Generalized functions
Tutorial notes

Tutorial 4

4.1. Frćhet spaces.

DEFINITION 4.1. (1) A topological space X is metrizable if there exists a metric d on X such that the open balls with respect to d comprise a base for X;
(2) X is said to be S_1 or first countable, if every point has a countable basis;
(3) A Frèchet space is a locally convex complete metrizable topological vector space.

EXERCISE 4.2. *The following are equivalent for a locally compact topological vector space:*
(1) *metrizability;*
(2) *first countable;*
(3) *countable collection of seminorms*

In particular: complete+either of the above implies Frèchet.

SOLUTION. Home exercise. □

EXERCISE 4.3. *Show that* $C^\infty(\mathbb{S}^1) \simeq \mathrm{SW}(\mathbb{Z})$*, where* $\mathrm{SW}(\mathbb{Z})$ *denotes the space of rapidly decaying sequences* $\{(x_n)_{n\in\mathbb{Z}} : \lim_{n\to\infty} |x_n| n^\alpha < \infty$ *for all* $\alpha \in \mathbb{N}\}$ *indexed by* $\mathbb{Z}$*.*

SOLUTION. Both spaces are Freèchet: $C^\infty(\mathbb{S}^1)$ is endowed with the countable family of seminorms $\left\{\eta_j(f) = \sup_{x\in\mathbb{S}^1} \left|f^{(j)}(x)\right| : i = 0, 1, \ldots\right\}$ and $\mathrm{SW}(\mathbb{Z})$ with the countable family $\{nu_i((c_n)_n) = \sup_{n\in\mathbb{Z}} |n^i c_n| : i = 0, 1, \ldots\}$. The Fourier transform map $\mathcal{F} : C^\infty(\mathbb{S}^1) \to \mathrm{SW}(\mathbb{Z})$, defined by by $\mathcal{F}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-nxi}dx\right)_{n\in\mathbb{Z}}$, with inverse $\mathcal{F}^{-1}((c_n)_{n\in\mathbb{Z}})(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, is a bijection between the two spaces. To prove $\mathcal{F}$ is a homeomorphism, we need to prove that both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are bounded with respect to these seminorms, i.e. for any j, there exists $M_1, M_2 > 0$ and $k_1(j), k_2(j)$ such that

$$\nu_j(\mathcal{F}(f)) \le M_1\eta_{k_1(j)}(f) \quad \text{and} \quad \eta_j(\mathcal{F}^{-1}(c_n)_{n\in\mathbb{Z}}) \le M_2\eta_{k_2(j)}((c_n)),$$

for any $f \in C^\infty(\mathbb{S}^1)$ and $(c_n) \in \mathrm{SW}(\mathbb{Z})$.

Given $f \in C^\infty(\mathbb{S}^1)$ with $\mathcal{F}(f) = (c_n)_{n\in\mathbb{Z}}$, and $k \in \mathbb{Z}$, we have that

$$\begin{aligned}\left|k^j c_k\right| = \left|\frac{1}{2\pi}\int_0^{2\pi} k^j f(x)e^{-ikx}dx\right| &\le \frac{1}{2\pi}\int_0^{2\pi} \left|k^j f(x)e^{-ikx}\right| dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|f^{(j)}(x)e^{-ikx}\right| dx \le \frac{1}{2\pi}\int_0^{2\pi} \sup_{x\in\mathbb{S}^1} \left|f^{(j)}(x)\right| dx = \eta_j(f)\end{aligned}$$

where the penultimate equality is justified by a simple inductive argument, using integration by parts.

In particular, we have that $\nu_j(\mathcal{F}) \le \eta_j(f)$. On the other hand, we have that

$$\eta_j(f) = \sup_{x\in\mathbb{S}^1} \left|f^{(j)}(x)\right| = \sup_{x\in\mathbb{S}^1} \left|\sum_{n\in\mathbb{Z}} c_n \cdot (ni)^j \cdot e^{inx}\right| \le$$

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}\left|c_n n^{j+2}\right| \le \sup_{k\in\mathbb{Z}}\left|k^{j+2}c_k\right| \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = \frac{2\pi^2}{6}\nu_{j+2}((c_n)).$$

