Generalized functions
Tutorial notes

Tutorial 4

4.1. Frćhet spaces.

Definition 4.1. (1) Metrizable space;
(2) S_1 or first countable, if every point has a countable basis;
(3) Fréchet =locally convex complete metrizable tvs.

Example 4.2. $C^\infty(K)$ is Frèchet, but not normed (with the norm of uniform convergence of all derivatives).

Exercise 4.3. *TFAE for a lctvs:*
(1) *metrizability;*
(2) *first countable;*
(3) *countable collection of seminorms*

In particular: complete+either of the above gives Frèchet.

Exercise 4.4. *Show that* $C^\infty(\mathbb{S}^1) \simeq \mathrm{SW}(\mathbb{Z})$*, where* $\mathrm{SW}(\mathbb{Z})$ *denotes the space of rapidly decaying sequences* $\{(x_n)_{n\in\mathbb{Z}} : \lim_{n\to\infty} |x_n|\, n^\alpha < \infty \text{ for all } \alpha \in \mathbb{N}\}$ *indexed by* $\mathbb{Z}$*.*

Solution. Both spaces are Freèchet: $C^\infty)(\mathbb{S}^1)$ is endowed with the countable family of seminorms $\eta_j(f) = \sup_{x\in\mathbb{S}^1} \left|f^{(j)}(x)\right|$ and $\mathrm{SW}(\mathbb{Z})$ with the countable family $\nu_j((c_n)_n) = \sup_{n\in\mathbb{Z}} |n^j c_n|$ (here $j \in \mathbb{N}$), and the Fourier transform map $\mathcal{F} : C^\infty(\mathbb{S}^1) \to \mathrm{SW}(\mathbb{Z})$ defines a bijection of the two spaces. To prove $\mathcal{F}$ is a homeomorphism, we need to prove that both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are bounded with respect to these seminorms, i.e. for any j, there exists $M_1, M_2 > 0$ and $k_1(j), k_2(j)$ such that

$$\nu_j(\mathcal{F}(f)) \le M_1 \eta_{k_1(j)}(f) \quad \text{and} \quad \eta_j(\mathcal{F}^{-1}(c_n)) \le M_2 \eta_{k_2(j)}((c_n)).$$

Recall that $\mathcal{F}$ is given by $\mathcal{F}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-nxi}dx\right)_{n\in\mathbb{Z}}$, with inverse $\mathcal{F}^{-1}((c_n)_{n\in\mathbb{Z}}) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. Let $\eta_j(f) := \sup_{x\in\mathbb{S}^1} \sup \left|f^{(j)}(x)\right|$ and $\nu_j((c_n)_n) = \sup_{n\in\mathbb{Z}} |n^j c_n|$.

Given $f \in C^\infty(\mathbb{S}^1)$ with $\mathcal{F}(f) = (c_n)_{n\in\mathbb{Z}}$, and $k \in \mathbb{Z}$, we have that

$$\begin{aligned}
\left|k^j c_k\right| = \left|\langle f, e^{-ikx}\rangle\right| &= \left|\sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_0^{2\pi} c_n n^j e^{i(n-k)x} dx\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}} c_n n^j e^{i(n-k)x}\right| dx = \frac{1}{2\pi}\int_0^{2\pi} \left|f^{(j)} e^{-ikx}\right| dx \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \sup_{x\in\mathbb{S}^1} \left|f^{(j)}(x)\right| dx = \eta_j(f)
\end{aligned}$$

In particular, we have that $\nu_j(\mathcal{F}) \leq \eta_j(f)$. On the other hand, we have that

$$\eta_j(f) = \sup_{x\in\mathbb{S}^1} \left|f^{(j)}(x)\right| = \sup_{x\in\mathbb{S}^1} \sum_{n\in\mathbb{Z}} c_n (ni)^j e^{inx} \leq$$

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} \left|c_n n^{j+2}\right| \leq \sup_{k\in\mathbb{Z}} \left|k^{j+2} c_k\right| \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = \frac{2\pi^2}{6} \nu_{j+2}((c_n)).$$

□

Theorem 4.5 (Banach-Steinhaus for Frèchet spaces). *Let X be a Frèchet space and Y a normed space, and H a family of continuous linear maps form X to Y. Assume $\sup_{f\in H} \|f(x)\| < \infty$ for any $x \in X$. Then the family H is equicontinuous, i.e. for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(x_1) - f(x_2)\| < \epsilon$ for any $f \in F$ and $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$.*

