Algebraic Geometry 2
Tutorial session 5

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Proposition (2.3 in Hartshorne)

1. *Let* $\varphi : A \to B$ *be a homomorphism of rings. Then* φ *induces a natural morphism of locally ringed spaces*

$$(f, f^\sharp) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}).$$

2. *Conversely, any morphism of locally ring spaces* $(f, f^\sharp)$ *as above is induced from a ring homomorphism* $\varphi : A \to B$.

Example

Let $R = \overline{\mathbb{F}_p}[[t]]$, and $\varphi : R \to R$ defined by $\varphi(\sum a_i t^i) = \sum a_i^q t^i$, for $q = p^\alpha$.

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- The associated map $f = \varphi^{-1} : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is the identity map.
- Given $V \subseteq \mathrm{Spec}(R)$ and $s \in \mathcal{O}_{\mathrm{Spec}(R)}(V)$, we have

$$f^\sharp(s)(\cdot) = (s(\cdot))^q.$$

Corollary (of the proposition)

Let A, B be rings. Then A and B are isomorphic if and only if $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are isomorphic as locally ringed spaces.

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More precisely, the proposition says that

$$A \mapsto \operatorname{Spec}(A)$$

is a fully faithful functor from the category of commutative unital rings to the category of schemes; the image of this functor is the sub-category of *affine schemes*.

Definition

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Here $\mathcal{O}_X |_{U_\alpha}$ denotes the restricted sheaf $\mathcal{O}_X |_{U_\alpha} (V) = \mathcal{O}_X(V)$ for $V \subseteq U_\alpha$ open.

Example

Let K be an infinite field and consider

$$X = ((\{0\} \times \mathbb{A}_K) \cup (\mathbb{A}_K \times \{1\})/ \sim,$$

where $\sim$ is the smallest equivalence relation such that $(0, x) \sim (x, 1)$ for all $0 \neq x \in K$.

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X is also endowed with a structure sheaf $\mathcal{O}_X$:
Let X_1, X_2 be the images of $\{0\} \times \mathbb{A}_K$ and $\mathbb{A}_K \times \{1\}$ in X, respectively. Given $U \subseteq X$ open, write $U_i = U \cap X_i$ and define

$$\mathcal{O}_X(U) = \{(s_1, s_2) \in \mathcal{O}_{X_1}(U_1) \times \mathcal{O}_X(U_2) : s_1 \mid_{U_1 \cap U_2} = s_2 \mid_{U_1 \cap U_2}\}.$$

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Answer

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$$\mathcal{O}_X(X) := \{(s_1, s_2) \in \mathcal{O}_{X_1}(X_1) \times \mathcal{O}_{X_2}(X_2) : s_1 \mid_{X_1} = s_2 \mid_{X_2}\},$$

i.e. is the set of polynomial functions on $X_i = \mathbb{A}_K$ which agree everywhere except zero. Since K is infinite, this can only occur if $s_1 = s_2$ as polynomials.

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Let $I \triangleleft A$ be an ideal and $\varphi : A \to A/I$ the quotient map. The associated map $f = \varphi^{-1} : \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is a closed injective map whose image is $V(I)$.

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For example, for $A = \mathbb{C}[x, y]/(xy)$, $\mathrm{Spec}(A)$ is naturally identified with a subset of $\mathbb{A}^2_{\mathbb{C}}$ (the cross $xy = 0$), with its minimal primes (x) and (y) corresponding the (generic points of the) irreducible components of $\mathrm{Spec}(A)$. All other points are closed and of the form $(x - a, y)$ or $(x, y - b)$.

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Fact

The map $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$ is an order-preserving (wrt inclusion) bijection of the primes of $S^{-1}A$ onto the primes $\mathfrak{q} \lhd A$ such that $\mathfrak{q} \cap S = \varnothing$.

