On Regular Representations of Groups

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Introduction

Jaikin's Example- $\zeta_{\mathrm{SL}_2(\mathfrak{o})}$

Norm-1 Subgroups of Local Division Algebras

Regular Representations

Motivation

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- **Mission-** Understand $\mathrm{Irr}(\Gamma)$.

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- **Possible Answer -**

Understand Irr(Γ)?

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  - Parametrize all irreducible representations of Γ according to some known parametrizing space
  $$X \leftrightarrow \mathrm{Irr}(\Gamma).$$
  - For each class $[\rho] \in \mathrm{Irr}(\Gamma)$, present a complex finite dimensional vector space V_ρ, and the homomorphism $\rho : \Gamma \to \mathrm{Aut}(V_\rho)$.

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  - For each class $[\rho] \in \mathrm{Irr}(\Gamma)$, present a complex finite dimensional vector space V_ρ, and the homomorphism $\rho : \Gamma \to \mathrm{Aut}(V_\rho)$.
- **Another Option**- Describe all characters of Γ explicitly

$$\chi_\rho(\gamma) = \mathrm{Tr}(\rho(\gamma)) \quad (\gamma \in \Gamma).$$

Understand $\mathrm{Irr}(\Gamma)$?

- **Asymptotics-** For any $n \in \mathbb{N}$, define

$$r_n(\Gamma) := \#\left\{[\rho] \in \mathrm{Irr}(\Gamma) \mid \dim(\rho) = n\right\} \in \mathbb{N} \cup \{0, \infty\}\,.$$

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E.g.- Do there exist $c_1, c_2 \in \mathbb{R}$ such that

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The group Γ is said to be **rigid** if $r_n(\Gamma) < \infty$ for all n.

Arithmetic Groups

Let Γ be an arithmetic group, e.g. $\Gamma = \mathrm{SL}_3(\mathbb{Z})$. The group Γ is said to have **the congruence subgroup property** (CSP) if any finite-index normal subgroup of Γ contains a principal congruence subgroup.
For example, the principal congruence subgroups of $\mathrm{SL}_3(\mathbb{Z})$ are all subgroups of the form $\mathrm{Ker}\left(\mathrm{SL}_3(\mathbb{Z}) \to \mathrm{SL}_3(\mathbb{Z}/n\mathbb{Z})\right)$, for some $n \in \mathbb{N}$.

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The same is true for $\mathrm{SL}_k(\mathbb{Z})$ *for all* $k \geq 3$.
The group $\mathrm{SL}_2(\mathbb{Z})$*, however, does not have CSP.*

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An arithmetic group over a number field has polynomial representation growth iff it has CSP.

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The **representation zeta function** of Γ is

$$\zeta_\Gamma(s) = \sum_{n=1}^{\infty} r_n(\Gamma) n^{-s} = \sum_{[\rho]\in \mathrm{Irr}(\Gamma)} \dim(\rho)^{-s}, \quad (s \in \mathbb{C}). \qquad (*)$$

Arithmetic Groups

The representation zeta function of Γ admits an Euler product-

$$\zeta_{\mathrm{SL}_3(\mathbb{Z})}(s) = \zeta_{\mathrm{SL}_3(\mathbb{C})}(s) \cdot \prod_{p \text{ is prime}} \zeta_{\mathrm{SL}_3(\mathbb{Z}_p)}(s).$$

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Here-

- $\zeta_{\mathrm{SL}_3(\mathbb{Z})}$ enumerates arbitrary representations of Γ.
- The archimedean factor $\zeta_{\mathrm{SL}_3(\mathbb{C})}$ enumerates rational representations of $\mathrm{SL}_3(\mathbb{C})$.