Exercise 4.6. *Show that $C^{-\infty}(\mathbb{R})$ is weakly sequentially complete.*

Proof. Let (ξ_n) be a weakly Cauchy sequence in $(C_c^\infty(\mathbb{R}))^*$. We have a natural candidate for the limit of (ξ_n), which is defined by

$$\langle \xi, f\rangle := \lim_{n\to\infty} \langle \xi_n, f\rangle,$$

which exists by definition of weakly Cauchy. ξ is clearly linear, but not obviously continuous.

Let $K \subseteq \mathbb{R}$ be a compact set. We consider the restrictions of the ξ_n's and ξ to $C^\infty(K)$. More specifically, we consider $F = \left\{\xi_n |_{C^\infty(K)}, \xi |_{C^\infty(K)}\right\} \subseteq \mathrm{Hom}(C^\infty(K), \mathbb{R})$. By weak convergence, F is pointwise bounded, and hence, by Banach-Steinhaus, it is equicontinuous. Let f_n be a sequence in $C^\infty(K)$, converging to f (note that $C^\infty(K)$ is sequentially complete with respect to the topology of uniform convergence of all derivatives, so $f \in C^\infty(K)$). Then $d(f_n, f) \xrightarrow{n\to\infty} 0$, and hence given $\epsilon > 0$, for any $n \gg 0$ we have that $d(f_n, f) < \delta$ and hence

$$|\langle \xi, f - f_n\rangle| \leq \sup_{\nu\in F} |\langle \nu, f - f_n\rangle| < \epsilon$$

for all $n \gg 0$. □

As seen in the lecture, we have a description of $C_c^\infty(\mathbb{R})$ as a direct limit of Frèchet spaces:

$$C_c^\infty(\mathbb{R}) = \varinjlim_{K\subseteq\mathbb{R} \text{ compact}} C_K^\infty(\mathbb{R}).$$

The following exercise explicitly describes the topology of this space.

Exercise 4.7. *Given $n \in \mathbb{N}$, $k_n \in \mathbb{Z}_{\geq 0}$ and $\epsilon_n > 0$, put*

$$A_{k_n,\epsilon_n} = \left\{ f \in C^\infty(\mathbb{R}) : \mathrm{Supp}(f) \subseteq [-n, n] \text{ and } \sup_{x\in\mathbb{R}} \left|f^{(k_n)}(x)\right| < \epsilon_n \right\},$$

the open ball of radius ϵ_n with respect to $\nu_{k_n}(f) = \sup \left|f^{(k_n)}\right|$ on the space $C_{[-n,n]}^\infty(\mathbb{R})$. Define

$$U_{(k_n,\epsilon_n)_n} := \sum_{n\in\mathbb{N}} A_{k_n,\epsilon_n},$$

where (k_n, ϵ_n) is a sequence of pairs in $\mathbb{Z}_{\geq 0} \times \mathbb{R}_{>0}$.

Show that a sequence (f_n) in $C_c^\infty(\mathbb{R})$ converges to f with respect to the topology generated by the $U_{(k_n,\epsilon_n)}$'s if and only if it converges in the sense defined in the first lecture.

Solution. Let $(f_n)_n$ be a sequence in $C_c^\infty(\mathbb{R})$. Assume $f_n \to f$ in the sense defined previously, so that there exists $K \subseteq \mathbb{R}$ such that $\mathrm{Supp}(f_n) \cup \mathrm{Supp}(f) \subseteq K$ and $f_n^{(k)} \to f^{(k)}$ uniformly on K for all $k \in \mathbb{N}$. In particular, given $k \in \mathbb{N}$ and $\epsilon > 0$, it holds that $\sup_{x\in K} \left| f_n^{(k)}(x) - f^k(x) \right| < \epsilon$ for all but finitely many n's.

Let $U = U_{(k_n,\epsilon_n)}$ be an open set as above, and let n_0 be large enough so that $K \subseteq [-n_0, n_0]$. Then, by the previous paragraph, we have that $\left\| f_m^{(k_{n_0})} - f^{(k_{n_0})} \right\|_\infty < \epsilon_{n_0}$, and thus $f_m - f \in A_{k_{n_0},\epsilon_{n_0}} \subseteq U$ for all but finitely many m's. Therefore, we have that (f_n) converges to f in the given topology.

