# Generalized functions
# Tutorial notes

## Tutorial 6

### 6.1. Complements regarding distributions on $\mathbb{R}^n$.

DEFINITION 6.1. Given a closed subspace $W \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}$ define

$$V_m(C_c^\infty(\mathbb{R}^n), W) := \left\{ f \in C_c^\infty(\mathbb{R}^n) : \frac{\partial^\alpha}{(\partial x)^\alpha} f \mid_W \equiv 0, |\alpha| \leq m \right\}.$$

Defined similarly for $W$ a subset.

EXERCISE 6.2. *Let $W$ be a $k$-dimensional subspace of $\mathbb{R}^n$ and $U = \mathbb{R}^n \setminus W$. Show that*

$$\overline{C_c^\infty(U)} = \bigcap_{m=0}^{\infty} V_m(C_c^\infty(\mathbb{R}^n), W).$$

SOLUTION. Home exercise. $\square$

DEFINITION 6.3. Define $F_m((C_c(\mathbb{R}^n)^*, W) := \{\xi \in C_c^\infty(\mathbb{R}^n)^* : \xi \mid_{V_m} \equiv 0\}$, where $V_m$ is as above.

EXERCISE 6.4. *Prove that for any $U \subseteq \mathbb{R}^n$ with compact closure and any $\xi \in C_W^{-\infty}(\mathbb{R}^n)$, there exists $\xi' \in F_m$ such that $\xi \mid_U \equiv \xi' \mid_U$.*

SOLUTION. Let $V$ be an open subset with compact closure, properly containing $\bar{U}$, and let $\xi' = \xi \cdot I_U$, where $I_U$ is some smooth function such that which attains the value 1 on $\bar{U}$ and 0 on $V^c$. Clearly, $\xi \mid_U \equiv \xi' \mid_U$, and we need to verify $\xi' \in F_m$ for some $m \in \mathbb{N}$. Assume this is false- then for any $m \in \mathbb{N}$ we may find $f_m \in V_m$ such that $\langle \xi', f_m \rangle \neq 0$. Up to multiplying each $f_m$ by a constant (namely- the constant $\langle \xi', f_m \rangle^{-1}$), we may assume $\langle \xi', f_m \rangle = 1$ for all $m$. Furthermore, since $\xi'$ is compactly supported, without loss of generality, up to mulitiplication by a test function $\psi$ with $\psi_{\mathrm{cl}(V)} \equiv 1$, we may assume that the $f_m$'s are all supported on a compact set containing $\bar{V}$. Finally we note that, by definition, we have that, for any $\alpha \in \mathbb{N}_0^m$, the sequence $(\frac{\partial^\alpha}{\partial x^\alpha} f_m)_m$ is, apart from finitely many terms, constant zero, and therefore converges uniformly to 0. In particular $f_m \xrightarrow{m \to \infty} 0$ in $C_c^\infty(U)$ and thus $\langle \xi', f_m \rangle \xrightarrow{m \to \infty} 0$. A contradiction, since $\langle \xi', f_m \rangle = 1$ by definition. $\square$

EXERCISE 6.5. *Compute $\overline{C_c^\infty(\mathbb{R}^n \setminus \{0\})}$.*

SOLUTION. $\overline{C_c^\infty(\mathbb{R}^n \setminus \{0\})}$ is the set of compactly supported smooth functions such that $\frac{\partial^\alpha}{\partial x^\alpha} f(0) = 0$, for all $\alpha \in \mathbb{N}_0^n$. Note that, for example, any function of the form $e^{-1/\|x\|^2} \cdot \psi$, where $\psi$ is smooth function supported on a compact neighbourhood of 0, lies in $\overline{C_c^\infty(\mathbb{R}^n)}$, but is supported on $\mathrm{Supp}(\psi) \ni 0$. $\square$

**6.2. Complements regarding $\ell$-spaces.** Recall the following definition:

DEFINITION 6.6. An $\ell$-space $X$ is a locally compact, totally disconnected and Hausdorff space. $X$ is said to be *countable* at infinity, or $\sigma$-compact, if it is a countable union of compact sets.

EXERCISE 6.7. *Find a compact $\ell$-space $X$ and an open subset $U \subseteq X$ such that $U$ is not $\sigma$-compact.*

SOLUTION. Let $X = \mathbb{R} \cup \{\infty\}$, with the topology defined so that $V \subseteq X$ is open if $\infty \notin V$ or otherwise, if $\infty \in V$ and $\mathbb{R} \setminus V$ is finite. One easily verifies that $X$ is compact, Hausdorff and totally disconnected. Moreover, for $U = \mathbb{R} \subseteq X$, we know that the topology on $U$ is simply the discrete topology on a space of cardinality $2^{\aleph_0}$, and in particular, cannot be $\sigma$-compact. $\square$

Recall that, in the previous tutorial, it was shown that any $\ell$-space is (topologically) zero dimensional, i.e. it has a basis of clopen sets. In the case where $G$ is an $\ell$-group (i.e. a topological group which is an $\ell$-space), we have the following:

LEMMA 6.8 (van-Danzig's Theorem). *Let $G$ be an $\ell$-group, i.e. a topological group which is an $\ell$-space. Then $G$ has a neighborhood base at $1$ of compact open subgroups.*

PROOF. Since we already saw in the previous tutorial that any $\ell$-space has a basis of compact open neighbourhoods of 1, it would suffice to prove that any compact open subset $1 \in K \subseteq G$ contains a compact open subgroup.

