Algebraic Geometry 2
Tutorial session 2

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Presheaves and sheaves

Definition (Presheaf)

Let X be a topological space. A presheaf $\mathcal{F}$ of abelian groups on X consists of the following data:

1. for any open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$, and
2. for every inclusion $V \subseteq U$ of open sets in X, a (restriction) homomorphism $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,

such that the following hold:

1. $\mathcal{F}(\emptyset) = 0$,
2. $\mathrm{res}_{UU} = \mathbf{1}_{\mathcal{F}(U)}$, and
3. $\mathrm{res}_{UW} = \mathrm{res}_{VW} \circ \mathrm{res}_{UV}$ for any $W \subseteq V \subseteq U$ open.

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The elements of $\mathcal{F}$ are called *sections* of $\mathcal{F}$ over U. It is sometimes convenient to write $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$.

Equivalently, a presheaf is a contravariant functor

$$\mathcal{F} : \underline{\textbf{Open}}(X) \to \underline{\textbf{AbGps}},$$

where $\underline{\textbf{Open}}(X)$ is the category of open sets of X with

$$\mathrm{Hom}(V, U) = \begin{cases} \{i_{VU}\} & \text{if } V \subseteq U \\ \emptyset & \text{otherwise.} \end{cases}$$

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1. (locality) for any $s \in \mathcal{F}(U)$, if $s\,|_{V_\alpha} = 0$ for all α then $s = 0$, and
2. (gluing) given sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $s_\alpha\,|_{V_\alpha \cap V_\beta} = s_\beta\,|_{V_\alpha \cap V_\beta}$ for all α, β, there exists $s \in \mathcal{F}(U)$ such that $s\,|_{V_\alpha} = s_\alpha$ for all α.

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Given an inclusion $V \subseteq U$ of open sets and $s \in \mathcal{F}(U)$, we abbreviate $s\,|_V = \mathrm{res}_{UV}(s)$.

The sheaf axioms can be described by a sequence:

$$1 \to \mathcal{F}(U) \xrightarrow{s \mapsto (s|_{V_\alpha})_\alpha} \prod_\alpha \mathcal{F}(V_\alpha) \xrightarrow{(s_\alpha)_\alpha \mapsto (s_\alpha|_{V_{\alpha,\beta}} - s_\beta|_{V_{\alpha,\beta}})_{\alpha,\beta}} \prod_{\alpha,\beta} \mathcal{F}(V_{\alpha,\beta})$$

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Remark. In different contexts, e.g. if $\mathcal{F}$ is a sheaf of sets, the last arrow is often replaced by an equalizer arrow.

Some examples

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2. Given an abelian group, the *constant* presheaf on X is the assignment $\mathcal{F}(U) = \{\text{constant functions } U \to A\}$, for any $U \subseteq X$ open, with function restriction. This is *not* a presheaf.

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2. Given an abelian group, the *constant* presheaf on X is the assignment $\mathcal{F}(U) = \{\text{constant functions } U \to A\}$, for any $U \subseteq X$ open, with function restriction. This is *not* a presheaf.
3. In the same setting, the assignment

$$\mathcal{F}(U) = \{\varphi : U \to A : \varphi \text{ is locally constant}\}$$

is the presheaf of *locally constant functions*. It is a sheaf.

Some examples

In general, given a field K and a property (P) of functions with values to K (e.g. continuity, differentiability, integrability, boundedness etc.), any assignment of the form

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gives rise to a persheaf on X.
Heuristically, the sheaf axioms tell us that the property (P) is of local nature.

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To show that the preasheaf of locally constant functions is a sheaf, we need to verify the locality and gluing axioms. The locality axiom holds for any sheaf of functions. The gluing axiom is also easy. $\square$

Stalks

Definition

Let X be a topological space and $\mathcal{F}$ a presheaf on X. Given a point $p \in X$, the stalk of $\mathcal{F}$ is defined to be

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The set $\mathcal{F}_p$ can be identified with the set equivalence classes of of pairs

$$\{(s, U) : p \in U \subseteq X \text{ open}, s \in \mathcal{F}(U)\},$$

with respect to the relation

$$(s, U) \sim (t, V) \iff \exists p \in W \subseteq U \cap V \text{ such that } s|_W = t|_W .$$

Another example - Skyscraper sheaves

Let X be a topological space, $p \in X$ a point and A an abelian group. Define, for any $U \subseteq X$ open,

$$i_{p,A}(U) = \begin{cases} A & \text{if } p \in U \\ 0 & \text{otherwise,} \end{cases}$$

with $\mathrm{res}_{U,V} = \mathbf{1}_A$ if $p \in V \subseteq U$ and 0 otherwise.

Exercise

Show that $i_{p,A}$ is a sheaf. What are its stalks?