Conversely, assume we have convegence in topology of f_n to f, and let $k \in \mathbb{N}$ and $\epsilon > 0$ be arbitrary. We want to show $\left\| f_n^{(k)} - f^{(k)} \right\|_\infty < \epsilon$ for all but finitely many n's. To show this, it is (more than) enough to consider the set $U_{(k_n,\epsilon_n)}$, with $k_n = k$ and $\epsilon_n = \epsilon$ for all n, and use the convergence in topology to obtain that all but finitely many elements of the sequence $(f_n - f)$ are in this set and therefore, in particular, satisfy the desired inequality. $\square$

Remark 4.8. One can also show that the topology on $C_c^\infty(\mathbb{R})$ is generated by the convex hulls of the $U_{(k_n,\epsilon_n)}$, thereby giving a direct proof of local convexity.

Exercise 4.9. *Let $S \in C_c^\infty(\mathbb{R})$ be bounded. Then there exists K compact such that $S \subseteq C_K^\infty(\mathbb{R})$.*

Solution. We need the following lemma:

Lemma 4.10. *Let V be a locally convex topological vector space. A set $S \subseteq V$ is bounded if and only if $\eta(S) < \infty$ for any continuous seminorm η on V.*

Proof. If S is bounded and η is a countinuous seminorm with $B = \eta^{-1}(-\infty, 1)$, the open ball around 0 of radius 1, then there exists $\lambda > 0$ such that $S \subseteq \lambda B$ and hence $\eta(S) \subseteq \eta(\lambda B) \subseteq [0, \lambda]$.

Conveserly, if S is bounded with respect to all continuous seminorms, let $0 \in U$ be open and let $0 \in C \subseteq U$ be a OCB set. Put $\lambda = \sup N_C(S) < \infty$, then $S \subseteq \lambda C \subseteq \lambda U$. $\square$

Assume S is not included in $C_K^\infty(\mathbb{R})$ for any compact K. In particular, this means that there exists a sequence (f_n) of elements of S and a sequence (x_n) in $\mathbb{R}$ without accumulation points such that $f_n(x_n) \neq 0$ for all $n \in \mathbb{N}$. Wlog, we may assume $x_n \in [-n, n]$ for all $n \in \mathbb{N}$. Define

$$\eta(f) = \sup_{n\in\mathbb{N}} \frac{n\,|f(x_n)|}{|f_n(x_n)|}.$$

Then η is continuous (home exercise), because $\eta(f) \leq \alpha_n \|f\|_\infty$ for any $f \in C_c^\infty(\mathbb{R})$ for suitable α_n dependent on $\mathrm{Supp} f$, and $\eta(S)$ is unbounded, because $\eta(f_n) = n$ for any $n \in \mathbb{N}$. $\square$

4.2. Topologies on $C^{-\infty}(\mathbb{R})$.

Definition 4.11. Let V be a topologival vector space, and V^* its continuous dual. Given $S \subseteq V$ and $\epsilon > 0$, define $U_{S,\epsilon} := \{\xi \in V^* : \forall f \in S,\ |\langle \xi, f\rangle| < \epsilon\}$.

1. A set $B \subseteq V$ is bounded if for any open set $0 \in U \subseteq V$, there exists $\lambda \in \mathbb{R}$ such that $B \subseteq \lambda U$.
2. The *weak*(-$*$) topology on V^* has as a neighborhood basis at 0 the collection $\mathcal{B}_w := \{U_{\epsilon,S} : \epsilon > 0 \text{ and } S \text{ finite}\}$.

(3) The *strong* topolgy has as a neighborhood basis at 0 the collection $\mathcal{B}_s := \{U_{\epsilon,S} : \epsilon > 0 \text{ and } S \text{ bounded}\}$.

EXERCISE 4.12. *Prove the following variant of the Banach-Steinhaus Theorem.*
Let X be a Frèchet space and Y a normed space, and let H be a family of bounded linear operators. Let $\{\eta_i\}_{i=1}^{\infty}$ be a family of seminorms generating the topology on X. Assume $\sup_{f\in H} \|f(x)\| < \infty$ for all $x \in X$. Then $\sup_{f\in H,\, \eta_i(x)=1} \|f(x)\| < \infty$ for all but finitely many i's.