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- The non-archimedean factors $\zeta_{SL_3(\mathbb{Z}_p)}$ enumerate continuous representations.

p-adic Groups

Theorem (Jaikin-Zapirain, '07)

Fix an odd prime $p \in \mathbb{Z}$. Let G be a rigid compact p-adic analytic group. There exist rational functions $f_1(t), \ldots, f_k(t) \in \mathbb{Q}(t)$ and numbers $n_1, \ldots, n_k \in \mathbb{N}$ such that

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Example (AKOV) Assume $p > 2$ and let $G \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ be given by $\mathrm{SL}_2(\mathbb{Z}_p) \cap (1 + p\mathrm{M}_2(\mathbb{Z}_p))$. Then

$$\zeta_G(s) = p^3 \frac{1 - p^{-2-s}}{1 - p^{1-s}}.$$

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3. Model theoretic proof of the rationality of such integrals (Denef).

Jaikin's Example

Computing $\zeta_{\mathrm{SL}_2(\mathfrak{o})}$

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- Fix $G = \mathrm{SL}_2(\mathfrak{o})$.

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- Fix $G = \mathrm{SL}_2(\mathfrak{o})$. The subgroups $G^m = \mathrm{SL}_2(\mathfrak{o}) \cap (1 + \pi^m \mathrm{M}_2(\mathfrak{o}))$ are called the **principal congruence subgroups** of G. The sequence $\{G^m\}$ is a neighbourhood basis at 1.

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Definition

The **level** of a representation ρ of G is the minimal $k \in \mathbb{N}_0$ such that $G^{k+1} \subseteq \mathrm{Ker}(\rho)$.

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For example, since the map $G \to \mathrm{SL}_2(\mathbb{F}_q)$ is surjective, there is a natural identification

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Put $\mathrm{Irr}^k(G) := \{\rho \in \mathrm{Irr}(G) \mid \rho \text{ of level } k\}$. Then

$$\dot{\bigcup}_{m\leq k} \mathrm{Irr}^m(G) = \mathrm{Irr}\left(\mathrm{SL}_2(\mathfrak{o}/\mathfrak{p}^{k+1})\right).$$

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Computing $\zeta_{\mathrm{SL}_2(\mathfrak{o})}$- Strategy

- In what follows, we will describe an explicit map between the set $\mathrm{Irr}^k(G)$ that the space of orbits in a finite G-space.

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- In what follows, we will describe an explicit map between the set $\mathrm{Irr}^k(G)$ that the space of orbits in a finite G-space.
- Using this map, we can compute the cardinality of $\mathrm{Irr}^k(G)$, as well as the dimensions of its elements, thereby allowing us to compute

$$\zeta_{\mathrm{SL}_2(\mathfrak{o}/\mathfrak{p}^{k+1})}(s) = \sum_{\substack{\rho\in\mathrm{Irr}^m(G)\\ m\leq k}} \dim(\rho)^{-s}.$$

Computing $\zeta_{\mathrm{SL}_2(\mathfrak{o})}$- Strategy (contd.)

Applying the process above for all $k \in \mathbb{N}$ we obtain-

Theorem (Jaikin-Zapirain, '07)

$$\begin{aligned}\zeta_{\mathrm{SL}_2(\mathfrak{o})}(s) =&\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s) \\ &+ \frac{4q\left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \left(\frac{q-1}{2}\right)^2(q^2+q)^{-s}}{1-q^{1-s}}.\end{aligned}$$

General Notation

- G^r- superscript indices stand for congruence subgroups.
- G_m- subscript indices stand for congruence quotients- i.e. quotients of G by a subgroup G^m.
- G^r_m- both super- and subscript indices signify congruence subquotients, G^r/G^m.

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Similar notation will be used for Lie-algebras. From here on we fix $K \supseteq \mathfrak{o} \supseteq \mathfrak{p} = \pi\mathfrak{o}$.

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- Put $r = \lceil \frac{k}{2} \rceil$, and consider the restriction of ρ to the group G_k^r.