Let $1 \in K$ be a compact open subset. Since group multiplication is continuous and $K$ is open, for any $x \in K$, we may find a neighbourhood $V_x \subseteq K$ of 1 such that $x \cdot V_x \subseteq K$. Furthermore, again, using continuity, for any $x \in K$, we may find an open neighbourhood $1 \in L_x \subseteq V_x$ such that $L_x^2 \subseteq V_x$. Clearly, $K = \bigcup_{x \in K} xL_x = \bigcup_{x \in K} xV_x$, and, by compactness, there exist $x_1, \ldots, x_r$ such that $K = \bigcup_{j=1}^r x_j L_{x_j}$. Put $L = \bigcap_{j=1}^r L_{x_j}$. Then $L$ is open and satisfies

$$K \cdot L = \bigcup_{j=1}^{r} x_j L_{x_j} L \subseteq \bigcup_{j=1}^{r} x_j L_{x_j}^2 \subseteq \bigcup_{j=1}^{r} x_j V_{x_j} \subseteq K.$$

Note that we may assume $L$ is symmetric, i.e. $L^{-1} = L$. Otherwise, just take the intersection of $L$ and $L^{-1}$. Let $H$ be the subgroup of $G$ generated by $L$. Then $H = \bigcup_{i=0}^{\infty} \underbrace{L \cdots L}_{m \text{ times}}$ is the union of open sets and thus open; it is closed in $K$ and hence compact, since any the complement of an open subgroup is a union of cosets, which is also open; and, by the defining property of $L$, it satisfies $K \cdot H \subseteq K$, and hence $H = 1 \cdot H \subseteq K$, as wanted. $\square$

REMARK 6.9. If in addition $G$ is compact, we may take the subgroups in the lemma to be normal in $G$. This is proved in the home-exercise.

We will now prove an important result on $\ell$-groups.

THEOREM 6.10. *Let $G$ be an $\ell$-group. There exists, up to multiplication by a constant, a unique left invariant distribution $\mu_G \in (S(G)^*)^G$- i.e., such that*

$$\langle g_0 \mu_G, f \rangle = \int_G f(g_0 g) d\mu_G(g) = \int_G f(g) d\mu_G(g) = \langle \mu_G, f \rangle,$$

*for all $g_0 \in G$ and $f \in S(G)$. Furthermore, we may assume $\mu$ is positive. Such a distribution is called a* Haar measure *on $G$.*

Proof. We begin by showing uniqueness. The essential step in this case is to note that the measure $\mu_G$ is completely determined by its value on the indicator function of a small enough subgroup.

Let $\mathcal{H} = \{H_\alpha\}_{\alpha \in I}$ be a basis of compact open subgroups of $G$. Note that, up to replacing all $H_\alpha$'s with $H_\alpha \cap H_{\alpha_0}$ for some $\alpha_0$, we may assume $\bigcup \mathcal{H} \subseteq H_{\alpha_0}$.

Step 1. *Uniquness.* Since $S(X) = \mathrm{Span}_{\mathbb{R}} \{I_H : H \in \mathcal{H}\}$ and $\mu$ is non-zero, there necessarily exists $H \in \mathcal{H}$ such that $\mu(H) \neq 0$. We claim that this implies $\mu(H') \neq 0$ for *all* $H' \in \mathcal{H}$. Indeed, assume towards a contradiction that $\mu(H') = 0$ for some $H' \in \mathcal{H}$ and put $K = H' \cap H$. Then $K$ is again a compact open subgroup, and, since its translates cover both $H$ and $H'$, it is of finite index in both groups. Now, by the assumption $\mu(H') = 0$ and $\mu$ is translation invariant, it follows that

$$0 = \mu(H') = [H' : K]\mu(K),$$

and hence $\mu(K) = 0$. On the other hand, the same argument shows that

$$0 \neq \mu(H) = [H : K]\mu(K) = [H : K] \cdot 0 = 0.$$

A contradiction.

Moreover, this computation shows that

$$\mu(H') = [H' : K]\mu(K) = \frac{[H' : K]}{[H : K]}\mu(H)$$

for any $H' \in \mathcal{H}$. Thus, the values of $\mu$ on the $I_{H'}$'s. and hence on $S(X)$, is determined by its values on any compact subgroup $H \in \mathcal{H}$.

Step 2. *Existence.* This will be proved in the next tutorial.

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