Ben-Gurion University of the Negev
The Faculty of Natural Sciences
Department of Mathematics

The Representation Zeta Function for the Special Linear Group of Certain Division Algebras Over a Local Field

Thesis Submitted in Partial Fulfillment of the Requirements for the Master of Sciences Degree

By: Shai Shechter

Under the Supervision of: Dr. Uri Onn

Be'er Sheva, October 2013

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Abstract

Let D be a central division algebra over a p-adic field F. Let $\ell \in \mathbb{N}$ be the degree of D, that is to say, $\dim_F D = \ell^2$. The reduced norm of an element in D is given by a certain power of the determinant of its image in an appropriate matrix algebra (e.g. an ℓ-th root of the determinant in the left-regular representation of D). In this thesis we study the representation zeta function of the pro-finite group $SL_1(D)$ of elements of reduced norm 1 in D.

Avni, Klopsch, Onn and Voll introduced a formalism based on p-adic integration for calculating the representation zeta functions of a class of groups (viz. potent and saturable groups), which arise naturally from (global) Lie-lattices.

We employ this method in order to study the representation zeta function of some congruence pro-p subgroups of $SL_1(D)$, in the case where ℓ is a prime number and F contains elements of multiplicative order ℓ. We present explicit formulae for these zeta functions in the special case $\ell = 3$.

In the last part of the thesis, we invoke Clifford theory in order to extend the calculation to the group $SL_1(D)$ itself, under some mild assumptions on the ramification index of the ground field F over the p-adic numbers. We present a general argument, setting the ground for performing this extension for any prime ℓ, and perform the extension explicitly for the case $\ell = 3$, based on our previous calculation.

We conclude this thesis with a conjecture regarding the general form of the representation zeta function of congruence subgroups of $SL_1(D)$ for ℓ prime. We discuss optional strategies for proving this conjecture.

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Contents

Introduction

0.1 History and Background

0.1.1 Representation Zeta Functions

Let G be a topological group, and for any $n \in \mathbb{N}$, let $r_n(G)$ denote the number of isomorphism classes of irreducible n-dimensional continuous complex representations of G. Finite groups are regarded here as topological groups with the discrete topology. We call a group G (representation) rigid if the value $r_n(G)$ is finite for all $n \in \mathbb{N}$. In the study of representation growth, one studies the arithmetic properties of the sequence $\{r_n(G)\}_{n\in\mathbb{N}}$ and its asymptotic behaviour, as n tends to infinity. The group G is said to have polynomial representation growth (PRG) if the sequence $r_n(G)$ is bounded above by some polynomial in n. In the case where G is has PRG one can define the representation zeta function of G to be the Dirichlet generating function

$$\zeta_G(s) := \sum_{n=1}^{\infty} r_n(G)n^{-s},\ s \in \mathbb{C}.$$

In this thesis we specialize to the case where G is a compact p-adic analytic group. A compact p-adic analytic group is rigid if and only if it is FAb, that is, if every open subgroup of G has a finite abelianization [BLMM06]. Jaikin-Zapirin proved rationality results regarding the representation zeta function of FAb p-adic analytic groups, using tools from model theory [JZ06]. In particular the representation zeta function of a FAb p-adic analytic group is a rational function in p^{-s}.

Avni, Klopsch, Onn and Voll introduced a p-adic formalism for the representation zeta function of potent and saturable pro-p groups (cf. Chapter 1 for definitions), applying new methods from p-adic integration. Potent and saturable pro-p groups are notable for the fact that they can be naturally associated to a $\mathbb{Z}_p$ -Lie lattice, and for the fact that the characters of a potent and saturable pro-p group G can be described explicitly as averages over G-orbits in the Pontryagin dual of the associated $\mathbb{Z}_p$-Lie-lattice [GS09].

0.1.2 The Group $SL_1(D)$

Fix p to be an odd prime number. Let D be a central division algebra over a p-adic field F , and let ℓ denote the degree of D. It is known [Pie82, Chapter 17], that D contains a subfield E such that the extension E/F is an unramified cyclic Galois extension of degree ℓ. The reduced norm and trace of an element $x \in D$ are defined to be the determinant and trace of its image in the matrix algebra $\mathrm{Mat}_\ell(E)$, and are denoted by $\mathrm{Nrd}_{D/F}(x)$ and $\mathrm{Trd}_{D/F}(x)$ (cf. § 3.1.1).

The algebra D is also endowed with an absolute value, which satisfies the equality

$$|x|_\wp = \left|\mathrm{Nrd}_{D/F}(x)\right|_\wp^{1/\ell}, \quad \forall x \in D,$$

where $|\cdot|_\wp$ denotes the absolute value given on F. In view of this, we denote the absolute value on D by $|\cdot|_\wp$ as well. Let $\mathcal{O}$ denote maximal compact sub-ring of D, that is $\mathcal{O} := \left\{x \in D \mid |x|_\wp \leq 1\right\}$.

We aim to study the representation zeta function of the group G and its congruence subgroups. The group G itself, along with its congruence subgroups were studied extensively by Riehm in [Rie70]. Being an algebraic group defined over F , $SL_1(D)$ can be associated to an F -Lie-algebra, which can be realized as the Lie-algebra $\mathfrak{sl}_1(D)$ of elements of reduced trace 0 in D. It is known that the congruence subgroups $G_m := SL_m^1(\mathcal{O})$ are potent and saturable for sufficiently large $m \in \mathbb{N}$. Moreover, given some mild assumptions on the ramification index of F over the field of p-adic numbers [AKOV12, Proposition 4.2] it is known that the first congruence subgroup $SL_1^1(\mathcal{O}) := \left\{x \in SL_1(D) \mid |x-1|_\wp < 1\right\}$, is a potent and saturable pro-p group, with an associated $\mathbb{Z}_p$ -Lie-lattice $\mathfrak{sl}_1^1(\mathcal{O}) := \left\{x \in \mathfrak{sl}_1(D) \mid |x|_\wp < 1\right\}$. In this thesis, we will first investigate the representation zeta function of some congruence subgroups of $SL_1(D)$, using the p-adic formalism described in [AKOV13], under some assumptions on the ground field F. Once this is done, we will be able to extend this computation to the group $SL_1(D)$.

0.2 The Current Research

Let F be a p-adic field, with $\mathscr{O}$ its maximal compact subring, and $\wp$ the maximal ideal of $\mathscr{O}$. Let q denote the cardinality of $\mathscr{O}/\wp$. Let G be a potent and saturable pro-p group, and let $\mathfrak{g}$ be the associated potent and saturable $\mathbb{Z}_p$-Lie lattice. Assume further that $\mathfrak{g}$ is an $\mathscr{O}$-Lie lattice. The p-adic formalism described in [AKOV13] ties the calculation of the representation zeta function of G with an analysis of the determinantal ideals defined by a certain $d \times d$ structure matrix of $\mathfrak{g}$,viz. **the commutator matrix** (cf. § 2.2), where $d := \mathrm{rk}_{\mathscr{O}}\mathfrak{g}$. This matrix is defined over the ring of polynomials in d variables over $\mathscr{O}$, and denoted as $\mathcal{R}(\mathbf{Y})$.

To be more precise, for any $j = 1, \ldots, d/2$, one defines the set

$$F_j(\mathbf{Y}) := \{f \in \mathscr{O}[\mathbf{Y}] \mid f \text{ is a } 2j \times 2j \text{ minor of } \mathcal{R}(\mathbf{Y})\},$$

and puts, for any $\mathbf{y} \in \mathscr{O}^d$, and $j \in \{1, \ldots, d/2\}$,

$$\|F_j(\mathbf{y})\|_\wp = \max\left\{|f(\mathbf{y})|_\wp \mid\in F_j(\mathbf{Y})\right\}.$$

The representation zeta function of G is dependent on the computation of the integral

$$\mathcal{Z}(r,t) = \int_{(x,\mathbf{y})\in\wp\times W(\mathscr{O})} |x|_\wp^r \prod_{j=1}^{\rho} \frac{\|F_j(\mathbf{y}) \cup x^2 F_{j-1}(\mathbf{y})\|_\wp^{\,t}}{F_{j-1}(\mathbf{y})}\, d\mu(x,\mathbf{y}), \quad r,t \in \mathbb{C},$$

where $W(\mathscr{O})$ is the set $\mathscr{O}^d \setminus \wp \cdot \mathscr{O}^d$, μ is the probability Haar measure on $\mathscr{O}^d$ and $\rho := \max\{\mathrm{rk}_F \mathcal{R}(\mathbf{y}) \mid \mathbf{y} \in W(\mathscr{O})\}$.

In the first part of our research, we investigate the commutator matrix arising from certain congruence subgroups of $G = SL_1(D)$, where D is a central division algebra of degree ℓ over F . We assume throughout this part and the subsequent parts that $\ell \neq p$ is a prime number, and that F contains a primitive root unity of order ℓ.

For any $j = 1, ..., \rho$, we give a construction of specific $2j \times 2j$ minors of $\mathcal{R}(\mathbf{Y})$ and present a calculation of their absolute values, once the variables $\mathbf{Y}$ are substituted for an element $\mathbf{y} \in W(\mathscr{O})$. This calculation gives us a lower bound for the value of $\|F_j(\mathbf{y})\|_\wp$, for any $\mathbf{y} \in W(\mathscr{O})$. By considering also a naïve upper bound for the values of $\|F_j(\mathbf{y})\|_\wp$, we obtain the following:

Statement. *The set $W(\mathscr{O})$ can be partitioned into ℓ disjoint subsets*

$$W^{[0]}(\mathscr{O}) \sqcup W^{[1]}(\mathscr{O}) \sqcup \ldots \sqcup W^{[\ell-1]}(\mathscr{O}),$$

Such that:

1. *For any* $j = 1, \ldots, \rho$ *and* $\mathbf{y} \in W^{[0]}(\mathcal{O})$, $\|F_j(\mathbf{y})\|_{\wp} = 1$.
2. *For any* $j = 1, \ldots, \rho$ *and* $\mathbf{y} \in W^{[1]}(\mathcal{O})$, $\|F_j(\mathbf{y})\|_{\wp} = \min\{1, q^{-2j}\}$.
3. *Let* $\gamma \in \{2, \ldots, \ell - 1\}$. *For any* $\mathbf{y} \in W^{[\gamma]}(\mathcal{O})$ *and* $j = 1, \ldots, \rho$,

$$\min\left\{1, q^{-2j+(\ell-1)(\gamma-1)}\right\} \leq \|F_j(\mathbf{y})\|_{\wp} \leq \min\left\{1, q^{-2j+\ell(\gamma-1)-1}\right\}. \qquad \text{(0.A)}$$

This statement is proven in detail in Chapter 7.

Note that the inequalities of 0.A are in fact an equality in the case where $\gamma = 2$. This fact allows us to complete the computation of the representation zeta function of the aforementioned congruence subgroups, for the case $\ell = 3$. In particular, we have

Theorem A. *Let* F *be a non-archemedian local field of residue field characteristic* $p > 3$, *and let* D *be a division algebra of degree* 3 *over* F. *Assume* F *contains a non-trivial cube root of unity, and that* $SL_1^1(\mathcal{O})$ *is potent and saturable (cf. § 1.2).*

Then

$$\zeta_{SL_1^1(D)}(s) = q^3 \cdot \frac{1 + q^{-(s-1)}(1 - q^{-3}) - q^{-3s-3}}{1 - q^{-3s+2}}.$$

Following this part of the thesis, the third part of this thesis deals with extending the calculation of the representation zeta function to the group $G = SL_1(D)$ itself, under the assumption that the group $SL_1^1(\mathcal{O})$ is potent and saturable. The main tool used in this part of the thesis is an application of Clifford theory, in order to investigate the connection between the representation zeta function of $SL_1^1(\mathcal{O})$, and that of G. The primary step towards investigating this connection will be to compute the so-called inertia subgroups (cf. § 9.1) of an arbitrary character ϑ of $SL_1^1(\mathcal{O})$. In chapter 10 we will show that the number of options for such an inertia subgroup is extremely limited. In particular, if ϑ is a character of $SL_1^1(\mathcal{O})$, and $I_G(\vartheta) \subseteq G$ is its inertia subgroup, then either $I_G(\vartheta) = G$ or $I_G(\vartheta)$ equals the product of $SL_1^1(\mathcal{O})$ and a cyclic group of order ℓ. Furthermore, this dichotomy is shown to be related to the partition of $W(\mathcal{O})$ shown in the previous part.

Specializing to the case where $\ell = 3$ we obtain the following:

Theorem B. *Let* F *and* D *be as in Theorem A, and that* $SL_1^1(\mathcal{O})$ *is potent and saturable.*

Then

$$\zeta_{SL_1(\mathbb{O})}(s) = \frac{(1+q+q^2)(1-q^{-3s}) + 9\cdot\left(\frac{q^2+q+1}{3}\right)^{-s}(q^{-s+1}+1)(q-1)}{1-q^{-3s+2}}.$$

The discussion about the representation zeta function of the congruence subgroups of $SL_1(D)$ is complemented in Chapter 12, in which we pose the conjecture that the lower bounds described in Chapter 7 actually give the precise values of the function $\|F_j(\mathbf{y})\|_{\wp}$.

Conjecture A. *Let* $\mathbf{y} \in W^{[\gamma]}(\mathscr{O})$ *be arbitrary, then*

$$F_j(\mathbf{y}) = \begin{cases} 1 & \text{, if } \gamma = 0 \\ \min\left\{1, q^{-2j+(\ell-1)(\gamma-1)}\right\} & \text{, if } \gamma \in \{1,\ldots,\ell-1\}, \end{cases}$$

for all $j = 1, \ldots, \rho$.

Additionally, we discuss strategies for proving this conjecture and suggest two conjectures whose proof will imply conjecture A.

Part I

Preliminaries

Chapter 1

Orbit Method

The orbit method, as originally discovered by Alexandre Kirillov, establishes a connection between the set isomorphism classes of irreducible unitary representations of certain Lie-groups and the set of the co-adjoint orbits of the group.

The orbit method can be generalized to certain pro-p groups as well. In this chapter we shall describe the main definitions and theorems, as given by Boyarchenko and Sabitova in [BS08] in the generality of pro-finite groups. Following this, we will present a characterization of a class of p-adic groups to which one can effectively apply the orbit method, viz. potent and saturable groups.

1.1 Profinite Groups

Given a pro-finite group Π, with a Haar probability measure μ_Π, we consider the space $L^2(\Pi)$ of complex valued square summable functions on Π as a $\mathbb{C}$-algebra with the convolution operation

$$f_1 * f_2(\gamma) = \int_\Pi f_1(h) f_2(h^{-1}\gamma) d\mu_\Pi(h), \quad \gamma \in \Pi,$$

as multiplication. The set $\mathrm{Fun}(\Pi)$ denotes the set of functions which are bi-invariant with respect to a sufficiently small compact-open subgroup, and the set $\mathrm{Fun}(\Pi)^\Pi$ will denote the subspace of functions which are invariant with respect to the conjugation action of Π.

Theorem 1 (Abstract orbit method [BS08, Theorem 1.1]). *Let G be a pro-finite group, and suppose that there exists a pro-finite abelian group $\mathfrak{g}$ and a homeomorphism $\exp : \mathfrak{g} \to G$ such that the following two conditions hold:*

1. *For each $g \in G$, the map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$, given by $x \mapsto \log(g \exp(x) g^{-1})$ is a group automorphism, where $\log$ is the inverse of $\exp$; and*
2. *The pullback map $\exp^* : \mathrm{Fun}(G)^G \xrightarrow{\simeq} \mathrm{Fun}(\mathfrak{g})^G$, defined by $\exp^*(f)(x) = f(\exp(x))$ for any $x \in \mathfrak{g}$, satisfies*

$$\exp^*(f_1 * f_2) = \exp^*(f_1) * \exp^*(f_2), \quad \forall f_1, f_2 \in \mathrm{Fun}(G)^G.$$

Let $\hat{\mathfrak{g}} := \mathrm{Hom}(\mathfrak{g}, \mathbb{C}^\times)$ denote the Pontryagin dual of $\mathfrak{g}$. Then any G-orbit $\Omega \subseteq \hat{\mathfrak{g}}$ (under the action induced from $g \mapsto \mathrm{Ad(g)}$) is finite. Moreover, there is a bijection between the set of such orbits $\hat{\mathfrak{g}}/G$, and the set of equivalence classes of irreducible continuous representations of G, such that the irreducible character corresponding to an orbit Ω is given by

$$\chi_\Omega(\exp(x)) = (\#\Omega)^{-1/2} \sum_{f \in \Omega} f(x), \quad \forall x \in \mathfrak{g}.$$

In particular, the set $\{\chi_\Omega \mid \Omega \text{ is a } G\text{-orbit in } \hat{\mathfrak{g}}\}$ is a basis for the space of class functions on G.

1.2 Potency and Saturablity

Fix an odd prime number $p \in \mathbb{Z}$.

The notion of saturablilty dates back to the work of Lazard, who proved that a topological group is p-adic analytic if and only if it contains a saturble pro-p group as an open subgroup. Saturable groups are notable for the fact that they can be naturally associated to a Lie-lattice over the ring $\mathbb{Z}_p$ of p-adic integers. The importance of this association is exemplified in a tight correspondence between the subgroups (resp. normal subgroups) of the group and the sub-Lie-lattices (resp. Lie-ideals) of the corresponding $\mathbb{Z}_p$-Lie-lattice.

We shall use the following definition of saturability, due to Gonzáles-Sánchez, of finitely generated pro-p groups.

Theorem 2 (cf. [GS07, Theorem A.]). *Let G be a torsion free finitely generated pro-p group. Then G is **saturable** if and only if it admits a **potent filtration**. That is to say, there exists a sequence $\{N_i\}_{i\in\mathbb{N}}$ of normal subgroups of G, such that*

1. *$N_1 = G$ and $\bigcap_{i\in\mathbb{N}} N_i = \{1\}$,*
2. *$[N_{i+1}, G] \subseteq N_i$ for all $i \in \mathbb{N}$, and*
3. *$[N_i,_{p-1} G] \subseteq N_{i+1}^p$ for all $i \in \mathbb{N}$, where N_{i+1}^p is the group generated by all p-th powers of elements in N_{i+1}, and $[N_i,_{p-1} G]$ is shorthand for commutator subgroup $[\ldots[[N_i, G], G]\ldots, G]$, in which G appears $p-1$ times.*

Similarly, one defines the concept of a **saturable Lie-algebra**. For our purposes, a $\mathbb{Z}_p$-Lie algebra $\mathfrak{g}$ is said to be saturable if there exists a series of Lie-ideals $\{\mathfrak{n}_i\}_{i\in\mathbb{N}}$, such that

1. $\mathfrak{n}_1 = \mathfrak{g}$ and $\bigcap_{i\in\mathbb{N}} \mathfrak{n}_i = \{0\}$,
2. $[\mathfrak{n}_{i+1}, \mathfrak{g}] \subseteq \mathfrak{n}_i$ for all $i \in \mathbb{N}$, and
3. $[\mathfrak{n}_i,_{p-1} \mathfrak{g}] \subseteq p\cdot\mathfrak{n}_{i+1}$, where $[\mathfrak{n}_i,_{p-1} \mathfrak{g}]$ denotes the Lie-sub-algebra $[\ldots[\mathfrak{n}_i, \mathfrak{g}], \ldots, \mathfrak{g}]$.

Any saturable $\mathbb{Z}_p$-Lie algebra $\mathfrak{g}$ can be endowed with the structure of a saturable pro-p group, where the multiplication operation is given by

$$\Phi(x, y) = \log(\exp(x)\exp(y)), \quad \forall x, y \in \mathfrak{g},$$

where Φ is the Baker-Campbell-Hausdorff series, defined over the ring of power series with rational coefficients in two non-commuting variables. We write $\exp(\mathfrak{g})$ to denote the group structure of $\mathfrak{g}$, and write $\exp(x)$ when considering an element x as an element of $\exp(\mathfrak{g})$.

Conversely, any saturable pro-p group G can be endowed with the structure of a saturable $\mathbb{Z}_p$-Lie algebra (cf. [GS07, § 4]). We denote this Lie-algebra by $\log(G)$.

The correspondences $\mathfrak{g} \mapsto \exp(G)$ and $G \mapsto \log(G)$ induce a functorial isomorphism between the category of saturable $\mathbb{Z}_p$-Lie algebras and the category of saturable pro-p groups [GS09, Theorem 2.2]. Furthermore, we have that

$$\exp(\log(G)) = G \quad \text{and} \quad \log(\exp(\mathfrak{g})) = \mathfrak{g}.$$

The conjugation action of G on itself induces an action G on the $\mathbb{Z}_p$-Lie-algebra $\mathfrak{g} := \log(G)$. As in the case of pro-finite groups, this induces an action of G on the Pontryagin dual of the additive group of $\mathfrak{g}$, namely- the **co-adjoint action** of G on $\hat{\mathfrak{g}}$.

To any G-orbit $\Omega \subseteq \hat{\mathfrak{g}}$, we associate the function

$$\chi_\Omega(\exp(x)) = (\#\Omega)^{-1/2} \sum_{f \in \Omega} f(x), \quad \forall x \in \mathfrak{g}. \tag{1.A}$$

By the definition as an average over a G-orbit, it is clear that χ_Ω is a class-function of G. Moreover, it is known (cf. [GS09, Proposition 4.1]) that the functions $\{\chi_\Omega \mid \Omega \in \hat{\mathfrak{g}}/G\}$ are orthonormal with respect to the standard inner product on the space of class functions on G.

A pro-p group is called **potent** if it has the property that

$$\gamma_{p-1}(G) \subseteq G^p,$$

where $\gamma_{p-1}(G)$ is the $(p-1)$-th term in the lower central series of G, and G^p is the group generated by the p-th powers in G. Similarly, a $\mathbb{Z}_p$-Lie algebra is called potent if $\gamma_{p-1}(\mathfrak{g}) \subseteq p\mathfrak{g}$.

In the case where G is both potent and saturable we have the following:

Theorem 3 ([GS09, Theorem 5.2]). *Let G be a potent and saturable pro-p group. Then the set of irreducible complex characters of G coincides with the set*

$$\{\chi_\Omega \mid \Omega \textit{ is a } G\textit{-orbit in } \hat{\mathfrak{g}}\}.$$

Specifically, since $\{\chi_\Omega \mid \Omega \in \hat{\mathfrak{g}}/G\}$ is an orthonormal basis for the space of class functions of G, we obtain that for a potent and saturable group the correspondence $\Omega \mapsto \chi_\Omega$ is a bijection between the co-adjoint orbits in $\hat{\mathfrak{g}}$ and the characters of G.

Chapter 2

AKOV Machinery

In this chapter we give an overview a method, developed by Avni, Klposch, Onn and Voll in [AKOV13], for calculating the representation zeta function of certain p-adic analytic pro-p groups, which arise from 'global' Lie-lattices.

For any topological group G, we define the representation zeta function of G by

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s},$$

where $r_n(G) \in \mathbb{N} \cup \{0, \infty\}$ is the number of isomorphism classes of irreducible continuous representations of dimension n, and s is a complex variable. In general, it is not true that $r_n(G) < \infty$ for all n's, or that the sum $\zeta_G(s)$ is convergent. Nonetheless, the first assertion is known to hold whenever G is a saturable pro-p group with a finite abelianization (cf. [AKOV13, Proposition 2.1]).

Suppose that G is a potent and saturable pro-p group, and put $\mathfrak{g} := \log(G)$. As seen in previous section, the dimension of any irreducible representation of G can be realized as the square-root of the size of a co-adjoint orbit in $\hat{\mathfrak{g}}/G$, by evaluating the characters χ_Ω (cf. Equation 1.A) at $x = 1$. This value can in fact be realized as the index of a distinguished $\mathbb{Z}_p$-sub-module of $\mathfrak{g}$. This allows us to transfer the question of enumerating representation degrees of G into the setting of a linear space (viz. the $\mathbb{Z}_p$-module $\mathfrak{g}$).

As it turns out, one can give a concrete realization of this sub-$\mathbb{Z}_p$-module as the space of solutions to a predetermined set of linear equations, defined over some finite ring-extension of $\mathbb{Z}_p$, modulo an appropriate ideal (cf. Lemma 2.2.1). Once this realizations has been made, the representation zeta function of a potent and saturable pro-p group can be thought of as a Poincaré series, whose coefficients arise as the number of solutions to the aforementioned set of linear equations. At this point, one can now derive an integral formula, connecting the representation zeta function of G to an Igusa local zeta integral.

2.1 Preparations

Let $p \geq 3$ be a fixed prime number, and let $\mathscr{o}$ be a compact discrete valuation ring of characteristic 0, with residue field cardinality $q = p^\alpha$, for some $\alpha \in \mathbb{N}$. Let $\wp$ be the unique maximal ideal of $\mathscr{o}$, and let π be a uniformizing parameter at $\wp$.

Notation. Let M be an $\mathscr{o}$-module. Throughout this essay, we write M^* to denote the set $M \smallsetminus \wp \cdot M$.

Let G be a potent an saturable pro-p group, with $\mathfrak{g} = \log(G)$ the associated potent and saturable $\mathbb{Z}_p$-Lie algebra. Assume further that $\mathfrak{g}$ is in fact an $\mathscr{o}$-Lie lattice. By Theorem 3 and equation (1.A), if $\chi \in \mathrm{Irr}(G)$ is an irreducible character of G corresponding to the co-adjoint orbit $\Omega \in \hat{\mathfrak{g}}/G$, then the dimension of a representation corresponding to χ is

$$\chi(1) = \chi(\exp(0)) = (\#\Omega)^{-1/2} \sum_{f \in \Omega} f(0) = (\#\Omega)^{1/2} .$$

Furthermore, we obtain the following equalities:

$$\begin{aligned} \zeta_G(s) &= \sum_{n=0}^{\infty} r_n(G) n^{-s} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s} \\ &= \sum_{\Omega \in \hat{\mathfrak{g}}/G} |\Omega|^{-s/2} = \sum_{\omega \in \hat{\mathfrak{g}}} |G.\omega|^{-s/2-1} \\ &= \sum_{\omega \in \hat{\mathfrak{g}}} |G : C_G(\omega)|^{-(s+2)/2} , \end{aligned}$$

where $C_G(\omega) := \{g \in G \mid g.\omega(y) = \omega(y),\ \forall y \in \mathfrak{g}\}$ is the stabilizer of ω in G. The Lie correspondence for saturable pro-p groups states that the subgroup $C_G(\omega)$ coincides (as a set) with the Lie-sub-algebra of $\mathfrak{g}$ consisting of elements of $\mathfrak{g}$ which trivialize ω, via a suitable action of $\mathfrak{g}$ on $\hat{\mathfrak{g}}$ (see [GS09, Lemma 3.2]). This sub-algebra is given explicitly by

$$\mathrm{Rad}(\omega) := \{x \in \mathfrak{g} \mid \omega([x, y]) = 1,\ \forall y \in \mathfrak{g}\} .$$

Additionally, the Lie correspondence is index preserving. Thus, we have the equality

$$\zeta_G(s) = \sum_{\omega \in \hat{\mathfrak{g}}} |\mathfrak{g} : \mathrm{Rad}(\omega)|^{-(s+2)/2}. \tag{2.A}$$

Definition 2.1.1 ([AKOV13, §2.2]). The **level** of an element $\omega \in \hat{\mathfrak{g}}$ is defined to be the minimal $n \in \mathbb{N} \cup \{0\}$ such that $\wp^n \cdot \mathfrak{g} \subseteq \mathrm{Ker}(\omega)$. The set of elements of level n is denoted as $\mathrm{Irr}_n(\mathfrak{g})$.

Let us recall an important lemma regarding the set $\hat{\mathfrak{g}}$.

Lemma 2.1.2 ([AKOV13, Lemma 2.4])**.** The Pontryagin dual $\hat{\mathfrak{g}}$ can be written as a disjoint union

$$\hat{\mathfrak{g}} = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \mathrm{Irr}_n(\mathfrak{g}), \quad \text{where} \quad \mathrm{Irr}_n(\mathfrak{g}) \cong \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O}/\wp^n)^*.$$

The basic yoga underlying this lemma lies in the fact that any element $\omega \in \hat{\mathfrak{g}}$ has an open kernel, and hence can be regarded as an element of $\widehat{\mathfrak{g}/\wp^n\mathfrak{g}}$ for some power of the ideal $\wp$. Once this is known, the lemma follows by constructing an explicit bijection between $\widehat{\mathfrak{g}/\wp^n\mathfrak{g}}$ a the set $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O}/\wp^n)^*$, for any $n \in \mathbb{N} \cup \{0\}$.

An important consequence of the proof of Lemma 2.1.2 is that this bijection is G-equivariant. That is to say, if $\omega \in \hat{\mathfrak{g}}$ is mapped via this bijection to a homomorphism $w \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O}/\wp^n)$ for some $n \in \mathbb{N} \cup \{0\}$, then for any $g \in G$ the element $g.\omega \in \hat{\mathfrak{g}}$ is mapped by to the element $g.w \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O}/\wp^n)$ (where G acts on the sets $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O}/\wp^n)$ by the co-adjoint action).

2.2 The Commutator Matrix

By equation (2.A), we know that the problem of enumerating representation of G of a given dimension can be reduced to a calculation of the indices of the Lie-sub-modules $\mathrm{Rad}(\omega)$ for all $\omega \in \hat{\mathfrak{g}}$. Examining the definitions, one might hope to be able to realize $\mathrm{Rad}(\omega)$ as the left-kernel of an $\mathcal{O}$-bilinear form, namely that of the map $(x, y) \mapsto \omega([x, y])$. The basic fault in this reasoning is that this map is not $\mathcal{O}$-bilinear, but rather a bi-additive map (i.e. $\omega([x_1+x_2, y]) = \omega([x_1, y])\omega([x_2, y])$, and likewise for the right coordinate). Nonetheless, on the grounds of Lemma 2.1.2, one can utilize this bi-additive form, in order to conjure up a sufficient supplement, as we shall see in this section.

From this point on we fix a basis $\mathcal{B} := \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ of $\mathfrak{g}$. The **structure constants** of $\mathfrak{g}$ with respect to the basis $\mathcal{B}$ are the elements $\lambda_{i,j}^1, \dots, \lambda_{i,j}^d \in \mathcal{O}$ satisfying

$$[\mathbf{e}_i, \mathbf{e}_j] = \sum_{h=1}^{d} \lambda_{i,j}^h \mathbf{e}_h,$$

where $i, j \in \{1, \dots, d\}$.

We define the **commutator matrix** of $\mathfrak{g}$ with respect to the basis $\mathcal{B}$ by

$$\mathcal{R}(\mathbf{Y}) := \mathcal{R}_{\mathfrak{g},\mathcal{B}}(\mathbf{Y}) = \left(\textstyle\sum_{h=1}^{d} \lambda_{i,j}^h Y_h \right)_{i,j=1}^{d} \in \mathrm{Mat}_d(\mathcal{O}[\mathbf{Y}])$$

whose entries are linear forms in independent variables $Y_1, \dots, Y_d$.

Remark. Note that in particular, if $\mathbf{y} = (y_1, \dots, y_d) \in \mathcal{O}^d$ is the coordinate vector of an element $\varphi \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})$ with respect to the dual basis of $\mathcal{B}$, then the matrix $\mathcal{R}(\mathbf{y})$ represents the anti-symmetric bilinear form

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\varphi} := \varphi\left([\mathbf{v}_1, \mathbf{v}_2]\right),$$

defined on $\mathfrak{g} \times \mathfrak{g}$.

For any $n \in \mathbb{N}$, there exists a natural surjection from the set $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})^*$ onto the set $\mathrm{Irr}_n(\mathfrak{g})$. We call $w \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})^*$ a **representative** of $\omega \in \hat{\mathfrak{g}}$ if ω has level n, and w is mapped by this surjection to ω.

The $\mathcal{O}$-basis $\mathcal{B}$ yields an explicit coordinate system for $\mathfrak{g}$, and hence an identification (as $\mathcal{O}$-modules) with $\mathcal{O}^d$. Furthermore, the dual basis of $\mathcal{B}$ yields a coordinate system for $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})$, in which the set $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})^*$ is encoded in the set

$$W(\mathcal{O}) := \left\{ (y_1, \dots, y_d) \in \mathcal{O}^d \mid y_i \in \mathcal{O}^{\times} \text{ for at least one } i \right\}.$$

The connection between the radical of an element $\omega \in \hat{\mathfrak{g}}$ and the commutator matrix is described in the following lemma.

Lemma 2.2.1 ([AKOV13, Lemma 3.3])**.** Let $\omega \in \hat{\mathfrak{g}}$ have level n, and let $w \in \mathrm{Hom}(\mathfrak{g}, \mathcal{O})^*$ represent ω. Then for any $z \in \mathfrak{g}$ we have

$$z \in \mathrm{Rad}(\omega) \quad \Longleftrightarrow \quad \underline{z} \cdot \mathcal{R}(\underline{w}) \equiv 0 (\bmod\ \wp^n),$$

where $\underline{z} \in \mathcal{O}^d$ and $\underline{w} \in W(\mathcal{O})$ are the d-tuples of coordinates for z and w respectively.

2.3 Poincaré Series

Assume from here on that $d := \mathrm{rk}_{\mathcal{O}}(\mathfrak{g})$ is even [1]. Given a d-tuple $\mathbf{y} \in W(\mathcal{O})$, the matrix $\mathcal{R}(\mathbf{y})$ is an antisymmetric $d \times d$ matrix over $\mathcal{O}$. Therefore, it can be brought by simultaneous row and column operations to an anti-symmetric normal form

$$\mathrm{ASNF}(a_1, \ldots, a_{d/2}) := \begin{pmatrix} 0 & \pi^{a_1} & & & & & \\ -\pi^{a_1} & 0 & & & & & \\ & & 0 & \pi^{a_2} & & & \\ & & -\pi^{a_2} & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \pi^{a_{d/2}} \\ & & & & & -\pi^{a_{d/2}} & 0 \end{pmatrix},$$

where $a_i \in \mathbb{N} \cup \{0, \infty\}$, $0 \leq a_1 \leq \ldots \leq a_{d/2}$ and we agree to the convention $\pi^\infty = 0$.

Define $\nu(\mathcal{R}(\mathbf{y})) := (a_1, \ldots, a_{d/2}) \in (\mathbb{N} \cup \{0, \infty\})^{d/2}$. For a given $n \in \mathbb{N} \cup \{0\}$, let $\overline{\mathbf{y}}$ denote the image of $\mathbf{y}$ in $W_n(\mathcal{O}) := \left((\mathcal{O}/\wp^n)^d\right)^*$. Then $\mathcal{R}(\overline{\mathbf{y}})$ is an antisymmetric $d \times d$ matrix over $\mathcal{O}/\wp^n$, and thus is congruent to a unique matrix as above, whose elementary divisors (the exponents a_i), are encoded in the $d/2$-tuple

$$\nu_n(\mathcal{R}(\mathbf{y})) := \nu(\mathcal{R}(\overline{\mathbf{y}})) = (\min\{a_i, n\})_{i=1,\ldots,d/2}.$$

As seen in Lemma 2.2.1, given an element $\omega \in \hat{\mathfrak{g}}$ of level n, we can identify its radical with the set of solutions modulo $\wp^n$ to a system of linear equations (i.e. the left kernel modulo $\wp^n$ of $\mathcal{R}(\underline{w})$). The connection between the $d/2$-tuples $\nu_n(\mathcal{R}(\mathbf{y}))$ and the index $|\mathfrak{g} : \mathrm{Rad}(\omega)|$ is given explicitly in the following lemma.

Lemma 2.3.1 ([AKOV13, Lemma 3.4]). Let $\omega \in \hat{\mathfrak{g}}$ have level n, and let $w \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{g}, \mathcal{O})^*$ represent ω. Let $\underline{w} \in W(\mathcal{O})$ be the coordinate d-tuple of w, and let $\mathbf{a} = (a_1, \ldots, a_{d/2}) = \nu_n(\mathcal{R}(\underline{w}))$. Then

$$|\mathfrak{g} : \mathrm{Rad}(\omega)| = q^{2\sum_{i=1}^{d/2} n - a_i}.$$

Note that the value obtained in the lemma does not depend on the choice of the representative w. Thus, the question of calculating the representation zeta function of G now reduces to the following question:

[1] A similar description exists when d is odd. We will not encounter odd-dimensional Lie-algebras in this essay, and thus we ignore it, for convenience of notation.

