Generalized functions
Exercise sheet 1

Solve the following exercises. Exercises marked with $*$ are optional. A good additional reference is Terence Tao's notes, available at: `https://terrytao.wordpress.com/2009/04/19/245c-notes-3-distributions/`

Exercise 1. Fix $1 \leq p < \infty$.

(a) Show that $L^1(\mathbb{R}) * L^p(\mathbb{R}) \subseteq L^p(\mathbb{R})$.

(b) Show that compactly supported bounded functions form a dense subset of $L^p(\mathbb{R})$, with respect to $\|\cdot\|_p$. *Hint*: Given $f \in L^p(\mathbb{R})$, consider the functions $f_n = f \cdot I_{\{x:|x|<n\, f(x)<n\}}$, where $I_\bullet$ denotes the indicator function.

(c) Prove that $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, with respect to $\|\cdot\|_p$.

(d) $(*)$ Is the inclusion $C_c(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ continuous with respect to the uniform convergence topology on the domain?

Exercise 2. Compute the generalised function $x^n \delta_0^{(m)}$, for any $n, m \in \mathbb{N}$.

Exercise 3. $(*)$ We showed in class that given $f, g \in C_c^\infty(\mathbb{R})$, we have an inclusion $\mathrm{Supp}(f * g) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(g)$. Find an example of $f, g \in C_c^\infty(\mathbb{R})$ for which this inclusion is strict.

Exercise 4. Let $\xi \in C^{-\infty}(\mathbb{R})$. Given $U \subseteq \mathbb{R}$, the notation $\xi\mid_U \equiv 0$ means $\langle \xi, f\rangle = 0$ for all $f \in C_c^\infty(U)$.

(1) Let $U_1, U_2 \subseteq \mathbb{R}$ be open. Show that if $\xi\mid_{U_1} \equiv \xi\mid_{U_2} \equiv 0$ the $\xi\mid_{U_1 \cup U_2} \equiv 0$.

(2) Show that if $\{U_\alpha\}_{\alpha_I}$ is a collection of arbitrary cardinality of open subsets of $\mathbb{R}$ with compact closures and $\xi\mid_{U_\alpha} \equiv 0$ for all $\alpha \in I$, then $\xi\mid_{\bigcup_\alpha U_\alpha} \equiv 0$.

Exercise 5.

(1) Given $D \subseteq \mathbb{R}$ compact and $k \geq 0$, the C^k-norm on $C^k(D)$ is defined by $\|f\|_{C^k} := \sup_{x\in D} \sum_{i=1}^k \left\|f^{(i)}(x)\right\|$. Show that a functional $\xi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ is continuous if and only if there exists $k \geq 0$ and $c > 0$ such that

$$|\langle \xi, f\rangle| \leq c\, \|f\|_{C^k} \quad \text{for all } f \in C^\infty(D).$$

Deduce that for any $\xi \in C^{-\infty}(\mathbb{R})$ and $D \subseteq \mathbb{R}$ compact, there exists $k_D \geq 0$ and $c_D > 0$ such that $|\langle \xi, f\rangle| \leq c_D \|f\|_{C^{k_D}}$, whenever $\mathrm{Supp}(f) \subseteq D$.

(2) Let ξ be a distribution supported on $\{0\}$.

(a) Let $\psi \in C_c^\infty(\mathbb{R})$ with $\mathrm{Supp}(\psi) \subseteq [-1, 1]$ and such that $\psi\mid_U \equiv 1$ for some open $U \ni 0$. Show that $\langle \xi, g\rangle \leq c \sup_{x\in[-1,1]} \sum_{i=1}^k \left|(g\psi)^{(i)}(x)\right|$, for all $g \in C_c^\infty(\mathbb{R})$, for suitable $k \geq 0$ and $c > 0$.

(b) Prove that there exist $k \geq 0$ such that $x^k \xi \equiv 0$.
Hint: Use the previous item to bound the value $\langle x^k \xi, f\rangle = \langle \xi, x^k f\rangle$ using test functions of the form $\psi_\epsilon(x) = \psi(\epsilon^{-1}x)$, with ψ as above.

Exercise 6. Let $\xi \in C^{-\infty}(\mathbb{R})$ and $f \in C^\infty(\mathbb{R})$. Prove the following assertions.

(1) If f has compact support then $\xi * f$ smooth.

(2) If ξ has compact support then $\xi * f$ is smooth.

Exercise 7. Let A be a differential operator with constant coefficients.

(1) Describe the Green function (G_A such that $A(G_A) = \delta_0$) without using generalized functions.

(2) Set

$$A_{G_A}(g)(y) = \int_{-\infty}^{\infty} G_A(x, y) g(x)\, dx.$$

Show that $A(A_{G_A}(g)) = g$ for every $g \in C_c^\infty(\mathbb{R})$.