Generalized functions
Exercise sheet 4

Exercises marked with * are optional.

Exercise 1. Prove the product formula

$$|x|_\infty \cdot \prod_{p \text{ prime}} |x|_p = 1, \quad \text{for all } x \in \mathbb{Q},$$

where, for each prime p, $|\cdot|_p$ is normalized so that $|p|_p = p^{-1}$.

Exercise 2. Ostrowki's Theorem. Let $|\cdot|$ be a non-trivial absolute value on $\mathbb{Q}$.

(1) Show that $|\cdot|$ is ultra-metric if and only if $|x| \leq 1$ for all $x \in \mathbb{Z}$.
(2) Assume $|\cdot|$ is ultra-metric.
  (a) Show that $\mathfrak{a} = \{x \in \mathbb{Z} : |x| < 1\}$ is a maximal ideal of $\mathbb{Z}$.
  (b) Write $p = |\mathbb{Z} : \mathfrak{a}|$, and let $x = p^m \frac{a}{b}$, with a, b coprime to p. Show that $|x| = |p|^m$.
(3) Assume $|\cdot|$ is not ultra-metric.
  (a) Show that $|x| \geq 1$ for all $x \in \mathbb{Z}$. (*Hint.* If there exists $n_0 \in \mathbb{Z}$ such that $|n_0| < 1$, use base-n_0 representations of integers to show that $|\mathbb{Z}|$ is bounded).
  (b) Let $m, n \in \mathbb{Z}$. Using the base-n representation of m, prove that $|m| \leq (1 + \frac{\log m}{\log n}) n \cdot |n|^{\log m / \log n}$.
  (c) Substitute m^k for m in the previous item, and deduce that $m^{1/\log m} \leq n^{1/\log n}$, for any $m, n \in \mathbb{Z}$.
(4) Prove *Ostrowski's theorem.* Show that $|\cdot|$ is equivalent to either the standard absolute value, or a p-adic absolute value on $\mathbb{Q}$.

Exercise 3. Given $a \in \mathbb{Q}_p$ and $\epsilon > 0$, write $B_\epsilon(a) = \left\{ x \in \mathbb{Q}_p : |x - a|_p < \epsilon \right\}$ for the open ball of radius ϵ around a.

(1) Show that $B_\epsilon(a)$ is open and closed. * Show in addition that it is compact.
(2) Show that $B_\epsilon(a) = B_\epsilon(b)$ whenever $|a - b|_p < \epsilon$.
(3) Prove that $\{B_\epsilon(0) : \epsilon > 0\}$ is countable.

Exercise 4. Let C denote the Cantor set.

(1) Show that $\mathbb{Z}_p$ is homeomorphic to C.
(2) Show that $\mathbb{Q}_p$ is homeomorphic to $C \times \mathbb{N} \simeq C \setminus \{0\}$.
(3) * Show that $\mathbb{Q}_p^n$ is homeomorphic to $\mathbb{Q}_p$.
(4) * Show that any open subset $U \subseteq \mathbb{Q}_p$ is homeomorphic to either C or $C \setminus \{0\}$.

Definition 4.1. Recall tha an ℓ-space is a topological space which is totally-disconnected locally-compact and Hausdorff.

Exercise 5. Let X be an ℓ-space.

(1) * Show that there exists a basis of open compact sets for the topology of X.
(2) Let $K \subseteq X$ be a compact set, and let $K = \bigcup_\alpha U_\alpha$ be an open cover. Show that there exist disjoint compact open sets $V_1, \ldots, V_n$ such that $K \subseteq \bigcup_{i=1}^n V_i$ and such that for any $i = 1, \ldots, n$ there exists α such that $V_i \subseteq U_\alpha$.

Exercise 6. Let X be an ℓ-space. In this exercise, we show that $S^*(X)$ is a sheaf.

(1) *Locality axiom.* Let $X = \bigcup_{i \in I} U_i$ be an open cover of X and let $\xi \in S^*(X)$ be such that $\xi|_{U_i} \equiv 0$ for all $i \in I$ (i.e. $\xi(f) = 0$ for any $f \in S(U_i)$ and $i \in I$). Show that $\xi \equiv 0$.
(2) *Gluing axiom.* Let $X = \bigcup_{i \in I} U_i$ be an open cover and, for any i, let $\xi_i \in S^*(U_i)$. Assume $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ for any $i, j \in I$. Show that there exists $\xi \in S^*(X)$ such that $\xi|_{U_i} = \xi_i$ for any $i \in I$.

Exercise 7. * Basic properties of topological groups. Let G be a topological group.

(1) Given $x \in G$, show that the map $g \mapsto xg$ and $g \mapsto gx$ define homeomorphisms of G. Show that the inversion map $g \mapsto g^{-1}$ is also a homeomorphism.
(2) Let $H \subseteq G$ be a subgroup. Show that $\overline{H}$ (the closure of H) is also a subgroup of G. Show that, if H is normal in G, then so is $\overline{H}$.
(3) Show that an open subgroup is closed.
(4) Assume $H \subseteq G$ is open and of finite index, say $|G : H| = n$. Define a homomorphism $\Phi : G \to \mathrm{Sym}(G/H)$ by $\Phi(g)(g'H) = gg'H$. Show that $N = \mathrm{Ker}\Phi$ is a normal open subgroup of G of index $\leq n!$.

(Here $\mathrm{Sym}(X)$ is the group of permutations of a finite set X.)