SOLUTION.

□

EXERCISE 4.13. *Prove that the inclusion $f \mapsto \xi_f : C_c^{\infty}(\mathbb{R}) \to C^{-\infty}(\mathbb{R})$ is dense with respect to the* strong *topology.*

REMARK 4.14. Recall that we considered the weak topology in the first tutorial. The following will give an alternative proof of this fact.

SOLUTION. Let us first show that this inclusion is dense in $C_c^{-\infty}(\mathbb{R})$, the space of compactly supported distributions. Following this we will show that $C_c^{-\infty}(\mathbb{R})$ is dense in $C^{-\infty}(\mathbb{R})$ in both topologies.

Recall that, given $\xi \in C^{-\infty}$ and $\psi \in C_c^{\infty}(\mathbb{R})$, we defined $\xi * \psi(x) = \langle \xi, L_x\bar{\psi}\rangle$. Moreover, assuming Supp(ξ) is compact, we have that $\xi * \psi \in C_c^{\infty}(\mathbb{R})$. Essentially, by design and continuity of ξ, we have that $\xi_{\xi*\psi} = \xi * \xi_{\psi}$ (here the RHS is the distribution of two compactly supported distributions), i.e.

$$\langle \xi * \xi_{\psi}, f\rangle = \langle \xi, \overline{\xi_{\psi} * \overline{f}}\rangle = \int_{\mathbb{R}} \xi * \psi(x) f(x)dx.$$

[FIND REFERENCE.]

Let $\xi \in C_c^{-\infty}(\mathbb{R})$, and let ψ_n be an approximation of unity with Supp$\psi_n \subseteq [-1/n, 1/n]$. We will show that $\xi * \xi_{\psi_n} \to \xi$ both in the weak and the strong topology.

- Weakly: We need only to show that $\langle \xi * \xi_{\psi_n}, f\rangle \to \langle \xi, f\rangle$ as $n \to \infty$. This holds since $\langle \xi * \xi_{\psi_n}, f\rangle = \langle \xi, \overline{\psi_n * \overline{f}}\rangle$ and $\psi_n * f$ converges to f in $C_c^{\infty}(\mathbb{R})$ (proof uses that $\psi_n * f$ is supported on $[1, 1] + \text{Supp}(f)$ and $(\psi_n * f)^{(k)} = \psi_n * f^{(k)}$ converges to $f^{(k)}$ uniformly).
- Strongly: This is a bit harder; it essentially uses the fact (Lagrange MVT) that the $k+1$-th derivative gives bounds on the value of the k-derivative. Fix a set $U_{S,\epsilon} = \{\xi : \forall f \in S, |\langle \xi, f\rangle| < \epsilon\}$ as above with $S \subseteq C_c^{\infty}(\mathbb{R})$ bounded. We need to prove $()\xi * \psi_n - \xi) \in U_{S,\epsilon}$ for all but finitely many n's.

  Let's set up some notation: $\xi_n = \xi * \psi_n$ and $\|f\|_k = \sup_{x\in\mathbb{R}} \left|f^{(k)}\right|$. By continuity of ξ, we have that

  $$|\langle \xi - \xi_n, f\rangle| = \left|\xi, f - \overline{\psi_n * \overline{f}}\right| \le C \left\|f - \overline{\psi_n * \overline{f}}\right\|_k$$

  for some k and $C > 0$.

  Using Lagrange's MVT, we have that

$$\left|f^{(k)}(x) - \overline{\psi_n * \overline{f^{(k)}}}(x)\right| = \int_{-1/n}^{1/n} (f^{(k)}(x) - f^{(k)}(x+t))\psi_n(t)dt$$
$$\le \sup_{|t|\le 1/n} \left|f^{(k)}(x) - f^{(k)}(x+t)\right| \le \sup_{|t|\le 1/n} \left|f^{(k+1)}(c)t\right| \le \frac{\|f\|_{k+1}}{n}$$

for some $c \in [x, x+t]$. By boundedness of S, there exists $\lambda > 0$ such that $S \subseteq \lambda B_{\|\cdot\|_{k+1}}(0,1)$. In particular, if $f \in S$ we have that $\|f\|_{k+1} \leq \lambda$, so that

$$\left| f^{(k)}(x) - \overline{\psi_n * \overline{f^{(k)}}}(x) \right| \leq \frac{\lambda}{n}.$$

In particular, taking $n \gg 0$, we have that the RHS is smaller than epsilon for all $f \in S$, and hence $\xi_n \in U_{S,\epsilon}$.