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- **Case 1**. Consider $q = p$. Then, for any open neighbourhood $p \in U$ we have $i_{p,A}(U) = A$, and for two pairs (s, U) and (t, V) as above, we have that $(s, U) \sim (t, V)$ iff there exists $W \subseteq U \cap V$ such that $s = s\,|_W = t\,|_W = t$. That is, $(s, U) \sim (t, V)$ iff $s = t$, which implies $(i_{p,A})_p = A$.

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- **Case 2**. Assume there exists $q \in U \subseteq X$ open such that $p \notin q$. Then, for any $q \in V \subseteq U$ we necessarily have that $i_{p,A}(V) = 0$, since $p \notin V$. Thus, the stalk in this case is 0.

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- **Case 3**. Otherwise, any $q \in U \subseteq X$ open necessarily contains p as well, and hence $i_{p,A}(U) = A$. It follows, as in the first case, that $(i_{p,A})_q = A$ in this case as well.

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Example

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The following table describes the stalk $(i_{p,\mathbb{Z}})_q$ for different values of p and q:

$q \backslash p$	1	2	3
1	$\mathbb{Z}$	0	0
2	$\mathbb{Z}$	$\mathbb{Z}$	0
3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

Morphisms of presheaves

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A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ between two presheaves on a topological space X is a collection $(\varphi_U)_{U \subseteq X \text{ open}}$ such that the following diagram commutes for any $V \subseteq U \subseteq X$:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{U,V}"] \\
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Equivalently, a morphism of presheaves is the same as a natural transformation between $\mathcal{F}$ and $\mathcal{G}$, when considered as functors $\underline{\textbf{Open}}(X) \to \underline{\textbf{AbGps}}$.

Exercise

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that the presheaf $\mathrm{Ker}(\varphi)$, defined by $\mathrm{Ker}(\varphi)(U) = \mathrm{Ker}(\varphi_U)$ with the restriction maps induced from $\mathcal{F}$, is a sheaf on X.

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The fact that $\mathrm{Ker}(\varphi)$ is a presheaf follows from the presheaf axioms of $\mathcal{F}$. For example, by definition, $\mathrm{res}_{U,U}^{\mathrm{Ker}(\varphi)} = \mathrm{res}_{U,U}^{\mathcal{F}} |_{\mathrm{Ker}(\varphi_U)}$, which is simply $\mathbf{1}_{\mathrm{Ker}(\varphi_U)}$.

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1. φ is said to be *injective* if $\mathrm{Ker}(\varphi)$ is the zero sheaf.
2. Defining surjective morphisms requires the notion of a sheafification, to be defined in the next lecture.
3. However, we have another characterization of surjective and injective morphisms using stalks (to be proved in the home exercise).

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Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a top space X.

1. φ is said to be *injective* if $\mathrm{Ker}(\varphi)$ is the zero sheaf.
2. Defining surjective morphisms requires the notion of a sheafification, to be defined in the next lecture.
3. However, we have another characterization of surjective and injective morphisms using stalks (to be proved in the home exercise).

Fact

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ as above is injective (resp surjective) iff, for any $p \in X$, φ induces an injective (resp surjective) homomorphism $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$.

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Example

Let $X = \mathbb{R}$ with the standard topology, and let $\mathcal{F}$ be the sheaf of locally constant functions on X with values in $\mathbb{Z}$, and $\mathcal{G} = i_{0,\mathbb{Z}} \oplus i_{1,\mathbb{Z}}$ (direct sum of two skyscraper sheaves).

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Let $\varphi : \mathcal{F} \to \mathcal{G}$ be defined, for any $U \subseteq X$ open, by

$$\varphi_U(s) = (\widetilde{s}(0), \widetilde{s}(1)) \quad (s \in \mathcal{F}(U)),$$

where $\widetilde{s} : \mathbb{R} \to \mathbb{Z}$ is such that $\widetilde{s}\,|_U \equiv s$ and $\widetilde{s}\,|_{U^c} \equiv 0$.

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However, $\mathcal{F}(X) = \mathbb{Z}$ and $\mathcal{G}(X) = \mathbb{Z} \oplus \mathbb{Z}$, and the map φ_X is *not* surjective.

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Definition

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   $\subseteq$: Since $I_{\alpha_0} \subseteq \sum I_\alpha$ for all α_0, $\mathfrak{p} \in V(\sum I_\alpha)$ implies $\mathfrak{p} \in V(I_{\alpha_0})$ for all α_0.

The collection $\{V(I) : I \lhd R\}$ is the set of closed sets for a topology on $\mathrm{Spec}(R)$, which is known as the *Zariski Topolgy* of R.

Exercise

Let R be a ring.

1. Show that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and, in particular, that $\{\mathfrak{p}\}$ is closed iff $\mathfrak{p}$ is maximal.
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2. Note: $(0) \in \operatorname{Spec}(R)$ iff R is a domain, in which case $V(0) = R$.

□

It still remains to define a sheaf structure on $\mathrm{Spec}(R)$. We will do this in the next tutorial.

Questions?