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- Put $r = \lceil \frac{k}{2} \rceil$, and consider the restriction of ρ to the group G_k^r. The following hold
  - The group G_k^r is abelian.
  - The map $1 + \pi^r x \mapsto x$ is an isomorphism of abelian groups between G^r/G^k and the finite Lie-ring $\mathfrak{g}_{k-r} := \mathfrak{sl}_2(\mathfrak{o}/\mathfrak{p}^{k-r})$ of traceless 2×2 matrices over $\mathfrak{o}/\mathfrak{p}^{k-r}$.
  - There is a G-equivariant bijection of $\mathfrak{g}_{k-r}$ with its Pontryagin dual $\widehat{\mathfrak{g}_{k-r}}$,

$$y \mapsto \phi_y(x) := \chi\left(\mathrm{Tr}(xy)\right).$$

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Get a G-equivariant bijection

$$\begin{aligned} \mathfrak{g}_{k-r} &\xrightarrow{1-1} \widehat{G_k^r} = \mathrm{Irr}(G_k^r) \\ y &\mapsto \theta_y(1+\pi^r x) = \phi_y(x). \end{aligned}$$

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By Clifford Theory, there exists $y \in \mathfrak{g}_{k-r}$ which is unique up to G_k-conjugation and a fixed $e \in \mathbb{N}$, such that

$$\rho_{G_k^r} = \left(\bigoplus_{y' \in \mathrm{Ad}(G_k)y} \theta_{y'} \right)^{\oplus e}.$$

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We obtain a mapping $\rho \mapsto \mathrm{Ad}(G_k)y$.

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More generally- Suppose that there exists $\Theta \in \mathrm{Irr}(S)$ whose restriction to G_k^r lies above θ_y and under ρ. Then the induced representation Θ^{G_k} is irreducible, and furthermore

$$\rho \simeq \Theta^{G_k}.$$

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The question of computing the dimension of ρ is now converted to analysis of S and of the existence of extensions of θ_y to S.

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- The following facts are true-
  - If y is non-zero modulo $\mathfrak{p}$, then $\mathbf{C}_{G_k}(y)$ is abelian.
  - In fact, $\mathbf{C}_{G_k}(y)$ belongs to one of 3 isomorphism types of abelian subgroups of G_k.
  - The inclusion $(**)$ is an equality.

Extending θ_y- Even Level

$$\begin{tikzcd}
& G_k \arrow[d, dash] & \\
& S \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{C}_{G_k}(y) \arrow[dr, dash] & & G_k^r \arrow[dl, dash] \\
& \text{intersection} \arrow[d, dash] & \\
& \{1\} &
\end{tikzcd}$$

Assume $k = 2r$.

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Assume $k = 2r$. By gluing θ_y to any suitable representation of $\mathbf{C}_{G_k}(y)$, we obtain the desired extension $\Theta \in \mathrm{Irr}(S)$.

Extending θ_y- Odd Level

$$
\begin{tikzcd}
 & G_k & \\
 & S \arrow[u, no head] & \\
\mathbf{C}_{G_k}(y) \arrow[ur, no head] & & G_k^r \arrow[ul, no head] \\
 & & G_k^{r+1} \arrow[u, dashed, no head] \\
 & \text{intersection} \arrow[uul, no head] \arrow[ur, no head] & \\
 & \{1\} \arrow[u, no head] &
\end{tikzcd}
$$

Assume $k = 2r + 1$.

Extending θ_y- Odd Level

In order to be able to extend θ_y to S, it is necessary to understand the set of representations of G^r/G^k lying above θ_y. This is done using the method of **Heisenberg lifts**.

$$
\begin{tikzcd}
 & S \arrow[dl, dash] \arrow[dr, dash] & & & \\
\mathbf{C}_{G_k}(y) \arrow[dddr, dash] & & G_k^r \arrow[r, dashed, dash, "\log"] \arrow[d, dash] & \mathfrak{g}_{k-r} \arrow[r, two heads] \arrow[d, dash] & L \arrow[d, dash] \\
 & & RG_k^{r+1} \arrow[r, dashed, dash] \arrow[d, dash] & \mathfrak{R} \arrow[r, two heads] \arrow[d, dash] & R \arrow[d, dash] \\
 & & G_k^{r+1} \arrow[r, dash] \arrow[dl, dash] & \mathfrak{g}_{k-r-1} \arrow[r, two heads] & \{0\} \\
 & \text{intersection} \arrow[d, dash] & & & \\
 & \{1\} & & &
\end{tikzcd}
$$

