Generalized functions
Exercise sheet 2

Exercise 1. Let V be a finite dimensional topological vector space over $\mathbb{R}$. Prove that, if V is Hausdorff, then V is homeomorphic to $\mathbb{R}^{\dim V}$.

Exercise 2. Let V be a locally convex topological vector space. Show that the following conditions are equivalent for V Hausdorff:

(1) V is metrizable.
(2) V is first countable; i.e. , that each point $v \in V$ has a countable neighborhood basis.
(3) There exists a countable family of seminorms $\{\nu_i\}_{i\in\mathbb{N}}$ which defines the topology on V.

Conclude that a locally convex topological vector space$V$ is Fréchet if and only if it is complete and satisfies either of the conditions above.

Hint for the implication (3)⇒(2): show that a seminorm ν on V gives rise to a *bounded* map $d_\nu(x,y) = \frac{\nu(x-y)}{1+\nu(x-y)}$ on V, which satisfies all metric axioms except for the fact that $d_\nu(x,y) = 0$ does not imply $x \neq y$. What can you say about $\sum_{i=1}^{\infty} 2^{-i} d_{\nu_i}(\cdot,\cdot)$?

Exercise 3. Let V be a topological vector space over $\mathbb{R}$, and $0 \in U \subseteq V$ a open and convex.

(1) Show that U is balanced (i.e. $\lambda U \subseteq U$ whenever $|\lambda| \leq 1$) if and only if U is symmetric (i.e. $U = -U$).
(2) Show that there exists an open, convex and *balanced* set $W \subseteq U$ with $0 \in W$.

Exercise 4. Let V a topological vector space over $\mathbb{R}$, and let $0 \in C$ be an open, balanced and convex subset.

(1) Show that $N_C(v) := \inf\left\{\lambda \in \mathbb{R}_{>0} : \lambda^{-1}v \in C\right\}$ defines a seminorm on V.
(2) Assume further that V is normed and C is bounded. Show that N_C defines a norm on V.
(3) With the assumptions of (2), let $\|\cdot\|$ denote the norm of V. Show that N_C is equivalent to $\|\cdot\|$, i.e. there exist $c_1, c_2 > 0$ such that $c_1 N_C(v) \leq \|v\| \leq c_2 N_C(v)$ for all $v \in V$.
    Hint: Consider the values $\sup\{\|u\| : u \in C\}$ and $\inf\{\|u\| : u \notin C\}$.
(4) Find a locally convex topological vector space that admits no continuous norm.

Exercise 5. Given a topological vector space V over $\mathbb{R}$, let $V^\sharp$ denote the complete dual of V (i.e. the space of all linear maps $V \to \mathbb{R}$) and V^* denote the continuous dual of V.

Let $W \subseteq V$ be locally convex topological vector spaces.

(1) Show that the restriction map $V^\sharp \to W^\sharp$ is surjective.
(2) Show that the restriction map $V^* \to W^*$ is surjective.

Exercise 6. Let $V, \bar{V}$ be topological vector spaces, with $\bar{V}$ complete , and a continuous linear map $\iota : V \to \bar{V}$.

(1) * Show that a subspace $W \subseteq \bar{V}$ is closed if and only if it is complete.
(2) Show that the following are equivalent.
    (a) ι is a homeomorphism of V onto $\iota(V)$, and $\iota(V)$ is dense in $\bar{V}$.
    (b) For every complete space W and any map $f : V \to W$, there exists a unique map $F : \bar{V} \to W$ such that $f = F \circ \iota$.

    A space $\bar{V}$ satisfying the above conditions is called the *completion* of V.

    Remark: You may use the conclusions of items (1) and (3) in the proof of (2), even if you choose not to submit them.
(3) * Construct the completion of V (see [**1**, Theorem 5.2]).

Exercise 7.

(1) * Let V be a locally convex topological vector space. Show that if V is first countable and sequentially complete, then V is complete.
(2) Prove that $C^\infty(\mathbb{R})$ is complete.

References

[1] Trèves, François, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967.