Examples of morphisms of affine schemes

- If $S = A \setminus \mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(A)$, then $f : \mathrm{Spec}(A_\mathfrak{p}) \to \mathrm{Spec}(A)$ is the inclusion of $\overline{\mathfrak{p}} = V(\mathfrak{p})$ in $\mathrm{Spec}(A)$.

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  For example, take $A = \mathbb{C}[x, y]$ and $\mathfrak{p} = (x)$. Then

  $$S^{-1}A = \mathbb{C}[x, y]_{(x)} = \{f(x, y)/g(x, y) \mid g(0, y) \not\equiv 0\}$$

  and f is given by $f(\mathfrak{q}) = \mathfrak{q} \cap A$, for $\mathfrak{q} \in \operatorname{Spec}(S^{-1}A)$. It maps $\operatorname{Spec}(S^{-1}A)$ homeomorphically onto the affine line $V((x)) \subseteq \mathbb{A}^2$,

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  For example, for $A = \mathbb{C}[x]$ and $a = (x)$, we get $S^{-1}A = A_x = \mathbb{C}[x, x^{-1}]$ and f is the open inclusion of $D(x) = \mathbb{A}^1 \setminus V(x)$ (line without the origin) into $\mathbb{A}^1$.

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We have an inclusion

$$\mathrm{Spec}(k(x)) \xrightarrow{\text{closed embedding}} \mathrm{Spec}(\mathcal{O}_{X,x}) \xrightarrow{\text{homeo onto } V(x)} \mathrm{Spec}(A)$$

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That is-

Corollary

For any scheme X and $x \in X$, there exists a field $k = k(x)$ and a morphism $\mathrm{Spec}(k) \to X$ whose image is precisely x.

Exercise (Hartshorne, Ex 2.7)

Let X be a scheme and K a field. Show that to give a morphism $\mathrm{Spec}(K) \to X$ is equivalent to specifying a point $x \in X$ and an inclusion of fields $k(x) \to K$.

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Proof.

Assume given a morphism $f : \mathrm{Spec}(K) \to X$ and let x be the image of the unique point $* \in \mathrm{Spec}(K)$.

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Assume given a morphism $f : \mathrm{Spec}(K) \to X$ and let x be the image of the unique point $* \in \mathrm{Spec}(K)$. The associated sheaf morphism $f^\sharp$ induces a *local* homomrphism

$$f_x^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{\mathrm{Spec}(K),*} \simeq K,$$

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i.e. $\mathfrak{m}_x = \mathrm{Ker}(f)$. Thus $f^\sharp$ reduces to an inclusion $k(x) \hookrightarrow K$. Conversely, given a map $k(x) \to K$, we get a morphism

$\mathrm{Spec}(K) \to \mathrm{Spec}(k(x))$, which we can then compose with the map $\mathrm{Spec}(k(x)) \to X$ □

The functor of points approach

Tell me who your friends are and I'll tell you who you are

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$$h^A(B) = \mathrm{Mor}_{\underline{\mathbf{C}}}(B, A),$$

acting on a morphism $f : B \to C$ by

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Definition

The **Yoneda functor** $I : \underline{\mathbf{C}} \to \mathrm{Mor}_{\underline{\mathbf{Cat}}}(\underline{\mathbf{C}}^{\mathrm{op}}, \underline{\mathbf{Set}})$ is defined by

$$I(A) = h^A \quad \text{and} \quad I(\psi)(\varphi) = \psi \circ \varphi$$

for $A, A', B \in \underline{\mathbf{C}}$, $\psi : A \to A'$ and $\varphi \in \mathrm{Mor}_{\underline{\mathbf{C}}}(A, B)$.

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Proof of fullness.

Let $\varphi : h^A \to h^{A'}$, $C \in \underline{\mathbf{C}}$ and $\tau \in \mathrm{Mor}_{\underline{\mathbf{C}}}(C, A)$ be given. We seek $\psi \in \mathrm{Mor}_{\underline{\mathbf{C}}}(A, A')$ such that $\varphi_C(\tau) = I(\psi)(\tau) = \psi \circ \tau$.

