Algebraic Geometry 2
Exercise sheet 3

Exercise 1. The Yoneda Lemma. Let $\underline{\mathsf{C}}$ be a category. Recall that, given an object A in $\underline{\mathsf{C}}$ we have a functor $h^A : \underline{\mathsf{C}}^{\mathrm{op}} \to \underline{\mathsf{Set}}$, defined by

$$h^A(B) = \mathrm{Mor}_{\underline{\mathsf{C}}}(B, A) \quad \text{for } B \in \underline{\mathsf{C}}.$$

(1) The assignment $A \mapsto h^A$ gives rise to a functor $I : \underline{\mathsf{C}} \to \mathrm{Funct}(\underline{\mathsf{C}}^{\mathrm{op}}, \underline{\mathsf{Set}})$. Describe the action of I on morphisms in $\underline{\mathsf{C}}^{\mathrm{op}}$.

(2) Show that I is fully faithful; that is, given objects A, A' of $\underline{\mathsf{C}}$ and a natural transformation $\eta : h^A \to h^{A'}$, show that there exists a *unique* morphism $\tau \in \mathrm{Mor}_{\underline{\mathsf{C}}}(A, A')$ such that $\eta = I(\tau)$.

Exercise 2. Let X be a scheme. Given pairs (K_1, p_1), (K_2, p_2), with K_1, K_2 fields and $p_i : \mathrm{Spec}(K_i) \to X$ a morphism, we write $(K_1, p_1) \sim (K_2, p_2)$ is there exists a field with inclusion maps $K_2, K_1 \to L$ such that the diagram

$$\begin{tikzcd}
\mathrm{Spec}(L) \arrow[r] \arrow[d] & \mathrm{Spec}(K_1) \arrow[d, "p_1"] \\
\mathrm{Spec}(K_2) \arrow[r, "p_2"'] & X
\end{tikzcd}$$

commutes. Shows that $\sim$ defines an equivalence relation on such pairs, and that the its equivalence classes are in bijection with the points of X.

Exercise 3. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ be affine schemes. Let $\varphi : A \to B$ be a ring homomorphism, and $f : Y \to X$ the corresponding morphism of spectra.

(1) Show that φ is injective if and only if the associated map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective (i.e. $\mathrm{Ker}(f^\sharp$ is the zero sheaf). In addition, show that, under this assumption, $f(Y)$ is dense in X.

(2) Show that, if φ is surjective, then f is a homeomorphism onto a closed subset of X and $f^\sharp$ is surjective.

(3) Conversely, assume f a homeomorphism onto a closed set and $f^\sharp$ is surjective, and show that φ is surjective.

(4) * Find an example where f is a homeomorphism onto a closed set and φ is not surjective.

Exercise 4. Let X, Y, Z be affine schemes and $f : X \to Y$, $g : Z \to Y$ morphisms of schemes. Let A, B and C denote the rings of global sections of X, Y and Z, respectively.

(1) Show that $X \times_Y Z \simeq \mathrm{Spec}(A \otimes_C B)$.

Hint. *Apply Exercise 4 of Worksheet 2 in proving that* $\mathrm{Spec}(A \otimes_C B)$ *satisfies the required universal property.*

(2) Let $\Delta : X \to X \times_Y X$ be the unique morphism given by the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Show that, for X and Y affine, Δ is a closed embedding.

Hint: Apply Ex 3.(2) above.

(3) * Show that the scheme defined in Example 2.3.6 in Hartshorne (a line with doubled origin) is not affine.

Exercise 5. Let R be a ring and $\underline{\mathsf{Mod}}_R$ the category of R-modules.

(1) Let P be an R-module, and $A \to B \to C \to 0$ a short exact sequence of R-modules. Show that
$$0 \to \mathrm{Hom}(C, P) \to \mathrm{Hom}(B, P) \to \mathrm{Hom}(A, P)$$
is exact. That is, show that the functor $\mathrm{Hom}(\cdot, P)$ is right exact, considered as a functor on $\underline{\mathsf{Mod}}_R^{\mathrm{op}}$.

(2) Conversely, let $A \to B \to C \to 0$ be a sequence such that for any R-module P, the sequence $0 \to \mathrm{Hom}(C, P) \to \mathrm{Hom}(B, P) \to \mathrm{Hom}(A, P)$ is exact. Prove that $A \to B \to C \to 0$ is exact.
   Hint. Consider the cases $P = C$ and $P = A/\mathrm{Im}(A \to B)$.

(3) Let $F, G : \underline{\mathsf{Mod}}_R \to \underline{\mathsf{Mod}}_R$ be adjoint functors, i.e. such that there exists a natural bijection
$$\mathrm{Hom}(FA, B) \simeq \mathrm{Hom}(A, GB) \quad \text{for any } A, B \in R - \underline{\mathsf{Mod}}.$$
Show that F is left exact and G is right exact.