Finally, since the weak topology is coarser than the strong topology, it suffices to show that $C_c^{-\infty}(\mathbb{R})$ is strongly dense in $C^{-\infty}(\mathbb{R})$. This follows from a previous exercise, namely, given $S \subseteq C_c^{\infty}(\mathbb{R})$ bounded, there exists $K \subseteq \mathbb{R}$ compact such that $S \subseteq C_K^{\infty}(\mathbb{R})$. Then, given $\xi \in C^{-\infty}(\mathbb{R})$, we have that $\xi \mid_S \equiv (\xi \cdot I_K)_S$, where I_K is the indicator of K, so that $\xi \cdot I_K \in C_c^{-\infty}(\mathbb{R})$ is an element of $\xi + U_{S,\epsilon}$ for all $\epsilon > 0$.

□

EXERCISE 4.15. (1) *Show that $C^{-\infty}(\mathbb{R})$ is not complete in the weak topology.*
(2) *Show that the weak completion of $C^{-\infty}(\mathbb{R})$ is $(C_c(\mathbb{R}))^\sharp$, the abstract dual.*
(3) *Show that $C^{-\infty}(\mathbb{R})$ is* strongly *complete.*

SOLUTION. For (1), define the waek topology on $(C_c^{\infty}(\mathbb{R}))^\sharp$ (makes sense), and show that the embedding $C^{-\infty}(\mathbb{R}) \to (C_c^{\infty}(\mathbb{R}))^\sharp$ is strict with dense image.
Try to prove the universal property for (2).
(3)? □

DEFINITION 4.16. Given a closed subspace $W \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}$ define

$$V_m(C_c^{\infty}(\mathbb{R}^n), W) := \left\{ f \in C_c^{\infty}(\mathbb{R}^n) : \frac{\partial^\alpha}{(\partial x)^\alpha} f \mid_W \equiv 0, |\alpha| \leq m \right\}.$$

Defined similarly for W a subset.

EXERCISE 4.17. *Let W be a k-dimensional subspace of $\mathbb{R}^n$ and $U = \mathbb{R}^n \setminus W$. Show that*

$$\overline{C_c^{\infty}(U)} = \bigcap_{m=0}^{\infty} V_m(C_c^{\infty}(\mathbb{R}^n), W).$$

REMARK 4.18. This exercise is true for a general open U.

SOLUTION.

- $\subseteq$ Enough to show $C_c^{\infty}(U) \subseteq \bigcap V_m$, since the RHS is clearly closed. Let $f \in C_c^{\infty}(U)$. Then f vanishes on W, and in particular so do all partial derivarives of f of any order, at any point $x \in W$.
- $\supseteq$ Take $f \in \bigcap V_m$, we want to show that f is the limit of a sequence of functions with support on U. We can use cutoff functions, which are identically zero in a small neighbourhood of W, and put $f_n = f \cdot I_n$.

□

DEFINITION 4.19. Define $F_m((C_c(\mathbb{R}^n)^*, W) := \{\xi \in C_c^{\infty}(\mathbb{R}^n)^* : \xi \mid_{V_m} \equiv 0\}$, where V_m is as above.

EXERCISE 4.20. *Show that $C_W^{\infty}(\mathbb{R}^n) \neq \cup_m F_m$.*

SOLUTION. Take a comb of derivatives of δ at a discrete set of points. $\square$

EXERCISE 4.21. *Prove that for any $U \subseteq \mathbb{R}^n$ with compact closure and any $\xi \in C_W^{-\infty}(\mathbb{R}^n)$, there exists $\xi' \in F_m$ such that $\xi\mid_U \equiv \xi'\mid_U$.*

EXERCISE 4.22. *Compute $\overline{C_c^\infty(\mathbb{R}^n \setminus \{0\})}$.*