□

THEOREM 4.4 (Banach-Steinhaus for Frèchet spaces). *Let X be a Frèchet space and Y a normed space, and H a family of continuous linear maps form X to Y. Assume $\sup_{f\in H}\|f(x)\| < \infty$ for any $x \in X$. Then the family H is equicontinuous, i.e. for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(x_1) - f(x_2)\|_Y < \epsilon$ for any $f \in F$ and $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$.*

EXERCISE 4.5. *Show that $C^{-\infty}(\mathbb{R})$ is weakly sequentially complete.*

PROOF. Let (ξ_n) be a weakly Cauchy sequence in $(C_c^\infty(\mathbb{R}))^*$. We have a natural candidate for the limit of (ξ_n), which is defined by

$$\langle \xi, f\rangle := \lim_{n\to\infty}\langle \xi_n, f\rangle,$$

which exists by definition of weakly Cauchy. ξ is clearly linear, but not obviously continuous.

Let $K \subseteq \mathbb{R}$ be a compact set. We consider the restrictions of the ξ_n's and ξ to $C_K^\infty(\mathbb{R})$, the space of smooth maps supported on K. More specifically, we consider $F = \left\{\xi_n|_{C_K^\infty(\mathbb{R})}, \xi|_{C_K^\infty(\mathbb{R})}\right\} \subseteq \mathrm{Hom}(C_K^\infty(\mathbb{R}), \mathbb{R})$. By weak convergence, F is pointwise bounded, and hence, by Banach-Steinhaus, it is equicontinuous. Let f_n be a sequence in $C_K^\infty(\mathbb{R})$, converging to f. Then $d(f_n, f) \xrightarrow{n\to\infty} 0$, for d a metric on $C_K^\infty(\mathbb{R})$, and hence given $\epsilon > 0$, there exists $\delta > 0$ such that for any $n \gg 0$ we have that $d(f_n, f) < \delta$. Thus

$$|\langle \xi, f - f_n\rangle| \le \sup_{\nu\in F}|\langle \nu, f - f_n\rangle| < \epsilon$$

for $n \gg 0$. Since K is an arbitrary compact set in $\mathbb{R}$, by the definition of the topology on $C_c^\infty(\mathbb{R})$, it follows that ξ is continuous. □

As seen in the lecture, we have a description of $C_c^\infty(\mathbb{R})$ as a direct limit of Frèchet spaces:

$$C_c^\infty(\mathbb{R}) = \varinjlim_{K\subseteq\mathbb{R}\text{ compact}} C_K^\infty(\mathbb{R}).$$

The following exercise explicitly describes the topology of this space.

EXERCISE 4.6. *Given $n \in \mathbb{N}$, $k_n \in \mathbb{Z}_{\ge 0}$ and $\epsilon_n > 0$, put*

$$A_{k_n,\epsilon_n} = \left\{ f \in C^\infty(\mathbb{R}) : \mathrm{Supp}(f) \subseteq [-n, n] \text{ and } \sup_{x\in\mathbb{R}}\left|f^{(k_n)}(x)\right| < \epsilon_n \right\},$$

the open ball of radius ϵ_n with respect to $\nu_{k_n}(f) = \sup\left|f^{(k_n)}\right|$ on the space $C_{[-n,n]}^\infty(\mathbb{R})$. Define

$$U_{(k_n,\epsilon_n)_n} := \sum_{n\in\mathbb{N}} A_{k_n,\epsilon_n},$$

where (k_n, ϵ_n) is a sequence of pairs in $\mathbb{Z}_{\ge 0} \times \mathbb{R}_{>0}$.

Show that a sequence (f_n) in $C_c^\infty(\mathbb{R})$ converges to f with respect to the topology generated by the $U_{(k_n,\epsilon_n)}$'s if and only if it converges in the sense defined in the first lecture.

SOLUTION. Home exercise. $\square$

REMARK 4.7. One can also show that the topology on $C_c^\infty(\mathbb{R})$ is generated by the convex hulls of the $U_{(k_n,\epsilon_n)}$, thereby giving a direct proof of local convexity, this will also be proved in the home exercise.

4.2. Topologies on $C^{-\infty}(\mathbb{R})$.

DEFINITION 4.8. Let V be a topological vector space over $\mathbb{R}$. A subset $S \subseteq V$ is said to be *bounded* if for any open set $0 \in U$ there exists $\lambda > 0$ such that $S \subseteq \lambda V$.

