Generalized functions
Exercise sheet 1

Exercise 1. Fix $1 \leq p < \infty$.

(a) Show that $L^1(\mathbb{R}) * L^p(\mathbb{R}) \subseteq L^p(\mathbb{R})$.

(b) Show that compactly supported bounded functions form a dense subset of $L^p(\mathbb{R})$, with respect to $\|\cdot\|_p$. *Hint*: Given $f \in L^p(\mathbb{R})$, consider the functions $f_n = f \cdot I_{\{x:|x|<n\, f(x)<n\}}$, where $I_\bullet$ denotes the indicator function.

(c) Prove that $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, with respect to $\|\cdot\|_p$.

(d) ($*$) Is the inclusion $C_c(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ continuous with respect to the uniform convergence topology on the domain?

Solution.

(1) Given $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ we have that

$$\begin{aligned}\int_{\mathbb{R}} |f * g(x)|^p \, dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t)g(x-t)dt \right|^p dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t)|^p \, |g(x-t)|^p \, dx dt \int_{\mathbb{R}} \|g\|_p^p \, |f(t)| \, dt = \|g\|_p^p \, \|f\|_1\end{aligned}$$

where the inequality is justified by Minkowski integral inequality.

(2) Given $f \in L^p(\mathbb{R})$ the function $f_n = f \cdot I_{\{x:|x|<n,\, |f(x)|<n\}}$ is clearly bounded and supported on a subset of $[-n, n]$, hence compactly supported. We only need to prove $f_n \to f$ in the L^p-norm. Note that, for any $n \in \mathbb{N}$ we have that $|f - f_n|^p \leq (|f| + |f_n|)^p \leq 2^p \, |f|^p$, and hence, by Dominated Convergence,

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n|^p \, dx = \int_{\mathbb{R}} \lim_{n\to\infty} |f - f_n| \, dx = 0.$$

(3) By the previous item, it suffices to show that $C_c^\infty(\mathbb{R})$ is dense in the space of compactly supported bounded functions with the L^p-norm. Let f be compactly supported and bounded, and let, for any $n \in \mathbb{N}$, $\chi_n \in C_c^\infty(\mathbb{R})$ be non-negative with $\mathrm{Supp}(\chi_n) \subseteq [-1/n, 1/n]$ and $\int_{\mathbb{R}} |\chi_n(x)| \, dx = 1$. Consider $f_n = f * \chi$. Note that, for any $n \in \mathbb{N}$,

$$\begin{aligned}|f(x) - f_n(x)|^p &= \\ \left| \int_{-1/n}^{1/n} (f(x) - f(x-t))\chi_n(t) dt \right|^p &\leq \int_{-1/n}^{1/n} |f(x) - f(x-t)|^p \, |\chi_n(t)|^p \, dt \\ &\leq 2^p \sup |f|^p \, \|\chi_n\|_p < \infty.\end{aligned}$$

Therefore, by dominated convergence, $\lim_{n\to\infty} \|f - f_n\|_p^p = \int_{\mathbb{R}} \lim_{n\to\infty} |f - f_n|^p \, (x) dx = 0$.

(4) The sequence $(2^{-n/p} I_{[-2^n, 2^n]})_{n=1}^\infty$ converges uniformly to the zero function, but not in the L^p-norm. Thus, the inclusion is not continuous.

$\square$

Exercise 2. Compute the generalised function $x^n \delta_0^{(m)}$, for any $n, m \in \mathbb{N}$.