The Norm-One Subgroups of Local Division Algebras

Division Algebras- General Properties

Let D be a division algebra, with $\mathbf{Z}(D) = K$ a local field of odd residual characteristic. Let $\ell = \deg D = (\dim_K D)^{1/2}$.

Let L/K be an unramified field extension of degree ℓ.

Division Algebras- General Properties

Let D be a division algebra, with $\mathbf{Z}(D) = K$ a local field of odd residual characteristic. Let $\ell = \deg D = (\dim_K D)^{1/2}$.
Let L/K be an unramified field extension of degree ℓ.

The algebra D embeds as a subalgebra of the matrix algebra $\mathrm{M}_\ell(L)$, where an element $x = x_0 + \nu x_1 + \ldots + \nu^{\ell-1} x_{\ell-1}$ is mapped to the matrix

$$\Lambda_x = \begin{pmatrix} x_0 & \pi\sigma(x_{\ell-1}) & \ldots & \pi\sigma^{\ell-1}(x_1) \\ x_1 & \sigma(x_0) & \ldots & \pi\sigma^{\ell-1}(x_2) \\ \vdots & & \ddots & \vdots \\ x_\ell & \sigma(x_{\ell-1}) & \ldots & \sigma^{\ell-1}(x_0) \end{pmatrix},$$

where $\sigma : L \to L$ is a generates the Galois groups $\mathbf{Gal}(L/K)$.

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where $\sigma : L \to L$ is a generates the Galois groups $\mathbf{Gal}(L/K)$.
Define $\mathrm{Nrd}(x) = \det \Lambda_x$ and $\mathrm{Trd}(x) = \mathrm{Tr}\Lambda_x$.

The Norm-One Subgroup of $D^\times$

Let $\mathfrak{o}_L$ denote the valuation ring of L, and let $\mathcal{O} := D \cap \mathrm{M}_\ell(\mathfrak{o}_L)$.

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Let $\mathfrak{o}_L$ denote the valuation ring of L, and let $\mathcal{O} := D \cap \mathrm{M}_\ell(\mathfrak{o}_L)$. Let

$$G = \mathrm{SL}_1(D) = \{x \in D \mid \mathrm{Nrd}(x) = 1\} \subseteq \mathcal{O}^\times.$$

The congruence subgroups of G are defined by

$$G^m := G \cap \left(1 + \nu^m \mathrm{M}_\ell(\mathfrak{o}_L)\right),$$

where ν is the element of $\mathcal{O}$ which is identified with the matrix

$$\begin{pmatrix} 0 & & \dots & \pi \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathrm{M}_\ell(\mathfrak{o}_L).$$

The **level** of an irreducible representation $\rho \in \mathrm{Irr}(G)$ is defined as in the case of $\mathrm{SL}_2(\mathfrak{o})$, i.e. the minimal $m \in \mathbb{N}$ such that $G^{m+1} \subseteq \mathrm{Ker}(\rho)$.

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Owing to the work of H. Koch, the centralizer in G_k of an element $y \in G_k^r$ can be effectively described. An explicit description of the centralizer was presented in the work of V. Singh. In particular- in the case where $\ell = \deg D$ is a prime number, all such centralizers are of the form

$$((L')^\times \cap G) \cdot G^{m'},$$

where $L' \supseteq K$ is a field extension, and $m' \in \mathbb{N}$ is dependent on y.

The case where ℓ is prime is amenable to a similar treatment as $\mathrm{SL}_2(\mathfrak{o})$.