EXERCISE 4.9. *Let $S \subseteq C_c^\infty(\mathbb{R})$ be bounded. Then there exists K compact such that $S \subseteq C_K^\infty(\mathbb{R})$.*

LEMMA 4.10. *Let V be a locally convex topological vector space. A set $S \subseteq V$ is bounded if and only if $\sup_{s \in S} \eta(s) < \infty$ for any continuous seminorm η on V.*

PROOF OF LEMMA. If S is bounded and η is a countinuous seminorm with $B = \eta^{-1}(-\infty, 1)$, the open ball around 0 of radius 1, then there exists $\lambda > 0$ such that $S \subseteq \lambda B$ and hence $\eta(S) \subseteq \eta(\lambda B) \subseteq [0, \lambda]$.

Conveserly, if S is bounded with respect to all continuous seminorms, let $0 \in U$ be open and let $0 \in C \subseteq U$ be an open convex and balanced set. Put $\lambda = \sup_{s \in S} N_C(s) < \infty$, then $S \subseteq \lambda C \subseteq \lambda U$. $\square$

SOLUTION OF EXERCISE 4.9. Assume S is not included in $C_K^\infty(\mathbb{R})$ for any compact K. In particular, this means that there exists a sequence (f_n) of elements of S and a sequence (x_n) in $\mathbb{R}$ without accumulation points such that $f_n(x_n) \neq 0$ for all $n \in \mathbb{N}$. Define

$$\eta(f) = \sup_{n \in \mathbb{N}} \frac{n\,|f(x_n)|}{|f_n(x_n)|}.$$

Then η is continuous, because given $K \subseteq \mathbb{R}$ compact, there exists $\alpha_K \in \mathbb{R}$ such that $\eta(f) \leq \alpha_K \sup_{x \in \mathbb{R}} f(x)$ for all $f \in C_K^\infty(\mathbb{R})$ (*home exercise*: verify that this indeed implies continuity of η). Furthermore, $\eta(S)$ is unbounded, because $\eta(f_n) = n$ for any $n \in \mathbb{N}$, as required. $\square$

DEFINITION 4.11. Let V be a topological vector space over $\mathbb{R}$, and V^* its continuous dual. Given $S \subseteq V$ and $\epsilon > 0$, define $U_{S,\epsilon} := \{\xi \in V^* : \forall f \in S,\ |\langle \xi, f \rangle| < \epsilon\}$.

(1) The *weak*(-$*$) topology on V^* is generated by the neighborhood basis at 0 given by the collection $\mathcal{B}_w := \{U_{\epsilon,S} : \epsilon > 0 \text{ and } S \text{ finite}\}$.
(2) The *strong* topolgy has as a neighborhood basis at 0 the collection $\mathcal{B}_s := \{U_{\epsilon,S} : \epsilon > 0 \text{ and } S \text{ bounded}\}$.

EXERCISE 4.12. *Show that the space $C_c^{-\infty}(\mathbb{R})$, of compactly supported distributions, is dense in $C^{-\infty}$ with respect to both the weak and strong topology.*

SOLUTION. Since the weak topology is coarser than the strong topology, it suffices to show that $C_c^{-\infty}(\mathbb{R})$ is strongly dense in $C^{-\infty}(\mathbb{R})$. This follows from a previous exercise.

Given $S \subseteq C_c^\infty(\mathbb{R})$ bounded, there exists $K \subseteq \mathbb{R}$ compact such that $S \subseteq C_K^\infty(\mathbb{R})$. Then, given $\xi \in C^{-\infty}(\mathbb{R})$, we have that $\xi\,|_S \equiv (\xi \cdot I_K)_S$, where I_K is the indicator of K, so that $\xi \cdot I_K \in C_c^{-\infty}(\mathbb{R})$ is an element of $\xi + U_{S,\epsilon}$ for all $\epsilon > 0$. $\square$

EXERCISE 4.13. *Prove that the inclusion $f \mapsto \xi_f : C_c^\infty(\mathbb{R}) \to C^{-\infty}(\mathbb{R})$ is dense with respect to both the weak and the strong topology.*

Remark 4.14. We have proved the density with respect to the weak topology in the first tutorial. Since the weak topology is coarser than the strong topology, this exercise gives us an alternative proof of this fact.

Solution. By the previous exercise, it would suffice to show that this inclusion is dense in $C_c^{-\infty}(\mathbb{R})$, the space of compactly supported distributions.