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$$\begin{tikzcd} h^A(A) \arrow[r, "h^A(\tau)"] \arrow[d, "\varphi_A"'] & h^A(C) \arrow[d, "\varphi_C"] \\ h^{A'}(A) \arrow[r, "h^{A'}(\tau)"] & h^{A'}(C). \end{tikzcd}$$

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Specifically:

$$\varphi_C(\tau) = \varphi_C \circ h^A(\tau)(\mathbf{1}_A) = h^{A'}(\tau) \circ \varphi_A(\mathbf{1}_A) = \varphi(\mathbf{1}_A) \circ \tau.$$

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In fact, the Yoneda lemma is even more general (see Wikipedia), but we will not require this generality at the moment.

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New definition of (affine) schemes)

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This is called the *functor of points* of X. By an exercise, a scheme is affine iff F_X is representable; specifically, $F_X(\cdot) = \mathrm{Hom}(\mathcal{O}_X(X), \cdot)$.

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A general scheme, in this setting, would be a functor which is “locally representable”.

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The associated functor of points, however, is very easy. Namely-

$$F_{\mathrm{Spec}(A)}(R) = \mathrm{Hom}_{\underline{\textbf{Rings}}}(\mathbb{Z}[t, t^{-1}], R) = R^{\times},$$

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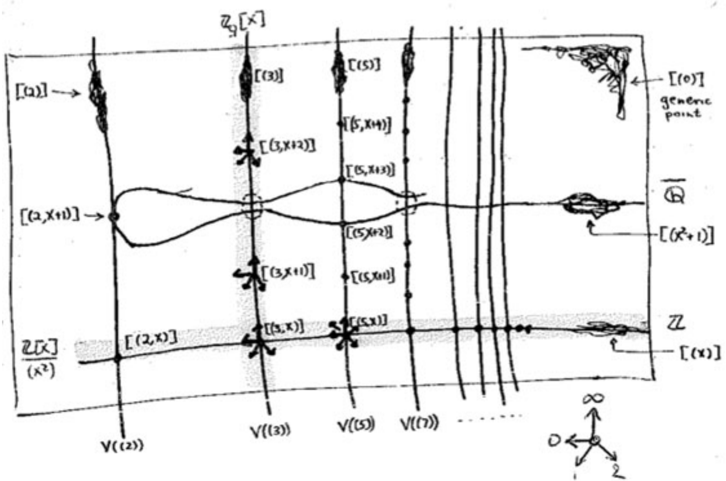
In fact, we can also encode the group laws of $R^{\times}$ in terms natural transformations of $F_{\mathrm{Spec}(A)}$, giving rise to $F_{\mathrm{Spec}(A)}$ as a *group scheme*. For example, the inversion in $R^{\times}$ is "encoded" in the map

$$t \mapsto t^{-1} : A \to A.$$

Another very nice example is Mumford's doodle of $\mathrm{Spec}(\mathbb{Z}[t])$:



See http://www.neverendingbooks.org/grothendiecks-functor-of-points for more information.

In a sense, the functor of points approach and the spectrum manifest two possible interpretations of a phenomenon from classical algebraic topology. Given an affine variety $V \subseteq \mathbb{C}^n$, it is given as the solution set of a set of polynomials. However this set is not uniquely defined. What *is* uniquely defined is the (radical of) the ideal generated by these polynomials, or, equivalently, the associated coordinate algebra. If we start with the coordinate algebra $A = \mathbb{C}[V]$, instead of the variety, there are ways to reconstruct the variety:

1. Consider all maximal ideals of A; or
2. Consider all homomorphisms $A \to \mathbb{C}$.

Over $\mathbb{C}$, Nullstellensatz tells us that these two give the "same" answer and are sufficient to describe V.
If we want to consider non-ac fields, or rings, these two methods diverge. The first generalizes to the spectum, and the second to the functor of points.

Questions?