Results

Let D/K be a division algebra of prime degree ℓ, and put $\epsilon = \gcd(\ell, q-1) = \#\mu_\ell(K)$.

Theorem

1. *There exist rational polynomials $a_m^\epsilon(t),\ d_m^\epsilon(t) \in \mathbb{Q}(t)$ ($m \in \mathbb{N}_0$), such that for any $m \in \mathbb{N}_0$, $\mathrm{SL}_1(D)$ has exactly $a_m^\epsilon(q)$ irreducible representations of level m. All such representations are of dimension $d_m^\epsilon(q)$.*

Theorem

2.

$$\zeta_{\mathrm{SL}_1(D)}(s) = \zeta_{\mathrm{SL}_1(\mathbb{F}_{q^\ell}|\mathbb{F}_q)}(s) + \sum_{m=1}^{\infty} a_m^{\epsilon}(q) d_m^{\epsilon}(q)^{-s}$$

$$= \frac{\frac{q^\ell-1}{q-1}\left(1 - q^{-\binom{\ell}{2}s}\right) + \left(\frac{q^\ell-1}{\epsilon(q-1)}\right)^{-s}\epsilon^2\,(q-1)\left(\sum_{\lambda=0}^{\ell-2} q^{\lambda(1-\frac{\ell-1}{2}s)}\right)}{1 - q^{(\ell-1)-\binom{\ell}{2}s}}.$$

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3. *If $\ell > 2$ then all irreducible representations of $\mathrm{SL}_1(D)$ are induced from one-dimensional representations of congruence subgroups.*

Regular Representations

Generalizing to Other Groups

Let $\mathbf{G} \subseteq \mathrm{GL}_n$ be a connected reductive algebraic group, which is defined over $\mathfrak{o}$. Put $G = \mathbf{G}(\mathfrak{o})$.

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and under some mild assumptions on p, we have that

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We retain the notation G_m^r for the subgroups G^r/G^m.

Generalizing to Other Groups

By virtue of $\mathbf{G}$ being reductive, it is possible to apply the process described above for $\mathbf{G} = \mathrm{SL}_2$ and to associate any representation $\rho \in \mathrm{Irr}(G)$ of level m with an orbit in a finite Lie-ring $\mathfrak{g}_{m-r}$, as above.

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However, the description of stabilizers of the associated characters $\theta_y \in \mathrm{Irr}(G_m^r)$ is seldom as straightforward as in the previous cases.

Regular Representations

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}(\mathfrak{o})$, and put $\mathfrak{g}^m := \pi^m \mathfrak{g}$ and $\mathfrak{g}_m = \mathfrak{g}/\mathfrak{g}^m$.

Definition (Steinberg, Hill)

- An element $y \in \mathfrak{g}_1 = \mathfrak{g}(\mathbb{F}_q)$ is called regular if its centralizer $\mathbf{C}_{\mathbf{G}(\mathbb{F}_q)}(y)$ in G_1 is of minimal dimension.
- An element $y \in \mathfrak{g}_m$ is called regular if its image in $\mathfrak{g}_1$ is regular.
- A representation $\rho \in \mathrm{Irr}(G)$ of level m is called regular, if the orbit associated to it is regular.

Regular Representations

For example, in the case $\mathbf{G} = \mathrm{GL}_n$ it is true that an element $y \in \mathfrak{gl}_n(\mathfrak{o}/\mathfrak{p}^m)$ is regular if and only if its minimal polynomial is equal to its characteristic polynomial.

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It is also true in this case that the centralizer $\mathbf{C}_{G_m}(y)$ is an abelian group, and in fact equals to the group

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$$(\mathfrak{o}/\mathfrak{p}^m)[y] \cap \mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}^m).$$

In addition, if y is taken to be in the orbit associated to a regular representation $\rho \in \mathrm{Irr}(\mathrm{GL}_n(\mathfrak{o}))$ of level m, and $\theta_y \in \mathrm{Irr}(G_m^r)$ is the associated character, then

$$Stab^{G_m}(\theta_y) = \mathbf{C}_{G_m}(y) \cdot \mathbf{G}_m^{m-r},$$

as in the case of SL_2.