> For a given $d/2$-tuple $\mathbf{a}$ and $n \in \mathbb{N}$, how many elements $\overline{\mathbf{y}} \in W_n(\mathscr{O})$ are such that $\nu_n(\mathcal{R}(\mathbf{y})) = \mathbf{a}$?

Note that by the definition of ν_n the answer to this question is 0 whenever $\mathbf{a} = (a_1, \ldots, a_{d/2})$ does not satisfy $a_1 \leq \ldots \leq a_{d/2} \leq n$.

The tools to give an answer to this question will be provided in the upcoming section. For now, we define for any $n \in \mathbb{N} \cup \{0\}$ and $\mathbf{a} \in (\mathbb{N} \cup \{0\})^{d/2}$,

$$\mathcal{N}^{\mathscr{O}}_{n,\mathbf{a}} := \# \left\{ \overline{\mathbf{y}} \in W_n(\mathscr{O}) \mid \nu(\mathcal{R}(\overline{\mathbf{y}})) = \mathbf{a} \right\}.$$

Additionally, we define the associated Poincaré series, by

$$\mathcal{P}_{\mathcal{R},\mathscr{O}}(s) := \sum_{\substack{n \in \mathbb{N} \cup \{0\} \\ \mathbf{a} \in (\mathbb{N} \cup \{0\})^{d/2}}} \mathcal{N}^{\mathscr{O}}_{n,\mathbf{a}} \cdot q^{-\left(\sum_{i=1}^{d/2} (n - a_i)\right) s}.$$

By Lemmas 2.2.1 and 2.3.1, we have that

$$\begin{aligned}
\zeta_G(s) &= \sum_{\omega \in \hat{\mathfrak{g}}} |\mathfrak{g} : \mathrm{Rad}(\omega)|^{-(s+2)/2} \\
&= 1 + \sum_{n=1}^{\infty} \sum_{\omega \in \mathrm{Irr}_n(\mathfrak{g})} |\mathfrak{g} : \mathrm{Rad}(\omega)|^{-(s+2)/2} \\
&= 1 + \sum_{n=1}^{\infty} \sum_{\mathbf{a} \in (\mathbb{N} \cup \{0\})^{d/2}} \mathcal{N}^{\mathscr{O}}_{n,\mathbf{a}} \cdot q^{-\left(\sum_{i=1}^{d/2} n - a_i\right)(s+2)} = \mathcal{P}_{\mathcal{R},\mathscr{O}}(s+2).
\end{aligned}$$

Remark. The last equality is in fact Proposition 3.1 of [AKOV13]. In the original paper this proposition appears in greater generality, for a family of congruence subgroups of G which are potent and saturable.

2.4 Integral Formula

So far, we've transferred the question of representation enumeration into a Poincaré series whose coefficients are given by the set of linear equations modulo powers of the prime ideal $\wp$. In this section we will present the final development of AKOV machinery, formulating the zeta function of a potent and saturable pro-p group as a $\wp$-adic integral.

2.4.1 Evaluating Elementary Divisors

In the most general setting, given any $m \times m$ ($m \in \mathbb{N}$) matrix $A \in \mathrm{Mat}_m(\mathcal{O})$ and $0 \le k \le m$ one can define an ideal $\mathcal{D}_k(A) \subseteq \mathcal{O}$ generated in $\mathcal{O}$ by all $k \times k$ minors of A. Such an ideal is called a **determinantal ideal** associated to A.

Note that $\mathcal{D}_{k+1}(A) \subseteq \mathcal{D}_k(A)$ by the minor formula for the determinant. Moreover, if $B \in \mathrm{Mat}_m(\mathcal{O})$ can be obtained from A by elementary row and column operations over $\mathcal{O}$, then $\mathcal{D}_k(A) = \mathcal{D}_k(B)$ for all $k = 1, \dots, m$.

Similarly, we let $\mathcal{D}_k^{\mathrm{Pr}}(A)$ denote the ideal generated by all principal $k \times k$ minors of A. Clearly, $\mathcal{D}_k^{\mathrm{Pr}}(A) \subseteq \mathcal{D}_k(A)$, for all k's.

Returning to our case, for any $j = 1, \dots, \frac{d}{2}$, let $F_j(\mathbf{Y}) \subseteq \mathcal{O}[\mathbf{Y}]$ denote the set of all $2j \times 2j$ minors of $\mathcal{R}(\mathbf{Y})$. For any $\mathbf{y} \in W(\mathcal{O})$ and $j = 1, \dots, \frac{d}{2}$ put

$$\|F_j(\mathbf{y})\|_{\wp} := \max\left\{ |f(\mathbf{y})|_{\wp} \mid f \in F_j(\mathbf{Y}) \right\},$$

and $\|F_0(\mathbf{y})\|_{\wp} = 1$.

Note that $\|F_j(\mathbf{y})\|_{\wp}$ is the absolute value of any generating element of the ideal $\mathcal{D}_{2j}\left(\mathcal{R}(\mathbf{y})\right)$. Put $\nu(\mathbf{y}) = (a_1, \dots, a_{d/2})$. Since $\mathcal{R}(\mathbf{y})$ is congruent to the matrix $\mathrm{ASNF}(\nu(\mathbf{y}))$ their determinantal ideals are the same. Thus, one easily verifies that

$$\|F_j(\mathbf{y})\|_{\wp} = q^{-2\sum_{i=1}^{j} a_i}, \quad \forall j = 1, \dots, \tfrac{d}{2}.$$

Moreover, if we fix $x \in \wp$ with $|x|_{\wp} = q^{-n}$ for some $n \in \mathbb{N}$ we have that

$$\left\|F_j(\mathbf{y}) \cup x^2 F_{j-1}(\mathbf{y})\right\|_{\wp} = q^{-2\sum_{i=1}^{j-1} a_i - 2\min\{n, a_j\}}.$$

Combining the two above equalities, we obtain the following equation:

$$\prod_{j=1}^{d/2} \frac{\|F_j(\mathbf{y}) \cup x^2 F_{j-1}(\mathbf{y})\|_{\wp}}{\|F_{j-1}(\mathbf{y})\|_{\wp}} = q^{-2\sum_{i=1}^{d/2} \min\{n, a_i\}}, \quad \forall x \in \wp^n \smallsetminus \wp^{n+1} \tag{2.B}$$

Before proceeding to the final step of this computation, we remark that in our case the value of $\|F_j(\mathbf{y})\|_{\wp}$ is obtained on the subset of principal minors of $\mathcal{R}(\mathbf{y})$. This can be seen easily from the fact that for any anti-symmetric matrix the determinantal ideals of minors of even order coincide with the ideals generated by all principal minors, as is evident from their normal forms.

2.4.2 Poincaré Series as a $\wp$-adic Integral

Put

$$\rho := \frac{1}{2}\max\left\{\mathrm{rk}_F(\mathcal{R}(\mathbf{y})) \mid \mathbf{y} \in W(\mathscr{O})\right\},$$

where F is the fraction field of $\mathscr{O}$, and let

$$P(x,\mathbf{y}) := \prod_{j=1}^{\rho} \frac{\|F_j(\mathbf{y}) \cup x^2 F_{j-1}(\mathbf{y})\|_{\wp}}{\|F_{j-1}(\mathbf{y})\|_{\wp}}.$$

By definition, it is clear that for any $\mathbf{y}$, with $\nu(\mathbf{y}) = (a_1, \dots a_{d/2})$, one has $a_i = \infty$ whenever $i > \rho$. Thus, by equation (2.B), we have the equality

$$q^{-2\sum_{i=1}^{d/2}\min\{n,a_i\}} = P(x,\mathbf{y}) \cdot q^{-2(d/2-\rho)n} = P(x,\mathbf{y}) \cdot |x|_{\wp}^{d-2\rho},$$

whenever $|x|_{\wp} = q^{-n}$.

Let

$$\mathcal{Z}_{\mathscr{O}}(r,t) := \int_{(x,\mathbf{y})\in\wp\times W(\mathscr{O})} |x|_{\wp}^{t} P(x,\mathbf{y})^{r} d\mu(x,\mathbf{y}),$$

where μ is the probability Haar measure on $\mathscr{O}^{d+1}$.

Proposition 2.4.3 ([AKOV13, §3.2]).

$$\mathscr{P}_{\mathcal{R},\mathscr{O}}(s) = 1 + (1-q^{-1})^{-1}\mathcal{Z}_{\mathscr{O}}(-s/2, \rho s - d - 1).$$

Proof.

$$(1-q^{-1})^{-1}\mathcal{Z}_{\mathscr{O}}(-s/2,\rho s-d-1) =$$

$$= \int_{x\in\wp} \frac{1}{(1-q^{-1})\,|x|_{\wp}} \int_{\mathbf{y}\in W(\mathscr{O})} |x|_{\wp}^{\rho s-d} P(x,\mathbf{y})^{-s/2} d\mathbf{y}dx$$

$$= \sum_{n=1}^{\infty} \int_{x\in\wp^n\smallsetminus\wp^{n+1}} \frac{1}{(1-q^{-1})\,|x|_{\wp}} \int_{\mathbf{y}\in W(\mathscr{O})} |x|_{\wp}^{-d} \left(|x|_{\wp}^{-d}\,|x|_{\wp}^{d-2\rho} P(x,\mathbf{y})\right)^{-s/2} d\mathbf{y}dx$$

$$= \sum_{n=1}^{\infty} \int_{x\in\wp^n\smallsetminus\wp^{n+1}} \frac{1}{\mu(\wp^n\smallsetminus\wp^{n+1})} \int_{\mathbf{y}\in W(\mathscr{O})} \frac{1}{\mu\left((\wp^n)^{(d)}\right)} \cdot q^{-\left(\sum_{i=1}^{d/2}(n-\min\{n,a_i\})\right)s} d\mathbf{y}dx$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{\overline{\mathbf{y}}\in W_n(\mathscr{O})\\ \nu(\overline{\mathbf{y}})=(a_1,\dots,a_{d/2})}} q^{-\left(\sum_{i=1}^{d/2}(n-\min\{n,a_i\})\right)s}$$

$$= \sum_{n=1}^{\infty} \sum_{\mathbf{a}\in(\mathbb{N}\cup\{0\})^d} \mathcal{N}_{n,\mathbf{a}} q^{-\left(\sum_{i=1}^{d/2}(n-\min\{n,a_i\})\right)s} = \mathcal{P}_{\mathcal{R},\mathscr{O}}(s) - 1$$

□

2.4.4 A Precise Statement (cf. [AKOV13, Corollary 3.7])

Let G be a potent and saturable pro-p group, with $\mathfrak{g}$ its associated $\mathbb{Z}_p$-Lie-algebra. Then

$$\zeta_G(s) = 1 + (1 - q^{-1})^{-1} \mathcal{Z}_{\mathscr{O}}(-s/2 - 1, \rho(s + 2) - d - 1),$$

where $\mathcal{Z}_{\mathscr{O}}(r, t)$ is the $\wp$-adic integral described in the previous section.

Chapter 3

Division Algebras

In this chapter, we will review the basic properties of finite dimensional division algebras over local fields. When discussing local fields, we will implicitly assume that the field is non-archemedian and of characteristic 0 (i.e. a finite extension of $\mathbb{Q}_p$).

The main theorem of this chapter gives a complete description of the structure of such algebras. Namely, we will see that any division algebra contains an unramified extension of its center as a maximal subfield, and is determined by a choice of an element of the Galois group of a this unramified extension of the ground field. It follows that up to isomorphism, the number of choices of a division algebra of a given degree is finite.

We start off with some general definitions and facts, regarding associative algebras over arbitrary fields.

3.1 Background

3.1.1 Central Simple Algebras

Let F be a field, and let A be an algebra over F. Recall that the algebra A is called **simple** (or **irreducible**) if the only proper two-sided ideal of A is $\{0\}$. The algebra A is called **central** over F if the center of A coincides with F (i.e. $Z(A) = F$). Following [Pie82], we use the notation $\mathfrak{S}(F)$ to denote the set of isomorphism classes of finite-dimensional algebras which are central simple over F.

A **subfield** of A is simply a sub-F-algebra $E \subseteq A$ which is also a field. Note that any subfield of A is a field extension of the ground field F.

Proposition 3.1.2 ([Pie82, Proposition 13.1])**.** Let $A \in \mathfrak{S}(F)$, and let $E \subseteq A$ be a subfield, with $|E : F| = k$. The following conditions are equivalent

1. E is a maximal subfield of A (i.e. if $K \subseteq A$ is a subfield which contains E, then $E = K$),
2. $C_A(E) := \{x \in A \mid \ xa = ax, \ \forall a \in E\}$ is isomorphic to the matrix algebra $\mathrm{Mat}_n(E)$, for some $n \in \mathbb{N}$, and E has no proper field extension $K \supseteq E$ such that $|K : E|$ divides n.

If both conditions are satisfied, then $\dim_F A = (kn)^2$.

For any $A \in \mathfrak{S}(F)$, we define the **degree** of A by

$$\deg A := (\dim_F A)^{1/2}.$$

Note that by Proposition 3.1.2 $\deg A$ is always a positive integer, and the degree of any subfield of A divides $\deg A$ (cf. [Pie82, Corollary 13.1.a]).

The Reduced Norm and Trace

Let $A \in \mathfrak{S}(F)$. A **splitting representation** is any F-algebra homomorphism $\phi : A \to \mathrm{Mat}_m(K)$, in which $m = \deg A$ and $K \supseteq F$ is a field extension. Given a splitting representation, we define the reduced norm and trace of an element in A by

$$\mathrm{Nrd}_{A/F}(x) = \det \phi(x), \quad \text{and} \quad \mathrm{Trd}_{A/F}(x) = \mathrm{Tr}\phi(x), \quad \forall x \in A.$$

This definition is shown in [Pie82, §16.1] to be independent of the choice of K and ϕ. Furthermore, the functions $\mathrm{Nrd}_{A/F}$ and $\mathrm{Trd}_{A/F}$ are shown to be well-behaved with respect to any F-homomorphism which maps A into a matrix algebra over a field, in the following sense.

Lemma 3.1.3 ([Pie82, Corollary 16.1.a])**.** Let $A \in \mathfrak{S}(F)$ have degree n, and let $\psi : A \to \mathrm{Mat}_m(L)$ be an F-algebra homomorphism (i.e a multiplicative F-linear map, such that $\psi(1_A)$ is the identity matrix of order m), with $L \supseteq F$ a field extension. Then n divides m, and

$$\det \psi(x) = \mathrm{Nrd}_{A/F}(x)^{m/n}, \quad \text{and} \quad \mathrm{Tr}\psi(x) = \frac{m}{n}\mathrm{Trd}_{A/F}(x), \quad \forall x \in A.$$

3 Division Algebras

In the next section we will exhibit one such splitting representation for a class of F-algebras, which includes all division algebras over local fields. Once this is done, we will be able to extract an explicit formula for the reduced trace of an arbitrary element.

Let us finish this introductory section by citing a fundamental structure theorem for central simple algebras over F.

Theorem 4 (Wedderburn, [Pie82, Theorem 3.5])**.** *Let A be a central simple F algebra. Then there exists a number $m \in \mathbb{N}$ and a division algebra D over F, such that A is isomorphic to the matrix algebra $\mathrm{Mat}_m(D)$.*

Corollary 3.1.4. Suppose $A \in \mathfrak{S}(F)$ has $\deg A = \ell$, where ℓ is a prime number. Then only one of the following cases can occur

1. $A \cong \mathrm{Mat}_\ell(F)$,
2. A is a division algebra over F.

3.1.5 Cyclic Algebras

Let F be a field, and let $A \in \mathfrak{S}(F)$.

Definition 3.1.6. The algebra A is called **cyclic**, if A contains a maximal subfield E, such that E/F is a cyclic Galois extension.

The main appeal of cyclic algebras is that their structure is fairly well-understood, as is evident in the following proposition.

Proposition 3.1.7 ([Pie82, Proposition 15.1.a])**.** Let E/F be a cyclic Galois extension of degree n, with Galois group $\mathbf{G}(E/F)$ generated by the element σ. Let $A \in \mathfrak{S}(F)$ contain E as a maximal subfield. Then there exist elements $u \in A^\times$ and $a \in F^\times$, such that

1. $A = \bigoplus_{j=0}^{n-1} u^j E$,
2. $u^n = a$, and
3. $u^{-1} d u = \sigma(d)$, $\forall d \in E$.

Conversely, if $A \in \mathfrak{S}(F)$ is defined by the above three conditions then A is cyclic, with $E \subseteq A$ a maximal subfield.

Remark. From the perspective of cohomology of cyclic groups, one can view the algebras defined in Proposition 3.1.7 as a crossed product $(E, \mathbf{G}(E/F), \alpha)$, where the cocycle condition α is given by

$$\alpha(\sigma^i, \sigma^j) = \begin{cases} 1 & \text{if } i+j < n \\ a & \text{if } i+j \geq n \end{cases}, \quad \forall 0 \leq i, j < n,$$

and the action is the Galois action.

To simplify notation, we will write (E, σ, a) to denote the the F-algebra defined by conditions (1), (2) and (3) of Proposition 3.1.7.

A Splitting Representation for Cyclic Algebras

Let $A \in \mathfrak{S}(F)$ be the cyclic F-algebra (E, σ, a), with $\deg A = n$. Note that the underlying vector space of A can be identified with the space E^n. Thus, any element $x \in A$ is associated to a linear transformation $\lambda_x : A \to A$ given by $\lambda_x(y) = xy$. Consequently, we obtain the following proposition.

Proposition 3.1.8. The map $x \mapsto \lambda_x : A \to End(E^n) = M_n(E)$ is a splitting representation of A. The matrix representation of an element $x = \sum_{j=0}^{n-1} u^j x_j$ $(x_0, \ldots, x_{n-1} \in E)$ with respect to the basis $\{1, u, \ldots, u^{n-1}\}$ is given by

$$\begin{pmatrix} x_0 & a\sigma(x_{n-1}) & a\sigma^2(x_{n-2}) & & a\sigma^{n-1}(x_1) \\ x_1 & \sigma(x_0) & a\sigma^2(x_{n-1}) & \ldots & a\sigma^{n-1}(x_2) \\ x_2 & \sigma(x_1) & \sigma^2(x_0) & & a\sigma^{n-1}(x_3) \\ \vdots & & & \ddots & \vdots \\ x_{n-1} & \sigma(x_{n-2}) & \sigma^2(x_1) & \ldots & \sigma^{n-1}(x_0) \end{pmatrix}$$

(where the a's occur in all entries strictly above the main diagonal).

Proof. The fact that λ_x is a linear operator, and that $\lambda_{\mu x + \nu y} = \mu\lambda_x + \nu\lambda_y$ for any $x, y \in A$ and $\mu, \nu \in F$ are simply the distributive laws of multiplication in A, so that $x \mapsto \lambda_x$ is an F-linear transformation. Also, note that for any $x, y \in A$ and $z \in A$,

$$\lambda_x \circ \lambda_y(z) = \lambda_x(y \cdot z) = x(y \cdot z) = (x \cdot y)z = \lambda_{x \cdot y}(z)$$

so that this map is in fact an F-algebra homomorphism. Since $n = \deg A$, this is a splitting representation.

To show the final assertion, let $x = \sum_{j=0}^{n-1} u^j x_j$ and $k \in \{0, \ldots, n-1\}$ be arbitrary. Then

$$\begin{aligned}\lambda_x(u^k) &= \sum_{j=0}^{n-1} u^j x_j u^k = \sum_{j=0}^{n-1} u^{j+k}\sigma^k(x_j)\\ &= \sum_{j=0}^{k-1} a u^j \sigma^k(x_{n+j-k}) + \sum_{j=k}^{n-1} u^j \sigma^k(x_{j-k})\end{aligned}$$

In coordinates, this means that the vector represented by $\left(0, \ldots, 0, 1, 0, \ldots, 0 \right)^T$ in the basis $\{1, u, \ldots, u^n\}$ (where 1 appears only in the k-th entry), is mapped by λ_x to the vector

$$\begin{pmatrix} a\sigma^k(x_{n-k}) \\ \vdots \\ a\sigma^k(x_{n-1}) \\ \sigma^k(x_0) \\ \vdots \\ \sigma^k(x_{n-k-1}) \end{pmatrix},$$

in accordance with the proposed matrix representation. □

In particular, we obtain a simple formula for the reduced trace of an element in A. Let $x = \sum_{j=0}^{n-1} x_j u^j$, then

$$\operatorname{Trd}_{A/F}(x) = \operatorname{Tr}_{E/F}(x_0). \tag{3.A}$$

Criterion for Isomorphism of Cyclic Algebras

Proposition 3.1.9. Let $A_1 = (E, \sigma, a_1)$, $A_2 = (E, \sigma, a_2) \in \mathfrak{S}(F)$ be two cyclic algebras of degree n. Then $A_1 \cong A_2$ if and only if there exists an element $c \in E^\times$ such that $a_1 = \operatorname{Nr}_{E/F}(c) a_2$.

Proof. Write $A_1 = \bigoplus_{j=0}^{n-1} u^j E$ and $A_2 = \bigoplus_{j=0}^{n-1} v^j E$, with u, v as in Proposition 3.1.7, such that $u^n = a_1$ and $v^n = a_2$.

Assume $\psi : A_1 \to A_2$ is an isomorphism of F-algebras. Note that since ψ is F linear, its restriction to E is an F-automorphism. Since the extension E/F is Galois (and in particular normal), this implies that $\psi_{|E} \in \mathbf{G}(E/F)$. Let $0 \leq r < n$ be such that $\psi_{|E} = \sigma^r$. Then, for any $d \in E$, we have that

$$\psi(u)^{-1} v d = \psi(u^{-1})\sigma^{-1}(d)v = \psi(u^{-1}\sigma^{-r-1}(d))v = \psi(\sigma^{-r}(d)u^{-1})v = d\psi(u)^{-1}v,$$

and hence $\psi(u)^{-1}v \in C_{A_2}(E) = E$. Let $c \in E$ be such that $\psi(u) = cv$, then

$$a_1 = \psi(u^n) = (cv)^n = c\sigma(c)\dots\sigma^{n-1}(c)v^n = \mathrm{Nr}_{E/F}(c)a_2.$$

For the converse, one simply checks that the map $u \mapsto cv$ induces an isomorphism of F-algebras. $\square$

3.2 Division Algebras Over a Local Field

From here we let $F \supseteq \mathbb{Q}_p$ be a local field, with $\mathscr{O}$ its maximal compact subring, and $\wp$ its unique maximal ideal. Recall that a finite extension E/F is called **unramified** if $\wp_E = \wp \cdot \mathscr{O}_E$, where $\mathscr{O}_E$ and $\wp_E$ are the compact subring and maximal ideal of E. Let us recall an important fact about unramified extensions of local field.

Proposition 3.2.1 ([Pie82, Corollary 17.8.a])**.** Let Ω denote an algebraic closure of F. For any $n \in \mathbb{N}$, there is a unique field $K_n \subseteq \Omega$ such that $|K_n : F| = n$ and K_n/F is unramified. Furthermore, K_n/F is a cyclic Galois extension.

In this section we will review the proof of the following structure theorem for division algebras over local fields.

Theorem 5. *Let $D \in \mathfrak{S}(F)$ be a division algebra, with $\deg D = n$, and let $E = K_n \supseteq F$ be the unique unramified extension of degree n. Then D is a cyclic algebra over F, and contains E as a maximal subfield.*

Taking into account Theorem 5, along with Proposition 3.1.9 and corollary 3.1.4 we will obtain the following corollary.

Corollary 3.2.2. Suppose $\deg D = \ell$ is prime. There are precisely $\ell - 1$ isomorphism classes of division algebras over F, and a complete set of representatives for those classes is given by

$$(E, \sigma, \pi), \quad \sigma \in \mathbf{G}(E/F) \smallsetminus \{1\},$$

where E is the unramified extension of F of degree ℓ.

Before proceeding to the proof, let us fix some notations that we will use throughout the remainder of this chapter.

- As before, $F \supseteq \mathbb{Q}_p$ is a local field, $\mathscr{O}$ denotes the maximal compact subring of F and $\wp$ the maximal ideal of $\mathscr{O}$. Let $\mathfrak{f} := \mathscr{O}/\wp$ denote the residue class field of F, and q be its cardinality.
- The letter π stands for a uniformizing parameter at $\wp$.
- $|\cdot|_\wp$ denotes the normalized absolute value on F, such that $|\pi|_\wp = q^{-1}$.
- $D \in \mathfrak{S}(F)$ is a fixed division algebra.

3.2.3 Local Structure of D

An Absolute Value

Similar to the case of local fields, one can define a (non-archemedian) absolute value on D to be a map $|\cdot| : D \to \mathbb{R}^{\geq 0}$ satisfying the conditions

1. $|x| \geq 0$ for any $x \in D$, with equality iff $x = 0$,
2. $|xy| = |x|\,|y|\,,\ \forall x, y \in D$,
3. $|x+y| \leq \max\{|x|\,,|y|\}\,,\ \forall x, y \in D$.

It is known [Pie82, Theorem 17.1] that any absolute value on D satisfies the equality

$$|x| = \left|\mathrm{Nrd}_{D/F}(x)\right|^{1/n}, \quad \forall x \in D,$$

and hence is determined by its values on F. Since the absolute value on F is unique (up to exponentiation), we may assume that the absolute value on D coincides with $|\cdot|_\wp$ on F, and denote it by $|\cdot|_\wp$ as well. Additionally, it follows from the above formula that any F-algebra automorphism of D is an isometry with respect to $|\cdot|_\wp$.

The absolute value on D endows it with the topology of a locally compact space [Pie82, Proposition 17.6]. Let

$$\mathcal{O} := \left\{x \in D \mid |x|_\wp \leq 1\right\}, \quad \text{and} \quad \mathcal{P} := \left\{x \in D \mid |x|_\wp < 1\right\}$$

denote the closed and open unit balls of D. Note that $\mathcal{O}$ is compact. Furthermore, it holds that $\mathcal{O}$ is a local non-commutative ring and $\mathcal{P}$ is the unique maximal two-sided ideal of $\mathcal{O}$.

A Maximal Unramified Subfield

The ring $\mathcal{O}/\mathcal{P}$ is a division algebra (as the quotient of a local ring by its maximal ideal). Moreover, the compactness of $\mathcal{O}$ implies that $\mathcal{O}/\mathcal{P}$ is finite and hence, by Wedderburn's finite division algebra theorem [Pie82, §13.6], a finite field. Let us denote this field by $\mathfrak{e}$, and call it the **residue class field** of D. Note that since the ideal $\wp$ is contained in $\mathcal{P}$, there is a natural inclusion $\mathfrak{f} \subseteq \mathfrak{e}$. Let

$$f(D/F) := \dim_{\mathfrak{f}} \mathfrak{e}$$

be the **residue class degree** of D.

The residue class field has an important part in the following generalization of Hensel's Lemma.

Proposition 3.2.4 (Hensel's Lemma, [Pie82, §17.4])**.** Let D be a finite dimensional division algebra over F, and let $p(T) \in \mathscr{O}[T]$ be a polynomial. Let $\overline{p}(T)$ and $\overline{p}'(T)$ denote the respective images of p and its formal derivative p' in $\mathfrak{f}(T)$, and assume that $\gcd(\overline{p}(T), \overline{p}'(T)) = 1$.

If $\xi \in \mathfrak{e}$ is a root of $\overline{p}(T)$, then there exists an element $x \in \mathcal{O}$ such that $p(x) = 0$ and $x + \wp = \xi$.

Let us extend the definition of ramification index to division algebras in the following manner. Put $\Gamma_D := \left\{ |x|_\wp \mid x \in D^\times \right\}$, and $\Gamma_F := \left\{ |x|_\wp \mid x \in F^\times \right\}$. It holds that $\Gamma_D \supseteq \Gamma_F$, and that both sets are infinite cyclic subgroups of $\mathbb{R}^{>0}$. Let

$$e(D/F) = |\Gamma_D : \Gamma_F|$$

be the **ramification index** of D/F. Note that this definition coincides with the standard definition of ramification in the case where D is a field.

We are now in a position to state the main proposition in the proof of Theorem 5.

Proposition 3.2.5 ([Pie82, Proposition 17.7])**.** Let D be a finite dimensional division algebra over a local field F.

1. $\dim_F D = e(D/F) f(D/F)$.

2. There is a subfield E of D such that the extension E/F is unramified, with residue class field equal $\mathfrak{e}$, and such that $|E : F| = f(D/F)$.

Furthermore, if $F = Z(D)$ then E is a maximal subfield.

We review the proof of item (2). By the theory of finite fields, $\mathfrak{e}$ is the splitting field of an irreducible polynomial $p(T) \in \mathfrak{f}[T]$. By a simple arithmetic argument one sees that any polynomial $\tilde{p}(T) \in \mathscr{O}[T]$, which is mapped to $p(T)$ by the projection map, is irreducible in F. Fix such a polynomial $\tilde{p}(T) \in \mathscr{O}[T]$. By Hensel's Lemma, since p has roots in $\mathfrak{e}$, the splitting field of $\tilde{p}(T)$ is contained in D.

Let $E \subseteq D$ be the splitting field of such a polynomial $\tilde{p}(T)$. One easily verifies that E indeed satisfies all assertions of the proposition. To prove that E is maximal in the case where $F = Z(D)$, one shows that in fact $f(D/F) = \deg D$, and invokes Proposition 3.1.2.

Proof of Corollary 3.2.2

Let us summarize the fact thus far. Our division algebra $D \in \mathfrak{S}(F)$ has F as its center, and thus contains a maximal subfield E. Furthermore, since E/F is unramified, by Proposition 3.2.1 we have that E/F is a cyclic Galois extension. Thus, by definition, D is cyclic and Theorem 5 holds.

At this point we are ready to present a proof of corollary 3.2.2. Let $\mathscr{O}_E$ denote the compact subring of E, and write $\mathscr{O}_F := \mathscr{O}$. Let $\sigma \in \mathbf{G}(E/F)$ be a fixed of the Galois group. In accordance with 3.2.2 we assume from here on that $\deg D = \ell$, is a prime number.

Let recall a basic facts about field norms.

Lemma 3.2.6 ([Pie82, Lemma 17.9.b])**.** Assuming E/F is unramified of degree n, the norm map $\mathrm{Nr}_{E/F} : E \to F$ induces a surjection $\mathscr{O}_E^\times \twoheadrightarrow \mathscr{O}_F^\times$.

From this, we obtain the following

Proposition 3.2.7. Let D be a division algebra of prime degree ℓ over F, then

$$D \cong (E, \sigma, \pi^k),$$

for some $k = 1, \dots, \ell - 1$.

Proof. Applying Lemma 3.2.6 and Proposition 3.1.9, we have that if $D = (E, \sigma, a)$ for some $a \in F$, and $n \in \mathbb{N}$ is such that $|\pi^n|_\wp = |a|_\wp$, then

$$D \cong (E, \sigma, \pi^n).$$

Taking into account the fact that $\pi^\ell = \mathrm{Nr}_{E/F}(\pi)$, and applying Lemma 3.2.6 again, we deduce that any division algebra over F is isomorphic to one of the cyclic algebras

$$(E, \sigma, \pi^j), \quad j = 1, \dots, \ell - 1.$$

□

Corollary 3.2.2 now follows from the following lemma:

Lemma 3.2.8. Let $k, j \in \{1, \dots, \ell - 1\}$ be such that $jk \equiv 1 (\mod \ell)$. Then

$$(E, \sigma, \pi^k) \cong (E, \sigma^j, \pi).$$

Proof. Put $D_1 := (E, \sigma, \pi^k)$ and $D_2 := (E, \sigma^j, \pi)$. Let $u \in D_1^\times$ and $v \in D_2^\times$ be such that

$$D_1 = \bigoplus_{i=0}^{\ell-1} u^i E, \quad D_2 = \bigoplus_{i=0}^{\ell-1} v^i E,$$

with $u^\ell = \pi^k$, $v^\ell = \pi$ and such that

$$u^{-1} du = \sigma(d) \quad \text{and} \quad v^{-1} dv = \sigma^j(d), \quad \forall d \in E.$$

Define a map $\psi : D_1 \to D_2$ by

$$\psi(u) = v^k,$$

and $\psi(d) = d$ for any $d \in E$. Extend ψ to D_1 in the obvious manner, by setting

$$\psi(\sum_{i=0}^{\ell-1} u^i x_i) = \sum_{i=0}^{\ell-1} \psi(u)^i \psi(x_i) = \sum_{i=0}^{\ell-1} v^{ki} x_i,$$

where $x_0, \dots, x_{\ell-1} \in E$. It is clear that ψ is an F-linear transformation, and that $\psi(u^m) = \psi(u)^m$ for any $m \in \mathbb{Z}$. Note that for any $d \in D_1$ we have that

$$\psi(u)^{-1} \psi(d) \psi(u) = v^{-k} d v^k = \sigma^{jk}(d) = \sigma(d) = \psi(u^{-1} du).$$

From this it follows easily that $\psi(xy) = \psi(x)\psi(y)$ for any $x, y \in D_1$, and hence ψ is an F-algebra homomorphism. Thus, by the fact that D_1 is simple, and that $\dim_F D_1 = \dim_F D_2$, ψ an isomorphism.

Since $\{\sigma^j \mid j = 1 \dots \ell - 1\} = \mathbf{G}(E/F) \smallsetminus \{1\}$ for any $\sigma \neq 1$, the claim follows. □

Chapter 4

The Group $SL_1(D)$

4.1 Congruence Subgroups

Let ℓ, p be two distinct prime numbers, with $p \geq 3$, and let $F \supseteq \mathbb{Q}_p$ be a local field. Let D be a division algebra of degree ℓ over F, and let $E \subseteq D$ be the maximal unramified extension of F in D. We retain the notations for the maximal sub-rings and maximal ideals of F, E and D defines in Section 3.2.3. By Corollary 3.2.2, we have that

$$D = (E, \sigma, \pi),$$

for π a uniformizing parameter at $\wp_F$, and σ some element of $\mathbf{G}(E/F)$.

Let

$$G = SL_1(D) := \left\{ x \in D \mid \mathrm{Nrd}_{D/F}(x) = 1 \right\},$$

and put, for any $m \in \mathbb{N}$,

$$G_m = SL_1^m(\mathcal{O}) := \left\{ x \in SL_1(D) \mid x \equiv 1 \bmod \pi^{m-1}\mathcal{P} \right\} = G \cap (1 + \pi^{m-1}\mathcal{P}).$$

Then G is a closed subgroup of the group $\mathcal{O}^\times$, and hence is compact. Furthermore, the groups G_i are normal subgroups of G, and comprise a basis of normal open subgroups at 1, in the topology induced from D. In particular G is a pro-finite group.

In general, the group G itself is not pro-p (e.g. in our essay, the quotient G/G_1 is isomorphic to a group of order prime to p, cf. § 9.2). Nonetheless, the groups G_m are all subgroups of the pro-p group $1 + \mathcal{P}$, and thus are pro-p.

The groups G_m are called the **congruence subgroups of** G. We have the following

Proposition 4.1.1 ([AKOV12, Proposition 4.2])**.** In the setting described above, $G_1 = SL_1^1(\mathcal{O})$ is an insoluble maximal p-adic analytic just infinite pro-p group. Moreover, if $e(F, \mathbb{Q}_p) \cdot \ell < p - 1$ then G_1 is potent and saturable.

Furthermore, it is known that the subgroups G_m are potent and saturable for $m \in \mathbb{N}$ sufficiently large , regardless of the value of $e(F/\mathbb{Q}_p)$ (cf. [AKOV13, Proposition 2.3]).

4.2 Associated $\mathscr{O}_F$-Lie Algebras

The left regular representation of D on F^{ℓ^2}, given by $x \mapsto \Lambda_x$ where $\Lambda_x(y) = x \cdot y$ for any $y \in D \cong F^{\ell^2}$, embeds G as a closed subgroup of $SL_{\ell^2}(F)$ (by Lemma 3.1.3), and hence endows it with the structure of an algebraic group. As such, it is naturally associated to a Lie algebra over F, denoted as $\mathscr{G}$.

Applying a similar argument to [Wat79, § 12.3 (b)], we identify $\mathscr{G}$ with the F-Lie algebra of elements $x \in D$ with $\mathrm{Trd}_{D/F}(x) = 0$.

