.2. If $|a|_p > 1$ then there is p^m s.t. $|p^m a|_p \leq 1$ hence $a = \sum_{i=-m}^{\infty} b_i p^i$

THEOREM 6.2.2. Balles are disjoint or coincide

PROOF. Exercise. □

LEMMA 6.2.3. Any ultrametric absolute value on a field induces a totally disconnected topology, i.e. each closed ball is open.

PROOF. Leave as exercise □

REMARK 6.2.1. Properties of $\mathbb{Q}_p$:

- $\mathbb{Q}_p$ is locally compact. (Since $\mathbb{Z}_p$ is compact - we will see this next section-)
- $|\mathbb{Q}_p$ is totally disconected (general argument for non-archemedian metric)
- $\mathbb{Q}_p$ non algebraically closed. (for example $p = 5$ we get that $\sqrt{7} \notin \mathbb{Q}_5$ since $|7|_5 = 1$ so $|\sqrt{7}|_5 = 1$ but no integer is a square root mod 5)

DEFINITION 6.2.2. $\mathbb{Z}_p = \{a \in \mathbb{Q}_p \ : \ |a| < 1\}$, it is a ring.

6.3. Inverse limits. Given a directed-poset (called "inverse sysltem) we we construct the limit of this system.
If we have a system of objects $\{A_i\}$ and morphsims $f_j^i : A_i \to A_j$ for any $i \leq j$ together with $f_k^j \circ f_j^i = f_k^i$ this system of objects is called an inverse system. We assume for simplicity that our system is over $\mathbb{N}$. we get the limit object

$$A = \lim_{\leftarrow} A_i = \{\vec{a} \in \prod_i A_i \ : \ a_j = f_j^i(a_i) \ \ \forall j \leq i\}$$

We obtain projections $\pi_i : A \to A_i$
We give the limit a initial topology, i.e. the minimal topology on A s.t.

π_i are continuous. Explicitly, a basis of open sets is $\pi_i^{-1}(U)$ (cylinder sets...). **Fact**: The category of rings has limits. So lets build the p-adic integers:
define $A_i = \mathbb{Z}/p^i\mathbb{Z}$ and the transition maps

$$f_j^i : A_i \to A_j, \qquad f_j^i(a \mod (p^i)) = (a \mod (p^j))$$

Then $A = \lim_{\leftarrow} A_i = \{\vec{a} \in \prod_i \mathbb{Z}/p^i\mathbb{Z} \ : \ a_j = a_i \mod (p^j) \ \ \forall j \leq i\}$

THEOREM 6.3.1. $\mathbb{Z}_p \cong A$ (as topological rings)

PROOF. the map os just $\phi : \quad a = \sum_i a_i p^i \mapsto (\sum_{j=0}^i a_j p^j)_i$ check that this is indeed a map of rings (modolu commutes with addition and multiplication).
Injective: if $\phi(a) = 0$ then $a = 0 \mod (p^n)$ for any n so is 0.
Surjective: Let $\{a_i\}$ be a sequence in the limit, we show that it is a image of $a = \sum_i b_i p^i$ with $b_0 = a_0$, $b_i = (a_{i+1} - a_i)/p^i$ since $a_{i+1} - a_i = 0 \mod (p^i)$ so is well defined.
Continuous: Let $U = \pi_n^{-1} a_n$ be a cylinder set, $U = \pi_n^{-1}(a_n) = \{(a_i) \in A \ : \ a_i = a_n \mod (p^n)\}$ and thus $\phi^{-1}(U) = \{x \in \mathbb{Z}_p \ : \ x - a_n \in p^n\mathbb{Z}_p\} = a_n + p^n\mathbb{Z}_p$ thus is open.
Open map: a basic open set around 0 is $p^n\mathbb{Z}_p$ which maps to $\pi_n^{-1}(0)$ $\square$

COROLLARY 6.3.1. $\mathbb{Z}_p$ is compact

6.4. tdlc. The non-archemedian p-adic fields are an example of a space with the following properties:

DEFINITION 6.4.1. An l-space is a locally compact totally disconnected housdorff topological space.

LEMMA 6.4.1. Let X be a l space- it has a basis of open compact neighborhoods

PROOF. Since is locally compact we may assume X is compact.
To show that there is a system of compact open neighborhoods we show that for any $x \in X$ the intersection of all compact open sets that contain x is exactly x. We do so by showing this intersection is connected. Denote this intersection by K, assume is not connected, thus $K = K_1 \cup K_2$ (disjoint union of non empty relatively open sets). w.l.o.g assume $x \in K_1$. Notice that K_i are closed in K thus compact, hence by regularity of X (Hausdorff + locally compact implies regular) there is open disjoint U_i containing K_i.
The set $F \coloneqq X \setminus (U_1 \cup U_2)$ does not contain x, it is closed hence compact.
Adding F to the collection of neighborhoods above makes the intersection of the collection empty thus there is finite $C_i \in K$ s.t. $F \cap_i C_i = \emptyset$, denote $L = \cap_i C_i$.
We have: $x \in L$, $L \cap F = \emptyset \Rightarrow L \subset U_1 \cup U_2$ so let $L_i \coloneqq U_i \cap L$ is open and $x \in L_1$.

Observe- L_1 is closed, $\overline{L_1} = \overline{U_1 \cap L} \subset \overline{U_1} \cap L \subset U_2^c \cap (U_1 \cup U_2) \subset U_1$ thus since L_1 is closed compared to U_1 it is closed.
We get $K \subset L \subset U_1$ thus $K_2 = \emptyset$ in contradiction!
Why do we have a neighborhood ?
Since if $x \in V$ some open set, then define $F = K \setminus V$ and obtain clopen $L \subset V$ containing x as above. $\square$

THEOREM 6.4.1. If X is a compact l-group there is such basis (around the identity) of normal subgroups. And is an l-group is such (compact) iff is pro-finite