Algebraic Geometry 2
Tutorial session 8

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Some properties of schemes

Exercise

Let A be a ring. Show that the following are equivalent:

1. $\mathrm{Spec}(A)$ is disconnected
2. A has non-trivial idempodents, i.e $\exists e \in A \setminus \{0, 1\}$ such that $e^2 = e$
3. $A \simeq A_1 \times A_2$ for A_1, A_2 non-zero subrings.

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Solution.

- $\underline{(1) \Rightarrow (2)}$. Let $\mathrm{Spec}(A) = U_1 \sqcup U_2$ be a cover by two open and disjoint non-trivial open sets, and let $s_e \in \Gamma(\mathrm{Spec}(A), \mathcal{O}_A)$ be defined by

$$s_e(\mathfrak{p}) = \begin{cases} 1 \in A_{\mathfrak{p}} & \text{if } \mathfrak{p} \in U_1 \\ 0 \in A_{\mathfrak{p}} & \text{if } \mathfrak{p} \in U_2. \end{cases}$$

  Then the section s_e is a non-trivial idempodent of $\mathcal{O}_A(\mathrm{Spec}(A)) \simeq A$.

$\square$

Solution.

- $\underline{(2) \Rightarrow (3)}$. Note that, if $e \in A$ is an idemopotent, then $(1-e)$ is an idempotent as well:

$$(1-e)^2 = 1 - 2e + e^2 = 1 - e.$$

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- (3) $\Rightarrow$ (1) Note that, assuming $A = A_1 \times A_2$ then A_1, A_2 are both ideals, and $\mathrm{Spec}(A) = V(A_1) \sqcup V(A_2)$.

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- X is *reduced* if, for any $U \subseteq X$ open, the ring $\mathcal{O}_X(U)$ has no nilpotents.
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$\Leftarrow$ If $\mathcal{O}_X(U)$ has a domain then it has no nilpotents, thus we have reducedness. Furthermore, if X is not irreducible then it contains two non-empty disjoint open subsets U_1, U_2, and $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$, which is not a domain.

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- $\Rightarrow$ Assume X is reduced and irreducible, and let $U \subseteq X$ be open. Let $f, g \in \mathcal{O}_X(U)$ be such that $fg = 0$, and put $Y = \{x \in U : f_x \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}\}$, $Z = \{x \in U : g_x \in \mathfrak{m}_x\}$. Then Y, Z are closed (home exercise), and $U = Y \cup Z$. By irreducibility, wlog, $Y = U$.

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$\square$

Exercise

Let $X = \operatorname{Spec}(A)$ be affine and let $\operatorname{nil}(A)$ denote the nilradical of A (=ideal given by the set of all nilpotents). Then

1. X is irreducible iff $\operatorname{nil}(A)$ is prime;
2. X is reduced iff $\operatorname{nil}(A) = 0$;
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The first assertion here is a correction to a false statement from a previous tutorial.

Solution.

1. If A is irr and $f_1, f_2 \in A$ are such that $f_1 f_2 \in \mathrm{nil}(A)$, then $V(f_1) \cup V(f_2) = V(f_1 f_2) = \mathrm{Spec}(A) = X$ and hence $X = V(f_1)$ or $X = V(f_2)$. Consequenly, $f_1 \in \mathrm{nil}(A)$ or $f_2 \in \mathrm{nil}(A)$.

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2. Note that, if $\mathrm{nil}(A) = 0$ then $\mathrm{nil}(A_f) = 0$ for all $f \in A$. In particular, $\mathcal{O}_X(D(f))$ has no nilpotents for all f, and reducedness follows from locality.

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3. Note that A is a domain iff (0) is prime, in which case it equals $\mathrm{nil}(A)$. The assertion follows.

□

The reduced scheme associated to a scheme

Exercise

Let X be a scheme.

1. Show that X is reduced iff, for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ has no nilpotents.
2. Let $\mathcal{O}_{\mathrm{red}}$ be the sheafification of the presheaf $\widetilde{\mathcal{O}_{\mathrm{red}}}(U) = \mathcal{O}_X(U)_{\mathrm{red}}$, where $A_{\mathrm{red}} := A/\mathrm{nil}(A)$, for $U \subseteq X$ open. Show that $X_{\mathrm{red}} = (X, \mathcal{O}_{\mathrm{red}})$ is a scheme, and there is a natural morphism $X_{\mathrm{red}} \to X$, which is a homeo on the underlying topological spaces.

Solution.

1. Assume X is reduced and let $x \in X$ be arbitrary. Assume $f_x \in \mathcal{O}_{X,x}$ is nilpotent, with $n_x \in \mathbb{N}$ such that $f_x^{n_x} = 0$. Take $x \in U$ open and $f \in \mathcal{O}_X(U)$ st $[U, f] \equiv f_x$ in $\mathcal{O}_{X,x}$.

