Algebraic Geometry 2
Exercise sheet 4

Exercise 1. Let $(X, \mathcal{O}_X)$ be a scheme.

(1) Show that $(X, \mathcal{O}_X)$ is reduced iff for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ has no nilpotent elements.
(2) Let $(\mathcal{O}_X)_{\text{red}}$ be the sheafification of the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$, where $A_{\text{red}} := A/\text{nil}(A)$ for any ring A. Show that $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme, and that there is a natural injective map $(X, (\mathcal{O}_X)_{\text{red}}) \to (X, \mathcal{O}_X)$.
(3) Let $f : X \to Y$ be a morphism of schemes, and assume X is reduced. Show that there exists a unique morphism $g : X \to Y_{\text{red}}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "\exists! g"] & Y \\ & Y_{\text{red}} \arrow[u] \end{tikzcd}$$

commutes. (Here Y_{red} is shorthand for $(Y, (\mathcal{O}_Y)_{\text{red}})$.)

Exercise 2. Let $f : X \to Y$ be a morphism of schemes and $y \in |Y|$. Show that the underlying topological space of the fiber $X_y := k(y) \times_Y X$ is homeomorphic to $f^{-1}(y)$ with the topology induced from $|X|$. (Here $k(y)$ denotes the residue field $\mathcal{O}_{Y,y}/\mathfrak{m}_y$.)

Remark. We have proved this in the case where X is an affine scheme in the tutorial. It is recommended (though not necessary) that you try to reprove this yourselves. For X non-affine, use the fact that X is covered by affine open subsets, in order to glue the fiber X_y from its corresponding affine subsets.

Exercise 3. The affine communication lemma. Let X be a scheme.

(1) Let $\text{Spec}(A)$ an $\text{Spec}(B)$ be affine open subschemes of X. Given $x \in \text{Spec}(A) \cap \text{Spec}(B)$, show that there exist elements $f \in A$ and $g \in B$ such that the open sets $D^A(f) \subseteq \text{Spec}(A)$ and $D^B(g) \subseteq \text{Spec}(V)$ coincide and contain x. Here $D^R(f) = \{\mathfrak{p} \in \text{Spec}(R) : f \notin \mathfrak{p}\}$, for any ring R and $f \in R$.
(2) Let $(\mathbf{P})$ be a property enjoyed by some open affine subsets of X, such that
  (a) if an affine open subset $\text{Spec}(A) \hookrightarrow X$ has $(\mathbf{P})$ then, for any $f \in A$, the principal open subset $D^A(f) = \text{Spec}(A_f)$ has $(\mathbf{P})$ as well; and
  (b) if $\text{Spec}(A) \hookrightarrow X$ and $f_1, \dots, f_N \in A$ are such that $A = Af_1 + \cdots + Af_N$ and such that $\text{Spec}(A_{f_i}) \hookrightarrow X$ has $(\mathbf{P})$ for all $i = 1, \dots, N$, then $\text{Spec}(A)$ has $(\mathbf{P})$ as well.

  Assume $X = \bigcup_{i \in I} \text{Spec}(A_i)$, where $\text{Spec}(A_i)$ has property $(\mathbf{P})$ for all i. Then any open affine subscheme of X has $(\mathbf{P})$.

  Remark. A property $(\mathbf{P})$ as above is called an *affine-local property.*

(3) Verify that the following properties of an open affine $V \subseteq X$ are affine-local:
  (a) V is the spectrum of a noetherian ring.
  (b) V is a reduced scheme.
  (c) V is the spectrum of a finitely generated B algebra, for a fixed ring B.
(4) Show that a morphism $f : X \to Y$ is locally of finite type (as defined in Hartshorne, p 84) if and only if for any affine open $V = \text{Spec}(B) \subseteq Y$ the preimage $f^{-1}(V)$ can be covered by affine open subsets $U_i = \text{Spec}(A_i)$ with A_i a finitely generated B-algebra.

Exercise 4. Stability under base change. Show that the following properties of morphisms of schemes are stable under base change; that is given a morphism of schemes $f : X \to S$ with one the properties listed below, and another morphism $g : S' \to S$, show that the morphism $X \times_S S' \to S'$ has the same property.

(1) Open immersion

(2) Closed immersion
(3) Quasicompact (i.e. $f^{-1}(V)$ is quasicompact for all open affine $V \subseteq S$)
(4) Finite
(5) Locally of finite type
(6) Finite type

Exercise 5. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes.

(1) Show that, if f is quasicompact and $g \circ f$ is of finite type, then f is of finite type.
(2) Assume Y is a noetherian scheme and f is of finite type. Show that X is noetherian.

Exercise 6.

(1) Let $S = \bigoplus_{d \in \mathbb{N} \cup \{0\}} S_d$ be a graded ring. Show that $\mathrm{Proj}(S) = \varnothing$ if and only if every $f \in S_+ = \bigoplus_{d>0} S_d$ is nilpotent.
(2) Let $\varphi : S \to T$ be a degree preserving homomorphism between graded rings. Let
$$U = \{\mathfrak{p} \in \mathrm{Proj}(T) : \mathfrak{p} \not\supseteq \varphi(S_+)\}\,.$$
Show that $U \subseteq \mathrm{Proj}(T)$ is an open subset and that φ determines a natural morphism $f : U \to \mathrm{Proj}(S)$.
(3) * In the notation of the previous item, show that f can be an isomorphism even if φ is not.