Regular Representation

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A possible application to the study of regular representation is to the analysis of the regular part of the representation zeta function of G, in view of the decomposition-

$$\zeta_G(s) = \zeta_{\mathbf{G}(\mathbb{F}_q)}(s) + \zeta_G^{\text{reg.}}(s) + \zeta_G^{\text{irreg.}}(s).$$

Regular Zeta Functions of Groups of Type A_{n-1}

Let $\mathbf{G}_{+1} = \mathrm{GL}_n$ and $\mathbf{G}_{-1} = \mathrm{U}_n$ the unitary group attached to an unramified quadratic extension L/K. Let $\epsilon \in \{\pm 1\}$.

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As was recently shown by Krokovski, Onn and Singla, there exist

- $\mathcal{A}_n$ a 'nice' set,
- an explicit map from the regular part of $\mathrm{Irr}^m(\mathbf{G}_\epsilon(\mathfrak{o}))$ to $\mathcal{A}_n$,

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As was recently shown by Krokovski, Onn and Singla, there exist

- $\mathcal{A}_n$ a 'nice' set,
- an explicit map from the regular part of $\mathrm{Irr}^m(\mathbf{G}_\epsilon(\mathfrak{o}))$ to $\mathcal{A}_n$,
- polynomials $u_\epsilon^\tau(t)$, for all $\tau \in \mathcal{A}_n$

such that, if $\rho \in \mathrm{Irr}^m(\mathbf{G}_\epsilon(\mathfrak{o}))$ is mapped to $\tau \in \mathcal{A}_n$ then

$$\dim(\rho) = q^{m\binom{n}{2}} \left(\frac{|\mathbf{G}_\epsilon(\mathbb{F}_q)|}{u_\epsilon^\tau(q)} \right).$$

The number of regular representations of a given type $\tau \in \mathcal{A}_n$ can also be described, using the polynomial $w_d(q)$, which gives the number of irreducible polynomials of degree d over $\mathbb{F}_q$.

The number of regular representations of a given type $\tau \in \mathcal{A}_n$ can also be described, using the polynomial $w_d(q)$, which gives the number of irreducible polynomials of degree d over $\mathbb{F}_q$. From this, one obtains the regular zeta function of $\mathbf{G}_\epsilon(\mathfrak{o})$.

Theorem (Krakovski, Onn, Singla, '16)

$$\zeta^{\text{reg.}}_{\mathbf{G}_\epsilon(\mathfrak{o})}(s) = \frac{1}{1 - q^{n - \binom{n}{2}s}} \cdot \sum_{\tau \in \mathcal{A}_n} u^\tau_\epsilon(q) \prod_{i=1}^{n} \binom{\sum_j \tau_{i,j}}{\tau_{i,1}, \tau_{i,2}, \ldots, \tau_{i,n}} \binom{w_d(q)}{\sum_j \tau_{i,j}} \left(\frac{|\mathbf{G}_\epsilon(\mathbb{F}_q)|}{u^\tau_\epsilon(q)} \right)^{-s}.$$

Regular Zeta Function for Classical Groups

A similar analysis has be thus far accomplished for classical groups defined over $\mathfrak{o}$, i.e. linear algebraic groups of the form

$$\left\{\mathbf{X} = (x_{i,j}) \in \mathrm{M}_n \mid \mathbf{X}^T\mathbf{J}\mathbf{X} = \mathbf{J}\right\},$$

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where $\mathbf{J} \in \mathrm{GL}_n(\mathfrak{o})$ is a fixed symmetric or anti-symmetric matrix. In the specific cases of the symplectic group Sp_{2n} and the odd-orthogonal group SO_{2n+1}, the regular representation zeta function has been computed.

Questions?

Thank You.