Recall that, given $\xi \in C^{-\infty}$ and $\psi \in C_c^{\infty}(\mathbb{R})$, we defined $\xi * \psi(x) = \langle \xi, L_x \bar{\psi} \rangle$. Moreover, assuming $\mathrm{Supp}(\xi)$ is compact, we have that $\mathrm{Supp}(\xi * \psi) \subseteq \mathrm{Supp}(\xi) + \mathrm{Supp}(\psi)$ and hence $\xi * \psi \in C_c^{\infty}(\mathbb{R})$. Finally, by definition, we have that $\xi_{\xi * \psi} = \xi * \xi_\psi$ (here the RHS is the distribution of two compactly supported distributions), i.e.

$$\langle \xi * \xi_\psi, f \rangle = \langle \xi, \overline{\xi_\psi * \overline{f}} \rangle = \int_{\mathbb{R}} \xi * \psi(x) f(x) dx.$$

For further discussion, see [**1**, p. 109].

Let $\xi \in C_c^{-\infty}(\mathbb{R})$, and let ψ_n be an approximation of unity with $\mathrm{Supp}\psi_n \subseteq [-1/n, 1/n]$. We will show that $\xi * \xi_{\psi_n} \to \xi$ both in the weak[1] and the strong topology.

- Weakly: We need only to show that $\langle \xi * \xi_{\psi_n}, f \rangle \to \langle \xi, f \rangle$ as $n \to \infty$. This holds since $\langle \xi * \xi_{\psi_n}, f \rangle = \langle \xi, \overline{\psi_n * \overline{f}} \rangle$ and $\psi_n * f$ converges to f in $C_c^{\infty}(\mathbb{R})$ (home exercise: complete the details of the proof; use the fact that $\psi_n * f$ is supported on $[1, 1] + \mathrm{Supp}(f)$ and $(\psi_n * f)^{(k)} = \psi_n * f^{(k)}$ converges to $f^{(k)}$ uniformly).
- Strongly: This is a bit harder; it essentially uses the fact that the $(k+1)$-st derivative gives bounds on the value of the k-derivative (Lagrange Mean Value Theorem). Fix a set $U_{S,\epsilon} = \{\xi : \forall f \in S, |\langle \xi, f \rangle| < \epsilon\}$ as above with $S \subseteq C_c^{\infty}(\mathbb{R})$ bounded and $\epsilon > 0$. We need to prove $\xi * \psi_n - \xi \in U_{S,\epsilon}$ for all but finitely many n's.

  Put $\xi_n = \xi * \psi_n$ and $\|f\|_k = \sup_{x \in \mathbb{R}} \left| f^{(k)} \right|$. By continuity of ξ, we have that

  $$|\langle \xi - \xi_n, f \rangle| = \left| \langle \xi, f - \overline{\psi_n * \overline{f}} \rangle \right| \leq C \left\| f - \overline{\psi_n * \overline{f}} \right\|_k$$

  for some k and $C > 0$.

  Using Lagrange's MVT, we have that

  $$\left| f^{(k)}(x) - \overline{\psi_n * \overline{f^{(k)}}}(x) \right| = \left| \int_{-1/n}^{1/n} (f^{(k)}(x) - f^{(k)}(x+t)) \psi_n(t) dt \right|$$
  $$\leq \sup_{|t| \leq 1/n} \left| f^{(k)}(x) - f^{(k)}(x+t) \right| \leq \sup_{|t| \leq 1/n} \left| f^{(k+1)}(c) t \right| \leq \frac{\|f\|_{k+1}}{n}$$

  for some $c \in [x, x+t]$. By boundedness of S, there exists $\lambda > 0$ such that $S \subseteq \lambda B_{\|\cdot\|_{k+1}}(0,1)$. In particular, if $f \in S$ we have that $\|f\|_{k+1} \leq \lambda$, so that

  $$\left| f^{(k)}(x) - \overline{\psi_n * \overline{f^{(k)}}}(x) \right| \leq \frac{\lambda}{n}.$$

  In particular, taking $n \gg 0$, we have that the RHS is smaller than ϵ for all $f \in S$, and hence $\xi_n \in U_{S,\epsilon}$.

□

[1] As already mentioned, proving weak convergence is superfluous, but it is very simple so we'll do it anyway

References

[1] Kanwal, Ram P., *Generalized functions, Theory and applications*, Birkhäuser Boston, Inc., Boston, MA, 2004