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Conversely, given $U \subseteq X$ open, we have that

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Conversely, given $U \subseteq X$ open, we have that

$$\mathcal{O}_X(U) \hookrightarrow \prod_{x \in X} \mathcal{O}_X(U).$$

The fact that $\mathcal{O}_X(U)$ has no non-zero nilpotents follows, since having no nilpotents is preserved under taking products and passing to a subring.

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In particular $\widetilde{\mathcal{O}_{\mathrm{red}}}(D(f)) \simeq \Gamma(D(f), \mathrm{Spec}(A_{\mathrm{red}}))$, and hence the two presheaves agree on a basis for the topology.

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   Assume now that $X = \bigcup U_\alpha$ with $(U_\alpha, \mathcal{O}_{U_\alpha} := \mathcal{O}_X|_{U_\alpha})$ affine. We need to verify that $(\mathcal{O}_{U_\alpha})_{\mathrm{red}}|_{U_\alpha \cap U_\beta} \simeq (\mathcal{O}_{U_\beta})_{\mathrm{red}}|_{U_\alpha \cap U_\beta}$, and that these isomorphisms agree on triple intersection (home exercise; similar to affine case). Therefore, the schemes $(\mathcal{O}_{U_\alpha})_{\mathrm{red}}$ glue uniquely to a scheme on X.

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X is *noetherian* if it is locally noetherian and quasi-compact.
Equivalently, if $X = \bigcup_{i=1}^{n} \mathrm{Spec}(A_i)$ for A_i noetherian.

By a theorem proved in class, if X is noetherian and $U =\subseteq X$ is open affine, then $A = \Gamma(U, \mathcal{O}_X)$ is noetherian. In particular, for $X = \mathrm{Spec}(A)$ affine, X is noetherian iff A is noetherian.

Noetherity vs noetherity

Recall that a topological space Ω is for any decreasing sequence $\Omega \supseteq F_1 \supseteq F_2 \supseteq \ldots$ of closed sets, there exists $n \in \mathbb{N}$ such that $V_{n+k} = V_n$ for all $k > 0$.

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Exercise

Show that if X is a noetherian scheme then $|X|$ is noetherian. Show that the converse is false.

Solution.

Let $X = \bigcup_{i=1}^{n} \mathrm{Spec}(A_i)$ be a finite cover with A_i noetherian, and let $F_1 \supseteq F_2 \supseteq \dots$ be a decreasing sequence of closed sets. Then $F_j \cap \mathrm{Spec}(A_i)$ is closed and hence corresponds to an ideal I_j^i with $I_1^i \subseteq I_2^i \subseteq \dots$. By noetherity of the A_i's, each such sequence stabilizes at some $j(i)$, and hence the sequence $(F_j = \bigcup_{i,j}(F_j \cap \mathrm{Spec}(A_i)))_{j \geq 1}$ stabilizes as well. $\square$

For an example of a scheme whose topological space is noetherian while the scheme is not, consider

$$A = \mathbb{C}[x_n : n = 1, 2, \ldots]/(x_n^n : n = 1, 2, \ldots).$$

and $X = \mathrm{Spec}(A)$. Then, as all variables x_n are nilpotent, we have that $\mathrm{Spec}(A) = \mathrm{Spec}(A_{\mathrm{red}}) = \mathrm{Spec}(\mathbb{C})$, which is a point and, consequently, noetherian.

On the other hand, the ideals $I_n = (x_1, \ldots, x_n)$ comprise a non-stabilizing increasing sequence. Therefore A is not noetherian and, by a theorem from class, $X = \mathrm{Spec}(A)$ is not noetherian.

Exercise (Noetherian induction)

Let X be a noetherian topological space, and let (P) be a property of closed subsets of X. Assume that, for any $Y \subseteq X$ closed, if (P) holds for all proper closed subsets of Y, then (P) holds for Y. Then (P) holds for X.

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Solution.

Assume the statement is false and let (P) be the property: "*is equal to the union of finitely many irreducible subspaces*". Let $\mathscr{S}$ be the set of all closed subset of X for which (P) does not hold. By noetherian induction $\mathscr{S}$ is not empty, and, by noetherity, it has a minimal element Y. Since (P) holds for irreducible sets, Y is not irreducible, and hence $Y = Y_1 \cup Y_2$ for Y_1, Y_2 distinct proper closed subsets. But (P) *does hold* for Y_1 and Y_2, hence also for Y. A contradiction. □

Open and closed embeddings

Definition

An *open subscheme* of X is a scheme $(U, \mathcal{O}_U)$ where $U \subseteq X$ is open and $\mathcal{O}_U \simeq \mathcal{O}_X\,|_U$.