Solution. Given $f \in C_c^\infty(\mathbb{R})$, we have that

$$\langle x^n \delta_0^{(m)}, f \rangle = \langle \delta_0^{(m)}, x^n f \rangle = (-1)^m \langle \delta_0, (x^n f)^{(m)} \rangle.$$

Using the formula (proved, e.g., by induction on m)

$$(x^n f)^{(m)} = \sum_{j=0}^{\min\{m,n\}} \binom{m}{j} \frac{n!}{(n-j)!} x^{n-j} f^{(m-j)}(x)$$

we deduce that

$$\langle x^n \delta_0^{(m)}, f \rangle = \begin{cases} 0 & m < n \\ \frac{m!}{(m-n)!} f^{(m-n)}(0) & \text{if } m \geq n. \end{cases}$$

□

Exercise 3. (*) We showed in class that given $f, g \in C_c^\infty(\mathbb{R})$, we have an inclusion $\operatorname{Supp}(f * g) \subseteq \operatorname{Supp}(f) + \operatorname{Supp}(g)$. Find an example of $f, g \in C_c^\infty(\mathbb{R})$ for which this inclusion is strict.

Solution. Let $\varphi \in C_c^\infty(\mathbb{R})$ to be an *even* function such that $\varphi(x) = 1$ if $|x| < 1$, $\varphi(x) > 0$ for $1 \leq |x| \leq 2$ and $\varphi(x) = 0$ otherwise (using the smooth version of Urysohn's lemma to find φ), and put $f(x) = x \cdot \varphi(x)$ and $g(x) = \varphi(\frac{x}{3})$ then

$$\operatorname{Supp}(f) = [-2, 2],\ \operatorname{Supp}(g) = [-6, 6],\ \text{and} \operatorname{Supp}(f) + \operatorname{Supp}(g) = [-8, 8],$$

but, for any $t \in (-1, 1)$

$$\begin{aligned} (f * g)(t) &= \int_{-6}^{6} f(t-x)g(x)dx \\ &= \underbrace{\int_{-6}^{-3} f(t-x)g(x)dx}_{t-x>2} + \underbrace{\int_{-3}^{3} f(t-x)g(x)dx}_{g(x)=1} + \underbrace{\int_{3}^{6} f(t-x)g(x)dx}_{t-x<-2}. \end{aligned}$$

Noting that the first and third integral vanish, since $t - x \notin \operatorname{Supp}(f)$, and that the second integral vanishes since $\{t - x : x \in (-3, 3)\} \supseteq [-2, 2]$ for any $t \in (-1, 1)$, and f is an odd function. □

Exercise 4. Let $\xi \in C^{-\infty}(\mathbb{R})$. Given $U \subseteq \mathbb{R}$, the notation $\xi|_U \equiv 0$ means $\langle \xi, f \rangle = 0$ for all $f \in C_c^\infty(U)$.

(1) Let $U_1, U_2 \subseteq \mathbb{R}$ be open. Show that if $\xi|_{U_1} \equiv \xi|_{U_2} \equiv 0$ the $\xi|_{U_1 \cup U_2} \equiv 0$.
(2) Show that if $\{U_\alpha\}_{\alpha_I}$ is a collection of arbitrary cardinality of open subsets of $\mathbb{R}$ with compact closures and $\xi|_{U_\alpha} \equiv 0$ for all $\alpha \in I$, then $\xi|_{\bigcup_\alpha U_\alpha} \equiv 0$.

Solution. (1) We'll prove the claim for $U = U_1 \cup \cdots \cup U_n$ a finite union of open sets. We saw in class that, in this situation, there exist non-negative functions $\rho_1, \ldots, \rho_n \in C_c^\infty(\mathbb{R})$ with $\operatorname{Supp}(\rho_i) \subseteq U_i$, for all i, such that $(\sum_{i=1}^n \rho_i)|_U \equiv 1$. Let $f \in C_c^\infty(U)$, and put $f_i = f \cdot \rho_i$ for $i = 1, \ldots, n$. Then $f = f_1 + \cdots + f_n$ and $\operatorname{Supp}(f_i) \subseteq U_i$. Additionally

$$\langle \xi, f \rangle = \sum_{i=1}^{n} \langle \xi, f_i \rangle = \sum_{i=1}^{n} \langle \xi|_{U_i}, f_i \rangle = 0.$$

That is, ξ vanishes on all elements of $C_c^\infty(U)$, and hence $\xi|_U \equiv 0$.