For any $m \in \mathbb{N}$ we put

$$\mathfrak{g}_m := \mathfrak{sl}_1^m(\mathcal{O}) = \mathscr{G} \cap \pi^{m-1}\mathcal{P}.$$

For m sufficiently large we have that $\mathfrak{g}_m$ is potent and saturable $\mathbb{Z}_p$-Lie algebra. In this case the exponential series $x \mapsto \sum_{j=0}^{\infty} \frac{x^j}{j!}$ is convergent on $\mathfrak{g}_m$ and induces an isomorphism of the group $\exp(\mathfrak{g})$ (defined in § 1.2), and the subgroup G_m, which is potent and saturable. In the sequel, we will identify G_m with $\exp(\mathfrak{g}_m)$ and $\mathfrak{g}_m$ with $\log(G_m)$.

We conclude this chapter with a calculation of an invariant value of the Lie-algebras $\mathfrak{g}_m$, which will be important in the course of our computation.

4.2.1 The Parameter ρ

In this section we calculate the value of the parameter ρ, defined in Section 2.4.2 for the $\mathscr{O}_F$-Lie algebras $\mathfrak{sl}_1^m(\mathcal{O})$. Let $\mathfrak{g} := \mathfrak{sl}_1^m(\mathcal{O})$ for some $m \in \mathbb{N}$. Recall that the parameter ρ is defined as

$$\rho := \frac{1}{2} \max \{\mathrm{rk}_F \mathcal{R}(\mathbf{y}) \mid \mathbf{y} \in W(\mathscr{O}_F)\},$$

where $\mathcal{R}(\mathbf{y})$ is the commutator matrix, defined in Section 2.2. Note that a-priori, since the definition of $\mathcal{R}(\mathbf{y})$ depends on the choice of an $\mathscr{O}_F$-module basis for $\mathfrak{sl}_1^m(\mathcal{O})$, it is not clear that ρ is an invariant of $\mathfrak{g}$. Nonetheless, it is not hard to verify that ρ is independent of the choice of basis.

Indeed, if $\mathcal{B}$ and $\mathcal{C}$ are two $\mathcal{O}_F$-module, let $\mathcal{B}^\vee$, $\mathcal{C}^\vee$ denote their respective dual bases, with $\mathsf{M}^\vee$ the transformation matrix from $\mathcal{B}^\vee$ to $\mathcal{C}^\vee$. Then, for any $\mathbf{y} \in W(\mathcal{O}_F)$ we have that $\mathcal{R}_{\mathfrak{g},\mathcal{B}}(\mathbf{y})$ and $\mathcal{R}_{\mathfrak{g},\mathcal{C}}(\mathsf{M}^\vee \cdot \mathbf{y})$ (see § 2.2) represent the same anti-symmetric bilinear form, and thus have the same F-rank. It follows that

$$\max\left\{\mathrm{rk}_F \mathcal{R}_{\mathfrak{g},\mathcal{B}}(\mathbf{y}) \mid \mathbf{y} \in W(\mathcal{O}_F)\right\} \leq \max\left\{\mathrm{rk}_F \mathcal{R}_{\mathfrak{g},\mathcal{C}}(\mathbf{y}) \mid \mathbf{y} \in W(\mathcal{O}_F)\right\}.$$

By the symmetry of the argument, we obtain that ρ is independent of the choice of basis.

Proposition 4.2.2 (Cf. [AKOV13, § 5])**.** Put $d := \mathrm{rk}_{\mathcal{O}_F}\mathfrak{g} = \ell^2 - 1$, and let $\mathcal{G}$ denote the F-Lie algebra $\mathfrak{g} \otimes_{\mathcal{O}_F} F = \mathfrak{sl}_1(D)$. Then

$$d - 2\rho = \min\left\{\dim_F C_{\mathcal{G}}(y) \mid y \in \mathcal{G} \smallsetminus \{0\}\right\},$$

where $C_{\mathcal{G}}(y) := \{x \in \mathcal{G} \mid [y,x] = 0\}$.

Proof. Let $\kappa : \mathcal{G} \times \mathcal{G} \to F$ denote the Killing form of $\mathcal{G}$, given by $\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$, where ad_x is defined, for any $x \in \mathcal{G}$, by $\mathrm{ad}_x(z) = [x,z]$. The Killing form of $\mathcal{G}$ is non-degenerate (see Appendix B), and induces an isomorphism of $\mathcal{G}$ with its dual space $\mathrm{Hom}_F(\mathcal{G}, F)$, where any $x \in \mathcal{G}$ is mapped to the functional $\iota(x)(y) = \kappa(x,y)$.

Let $\mathbf{y} \in W(\mathcal{O}_F)$ be arbitrary. Fix an $\mathcal{O}_F$-module basis $\mathcal{B}$ of $\mathfrak{g}$, and let $\mathcal{B}^\vee$ denote its dual basis. Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_F}(\mathfrak{g}, \mathcal{O}_F)^*$ be such that $\mathbf{y}$ is the coordinate vector of φ with respect to $\mathcal{B}^\vee$. Let $y \in \mathcal{G}$ be such that $\iota(y) = \varphi$.

As mentioned in Section 2.2, the matrix $\mathcal{R}(\mathbf{y})$ represents the antisymmetric bilinear form

$$\langle x, y\rangle_\varphi = \varphi([x,y]), \quad \forall x, y \in \mathcal{G}.$$

Let $x \in \mathcal{G}$ be arbitrary, and let $\mathbf{x}$ be the coordinate vector of x with respect to $\mathcal{B}$. Then

$$\begin{aligned}
\mathbf{x} \cdot \mathcal{R}(\mathbf{y}) = 0 \quad &\iff \quad \varphi([x,z]) = 0, \quad \forall z \in \mathcal{G}, \\
&\iff \quad \iota(y)(\mathrm{ad}_x(z)) = 0, \quad \forall z \in \mathcal{G} \\
&\iff \quad \kappa(y, \mathrm{ad}_x(z)) = \kappa(\mathrm{ad}_y(x), z), \quad \forall z \in \mathcal{G} \\
&\iff \quad \mathrm{ad}_y(x) = [y,x] = 0.
\end{aligned}$$

Thus, $\mathbf{x}$ is in the left-kernel of $\mathcal{R}(\mathbf{y})$ iff $x \in C_{\mathcal{G}}(y)$.

It follows that

$$\mathrm{rk}_F \mathcal{R}(\mathbf{y}) = d - \dim_F \mathrm{Ker}\mathcal{R}(\mathbf{y}) = d - \dim_F C_{\mathscr{G}}(y).$$

In particular

$$\{\mathrm{rk}_F \mathcal{R}(\mathbf{y}) \mid y \in W(\mathscr{O}_F)\} \subseteq \{d - \dim C_{\mathscr{G}}(y) \mid y \in \mathscr{G} \smallsetminus \{0\}\}.$$

This is in fact an equality, by observing that for any non-zero $y \in \mathscr{G}$, there exists some $n \in \mathbb{Z}$ such that the restriction of $\pi^n \iota(y)$ to $\mathfrak{g}$ is an element $\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{g}, \mathscr{O}_F)^*$. Thus

$$2\rho = \max\{\mathrm{rk}_F \mathcal{R}(\mathbf{y}) \mid y \in W(\mathscr{O}_F)\} = d - \min\{\dim C_{\mathscr{G}}(y) \mid y \in \mathscr{G} \smallsetminus \{0\}\}.$$

□

Proposition 4.2.3. In the setting of the previous proposition. Then

$$\dim_F C_{\mathscr{G}}(y) = \ell - 1, \quad \text{for any } y \in \mathscr{G} \smallsetminus \{0\}.$$

Proof. Let $y \in \mathscr{G}$ be arbitrary. We have that

$$C_{\mathscr{G}}(y) = \{x \in \mathscr{G} \mid [x, y] = xy - yx = 0\} = C_D(y) \cap \mathscr{G},$$

where $C_D(y)$ is the centralizer of y in D (cf. § 3.1.2). Note that $C_D(y) \subseteq D$ is an F-division algebra. Indeed, it clearly holds that $1 \in C_D(y)$, and if $a, b \in C_D(y)$ then

- $(\lambda a + \mu b)y = \lambda ay + \mu by = y\lambda a + y\mu b = y(\lambda a + \mu b)$ for any $\lambda, \mu \in F$,
- $(ab)y = a(yb) = (ya)b = y(ab)$, and
- By multiplying the equality $ay = ya$ by a^{-1} from the left and from the right, we have that
$$a \in C_D(y) \Rightarrow a^{-1} \in C_D(y).$$

Thus, we have that $\dim_F C_D(y)$ divides ℓ^2, and hence is an element of $\{1, \ell, \ell^2\}$. Note that since $\{1, y\} \in C_D(y)$ is linearly independent over F, we have that $\dim_F C_D(y) \neq 1$.

Additionally, note that

$$\dim_F C_D(y) = \ell^2 \iff C_D(y) = D \iff y \in Z(D) = F,$$

which would imply that $y \in \mathscr{G} \cap F = \{0\}$.

So that $\dim_F C_D(y) = \ell$, whenever $y \neq 0$. Note that $C_{\mathscr{G}}(y) = C_D(y) \cap \mathscr{G}$ is the kernel of the restriction of the map $\mathrm{Trd}_{D/F}$ to $C_D(y)$, and hence

$$\dim_F C_{\mathscr{G}}(y) = \dim_F C_D(y) - 1 = \ell - 1,$$

since $\mathrm{Trd}_{D/F}$ is an F-linear functional. □

From this we have that

$$\rho = \frac{1}{2}\left(d - (\ell - 1)\right) = \frac{\ell(\ell - 1)}{2}.$$

Part II

Representation Zeta Function for Congruence Subgroups of $SL_1(D)$

Basic Notation and Outline

This part of our essay is dedicated to the analysis of the commutator matrix of the congruence subgroups of $SL_1(D)$, where D is a division algebra of prime degree over a non-archemdian local field of characteristic zero, under some additional assumptions on the ground field. The outline of this part is as follows:

- In chapter 5 we will review the $\mathscr{O}$-Lie algebras $\mathfrak{sl}_1^m(\mathcal{O})$, in order to obtain a specific $\mathscr{O}_F$-module basis for $\mathfrak{sl}_1^m(\mathcal{O})$ from which we will be able to calculate their commutator matrices.

  In the final section of this chapter we will break the commutator matrix down into sub-matrices, in order to obtain a coordinate free interpretation of the matrix $\mathcal{R}(\mathbf{y})$ for any $\mathbf{y} \in W(\mathscr{O})$ (cf. § 2.2 for the definition).

- In chapter 6 we will prove a combinatorial Lemma, which will provide us with a tool to calculate the determinant absolute values of certain $\wp$-adic matrices. These matrices occur naturally as sub-matrices of the commutator matrix we wish to analyse.

- In chapter 7 we will return to our coordinate free interpretation of $\mathcal{R}(\mathbf{y})$, and compute the ranks the distinguished sub-matrices describe in Section 5.4.

  Given this data, along with the above mentioned combinatorial lemma, we will show that the set $W(\mathscr{O})$ can be partitioned into ℓ disjoint subsets, on which the values of the function $\|F_j(\mathbf{y})\|_\wp$ have a common lower bound.

- Chapter 8 will be devoted to the calculation of the representation zeta function of $SL_1^1(\mathcal{O})$, where $\deg D = 3$, under the assumption that $SL_1^1(\mathcal{O})$ is potent and saturable, on the basis of the aforementioned lower bound and a naïve upper bound.

Basic Notation

We fix two distinct prime integers p and ℓ, with $p \geq 3$. Let $F \supseteq \mathbb{Q}_p$ be a local field, and $E \supseteq F$ the unique unramified extension of degree ℓ.

Retaining the notations of Section 3.2.3, we write $\mathscr{O}_F$, $\mathscr{O}_E$ and $\wp_F, \wp_E$ to denote the respective local compact subrings of E and F and their respective maximal ideals, and let π denote a fixed uniformizing parameter at $\wp_F$. The symbols $\mathfrak{f}$ and $\mathfrak{e}$ denote the respective residue class fields of F and E, and q is the cardinality of $\mathfrak{f}$. For any $x \in \mathscr{O}_E$, we will write $\overline{x} := x + \wp_E$ to denote the image of x in $\mathfrak{e}$. We continue to write $|\cdot|_\wp$ to denote the normalized absolute value on F, such that $|\pi|_\wp = q^{-1}$.

We fix D to be the division algebra (E, σ, π), as defined in Section 3.1.5, where σ is the local Frobenius automorphism of E/F (defined in Section 5.1). We denote by $\mathcal{O}$ and $\mathcal{P}$ the maximal compact local subring of D and its maximal ideal. Note that the element u, defined in Proposition 3.1.7, is a uniformizing parameter at $\mathcal{P}$.

In general, wherever there is a risk of confusion, we will use the notation $\wp^{(n)}$ to denote the n-fold Cartesian product of an ideal $\wp$, whereas the notation $\wp^n$ is reserved for the ideal

$$\wp^n = \left\{ x_1 \cdot \ldots \cdot x_n \mid (x_1, \ldots, x_n) \in \wp^{(n)} \right\}.$$

Remark. The choice of the specific element $\sigma \in \mathbf{G}(E/F)$ is done mainly for aesthetic reasons, in order to simplify notation in the sequel. As can be easily verified, the modifications needed in order to substitute σ for some $\sigma^j \in \mathbf{G}(E/F) \smallsetminus \{0\}$ are slight, and have no effect on the values calculated below. Thus, in view of Corollary 3.2.2, the computation for $D = (E, \sigma, \pi)$ generalizes easily to all division algebras of degree ℓ over F.

Chapter 5

Structure Theory of $\mathfrak{sl}_1^m(\mathcal{O})$

5.1 Some Field Theoretic Preparations

The Galois group $\mathbf{G}(\mathfrak{e}/\mathfrak{f})$ is cyclic of order ℓ, and is generated by the Frobenius automorphism $\mathfrak{fr}$, defined by

$$\mathfrak{fr}(\xi) = \xi^q, \quad \forall \xi \in \mathfrak{e}.$$

Let $\sigma \in \mathbf{G}(E/F)$ be the unique F-automorphism such that

$$\overline{\sigma(x)} = \mathfrak{fr}(\overline{x}), \quad \forall x \in \mathcal{O}_E,$$

(cf. [Gol71, §4.5]). Such an element $\sigma \in \mathbf{G}(E/F)$ is called a **local Frobenius automorphism** of E/F.

From here on we assume that F contains a primitive ℓ-th root of unity, denoted by $\omega \in F$, and let ϖ denote its image in $\mathfrak{f}$. This assumption allows us to invoke Kummer theory in order to find an element $b \in F$ such that E is the splitting field of the polynomial $T^\ell - b$. Since the extension E/F is unramified, we may assume that $b \in \mathcal{O}_F^\times$.

Let $\theta \in \mathcal{O}_E^\times$ be a solution of $T^\ell - b$ which satisfies $\sigma(\theta) = \omega \cdot \theta$. Such an element θ exists, since σ permutes the set $\{\omega^j\theta \mid j = 1, \ldots, \ell\}$ of ℓ-th roots of b. The set $\Theta := \{1, \theta, \ldots, \theta^{\ell-1}\}$ is an F-basis of E.

Furthermore, we have the following

Lemma 5.1.1.

$$\mathcal{O}_E = \mathcal{O}_F \oplus \theta\mathcal{O}_F \oplus \ldots \oplus \theta^{\ell-1}\mathcal{O}_F.$$

Proof. In view of [Gol71, Proposition 4.3.5], it suffices to prove that

$$\Delta(\Theta) := \det M^2 \in \mathscr{O}_F^\times,$$

where $M = \left(\sigma^i(\theta^j)\right)_{i,j=0}^{\ell-1}$. Note that $\det M^2 = \det(M \cdot M^T)$, and that

$$M \cdot M^T = \left(\sum_{i=0}^{\ell-1} \sigma^i(\theta^r \theta^m)\right)_{r,m=0}^{\ell-1} = \left(\mathrm{Tr}_{E/F}(\theta^{r+m})\right)_{r,m}.$$

Additionally, it holds that

$$\mathrm{Tr}_{E/F}(\theta^{r+m}) = \begin{cases} \ell & \text{if } r = m = 0 \\ \ell \cdot b & \text{if } r + m = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$\Delta(\Theta) = \det \begin{pmatrix} \ell & & & \\ & & & \ell b \\ & & \iddots & \\ & \ell b & & \end{pmatrix} = -\ell^\ell \cdot b^{\ell-1} \in \mathscr{O}_F^\times,$$

as $b, \ell \in \mathscr{O}_F^\times$. $\square$

From this we deduce that $\overline{\Theta} := \left\{\overline{1}, \overline{\theta}, \ldots, \overline{\theta^{\ell-1}}\right\}$ spans $\mathfrak{e}$ over $\mathfrak{f}$, and hence is an $\mathfrak{f}$-basis.

5.2 Congruence Sub-Lie-Algebras of $\mathfrak{sl}_1(D)$

As mentioned before (§ 4), we put $\mathscr{G} := \mathfrak{sl}_1(D)$ to denote the F-Lie-algebra of elements $x \in D$ such that $\mathrm{Trd}_{D/F}(x) = 0$.

Let $\mathfrak{sl}_1^1(\mathcal{O}) := \mathscr{G} \cap \mathcal{P} = \left\{x \in \mathcal{P} \mid \mathrm{Trd}_{D/F}(x) = 0\right\}$. For any $m \in \mathbb{N}$ put

$$\mathfrak{g}_m := \mathscr{G} \cap \pi^{m-1}\mathcal{P} = \left\{x \in \pi^{m-1}\mathcal{P} \mid \mathrm{Trd}_{D/F}(x) = 0\right\},$$

as in Section 4.2.

If $\mathcal{B}$ is an $\mathscr{O}_F$-module basis for $\mathfrak{g}_1$, then $\pi^{m-1}\mathcal{B}$ is clearly an $\mathscr{O}_F$-module basis for $\mathfrak{g}_m$ for any $m > 0$. It follows that

$$\mathcal{R}_{\mathfrak{g}_m,\pi^{m-1}\mathcal{B}}(\mathbf{Y}) = \pi^{m-1}\mathcal{R}_{\mathfrak{g}_1,\mathcal{B}}(\mathbf{Y}), \quad \forall m > 0.$$

Knowing this relation between the commutator matrices, in the sequel we will perform all calculations with respect to a monomial basis and the commutator matrix of $\mathfrak{g}_1$, as these invariants are independent of the potency and saturability of $\mathfrak{g}_1$.

The first step we take towards this calculation will be to find an $\mathscr{O}_F$-module basis for $\mathfrak{g}_1$, which will be overtly suitable for this computation, in the sense that it is conformally closed under the bracket operation.

5.2.1 Monomial Bases for $\mathfrak{sl}_1^m(\mathcal{O})$

Let us recall that the structure constants of $\mathfrak{g}_m$ with respect to a basis $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ are the elements $\left\{\lambda_{i,j}^h \mid i, j, h = 1, \ldots, d\right\} \subseteq \mathscr{O}_F$, where $d := \mathrm{rk}_{\mathscr{O}_F}\mathfrak{g}_m = \ell^2 - 1$, such that

$$[\mathbf{e}_i, \mathbf{e}_j] = \sum_{h=1}^{d} \lambda_{i,j}^h \mathbf{e}_h, \quad \forall i, j = 1, \ldots, d.$$

We call a basis $\mathcal{B}$ a **monomial basis** of $\mathfrak{g}_m$, if for every $1 \le i, j \le d$ there exists at most one $h \in \{1, \ldots, d\}$ such that $\lambda_{i,j}^h \neq 0$. Given such a monomial basis, we define $h(i, j)$ to be the unique $1 \le h \le d$ such that $\lambda_{i,j}^h \neq 0$ if it exists (for any i, j), and choose $h(i, j) \in \{1, \ldots, d\}$ arbitrarily otherwise. Also, we write

$$\lambda_{i,j} := \lambda_{i,j}^{h(i,j)}.$$

We have that

$$[\mathbf{e}_i, \mathbf{e}_j] = \lambda_{i,j}\mathbf{e}_{h(i,j)}, \quad \forall i, j = 1, \ldots, d$$

(note that this equality holds trivially if $[\mathbf{e}_i, \mathbf{e}_j] = 0$).

Note that if $\mathcal{B}$ is a monomial basis for $\mathfrak{g}_1$, then $\pi^{m-1}\mathcal{B}$ is a monomial basis of $\mathfrak{g}_m$, with the associated structure constants being $\pi^{m-1}\lambda_{i,j}$. Thus, in this section it would suffice to find a monomial basis for $\mathfrak{g}_1$. Additionally, we will find explicit formulas for $h(i, j)$ and $\lambda_{i,j}$, for all i, j, for a specific monomial basis.

Lemma 5.2.2.

$$\mathcal{P} = \bigoplus_{j=1}^{\ell} u^j \mathscr{O}_E = \pi\mathscr{O}_E \oplus \bigoplus_{j=1}^{\ell-1} u^j \mathscr{O}_E.$$

Proof. The inclusion $\supseteq$ of all summands is immediate. Moreover, if $0 \neq x \in u^j \mathscr{O}_E \cap u^k \mathscr{O}_E$ for some $1 \leq j, k \leq \ell$ then there exist $x_1, x_2 \in \mathscr{O}_E$ such that $u^j x_1 = u^k x_2 = x$, and hence $u^{j-k} = \sigma^k(x_1^{-1}) x_2 \in F$. In particular, this means that $\left|u^{j-k}\right|_\wp$ is an integral power of q, which is only possible if $j = k$, so that this sum is indeed direct.

To prove the inclusion $\subseteq$, assume towards a contradiction that $x \in \mathcal{P}$ is such that $x \notin \oplus_{j=1}^{\ell} u^j \mathscr{O}_E$. Let $x_0, \ldots, x_{\ell-1} \in E$ be such that

$$x = \pi \cdot x_0 + \sum_{j=1}^{\ell-1} u^j x_j,$$

(such elements exist since $D = \bigoplus_{j=0}^{\ell-1} u^j E$).

Initially, we assume that $|x_0|_\wp = \max \left\{ |x_j|_\wp \mid j = 0, \ldots, \ell - 1 \right\} \geq q$, and consider the element

$$x_0^{-1} x = \pi + \sum_{j=1}^{\ell-1} u^j \sigma^j(x_0^{-1}) x_j.$$

Note that since σ is an isometry of $|\cdot|_\wp$, we have that

$$\left|\sigma^j(x_0^{-1}) x_j\right|_\wp = \left|x_0^{-1} x_j\right|_\wp \leq 1, \quad \forall j = 1, \ldots, \ell - 1.$$

Thus, we have that for any $j = 1, \ldots, \ell - 1$, the value of $\left|u^j \sigma^j(x_0^{-1}) x_j\right|_\wp$ can be either $q^{-j/\ell}$ (in the case where $\left|x_0^{-1} x_j\right|_\wp = 1$), or be strictly smaller then q^{-1}. Taking into account the fact that π occurs with coefficient 1 in $x_0^{-1} x$, we now have that

$$\left|x_0^{-1} x\right|_\wp \in \left\{ q^{-1/\ell}, q^{-2/\ell}, \ldots, q^{-1} \right\}.$$

But this implies that

$$|x|_\wp = |x_0|_\wp \cdot \left|x_0^{-1} x\right|_\wp \geq q \cdot q^{-1} = 1,$$

in contradiction to the assumption $x \in \mathcal{P}$.

As for the case where $|x_0|_\wp$ is not maximal, suppose i_0 is such that

$$|x_{i_0}|_\wp = \max \left\{ |x_j|_\wp \mid j = 0, \ldots, \ell - 1 \right\} \geq q,$$

and consider the element $x' = u^{\ell - i_0} x \in u^{\ell - i_0} \mathcal{P}$. Then

$$x' = \pi \cdot x_{i_0} + \sum_{j=1}^{\ell - i_0} u^j \pi x_{i_0 + j} + \sum_{j=\ell - i_0 + 1}^{\ell - 1} u^j x_{j + i_0 - \ell},$$

and admits a similar contradiction to the previous case, by showing that $x' \notin u^{\ell - i_0} \mathcal{P}$. $\square$

Lemma 5.2.2, in conjunction with Lemma 5.1.1, implies the following:

Lemma 5.2.3. The set

$$\left\{\pi, \pi\theta, \ldots, \pi\theta^{\ell-1}, u^j\theta^k \mid j = 1, \ldots, \ell-1,\ k = 0, \ldots, \ell-1\right\}$$

is an $\mathscr{O}_F$-module basis for $\mathcal{P}$.

From this, we can extract an $\mathscr{O}_F$-module basis for $\mathfrak{g}_1$. Let

$$\mathcal{B} := \left\{\pi\theta, \ldots \pi\theta^{\ell-1}, u^j\theta^k \mid j = 1, \ldots, \ell-1,\ k = 0, \ldots, \ell-1\right\}.$$

Proposition 5.2.4. The set $\mathcal{B}$ is an $\mathscr{O}_F$-module basis for $\mathfrak{g}_1 = \mathfrak{sl}_1(D) \cap \mathcal{P}$.

Proof. We already know that $\mathcal{B}$ is linearly independent over $\mathscr{O}_F$. Moreover, we have that $\mathrm{Trd}_{D/F}(\theta^j) = \mathrm{Tr}_{E/F}(\theta^j) = 0$ for any $j = 1, \ldots, \ell-1$, and by equation 3.A, that $\mathrm{Trd}_{D/F}(u^j\theta^k) = 0$ for all $j = 1, \ldots, \ell-1$ and $k = 0, \ldots, \ell-1$. Thus, we have that $\mathcal{B} \subseteq \mathfrak{sl}_1(D) \cap \mathcal{P}$.

To show that $\mathcal{B}$ spans $\mathfrak{g}_1$, let $x = \pi x_0 + \sum_{j=1}^{\ell-1} u^j x_j \in \mathcal{P}$ have reduced trace 0 (where $x_0, \ldots, x_{\ell-1} \in \mathscr{O}_E$). Then, by Equation 3.A,

$$\mathrm{Trd}_{D/F}(x) = \pi \mathrm{Tr}_{E/F}(x_0) = 0.$$

Let $\alpha_0, \ldots, \alpha_{\ell-1} \in \mathscr{O}_F$ be such that $x_0 = \sum_{j=0}^{\ell-1} \alpha_j\theta^j$. Then

$$\begin{aligned} 0 &= \mathrm{Tr}_{E/F}\Big(\sum_{j=0}^{\ell-1} \alpha_j\theta^j\Big) \\ &= \sum_{j=0}^{\ell-1} \alpha_j \mathrm{Tr}_{E/F}(\theta^j) = \alpha_0 \mathrm{Tr}_{E/F}(1) = \ell \cdot \alpha_0 \end{aligned}$$

Thus, $\alpha_0 = 0$ and $x \in \mathit{Span}_{\mathscr{O}_F}\mathcal{B}$. □

An Interlude for the Sake of Indices

The next goal of this section will be to show that $\mathcal{B}$ is monomial. Before approaching this task, it will be convenient to fix an index system for $\mathcal{B}$ which is more natural for our basis. Namely, we wish for it to be an ordered set of d elements, which is compatible relations $u^\ell = \pi$ and $\theta^\ell = b$. Though not necessary, we require the order to be linear, for the sake of straightforward matrix notation.

Let $\mathbb{F}_\ell$ denote the field of ℓ elements. Let $\mathcal{J}$ denote the ring $\mathbb{F}_\ell \times \mathbb{F}_\ell$ with coordinate-wise addition and multiplication. As a notational convention, we write $\mathcal{J}^* := \mathcal{J} \setminus \{(0,0)\}$. For any $\alpha \in \mathbb{F}_\ell$ write $r(\alpha)$ to denote the unique element of $\{0, \ldots, \ell-1\} \subseteq \mathbb{Z}$ such that $r(\alpha) \equiv \alpha (\mod \ell)$.

We fix a bijection between the set $\{1, \ldots, d\}$ and $\mathcal{J}^*$ by mapping any $1 \leq t \leq d$ to the unique pair $\underline{\alpha}_t = (\alpha_1, \alpha_2) \in \mathbb{F}_\ell \times \mathbb{F}_\ell$ such that

$$t = r(\alpha_1)\ell + r(\alpha_2).$$

From this we derive a linear order on $\mathcal{J}^*$, by defining

$$\underline{\alpha}_t < \underline{\alpha}_s \quad \iff \quad t < s.$$

We also define a linear order on the field $\mathbb{F}_\ell$ by $\alpha < \beta \iff r(\alpha) < r(\beta)$.

For any $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathcal{J}^*$, let

$$\mathbf{e}_{\underline{\alpha}} := \begin{cases} \pi\theta^{r(\alpha_2)} & \text{if } \alpha_1 = 0, \\ u^{r(\alpha_1)}\theta^{r(\alpha_2)} & \text{otherwise.} \end{cases}$$

In this notation, we have

$$\mathcal{B} = \left\{ \mathbf{e}_{\underline{\alpha}_t} \mid t = 1, \ldots, d \right\}.$$

Commutator Relations in $\mathcal{B}$

Let $\underline{\alpha} = (\alpha_1, \alpha_2),\ \underline{\beta} = (\beta_1, \beta_2) \in \mathcal{J}^*$. We have the following commutator relations

- $[\mathbf{e}_{\underline{\alpha}}, \mathbf{e}_{\underline{\beta}}] = 0$, if $\alpha_1 = \beta_1 = 0$.
- $[\mathbf{e}_{\underline{\alpha}}, \mathbf{e}_{\underline{\beta}}] = -[\mathbf{e}_{\underline{\beta}}, \mathbf{e}_{\underline{\alpha}}] = \pi\left(\omega^{r(\alpha_2)r(\beta_1)} - 1\right) u^{r(\beta_1)}\theta^{r(\alpha_2)+r(\beta_2)}$, if $\alpha_1 = 0$ and $\beta_1 \neq 0$.
- $[\mathbf{e}_{\underline{\alpha}}, \mathbf{e}_{\underline{\beta}}] = \left(\omega^{r(\alpha_2)r(\beta_1)} - \omega^{r(\alpha_1)r(\beta_2)}\right) u^{r(\alpha_1)+r(\beta_1)}\theta^{r(\alpha_2)+r(\beta_2)}$, if $\alpha_1 \neq 0$ and $\beta_1 \neq 0$.

In particular, we have obtained that $\mathcal{B}$ is monomial.

The above commutator relations can be brought into a closed form. Firstly, to simplify notation, we write $\omega^\gamma := \omega^{r(\gamma)}$ for any $\gamma \in \mathbb{F}_\ell$. Note that this simplification is well defined, since

$$\omega^{r(\alpha)+r(\beta)} = \omega^{r(\alpha+\beta)}, \quad \text{and} \quad \omega^{r(\alpha)r(\beta)} = \omega^{r(\alpha\beta)}$$

for any $\alpha, \beta \in \mathbb{F}_\ell$, as $\omega^\ell = 1$.

Now, note that

$$\theta^{r(\alpha)+r(\beta)} = b^{\epsilon(\alpha,\beta)}\theta^{r(\alpha+\beta)}, \quad \forall \alpha, \beta \in \mathbb{F}_\ell,$$

where $\epsilon(\alpha,\beta) := \begin{cases} 1 & \text{if } r(\alpha)+r(\beta) \geq \ell \\ 0 & \text{otherwise} \end{cases}.$

A similar equality holds for the powers of u. A slight change is needed in order to take into account that the basis elements $\mathbf{e}_{(0,\alpha)}$ already have π as a coefficient of $\theta^{r(\alpha)}$. To do so, we define

$$\delta(\alpha,\beta) = \begin{cases} 1 & \text{if } r(\alpha)+r(\beta) > \ell, \\ & \quad \text{or } r(\alpha) = 0, \text{ or } r(\beta) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\alpha, \beta \in \mathbb{F}_\ell$.

Then, the above commutator relations reduce to the equality

$$\cdot \left[\mathbf{e}_{\underline{\alpha}}, \mathbf{e}_{\underline{\beta}}\right] = \pi^{\delta(\alpha_1,\beta_1)} b^{\epsilon(\alpha_2,\beta_2)} \left(\omega^{\alpha_2\beta_1} - \omega^{\alpha_1\beta_2}\right) \mathbf{e}_{\underline{\alpha}+\underline{\beta}}. \tag{5.A}$$

Keeping this equation in mind, we put $h(s,t)$ to be the element of $\{1,\ldots,d\}$ which corresponds to $\underline{\alpha}_t + \underline{\alpha}_s$, and $\lambda_{s,t}$ to be the coefficient of $\mathbf{e}_{\underline{\alpha}_{h(s,t)}}$ in equation 5.A. Explicitly, if we write $s = \alpha_1\ell + \alpha_2$ and $t = \beta_1\ell + \beta_2$, we have that

$$\lambda_{s,t} = \pi^{\delta(\alpha_1,\beta_1)} b^{\epsilon(\alpha_2,\beta_2)} \left(\omega^{\alpha_2\beta_1} - \omega^{\alpha_1\beta_2}\right),$$

$$\text{and} \qquad h(s,t) = r(\alpha_1+\beta_1)\ell + r(\alpha_2+\beta_2).$$

In conclusion, we have that

Proposition 5.2.5. The set $\mathcal{B} = \left\{\mathbf{e}_{\underline{\alpha}_t} \mid t = 1,\ldots,d\right\}$ is a monomial $\mathcal{O}_F$-basis for $\mathfrak{g}_1$, with the values $h(s,t)$ and $\lambda_{s,t}$ given above.

Likewise, a monomial basis for $\mathfrak{g}_m$ is obtained from $\pi^{m-1}\mathcal{B}$ with the appropriate structure constants.

5.3 The Commutator Matrix

The commutator matrix $\mathcal{R}(\mathbf{Y}) = \mathcal{R}_{\mathfrak{g}_1,\mathcal{B}}(\mathbf{Y})$ is defined over the ring of polynomials $\mathcal{O}_F(\mathbf{Y})$ in d variables (cf. § 2.2) and is given by the formula

$$\mathcal{R}(\mathbf{Y}) = \left(\lambda_{s,t} Y_{h(s,t)}\right)_{s,t=1}^{d}.$$

We partition the matrix $\mathcal{R}(\mathbf{Y})$ into sub-matrices, on which the first coordinates of $\underline{\alpha}_s$ and $\underline{\alpha}_t$ are fixed. Specifically, for any $\alpha_1, \beta_1 \in \mathbb{F}_\ell^\times$, we define the matrix

$$\mathsf{C}_{\alpha_1,\beta_1}(\mathbf{Y}) = \pi^{\delta(\alpha_1,\beta_1)} \left(b^{\epsilon(\alpha_2,\beta_2)} \left(\omega^{\alpha_2\beta_1} - \omega^{\alpha_1\beta_2}\right) Y_{r(\alpha_1+\beta_1)\ell + r(\alpha_2+\beta_2)} \right)_{\alpha_2,\beta_2\in\mathbb{F}_\ell} \in \mathrm{Mat}_\ell\left(\mathscr{O}_F[\mathbf{Y}]\right).$$

Similarly, for any $\beta_1 \in \mathbb{F}_\ell^\times$ we define

$$\mathsf{B}_{\beta_1}(\mathbf{Y}) = \pi^{\delta(0,\beta_1)} \left(b^{\epsilon(\alpha_2,\beta_2)} \left(\omega^{\alpha_2\beta_1} - 1\right) Y_{r(\alpha_1+\beta_1)\ell + r(\alpha_2+\beta_2)} \right)_{\substack{\alpha_2\in\mathbb{F}_\ell^\times \\ \beta_2\in\mathbb{F}_\ell}} \in \mathrm{Mat}_{(\ell-1)\times\ell}\left(\mathscr{O}_F[\mathbf{Y}]\right).$$

The matrix $\mathcal{R}(\mathbf{Y})$ can be visualized as the block matrix

$$\mathcal{R}(\mathbf{Y}) = \left(\begin{array}{c|c|c|c|c} \mathsf{0}_{\ell-1} & \mathsf{B}_1(\mathbf{Y}) & \mathsf{B}_2(\mathbf{Y}) & \cdots & \mathsf{B}_{\ell-1}(\mathbf{Y}) \\ \hline -\mathsf{B}_1(\mathbf{Y})^T & \mathsf{C}_{1,1}(\mathbf{Y}) & \mathsf{C}_{1,2}(\mathbf{Y}) & \cdots & \mathsf{C}_{1,\ell-1}(\mathbf{Y}) \\ \hline -\mathsf{B}_2(\mathbf{Y})^T & \mathsf{C}_{2,1}(\mathbf{Y}) & \mathsf{C}_{2,2}(\mathbf{Y}) & \cdots & \mathsf{C}_{2,\ell-1}(\mathbf{Y}) \\ \hline \vdots & \vdots & \vdots & \iddots & \vdots \\ \hline -\mathsf{B}_{\ell-1}(\mathbf{Y})^T & \mathsf{C}_{\ell-1,1}(\mathbf{Y}) & \mathsf{C}_{\ell-1,2}(\mathbf{Y}) & \cdots & \mathsf{C}_{\ell-1,\ell-1} \end{array}\right), \tag{5.B}$$

where $\mathsf{0}_{\ell-1}$ is the $(\ell-1)\times(\ell-1)$ zero matrix.

