Lösningsförslag tenta 20210318.

1. V i watt, M mätt i kg.

$$V = 2{,}652 M^{0{,}713}$$

a) Om $V = 130$, vad är M?

$$130 = 2{,}652 M^{0{,}713}$$

$$M^{0{,}713} = \frac{130}{2{,}652}$$

$$\left(M^{0{,}713}\right)^{1/0{,}713} = \left(\frac{130}{2{,}652}\right)^{1/0{,}713}$$

$$M = \left(\frac{130}{2{,}652}\right)^{1/0{,}713} \approx 234{,}8$$

b) V ökar med 50%, hur många procent ökar M?

Låt $V_2 = 1{,}5 V_1$

$$V_2 = 2{,}652 M_2^{0{,}713}$$

$$1{,}5 = \frac{V_2}{V_1} = \frac{\cancel{2{,}652} M_2^{0{,}713}}{\cancel{2{,}652} M_1^{0{,}713}} = \left(\frac{M_2}{M_1}\right)^{0{,}713}$$

$$\frac{M_2}{M_1} = (1{,}5)^{1/0{,}713} \approx 1{,}77$$

$M_2 = 1{,}77 M_1$. Svar: den ökar med 77 %.

c) Om $M_2 = 1{,}5 M_1$, vad $\frac{V_2}{V_1}$?

$$\frac{V_2}{V_1} = \frac{2{,}652 M_2^{0{,}713}}{2{,}652 M_1^{0{,}713}} = \left(\frac{1{,}5 \cancel{M_1}}{\cancel{M_1}}\right)^{0{,}713}$$

$\approx 1{,}36$. Svar: Med ca 36 %.

2. $f(x) = \sqrt{\frac{2x^3}{3} - \frac{x}{2}}$

a) $f'(x) = \dfrac{1}{2\sqrt{\frac{2x^3}{3} - \frac{x}{2}}} \cdot \left(2x^2 - \frac{1}{2}\right)$

$= \dfrac{2x^2 - 1/2}{2\sqrt{\frac{2x^3}{3} - \frac{x}{2}}} = \dfrac{x^2 - 1/4}{\sqrt{\frac{2x^3}{3} - \frac{x}{2}}}.$

b) Kritiska punkter:

$f'(x) = 0$ betyder att

$x^2 - 1/4 = 0 \iff x = \pm\frac{1}{2}.$

Båda utanför $[1, 5]$.

Ändpunkter: $f(1) \approx 0{,}41$,

$f(5) \approx 9.$

Svar: $f(1) \approx 0{,}41$ är minst och

$f(5) \approx 9$ är stort.

c) Tangent: $y = kx + m$

$k = f'(2) \approx 1{,}8$. Vet att

$f(2) = 1{,}8 \cdot 2 + m$

$2{,}1 = 1{,}8 \cdot 2 + m$

$m = 2{,}1 - 1{,}8 \cdot 2 = -1{,}5$.

Svar: Tangenten har ekv. $y = 1{,}8x - 1{,}5$.

3. $A = \begin{pmatrix} 1/2 & -2 \\ -1 & 3/2 \end{pmatrix}$

a) Visa att $\lambda_1 = -1/2$, $\lambda_2 = 5/2$.

Hitta egenvärden: Lös ekv. $\det(A-\lambda E)=0$.

$$A-\lambda E = \begin{pmatrix} 1/2 & -2 \\ -1 & 3/2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1/2-\lambda & -2 \\ -1 & 3/2-\lambda \end{pmatrix}$$

$$\det(A-\lambda E) = \begin{vmatrix} 1/2-\lambda & -2 \\ -1 & 3/2-\lambda \end{vmatrix}$$

$$= \left(\frac{1}{2}-\lambda\right)\left(3/2-\lambda\right) - 2 = \frac{3}{4} - \frac{\lambda}{2} - \frac{3\lambda}{2} - 2$$

$\frac{8}{4}$

$$+\lambda^2 = \lambda^2 - \frac{4\lambda}{2} - \frac{5}{4} = \lambda^2 - 2\lambda - \frac{5}{4}$$

Lös $\lambda^2 - 2\lambda \ \frac{5}{4} = 0$:

$$(\lambda-1)^2 - 1 - \frac{5}{4} = 0$$

$$(\lambda-1)^2 = \frac{9}{4}$$

$$\lambda - 1 = \pm \frac{3}{2}$$

$$\lambda = 1 \pm \frac{3}{2}, \quad \lambda_1 = -1/2, \ \lambda_2 = 5/2.$$

b) Hitta egenvektorerna:

Löser $(A-\lambda E)v = 0$ för

$\lambda = \lambda_1 = -1/2$ och $\lambda = \lambda_2 = 3/2$.

$\lambda_1 = -1/2$:

$$A-\lambda E = \begin{pmatrix} 1/2-(-1/2) & -2 \\ -1 & 3/2-(-1/2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \qquad \text{Om } v_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

Motsvarar $(A-\lambda E)v_1 = 0$ systemet

$$\begin{aligned} x - 2y &= 0 \\ -x + 2y &= 0. \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right) (+1)$$

$$\sim \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Då är $x = 2y$ och $y = t$.