(2) Let $f \in C_c^\infty(U)$. As $\operatorname{Supp}(f)$ is compact, there exist $\alpha_1, \ldots, \alpha_n$ such that $\operatorname{Supp}(f) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ and hence, for $U' = \bigcup_{i=1}^n U_{\alpha_i} \subseteq U$, we have that $f \in C_c^\infty(U')$ and $\xi|_{U'} \equiv 0$. By the previous item, it follows that $\langle \xi, f \rangle = 0$.

□

Exercise 5.

(1) Given $D \subseteq \mathbb{R}$ compact and $k \geq 0$, the C^k-norm on $C^k(D)$ is defined by $\|f\|_{C^k} := \sup_{x \in D} \sum_{i=0}^k \left\| f^{(i)}(x) \right\|$. Show that a functional $\xi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ is continuous if and only if there exists $k \geq 0$ and $c > 0$ such that

$$|\langle \xi, f \rangle| \leq c \|f\|_{C^k} \quad \text{for all } f \in C^\infty(D).$$

Deduce that for any $\xi \in C^{-\infty}(\mathbb{R})$ and $D \subseteq \mathbb{R}$ compact, there exists $k_D \geq 0$ and $c_D > 0$ such that $|\langle \xi, f \rangle| \leq c_D \|f\|_{C^{k_D}}$, whenever $\operatorname{Supp}(f) \subseteq D$.

(2) Let ξ be a distribution supported on $\{0\}$.
   (a) Let $\psi \in C_c^\infty(\mathbb{R})$ with $\operatorname{Supp}(\psi) \subseteq [-1, 1]$ and such that $\psi|_U \equiv 1$ for some open $U \ni 0$. Show that $\langle \xi, g \rangle \leq c \sup_{x \in [-1,1]} \sum_{i=1}^k \left| (g\psi)^{(i)}(x) \right|$, for all $g \in C_c^\infty(\mathbb{R})$, for suitable $k \geq 0$ and $c > 0$.
   (b) Prove that there exist $k \geq 0$ such that $x^k \xi \equiv 0$.
   Hint: Use the previous item to bound the value $\langle x^k \xi, f \rangle = \langle \xi, x^k f \rangle$ using test functions of the form $\psi_\epsilon(x) = \psi(\epsilon^{-1} x)$, with ψ as above.

SOLUTION. (1) $\Leftarrow$ Assume the assumption holds for any compact set $D \subseteq \mathbb{R}$, and let $(f_n)_n$ be a sequence in $C_c^\infty(\mathbb{R})$ converging to $f \in C_c^\infty(\mathbb{R})$. By definition, there exits a compact $D \subseteq \mathbb{R}$ such that $\mathrm{Supp}(f_n), \mathrm{Supp}(f) \subseteq D$ and $\left\| f_n^{(k)} - f^{(k)} \right\|_{C^0} \xrightarrow{n\to\infty} 0$ (i.e. uniform convergence) for all $k \in \mathbb{Z}_{\geq 0}$. Let $c_D > 0$ and $k_D \geq 0$ be as in the assumptions. Then

$$|\langle \xi, f_n - f\rangle| \leq c_D \cdot \|f - f_n\|_{C^{k_D}} \leq \sum_{i=0}^{k_D} \left\| f_n^{(i)} - f^{(i)} \right\|_{C^0} \xrightarrow{n\to\infty} 0,$$

where the last equality is justified since k_D is a fixed finite number, and the penultimate equality holds since

$$\|\varphi\|_{C^k} = \sup_{x\in D} \sum_{i=0}^{k} \left|\varphi^{(i)}(x)\right| \leq \sum_{i=0}^{k} \sup_{x\in D} \left|\varphi^{(i)}(x)\right| = \sum_{i=0}^{k} \left\|\varphi^{(i)}\right\|_{C^0},$$

for any $\varphi \in C^\infty(D)$ and $k \geq 0$.