Remark. For the sake of visualization, we coloured the matrices $\mathsf{C}_{\alpha,\beta}(\mathbf{Y})$ for which $\delta(\alpha,\beta) = 1$ in light gray. Note that all matrices $\mathsf{B}_\beta(\mathbf{y})$ are multiples of π as well (since $\delta(0,\beta) = 1$ by definition). Additionally, note that the condition $\epsilon(\alpha,\beta) = 1$ simply means that the (α,β)-th entry of a sub-matrix lies strictly beneath the secondary diagonal of the sub-matrix.

Note that for any $\alpha,\beta \in \mathbb{F}_\ell$ the entries of the matrices $\mathsf{C}_{\alpha,\beta}(\mathbf{Y})$ are monomials with variables in the set $\{Y_t \mid \underline{\alpha}_t \in \{\alpha+\beta\}\times\mathbb{F}_\ell\}$. Likewise, for any $\beta \in \mathbb{F}_\ell$, the entries of the matrix $\mathsf{B}_\beta(\mathbf{Y})$ are monomials with variables in the set $\{Y_t \mid \underline{\alpha}_t \in \{\beta\}\times\mathbb{F}_\ell\}$.

5.4 Coordinate-Free Interpretation

The current realization of the commutator matrix is simply as a table of $d \times d$ linear forms. The main step up to now was to partition the matrix into sub-matrices in which the entries are all forms in known subsets of the set of variables $\{Y_t \mid t = 1,\ldots,d\}$.

The next step we wish to make is to represent these sub-matrices as simple algebraic expressions in a predetermined family of matrices defined over $\mathscr{O}_F$. The main benefit of this description is that it will allow us to understand the matrices $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$ and $\mathsf{B}_{\beta}(\mathbf{y})$ (once the variables $\mathbf{Y}$ have been substituted for an element $\mathbf{y} \in W(\mathscr{O}_F)$), as $\mathscr{O}_F$-linear transformations, defined on ${\mathscr{O}_F}^{\ell}$. Using this description, in Chapter 7 we will be able to calculate the ranks of these operators over $\mathfrak{f}$ and consequently, to derive a lower bound for the values of $\|F_j(\mathbf{y})\|_{\wp}$.

Let us introduce the aforementioned family of $\ell \times \ell$ matrices. Recall that the set $\Theta = \{1, \theta, \ldots, \theta^{\ell-1}\}$ if an F-basis for the underlying vector space of E.

- Let $\boldsymbol{\sigma}$ denote the matrix representing the operator $\sigma : E \to E$ in the basis Θ. Then

$$\boldsymbol{\sigma} := [\sigma]_{\Theta} = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{\ell-1} \end{pmatrix}.$$

- Let T denote the matrix representing the reflection operator defined on basis elements by $\tau(\theta^j) = \theta^{\ell-1-j}$ for any $j = 0, \ldots, \ell-1$, in the basis Θ. Then

$$\mathsf{T} := [\tau]_{\Theta} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}.$$

- For any ℓ-tuple $\mathbf{x} = (x_0, \ldots, x_{\ell-1}) \in F^{\ell}$ let $\mathsf{L}_{\mathbf{x}}$ denote the matrix representing the operator $z \mapsto (\sum_{j=0}^{\ell-1} x_j \theta^j) \cdot z$, in the basis Θ. Then

$$\mathsf{L}_{\mathbf{x}} := \begin{pmatrix} x_0 & b \cdot x_{\ell-1} & b \cdot x_{\ell-2} & \ldots & bx_1 \\ x_1 & x_0 & bx_{\ell-1} & \ldots & bx_2 \\ x_2 & x_1 & x_0 & \ldots & bx_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{\ell-1} & x_{\ell-2} & x_{\ell-3} & \ldots & x_0 \end{pmatrix}.$$

Note that $\boldsymbol{\sigma}$, T and the elements of $\{\mathsf{L}_{\mathbf{x}} \mid \mathbf{x} \in \mathscr{O}_F^{\ell}\}$ are all matrices defined over $\mathscr{O}_F$.

Notation. For any $\mathbf{y} \in W(\mathscr{O}_F)$ and $\alpha_1 \in \mathbb{F}_\ell \smallsetminus \{0\}$, we write $\mathbf{y}_{|\alpha_1}$ to denote the ℓ-tuple

$$\mathbf{y}_{|\alpha_1} = \begin{pmatrix} y_{r(\alpha_1)\ell} \\ y_{r(\alpha_1)\ell+1} \\ \vdots \\ y_{r(\alpha_1)\ell+(\ell-1)} \end{pmatrix}, \quad \text{and put} \quad \mathbf{y}_{|0} := \begin{pmatrix} 0 \\ y_1 \\ \vdots \\ y_{\ell-1} \end{pmatrix}.$$

Proposition 5.4.1. Let $\mathbf{y} \in W(\mathscr{O}_F)$, and let $\alpha_1, \beta_1 \in \mathbb{F}_\ell^\times$ be arbitrary. Put $\gamma = \alpha_1 + \beta_1 \in \mathbb{F}_\ell$. The following equalities hold

1. $\mathsf{C}_{\alpha_1,\beta_1}(\mathbf{y}) = \pi^{\delta(\alpha_1,\beta_1)} \left(\boldsymbol{\sigma}^{\beta_1} \mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\gamma}} - \mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\gamma}} \boldsymbol{\sigma}^{\alpha_1} \right)$,

2. $\mathsf{B}_{\beta_1}(\mathbf{y}) = \mathsf{P}{\cdot}\pi \left(\boldsymbol{\sigma}^{\beta_1} \mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\beta_1}} - \mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\beta_1}} \right)$, where $\mathsf{P} = \begin{pmatrix} 0 & 1 & & \\ 0 & & 1 & \\ \vdots & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \in \mathrm{Mat}_{(\ell-1)\times\ell}(\mathscr{O}_F)$.

Proof. Consider the matrix

$$\begin{aligned} H &:= \left(b^{\epsilon(\alpha_2,\beta_2)} y_{r(\gamma)\ell + r(\alpha_2+\beta_2)} \right)_{\alpha_2,\beta_2 \in \mathbb{F}_\ell} \\ &= \begin{pmatrix} y_{r(\gamma)\ell} & y_{r(\gamma)\ell+1} & \cdots & y_{r(\gamma)\ell+(\ell-2)} & y_{r(\gamma)\ell+(\ell-1)} \\ y_{r(\gamma)\ell+1} & y_{r(\gamma)\ell+2} & \cdots & y_{r(\gamma)\ell+(\ell-1)} & by_{r(\gamma)\ell} \\ y_{r(\gamma)\ell+2} & y_{r(\gamma)\ell+3} & \cdots & by_{r(\gamma)\ell} & by_{r(\gamma)\ell+1} \\ \vdots & & \iddots & & \vdots \\ y_{r(\gamma)\ell+(\ell-1)} & by_{r(\gamma)\ell} & \cdots & by_{r(\gamma)\ell+(\ell-3)} & by_{r(\gamma)\ell+(\ell-2)} \end{pmatrix}. \end{aligned}$$

Note that $\mathsf{T}^2 = \mathbf{1}_\ell$, and that multiplying from the left by T reflects the matrix along the horizontal line. Thus, the matrix above equals

$$\begin{aligned} H = \mathsf{T} \cdot (\mathsf{T} \cdot H) = \mathsf{T} \cdot &\begin{pmatrix} y_{r(\gamma)\ell+(\ell-1)} & by_{r(\gamma)\ell} & \cdots & by_{r(\gamma)\ell+(\ell-2)} \\ y_{r(\gamma)\ell+(\ell-2)} & y_{r(\gamma)\ell+(\ell-1)} & \cdots & by_{r(\gamma)\ell+(\ell-3)} \\ \vdots & \vdots & \ddots & \vdots \\ y_{r(\gamma)\ell} & y_{r(\gamma)\ell+1} & \cdots & y_{r(\gamma)\ell+(\ell-1)} \end{pmatrix} \\ = \mathsf{TL}_{\left(y_{r(\gamma)\ell+(\ell-1)}, y_{r(\gamma)\ell+(\ell-2)}, \ldots, y_{r(\gamma)\ell}\right)} &= \mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\gamma}} \end{aligned}$$

Now, recall from the definition that

$$\begin{aligned}
\mathsf{C}_{\alpha_1,\beta_1}(\mathbf{y}) &= \pi^{\delta(\alpha_1,\beta_1)}\left(b^{\epsilon(\alpha_2,\beta_2)}\left(\omega^{\alpha_2\beta_1}-\omega^{\alpha_1\beta_2}\right)y_{r(\gamma)\ell+r(\alpha_2+\beta_2)}\right)_{\alpha_2,\beta_2} \\
&= \pi^{\delta(\alpha_1,\beta_1)}\left(\left(\omega^{\alpha_2\beta_1}b^{\epsilon(\alpha_2,\beta_2)}y_{r(\gamma)\ell+r(\alpha_2+\beta_2)}\right)_{\alpha_2,\beta_2}-\left(\omega^{\alpha_1\beta_2}b^{\epsilon(\alpha_2,\beta_2)}y_{r(\gamma)\ell+r(\alpha_2+\beta_2)}\right)_{\alpha_2,\beta_2}\right) \\
&= \pi^{\delta(\alpha_1,\beta_1)}\left(\begin{pmatrix}1 & & & \\ & \omega^{\beta_1} & & \\ & & \ddots & \\ & & & \omega^{\beta_1(\ell-1)}\end{pmatrix}H - H\begin{pmatrix}1 & & & \\ & \omega^{\alpha_1} & & \\ & & \ddots & \\ & & & \omega^{\alpha_1(\ell-1)}\end{pmatrix}\right) \\
&= \pi^{\delta(\alpha_1,\beta_1)}\left(\boldsymbol{\sigma}^{\beta_1}\mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\gamma}}-\mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\gamma}}\boldsymbol{\sigma}^{\alpha_1}\right),
\end{aligned}$$

as wanted. Thus equality (1) is proven. Similarly, for equality (2), we have that

$$\begin{aligned}
\mathsf{B}_{\beta_1}(\mathbf{y}) &= \mathsf{P}\pi\left(b^{\epsilon(\alpha_2,\beta_2)}\left(\omega^{\alpha_2\beta_1}-1\right)y_{r(\gamma)+r(\alpha_2+\beta_2)}\right)_{\alpha_2,\beta_2\in\mathbb{F}_\ell} \\
&= \mathsf{P}\pi\left(\boldsymbol{\sigma}^{\beta_1}\mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\gamma}}-\mathsf{TL}_{\mathsf{T}\mathbf{y}_{|\gamma}}\right).
\end{aligned}$$

$\square$

At this point we withhold from further investigation of $\mathcal{R}(\mathbf{y})$ and turn to prove a combinatorial result which will serve us in our computation.

Chapter 6

Coins on a Checker-Board- a Combinatorial Digression

Game. *Let us play a game.*

You are given an $n \times n$ checker-board, and a wallet with n coins. You are required to place the coins on the board so that no two coins appear on the same row or column of the board.

The squares on the board are marked by colours. The precise pattern of colouring will be given explicitly in the chapter, but for now let us just assume that the pattern is as in the following figure:

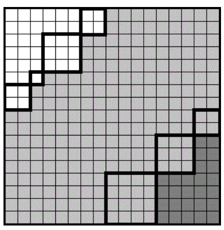

Figure 6.1: An example of a coloured 17×17 Checkerboard.

The basic rules are as follows: At the end of the game you are allowed to collect all coins which you have placed on a white square, to put back in your wallet. All other coins must remain on their squares. For any coin which is placed on one of dark gray squares in the bottom right corner of the board, you must place an additional coin on top of it at the end of the game.

That is to say, the profit obtained by placing a coin on as square varies with the color of the square, as in the following table

	White	Light Gray	Dark Gray
Value	+1	0	-1

.

Your mission, naturally, is to finish the game with as many coins as possible. There is one obvious way of doing this, simply by placing all coins in the boldfaced squares in the figure above. In the board above, the result would be a profit of 8 coins.

So let us pose another rule-

> ***Rule***. *If at the end of the game all coins are placed within the boldfaced region described above, the player must pay all coins given in the beginning of the game, thereby finishing the game with a profit of 0 coins.*

The question we will answer in this chapter is the following:

> ***Question***. *Given the additional rule, can you find a placement of the coins which will admit the same profit as the one obtained without the additional rule?*

It turns out that answering this sort of question has an immediate application to the calculation of absolute values of certain $\wp$-adic determinants.

Prior to stating and proving this application, we prove a purely combinatorial lemma, regarding finite sets and bijective functions.

6.1 A Combinatorial Lemma

Fix $n \in \mathbb{N}$, and let $X,\ Y$ be finite sets with n elements. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)$ be a composition of n (i.e. an ordered k-tuple of strictly positive integers such that $\sum_{j=1}^{k} \nu_j = n$), and fix some arbitrary $t \in \{1, \ldots, k\}$. Suppose $X_1, \ldots, X_k \subseteq X$ and $Y_1, \ldots, Y_k \subseteq Y$ are pairwise disjoint, and satisfy

$$\#X_j = \#Y_j = \nu_j, \quad \forall j = 1, \ldots k.$$

Then

$$X = \bigsqcup_{j=1}^{k} X_j, \quad Y = \bigsqcup_{j=1}^{k} Y_j.$$

Given such a pair $(\boldsymbol{\nu}, t)$, for any function $f : X \to Y$ we define a map $d_f : X \to \mathbb{N}$ by the following rule. For any $x \in X_i \subseteq X$, we let

$$d_f(x) := d_f^{\boldsymbol{\nu},t}(x) = \begin{cases} 0 & \text{if } i \leq t \text{ and } f(x) \in \bigcup_{j=i}^{t} Y_j \\ 2 & \text{if } i > t+1 \text{ and } f(x) \in \bigcup_{j=t+1}^{i-1} Y_j \\ 1 & \text{otherwise.} \end{cases}$$

That is to say, if $f(x) \in Y_j$ then $d_f(x) = \begin{cases} 0 & \text{if } i \leq j \leq t, \\ 2 & \text{if } t < j < i, \\ 1 & \text{otherwise.} \end{cases}$

Define a function T on the set of bijective functions from X to Y by

$$T(f) := T^{\boldsymbol{\nu},t}(f) = \sum_{x \in X} d_f(x),$$

and put $c_{\boldsymbol{\nu},t} := \sum_{j=t+1}^{k} \nu_j$.

Lemma 6.1.1. The function $T(f)$ attains its minimum at f if and only if $f(X_j) = Y_j$ for all $j = 1, \ldots, k$. Furthermore, this minimum equals $c_{\boldsymbol{\nu},t}$.

Remark. In our game example, one can think of X as the set of rows of the board, and Y as the set of columns. The composition $\boldsymbol{\nu}$ can be derived from the width of squares of the boldfaced region, where ν_1 is the width of the leftmost square and ν_k is the width of the rightmost square. The number t is the index of the uppermost square, when enumerated from left to right (e.g in Figure 6.1, we have $n = 17$, $\boldsymbol{\nu} = (2, 1, 3, 2, 4, 3, 2)$ and $t = 4$). The partitions of X and Y are defined in a suitable manner.

A bijective function $f : X \to Y$ defines a valid placement of the coins. The value of d_f is related to the profit from placing a coin on a square by the equation

$$\text{Value of the square } (x, f(x)) = 1 - d_f(x).$$

The total profit earned for choosing the coin-placement defined by f equals $n - T(f)$. The condition $f(X_j) = Y_j$ is equivalent in our example to a placement where all coins are placed in the boldfaced region.

Proof. Let $f : X \to Y$ be bijective. By the definition of d_f, it holds that $d_f(x) \geq 1$, for any $x \in \bigcup_{j=t+1}^{k} X_j$. Since the cardinality of $\bigcup_{j=t+1}^{k} X_j$ is $c_{\boldsymbol{\nu},t}$, we have that

$$T(f) = \sum_{x \in X} d_f(x) \geq c_{\boldsymbol{\nu},t}.$$

Additionally, if f has the property that $f(X_j) = Y_j$, by the definition of d_f we have that

$$d_f(x) = \begin{cases} 0 & \text{if } x \in X_j \text{ for } j \leq t \\ 1 & \text{if } x \in X_j \text{ for } j \geq t+1 \end{cases}$$

and thus $T(f) = c_{\boldsymbol{\nu},t}$. Hence $c_{\boldsymbol{\nu},t}$ is the minimum value of T, and the *if* direction is proven.

To show the *only if* direction, assume towards a contradiction that $f : X \to Y$ is such that $T(f) = c_{\boldsymbol{\nu},t}$ and such that $f(X_{i_0}) \neq Y_{i_0}$ for some $i_0 = 1, \ldots, k$.

Assume initially that $i_0 \leq t$, and let $x_0 \in X_{i_0}$ be such that $f(x_0) \notin Y_{i_0}$. Let i_1 be such that $f(x_0) \in Y_{i_1}$. Note that $i_0 < i_1 \leq t$. Otherwise $d_f(x_0) = 1$ by the definition of d_f, and

$$T(f) = \sum_{x \in X \smallsetminus \{x_0\}} d_f(x) + d_f(x_0) \geq c_{\boldsymbol{\nu},t} + 1$$

in contradiction to the assumption of minimality. Additionally, note that $f(x_0) \in Y_{i_1}$ implies that $f(X_{i_1}) \neq Y_{i_1}$. Otherwise, there would be some $x' \in X_{i_1}$ such that $f(x') = f(x_0)$ contradicting the injectivity of f.

Let $x_1 \in X_{i_1}$ be such that $f(x_1) \in Y_{i_2}$ for some $i_2 \neq i_1$. Then, as before $i_1 < i_2 \leq t$, and there is some $x_2 \in X_{i_2}$ such that $f(x_2) \notin Y_{i_2}$. Continuing inductively, for any $s \in \mathbb{N}$ we may find a sequence of elements $x_1, \ldots, x_s$ and indices $i_1, \ldots, i_s$ such that $f(x_j) \notin Y_{i_j}$, for all $j = 1, \ldots, s$, and such that

$$i_1 < i_2 < \ldots < i_s \leq t.$$

But then $(i_1, \ldots, i_s)$ is a strictly increasing bounded sequence of integers of arbitrary length. A contradiction.

The case where $i_0 > t$ is proven in a similar vein, by finding a strictly decreasing sequence of positive integers. $\square$

Corollary 6.1.2. In the setting of the lemma, let $f : X \to Y$ be bijective, and let $\delta_f : X \to \mathbb{R}$ be some map such that

$$\delta_f(x) \geq d_f(x), \quad \forall x \in X.$$

Then $\sum_{x\in X}\delta_f(x)\geq c_{\boldsymbol{\nu},t}$.

Furthermore, if there exists some $1\leq i_0\leq k$ such that $f(X_{i_0})\neq Y_{i_0}$, then this inequality is strict.

Proof. Clear, since

$$\sum_{x\in X}\delta_f(x)\geq\sum_{x\in X}d_f(x)=T(f)\geq c_{\boldsymbol{\nu},t},$$

and the latter inequality is a strict, unless $f(X_i)=Y_i$ for all i's. □

6.2 Application to $\wp$-adic Determinants

We retain the notations of the previous section, where $n\in\mathbb{N}$, $\boldsymbol{\nu}$ is a composition of n of length k, and $t\in\{1,\ldots,k\}$. Also, we let $X=Y=\{1,\ldots,n\}$ and let $X_1,\ldots,X_k\subseteq\{1,\ldots,n\}$, and $Y_1,\ldots,Y_k\subseteq\{1,\ldots,n\}$ be pairwise disjoint and such that $\#X_i=\#Y_i=\nu_i$ for all $i=1,\ldots,k$.

Let $M=\left(m_{\alpha,\beta}\right)_{\alpha,\beta=1}^n\in\mathrm{Mat}_n(\mathscr{O}_F)$. For any $1\leq i,j\leq k$ let $M_{i,j}$ denote the sub-matrix of M, which is obtained by taking all entries $m_{\alpha,\beta}$ such that $(\alpha,\beta)\in X_i\times Y_j$.

As before, we put $c_{\boldsymbol{\nu},t}=\sum_{j=t+1}^k\nu_j$.

Lemma 6.2.1. Let $t\in\{1,\ldots,k\}$. Suppose the sub-matrices $M_{i,j}$ have the following properties:

(1) The entries of the matrix $M_{i,j}$ are arbitrary elements of $\mathscr{O}_F$ whenever $i\leq j\leq t$. The entries of $M_{i,j}$ are elements of $\wp_F^2$, whenever $t<j<i\leq k$. In all other cases, the entries of $M_{i,j}$ are elements of $\wp_F$.

(2) For any $i\leq t$, the matrix $M_{i,i}$ is invertible over $\mathscr{O}_F$. For any $i>t$, $M_{i,i}=\pi\cdot M'_{i,i}$ where $M'_{i,i}\in\mathrm{Mat}_{\nu_i}(\mathscr{O}_F)$ is invertible over $\mathscr{O}_F$.

Then $\left|\det M\right|_\wp=q^{-c_{\boldsymbol{\nu},t}}$.

Proof. Let

$$\mathcal{F}:=\left\{f\in\mathrm{Sym}(n)\mid f(X_i)=Y_i,\quad\forall 1\leq i\leq k\right\}.$$

For any $f\in\mathrm{Sym}(n)$, let $\delta_f:\{1,\ldots,n\}\to\mathbb{R}$ be given by

$$\delta_f(x)=-\log_q\left|m_{x,f(x)}\right|_\wp.$$

(i.e. $\delta_f(x)$ is the valuation of the $(x, f(x))$-th entry of M).

By property (1) of M, we have that $\delta_f(x) \geq d_f(x)$ for any $x \in X$, where $d_f(x)$ is as defined in the previous section. Thus, by corollary 6.1.2, we have that for any $f \in \mathrm{Sym}(n) \setminus \mathcal{F}$

$$\sum_{x\in\{1,\dots,n\}} \delta_f(x) = -\log_q \left| \prod_{x\in\{1,\dots,n\}} m_{x,f(x)} \right|_{\wp} > c_{\boldsymbol{\nu},t}$$

Hence, for any $f \in \mathrm{Sym}(n) \setminus \mathcal{F}$,

$$\left| sgn(f) \prod_{x=1}^{n} m_{x,f(x)} \right|_{\wp} < q^{-c_{\boldsymbol{\nu},t}}.$$

Moreover, by property (2) of M, we have that

$$\begin{aligned}\left| \sum_{f\in\mathcal{F}} sgn(f) \prod_{x=1}^{n} m_{x,f(x)} \right|_{\wp} &= \left| \prod_{i=1}^{k} \det M_{i,i} \right|_{\wp} \\ &= \prod_{x=1}^{t} |\det M_{i,i}|_{\wp} \cdot \prod_{x=t+1}^{k} \left|\det(\pi M'_{i,i})\right|_{\wp} = q^{-c_{\boldsymbol{\nu},t}}.\end{aligned}$$

Combining the two equations, we have that

$$\begin{aligned}|\det M|_{\wp} &= \left| \sum_{f\in\mathrm{Sym}(n)} sgn(f) \prod_{x=1}^{n} m_{x,f(x)} \right|_{\wp} \\ &= \left| \sum_{f\in\mathcal{F}} sgn(f) \prod_{x=1}^{n} m_{x,f(x)} + \sum_{f\in\mathrm{Sym}(n)\setminus\mathcal{F}} sgn(f) \prod_{x=1}^{n} m_{x,f(x)} \right|_{\wp} = q^{-c_{\boldsymbol{\nu},t}}.\end{aligned}$$

□

Let us conclude this section by introducing a special type of matrices for which Lemma 6.2.1 is applicable, and which we will encounter in the sequel.

Given a composition $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k)$ of n, we can associate to it a partition $\{X_i^{\boldsymbol{\nu}}\}_{i=1}^{k}$ of $\{1, \dots, n\}$ by taking $X_1^{\boldsymbol{\nu}} = \{1, 2, \dots, \nu_1\}$, and

$$X_i^{\boldsymbol{\nu}} = \left\{ \sum_{r=1}^{i-1} \nu_r + 1, \sum_{r=1}^{i-1} \nu_r + 2, \dots, \sum_{r=1}^{i} \nu_r \right\}, \quad \forall i = 2, \dots, k.$$

Given a number $1 \leq t \leq k$, we define another partition $\{Y_i^{\boldsymbol{\nu}}\}_{i=1}^k$ of $\{1, \ldots, n\}$ in the following manner: Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ be the composition defined by

$$\mu_j = \begin{cases} \nu_{t-j+1} & \text{if } 1 \leq j \leq t \\ \nu_{k+t+1-j} & \text{if } t+1 \leq j \leq k. \end{cases}$$

That is to say, $\boldsymbol{\mu} = (\nu_t, \ldots, \nu_1, \nu_k, \nu_{k-1}, \ldots, \nu_{t+1})$. We associate to it the partition $X_1^{\boldsymbol{\mu}} = \{1, \ldots, \mu_1\}$ and

$$X_i^{\boldsymbol{\mu}} = \left\{ \sum_{r=1}^{i-1} \mu_r + 1, \sum_{r=1}^{i-1} \mu_r + 2, \ldots, \sum_{r=1}^{i} \mu_r \right\}, \quad \forall j = 2, \ldots, k.$$

Let $Y_i^{\boldsymbol{\nu}} = X_{t-i+1}^{\boldsymbol{\mu}}$ for any $i = 1, \ldots, t$ and $Y_i^{\boldsymbol{\nu}} = X_{k+t+1-i}^{\boldsymbol{\mu}}$ for all $i = t+1, \ldots, k$. Note that

$$\# X_i^{\boldsymbol{\nu}} = \# Y_i^{\boldsymbol{\nu}} = \nu_i, \quad \forall i = 1, \ldots, k.$$

Definition 6.2.2. We call a matrix $M \in \mathrm{Mat}_n(\mathcal{O}_F)$ $(\boldsymbol{\nu}, t)$**-graded**, if it satisfies the conditions of Lemma 6.2.1, with respect to the partitions $\left\{X_j^{\boldsymbol{\nu}}\right\}_{j=1}^k$, $\left\{Y_j^{\boldsymbol{\nu}}\right\}_{j=1}^k$ we have just defined.

Note that the partition $\left\{X_j^{\boldsymbol{\nu}}\right\}_{j=1}^k$ is 'increasing', in the sense that if $x_i \in X_i^{\boldsymbol{\nu}}$ and $x_j \in X_j^{\boldsymbol{\nu}}$ are arbitrary then $x_1 < x_2$ whenever $i < j$. The partition $\{Y_i^{\boldsymbol{\nu}}\}_{i=1}^k$, on the other hand, does not have this property (e.g. it is always the case that either $1 \notin Y_1^{\boldsymbol{\nu}}$ or $n \notin Y_k^{\boldsymbol{\nu}}$).

Example. Let $n = 17$, and $\boldsymbol{\nu} = (2, 1, 3, 2, 4, 3, 2)$. A $(\boldsymbol{\nu}, 4)$-graded matrix will have the form

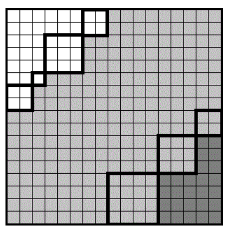

Figure 6.2: A $(\boldsymbol{\nu}, 4)$-graded matrix,

in which the entries in the white cells are elements of $\mathcal{O}_F$, the elements in the light gray cells are from $\wp_F$, and the elements in dark gray are in $\wp_F^2$.

Corollary 6.2.3. Let $M \in \mathrm{Mat}_n(\mathscr{O}_F)$ be a $(\boldsymbol{\nu}, t)$-graded matrix. Then

$$|\det M|_{\wp} = q^{-c_{\boldsymbol{\nu},t}},$$

where $c_{\boldsymbol{\nu},t} = \sum_{j=t+1}^{k} \nu_j$.

Chapter 7

Lower Bounds for $\left\|F_j(\mathbf{y})\right\|_\wp$

In this chapter we will pose a lower bound for the values of the function $\mathbf{y} \mapsto \|F_j(\mathbf{y})\|_\wp$ (cf. § 2.4), for any $j = 1, \ldots, \rho$, and $\mathbf{y} \in W(\mathscr{O}_F)$. As we shall see soon, the set $W(\mathscr{O}_F)$ can be partitioned into ℓ disjoint subsets on each of which a uniform lower bound can be put. Taking $\mathbf{y}$ to be in one of these subsets we will investigate the sub-matrices $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$ and $\mathsf{B}_\beta(\mathbf{y})$ (defined in § 5.3) and retrieve information regarding the absolute values of their minors by considering their images over $\mathfrak{f}$. Once this is done we will invoke our combinatorial lemma (cf. Corollary 6.2.3) in order to collate the data we have obtained and to calculate the absolute value of certain minors of $\mathcal{R}(\mathbf{y})$, thus providing a lower bound for $\|F_j(\mathbf{y})\|_\wp$.

In addition to these lower bounds, one can provide upper bound for the values of $\|F_j(\mathbf{y})\|_\wp$ in a rather simplistic manner. In the case where $\ell = 3$ these upper and lower bounds coincide, and thus allow us to complete the calculation of the representation zeta function of $SL_1^m(\mathcal{O})$ for all permissible m's. Unfortunately, to date we did not manage to prove, nor disprove, that these upper bounds can be lowered to coincide with the proposed lower bounds, in the case where $\ell > 3$.

7.1 A Closer Look at the Residue Field

Notation. For any $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathscr{O}_F)$, an $n \times n$ matrix defined over $\mathscr{O}_F$, let $\overline{A}$ denote the coordinate-wise image of A over $\mathfrak{f}$, i.e. the matrix $(\overline{a_{i,j}}) \in \mathrm{Mat}_n(\mathfrak{f})$.

Note that the map $A \mapsto \overline{A}$ is a ring homomorphism.

7 Lower Bounds for $\|F_j(\mathbf{y})\|_{\wp}$

Recall that $\overline{\Theta} = \left\{\overline{1}, \overline{\theta}, \ldots, \overline{\theta}^{\ell-1}\right\}$ is an $\mathfrak{f}$-basis of $\mathfrak{e}$. Throughout Section 7.1 we will fix $\mathbf{y} \in W(\mathscr{O}_F)$ and put, for any $\alpha \in \mathbb{F}_\ell^\times$,

$$\upsilon_{|\alpha} = \upsilon_{|\alpha}(\mathbf{y}) := \sum_{j=0}^{\ell-1} \overline{y_{r(\alpha)\ell+j}\theta^j},$$

and $\upsilon_{|0} := \upsilon_{|0}(\mathbf{y}) = \overline{y_1\theta} + \overline{y_2\theta^2} + \ldots + \overline{y_{\ell-1}\theta^{\ell-1}}$. Note that $\upsilon_{|\alpha}$ is the element $\mathfrak{e}$, whose coordinate vector with respect to $\overline{\Theta}$ is the image of $\mathbf{y}_{|\alpha}$ in $\mathfrak{f}^\ell$, for any $\alpha \in \mathbb{F}_\ell$ (cf.§ 5.4). Also, note that $\upsilon_{|\alpha} = 0$ if and only if $\mathbf{y}_{|\alpha} \in \wp_F^{(\ell)}$.

7.1.1 Sub-Matrices of $\overline{\mathcal{R}(\mathbf{y})}$

Let $\alpha, \beta \in \mathbb{F}_\ell$ be arbitrary. We let $\mathsf{c}_{\alpha,\beta}(\mathbf{y}) \in \mathrm{Mat}_\ell(\mathfrak{f})$ denote the matrix $\overline{\mathsf{C}_{\alpha,\beta}(\mathbf{y})}$ whenever $\delta(\alpha,\beta) = 0$ (i.e. $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$ is not a multiple of π in the obvious manner). By a slight abuse of notation, we write $\mathsf{c}_{\alpha,\beta}(\pi^{-1}\mathbf{y})$ to denote the matrix $\overline{\pi^{-1}\mathsf{C}_{\alpha,\beta}(\mathbf{y})}$, whenever $\delta(\alpha,\beta) = 1$, and we write $\mathsf{b}_\beta(\pi^{-1}\mathbf{y})$ for $\overline{\pi^{-1}\mathsf{B}_\beta(\mathbf{y})}$.

Recall that the Frobenius automorphism $\mathfrak{fr} : \mathfrak{e} \to \mathfrak{e}$ satisfies $\mathfrak{fr}(\overline{\theta}) = \overline{\varpi}\overline{\theta} = \overline{\omega\theta}$. Let F denote the matrix representing $\mathfrak{fr}$ in the basis $\overline{\Theta}$. Then

$$\mathsf{F} = \begin{pmatrix} 1 & & & \\ & \overline{\varpi} & & \\ & & \ddots & \\ & & & \overline{\varpi}^{\ell-1} \end{pmatrix}.$$

Note that $\mathsf{F} = \overline{\boldsymbol{\sigma}}$. As a notational convention, for any $\alpha \in \mathbb{F}_\ell$ we write

$$\mathsf{F}^\alpha := \mathsf{F}^{r(\alpha)}.$$

Note, that since the entries of $\mathsf{F}^{r(\alpha)}$ are powers of $\omega^{r(\alpha)}$, this notation is well-defined. Likewise, we will write $\mathfrak{fr}^\alpha := \mathfrak{fr}^{r(\alpha)}$, for all $\alpha \in \mathbb{F}_\ell$.

For any $\xi \in \mathfrak{e}$ we let L_ξ denote the matrix representing the operator $\nu \mapsto \xi \cdot \nu$ with respect to the basis $\overline{\Theta}$. Note that if $\mathbf{x} = (x_0, \ldots, x_{\ell-1}) \in {\mathscr{O}_F}^\ell$ is such that $\xi = \sum_{j=0}^{\ell-1} \overline{x_j\theta^j}$, then $\mathsf{L}_\xi = \overline{\mathsf{L}_\mathbf{x}}$.

Lastly, we keep the notation of T for the representative matrix of $\tau(\overline{\theta}^j) = \overline{\theta}^{\ell-1-j}$, as defined in Section 5.4. Note that T coincides with the coordinate-wise reduction of the matrix we denoted as T in Section 5.4.

The fact that $A \mapsto \overline{A}$ is a ring homomorphism provides us with the following analogue of Proposition 5.4.1:

Proposition 7.1.2. Let $\mathbf{y} \in W(\mathscr{O}_F)$, and let $\alpha, \beta \in \mathbb{F}_\ell$ be arbitrary. Put $\gamma = \alpha + \beta \in \mathbb{F}_\ell$. The following equalities hold

1. $\mathsf{c}_{\alpha,\beta}(\mathbf{y}) = \left(\mathsf{F}^\beta \mathsf{T} \mathsf{L}_{\tau(\upsilon_{|\gamma})} - \mathsf{T}\mathsf{L}_{\tau(\upsilon_{|\gamma})}\mathsf{F}^\alpha\right)$, whenever $\delta(\alpha, \beta) = 0$,

2. $\mathsf{c}_{\alpha,\beta}(\pi^{-1}\mathbf{y}) = \left(\mathsf{F}^\beta \mathsf{T} \mathsf{L}_{\tau(\upsilon_{|\gamma})} - \mathsf{T}\mathsf{L}_{\tau(\upsilon_{|\gamma})}\mathsf{F}^\alpha\right)$, whenever $\delta(\alpha, \beta) = 1$,

3. $\mathsf{b}_\beta(\pi^{-1}\mathbf{y}) = \mathsf{P}\left(\mathsf{F}^\beta \mathsf{T} \mathsf{L}_{\tau(\upsilon_{|\gamma})} - \mathsf{T}\mathsf{L}_{\tau(\upsilon_{|\gamma})}\right)$, where $\mathsf{P} = \begin{pmatrix} 0 & 1 & & \\ 0 & & 1 & \\ \vdots & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \in \mathrm{Mat}_{(\ell-1)\times\ell}(\mathfrak{f})$, and $\beta \neq 0$.

