Algebraic Geometry 2
Exercise sheet 1

Solve the following exercises. Exercises marked with $*$ are optional.

Exercise 1. Let X be a topological space, $p \in X$ a point and A an abalian group. Define, for $U \subseteq X$ open

$$i_{p,A}(U) = \begin{cases} A & \text{if } p \in U \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that $U \mapsto i_{p,A}(U)$ defines a sheaf of abelian groups on X, with the restriction maps defined, for $V \subseteq U$ open, by $\mathrm{res}_{U,V} = \mathrm{Id}_A$ if $p \in V$ and $\mathrm{res}_{U,V} = 0$ if $p \notin V$.

(2) $*$ Is it possible to replace the restriction maps $\mathrm{res}_{U,V}$ with different in order to obtain a sheaf structure on $i_{p,A}$?

Exercise 2. Let $X = \mathbb{C} \setminus \{0\}$ with the standard topology. Given $U \subseteq X$, let

$$\mathcal{F}(U) = \{\varphi : U \to \mathbb{C} \mid \varphi \text{ is holomorphic}\}$$

and

$$\mathcal{G}(U) = \{\varphi : U \to \mathbb{C} \mid \varphi \text{ is holomorphic and non-vanishing on } U\}\,.$$

(1) Show that $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups on X, where the group operation on $\mathcal{F}(U)$ is addition of functions, and on $\mathcal{G}(U)$ is multiplication.[1]

(2) Define a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ by setting $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ to be $\varphi_U(f)(z) = e^{f(z)}$.
   (a) Show that φ is a morphism of sheaves.
   (b) Describe $\mathrm{Ker}(\varphi)$; show that it is isomorphic to the sheaf of locally constant functions with values in $\mathbb{Z}$.
   (c) Let $U_1 = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $U_2 = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Show that φ_{U_1} and φ_{U_2} are surjective. Is $\varphi_{(U_1 \cup U_2)}$ surjective as well?
   (d) Deduce that the assignment $U \mapsto \mathrm{Im}(\varphi_U)$ defines a presheaf which is not a sheaf.

Exercise 3. Let X be a topological space and let $\mathcal{F}, \mathcal{G}$ be presheaves of abelian groups on X with $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves.

(1) Given $p \in X$, describe the induced morphism $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ on the stalks of $\mathcal{F}$ and $\mathcal{G}$ at p. Show that it is a group homomorphism.

(2) Show that φ is injective (i.e. $\mathrm{Ker}(\varphi)$ is the constant zero sheaf) if and only if φ_p is injective for all $p \in X$.

(3) Show that in the setting of Exercise 2.(2), the map $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all $p \in X$.

Exercise 4. Let $\pi : M \to X$ be a continuous surjective map of topological spaces.

(1) Given $U \subseteq X$ open, let $\mathcal{S}(U) = \{s : U \to M \mid \pi \circ s = \mathrm{Id}_U\}$. Show that $\mathcal{S}$ is a sheaf of sets over X.

(2) ($*$[2]) Assume X is a C^∞ manifold and $\pi : M \to X$ is a vector bundle. Let

$$\mathcal{S}^{\mathrm{diff}}(U) = \{s : U \to M \mid s \text{ is differentiable and } \pi \circ s = \mathrm{Id}_U\}\,.$$

Show that $\mathcal{S}^{\mathrm{diff}}$ is a sheaf of abelian groups on X.

[1] Recall that a function $\varphi : U \to \mathbb{C}$ is holomorphic iff it is analytic on U, i.e. if for any $x \in U$ there is a neighbourhood $x \in V \subseteq U$ such that $\varphi \mid_V$ is given by a power series. If you are not comfortable with either of these terms, you may replace the term 'holomorphic' with 'continuous' at each occurrence and solve the analogous question for continuous functions.

[2] This exercise is optional because acquaintance of vector bundles over C^∞ manifold is not a prerequisite of this course. If you *are* acquainted with this notion, or are willing to look them up, it is advisable to submit this exercise in any case.