$\Rightarrow$ Assume ξ is continuous but does not satisfy the assumption, and let $D \subseteq \mathbb{R}$ be a compact set for which there exist no c_D and k_D such that $|\langle \xi, f\rangle| \leq c_D \|f\|_{C^{k_D}}$ for all $f \in C^\infty(D)$. In particular, for any $n \in \mathbb{N}$, there exits $\varphi_n \in C^\infty(D)$ such that $|\langle \xi, \varphi_n\rangle| > n \|\varphi_n\|_{C^n}$. Note that multiplying φ_n by a constant results in multiplying both hands of the inequality by the absolute value of the same constant. Thus, we lose no generality by replacing φ_n with $\frac{1}{\|\varphi_n\|_{C^n}} \cdot \varphi_n$, thereby assuming $\|\varphi_n\|_{C^n} = \frac{1}{n}$. Thus, we have that $|\langle \xi, \varphi_n\rangle| > 1$ for all $n \in \mathbb{N}$. On the other hand, all φ_n's are supported on D, and one easily verifies that $\lim_{n\to\infty} \varphi_n = 0$ in $C_c^\infty(\mathbb{R})$. This contradicts the continuity of ξ.

(2) (a) Follows immediately from the previous item, applied to $D = [-1, 1]$ (note that $\mathrm{Supp}(g\psi) \subseteq [-1, 1]$ for all $g \in C_c^\infty(\mathbb{R})$).

(b) In the notation of the hint, let ψ be as in the previous item and put $\psi_\epsilon(\epsilon^{-1}x)$, for any $\epsilon > 0$. Note that $\mathrm{Supp}(\psi) \subseteq [-\epsilon, \epsilon]$ and $\psi_\epsilon \mid_{\epsilon U} \equiv 1$ for a given open set $0 \in U$.(Note that, given $f \in C_c^\infty(\mathbb{R})$, we have that, for any $\epsilon > 0$, $f - f \cdot \psi_\epsilon$ vanishes in a neighborhood of 0, and hence, since $\mathrm{Supp}(\xi) = 0$, we have that $\langle \xi, f - f \cdot \psi_\epsilon\rangle = 0$, i.e. $\langle \xi, f\rangle = \langle \xi, f\psi_\epsilon\rangle$.

Let $c > 0$ and $k \geq 0$ be as in item (a), and let us compute $\langle x^d\xi, f\rangle$ for $d \geq 0$ and an arbitrary function $f \in C_c^\infty(\mathbb{R})$. For $\epsilon > 0$ arbitrary, we have that

$$\begin{aligned}
\left|\langle x^d\xi, f\rangle\right| &= \left|\langle \xi, x^d f\rangle\right| = \left|\langle \xi, x^d\psi_\epsilon f\rangle\right| \\
&\leq c \sup_{x\in[-1,1]} \sum_{i=0}^{k} \left|(x^d\psi_\epsilon f)^{(i)}\right| \\
&= c \sup_{x\in[-1,1]} \sum_{i=0}^{k} \left| \sum_{\substack{j_1,j_2,j_3\geq 0\\ j_1+j_2+j_3=i}} \binom{i}{j_1, j_2, j_3} (x^d)^{(j_1)} \psi_\epsilon^{(j_2)} f^{(j_3)} \right| \\
&= c \sup_{x\in[-1,1]} \sum_{i=0}^{k} \left| \sum_{\substack{j_1,j_2,j_3\geq 0,\, j_1\leq k\\ j_1+j_2+j_3=i}} \binom{i}{j_1, j_2, j_3} \frac{k!}{j_1!} x^{d-j_1} \epsilon^{-j_2} \psi^{(j_2)}(\epsilon^{-1}x) f^{(j_3)} \right|
\end{aligned}$$

noting that the entire expression vanishes if $|x| > \epsilon$, we have the addtional upper bound

$$\leq c \sum_{i=0}^{k} \sum_{\substack{j_1,j_2,j_3\geq 0,\, j_1\leq k\\ j_1+j_2+j_3=i}} \binom{i}{j_1, j_2, j_3} \frac{k!}{j_1!} \epsilon^{d-j_1} \epsilon^{-j_2} \sup_{x\in[-\epsilon,\epsilon]} \left|\psi^{(j_2)}(\epsilon^{-1}x) f^{(j_3)}\right|.$$