7.1.3 Rank Calculation

As hinted before, our method of extracting information about the minors of $\mathcal{R}(\mathbf{y})$ will involve a calculation of the ranks of the matrices $\mathsf{c}_{\alpha,\beta}$ and b_β. In order to do so, in this section we will present an interpretation of the kernel of the operators $\mathsf{c}_{\alpha,\beta}(\mathbf{y})$ and $\mathsf{b}_\beta(\mathbf{y})$ as the space of solutions to certain functional equations defined over $\mathfrak{e}$.

Prior to this, we begin by listing several important properties of the matrices T, F and $\{\mathsf{L}_\xi \mid \xi \in \mathfrak{e}\}$.

First off, note that the map $\xi \mapsto \mathsf{L}_\xi$ is an $\mathfrak{f}$-algebra homomorphism. The proof of this fact is similar to that of Proposition 3.1.8. In particular, the set $\{\mathsf{L}_\xi \mid \xi \in \mathfrak{e}\}$ is a sub-field $\mathrm{Mat}_\ell(\mathfrak{f})$, and L_ξ is invertible whenever $\xi \neq 0$.

Additionally, by direct calculation we obtain the following identities:

Lemma 7.1.4.

1. $\mathsf{T}^2 = \mathbf{1}_\ell$, $\mathsf{F}^\ell = \mathbf{1}_\ell$, where $\mathbf{1}_\ell \in \mathrm{Mat}_\ell(\mathfrak{f})$ is the identity matrix.

2. $\mathsf{TFT} = \varpi^{-1} \cdot \mathsf{F}^{-1}$.

3. $\mathsf{F}\mathsf{L}_\xi\mathsf{F}^{-1} = \mathsf{L}_{\mathrm{fr}(\xi)}$ for any $\xi \in \mathfrak{e}$

Specifically, in the case where $\alpha + \beta = 0$ we obtain the following equality. Let $\xi \in \mathfrak{e}$ be arbitrary, then

$$\begin{aligned} \mathsf{F}^\beta\mathsf{T}\mathsf{L}_\xi - \mathsf{T}\mathsf{L}_\xi\mathsf{F}^\alpha &= \omega^{-\beta}\mathsf{T}\mathsf{F}^{-\beta}\mathsf{L}_\xi - \mathsf{T}\mathsf{L}_\xi\mathsf{F}^\alpha && \text{by Lemma 7.1.4, item (2),} \\ &= \mathsf{T}\left(\omega^\alpha \mathsf{L}_{\mathrm{fr}^\alpha(\xi)}\mathsf{F}^\alpha - \mathsf{L}_\xi\mathsf{F}^\alpha\right) && \text{by item (3), and since } \alpha+\beta=0, \\ &= \mathsf{T} \circ \mathsf{L}_{\omega^\alpha \mathrm{fr}^\alpha(\xi) - \xi} \circ \mathsf{F}^\alpha && \text{since } \xi \mapsto \mathsf{L}_\xi \text{ is a homomorphism.} \end{aligned}$$

In particular, since T and F are invertible, in the case where $\alpha + \beta = 0$ we have that $\mathsf{F}^{\beta}\mathsf{T}\mathsf{L}_{\xi} - \mathsf{T}\mathsf{L}_{\xi}\mathsf{F}^{\alpha}$ is invertible if and only if $\mathfrak{fr}^{\alpha}(\xi) \neq \omega^{-\alpha}\xi$, which is equivalent to

$$\mathfrak{fr}(\xi) \neq \omega^{-1}\xi.$$

Since $\mathfrak{fr}$ has ℓ distinct linearly independent eigenvectors (namely, $1, \overline{\theta}, \ldots, \overline{\theta}^{\ell-1}$), we have that the equality $\mathfrak{fr}(\xi) = \omega^{-1}\xi$ holds iff $\xi \in \overline{\theta}^{\ell-1} \cdot \mathfrak{f}$.

From this we obtain our first conclusion.

Proposition 7.1.5. Let $\mathbf{y} \in W(\mathscr{O}_F)$, and let $\alpha \in \mathbb{F}_{\ell} \setminus \{0\}$ and $\beta = -\alpha$. Then $\mathsf{c}_{\alpha,\beta}(\mathbf{y})$ is invertible if and only if $\mathbf{y}_{|0} \notin {\wp_F}^{(\ell)}$.

Proof. Put $\upsilon_{|0} = \sum_{j=1}^{\ell-1} \overline{y_j\theta^j}$, as defined above, and let $\xi := \tau(\upsilon_{|0})$. By Proposition 7.1.2, we have that

$$\mathsf{c}_{\alpha,\beta}(\mathbf{y}) = \mathsf{F}^{\beta}\mathsf{T}\mathsf{L}_{\xi} - \mathsf{T}\mathsf{L}_{\xi}\mathsf{F}^{\alpha}.$$

By the discussion above, we have that $\mathsf{c}_{\alpha,\beta}(\mathbf{y})$ is non-invertible iff $\xi \in \theta^{\ell-1}\mathfrak{f}$, which holds iff

$$\upsilon_{|0} = \tau(\xi) \in \mathfrak{f}.$$

By the definition of $\upsilon_{|0}$ (cf. § 7.1), this holds iff $\upsilon_{|0} = 0$ which is equivalent to $\mathbf{y}_{|0} \in {\wp_F}^{(\ell)}$.

□

In order to compute the ranks of $\mathsf{c}_{\alpha,\beta}$ whenever $\alpha + \beta \neq 0$, and of b_{β}, we will now show how to interpret the condition of being a kernel element of one of the above matrices, in terms of a functional equation defined over $\mathfrak{e}$.

Fix some $\xi \in \mathfrak{e}^{\times}$ and let $\alpha, \beta \in \mathbb{F}_{\ell}$ be such that $\alpha+\beta \neq 0$. By invoking the identities of Lemma 7.1.4 we have that

$$\begin{aligned} \mathsf{F}^{\beta}\mathsf{T}\mathsf{L}_{\xi} - \mathsf{T}\mathsf{L}_{\xi}\mathsf{F}^{\alpha} &= \omega^{-\beta}\mathsf{T}\mathsf{F}^{-\beta}\mathsf{L}_{\xi} - \mathsf{T}\mathsf{L}_{\xi}\mathsf{F}^{\alpha} && \text{by Lemma 7.1.4, item (2),} \\ &= \mathsf{T}\left(\mathsf{L}_{\omega^{-\beta}\mathfrak{fr}^{-\beta}(\xi)}\mathsf{F}^{-\beta} - \mathsf{L}_{\xi}\mathsf{F}^{\alpha}\right) && \text{by item (3).} \end{aligned}$$

Let $\upsilon \in \mathfrak{e}$ be arbitrary, and let $[\upsilon]_{\overline{\Theta}} \in \mathfrak{f}^{\ell}$ denote the coordinate vector of υ with respect to the basis $\overline{\Theta}$. Since T is invertible, by the above equality we have that

$$\left(\mathsf{F}^{\beta}\mathsf{T}\mathsf{L}_{\xi} - \mathsf{T}\mathsf{L}_{\xi}\mathsf{F}^{\alpha}\right)[\upsilon]_{\overline{\Theta}} = 0 \quad \Longleftrightarrow \quad \mathsf{L}_{\omega^{-\beta}\mathfrak{fr}^{-\beta}(\xi)}\mathsf{F}^{-\beta}[\upsilon]_{\overline{\Theta}} = \mathsf{L}_{\xi}\mathsf{F}^{\alpha}[\upsilon]_{\overline{\Theta}},$$

which, by definition, is equivalent to the condition

$$\omega^{-\beta}\mathrm{fr}^{-\beta}(\xi\upsilon) = \xi\mathrm{fr}^{\alpha}(\upsilon).$$

Multiplying by ω^{β} and applying fr^{β} to both sides of the equation, we have that

$$\left(\mathsf{F}^{\beta}\mathsf{TL}_{\xi} - \mathsf{TL}_{\xi}\mathsf{F}^{\alpha}\right)[\upsilon]_{\overline{\Theta}} = 0 \qquad \Longleftrightarrow \qquad \xi\upsilon = \omega^{\beta}\mathrm{fr}^{\beta}(\xi)\mathrm{fr}^{\alpha+\beta}(\upsilon).$$

Now, observe the following curious fact. Suppose $\upsilon_1, \upsilon_2 \in \mathfrak{e}$ are distinct, and satisfy the above equation, i.e.

$$\xi\upsilon_i = \omega^{\beta}\mathrm{fr}^{\beta}(\xi)\mathrm{fr}^{\alpha+\beta}(\upsilon_i) \quad \text{for } i = 1, 2.$$

Assume, without loss of generality, that $\upsilon_1 \neq 0$. Then, there exists an element $\lambda \in \mathfrak{e}$ such that $\upsilon_2 = \lambda\upsilon_1$ (namely $\lambda = \upsilon_1^{-1}\upsilon_2$). So the two equation reduce to :

$$\begin{cases} \xi\upsilon_1 & = & \omega^{\beta}\mathrm{fr}^{\beta}(\xi)\mathrm{fr}^{\alpha+\beta}(\upsilon_1) \\ \lambda\xi\upsilon_1 & = & \omega^{\beta}\mathrm{fr}^{\beta}(\xi)\mathrm{fr}^{\alpha+\beta}(\lambda\upsilon_1). \end{cases}$$

Since the left-hand side of the first equation is assumed invertible, we can multiply the second by its inverse, and obtain that

$$\lambda = \mathrm{fr}^{\alpha+\beta}(\upsilon_1\lambda)\mathrm{fr}^{\alpha+\beta}(\upsilon_1^{-1}) = \mathrm{fr}^{\alpha+\beta}(\lambda).$$

The assumption that $\alpha + \beta \neq 0$ implies that $\mathrm{fr}^{\alpha+\beta}$ generates the Galois group $\mathbf{G}(\mathfrak{e}/\mathfrak{f})$. Thus λ is fixed by $\mathbf{G}(\mathfrak{e}/\mathfrak{f})$, and hence is an element of $\mathfrak{f}$.

What we have learned is that if there exist two distinct elements $\upsilon_1, \upsilon_2 \in \mathfrak{e}$, such that

$$\left(\mathsf{F}^{\beta}\mathsf{TL}_{\xi} - \mathsf{TL}_{\xi}\mathsf{F}^{\alpha}\right)[\upsilon_1]_{\overline{\Theta}} = \left(\mathsf{F}^{\beta}\mathsf{TL}_{\xi} - \mathsf{TL}_{\xi}\mathsf{F}^{\alpha}\right)[\upsilon_2]_{\overline{\Theta}} = 0,$$

then $\upsilon_2 = \lambda\upsilon_1$ for some $\lambda \in \mathfrak{f}$. In particular, the kernel of the map $\mathsf{F}^{\beta}\mathsf{TL}_{\xi} - \mathsf{TL}_{\xi}\mathsf{F}^{\alpha}$, where $\xi \neq 0$, is at most one-dimensional. (In fact, we will see in the sequel that the kernel of such a map is in fact one-dimensional, whenever $\xi \neq 0$).

Consequently, we have the following.

Proposition 7.1.6. Let $\alpha, \beta \in \mathbb{F}_{\ell}$, be such that $\gamma := \alpha + \beta \neq 0$. Let $\mathbf{y} \in W(\mathscr{O}_F)$ have $\mathbf{y}_{|\gamma} \notin \wp_F{}^{(\ell)}$. Then:

1. If $\delta(\alpha, \beta) = 0$, then $\mathrm{rk}_{\mathfrak{f}}\mathsf{c}_{\alpha,\beta}(\mathbf{y}) \geq \ell - 1$.

2. If $\delta(\alpha,\beta)=1$, then $\mathrm{rk}_{\mathfrak{f}}\mathsf{c}_{\alpha,\beta}(\pi^{-1}\mathbf{y})\geq \ell-1$.
3. If $\alpha=0$, then $\mathrm{rk}_{\mathfrak{f}}\mathsf{b}_{\beta}(\pi^{-1}\mathbf{y})=\ell-1$.

Proof. Let $\upsilon_{|\gamma}=\sum_{j=0}^{\ell-1}\overline{y_{r(\gamma)\ell+j}\theta^j}\in\mathfrak{e}^\times$ and let $\xi=\tau(\upsilon_{|\gamma})$. By Proposition 7.1.2 we have that

$$\mathsf{c}_{\alpha,\beta}(\mathbf{y})=\mathsf{F}^\beta\mathsf{TL}_\xi-\mathsf{TL}_\xi\mathsf{F}^\alpha,$$

whenever $\delta(\alpha,\beta)=0$ and $\alpha\neq 0$. Hence, item (1) follows by the above discussion. The case of item (2), where $\delta(\alpha,\beta)=1$, follows in a similar vein.

As for item (3), assume $\alpha=0$ and consider the $\ell\times\ell$ matrix

$$\mathsf{F}^\beta\mathsf{TL}_\xi-\mathsf{TL}_\xi.$$

By the discussion above, we know that this matrix has rank $\geq \ell-1$. Also, an examination of its entries reveals that the first row of this matrix is the zero-row. Thus the rank of this matrix equals $\ell-1$. Clearly, the rank of this matrix is unchanged when multiplied on the left by $\mathsf{P}=\begin{pmatrix}0&1&&\\ \vdots&&\ddots&\\ 0&&&1\end{pmatrix}$. Thus, by Proposition 7.1.2 item (4)

$$\mathrm{rk}_{\mathfrak{f}}\mathsf{b}_\beta(\pi^{-1}\mathbf{y})=\mathrm{rk}_{\mathfrak{f}}\left(\mathsf{P}\circ\left(\mathsf{F}^\beta\mathsf{TL}_\xi-\mathsf{TL}_\xi\right)\right)=\ell-1.$$

□

Exact Rank of $\mathsf{c}_{\alpha,\beta}$, where $\alpha+\beta\neq 0$

We conclude this section with a complementary proposition which will prove that the inequalities in items (1) and (2) of Proposition 7.1.6 are in fact equalities. We remark that this proposition is not essential at this point for our results (Corollary 7.2.2 and Theorem A) to hold, and may be skipped if needed. Nonetheless, for the sake of completeness (and perhaps of future generalization), we add the proof here.

Before stating and proving the proposition we will require a lemma, which the reader may recognize as a specific instance of Hilbert's Theorem 90. As the proof for the case of finite fields is elementary, we add it here.

Lemma 7.1.7. Let $\gamma\in\mathbb{F}_\ell^\times$ be arbitrary. The map defined on $\mathfrak{e}^\times$ by

$$\xi\mapsto\mathfrak{fr}^\gamma(\xi)\cdot\xi^{-1}$$

is onto the group $SL_1(\mathfrak{e}/\mathfrak{f})$ of elements of field-norm 1 in $\mathfrak{e}^\times$.

Proof. Consider the kernel of this map. Since fr^γ generates $\mathbf{G}(\mathfrak{e}/\mathfrak{f})$, we have that

$$\mathrm{fr}^\gamma(\xi)\cdot\xi^{-1}=1 \quad\iff\quad \xi=\mathrm{fr}^\gamma(\xi) \quad\iff\quad \xi \text{ is fixed by } \mathbf{G}(\mathfrak{e}/\mathfrak{f}).$$

Thus, the kernel of this map equals $\mathfrak{f}^\times$, which implies that the image is of order

$$\left|\mathfrak{e}^\times:\mathfrak{f}^\times\right|=\frac{q^\ell-1}{q-1}.$$

Note that for any $\xi\in\mathfrak{e}^\times$ we have that

$$\mathrm{Nr}_{\mathfrak{e}/\mathfrak{f}}(\mathrm{fr}^\gamma(\xi)\cdot\xi^{-1})=\mathrm{Nr}_{\mathfrak{e}/\mathfrak{f}}(\mathrm{fr}^\gamma(\xi))\cdot\mathrm{Nr}_{\mathfrak{e}/\mathfrak{f}}(\xi)^{-1}=1,$$

and hence the image of this map is included in $SL_1(\mathfrak{e}/\mathfrak{f})$.

Taking into account the fact that $\#SL_1(\mathfrak{e}/\mathfrak{f})=|\mathfrak{e}^\times:\mathfrak{f}^\times|=\frac{q^\ell-1}{q-1}$ as well (since $SL_1(\mathfrak{e}/\mathfrak{f})$ is the kernel of the surjective map $\mathrm{Nr}_{\mathfrak{e}/\mathfrak{f}}$), this inclusion is in fact an equality. □

Proposition 7.1.8. Let $\mathbf{y}\in W(\mathscr{O}_F)$, and let $\alpha,\beta\in\mathbb{F}_\ell$ be such that $\gamma=\alpha+\beta\neq0$. Then

1. If $\upsilon_{|\gamma}\neq0$, and $\delta(\alpha,\beta)=0$ then $\mathrm{rk}_{\mathfrak{f}}\mathsf{c}_{\alpha,\beta}(\mathbf{y})=\ell-1$.
2. If $\upsilon_{|\gamma}\neq0$, and $\delta(\alpha,\beta)=1$ then $\mathrm{rk}_{\mathfrak{f}}\mathsf{c}_{\alpha,\beta}(\pi^{-1}\mathbf{y})=\ell-1$

Proof. We have already seen in Proposition 7.1.6 that the kernels of these matrices is at most 1-dimensional. Thus, to prove the proposition, we only need to show that a non-zero kernel element exists for all such $\mathsf{c}_{\alpha,\beta}$. Assume first that $\delta(\alpha,\beta)=0$, and put $\mu=\tau(\upsilon_{|\gamma})$. Note that the element $\varpi^\beta\cdot\mathrm{fr}^\beta(\mu)^{-1}\mu\in\mathfrak{e}^\times$ has field-norm 1, as

$$\mathrm{Nr}_{\mathfrak{e}/\mathfrak{f}}\left(\varpi^{-\beta}\cdot\mathrm{fr}^\beta(\mu)^{-1}\mu\right)=\varpi^{-\beta\ell}\cdot\mathrm{Nr}_{\mathfrak{e}/\mathfrak{f}}(\mathrm{fr}^\beta(\mu))^{-1}\cdot\mathrm{Nr}_{\mathfrak{e}/\mathfrak{f}}(\mu)=1.$$

By Lemma 7.1.7, there exists some $\xi\in\mathfrak{e}^\times$ such that

$$\mathrm{fr}^\gamma(\xi)\cdot\xi^{-1}=\varpi^{-\beta}\cdot\mathrm{fr}^\beta(\mu)^{-1}\mu.$$

Since $\mu,\xi\in\mathfrak{e}^\times$, we get from this that

$$\mu\cdot\xi=\varpi^\beta\mathrm{fr}^\beta(\mu)\mathrm{fr}^\gamma(\xi),=\varpi^\beta\mathrm{fr}^\beta(\mu)\mathrm{fr}^{\alpha+\beta}(\xi)$$

which translates, in the language of operators, to the condition

$$\mathsf{c}_{\alpha,\beta}(\mathbf{y})[\xi]_{\overline{\Theta}}=0.$$

The proof of the case where $\delta(\alpha,\beta)=1$ is similar. □

7.2 Calculation of Minors

At last, we arrive at the desired calculation of absolute values for specific minors of $\mathcal{R}(\mathbf{y})$. Our method will be as follows. To begin with, we recall the fact that the maximal rank of any minor of $\mathcal{R}(\mathbf{y})$ is $2\rho = \ell(\ell-1)$ (see § 4.2.1). Knowing this, for any $\mathbf{y} \in W(\mathscr{O}_F)$ we will use Propositions 7.1.5 and 7.1.6 in order to extract a sub-matrix of order $\ell(\ell-1)$ from which we can identify certain minors which are invertible over $\mathscr{O}_F$. The final touch will be to realize that these sub-matrices are actually $(\boldsymbol{\nu}, t)$-graded, for a suitable composition $\boldsymbol{\nu}$ and number t. Thus we will be able to calculate their determinant's absolute values by invoking corollary 6.2.3.

Prior to this, let us make a cosmetic change to the matrix $\mathcal{R}(\mathbf{Y})$, by passing to the matrix $\mathcal{R}'(\mathbf{Y}) := \mathsf{J}\mathcal{R}(\mathbf{Y})\mathsf{J}^T$, where

$$\mathsf{J} := \left(\begin{array}{ccc|ccccc} & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ \hline 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \end{array}\right) \in \mathrm{Mat}_d(\mathscr{O}),$$

in which the bottom-left block is $(\ell-1)\times(\ell-1)$, and the top-right block is $\ell(\ell-1)\times\ell(\ell-1)$.

Then $\mathcal{R}'(\mathbf{Y})$ can be visualized as the block matrix

$$\mathcal{R}'(\mathbf{Y}) = \left(\begin{array}{c|c|c|c|c} \mathsf{C}_{1,1}(\mathbf{Y}) & \mathsf{C}_{1,2}(\mathbf{Y}) & \cdots & \mathsf{C}_{1,\ell-1}(\mathbf{Y}) & -\mathsf{B}_1(\mathbf{Y})^T \\ \hline \mathsf{C}_{2,1}(\mathbf{Y}) & \mathsf{C}_{2,2}(\mathbf{Y}) & \cdots & \mathsf{C}_{2,\ell-1}(\mathbf{Y}) & -\mathsf{B}_2(\mathbf{Y})^T \\ \hline \vdots & \vdots & \ddots & & \vdots \\ \hline \mathsf{C}_{\ell-1,1}(\mathbf{Y}) & \mathsf{C}_{\ell-1,2}(\mathbf{Y}) & \cdots & \mathsf{C}_{\ell-1,\ell-1} & -\mathsf{B}_{\ell-1}(\mathbf{Y})^T \\ \hline \mathsf{B}_1(\mathbf{Y}) & \mathsf{B}_2(\mathbf{Y}) & \cdots & \mathsf{B}_{\ell-1}(\mathbf{Y}) & 0_{\ell-1} \end{array}\right),$$

It is clear (cf. § 2.4.1) that considering minors of $\mathcal{R}'(\mathbf{y})$ instead of $\mathcal{R}(\mathbf{y})$ will have no effect on the values of $\|F_j(\mathbf{y})\|_{\wp}$.

It will convenient for us to associate to each element $\mathbf{y} \in W(\mathscr{O}_F)$ an element of $\mathbb{F}_\ell$,

$$\mathtt{t}(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y}_{|0} \notin {\wp_F}^{(\ell)}, \\ \max\left\{\gamma \in \mathbb{F}_\ell \mid \mathbf{y}_{|\gamma} \notin {\wp_F}^{(\ell)}\right\} & \text{otherwise.} \end{cases}$$

Remark. Recall that in this essay $\mathbb{F}_\ell$ is endowed with a linear order, such that $\alpha < \beta$ iff $r(\alpha) < r(\beta)$ (cf. § 5.2.1). Thus, it makes sense for us to define $\mathtt{t}(\mathbf{y})$ as the maximum of a subset of $\mathbb{F}_\ell$.

Moreover, for any $\gamma \in \mathbb{F}_\ell$, we put $W^{[\gamma]}(\mathscr{O}_F) := \mathtt{t}^{-1}(\gamma)$. Explicitly, we have that

$$\begin{aligned} W^{[0]}(\mathscr{O}_F) &= \{(y_1, \ldots, y_d) \in W(\mathscr{O}_F) \mid \exists 1 \leq j \leq \ell, \text{ such that } y_j \notin \wp_F\} \\ &= {\mathscr{O}_F}^{(d)} \smallsetminus \left({\wp_F}^{(\ell-1)} \times {\mathscr{O}_F}^{(\ell^2-\ell)}\right), \end{aligned}$$

$$\begin{aligned} W^{[1]}(\mathscr{O}_F) &= \left\{\mathbf{y} = (y_1, \ldots, y_d) \in W(\mathscr{O}_F) \mid \mathbf{y} \notin W^{[0]}(\mathscr{O}_F), \text{ and } \forall j \geq 2\ell, \quad y_j \in \wp_F\right\} \\ &= \left({\wp_F}^{(\ell-1)} \times {\mathscr{O}_F}^{(\ell)} \times {\wp_F}^{(\ell^2-2\ell)}\right) \smallsetminus {\wp_F}^{(\ell^2-1)}, \end{aligned}$$

and in general, for any $\gamma \geq 1$

$$\begin{aligned} W^{[\gamma]}(\mathscr{O}_F) &= \left\{\mathbf{y} \in W(\mathscr{O}_F) \mid \mathbf{y} \notin W^{[\gamma-1]}(\mathscr{O}_F), \text{ and } \forall j \geq (r(\gamma)+1)\,\ell, \quad y_j \in \wp_F\right\} \\ &= \left({\wp_F}^{(\ell-1)} \times {\mathscr{O}_F}^{(r(\gamma)\ell)} \times {\wp_F}^{(\ell^2-(r(\gamma)+1)\ell)}\right) \smallsetminus \left({\wp_F}^{(\ell-1)} \times {\mathscr{O}_F}^{((r(\gamma)-1)\ell)} \times {\wp_F}^{(\ell^2-r(\gamma)\ell)}\right). \end{aligned}$$

Note that

$$W(\mathscr{O}_F) = \bigsqcup_{\gamma \in \mathbb{F}_\ell} W^{[\gamma]}(\mathscr{O}_F).$$

In this section we will prove the following proposition:

Proposition 7.2.1. Let $\mathbf{y} \in W(\mathscr{O}_F)$ and $j = 1, \ldots, \rho$ be arbitrary, and put $\gamma = \mathtt{t}(\mathbf{y})$.

1. If $\gamma = 0$ then there exists a $2j \times 2j$ sub-matrix M_j of $\mathcal{R}'(\mathbf{y})$ with $|\mathsf{M}_j|_\wp = 1$.
2. If $\gamma \neq 0$, then there exists a $2j \times 2j$ sub-matrix M_j of $\mathcal{R}'(\mathbf{y})$, such that

$$|\det \mathsf{M}_j|_\wp = \min\left\{1, q^{-2j+(\ell-1)(r(\gamma)-1)}\right\}.$$

Consequently, we will have

Corollary 7.2.2. Let $\gamma \in \mathbb{F}_\ell$ and $j = 1, \ldots, \rho$. Then

$$\|F_j(\mathbf{y})\|_\wp \geq \min\left\{1, q^{-2j+(\ell-1)(r(\gamma)-1)}\right\}.$$

Furthermore, if $\mathbf{y} \in W^{[0]}(\mathscr{O}_F)$ then $\|F_j(\mathbf{y})\|_\wp = 1$ for all j's.

Throughout the remainder of this chapter, we fix an element $\mathbf{y} \in W(\mathscr{O}_F)$ and put $\gamma = \mathtt{t}(\mathbf{y}) \in \mathbb{F}_\ell$. We shall prove the two parts of Proposition 7.2.1 separately. In both cases we will prove the case of $j = \rho$ explicitly, and deduce the rest of the cases from it. Prior to this, let us make two observations, which we will use throughout the proof.

Firstly, note that by the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Mat}_n(\mathscr{O}_F) \arrow[r, "A \mapsto \overline{A}"] \arrow[d, "\det"'] & \mathrm{Mat}_n(\mathfrak{f}) \arrow[d, "\det"] \\
\mathscr{O}_F \arrow[r, "\text{reduction mod } \wp"'] & \mathfrak{f}
\end{tikzcd}$$

we have that:

Observation 1. $|\det A|_\wp = 1 \quad \Longleftrightarrow \quad \overline{A}$ is invertible.

Secondly, by considering the rank of a matrix as the maximal number of linearly independent rows, we have the following:

Observation 2. Let $A \in \mathrm{Mat}_{n_1 \times n_2}(\mathfrak{f})$ have $\mathrm{rk}_{\mathfrak{f}} A = m \leq \min\{n_1, n_2\}$. Then for any $k \leq m$, A admits a $k \times k$ invertible minor.

With this at hand, we can get started. Let us take care of the easier case first.

7.2.3 Proposition 7.2.1 Part 1- The Case $\mathtt{t}(\mathbf{y}) = 0$

Consider the sub-matrix M_ρ which is obtained by deleting the last $\ell - 1$ columns and rows of $\mathcal{R}'(\mathbf{y})$ (i.e. by deleting the sub-matrices B_β). Then

$$\mathsf{M}_\rho = \left(\begin{array}{c:c:c:c}
\mathsf{C}_{1,1}(\mathbf{y}) & \mathsf{C}_{1,2}(\mathbf{y}) & \cdots & \mathsf{C}_{1,\ell-1}(\mathbf{y}) \\ \hdashline
\mathsf{C}_{2,1}(\mathbf{y}) & \mathsf{C}_{2,2}(\mathbf{y}) & \cdots & \mathsf{C}_{2,\ell-1}(\mathbf{y}) \\ \hdashline
\vdots & \iddots & \iddots & \\ \hdashline
\mathsf{C}_{\ell-1,1}(\mathbf{y}) & \mathsf{C}_{\ell-1,2}(\mathbf{y}) & \cdots & \mathsf{C}_{\ell-1,\ell-1}
\end{array}\right).$$

Recall, from Proposition 5.4.1, that all sub-matrices appearing in the bottom-right triangle of this matrix (that is, $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$ such that $\delta(\alpha,\beta)=1$) are multiples of π. Thus, by passing to the image of M_ρ over $\mathfrak{f}$, we see that

$$\overline{\mathsf{M}}_\rho = \begin{pmatrix} \mathsf{c}_{1,1}(\mathbf{y}) & \mathsf{c}_{1,2}(\mathbf{y}) & \cdots & \mathsf{c}_{1,\ell-1}(\mathbf{y}) \\ \mathsf{c}_{2,1}(\mathbf{y}) & \mathsf{c}_{2,2}(\mathbf{y}) & \cdots & 0_\ell \\ \vdots & \iddots & \iddots & \vdots \\ \mathsf{c}_{\ell-1,1}(\mathbf{y}) & 0_\ell & & 0_\ell \end{pmatrix}.$$

Since the sub-matrices $\mathsf{c}_{\alpha,\ell-\alpha}(\mathbf{y})$ (the sub-matrices appearing on the secondary diagonal) are invertible for any $\alpha \in \mathbb{F}_\ell$ (by Proposition 7.1.5), we see that in fact $\overline{\mathsf{M}}_\rho$ is invertible.

Consequently, by observation 1, $|\det \mathsf{M}_\rho|_\wp = 1$. Moreover, by observation 2, for any $j=1,\dots,\rho$, $\overline{\mathsf{M}}_\rho$ has an invertible sub-matrix, denoted as $\overline{\mathsf{M}}_j$. Taking M_j to be the sub-matrix of $\mathcal{R}(\mathbf{y})$ defined by the same coordinates, we see that $|\det \mathsf{M}_j|_\wp = 1$ as well.

Thus, we have that $\|F_j(\mathbf{y})\|_\wp \geq 1$. Furthermore, since any minor of $\mathcal{R}'(\mathbf{y})$ is an element of $\mathscr{O}_F$, it trivially has absolute value ≤ 1. Thus

$$\|F_j(\mathbf{y})\|_\wp = 1, \quad \forall j = 1,\dots,\rho.$$

7.2.4 Proposition 7.2.1 Part 2- The Case $\mathtt{t}(\mathbf{y}) \neq 0$

The assumption that $\gamma = \mathtt{t}(\mathbf{y}) \neq 0$ implies that $y_j \in \wp_F$ whenever $j \geq (r(\gamma)+1)\ell$ or $j \leq \ell$. In particular, if we visualize $\mathcal{R}'(\mathbf{y})$ as the block matrix:

$$\mathcal{R}'(\mathbf{y}) =$$

$$\begin{pmatrix}
\mathsf{C}_{1,1} & \cdots & & \boxed{\mathsf{C}_{1,\gamma-1}} & \mathsf{C}_{1,\gamma} & & & & & \mathsf{C}_{1,\ell-1} & -\mathsf{B}_1^T \\
\mathsf{C}_{2,1} & & \boxed{\mathsf{C}_{2,\gamma-2}} & \mathsf{C}_{2,\gamma-1} & & & & & & \mathsf{C}_{2,\ell-1} & -\mathsf{B}_2^T \\
\vdots & \iddots & \iddots & & & & & & & \vdots & \vdots \\
\boxed{\mathsf{C}_{\gamma-1,1}} & & & & & & & & & \mathsf{C}_{\gamma-1,\ell-1} & -\mathsf{B}_{\gamma-1}^T \\
\mathsf{C}_{\gamma,1} & & & & & & & & & \mathsf{C}_{\gamma,\ell-1} & \boxed{-\mathsf{B}_\gamma^T} \\
\mathsf{C}_{\gamma+1,1} & & & & & \iddots & & & & \boxed{\mathsf{C}_{\gamma+1,\ell-1}} & -\mathsf{B}_{\gamma+1}^T \\
\mathsf{C}_{\gamma+2,1} & & & & & & & & \boxed{\mathsf{C}_{\gamma+2,\ell-2}} & \mathsf{C}_{\gamma+2,\ell-1} & -\mathsf{B}_{\gamma+2}^T \\
\vdots & & & & & & & \iddots & \iddots & \iddots & \vdots \\
\mathsf{C}_{\ell-1,1} & & & & & & \boxed{\mathsf{C}_{\ell-1,\gamma+1}} & \mathsf{C}_{\ell-1,\gamma+2} & \cdots & \mathsf{C}_{\ell-1,\ell-1} & -\mathsf{B}_{\ell-1}^T \\
\mathsf{B}_1 & & & & & \boxed{\mathsf{B}_\gamma} & \mathsf{B}_{\gamma+1} & \mathsf{B}_{\gamma+2} & \cdots & \mathsf{B}_{\ell-1} & 0_{\ell-1}
\end{pmatrix},$$

we have that

- Any matrix $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$ where $\alpha+\beta>\gamma$ or $\alpha+\beta=0$ has all its entries in $\wp_F$.
- In addition, if $\delta(\alpha,\beta)=1$ and $\alpha+\beta>\gamma$ (i.e. $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$ appears in the dark-gray part of the above diagram), then all entries of $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$ are elements of $\wp_F^2$.
- Any matrix B_β with $\beta>\gamma$ also has all its entries in $\wp_F^2$ (by the assumption, and by Proposition 5.4.1).

Moreover, by Proposition 5.4.1, we know that all matrices $\mathsf{B}_\beta(\mathbf{y})$ and all $\mathsf{C}_{\alpha,\beta}(\mathbf{y})$'s with $\delta(\alpha,\beta)=1$ (i.e. appears underneath the secondary diagonal), are multiples of π.

To be a bit more schematic, the light gray cells in the above matrix all have entries in $\wp_F$, and the dark-gray ones have all entries in $\wp_F^2$. At this point a faint scent of corollary 6.2.3 should attract the attention of the reader. The only thing that is still missing is to replace the sub-matrices marked in boxes with invertible matrices (or multiples of π and an invertible matrix). To this end, we invoke Proposition 7.1.6.

First off, let $1\le\alpha<\gamma$. By item (1) of Proposition 7.1.6 we have that $\mathrm{rk}_{\mathfrak{f}}\mathsf{c}_{\alpha,\gamma-\alpha}(\mathbf{y})\ge\ell-1$. Thus, one can obtain an invertible $(\ell-1)\times(\ell-1)$ sub-matrix of $\mathsf{c}_{\alpha,\gamma-\alpha}$ by deleting one row and one column. Let r_α and c_α denote the respective indices in $\mathcal{R}'(\mathbf{y})$ of these row and columns. Note that since $\mathcal{R}'(\mathbf{y})$ is anti-symmetric we have that

$$\mathsf{c}_{\alpha,\gamma-\alpha}(\mathbf{y})=-\mathsf{c}_{\gamma-\alpha,\alpha}(\mathbf{y})^T.$$

In particular, we may assume that $c_\alpha=r_{\gamma-\alpha}$ for all $1\le\alpha<\gamma$. The sub-matrix of $\mathsf{C}_{\alpha,\gamma-\alpha}$ obtained by removing the row and column r_α and c_α has determinant of absolute value 1 (i.e. is invertible over $\mathscr{O}_F$).