För $t=1$ har vi då vektorn

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$\lambda_2 = 5/2$:

$$A - \lambda_2 E = \begin{pmatrix} 1/2 - 5/2 & -2 \\ -1 & 5/2 - 5/2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix}.$$

Vill lösa systemet $\left(\begin{array}{cc|c} -2 & -2 & 0 \\ -1 & -1 & 0 \end{array}\right) \begin{array}{c} \cdot(-\frac{1}{2}) \\ \leftarrow \end{array}$

$$\sim \left(\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \cdot -\frac{1}{2} \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x = -y$ och $y = t$. För $t=1$ har vi då

$$v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

c) Diagonalisera A. Dvs hitta C och D så att $A = CDC^{-1}$.

$$C = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & 5/2 \end{pmatrix}$$

Hitta C^{-1}:

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right) (-2)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right) \cdot -\frac{1}{3} \sim \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right) (-1)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right).$$

$$C^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

d) Bestäm A^9.

$$A^9 = (CDC^{-1})^9 = CD^9C^{-1}.$$

$$CD^9C^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1/2)^9 & 0 \\ 0 & (5/2)^9 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1/2)^9 \cdot \frac{1}{3} & (-1/2)^9 \cdot \frac{1}{3} \\ -(5/2)^9 \cdot \frac{1}{3} & (5/2)^9 \cdot \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3}(-1/2)^9 + \frac{1}{3}(5/2)^9 & \frac{2}{3}(-1/2)^9 - \frac{2}{3}(5/2)^9 \\ \frac{1}{3}(-1/2)^9 - \frac{1}{3}(5/2)^9 & \frac{1}{3}(-1/2)^9 - \frac{2}{3}(5/2)^9 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1272 & -2543 \\ -1271 & -2543 \end{pmatrix}.$$

4. $x_{n+2} = x_{n+1} + 6x_n$, $x_0 = 1$, $x_2 = 2$.

a) Explicit formel för x_n.

Lösa karakteristiska ekv. $r^2 = r + 6$

$$r^2 - r = 6$$

$$(r - 1/2)^2 - \frac{1}{4} = \frac{24}{4}$$

$$(r - 1/2)^2 = \frac{25}{4}$$

$$r = 1/2 \pm \frac{5}{2}. \quad r_1 = -\frac{4}{2} = -2. \quad r_2 = \frac{6}{2} = 3.$$

$$x_n = A r_1^n + B r_2^n = A(-2)^n + B\, 3^n.$$

Bestämmer A och B, nu ha x_0 och x_1.

För x_0 och x_1

$$1 = A + B \qquad A = 1 - B$$

$$2 = A \cdot (-2) + B \cdot 3 = 3B - 2A.$$

$$2 = 3B - 2(1 - B) = 3B - 2 + 2B$$

$$5B = 4, \quad B = \frac{4}{5}.$$

$$A = 1 - B = 1 - \frac{4}{5} = \frac{1}{5}.$$

$$x_n = \frac{1}{5}(-2)^n + \frac{4}{5} 3^n.$$

b) Beräkna $x_{12} + 6x_{11}$.

$$x_{13} = x_{12} + 6 \cdot x_{11}.$$

$$x_{13} = \frac{(-2)^{13}}{5} + \frac{4}{5} 3^{13} \approx 1273820.$$

5.

a) Les OBVP

$$\begin{cases} xy' = \dfrac{2xe^x + x}{y^2} \\ y(0) = 1 \end{cases}.$$

$$y' = \frac{2xe^x + x}{xy^2} = \underbrace{\frac{2xe^x + x}{x}}_{g(x)} \cdot \underbrace{\frac{1}{y^2}}_{h(y)}$$

$$\frac{dy}{dx} = (2e^x + 1) \cdot \frac{1}{y^2}$$

$$y^2 \, dy = 2e^x + 1 \, dx$$

$$\int y^2\,dy = \int 2e^x + 1\,dx$$

$$\frac{y^3}{3} = 2e^x + x + C$$: $y(0) = 1$ ger att

$$\frac{1}{3} = 2e^0 + C = 2 + C. \quad C = \frac{1}{3} - \frac{6}{3} = -\frac{5}{3}.$$

$$y = \left(3\left(2e^x + x - \frac{5}{3}\right)\right)^{1/3}.$$

2) Lös $y' = \sqrt{2x} + \frac{3}{2x^2} + 2x^3.$

$$\int y'\,dx = \int \sqrt{2}\sqrt{x} + \frac{3}{2}x^{-2} + 2x^3\,dx$$

$$y = \sqrt{2}\,\frac{1}{2\sqrt{x}} + \frac{3}{2}\,\frac{x^{-3}}{-3} + \frac{2x^4}{4} + C$$

$$y = \frac{1}{\sqrt{2}\sqrt{x}} - \frac{1}{2x^3} + \frac{x^4}{2} + C.$$

c) Beräkna:

$$\int_{-3}^{3} (x^4+2)^2 4x^3 \, dx = \left\{ \begin{array}{l} y = x^4+2 \\ \frac{dy}{dx} = 4x^3 \\ dy = 4x^3 dx \end{array} \right\}$$

$$= \int_{y(-3)}^{y(3)} y^2 \, dy .$$

$y(-3) = (-3)^4 + 2 = 83$

$y(3) = 3^4 + 2 = 83$

$$\int_{83}^{83} y^2 \, dy = 0 .$$