An *open embedding* $f : Y \to X$ is a morphism such that there exists an open subset $U \subseteq X$ and an isomorphism $Y \simeq U$ such that

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\simeq"'] & & X \\ & U \arrow[ur, "\subseteq"'] & \end{tikzcd}$$

Definition

A *closed embedding* is a morphism $f : X \to Y$ such that f induces a homeo of $|Y|$ on a closed subset of X, and such that the induced map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

The notion of closed subscheme is defined to be an equivalence class of of closed embeddings, under a suitable relation.

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Specifically, for X affine and $Y \subseteq X$ closed, let $A = \Gamma(X, \mathcal{O}_X)$ and $I \triangleleft A$ be an ideal such that $Y = V(I)$. Then one obtains a closed subscheme structure on Y by taking $f : Y \to X$ to be the map determine by the quotient $A \to A/I$.

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For example, consider $X = \mathbb{A}^2_k$ and $Y = V(x)$. Then Y may be endowed with the structure sheaf given from $k[x, y]/(x^n)$, for any $n = 1, 2, \ldots$.

Dimension and codimension

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$$\mathrm{codim}(Y, X) = \inf_{Z \subseteq Y \text{ irr}} \mathrm{codim}(Z, X)$$

Examples

1. $\dim \mathrm{Spec}(k) = 0$ and $\dim \mathbb{A}^n_k = n$ for any field k;
2. $\dim \mathrm{Spec}(\mathbb{Z}) = 1$;
3. More generally, if $X = \mathrm{Spec}(A)$ then $\dim(X) = \dim(A)$, where the RHS is the Krull dimesion, i.e the length of a maximal descending chain of prime ideals.
4. For a noetherian ring A, $\dim(\mathrm{Spec}(A[x_1, \ldots, x_n])) = \dim(A) + n$.

Exercise

Let X be an integral scheme of finite type over a field k.

1. For any closed point $x \in X$, $\dim X = \dim \mathcal{O}_{X,x}$
2. Given a closed subset $Y \subseteq X$, show that $\dim(Y) + \mathrm{codim}(Y, X) = \dim(X)$.
3. Let $U \subseteq X$ be a non-empty open subset. Show that $\dim(U) = \dim(X)$.

Solution.

1. In general, we have an bijective map $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$, with image within an affine open subset, which implies

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathrm{Spec}(\mathcal{O}_{X,x})) \leq \dim(X).$$

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Conversely, assume first that $X = \mathrm{Spec}(A)$ for A a f.g. domain over k. Then $x = \mathfrak{m}$ is a maximal ideal and, by Theorem 1.8A in Hartshorne

$$\dim(X) = \dim(A) = \mathrm{ht}(\mathfrak{m}) + \dim(A/\mathfrak{m}) = \dim(\mathcal{O}_{X,\mathfrak{m}}) + 0.$$

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In the more general case, we have that $X = \bigcup_{i=1}^{n} X_i$, a finite union of spectra of f.g. domains over k. We have that

$$\dim(X) = \max\{\dim X_i : i = 1, \ldots, n\},$$

from which the claim follows.

Solution- contd.

2. Assume first that Y is irreducible and $X = \mathrm{Spec}(A)$ for A f.g. domain over k. Then $Y = V(\mathfrak{p})$ for a prime $\mathfrak{p}$ and, by definition $\mathrm{codim}(Y, X) = \mathrm{ht}(\mathfrak{p})$ and $\dim(Y) = \dim(A/\mathfrak{p})$. The result then follows, again, from Theorem 1.8A:

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3. To prove the last item, it suffices to see that any non-empty open subset contains a closed point. For the case $X = \mathrm{Spec}(A)$ and $U = D(f) \neq \varnothing$, this is equivalent to finding a maximal ideal not containing f. But if no such maximal exists, then f is in the Jacobson radical of A, which is zero.

$\square$

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Solution.

Note that $\dim(X) = \dim(R) + 1 = 2$, since R is a dvr.
Consider $\mathfrak{p} = (xt - 1)$. Then $\mathfrak{p} \supseteq (x - 1, t - 1)$ is prime of height 1, hence

$$\dim(\mathcal{O}_{X,\mathfrak{p}}) = \dim(R[t]_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) = 1 < \dim X.$$

Moreover, taking $Y = V(\mathfrak{p})$, we have that $\mathrm{codim}(Y, X) = 1$.
However,

$$\dim(Y) = \dim \mathrm{Spec}(\mathbb{C}[[x]][t]/(xt - 1)) = \dim \mathrm{Spec}(\mathbb{C}((x))) = 0,$$

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Finally, the localization of $R[t]$ by x is a polynomial ring over the field $\mathbb{C}((x))$, hence one-dimensional. So $\dim(D(x)) = 1 < \dim(X)$.

Questions?