Likewise, for any $\gamma<\alpha\le\ell-1$ the matrix $\mathsf{c}_{\alpha,\ell-\gamma-\alpha}(\pi^{-1}\mathbf{y})$ has an invertible sub-matrix of order $(\ell-1)\times(\ell-1)$ (by item (2) of 7.1.6), obtained by deleting one row and one column. Let r_α and c_α denote the respective indices of these row and column in $\mathcal{R}'(\mathbf{y})$. As before, we may assume that $c_\alpha=r_{\ell+\gamma-\alpha}$ for all $\gamma<\alpha\le\ell-1$. The sub-matrix of $\mathsf{C}_{\alpha,\ell-\gamma-\alpha}(\mathbf{y})$ obtained by deleting the row and column r_α and c_α can be written as the product of π and an invertible $(\ell-1)\times(\ell-1)$ matrix over $\mathscr{O}_F$.

Lastly, the $\mathfrak{f}$-rank of the matrix $\mathsf{b}_\gamma(\pi^{-1}\mathbf{y})$ is $\ell-1$ (item (3) of 7.1.6), and hence it has an invertible $(\ell-1)\times(\ell-1)$ obtained by deleting a single column, denoted as c_γ. Again, the sub matrix of $\mathsf{B}_\gamma(\mathbf{y})$ which is obtained by deleting the column c_γ can be written as the product of π and an invertible matrix over $\mathscr{O}_F$.

Let M_ρ be the sub-matrix of $\mathcal{R}'(\mathbf{y})$ obtained by deleting the set of ℓ rows and columns with indices $\{c_\alpha\}_{\alpha\in\mathbb{F}_\ell}$. Additionally, for any $\alpha\in\mathbb{F}_\ell^\times\smallsetminus\{\gamma\}$ let $M_\alpha\in\mathrm{Mat}_{\ell-1}(\mathscr{O}_F)$ be the sub-matrix of $\mathsf{C}_{\alpha,\gamma-\alpha}(\mathbf{y})$ obtained by deleting the rows and columns $\{c_\alpha\}_{\alpha\in\mathbb{F}_\ell}$ from $\mathcal{R}'(\mathbf{y})$, and let $M_\gamma \in \mathrm{Mat}_{\ell-1}(\mathscr{O}_F)$ and $M_\ell \in \mathrm{Mat}_{\ell-1}(\mathscr{O}_F)$ be the respective sub-matrices of $\mathsf{B}_\gamma(\mathbf{y})$ and $-\mathsf{B}_\gamma(\mathbf{y})^T$, obtained by this deletion.

Consider the composition $\boldsymbol{\nu} = (\ell-1, \ell-1, \ldots, \ell-1)$ of length ℓ (that is, $\boldsymbol{\nu}$ is a composition of 2ρ), along with the number $t = r(\gamma)-1$. Applying the construction of Definition 6.2.2, one sees the M_ρ is $(\boldsymbol{\nu}, t)$ graded, with the sub-matrices M_α acting as the matrices $M_{r(\alpha),r(\alpha)}$ of Lemma 6.2.1, for any $\alpha \in \mathbb{F}_\ell$.

From this we obtain:

Lemma 7.2.5. Let $\mathbf{y} \in W^{[\gamma]}(\mathscr{O}_F)$ be such that $\gamma \neq 0$ and let M_ρ be the $2\rho \times 2\rho$ sub-matrix of $\mathcal{R}'(\mathbf{y})$ described above. Then

$$|\det \mathsf{M}_\rho|_\wp = q^{-c_{\boldsymbol{\nu},(r(\gamma)-1)}} = q^{-2\rho+(\ell-1)(r(\gamma)-1)},$$

as wanted.

From this point, the proof that for any $1 \le j \le \rho$ there exists a matrix M_j, which is a $2j \times 2j$ sub-matrix of M_ρ (and hence of $\mathcal{R}'(\mathbf{y})$) and has

$$|\det \mathsf{M}_j|_\wp = \min\left\{1, q^{-2j+(\ell-1)(r(\gamma)-1)}\right\}$$

should be clear. One simply needs to deduct values from the ultimate coordinates of $\boldsymbol{\nu}$ to get a composition of $2j$, and apply observation 2 to obtain a corresponding sub-matrix of M_ρ. Applying this, Proposition 7.2.1 follows.

Remark. As is evident from the above diagram, if $\mathtt{t}(\mathbf{y}) \neq 0$ there are at most $\ell\cdot(r(\gamma)-1)$ columns of $\mathcal{R}'(\mathbf{y})$ whose entries are not all elements of ${\wp_F}^d$. Thus, if M is a $2j \times 2j$ matrix of $\mathcal{R}'(\mathbf{y})$, then M has at least $2j - \ell((r(\gamma)-1)$ columns divisible by π (this number can be negative, that simply means that the upper bound for $|\det \mathsf{M}|_\wp$ is 1). Hence

$$\|F_j(\mathbf{y})\|_\wp \le \min\left\{1, q^{-2j+\ell(r(\gamma)-1)}\right\}.$$

This easily implies that the upper bound of $\|F_j(\mathbf{y})\|_\wp$ equals the lower bound in the case where $\mathtt{t}(\mathbf{y}) = 1$ as well.

In the case where $\mathtt{t}(\mathbf{y}) > 1$ this upper bound can be lowered a bit more, if we note that $\overline{\mathcal{R}'(\mathbf{y})}$ has an additional kernel vector, which can be obtained by regarding the non-zero kernel vector of $\mathsf{c}_{1,\gamma-1}(\mathbf{y})$ as an element of $\mathfrak{f}^d$ (with the obvious embedding). Thus

$$\|F_j(\mathbf{y})\|_\wp \le \min\left\{1, q^{-2j+\ell(r(\gamma)-1)-1}\right\}.$$

In general this reduction is not enough. In order for the upper bound of $\|F_j(\mathbf{y})\|_\wp$ to equal to the lower bound we have obtained, we need to reduce it by a factor of $q^{-r(\gamma)+1}$.

To date we do not know whether this reduction is possible or not. Nonetheless, the naïve upper bound for $\|F_j(\mathbf{y})\|_\wp$ we have described above is sufficient in order to calculate the representation zeta function of $SL_1^m(\mathcal{O})$ in the case where $\deg D = 3$, as we shall see in the next chapter.

Chapter 8

Example: The Representation Zeta Function of $SL_1^1(\mathcal{O})$, Where $\deg(D) = 3$

Theorem A. *Let $F \supseteq \mathbb{Q}_p$ be a local field, with $p > 3$, and let D be a division algebra of degree 3 over F. Assume that F contains a non-trivial cube root of unity, and that $\mathfrak{sl}_1^1(\mathcal{O})$ is a potent and saturable Lie-algebra.*

Then

$$\zeta_{SL_1^1(\mathcal{O})}(s) = q^3 \cdot \frac{1 + q^{-(s-1)}(1 - q^{-3}) - q^{-3s-3}}{1 - q^{-3s+2}}.$$

In this example, we retain all notations from the general case. For brevity, we write $\mathscr{O}$ and $\wp$ for $\mathscr{O}_F$ and $\wp_F$.

A monomial basis for $\mathfrak{sl}_1^1(\mathcal{O})$ is given by $\mathcal{B} = \left\{\pi\theta, \pi\theta^2, u^l\theta^k \mid l = 1, 2,\ k = 0, 1, 2\right\}$, and the commutator matrix $\mathcal{R}(\mathbf{Y})$ is given with respect to $\mathcal{B}$ by

$$\left[\begin{array}{cc:ccc:ccc}
0 & 0 & \pi(1-\omega)Y_4 & \pi(1-\omega)Y_5 & \pi b(1-\omega)Y_3 & \pi(1-\omega^2)Y_7 & \pi(1-\omega^2)Y_8 & \pi b(1-\omega^2)Y_6 \\
0 & 0 & \pi(1-\omega^2)Y_5 & \pi b(1-\omega^2)Y_3 & \pi b(1-\omega^2)Y_4 & \pi(1-\omega)Y_8 & \pi b(1-\omega)Y_6 & \pi b(1-\omega)Y_7 \\
\hdashline
\pi(\omega-1)Y_4 & \pi(\omega^2-1)Y_5 & 0 & (\omega-1)Y_7 & (\omega^2-1)Y_8 & 0 & (\omega-1)Y_1 & (\omega^2-1)Y_2 \\
\pi(\omega-1)Y_5 & \pi b(1-\omega^2)Y_3 & (1-\omega)Y_7 & 0 & b(\omega^2-\omega)Y_6 & (1-\omega^2)Y_1 & (\omega-\omega^2)Y_2 & 0 \\
\pi b(\omega-1)Y_3 & \pi b(\omega^2-1)Y_4 & (1-\omega^2)Y_8 & b(\omega-\omega^2)Y_6 & 0 & (1-\omega)Y_2 & 0 & b(\omega^2-\omega)Y_1 \\
\hdashline
\pi(\omega^2-1)Y_7 & \pi(\omega-1)Y_8 & 0 & (\omega^2-1)Y_1 & (\omega-1)Y_2 & 0 & \pi(\omega^2-1)Y_4 & \pi(\omega-1)Y_5 \\
\pi(\omega^2-1)Y_8 & \pi b(\omega-1)Y_6 & (1-\omega)Y_1 & (\omega^2-\omega)Y_2 & 0 & \pi(1-\omega^2)Y_4 & 0 & \pi b(\omega-\omega^2)Y_3 \\
\pi b(\omega^2-1)Y_6 & \pi b(\omega-1)Y_7 & (1-\omega^2)Y_2 & 0 & b(\omega-\omega^2)Y_1 & \pi(1-\omega)Y_5 & \pi b(\omega^2-\omega)Y_3 & 0
\end{array}\right].$$

8 Example: The Representation Zeta Function of $SL_1^1(\mathcal{O})$, Where $\deg(D) = 3$

The set $W(\mathscr{O})$ is partitioned into the following three subsets:

$$W^{[0]}(\mathscr{O}) := \left\{\mathbf{y} \in W(\mathscr{O}) \mid y_1 \in \mathscr{O}^\times \text{ or } y_2 \in \mathscr{O}^\times\right\} = W(\mathscr{O}) \smallsetminus (\wp^2 \times \mathscr{O}^6) = \mathscr{O}^8 \smallsetminus (\wp^2 \times \mathscr{O}^6),$$
$$W^{[1]}(\mathscr{O}) := \{\mathbf{y} \mid y_3, y_4 \text{ or } y_5 \text{ is invetible, and } y_1, y_2, y_6, y_7, y_8 \in \wp\} = (\wp^2 \times \mathscr{O}^3 \times \wp^3) \smallsetminus \wp^8,$$
$$W^{[2]}(\mathscr{O}) := \{\mathbf{y} \mid y_1, y_2 \in \wp \text{ and } y_6, y_7 \text{ or } y_8 \text{ is invertible}\} = (\wp^2 \times \mathscr{O}^6) \smallsetminus (\wp^2 \times \mathscr{O}^3 \times \wp^3).$$

The zeta function depends on the integral,

$$\mathcal{Z}_{\mathscr{O}}(r,t) = \int_{(x,\mathbf{y}) \in \wp \times W(\mathscr{O})} |x|_\wp^t \, P(x,\mathbf{y})^r d\mu(x,\mathbf{y}),$$

where

$$P(x,\mathbf{y}) := \prod_{j=1}^{\rho} \frac{\|F_j(\mathbf{y}) \cup x^2 F_{j-1}(\mathbf{y})\|_\wp}{\|F_{j-1}(\mathbf{y})\|_\wp},$$

and μ is the Haar probability measure on $\mathscr{O}^9$ (cf § 2.4, viz. Proposition 2.4.3).

We will now compute $P(x,\mathbf{y})$ for any $x \in \wp$ and $\mathbf{y} \in W(\mathscr{O})$. Note that $\rho = 3$ in this case.

- For $\mathbf{y} \in W^{[0]}(\mathscr{O})$ we know in general that it always holds that $P(x,\mathbf{y}) = 1$, regardless of the value of $x \in \wp$ (item (1) of Proposition 7.2.1).

- If $\mathbf{y} \in W^{[1]}(\mathscr{O})$, then in particular all entries of $\mathcal{R}(\mathbf{y})$ are in $\wp$. Consequently, any $2j \times 2j$ minor of $\mathcal{R}(\mathbf{y})$ is divisible by π^{2j}, and hence for any $j \in \{1,2,3\}$, $\|F_j(\mathbf{y})\|_\wp \leq q^{-2j}$.

  By item (2) of Proposition 7.2.1 we also know that for any j

  $$\|F_j(\mathbf{y})\|_\wp \geq q^{-2j}$$

  (since $\mathtt{t}(\mathbf{y}) - 1 = 0$). Thus, this is in fact an equality. From here it follows easily that

  $$P(x,\mathbf{y}) = q^{-6}$$

  for all $x \in \wp$.

  If $\mathbf{y} \in W^{[2]}(\mathscr{O})$, we have that $\mathcal{R}(\mathbf{y})$ has at most 3 columns which may have invertible entries. Thus

  $$\|F_1(\mathbf{y})\|_\wp \leq 1, \quad \|F_2(\mathbf{y})\|_\wp \leq q^{-1}, \quad \text{and } \|F_3(\mathbf{y})\|_\wp \leq q^{-3}.$$

But since we know that the value over $\|F_j(\mathbf{y})\|_\wp$ equals the maximal absolute value of the set of principal minors (which are all anti-symmetric), we are able to lower these upper bounds to the nearest square[1]. Thus, we have the formula

$$\|F_j(\mathbf{y})\|_\wp \leq \min\left\{1,\ q^{-2j+2}\right\}.$$

Similar to the previous item, we have from item (2) of Proposition 7.2.1 that in fact

$$\|F_j(\mathbf{y})\|_\wp = \min\left\{1,\ q^{-2j+2}\right\}, \quad \forall j = 1,2,3$$

and hence we have the equality

$$P(x,\mathbf{y}) = q^{-4}$$

for all $x \in \wp$.

From here on, the calculation of $\zeta_{SL_1^1(\mathbb{O})}(s)$ follows easily.

Note that the sets $W^{[0]}(\mathscr{O})$, $W^{[1]}(\mathscr{O})$, $W^{[2]}(\mathscr{O})$ have measures $1-q^2$, q^5-q^8 and q^2-q^5 respectively. We have that

$$\begin{aligned}
\mathcal{Z}_{\mathscr{O}}(r,t) &= \int_{(x,\mathbf{y})} |x|_\wp^t\, P(x,\mathbf{y}) d\mu(x,\mathbf{y}) \\
&= \int_{\wp\times W^{[1]}(\mathscr{O})} |x|_\wp^t\, d\mu + q^{-4r}\int_{\wp\times W^{[2]}(\mathscr{O})} |x|_\wp^t\, d\mu + q^{-6r}\int_{\wp\times W^{[3]}(\mathscr{O})} |x|_\wp^t\, d\mu \\
&= \left((1-q^{-2}) + q^{-4r}(q^{-2}-q^{-5}) + q^{-6r}(q^{-5}-q^{-8})\right)\int_\wp |x|_\wp^t\, dx.
\end{aligned}$$

Since

$$\begin{aligned}
\int_\wp |x|_\wp^t\, dx &= \sum_{n=1}^{\infty}\int_{\wp^n\setminus\wp^{n+1}} |x|_\wp^t\, dx \\
&= \sum_{n=1}^{\infty} q^{-nt}\cdot q^{-n}(1-q^{-1}) = \frac{q^{-t-1}(1-q^{-1})}{1-q^{-t-1}}
\end{aligned}$$

we have that

$$\mathcal{Z}_{\mathscr{O}}(r,t) = \frac{q^{-t-1}(1-q^{-1})}{1-q^{-t-1}}\left(1-q^{-2}+q^{-4r}(q^{-2}-q^{-5})+q^{-6r}(q^{-5}-q^{-8})\right)$$

[1]As discussed in the end of the previous chapter, this reduction of the upper-bound can also be explained by rank considerations.

8 Example: The Representation Zeta Function of $SL_1^1(\mathcal{O})$, Where $\deg(D) = 3$

Applying Corollary 3.7 of [AKOV13] we are now able to compute the representation zeta function of $SL_1^1(D)$ via the equation

$$\zeta_{SL_1^1(\mathcal{O})}(s) = 1 + (1 - q^{-1})^{-1} \mathcal{Z}_{\mathscr{O}}(-s/2 - 1, \rho(s+2) - d - 1),$$

which in our case has $d = 8$ and $\rho = 3$.

Thus,

$$\zeta_{SL_1^1(\mathcal{O})}(s) = q^3 \cdot \frac{1 + q^{-(s-1)}(1 - q^{-3}) - q^{-3s-3}}{1 - q^{-3s+2}}.$$

Remark. By applying the stronger version of [AKOV13, Corollary 3.7], one can derive the zeta function of $\mathfrak{sl}_1^m(\mathcal{O})$, whenever m is such that $\mathfrak{g}_m$ is potent and saturable. Namely, one has that

$$\zeta_{SL_1^m(\mathcal{O})}(s) = q^{8m-5} \cdot \frac{1 + q^{-(s-1)}(1 - q^{-3}) - q^{-3s-3}}{1 - q^{-3s+2}},$$

for all such m's.

We omit this calculation for the sake of conciseness, and since the final part of this essay only deals with the case where $\mathfrak{sl}_1^1(\mathcal{O})$ is potent and saturable.

Part III

The Representation Zeta Function of $SL_1(D)$

Basic Notation and Outline

This part of our essay will involve extending our results in order to calculate the representation zeta function of the compact p-adic analytic group $SL_1(D)$. The main tool we will invoke is an application of Clifford theory in order to derive information about the group $SL_1(D)$ from that of the normal subgroup $SL_1^1(\mathcal{O})$.

A basic assumption that will be necessary for our final calculation is that the group $SL_1^1(\mathcal{O})$ is potent and saturable. Recall (cf. Proposition 4.1.1), that a sufficient condition for this to hold is that the ramification index of the ground field F over $\mathbb{Q}_p$ is strictly smaller then $\frac{p-1}{\ell}$. Furthermore, we retain the assumption that F contains non-trivial ℓ-th root of unity, and note that this condition is equivalent to assumption that $q \equiv 1 (\mathrm{mod}\, \ell)$.

It will be convenient for us to modify some notation in order to fit the current setting. To this end, we let G denote the group $SL_1(D) = SL_1(\mathcal{O})$, and put $N := SL_1^1(\mathcal{O}) = G \cap (1 + \mathcal{P})$. The symbol $\mathfrak{g}$ will denote the $\mathscr{O}_F$-module $\mathfrak{sl}_1(\mathcal{O})$ of traceless element in $\mathcal{O}$, and $\mathfrak{n}$ will denote the $\mathscr{O}_F$-module $\mathfrak{sl}_1^1(\mathcal{O})$ (previously denoted as $\mathfrak{g}_1$). The F-Lie algebra $\mathfrak{sl}_1(D)$, of traceless elements in D, will be denoted by $\mathscr{G}$. Note that $\mathscr{G} = F \otimes_{\mathscr{O}} \mathfrak{g}$.

The outline of this part will be as follows:

- Chapter 9 acts as a preliminary chapter to this part of the essay. In this chapter we will recall the basics of Clifford theory, and state its consequences to the calculation of the representation zeta function of G, via the knowledge we have obtained regarding the representation theory of N, given the condition that G/N is a finite cyclic group. In the second section of this chapter we will show that G is in fact the semi-direct product of N and a finite cyclic group H.

- Chapter 10 aims towards the calculation of the inertia subgroups for irreducible characters of N (a precise definition will be given in chapter 9). In order to do so, we will first construct an explicit G-equivariant dictionary between the Pontryagin dual of $\mathfrak{n}$, and elements in certain sub-$\mathscr{O}_F$-modules of $\mathscr{G}$. Given this correspondence, it will be possible to interpret the co-adjoint action of G on $\hat{\mathfrak{n}}$, by considering orbits of the adjoint action of G on $\mathscr{G}$.

  In the last part of the chapter we will be involved in the actual calculation of the inertia subgroups of all characters of N. These subgroup, as it turns out, are closely related to the decomposition of $W(\mathscr{O}_F)$, described in Section 7.2

- To conclude this essay, we will return to the case $\deg D = 3$, and calculate the representation zeta function of $SL_1(D)$, under all the assumption stated above.

Chapter 9

The Groups $SL_1(D)$ and $SL_1^1(\mathcal{O})$

9.1 Clifford Theory

In this section we will review the basic theorems of Clifford theory, which will be used in order to extend our result to the computation of $\zeta_{SL_1(D)}(s)$. Throughout this section we let G denote an arbitrary compact group, and $N \subseteq G$ a closed normal subgroup of finite index. For any χ_1, χ_2 characters of G, we let $\langle \chi_1, \chi_2 \rangle_G$ denote the inner product

$$\langle \chi_1, \chi_2 \rangle_G := \int_G \chi_1(x)\overline{\chi_2(x)} d\mu_G(x),$$

where μ_G is the probability Haar measure on G. Similarly, we define $\langle \cdot, \cdot \rangle_N$ to be the inner product on characters of N.

For any $\vartheta \in \mathrm{Irr}(N)$, we let

$$\mathrm{Irr}(G, \vartheta) := \left\{ \chi \in \mathrm{Irr}(G) \mid \langle \chi_{|N}, \vartheta \rangle_N > 0 \right\}.$$

By Frobeinus reciprocity, $\mathrm{Irr}(G, \vartheta)$ is the set of irreducible characters of G which are constituents of $\mathrm{Ind}_N^G(\vartheta)$.

The group G acts on $\mathrm{Irr}(N)$ by

$$g.\vartheta(x) = \vartheta(g^{-1}xg) \quad \forall g \in G,\ \vartheta \in \mathrm{Irr}(N).$$

We define the **inertia subgroup of** ϑ **in** G to be

$$I_G(\vartheta) := \{ g \in G \mid g.\vartheta = \vartheta \}.$$

Note that $N \subseteq I_G(\vartheta) \subseteq G$, and that the G-orbit of ϑ is in bijection with the coset space $G/I_G(\vartheta)$.

Theorem 6 (Clifford, cf. [Hup98, 19.3])**.** *Let $\vartheta \in \mathrm{Irr}(N)$, and let $\chi \in \mathrm{Irr}(G, \vartheta)$. Then the irreducible constituents of $\chi_{|N}$ are all the G-conjugates of ϑ and occur with the same multiplicity. Specifically, if we put $e_\chi := \langle \chi_{|N}, \vartheta \rangle_N$, and let $\{g_1, \ldots, g_m\}$ be a set of representatives of $G/I_G(\vartheta)$, then*

$$\chi_{|N} = e_\chi \sum_{j=1}^{m} (g_j.\vartheta).$$

In particular,

$$\chi(1) = e_\chi \cdot |G : I_G(\vartheta)|\, \vartheta(1).$$

It follows that $\mathrm{Irr}(G, \vartheta) = \mathrm{Irr}(G, g.\vartheta)$ for all $g \in G$. Moreover, we obtain the following description of the representation zeta function of G via irreducible characters of N, by

$$\zeta_G(s) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s} = \sum_{\vartheta \in \mathrm{Irr}(N)} \vartheta(1)^{-s}\, |G : I_G(\vartheta)|^{-1-s}\, \zeta_{G,\vartheta}(s),$$

where $\zeta_{G,\vartheta}(s) := \sum_{\chi \in \mathrm{Irr}(G,\vartheta)} e_\chi^{-s} = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_{|N}, \vartheta \rangle_N^{-s}$.

It turns out that the function $\zeta_{G,\vartheta}(s)$ can be effectively computed in the case where G/N is a cyclic group.

Firstly, we have

Theorem 7 ([Hup98, 19.6.c])**.** *Suppose that there exists a character $\hat{\vartheta} \in \mathrm{Irr}(I_G(\vartheta))$ such that $\hat{\vartheta}_{|N} = \vartheta$. Then*

$$\mathrm{Ind}_N^G(\vartheta) = \sum_{j=1}^{t} \psi_j(1) \mathrm{Ind}_{I_G(\vartheta)}^G (\hat{\vartheta} \otimes \psi_j),$$

where $\mathrm{Irr}(I_G(\vartheta)/N) = \{\psi_1, \ldots, \psi_t\}$.

Additionally, the characters $\mathrm{Ind}_{I_G(\vartheta)}^G(\hat{\vartheta} \otimes \psi_j)$ are pairwise distinct and irreducible.

The case where G/N is cyclic is especially lucrative for our purposes. As it turns out, such an irreducible extension $\hat{\vartheta} \in \mathrm{Irr}(I_G(\vartheta))$ exists whenever $I_G(\vartheta)/N$ is cyclic [Hup98, Theorem 19.13]. If G/N is cyclic, then in particular, for any $\vartheta \in \mathrm{Irr}(N)$, $I_G(\vartheta)/N$ is cyclic as well. Thus, any irreducible character of N can be extended it its inertia subgroup in this case.

Let us compute $\zeta_{G,\vartheta}(s)$, for an arbitrary $\vartheta \in \mathrm{Irr}(N)$, assuming G/N is cyclic. Fix $\hat{\vartheta} \in \mathrm{Irr}(I_G(\vartheta))$ to be an extension of ϑ, and put $\chi_j := \mathrm{Ind}_{I_G(\vartheta)}^G(\hat{\vartheta} \otimes \psi_j)$ for any ψ_j, as in the theorem. By Theorem 7, and by Frobenius reciprocity, it follows that $\mathrm{Irr}(G, \vartheta) = \{\chi_1, \ldots, \chi_t\}$. Moreover, by Theorem 7, we have that for any j

$$e_{\chi_j} = \langle \chi_j, \mathrm{Ind}_N^G(\vartheta) \rangle_G = \psi_j(1) = 1,$$

as $I_G(\vartheta)/N$ is cyclic, and hence all irreducible characters of it are 1-dimensional. In particular, this implies that

$$\zeta_{G,\vartheta}(s) = \sum_{\chi \in \mathrm{Irr}(G,\vartheta)} e_\chi^{-s} = \sum_{j=1}^{t} 1^{-s} = |I_G(\vartheta) : N| \,.$$

Applying this to $\zeta_G(s)$, we have that

$$\zeta_G(s) = \sum_{\vartheta \in \mathrm{Irr}(N)} \vartheta(1)^{-s} \left|G : I_G(\vartheta)\right|^{-1-s} \cdot |I_G(\vartheta) : N| \,.$$

In view of this, in the upcoming section, we will be invested in proving that in the case of $G = SL_1(\mathcal{O})$ and $N = SL_1^1(\mathcal{O})$, the group G/N is indeed cyclic.

9.2 $SL_1(\mathcal{O})$ as a Semi-Direct Product

The group $D^\times$ contains a subgroup of **Teichmüller representatives** for $\mathfrak{e}^\times$ .That is to say, there exists a subgroup $\Phi \subseteq \mathcal{O}^\times$ such that the reduction map $\mathcal{O} \to \mathfrak{e}$ induces an isomorphism of multiplicative groups $\Phi \cong \mathfrak{e}^\times$. Thus, the exact sequence

$$1 \to 1 + \mathcal{P} \hookrightarrow \mathcal{O}^\times \twoheadrightarrow \mathfrak{e}^\times \to 1$$

has a section at $\mathcal{O} \twoheadrightarrow \mathfrak{e}$.

As mentioned in [Rie70, Theorem 7.iv], one can restrict this to obtain the exact sequence

$$1 \to SL_1^1(\mathcal{O}) \hookrightarrow SL_1(\mathcal{O}) \twoheadrightarrow H \to 1,$$

where H is the subgroup of Φ which is isomorphic to the group $SL_1(\mathfrak{e}/\mathfrak{f}) \subseteq \mathfrak{e}^\times$ of elements of field-norm 1. Thus $SL_1(\mathcal{O}) = SL_1^1(\mathcal{O}) \rtimes H$.

In this section we will overview a proof of this fact.

Consider the polynomial $p(T) = T^{q^\ell - 1} - 1 \in \mathscr{O}_F[T]$. The image $\overline{p}(T) \in \mathfrak{f}[T]$ has $q^\ell - 1$ distinct solutions in $\mathfrak{e}$, namely all elements of $\mathfrak{e}^\times$. Applying Hensel's Lemma, each $\xi_j \in \mathfrak{e}^\times$ lifts to a solution $x_j \in \mathscr{O}_E$ of $p(T)$, such that $x_j \equiv \xi_j (\bmod\ \wp_E)$. Let

$\Phi := \{x_1, \ldots, x_{q^\ell-1}\} \subseteq \mathscr{O}_E$ denote the set of these solutions to $p(T)$. Note that two elements of Φ are equal if and only if their images $\bmod \wp_E$ are equal, and that Φ is the complete set of solutions to $p(T)$.

Moreover, note that Φ is a commutative subgroup of $\mathcal{O}^\times$. Commutativity follows from $\Phi \subseteq \mathscr{O}_E^\times$. Additionally, for any $x_1, x_2 \in \Phi$ we have that

$$(x_1 x_2)^{q^\ell-1} = x_1^{q^\ell-1} x_2^{q^\ell-1} = 1.$$

Consequently $x_1 x_2 \in \mathscr{O}_E$ is a root of $p(T)$, and hence an element of Φ. Likewise, one shows that for any $x \in \Phi$, $x^{-1} \in \Phi$.

Lemma 9.2.1. The image of Φ modulo $1 + \mathcal{P}$ is isomorphic to $\mathfrak{e}^\times$.

Proof. Since $\Phi \subseteq \mathcal{O}^\times$, it is clear that the image of Φ in $\mathcal{O}^\times/(1+\mathcal{P}) = \mathfrak{e}^\times$ is isomorphic to a subgroup of $\mathfrak{e}^\times$. Let us show that the reduction map $\Phi \to \mathfrak{e}^\times$ is injective, and thus (since both groups have the same order), an isomorphism.

But this is immediate, once it is verified that

$$(1+\mathcal{P}) \cap \mathscr{O}_E = 1 + \wp_E.$$

Given this, if $x_1, x_2 \in \Phi$ are such that $x_1 + \mathcal{P} = x_2 + \mathcal{P}$, then $x_1 x_2^{-1} \in 1 + \mathcal{P} \cap \mathscr{O}_E$ and hence $x_1 x_2^{-1} \in 1 + \wp_E$. This implies that $x_1 + \wp_E = x_2 + \wp_E$ which in turn implies $x_1 = x_2$.

□

Thus, the exact sequence

$$1 \to 1 + \mathcal{P} \hookrightarrow \mathcal{O}^\times \twoheadrightarrow \mathfrak{e}^\times \to 1$$

splits, and hence $\mathcal{O}^\times = (1+\mathcal{P}) \rtimes \mathfrak{e}^\times$.

Note in addition that $\sigma(\Phi) = \Phi$. Moreover, we have that for any $x \in \Phi$

$$\overline{\sigma(x)} = \mathrm{fr}(\overline{x}) = \overline{x}^q \in \Phi.$$

Thus, we obtain that

$$\sigma(x) = x^q, \quad \forall x \in \Phi.$$

Since Φ is closed under multiplication, it follows that $\mathrm{Nr}_{E/F}(x) \in \Phi$ as well.

Put $H = \left\{x \in \Phi \mid \mathrm{Nr}_{E/F}(x) = x^{\sum_{j=0}^{\ell-1} q^j} = 1\right\} = \Phi \cap SL_1(D)$. Note that H is a cyclic group of $1 + q + \ldots + q^{\ell-1}$ elements, and hence is isomorphic (via reduction modulo $1 + \mathcal{P}$) to the group $SL_1(\mathfrak{e}/\mathfrak{f})$.

Lemma 9.2.2. Let $x \in 1 + \mathcal{P}$ be arbitrary. Then

$$\mathrm{Nrd}_{D/F}(x) \in 1 + \wp_F.$$

Proof. Let $x_0, \ldots, x_{\ell-1} \in \mathscr{O}_E$ be such that $x = \sum_{j=0}^{\ell-1} u^j x_j$. Since $x - 1 \in \mathcal{P}$ we have that $x_0 \in 1 + \wp_E$ (as all other summands $u^j x_j$'s are already elements of $\mathcal{P}$). By the matrix representation given in Proposition 3.1.8 we have that the reduced norm of x equals the determinant of the matrix

$$\begin{pmatrix} x_0 & \pi\sigma(x_{n-1}) & \pi\sigma^2(x_{n-2}) & & \pi\sigma^{n-1}(x_1) \\ x_1 & \sigma(x_0) & \pi\sigma^2(x_{n-1}) & \ldots & \pi\sigma^{n-1}(x_2) \\ x_2 & \sigma(x_1) & \sigma^2(x_0) & & \pi\sigma^{n-1}(x_3) \\ \vdots & & & \ddots & \vdots \\ x_{n-1} & \sigma(x_{n-2}) & \sigma^2(x_1) & \ldots & \sigma^{n-1}(x_0) \end{pmatrix}.$$

Since this matrix is lower-triangular modulo $\wp_E$, we have that

$$\mathrm{Nrd}_{D/F}(x) \equiv \prod_{i=0}^{\ell-1} \sigma^i(x_0) (\bmod\ \wp_F) \equiv \mathrm{Nr}_{E/F}(x_0) (\bmod\ \wp_E).$$

Moreover, since $\mathrm{Nrd}_{D/F}(x)$ and $\mathrm{Nr}_{E/F}(x_0)$ are both elements of F, and since $\wp_E \cap F = \wp_F$, it holds that

$$\mathrm{Nrd}_{D/F}(x) \equiv \mathrm{Nr}_{E/F}(x_0) (\bmod\ \wp_F).$$

It remains to show that $\mathrm{Nr}_{E/F}(x_0) \in 1 + \wp_F$. To do so, let $\alpha_0 \in \wp_E$ be such that $x_0 = 1 + \alpha_0$, and note that

$$\mathrm{Nr}_{E/F}(x_0) = (1 + \alpha_0)(1 + \sigma(\alpha_0)) \ldots (1 + \sigma^{\ell-1}(\alpha_0)) \in (1 + \wp_E) \cap F = 1 + \wp_F,$$

as $1 + \sigma^j(\alpha_0) \in 1 + \wp_E$ for all j's, and $\mathrm{Nr}_{E/F}(x_0) \in F$.

□

Proposition 9.2.3.

$$G = N \rtimes H,$$

for $G = SL_1(\mathcal{O})$ and $N = SL_1^1(\mathcal{O})$.

Proof. Let $z \in G$ be arbitrary. In particular we have that $z \in \mathcal{O}^\times$, and hence it can be written uniquely as $y \cdot x$ for $y \in 1 + \mathcal{P}$ and $x \in \Phi$.

By definition,

$$1 = \mathrm{Nrd}_{D/F}(z) = \mathrm{Nrd}_{D/F}(y) \cdot \mathrm{Nr}_{E/F}(x).$$

Multiplying by $\mathrm{Nr}_{E/F}(x)^{-1}$, we obtain

$$\mathrm{Nr}_{E/F}(x)^{-1} = \mathrm{Nrd}_{D/F}(y) \in \Phi \cap (1 + \wp_E) = \{1\},$$

by Lemma 9.2.2. It follows that $\mathrm{Nr}_{E/F}(x) \equiv 1 (\mod \wp_E)$, and hence $\mathrm{Nr}_{E/F}(x) = 1$ and $x \in H$. Consequently, $\mathrm{Nrd}_{D/F}(y) = \mathrm{Nr}_{E/F}(x)^{-1} = 1$ as well, and $y \in SL_1(\mathcal{O}) \cap (1 + \mathcal{P}) = N$.

Thus $G = N \cdot H$ and the assertion holds.

$\square$

In particular, we now have that $G/N \cong H$, is a finite cyclic group of order $\sum_{j=0}^{\ell-1} q^j$.

Chapter 10

Inertia Subgroups

In this chapter we will eventually calculate the inertia subgroup

$$I_G(\vartheta) := \{g \in G \mid g.\vartheta = \vartheta\}\,,$$

of any $\vartheta \in \mathrm{Irr}(N)$.

As it turns out, the number of possibilities for an inertia subgroup of an irreducible character of N is extremely limited. In fact, only two options for such a subgroup exist, and the distinction between the two is deeply connected with the partition of the set $W(\mathscr{O}_F)$, which we have described in Section 7.2.

In order to execute this calculation, we initially require an explicit description of the action of G on $\mathrm{Irr}(N)$. To do so, we will mimic the process described in [AKOV12, § 3.2 and§ 4.2] in order to translate between the co-adjoint orbits of G on $\hat{\mathfrak{n}}$ and the adjoint action of G on a suitable $\mathscr{O}_F$-Lie-sub-module of $\mathscr{G}$, via the normalized Killing form (to be described and calculated in the sequel).