Since the tem in absolute value is bounded above, say, by $M > 0$, we deduce that

$$\left|x^k\xi, f\right| \leq c \sum_{i=0}^{k} \sum_{\substack{j_1,j_2,j_3\geq 0,\, j_1\leq k\\ j_1+j_2+j_3=i}} \binom{i}{j_1, j_2, j_3} \frac{k!}{j_1!} \epsilon^{d-j_1-j_2} M.$$

Taking $d > k$, and recalling that $\epsilon > 0$ may be taken to be arbitrarily small, we deduce that $\langle x^d\xi, f\rangle = 0$.

□

Exercise 6. Let $\xi \in C^{-\infty}(\mathbb{R})$ and $f \in C^{\infty}(\mathbb{R})$. Prove the following assertions.

(1) If f has compact support then $\xi * f$ smooth.
(2) If ξ has compact support then $\xi * f$ is smooth.

SOLUTION. Recall the notation:

$$L_t f(x) = f(x+t) \quad \text{and} \quad \overline{f}(x) = f(-x)$$

for $f \in C(\mathbb{R})$ and $t \in \mathbb{R}$. By definition, we have that

$$(\xi * f)(x) = \langle \xi, \overline{L_{-x}\overline{f}} \rangle \quad \text{for any } x \in \mathbb{R}.$$

(1) Given a function $f \in C_c^{\infty}(\mathbb{R})$, say with $K = \mathrm{Supp}(f)$, we have that $f^{(k)} \in C_K^{\infty}(\mathbb{R})$ for all $k \geq 0$. Furthermore, by the definition of the derivative, for any $k \geq 0$, the net $\left\{ f_\epsilon^{(k)} = \frac{L_\epsilon f^{(k)} - f^{(k)}}{\epsilon} \right\}_{\epsilon \in (0,1)}$ lies in $C_c^{\infty}(\mathbb{R})$ (with support contained in $K + [0,1]$) converges in the supremum norm to $f^{(k+1)}$ as ϵ tends to 0. In particular, the convergence $f_\epsilon \xrightarrow{\epsilon \to 0} f^{(1)}$ is in the topology of $C_c^{\infty}(\mathbb{R})$ and hence, by definition of convolution of a distribution and a function

$$(\xi * f)'(x) = \lim_{\epsilon \to 0} \frac{\xi * f(x+\epsilon) - \xi * f(x)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\langle \xi, \overline{L_{-x-\epsilon}\overline{f}} \rangle - \langle \xi, \overline{L_{-x}\overline{f}} \rangle}{\epsilon}$$
$$= \lim_{\epsilon \to 0} \langle \xi, \frac{\overline{L_\epsilon(\overline{L_{-x}\overline{f}})} - \overline{L_{-x}\overline{f}}}{\epsilon} \rangle = \langle \xi, \overline{(\overline{L_{-x}\overline{f}})'} \rangle = \langle \xi, \overline{(\overline{L_{-x} f})'} \rangle = \langle \xi, \overline{L_{-x}(\overline{f'})} \rangle = (\xi * f')(x).$$

Since we can apply the same argument for each of the derivatives of f, we deduce that $\xi * f$ has continuous derivatives of all orders, and hence is smooth.

(2) Let $D = \mathrm{Supp}(\xi)$ and let $\psi \in C_c^{\infty}(\mathbb{R})$ be a smooth non-negative function with $\psi\mid_K \equiv 1$ for $K = D + [-1,1]$. We show that the equality $(\xi * f)' = \xi * (f')$ holds in this situation as well.

Note the following two facts:

(a) For any $g \in C^{\infty}(\mathbb{R})$ we have that $\langle \psi\xi, g \rangle = \langle \xi, \psi g \rangle = \langle \xi, g \rangle$. This holds since $g - \psi g$ is supported on the complement of $\mathrm{Supp}(\xi)$. [*In fact, this is the definition of* $\langle \xi, g \rangle$.]

