Algebraic Geometry 2 Exercise sheet 3

Exercise 1. The Yoneda Lemma. Let \underline{C} be a category. Recall that, given an object A in \underline{C} we have a functor $h^A : \underline{C}^{\mathrm{op}} \to \underline{\mathsf{Set}}$, defined by

$$h^A(B) = \operatorname{Mor}_{\mathsf{C}}(B, A) \text{ for } B \in \underline{\mathsf{C}}.$$

- (1) The assignment $A \mapsto h^A$ gives rise to a functor $I : \underline{\mathsf{C}} \to \operatorname{Funct}(\underline{\mathsf{C}}^{\operatorname{op}}, \underline{\mathsf{Set}})$. Describe the action of I on morphisms in $\underline{\mathsf{C}}^{\operatorname{op}}$.
- (2) Show that I is fully faithful; that is, given objects A, A' of \underline{C} and a natural transformation $\eta : h^A \to h^{A'}$, show that there exists a unique morphism $\tau \in \operatorname{Mor}_{\mathbf{C}}(A, A')$ such that $\eta = I(\tau)$.

Exercise 2. Let X be a scheme. Given pairs (K_1, p_1) , (K_2, p_2) , with K_1, K_2 fields and $p_i : \operatorname{Spec}(K_i) \to X$ a morphism, we write $(K_1, p_1) \sim (K_2, p_2)$ is there exists a field with inclusion maps $K_2, K_1 \to L$ such that the diagram

$$\operatorname{Spec}(L) \longrightarrow \operatorname{Spec}(K_1)
\downarrow \qquad \qquad \downarrow^{p_1}
\operatorname{Spec}(K_2) \longrightarrow^{p_2} X$$

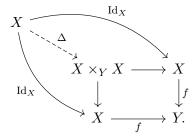
commutes. Shows that \sim defines an equivalence relation on such pairs, and that the its equivalence classes are in bijection with the points of X.

Exercise 3. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ be affine schemes. Let $\varphi : A \to B$ be a ring homomorphism, and $f: Y \to X$ the corresponding morphism of spectra.

- (1) Show that φ is injective if and only if the associated map $f^{\sharp}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective (i.e. $\operatorname{Ker}(f^{\sharp})$ is the zero sheaf). In addition, show that, under this assumption, f(Y) is dense in X.
- (2) Show that, if φ is surjective, then f is a homeomorphism onto a closed subset of X and f^{\sharp} is surjective.
- (3) Conversely, assume f a homeomorphism onto a closed set and f^{\sharp} is surjective, and show that φ is surjective.
- (4) * Find an example where f is a homeomorphism onto a closed set and φ is not surjective.

Exercise 4. Let X, Y, Z be affine schemes and $f: X \to Y, g: Z \to Y$ morphisms of schemes. Let A, B and C denote the rings of global sections of X, Y and Z, respectively.

- (1) Show that $X \times_Y Z \simeq \operatorname{Spec}(A \otimes_C B)$.
 - **Hint.** Apply Exercise 4 of Worksheet 2 in proving that $\operatorname{Spec}(A \otimes_C B)$ satisfies the required universal property.
- (2) Let $\Delta: X \to X \times_Y X$ be the unique morphism given by the diagram



Show that, for X and Y affine, Δ is a closed embedding.

Hint: Apply Ex 3.(2) above.

(3) * Show that the scheme defined in Example 2.3.6 in Hartshorne (a line with doubled origin) is not affine.

Exercise 5. Let R be a ring and $\underline{\mathsf{Mod}}_R$ the category of R-modules.

- (1) Let P be an R-module, and $A \to B \to C \to 0$ a short exact sequence of R-modules. Show that $0 \to \operatorname{Hom}(C,P) \to \operatorname{Hom}(B,P) \to \operatorname{Hom}(A,P)$
 - is exact. That is, show that the functor $\operatorname{Hom}(\cdot, P)$ is right exact, considered as a functor on $\operatorname{\underline{\mathsf{Mod}}}^{\operatorname{op}}_R$.
- (2) Conversely, let $A \to B \to C \to 0$ be a sequence such that for any R-module P, the sequence $0 \to \operatorname{Hom}(C,P) \to \operatorname{Hom}(B,P) \to \operatorname{Hom}(A,P)$ is exact. Prove that $A \to B \to C \to 0$ is exact. **Hint**. Consider the cases P = C and $P = A/\operatorname{Im}(A \to B)$.
- (3) Let $F, G: \underline{\mathsf{Mod}}_R \to \underline{\mathsf{Mod}}_R$ be adjoint functors, i.e. such that there exists a natural bijection $\mathsf{Hom}(FA,B) \simeq \mathsf{Hom}(A,GB)$ for any $A,B \in R-\underline{\mathsf{Mod}}$.

Show that F is left exact and G is right exact.