Given the assumption that N is potent and saturable, this will yield an explicit G-equivariant dictionary between the irreducible characters of N (via the orbit method), and a fairly accessible subset of $\mathscr{G}$, where the inertia subgroups are easily understood.

10.1 A G-Equivariant Dictionary

10.1.1 Step 1- From $\mathrm{Irr}(N)$ to $\hat{\mathfrak{n}}$

Fix a character $\vartheta \in \mathrm{Irr}(N)$. Since N is assumed potent and saturable, we have by Theorem 3, that there exists a unique orbit $\Omega \in \hat{\mathfrak{n}}/N$, such that

$$\vartheta(x) = \chi_\Omega(x) = (\#\Omega)^{-1/2} \sum_{f \in \Omega} f(\log(x)), \quad \forall x \in N.$$

By the normality of N in G, we have that G for any $x \in \mathfrak{n}$ and $g \in G$,

$$g \exp(x) g^{-1} \in N,$$

and hence G acts on $\mathfrak{n}$ via the adjoint map $\mathrm{Ad}_g(x) = \log(g \exp(x) g^{-1})$. This induces the co-adjoint action of G on $\hat{\mathfrak{n}}$, via $g.f(x) = f(\mathrm{Ad}_{g^{-1}}(x))$.

Note that for any $\nu \in N$, $g \in G$ and $f \in \hat{\mathfrak{n}}$, we have that $\nu.(g.f) = g.(\nu'.f)$ for some $\nu' \in N$. That is to say- for any N-orbit $\Omega \subseteq \hat{\mathfrak{n}}$, the set $g.\Omega$ is another N-orbit in $\hat{\mathfrak{n}}$. Thus, the action of G on $\hat{\mathfrak{n}}$ factors through the quotient space $\hat{\mathfrak{n}}/N$ and induces an action on the space of N-orbits of $\hat{\mathfrak{n}}$.

By the bijectivity of the Kirrilov correspondence, we have that

$$g.\vartheta = \vartheta \quad \Longleftrightarrow \quad g.\Omega = \Omega.$$

Thus $g \in I_G(\vartheta)$ if and only if g stabilizes the corresponding N-orbit.

Let $f \in \hat{\mathfrak{n}}$ be such that $\Omega = N.f$, and suppose $g \in G$ stabilizes Ω. Thus

$$g.f = \nu.f \in N.f,$$

for some $\nu \in N$. Thus, $\nu^{-1}g$ stabilizes f, and hence

$$g \in N \cdot Stab_G(f).$$

Since any element of the product $N \cdot Stab_G(f)$ clearly stabilizes Ω, we have that this group equals $I_G(\vartheta)$. We obtain:

Lemma 10.1.2. Let $\vartheta \in \mathrm{Irr}(N)$, and let $\Omega = N.f \in \hat{\mathfrak{n}}/N$ be such that $\vartheta = \chi_\Omega$. Then

$$I_G(\vartheta) = N \cdot Stab_G(f).$$

Hence, to calculate inertia subgroups it would be sufficient for us to be able to calculate the stabilizer in G of an arbitrary element $f \in \hat{\mathfrak{n}}$.

10.1.3 Step 2- From $\hat{\mathfrak{n}}$ to $\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^*$

In view of the previous section, we fix an element $f \in \hat{\mathfrak{n}}$. By Lemma 2.1.2 we have that

$$\hat{\mathfrak{n}} = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \mathrm{Irr}_n(\mathfrak{n}), \quad \text{where} \quad \mathrm{Irr}_n(\mathfrak{n}) \cong \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F/{\wp_F}^n)^*.$$

Hence there exists a unique pair (n, φ), where $n \geq 0$ is an integer and $\varphi \in \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F/{\wp_F}^n)^*$ is mapped by the bijection of the lemma to f.

We have already mentioned in Section 2.1 that this bijection is N-equivariant (in the sense described above). An inspection of the proof given in [AKOV13] confirms that this bijection is a-fortiori G-equivariant. That is to say, if $\varphi \in \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F/{\wp_F}^n)^*$ is mapped onto f and $g \in G$ is arbitrary, then $g.\varphi$ is mapped onto $g.f$, where G acts on $\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F/{\wp_F}^n)^*$ by the co-adjoint action (via the adjoint action on $\mathfrak{n}$). It follows that the stabilizer of f equals the stabilizer of φ.

Pulling back along the projection map $\mathscr{O}_F \twoheadrightarrow \mathscr{O}_F/{\wp_F}^n$, we have a surjection

$$\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^* \twoheadrightarrow \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F/{\wp_F}^n)^*.$$

Note that if $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^*$ have the same image under this surjection, then

$$(\phi_1 - \phi_2)(\mathfrak{n}) \subseteq {\wp_F}^n = \pi^n \mathscr{O}_F \quad \Longleftrightarrow \quad (\phi_1 - \phi_2)(\pi^{-n}\mathfrak{n}) \subseteq \mathscr{O}_F.$$

That is to say, any element $f \in \hat{\mathfrak{n}}$ corresponds to a pair (n, φ), where $n \in \mathbb{N} \cup \{0\}$, and $\phi \in \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^*$ is unique up to addition by an element in the $\mathscr{O}_F$-module $\mathrm{Hom}_{\mathscr{O}_F}(\pi^{-n}\mathfrak{n}, \mathscr{O}_F)$. Thus, we have:

Lemma 10.1.4. Let $\vartheta = \chi_\Omega$, for $\Omega = N.f$ and $f \in \hat{\mathfrak{n}}$ corresponds to a pair (n, φ), as described above, with $\varphi \in \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^*$. Then

$$I_G(\vartheta) = N \cdot Stab_G(f) = N \cdot \{g \in G \mid \varphi - g.\varphi \in \mathrm{Hom}_{\mathscr{O}_F}(\pi^{-n}\mathfrak{n}, \mathscr{O}_F)\}.$$

Before making the final step in this construction, let us place a notational convention.

Let $\mathcal{B}^\vee = \left\{ \mathbf{e}^\vee_{\underline{\alpha}_t} \mid t = 1, \ldots, d \right\}$ denote the dual basis to the monomial basis $\mathcal{B}$ (cf. Proposition 5.2.4). Recall, from Section 2.2, that the set $W(\mathscr{O}_F)$ gives a complete set of coordinates for $\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^*$, with respect to $\mathcal{B}^\vee$. Recall also, the partition of $W(\mathscr{O}_F)$ suggested in Section 7.2. We denote

$$W^{[\mathbb{F}_\ell^\times]}(\mathscr{O}_F) := \bigsqcup_{\gamma \in \mathbb{F}_\ell^\times} W^{[\gamma]}(\mathscr{O}_F) = \{\mathbf{y} = (y_1, \ldots, y_d) \in W(\mathscr{O}_F) \mid y_1, \ldots, y_{\ell-1} \in \wp_F\}$$

In the following sections we will see how the decomposition

$$W(\mathscr{O}_F) = W^{[0]}(\mathscr{O}_F) \sqcup W^{[\mathbb{F}_\ell^\times]}(\mathscr{O}_F)$$

governs the type of inertia subgroup associated to any functional of $\mathfrak{n}$.

10.1.5 Step 3- From $\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^*$ to sub-$\mathscr{O}_F$-modules of $\mathscr{G}$

Consider the F-Lie algebra $\mathscr{G} = \mathfrak{sl}_1(D)$. As any finite-dimensional vector space is, the space $\mathscr{G}$ is non-canonically isomorphic to its dual space $\mathrm{Hom}_F(\mathscr{G}, F)$. A standard way of obtaining such an isomorphism in the case of semi-simple Lie-algebras is via the Killing form $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$, which is a non-degenerate bilinear form on $\mathscr{G}$.

Let $\mathcal{M} := \{\mathbf{m}_{\underline{\alpha}_t} \mid t = 1, \ldots, d\}$ be defined for any $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathcal{I}^*$ by

$$\mathbf{m}_{\underline{\alpha}} = u^{r(\alpha_1)}\theta^{r(\alpha_2)}.$$

Note that $\mathcal{B} = \{\pi\mathbf{m}_{\underline{\alpha}_1}, \ldots, \pi\mathbf{m}_{\underline{\alpha}_{\ell-1}}, \mathbf{m}_{\underline{\alpha}_\ell}, \ldots, \mathbf{m}_{\underline{\alpha}_d}\}$, where $\mathcal{B}$ is the monomial basis of Section 5.2.1. It follows that $\mathcal{M}$ is a linearly independent set over F, with $\ell^2 - 1 = \dim_F \mathscr{G}$ elements, and hence is an F-basis of $\mathscr{G}$. The Killing form on $\mathscr{G} \times \mathscr{G}$ is computed explicitly in appendix B. As explained in [AKOV13, § 5], the normalization $\kappa_0 = (2h^\vee)^{-1}\kappa$ is defined over $\mathscr{O}_F$, where $h^\vee = \ell$ is the dual Coxeter number of $\mathfrak{sl}_1(D)$. The normalized Killing form is represented with respect to $\mathcal{M}$ by the matrix

$$[\kappa_0(\cdot,\cdot)]_{\mathcal{M}} = \left(\begin{array}{c:c:c:c:c} \mathsf{K}_1 & & & & \\ \hdashline & & & & \pi\cdot\mathsf{K}_\ell \\ \hdashline & & & \iddots & \\ \hdashline & & \pi\cdot\mathsf{K}_3 & & \\ \hdashline & \pi\cdot\mathsf{K}_2 & & & \end{array}\right) \in \mathrm{Mat}_{\ell^2-1}(\mathscr{O})$$

where $\mathsf{K}_1 \in \mathrm{Mat}_{\ell-1}(\mathscr{O}_F)$ and $\mathsf{K}_j \in \mathrm{Mat}_\ell(\mathscr{O}_F)$, are given by

$$\mathsf{K}_1 = \begin{pmatrix} & & \ell b \\ & \iddots & \\ \ell b & & \\ \ell b & & \end{pmatrix}, \quad \text{and} \quad \mathsf{K}_j = \begin{pmatrix} \ell & & & & \\ & & & & \ell b\omega^{-(\ell-1)j} \\ & & & \iddots & \\ & & \ell b\omega^{-2j} & & \\ & \ell b\omega^{-j} & & & \end{pmatrix},$$

for $j = 2, \dots, \ell - 1$.

The form κ_0 induces a G-equivariant isomorphism $\iota : \mathscr{G} \to \mathrm{Hom}_F(\mathscr{G}, F)$, by

$$\iota(x)(y) = \kappa_0(x, y), \quad \forall x, y \in \mathscr{G}.$$

In this section, we wish to employ this isomorphism in order to describe the elements $\varphi \in \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F/\wp_F{}^n)^*$ via certain sub-$\mathscr{O}_F$-modules of $\mathscr{G}$, and to describe their stabilizers under the adjoint action of G.

Recall that $\mathfrak{g} := \mathfrak{sl}_1(\mathcal{O})$. By repeating the proof of Proposition 5.2.4 practically verbatim, we have that $\mathfrak{g} = Span_{\mathscr{O}} \mathcal{M}$. We claim the following:

Lemma 10.1.6.

$$\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F) = \iota(\pi^{-1}\mathfrak{g}).$$

Proof. In practice, we need to prove two assertions. Firstly, we show that for any $x \in \pi^{-1}\mathfrak{g}$,

$$\iota(x)(y) \in \mathscr{O}_F, \quad \forall y \in \mathfrak{n}.$$

Secondly, we need to show that any element $\phi \in \mathrm{Hom}_F(\mathscr{G}, F)$ whose restriction to $\mathfrak{n}$ induces an $\mathscr{O}_F$-module homomorphism to $\mathscr{O}_F$, is an element of $\iota(\pi^{-1}\mathfrak{g})$.

Let $x \in \pi^{-1}\mathfrak{g}$ and $y \in \mathfrak{n}$ be arbitrary. The coordinate vectors of x and y with respect to the basis $\mathcal{M}$ are given by

$$[x]_{\mathcal{M}} = \begin{pmatrix} \pi^{-1}x_1 \\ \pi^{-1}x_2 \\ \vdots \\ \pi^{-1}x_d \end{pmatrix}, \quad \text{and} \quad [y]_{\mathcal{M}} = \begin{pmatrix} \pi y_1 \\ \vdots \\ \pi y_{\ell-1} \\ y_\ell \\ \vdots \\ y_d \end{pmatrix},$$

where $x_1, \dots, x_d, y_1, \dots, y_d \in \mathscr{O}_F$. It follows easily, from the representation of $\kappa_0(\cdot,\cdot)$ with respect to $\mathcal{M}$, that

$$\kappa_0(x, y) = [x]_{\mathcal{M}}^T \cdot [\kappa_0(\cdot,\cdot)]_{\mathcal{M}} \cdot [y]_{\mathcal{M}} \in \mathscr{O}_F.$$

As for the second statement, let $\phi \in \mathrm{Hom}_F(\mathscr{G}, F)$ be arbitrary, and let $x \in \mathscr{G}$ be such that $\iota(x) = \phi$. Assume that $x \notin \pi^{-1}\mathfrak{g}$ and let $n \in \mathbb{N}$ be minimal such that

$\pi^n x \in \pi^{-1}\mathfrak{g}$. Then

$$[\pi^n x]_{\mathcal{M}} = \begin{pmatrix} \pi^{-1}x'_1 \\ \pi^{-1}x'_2 \\ \vdots \\ \pi^{-1}x'_d \end{pmatrix},$$

where by minimality of n, $x'_i \in \mathscr{O}_F^\times$ for some $i = 1, \ldots, d$. From the structure of $[\kappa_0(\cdot,\cdot)]_{\mathcal{M}}$, one promptly verifies that there exists an element $y \in \mathfrak{n}$ such that

$$\kappa_0(\pi^n x, y) \in \mathscr{O}_F^\times.$$

By left-linearity of κ_0, it follows that $\kappa_0(x,y) = \phi(y) \in \pi^{-n}\mathscr{O}_F \smallsetminus \pi^{-n+1}\mathscr{O}_F$. Hence $\phi(\mathfrak{n}) \not\subseteq \mathscr{O}_F$.

□

From this we obtain the following corollary:

Corollary 10.1.7. For any $m \in \mathbb{Z}$,

$$\iota(\pi^m \mathfrak{g}) = \mathrm{Hom}_{\mathscr{O}_F}(\pi^{-m-1}\mathfrak{n}, \mathscr{O}_F).$$

Proof. Let $x \in \pi^m \mathfrak{g}$, and let $x' = \pi^{-m-1}x \in \pi^{-1}\mathfrak{g}$. Then by Lemma 10.1.6, $\iota(x') \in \phi(\mathfrak{n}, \mathscr{O}_F)$. By F-linearity of ι, it holds that

$$\iota(x) = \iota(\pi^{m+1}x') = \pi^{m+1}\iota(x') \in \pi^{m+1}\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F) = \mathrm{Hom}_{\mathscr{O}_F}(\pi^{-m-1}\mathfrak{n}, \mathscr{O}_F).$$

□

In particular, it follows that there exists a G-equivariant bijection between $\mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F)^* = \mathrm{Hom}_{\mathscr{O}_F}(\mathfrak{n}, \mathscr{O}_F) \smallsetminus \mathrm{Hom}_{\mathscr{O}_F}(\pi^{-1}\mathfrak{n}, \mathscr{O}_F)$ and the set $(\pi^{-1}\mathfrak{g}) \smallsetminus \mathfrak{g}$. Since that the adjoint of G on $\mathscr{G}$ action coincides with the conjugation $g.x = gxg^{-1}$, we have come to the following conclusion:

Proposition 10.1.8. For any $\vartheta \in \mathrm{Irr}(N)$, there exists a pair (n, x) where $n \in \mathbb{N}$ and $x \in (\pi^{-1}\mathfrak{g}) \smallsetminus \mathfrak{g}$ such that

$$I_G(\vartheta) = N \cdot Stab_G(f) = N \cdot \{g \in G \mid x - g^{-1}xg \in \pi^{n-1}\mathfrak{g}\},$$

where $f \in \hat{\mathfrak{n}}$ is such that $\vartheta = \chi_\Omega$, for $\Omega = N.f$. Moreover, for any $g \in G$, the pair $(n, g^{-1}xg)$ represents the character $g.\vartheta \in \mathrm{Irr}(N)$.

Thus, we have that

$$I_G(\vartheta) = N \cdot C_G(x + \pi^{n-1}\mathfrak{g}) = N \cdot C_H(x + \pi^{n-1}\mathfrak{g}),$$

where the latter equality is a direct consequence of the fact that $G = N \cdot H$.

Let us denote by $\mathfrak{g}^{[0]}$ and $\mathfrak{g}^{[\mathbb{F}_\ell^\times]}$ the subsets of $\pi^{-1}\mathfrak{g} \setminus \mathfrak{g}$ which are mapped by ι onto $W^{[0]}(\mathscr{O}_F)$ and $W^{[\mathbb{F}_\ell^\times]}(\mathscr{O}_F)$ respectively. Then

$$\mathfrak{g}^{[0]} = \left\{ x \in \pi^{-1}\mathfrak{g} \mid [x]_{\mathcal{M}} = \begin{pmatrix} \pi^{-1}x_1 \\ \pi^{-1}x_2 \\ \vdots \\ \pi^{-1}x_d \end{pmatrix}, \quad x_i \in \mathscr{O}_F^\times \text{ for at least one } i \in \{1, \ldots, \ell - 1\} \right\},$$

$$\text{and } \mathfrak{g}^{[\mathbb{F}_\ell^\times]} = \left\{ x \in \pi^{-1}\mathfrak{g} \setminus \mathfrak{g} \mid [x]_{\mathcal{M}} = \begin{pmatrix} \pi^{-1}x_1 \\ \pi^{-1}x_2 \\ \vdots \\ \pi^{-1}x_d \end{pmatrix}, \quad x_i \in \wp_F, \quad \forall i \in \{1, \ldots, \ell - 1\} \right\}.$$

10.2 Calculation of Inertia Subgroups

Let $\vartheta \in \mathrm{Irr}(N)$ be non-trivial and fixed, and let $f \in \hat{\mathfrak{n}}$ be such that $\vartheta = \chi_\Omega$ for $\Omega = N.f$. In accordance with Proposition 10.1.8, there exists a pair (n, x), where $x \in \pi^{-1}\mathfrak{g} \setminus \mathfrak{g}$ and $n \in \mathbb{N}$, such that

$$I_G(\vartheta) = N \cdot C_H(x + \pi^{n-1}\mathfrak{g}).$$

In this section we will prove the following:

Proposition 10.2.1. Let $\vartheta \in \mathrm{Irr}(N)$ be non-trivial, and let (n, x) be as above. Then

1. If $x \in \mathfrak{g}^{[0]}$ then $I_G(\vartheta) = N \cdot H = G$, and hence $|I_G(\vartheta) : N| = \sum_{j=0}^{\ell-1} q^j$.

2. If $x \in \mathfrak{g}^{[\mathbb{F}_\ell^\times]}$ then $I_G(\vartheta) = N \cdot \{1, \omega, \ldots, \omega^{\ell-1}\}$, and hence $|I_G(\vartheta) : N| = \ell$.

Let us prove the simpler case first.

10.2.2 The Case Where $x \in \mathfrak{g}^{[\mathbb{F}_\ell^\times]}$

The second part of Proposition 10.2.1 requires the use of several arithmetic facts, are proved in appendix A.

Note that the number $\sum_{j=0}^{\ell-1} q^j$ is divisible by ℓ (since $q \equiv 1(\mod \ell)$) and thus the ℓ-th root of unity $\omega \in F$ is a solution to the equation $T^{\sum_{j=0}^{\ell-1} q^j} - 1 = 0$, and hence $\omega \in H$. Since $\omega \in F = Z(D)$, it is obvious that

$$\mathbf{C}_\ell \cong \left\{\omega^j \mid j = 0, \dots, \ell-1\right\} \subseteq C_H(x + \pi^{n-1}\mathfrak{g}),$$

for any pair (x, n).

In addition, note that if $\mathfrak{m} \subseteq \mathscr{G}$ is any $\mathscr{O}_F$-module, such that $\mathfrak{m} \supseteq \pi^{n-1}\mathfrak{g}$, then

$$C_H(x + \pi^{n-1}\mathfrak{g}) \subseteq C_H(x + \mathfrak{m}).$$

Let $x \in \mathfrak{g}^{[\mathbb{F}_\ell^\times]}$ be arbitrary, and let $x_1, \dots, x_d \in \mathscr{O}_F$ be such that $x = \sum_{k=1}^d \pi^{-1} x_k \mathbf{m}_{\alpha_k}$. Let $1 \leq k \leq d$ be minimal such that $x_k \in \mathscr{O}_F^\times$. Note that by the assumption that $x \notin \mathfrak{g}^{[0]}$ we have that $k \geq \ell$.

Let $0 \leq r_1,\ r_2 \leq \ell - 1$ be such that $k = r_1\ell + r_2$, and put $\mathfrak{m} = \pi^{-1}u^{r_1+1}\mathfrak{g}$. Since ϑ is non-trivial, we have that the level of ϑ is positive and hence $n \geq 1$. Thus $\mathfrak{m} \supseteq \pi^{n-1}\mathfrak{g}$. We claim that the order of $C_H(x + \mathfrak{m})$ is ℓ, and hence the assertion.

By the minimality of r_1 we have that

$$x + \mathfrak{m} = \sum_{j=0}^{\ell-1} u^{r_1}\pi^{-1}x_{r_1\ell+j}\theta^j + \mathfrak{m} \neq 0 + \mathfrak{m}.$$

Let $\mu := \sum_{j=0}^{\ell-1} \pi^{-1}x_{r_1\ell+j}\theta^j \in E$. Then

$$x + \mathfrak{m} = u^{r_1}\mu + \mathfrak{m}.$$

Let $h \in H$ be arbitrary. Since $H \subseteq \mathscr{O}_E$ and $\mu \in E$, we have that $h\mu h^{-1} = \mu$. In addition, by Section 9.2 we know that $\sigma(h) = h^q$. Applying these two pieces of knowledge to the above equality, we have that

$$\begin{aligned} h\,(x + \mathfrak{m})\,h^{-1} &= hu^{r_1}\mu h^{-1} + \mathfrak{m} \\ &= h\sigma^{r_1}(h^{-1})u^{r_1}\mu + \mathfrak{m} \\ &= h^{1-q^{-r_1}}u^{r_1}\mu + \mathfrak{m} = h^{1-q^{r_1}}u^{r_1}\mu + \mathfrak{m}. \end{aligned}$$

The equality

$$u^{r_1}\mu + \mathfrak{m} = h^{-1}(u^{r_1}\mu + \mathfrak{m})h = h^{1-q^{r_1}}u^{r_1}\mu + \mathfrak{m}$$

holds if and only if $h^{1-q^{-r_1}} \in 1 + \mathcal{P}$. Since $H \cap (1 + \mathcal{P}) = \{1\}$ this implies that h is in the kernel of the map $h \mapsto h^{1-q^{\ell-r_1}}$, which is defined from H to H. By Lemma A.1 the order of this kernel is

$$\gcd(\#H, q^{r_1} - 1) = \gcd(\sum_{j=0}^{\ell-1} q^j, q^{r_1} - 1).$$

By Lemma A.4, and the fact that $r_1 \geq 1$, it follows that this order equals ℓ. Thus

$$\#C_H(x + \mathfrak{m}) = \ell,$$

and hence $C_H(x + \pi^{n-1}\mathfrak{g}) \cong \mathbf{C}_\ell$, as wanted.

10.2.3 The Case Where $x \in \mathfrak{g}^{[0]}$

The case where $x \in \mathfrak{g}^{[0]}$ requires some more preparations than the previous case. The main step towards proving the assertion in this case, will be to show that elements of $\mathfrak{g}^{[0]}$ are 'almost-diagonalizable', in the sense that any such element can be conjugated, by an element of N, to be arbitrarily close to an element in E. Thus, since H is a subgroup of E, we will have that H centralizes an N-conjugate of $x + \pi^{n-1}\mathfrak{g}$, and thus $H \cdot N \subseteq I_G(\vartheta)$, as wanted.

Lemma 10.2.4. Let $\lambda \in \mathcal{O}_E^\times$ have $\mathrm{Tr}_{E/F}(\lambda) = 0$, and let $\mu \in {\wp_E}^k \smallsetminus {\wp_E}^{k+1}$, for some $k \geq -1$. Let $j \in \{1, \dots, \ell - 1\}$ be arbitrary, and put

$$x = \pi^{-1}\lambda + u^j\mu.$$

Then, there exists an element $z \in 1 + \mathcal{P}$ such that

$$z^{-1}xz - \pi^{-1}\lambda \in u^{2j}{\wp_E}^k.$$

Proof. Firstly, note that the assumption that $\mathrm{Tr}_{E/F}(\lambda) = 0$ implies that $\lambda - \sigma^j(\lambda)$ is invertible in $\mathcal{O}_E^\times$. Indeed, if we assume towards a contradiction that $\lambda - \sigma^j(\lambda) \equiv 0(\bmod\ \wp_E)$, it follows that

$$\lambda \equiv \sigma^i(\lambda)(\bmod\ \wp_E), \quad \forall i = 0, \dots, \ell - 1.$$

But this implies that

$$0 = \mathrm{Tr}_{E/F}(\lambda) = \sum_{i=0}^{\ell-1} \sigma^i(\lambda) \equiv \ell \cdot \lambda(\bmod\ \wp_E),$$

and since $\ell \notin \wp_E$, it follows that $\lambda \in \wp_E$. A contradiction.

Let

$$z := 1 + \pi u^j (\lambda - \sigma^j(\lambda))^{-1}\mu \in 1 + \mathcal{P}.$$

Then

$$\begin{aligned}
xz &= \left(\pi^{-1}\lambda + u^j\mu\right)\left(1 + \pi u^j(\lambda - \sigma^j(\lambda))^{-1}\mu\right) \\
&= \pi^{-1}\lambda + u^j\sigma^j(\lambda)\left(\lambda - \sigma^j(\lambda)\right)^{-1}\mu + u^j\mu + \pi u^{2j}\left(\lambda - \sigma^j(\lambda)\right)^{-1}\mu\sigma^j(\mu) \\
&= \pi^{-1}\lambda + u^j\Big(\sigma^j(\lambda)\left(\lambda - \sigma^j(\lambda)\right)^{-1} + 1\Big)\mu + \pi u^{2j}\left(\lambda - \sigma^j(\lambda)\right)^{-1}\mu\sigma^j(\mu) \\
&= \pi^{-1}\lambda + u^j\left(\lambda\left(\lambda - \sigma^j(\lambda)\right)^{-1}\right)\mu + \pi u^{2j}\left(\lambda - \sigma^j(\lambda)\right)^{-1}\mu\sigma^j(\mu) \\
&= \left(1 + \pi u^j(\lambda - \sigma^j(\lambda))^{-1}\mu\right)\pi^{-1}\lambda + \pi u^{2j}(\lambda - \sigma^j(\lambda))^{-1}\mu\sigma^j(\mu) \\
&= z\pi^{-1}\lambda + \pi u^{2j}(\lambda - \sigma^j(\lambda))^{-1}\mu\sigma^j(\mu).
\end{aligned}$$

Multiplying from the left by z^{-1}, we have that

$$\begin{aligned}
\left|z^{-1}xz - \pi^{-1}\lambda\right|_{\wp} &= \left|z^{-1}\pi u^{2j}(\lambda - \sigma(\lambda))^{-1}\mu\sigma^j(\mu)\right|_{\wp} \\
&= |\pi|_{\wp}\left|u^{2j}\right|_{\wp}|\mu|_{\wp}|\sigma(\mu)|_{\wp} \\
&= q^{2j/\ell}q^{-2k-1} \leq q^{2j/\ell}q^{-k},
\end{aligned}$$

since $k \geq -1$. Thus

$$z^{-1}xz - \pi^{-1}\lambda \in u^{2j}{\wp_E}^k.$$

$\square$

In particular, in the above setting

$$\left|z^{-1}xz - \pi^{-1}\lambda\right|_{\wp} < \left|x - \pi^{-1}\lambda\right|_{\wp}.$$

Lemma 10.2.5. Let $z \in 1 + \mathcal{P}$ be arbitrary. There exists an element $\xi \in 1 + \wp_F$ such that

$$\mathrm{Nrd}_{D/F}(z) = \xi^\ell.$$

Proof. By Lemma 9.2.2, we have that the equation $T^\ell - \mathrm{Nrd}_{D/F}(z) = 0$ reduces modulo $\wp_F$ to the equation $T^\ell - \bar{1} = 0$. Since $\gcd(\ell, p) = 1$, the formal derivative of $T^\ell - 1$ is non-zero modulo $\wp_F$. Thus one can apply Hensel's Lemma in order to lift the solution $T = \bar{1}$ to an element $\xi \in 1 + \wp_F$, such that $\xi^\ell - \mathrm{Nrd}_{D/F}(z) = 0$. □

Let $x \in \mathfrak{g}^{[0]}$ be fixed, and let $x_1, \dots, x_d \in \mathscr{O}_F$ be such that $x = \sum_{t=1}^{d} \pi^{-1} x_t \mathbf{m}_{\underline{\alpha}_t}$. Let $\lambda = \sum_{j=1}^{\ell-1} x_j \theta^j$, and put, for any $r = 1, \dots, \ell - 1$,

$$\mu_r = \sum_{j=0}^{\ell-1} \pi^{-1} x_{r\ell+j} \theta^j \in \pi^{-1} \wp_E.$$

Suppose that there exists some $r \in \{1, \dots, \ell - 1\}$ such that $\mu_r \neq 0$.

Then $\lambda, \mu_1, \dots, \mu_{\ell-1}$ are such that

$$x = \pi^{-1} \lambda + \sum_{r=1}^{\ell-1} u^j \mu_r.$$

Moreover, $\mathrm{Tr}_{E/F}(\lambda) = 0$, and $\lambda \in \mathscr{O}_E^\times$ by assumption (since $\bar{\lambda} \neq \bar{0} \in \mathfrak{e}$). Let $1 \leq j \leq \ell - 1$ be such that $|u^j \mu_j|_\wp = \max\left\{ |u^i \mu_i|_\wp \mid i = 1, \dots, \ell - 1 \right\}$. Note that if $i, j \in \{1, \dots, \ell - 1\}$, then the equality $|u^i|_\wp = q^g |u^j|_\wp$ for $g \in \mathbb{Z}$, holds if and only if $i = j$. Since $|\mu_i|_\wp$ is either an integral power of q or 0, it follows that the maximal value of the set $\left\{ |u^i \mu_i|_\wp \mid i = 1, \dots, \ell - 1 \right\}$ is attained at a unique j (since at least one of the μ_i's is non-zero). Additionally,

$$\left| x - \pi^{-1} \lambda \right|_\wp = \left| u^j \mu_j \right|_\wp.$$

Put $x' = \pi^{-1} \lambda + u^j \mu_j$ and $x'' = x - x'$. Then

$$|x''|_\wp = \left| \sum_{i \neq j} u^i \mu_i \right|_\wp < \left| u^j \mu_j \right|_\wp.$$

Additionally, by Lemma 10.2.4, there exists a $z \in 1 + \mathcal{P}$, such that

$$\left| z^{-1} x' z - \pi^{-1} \lambda \right|_\wp < \left| x' - \pi^{-1} \lambda \right|_\wp = \left| u^j \mu^j \right|_\wp.$$

Applying Lemma 10.2.5, we can find an element $\xi \in 1 + \wp_F$ such that $\xi^\ell = \mathrm{Nrd}_{D/F}(z)$. Taking $z' = \xi^{-1} z$, we have that

$$\mathrm{Nrd}_{D/F}(z') = \mathrm{Nrd}_{D/F}(\xi^{-1} z) = \xi^{-\ell} \mathrm{Nrd}_{D/F}(z) = 1,$$

and that $z'^{-1}yz' = z^{-1}yz$ for any $y \in D$. Thus, we may assume that $z = z' \in SL_1(D) \cap (1 + \mathcal{P}) = N$.

We obtain the following:

Proposition 10.2.6. Let $x \in \mathfrak{g}^{[0]}$ be arbitrary, and let $\lambda \in \mathscr{O}_E^\times$ and $\mu_1, \ldots, \mu_{\ell-1} \in \pi^{-1}\wp_E$ be as above. Assume that $\mu_r \neq 0$ for some $r \in \{1, \ldots, \ell - 1\}$. There exists an element $z \in N$ such that

$$\left|z^{-1}xz - \pi^{-1}\lambda\right|_\wp < \left|x - \pi^{-1}\lambda\right|_\wp .$$

Proof. Let $j \in \{1, \ldots, \ell - 1\}$ be such that $\left|u^j\mu_j\right|_\wp$ is maximal, and put $x' = \pi^{-1}\lambda + u^j\mu_j$ and $x'' = x - x'$.

Then,

$$\begin{aligned}
\left|z^{-1}xz - \pi^{-1}\lambda\right|_\wp &= \left|\left(z^{-1}x'z - \pi^{-1}\lambda\right) + z^{-1}x''z\right|_\wp \\
&\leq \max\left\{\left|z^{-1}x'z - \pi^{-1}\lambda\right|_\wp, \left|z^{-1}x''z\right|_\wp\right\} \\
&\leq \max\left\{\left|z^{-1}x'z - \pi^{-1}\lambda\right|_\wp, \left|x''\right|_\wp\right\} \\
&< \left|u^j\mu_j\right|_\wp = \left|x - \pi^{-1}\lambda\right|_\wp .
\end{aligned}$$

□

Corollary 10.2.7. There exists a convergent sequence $\{x_t\}_{t=1}^\infty \subseteq \mathfrak{g}^{[0]}$ such that each x_t is an N-conjugate of x and

$$\lim_{t\to\infty} x_t = \pi^{-1}\lambda \in E.$$

Proof. If $x = \pi^{-1}\lambda$ for some $\lambda \in \mathscr{O}_E^\times \cap \mathscr{G}$, take $\{x_t\}$ to be the constant 1 sequence.

Otherwise, we claim that for any $t \in \mathbb{N}$, there exists an element $x_t \in \mathfrak{g}^{[0]}$ and $z \in N$ such that $x_t = z^{-1}xz$, and

$$\left|x_t - \pi^{-1}\lambda\right|_\wp < \left|x_{t-1} - \pi^{-1}\lambda\right|_\wp < \ldots < \left|x - \pi^{-1}\lambda\right|_\wp .$$

We prove this fact by induction on t.

For $t = 1$, let $x_1 = z_1^{-1}xz_1$, where $z_1 \in N$ is such that

$$\left|z_1^{-1}xz_1 - \pi^{-1}\lambda\right|_\wp < \left|x - \pi^{-1}\lambda\right|_\wp ,$$

as given by Proposition 10.2.6.

Assume that elements $x_1, \dots, x_{t-1} \in \mathfrak{g}^{[0]}$ have been found such that for any $i = 1, \dots, t-1$ there exists a $z_i \in N$ such that $x_i = z_i^{-1} x z_i$ and

$$\left|x_{t-1} - \pi^{-1}\lambda\right|_{\wp} < \dots < \left|x_1 - \pi^{-1}\lambda\right|_{\wp} < \left|x - \pi^{-1}\lambda\right|_{\wp}.$$

Apply Proposition 10.2.6, to find an element $z' \in N$ such that $x_t = {z'}^{-1} x_{t-1} z'$ and

$$\left|x_t - \pi^{-1}\lambda\right|_{\wp} = \left|{z'}^{-1} x_{t-1} z' - \pi^{-1}\lambda\right|_{\wp} < \left|x_{t-1} - \pi^{-1}\lambda\right|_{\wp}.$$

Then, for $z_t = z' \cdot z_{t-1}$ we have that

$$x_t = z_t^{-1} x z_t,$$

and x_t satisfies the wanted inequality. The fact that $x_t \in \mathfrak{g}^{[0]}$ is immediate.

