Generalized functions
Exercise sheet 6

Exercises marked with * are optional.

Exercise 1. Let $\varphi : M \to N$ be a smooth map of C^∞-manifolds and let $x \in M$.

(1) Show that if $d_x\varphi$ is injective (resp surjective) then there exists neighbourhoods $x \in U$ and $\varphi(x) \in V$ and diffeomorphisms $\psi : \mathbb{R}^n \to U$ and $\rho : \mathbb{R}^m \to V$ such that $\rho^{-1} \circ \varpi \circ \psi$ is injective (resp surjective).
(2) Show that a proper étale map is a covering map.
(3) Show that a covering map with finite fibers is proper and étale.

Exercise 2. Show that $C^\infty(\mathbb{R}^n, \mathbb{R}^k) = \{f : \mathbb{R}^n \to \mathbb{R}^k : f^*(\mu) := \mu \circ f \in C^\infty(\mathbb{R}^n) \text{ for all } \mu \in C^\infty(\mathbb{R}^k)\}$.

Exercise 3. Recall that, given an n-dimensional vector bundle (E, p) over M and open sets $U, V \subseteq M$ with trivializations $\varphi_U : U \times \mathbb{R}^n \to p^{-1}(U)$ and $\varphi_V : V \times \mathbb{R}^n \to p^{-1}(V)$, the associated *transition functions* $g^E_{U,V} : U \cap V \to \mathrm{GL}_n(\mathbb{R})$ are given by

$$\varphi_V^{-1} \circ \varphi_U(x, v) = (x, g^E_{U,V}(x)v).$$

Let M be a C^∞-manifold and $(E_1, p_1), (E_2, p_2)$ vector bundles over M. Let $M = \bigcup_{\alpha \in I} U_\alpha$ be an open cover trivializing both bundles, and write $g^{E_i}_{\alpha,\beta} := g^{E_i}_{U_\alpha,U_\beta}$ for the associated transition functions.

(1) Show that $E_1 \simeq E_2$ if and only if there exist functions $\psi_\alpha : U_\alpha \to \mathrm{GL}_n(\mathbb{R})$ such that $g^{E_1}_{\alpha,\beta}(x) = \psi_\alpha(x) g^{E_2}_{\alpha,\beta}(x) \psi_\beta(x)^{-1}$ for any $\alpha, \beta \in I$ and $x \in U_\alpha \cap U_\beta$.
(2) * Given an open cover $M = \bigcup_{\alpha \in I} U_\alpha$ and maps $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ which satisfy the cocycle condition
$$g_{\alpha,\alpha} \equiv \mathbf{1}_n \quad \text{and} \quad g_{\beta,\gamma}(x) g_{\alpha,\beta}(x) = g_{\alpha,\gamma}(x) \text{ for all } x \in U_\alpha \cap U_\beta \cap U_\gamma,$$
construct a vector bundle (E, p) over M for which $\bigcup_{\alpha \in I} U_\alpha$ is a trivializing open cover with associated transition functions $\{g_{\alpha,\beta}\}_{\alpha,\beta \in I}$.
(3) Given a functor $F : \mathsf{Vect} \to \mathsf{Vect}$ construct a vector bundle $(F(E_1), q)$ such that $q^{-1}(x) = F(p_1^{-1}(x))$ for all $x \in M$, and whose transition functions with respect to the trivializing cover $M = \bigcup_\alpha U_\alpha$ are $g^{F(E_1)}_{U_\alpha,U_\beta} = F(g^{E_1}_{U_\alpha,U_\beta})$.

Exercise 4. Show that every C^∞ manifold M has a Riemannian metric, i.e. a map

$$\langle \cdot, \cdot \rangle : TM \times TM \to \mathbb{R}$$

on the tangent bundle such that $\langle \cdot, \cdot \rangle_x := \langle \cdot, \cdot \rangle |_{T_xM}$ is an inner-product on T_xM, for any $x \in M$, and such that for any $s, t : M \to TM$ smooth sections, the map $x \mapsto \langle s(x), t(x) \rangle$ is a smooth map $M \to \mathbb{R}$.

Exercise 5.

(1) Let $f \in C(\mathbb{R}^n)$. Show that $f \in C_c^\infty(\mathbb{R}^n)$ if and only if $\|f\|_D = \sup_{x \in \mathbb{R}^n} |D(f)(x)| < \infty$ for any differential operator $D = \sum_{\alpha \in \mathbb{N}_0} g_\alpha \cdot \frac{\partial^\alpha}{\partial x^\alpha}$, with $g_\alpha \in C^\infty(\mathbb{R}^n)$, $g_\alpha = 0$ for all but finitely many α's.
(2) Show that the topology on $C_c^\infty(\mathbb{R}^n)$ is defined using the seminorms $\|\cdot\|_D$.

Exercise 6. * Given a manifold M and a vector bundle E over it show that the following two definitions of the topology on $C_c^\infty(M, E)$ are equivalent:

(1) The quotient toplogy induced from the surjective map
$$\bigoplus_{\alpha \in I} C_c^\infty(U_\alpha, E|_{U_\alpha}) \twoheadrightarrow C_c^\infty(M, E),$$
where $M = \bigcup_{\alpha \in I} U_\alpha$ is a trivializing cover, with $\phi_{U_\alpha} : \mathbb{R}^n \to U_\alpha$ and $E|_{U_\alpha} \simeq \mathbb{R}^n \times \mathbb{R}^k$, and the map $f \mapsto f \circ \varphi_{U_\alpha} : C_c^\infty(U_\alpha, E|_{U_\alpha}) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$ is a homeomorphism.
(2) The topology defined by the seminorms $\|f\|_D = \sum_{x \in M} |D(f)(x)|$, where $D \in \mathrm{Diff}(C^\infty(M, E), C^\infty(M, E))$.

Exercise 7. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ sheaves of abelian groups over X. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

(1) Show that φ defines a group homomorphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ on the stalk over x, for any $x \in X$.
(2) Show that φ is an isomorphism of sheaves if and only if φ_x is an isomorphism for all $x \in X$.