(b) Writing $f_\epsilon = \frac{L_\epsilon f - f}{\epsilon}$ as above, for any $\psi \in C_c^{\infty}(\mathbb{R})$, the net $\psi \cdot f_\epsilon$ is contained in $C_c^{\infty}(\mathbb{R})$ and converges to $\psi \cdot f'$ as ϵ tends to 0.

Using these two facts, we have that

$$(\xi * f)'(x) = \lim_{\epsilon \to 0} \frac{\langle \xi, \overline{L_{-x-\epsilon}\overline{f}} \rangle - \langle \xi, \overline{L_{-x}\overline{f}} \rangle}{\epsilon} = \lim_{\epsilon \to 0} \langle \xi, \psi \cdot \frac{\overline{L_{-x-\epsilon}\overline{f}} - \overline{L_{-x}\overline{f}}}{\epsilon} \rangle$$
$$= \langle \xi, \psi \cdot (\overline{L_{-x}\overline{f'}}) \rangle = \langle \xi, \overline{L_{-x}\overline{f'}} \rangle = (\xi * f')(x).$$

□

Exercise 7. Let A be a differential operator with constant coefficients.

(1) Describe the Green function (G_A such that $A(G_A) = \delta_0$) without using generalized functions.
(2) Set

$$A_{G_A}(g)(y) = \int_{-\infty}^{\infty} G_A(x-y) g(x) dx.$$

Show that $A(A_{G_A}(g)) = g$ for every $g \in C_c^{\infty}(\mathbb{R})$.

SOLUTION. (1) Write $A = \sum_{j=0}^{n} c_j \frac{d^j}{dx^j}$ for $c_0, \ldots, c_n \in \mathbb{R}$. By definition the Green function $G = G_A$ is the solution of the initial value problem

$$AG = \sum_{j=0}^{n} c_j G^{(j)}(x) = 0 \quad \text{for all } x \neq 0$$

and with all derivatives of order up to $n-2$ being continuous at zero with value 0, and $\lim_{\epsilon\to 0^+}(G^{(n-1)}(\epsilon)-G^{(n-1)}(-\epsilon)) = \frac{1}{c_n}$. Indeed,

$$\begin{aligned}
1 = \int_{\mathbb{R}} \delta_0(x)dx &= \int_{\mathbb{R}} \sum_{j=0}^{n} c_j \frac{d^j}{dx^j} c_j G^{(j)}(x)dx \\
&= \lim_{\epsilon\to 0^+} \left(\int_{-\infty}^{-\epsilon} \cdots dx + \int_{-\epsilon}^{\epsilon} \cdots dx \int_{\epsilon}^{\infty} \cdots dx \right) \\
&= \sum_{j=0}^{n} c_j \lim_{\epsilon\to 0^+} \int_{-\epsilon}^{\epsilon} G^{(j)}(x)dx \\
&= \lim_{\epsilon\to 0^+} c_n(G^{(n-1)}(\epsilon) - G^{(n-1)}(-\epsilon)).
\end{aligned}$$

REMARK 1. *Explicit description of the Green function may be obtained using standard methods for solving the homogeneous equation $AG = 0$ in the domain $\mathbb{R}\setminus\{0\}$. This was not expected as part of the solution, but is a welcome addition.*

(2)

$$\begin{aligned}
A(A_{G_A}(g))(y) = A\left(\int_{\mathbb{R}} G_A(y-x)g(x)\right) &= \sum_{j=0}^{n} \int_{\mathbb{R}} c_j \frac{d^j}{dy^j}(G_A(y-x)g(x))dx \\
&= \sum_{j=0}^{n} c_j \lim_{\epsilon\to 0^+} \int_{y-\epsilon}^{y+\epsilon} G_A^{(j)}(y-x)g(x)dx = g(y),
\end{aligned}$$

where the final equality may be proved, e.g., using the intermediate value theorem on the interval $[y-\epsilon, y+\epsilon]$, as $\epsilon \to 0$.

$\square$