To show that $\lim_{t\to\infty} x_t = \pi^{-1}\lambda$, note that the sequence of real numbers

$$\alpha_t = \left|x_t - \pi^{-1}\lambda\right|_{\wp}$$

is a strictly decreasing sub-sequence of the sequence $\left\{q^{-(s-\ell-1)/\ell}\right\}_{s=1}^{\infty}$ (which is simply the set of absolute values in $\pi^{-1}\mathfrak{g}$), and thus approaches 0 as t tends to infinity. □

Proof of Proposition 10.2.1 item (1). Let $\vartheta \in \mathrm{Irr}(N)$ be such that $\vartheta = \chi_\Omega$, with $\Omega = N.f$ for $f \in \hat{\mathfrak{n}}$. Suppose f corresponds to the pair (n, x), as describe above, with $x = \pi^{-1}\lambda + \sum_{i=1}^{\ell-1} u^i \mu_i \in \mathfrak{g}^{[0]}$. By corollary 10.2.7 there exists an element $\nu \in N$ such that

$$\left|\nu^{-1} x \nu - \pi^{-1}\lambda\right|_{\wp} < q^{-n+2},$$

and hence

$$\nu^{-1} x \nu \equiv \pi^{-1}\lambda (\bmod \pi^{n-1}\mathfrak{g}).$$

Since $H \subseteq \mathscr{O}_E^\times$, we have that for any $h \in H$,

$$h^{-1}(\nu^{-1} x \nu) h + \pi^{n-1}\mathfrak{g} = \nu^{-1} x \nu + \pi^{n-1}\mathfrak{g},$$

and hence $H \subseteq C_G(\nu^{-1} x \nu + \pi^{n-1}\mathfrak{g})$. Since the pair $(n, \nu^{-1} x \nu)$ represents the N-character $\nu.\vartheta = \vartheta$ (see Proposition 10.1.8), we have that

$$I_G(\vartheta) = N \cdot C_G(x + \pi^{n-1}\mathfrak{g}) = N \cdot C_G(\nu^{-1} x \nu + \pi^{n-1}\mathfrak{g}) \supseteq NH = G.$$

□

Chapter 11

Calculation of $\zeta_{SL_1(D)}(s)$, Where $\deg(D) = 3$

Theorem B. *Let $F \supseteq \mathbb{Q}_p$ be a local field with $p > 3$ and D a division algebra over F, with $\deg(D) = 3$. Assume that F contains a non-trivial cube root of unity, and that $SL_1^1(\mathcal{O})$ is potent and saturable. Then*

$$\zeta_{SL_1(D)}(s) = \frac{(1+q+q^2)(1-q^{-3s}) + 9 \cdot \left(\frac{q^2+q+1}{3}\right)^{-s}(q^{-s+1}+1)(q-1)}{1-q^{-3s+2}}.$$

Proof. In Theorem A, we have shown that

$$\zeta_{SL_1^1(D)}(s) = q^3 \cdot \frac{1+q^{-(s-1)}(1-q^{-3}) - q^{-3s-3}}{1-q^{-3s+2}}.$$

Retracing our steps in the calculation, we rewrite this equality as

$$\zeta_{SL_1^1(D)}(s) = 1 + \frac{q^{-3s+2}}{1-q^{-3s+2}}\left(\mu(W^{[0]}(\mathscr{O}_F)) + q^{2s+4}\mu(W^{[1]}(\mathscr{O}_F)) + q^{3s+6}\mu(W^{[2]}(\mathscr{O}_F))\right)$$

By Section 9.1 we know that

$$\zeta_{SL_1(D)}(s) = \sum_{\vartheta \in \mathrm{Irr}(N)} \vartheta(1)^{-s} \left|G : I_G(\vartheta)\right|^{-1-s} \zeta_{I_G(\vartheta)/N}(s)$$

Applying the inertia subgroup calculation we obtained, we have that:

- $I_G(\vartheta) = G$ whenever ϑ corresponds to the co-adjoint orbit of an element in $W^{[0]}(\mathscr{O}_F)$, or ϑ is the trivial character. Thus, in this case (by Proposition 9.2.3), $|G : I_G(\vartheta)| = 1$ and $\zeta_{I_G(\vartheta)/N}(s) = |I_G(\vartheta) : N| = 1+q+q^2$, for any s.

- $I_G(\vartheta) = \{1, \omega, \omega^2\} \cdot N$ whenever ϑ corresponds to the co-adjoint orbit of an element in $W^{[1]}(\mathscr{O}_F) \sqcup W^{[2]}(\mathscr{O}_F)$. In this case, $|G : I_G(\vartheta)| = \frac{1+q+q^2}{3}$ and $\zeta_{I_G(\vartheta)/N}(s) = |I_G(\vartheta) : N| = 3$.

Thus

$$
\begin{aligned}
\zeta_{SL_1(D)}(s) = &\sum_{\vartheta \in \mathrm{Irr}(N)} \vartheta(1)^{-s} \left|G : I_G(\vartheta)\right|^{-1-s} \zeta_{I_G(\vartheta)/N}(s) \\
=&\zeta_{\mathbf{C}_{q^2+q+1}}(s) + \frac{q^{-3s+2}}{1-q^{-3s+2}} \Bigg(\mu(W^{[0]}(\mathscr{O}_F)) 1^{-1-s} \zeta_{\mathbf{C}_{q^2+q+1}}(s) \\
&+ q^{2s+4} \mu(W^{[1]}(\mathscr{O}_F)) \left(\frac{q^2+q+1}{3}\right)^{-1-s} \zeta_{\mathbf{C}_3}(s) \\
&+ q^{3s+6} \mu(W^{[2]}(\mathscr{O}_F)) \left(\frac{q^2+q+1}{3}\right)^{-1-s} \zeta_{\mathbf{C}_3}(s) \Bigg) \\
=&\frac{(q^2+q+1)(1-q^{-3s+2}) + q^{-3s}(q^2-1)(q^2+q+1)}{1-q^{-3s+2}} \\
&+ \frac{3^{s+2} q^{2s} (q^{-1} + q^{-3} + q^{s+4} + q^s)(1+q+q^2)^{-1-s}}{1-q^{-3s+2}} \\
=&\frac{(1+q+q^2)(1-q^{-3s}) + 9 \cdot \left(\frac{q^2+q+1}{3}\right)^{-s} (q^{-s+1}+1)(q-1)}{1-q^{-3s+2}}.
\end{aligned}
$$

□

Chapter 12

Some Concluding Conjectures

In Chapter 7 we proved that the set of coordinates $W(\mathscr{O}_F)$ can be partitioned into ℓ disjoint subsets, on each of which the maps $\mathbf{y} \mapsto \|F_j(\mathbf{y})\|_\wp$ have a uniform lower bound, for $j = 1, \ldots, \rho$. Specifically, if $\mathbf{y}$ has $\mathtt{t}(\mathbf{y}) = \gamma \in \mathbb{F}_\ell$ (cf. § 7.2 for the definitions), then by Corollary 7.2.2 we have that $\|F_j(\mathbf{y})\|_\wp = 1$ whenever $\gamma = 0$, and

$$\|F_j(\mathbf{y})\|_\wp \geq \min\left\{1, q^{-2j+(\ell-1)(r(\gamma)-1)}\right\}$$

whenever $\gamma \neq 0$ for any $j \in \{1, \ldots, \rho\}$.

We conjecture the following:

Conjecture A. *Let $\mathbf{y} \in W^{[\gamma]}(\mathscr{O})$ be arbitrary, then*

$$\|F_j(\mathbf{y})\|_\wp = \begin{cases} 1 & \textit{if } \gamma = 0, \\ \min\left\{1, q^{-2j+(\ell-1)(r(\gamma)-1)}\right\} & \textit{otherwise}. \end{cases}$$

Assuming the conjecture, we will be able to bring the values of the function $P(x, \mathbf{y})$ (cf. § 2.4.2) into closed form, by the formula

$$P(x, \mathbf{y}) = \begin{cases} q^{-2\rho+(\ell-1)(r(\gamma)-1)} & \text{if } , \gamma \neq 0 \\ 1 & \text{otherwise} \end{cases}, \quad \forall x \in \wp_F.$$

As discussed at the end of Section 7.2, one can easily deduce the value of $\|F_j(\mathbf{y})\|_\wp$ in the case where $\mathbf{y} \in W^{[1]}(\mathscr{O}_F)$ by simply counting the number of columns of $\mathcal{R}(\mathbf{y})$ which are in $\wp_F{}^{(d)}$. Moreover, for the case where $\mathbf{y} \in W^{\gamma}(\mathscr{O}_F)$ for some $\gamma \in \mathbb{F}_\ell \setminus \{0, 1\}$, the value of $\|F_j(\mathbf{y})\|_\wp$ is bound from above by

$$\|F_j(\mathbf{y})\|_\wp \leq \min\left\{1, q^{-2j+\ell(r(\gamma)-1)-1}\right\}, \quad \forall j = 1, \ldots, \rho,$$

which trivially implies the equality

$$\left\|F_j(\mathbf{y})\right\|_{\wp} = \min\left\{1, q^{-2j+(\ell-1)(r(\gamma)-1)}\right\},$$

for the cases $\gamma = 1$ and $\gamma = 2$. So that the conjecture holds for $\gamma \in \{0, 1, 2\}$. Note that this equality is independent of ℓ. This equality allows us to complete the calculation of $\zeta_{SL_1^1(\mathcal{O})}(s)$ in the cases where $\deg(D) = 3$.

One can also apply certain manipulations to prove such an equality for $\gamma = 3$. The case $\gamma = 4$, for now, is beyond the computational scope of this essay, whence the case $\deg D = 5$ is not computed here.

Let us consider one more notable case, in which the conjecture holds. Let $0 \neq \gamma \in \mathbb{F}_\ell$ and let $\mathbf{y} = (y_1, \ldots, y_d) \in W^{[\gamma]}(\mathscr{O}_F)$ be such that $y_j \in \wp_F$ whenever $j < r(\gamma)\ell$ or $j \geq (r(\gamma)+1)\,\ell$. That is to say, if $\mathbf{y}' = (y_1', \ldots, y_d')$ is such that

$$y_j' = \begin{cases} y_j & \text{, if } r(\gamma)\ell \leq j \leq r(\gamma)\ell + (\ell-1), \\ 0 & \text{, otherwise,} \end{cases}$$

then $\mathbf{y}_{|\gamma} = \mathbf{y}'_{|\gamma} \in {\wp_F}^{(\ell)}$ and $\mathbf{y} - \mathbf{y}' \in {\wp_F}^{(d)}$.

Consider the matrix $\mathcal{R}(\mathbf{y})$. In view of the decomposition suggested in Section 5.3 we have that $\mathsf{C}_{\alpha,\beta}(\mathbf{y}) \in \mathrm{Mat}_\ell(\wp_F)$ whenever $\alpha + \beta \neq \gamma$, and $\mathsf{B}_\beta(\mathbf{y}) \in \mathrm{Mat}_{(\ell-1)\times\ell}(\wp_F)$ whenever $\beta \neq \gamma$. In particular, the image of $\mathcal{R}'(\mathbf{y})$ over $\mathfrak{f}$ has the following block decomposition:

$$\overline{\mathcal{R}'(\mathbf{y})} = \begin{pmatrix} & & & \mathsf{c}_{1,\gamma-1}(\mathbf{y}) & \\ & & \mathsf{c}_{2,\gamma-2}(\mathbf{y}) & & \\ & \iddots & & & \\ c_{\gamma-1,1}(\mathbf{y}) & & & & \\ & & & & \end{pmatrix},$$

where the blocks $\mathsf{c}_{\alpha,\gamma-\alpha}(\mathbf{y})$, for $\alpha \in \{1, \ldots, \gamma-1\}$, have $\mathrm{rk}_{\mathfrak{f}}\mathsf{c}_{\alpha,\gamma-\alpha}(\mathbf{y}) = \ell - 1$ by Lemma 7.1.8, and all other entries of $\overline{\mathcal{R}'(\mathbf{y})}$ are 0. In particular, it follows that

$$\mathrm{rk}\overline{\mathcal{R}(\mathbf{y})} = (r(\gamma)-1)\,(\ell-1),$$

whenever $\mathbf{y}$ is as assumed here. It follows that for any $2j \times 2j$ sub-matrix M of $\mathcal{R}'(\mathbf{y})$ has $\mathfrak{f}$-rank no greater then $r(\gamma)(\ell-1)$ and hence

$$|\mathsf{M}|_{\wp} \leq q^{-2j+(r(\gamma)-1)(\ell-1)}$$

It follows that the maximal absolute value over all $2j \times 2j$ minors of $\mathcal{R}'(\mathbf{y})$ can not exceed the value $q^{-2j+(r(\gamma)-1)(\ell-1)}$ as well (and it can not exceed the absolute value 1 trivially), and hence the conjecture holds for such distinguished $\mathbf{y}$'s.

This observation allows us to pose two claims, which will entail Conjecture A if proven to be correct.

Sub-Conjecture 12.1. Let $j \in \{1, \dots, \rho\}$. The function $\mathbf{y} \mapsto \|F_j(\mathbf{y})\|_\wp$, defined from $W(\mathscr{O}_F)$ to $\mathbb{R}$, is constant on the subsets $W^{[\gamma]}(\mathscr{O}_F)$, for $\gamma \in \mathbb{F}_\ell$.

Note that since we know the value of this function at some points in each of the subsets $W^{[\gamma]}(\mathscr{O}_F)$, this claim will imply Conjecture A immediately.

Another possible strategy for proof is the following claim:

Sub-Conjecture 12.2. Let $\mathbf{y}_1,\ \mathbf{y}_2 \in W(\mathscr{O}_F)$ be such that $\mathbf{y}_1 + \mathbf{y}_2 \in W(\mathscr{O}_F)$. Assume $\mathtt{t}(\mathbf{y}_2) < \mathtt{t}(\mathbf{y}_1)$. Then for any $j = 1, \dots, \rho$

$$\|F_j(\mathbf{y}_1 + \mathbf{y}_2)\|_\wp \leq \|F_j(\mathbf{y}_1)\|_\wp \,.$$

This claim will imply the conjecture in the following way. Suppose $\mathbf{y} = (y_1, \dots, y_d) \in W^{[\gamma]}(\mathscr{O}_F)$ for $\gamma \neq 0$. Let $\mathbf{y}' = (y_1', \dots, y_d')$ be given by

$$y_j' = \begin{cases} y_j & \text{, if } r(\gamma)\ell \leq j \leq r(\gamma)\ell + (\ell - 1), \\ 0 & \text{, otherwise,} \end{cases}$$

and put $\mathbf{y}'' := \mathbf{y} - \mathbf{y}'$. If $\mathbf{y}'' \in {\wp_F}^{(d)}$ the equality of Conjecture A follows from the above discussion. Otherwise, we have that $\mathtt{t}(\mathbf{y}'') < \gamma = \mathtt{t}(\mathbf{y}')$, and by the second claim we have that

$$\|F_j(\mathbf{y})\|_\wp = \|F_j(\mathbf{y}' + \mathbf{y}'')\|_\wp \leq \|F_j(\mathbf{y}')\|_\wp = \min\left\{1, q^{-2j+(r(\gamma)-1)(\ell-1)}\right\},$$

and hence the conjecture.

Appendices

Appendix A

Arithmetic Lemmas

In this section we will prove some arithmetic lemmas which are used throughout the last part of this essay.

Lemma A.1. *Let $k, m \in \mathbb{N}$ be arbitrary, and let $\mathbf{C}_k$ denote the cyclic group of k elements. Let $\phi_m : \mathbf{C}_k \to \mathbf{C}_k$ be defined by*

$$\phi_m(x) = x^m, \quad \forall x \in \mathbf{C}_k.$$

Then

$$\#\mathrm{Ker}\phi_m = \gcd(k, m).$$

Proof. Assume first that m divides k. In this case, note that if $x \in \mathbf{C}_k$ is of order k, then $\phi_m(x)$ is of order $\frac{k}{m}$. Consequently, the image of ϕ_m is isomorphic to $\mathbf{C}_{\frac{k}{m}}$, the cyclic group of $\frac{k}{m}$ elements, and the kernel is of order $m = \gcd(k, m)$.

Otherwise, let r be such that $m = \gcd(k, m) \cdot r$. Then $\gcd(r, k) = 1$, and hence $\phi_r(x) = x^r$ is an isomorphism of $\mathbf{C}_k$ (if $t \in \mathbb{N}$ is such that $rt \equiv 1(\mod k)$ then ϕ_t is the inverse of ϕ_r). Moreover, since $\mathbf{C}_k$ is abelian, we have that $\phi_r \circ \phi_{\gcd(r,k)} = \phi_m$ and hence

$$\mathrm{Ker}\phi_m = \phi_r^{-1}\left(\mathrm{Ker}\phi_{\gcd(k,m)}\right),$$

and thus

$$\#\mathrm{Ker}\phi_m = \#\phi_r^{-1}\left(\mathrm{Ker}\phi_{\gcd(k,n)}\right) = \#\mathrm{Ker}\phi_{\gcd(k,m)} = \gcd(k, m)$$

$\square$

Throughout the rest of this section, we fix a prime number p, and let $q = p^\alpha$ for some $\alpha \in \mathbb{N}$.

Recall that for for any $a, b \in \mathbb{Z}$ and $g, r \in \mathbb{Z}$ such that $a = g \cdot b + r$ and $0 \leq r < b$, we have that

$$\gcd(a, b) = \gcd(b, r).$$

Indeed, it can be easily seen that the sets of common divisors of $\{a, b\}$ and $\{b, r\}$ coincide.

Lemma A.2. *Let $m, k \in \mathbb{N}$ be arbitrary. Then*

$$\gcd\left(\sum_{j=0}^{m-1} q^j, \sum_{j=0}^{k-1} q^j\right) = 1 \qquad \Longleftrightarrow \qquad \gcd(m, k) = 1.$$

Proof. $\Rightarrow$ Assume $\gcd(m, k) > 1$. Note that if $t = r \cdot s$ ($s, r, t \in \mathbb{Z}^{\geq 0}$), then

$$\begin{aligned}
\frac{\sum_{j=0}^{t-1} q^j}{\sum_{j=1}^{r-1} q^j} &= \frac{\sum_{j=0}^{t-1} q^j}{\sum_{j=1}^{r-1} q^j} \cdot \frac{q-1}{q-1} \\
&= \frac{q^t - 1}{q^r - 1} = \frac{(q^r - 1)(1 + q^r + \ldots + q^{r(s-1)})}{q^r - 1} \\
&= 1 + q^r + \ldots q^{r(s-1)} \in \mathbb{Z}.
\end{aligned}$$

Thus, if $\gcd(m, k) = r > 1$ then $\sum_{j=1}^{r-1} q^j > 1$ and divides both $\sum_{j=0}^{m-1} q^j$ and $\sum_{j=0}^{k} q^j$.

$\Leftarrow$ Assume $\gcd(m, k) = 1$ and put $k = m \cdot t + r$ for $0 \leq r < m$. Then

$$\begin{aligned}
\sum_{j=0}^{k-1} q^j = \frac{q^k - 1}{q - 1} &= \frac{q^{mt+r} - q^r + q^r - 1}{q - 1} \\
&= q^r \frac{q^{mt} - 1}{q - 1} + \frac{q^r - 1}{q - 1} \\
&= q^r \left(1 + q^m + \ldots + q^{m(t-1)}\right) \frac{q^m - 1}{q - 1} + \frac{q^r - 1}{q - 1} \\
&= q^r \left(1 + q^m + \ldots + q^{m(t-1)}\right) \left(\sum_{j=0}^{m-1} q^j\right) + \sum_{j=0}^{r-1} q^j
\end{aligned}$$

and consequently

$$\gcd\left(\sum_{j=0}^{k-1} q^j, \sum_{j=0}^{m-1} q^j\right) = \gcd\left(\sum_{j=0}^{m-1} q^j, \sum_{j=0}^{r-1} q^j\right).$$

Additionally, we have that $\gcd(m,k) = \gcd(m,r) = 1$. Apply this process recursively. By the Euclidean algorithm we eventually terminate at $r = 1$ (since $\gcd(m,k) = 1$). Hence

$$\gcd\left(\sum_{j=0}^{k-1} q^j, \sum_{j=0}^{m-1} q^j\right) = 1.$$

□

Lemma A.3. *Let $\ell \in \mathbb{Z}$, and assume that $q \equiv 1 (\mod \ell)$. Then*

$$\gcd\left(\sum_{j=0}^{\ell-1} q^j, q-1\right) = \ell.$$

Proof. By assumption, it is clear that ℓ divides both $q-1$ and $\sum_{j=0}^{\ell-1} q^j$. In addition, one can verify that

$$\sum_{j=0}^{\ell-1} q^j = \left(\sum_{j=0}^{\ell-2} (\ell - j - 1) q^j\right)(q-1) + \ell,$$

and hence

$$\gcd\left(\sum_{j=0}^{\ell-1} q^j, q-1\right) = \gcd(q-1, \ell) = \ell.$$

□

Lemma A.4. *Let $\ell \in \mathbb{Z}$ be a prime, and assume $q \equiv 1 (\mod \ell)$. Let $m \in \mathbb{Z}$ be such that $0 < m < \ell$. Then*

$$\gcd\left(\sum_{j=0}^{\ell-1} q^j, q^m - 1\right) = \ell.$$

Proof. We have that $q^m - 1 = (q-1)\left(\sum_{j=0}^{m-1} q^j\right)$. By the assumption that $0 < m < \ell$ we have, by Lemma A.2, that

$$\gcd\left(\sum_{j=0}^{\ell-1} q^j, \sum_{j=0}^{m-1} q^j\right) = \gcd(m, \ell) = 1.$$

Thus, by applying Lemma A.3,

$$\gcd\left(\sum_{j=0}^{\ell-1} q^j, q^m - 1\right) = \gcd\left(\sum_{j=0}^{\ell-1} q^j, q-1\right) = \ell.$$

□

Appendix B

The Killing Form of $\mathfrak{sl}_1(D)$

In this section we calculate the Killing form $\kappa : \mathscr{G} \times \mathscr{G} \to F$, given by $\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$, with respect to the F-basis

$$\mathcal{M} := \left\{ \mathbf{m}_{\underline{\alpha}} := u^{\alpha_1}\theta^{\alpha_2} \mid \underline{\alpha} = (\alpha_1, \alpha_2) \in \mathcal{I}^* \right\}.$$

Let $\mathcal{M}^\vee = \left\{ \mathbf{m}_{\underline{\alpha}}^\vee \right\}_{\underline{\alpha} \in \mathcal{I}}$ denote the dual basis of $\mathcal{M}$ (i.e. $\mathbf{m}_{\underline{\alpha}}^\vee(\mathbf{m}_{\underline{\beta}}) = 1$ if $\underline{\alpha} = \underline{\beta}$ and 0 otherwise).

The commutator relations of elements in $\mathcal{M}$ are very similar to those described in Proposition 5.2.5, and are given by the equation

$$[\mathbf{m}_{\underline{\alpha}}, \mathbf{m}_{\underline{\beta}}] = \left(\omega^{\beta_2\alpha_1} - \omega^{\beta_1\alpha_2} \right) \pi^{\epsilon(\alpha_1,\beta_1)} b^{\epsilon(\alpha_2,\beta_2)} \mathbf{m}_{\underline{\alpha}+\underline{\beta}},$$

where $\underline{\alpha} = (\alpha_1, \alpha_2), \underline{\beta} = (\beta_1, \beta_2) \in \mathcal{I}^*$, and

$$\epsilon(\alpha, \beta) = \begin{cases} 0 & \text{if } r(\alpha) + r(\beta) < \ell \\ 1 & \text{otherwise} \end{cases}, \quad \forall \alpha, \beta \in \mathbb{F}_\ell.$$

Remark. Recall (§ 5.2.1) that $r : \mathbb{F}_\ell \to \mathbb{Z}$ maps any $\alpha \in \mathbb{F}_\ell$ to its natural representative in $\{0, \ldots, \ell - 1\}$.

We will compute

$$\kappa(\mathbf{m}_{\underline{\alpha}}, \mathbf{m}_{\underline{\beta}}) = \mathrm{Tr}\left(\mathrm{ad}_{\mathbf{m}_{\underline{\alpha}}} \circ \mathrm{ad}_{\mathbf{m}_{\underline{\beta}}} \right) = \sum_{\underline{\gamma} \in \mathcal{I}^*} \mathbf{m}_{\underline{\gamma}}^\vee \left([\mathbf{m}_{\underline{\alpha}}, [\mathbf{m}_{\underline{\beta}}, \mathbf{m}_{\underline{\gamma}}]] \right),$$

for any $\underline{\alpha}, \underline{\beta} \in \mathcal{I}^*$.

B The Killing Form of $\mathfrak{sl}_1(D)$

Let $\underline{\alpha} = (\alpha_1, \alpha_2),\ \underline{\beta} = (\beta_1, \beta_2), \underline{\gamma} = (\gamma_1, \gamma_2) \in \mathcal{I}^*$. We have

$$\begin{aligned}[\mathbf{m}_{\underline{\alpha}}, [\mathbf{m}_{\underline{\beta}}, \mathbf{m}_{\underline{\gamma}}]] &= \pi^{\epsilon(\beta_1,\gamma_1)} b^{\epsilon(\beta_2,\gamma_2)} \left(\omega^{\beta_2\gamma_1} - \omega^{\beta_1\gamma_2}\right) [\mathbf{m}_{\underline{\alpha}}, \mathbf{m}_{\underline{\beta}+\underline{\gamma}}] \\ &= \pi^{\epsilon(\beta_1,\gamma_1)+\epsilon(\alpha_1,\beta_1+\gamma_1)} b^{\epsilon(\beta_2,\gamma_2)+\epsilon(\alpha_2,\beta_2+\gamma_2)} \cdot \\ &\quad \left(\omega^{\beta_2\gamma_1} - \omega^{\beta_1\gamma_2}\right) \left(\omega^{\alpha_2(\beta_1+\gamma_1)} - \omega^{\alpha_1(\beta_2+\gamma_2)}\right) \mathbf{m}_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}}.\end{aligned}$$

This cumbersome equation already reveals one easy fact:

Conclusion. *If $\underline{\alpha} + \underline{\beta} \neq (0,0)$, then*

$$\mathbf{m}^\vee_{\underline{\gamma}} \Big([\mathbf{m}_{\underline{\alpha}}, [\mathbf{m}_{\underline{\beta}}, \mathbf{m}_{\underline{\gamma}}]]\Big) = 0, \quad \forall \underline{\gamma} \in \mathcal{I}^*,$$

and hence $\kappa(\mathbf{m}_{\underline{\alpha}}, \mathbf{m}_{\underline{\beta}}) = 0$.

Let us assume that $\underline{\alpha} + \underline{\beta} = (0,0)$. Note the following fact, regarding the exponents of π and b:

Lemma B.1. *Suppose $\alpha, \beta \in \mathbb{F}_\ell$ are such that $\alpha + \beta = 0$ and $\gamma \in \mathbb{F}_\ell$ is arbitrary. Then*

1. *If $\alpha = \beta = 0$ then $\epsilon(\beta, \gamma) = \epsilon(\alpha, \beta + \gamma) = 0$.*
2. *If $\alpha \neq 0$ (and hence $\beta = -\alpha \neq 0$), then $\epsilon(\beta, \gamma) + \epsilon(\alpha, \beta + \gamma) = 1$.*

Proof. The first item is immediate- if $\alpha = \beta = 0$ then

$$r(\beta) + r(\gamma) = r(\alpha) + r(\beta + \gamma) = r(\gamma) < \ell,$$

by definition of r. For the second item, divide into cases:

- If $r(\beta) + r(\gamma) < \ell$ then $\epsilon(\beta, \gamma) = 0$ and $r(\beta) + r(\gamma) = r(\beta + \gamma)$. Since $\alpha \neq 0$ we have that $r(\alpha) + r(\beta) = \ell$, and thus

  $$r(\alpha) + r(\beta + \gamma) = r(\alpha) + r(\beta) + r(\gamma) \geq \ell$$

  and thus $\epsilon(\alpha, \beta + \gamma) = 1$.

- If $r(\beta) + r(\gamma) \geq \ell$ then $\epsilon(\beta, \gamma) = 1$ and $r(\beta) + r(\gamma) = \ell + r(\beta + \gamma)$. Since $r(\alpha) + r(\beta) = \ell$, it follows that

  $$r(\alpha, \beta + \gamma) = r(\alpha) + r(\beta) + r(\gamma) - \ell = r(\gamma) < \ell.$$

  and thus $\epsilon(\alpha, \beta + \gamma) = 0$.

In both cases the assertion $\epsilon(\beta,\gamma)+\epsilon(\alpha,\beta+\gamma)=1$ holds. $\square$

We proceed to calculate by dividing into cases:

- Suppose $\alpha_1=\beta_1=0$. Since $(0,0)\notin \mathcal{I}^*$ this implies that $\alpha_2\neq 0$ and $\beta_2=\ell-\alpha_2$. Let $\underline{\gamma}=(\gamma_1,\gamma_2)\in\mathcal{I}^*$ be arbitrary. By Lemma B.1 we have that

$$\epsilon(\beta_1,\gamma_1)+\epsilon(\alpha_1,\beta_1+\gamma_1)=0,\quad\text{and}\quad \epsilon(\beta_2,\gamma_2)+\epsilon(\alpha_2,\beta_2+\gamma_2)=1.$$

  Thus

$$\begin{aligned}\mathbf{m}_{\underline{\gamma}}^{\vee}\left([\mathbf{m}_{\underline{\alpha}},[\mathbf{m}_{\underline{\beta}},\mathbf{m}_{\underline{\gamma}}]]\right)&=b\cdot\left(\omega^{\beta_2\gamma_1}-\omega^{\beta_1\gamma_2}\right)\left(\omega^{\alpha_2(\beta_1+\gamma_1)}-\omega^{\alpha_1(\beta_2+\gamma_2)}\right)\\&=b\cdot\left(\omega^{\beta_2\gamma_1}-1\right)\left(\omega^{\alpha_2\gamma_1}-1\right)\end{aligned}$$

  Note that this value is independent of the value of γ_2. It follows that

$$\begin{aligned}\kappa(\mathbf{m}_{\underline{\alpha}},\mathbf{m}_{\underline{\beta}})&=\ell\cdot b\cdot\sum_{\gamma\in\mathbb{F}_\ell}(1-\omega^{-\alpha_2\gamma_1})(1-\omega^{\alpha_2\gamma_1})\\&=\ell\cdot b\sum_{\gamma'\in\mathbb{F}_\ell}\left(2-\omega^{\gamma'}-\omega^{-\gamma'}\right)\\&=\ell b\cdot\left(2\ell-\sum_{\gamma'\in\mathbb{F}_\ell}\omega^{\gamma'}-\sum_{\gamma'\in\mathbb{F}_\ell}\omega^{-\gamma'}\right)=2\ell^2 b\end{aligned}$$

- Suppose $\alpha_2=\beta_2=0$. As in the previous item we have, by Lemma B.1, that

$$\epsilon(\beta_1,\gamma_1)+\epsilon(\alpha_1,\beta_1+\gamma_1)=1,\quad\text{and}\quad \epsilon(\beta_2,\gamma_2)+\epsilon(\alpha_2,\beta_2+\gamma_2)=0,$$

  for $\underline{\gamma}=(\gamma_1,\gamma_2)\in\mathcal{I}^*$ arbitrary.

  Thus,

$$\mathbf{m}_{\underline{\gamma}}^{\vee}\left([\mathbf{m}_{\underline{\alpha}},[\mathbf{m}_{\underline{\beta}},\mathbf{m}_{\underline{\gamma}}]]\right)=\pi\cdot\left(1-\omega^{\beta_1\gamma_2}\right)\left(1-\omega^{\alpha_1\gamma_2}\right)$$

  and by the same calculation as before,

$$\kappa(\mathbf{m}_{\underline{\alpha}},\mathbf{m}_{\underline{\beta}})=2\ell^2\pi.$$

- Suppose $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. In this case, By Lemma B.1 we have that

  $$\epsilon(\beta_1, \gamma_1) + \epsilon(\alpha_1, \beta_1 + \gamma_1) = \epsilon(\beta_2, \gamma_2) + \epsilon(\alpha_2, \beta_2 + \gamma_2) = 1.$$

  Additionally,

  $$\begin{aligned} \mathbf{m}_{\underline{\gamma}}^{\vee}\left([\mathbf{m}_{\underline{\alpha}}, [\mathbf{m}_{\underline{\beta}}, \mathbf{m}_{\underline{\gamma}}]]\right) &= \pi b \cdot \left(\omega^{\beta_2\gamma_1} - \omega^{\beta_1\gamma_2}\right)\left(\omega^{\alpha_2(\beta_1+\gamma_1)} - \omega^{\alpha_1(\beta_2+\gamma_2)}\right) \\ &= \pi b \cdot \left(\omega^{-\alpha_2\gamma_1} - \omega^{-\alpha_1\gamma_2}\right)\left(\omega^{\alpha_2(-\alpha_1+\gamma_1)} - \omega^{\alpha_1(-\alpha_2+\gamma_2)}\right) \\ &= \omega^{-\alpha_1\alpha_2}\pi b \cdot \left(\omega^{-\alpha_2\gamma_1} - \omega^{-\alpha_1\gamma_2}\right)\left(\omega^{\alpha_2\gamma_1} - \omega^{\alpha_1\gamma_2}\right) \\ &= \omega^{-\alpha_1\alpha_2}\pi b \cdot \left(2 - \omega^{-\alpha_1\gamma_2+\alpha_2\gamma_1} - \omega^{\alpha_1\gamma_2-\alpha_2\gamma_1}\right). \end{aligned}$$

  Thus,

  $$\begin{aligned} \kappa(\mathbf{m}_{\underline{\alpha}}, \mathbf{m}_{\underline{\beta}}) &= \omega^{-\alpha_1\alpha_2}\pi b \cdot \sum_{\underline{\gamma}\in\mathcal{I}^*}\left(2 - \omega^{-\alpha_1\gamma_2+\alpha_2\gamma_1} - \omega^{\alpha_1\gamma_2-\alpha_2\gamma_1}\right) \\ &= \omega^{-\alpha_1\alpha_2}\pi b\left(2\ell^2 - \sum_{\gamma_1\in\mathbb{F}_\ell}\left(\omega^{\alpha_2\gamma_1}\sum_{\gamma_2\in\mathbb{F}_\ell}\omega^{-\alpha_1\gamma_2} + \omega^{-\alpha_2\gamma_1}\sum_{\gamma_2\in\mathbb{F}_\ell}\omega^{\alpha_1\gamma_2}\right)\right) \\ &= 2\omega^{-\alpha_1\alpha_2}\pi b\ell^2 \end{aligned}$$

To summarize:

$$\kappa(\mathbf{m}_{\underline{\alpha}}, \mathbf{m}_{\underline{\beta}}) = \begin{cases} 0 & \text{if } \underline{\alpha} + \underline{\beta} \neq 0, \\ 2\ell^2 b & \text{if } \alpha_1 = \beta_1 = 0 \text{ and } \alpha_2 + \beta_2 = 0, \\ 2\ell^2\pi & \text{if } \alpha_2 = \beta_2 = 0 \text{ and } \alpha_1 + \beta_1 = 0, \\ 2\ell^2\omega^{-\alpha_1\alpha_2}\pi b & \text{otherwise.} \end{cases}$$

In a more pictorial manner, one can represent the killing form $\kappa(\cdot,\cdot)$ with respect to the basis $\mathcal{M}$ as a symmetric block matrix

$$[\kappa(\cdot,\cdot)]_{\mathcal{M}} = \left(\begin{array}{c:c:c:c} \mathsf{A}_1 & & & \\ \hdashline & & & \mathsf{A}_\ell \\ \hdashline & & \iddots & \\ \hdashline & & \mathsf{A}_3 & \\ \hdashline & \mathsf{A}_2 & & \end{array}\right) \in \mathrm{Mat}_{\ell^2-1}(\mathscr{O})$$

where $\mathsf{A}_1 \in \mathrm{Mat}_{\ell-1}(\mathscr{O}_F)$ is given by

$$\mathsf{A}_1 = 2\ell \cdot \begin{pmatrix} & & & \ell b \\ & & \iddots & \\ & \ell b & & \\ \ell b & & & \end{pmatrix},$$

and for any $j = 2, \ldots, \ell$, $\mathsf{A}_j \in \mathrm{Mat}_\ell(\mathscr{O}_F)$ is given by

$$\mathsf{A}_j = 2\ell \cdot \pi\ell \begin{pmatrix} 1 & & & & \\ & & & & b\omega^{-(\ell-1)j} \\ & & & \iddots & \\ & & b\omega^{-2j} & & \\ & b\omega^{-j} & & & \end{pmatrix}$$

Remark. We've factored out the common denominator 2ℓ in all of the above blocks, anticipating the normalization κ_0 which is obtained by dividing κ by the dual Coxeter number of $\mathfrak{sl}_1(D)$.

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