PhD Course in Probability and Statistics, Part II

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Note: *Please submit your solutions either as a pdf by email to shaobo.jin@math.uu.se or in person no later than* ***May 26th***. *You must hand in individual solutions. You are not allowed to discuss the tasks with anyone other than students enrolled in the course during the same year. You may reference and cite other sources outside of our course literature. Even though you reference to other literature, you still need to provide detailed steps in your solution. Using generative AI interfaces to solve tasks is strictly prohibited.*

1. (3p) Consider the follow distributions. Do they belong to the exponential family? If so, find also the minimal dimension s and determine whether it is full rank. Otherwise, state the reason.

$$
\begin{aligned}
&(a) \qquad p(x \mid \theta) = \frac{\theta}{x^{1+\theta}}, \qquad x \geq 1, \theta > 2, \\
&(b) \qquad p(x \mid \theta) = \frac{1}{2}\sin(x-\theta), \qquad \theta \leq x \leq \theta + \pi, \theta \in \mathbb{R}, \\
&(c) \qquad (X, Y), \text{where } X \sim \text{Bernoulli}(\theta) \text{ and } Y \sim \text{Bernoulli}\left(\theta^2\right), \qquad 0 < \theta < 1,
\end{aligned}
$$

2. (9p) Let X be a continuous random variable with density function

$$
p(x \mid \theta) = \sqrt{\frac{2}{\pi}} \frac{x^2}{\theta^3} \exp\left(-\frac{x^2}{2\theta^2}\right), \qquad x > 0, \theta > 0.
$$

It is known that $\mathrm{E}[X] = \theta\sqrt{8/\pi}$, $\mathrm{E}[X^2] = 3\theta^2$, and $\mathrm{Var}[X^2] = 6\theta^4$. Suppose that a random sample of n observations X_1, ..., X_n from this distribution is observed.

   (a) We have said that the expected value of the score function is zero under some regularity conditions. Show that the expected value of the score function of this model is zero by verifying the regularity conditions.

   (b) Show that $\hat{\theta}_{\mathrm{MLE}} = \sqrt{\frac{1}{3n}\sum_{i=1}^n x_i^2}$ is the MLE of θ .

   (c) Is the MLE an unbiased estimator of θ?

   (d) Is the MLE of θ a consistent estimator of θ.

   (e) Can we claim $\mathrm{E}\left[\hat{\theta}_{\mathrm{MLE}}\right] \to \theta$?

(f) Approximate the distribution of $\hat{\theta}^2_{\text{MLE}}$.

(g) Find a minimal sufficient statistic for θ.

(h) It is known that $\tilde{\theta} = \sqrt{\frac{\pi}{8}}\bar{X}$ is an unbiased estimator of θ. Can you find an unbiased estimator of θ such that it has a lower variance than your $\tilde{\theta}$?

(i) Is $\frac{1}{3n}\sum_{i=1}^n x_i^2$ an efficient estimator of θ^2?

3. (4p) Let X be a random variable with density

$$p(x \mid \theta) \;=\; \left[\frac{1}{2\pi x^3}\right]^{1/2} \exp\left\{-\frac{\theta^2}{2x}\left(x - \frac{1}{\theta}\right)^2\right\}, \qquad \text{for } x > 0 \text{ and } \theta > 0.$$

We know that $\mathrm{E}[X] = \theta^{-1}$, $\mathrm{Var}[X] = \theta^{-3}$, $\mathrm{E}[X^{-1}] = \theta + 1$, and $\mathrm{Var}[X^{-1}] = \theta + 2$. Suppose that a random sample of n observations X_1, ..., X_n from this distribution is observed. We also know that if the density of $Y = \sum_{i=1}^n X_i$ is

$$p(y \mid \theta) \;=\; \left[\frac{n^2}{2\pi y^3}\right]^{1/2} \exp\left\{-\frac{\theta^2}{2y}\left(y - \frac{n}{\theta}\right)^2\right\}.$$

(a) Suppose that θ has a prior distribution with the following density

$$\lambda(\theta) \;=\; \frac{1}{\Phi\left(\frac{\mu}{\sigma}\right)\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\theta-\mu)^2}{2\sigma^2}\right\}, \qquad \theta > 0,$$

where μ and σ^2 are constants, and $\Phi()$ is the distribution function of a $N(0,1)$ random variable. It is known that

$$\mathrm{E}[\theta] \;=\; \mu + \frac{\sigma^2}{\Phi\left(\frac{\mu}{\sigma}\right)\sqrt{2\pi}} \exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}.$$

Find the posterior distribution of θ.

(b) Consider the loss function $L(\theta, \delta) = (\theta - \delta)^2$. Find the Bayes estimator of θ.

(c) Consider another estimator $\tilde{\theta} = \frac{1}{n}\sum_{i=1}^n\left(\frac{1}{X_i} - 1\right)$. Find its frequentist risk.

(d) Is the Bayes estimator of θ an admissible estimator?

4. (3p) Consider still the distribution in Task 3. Suppose that a random sample of n observations X_1, ..., X_n from this distribution is observed.

(a) Find the UMP test of size α for testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

(b) Find the likelihood ratio test statistic for testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$. Find also the (approximated) distribution of the test statistic.

(c) Find a Wald test statistic for testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$. Find also the (approximated) distribution of the test statistic.

5. (2p) Consider the loss function $(\theta - d)^T W(\theta, x)(\theta - d)$, where $W(\theta, x)$ is a $p \times p$ symmetric and positive definite matrix. Suppose that there exists a decision rule with finite risk. Show that the Bayes decision rule is

$$\delta_B(X) = (\mathrm{E}[W(\theta, x) \mid X = x])^{-1} \mathrm{E}[W(\theta, x)\theta \mid X = x].$$

6. (2p) Prove the following Rao-Blackwell Theorem: Let T be a sufficient statistic for $\mathcal{P} = \{\mathrm{P}_\theta : \theta \in \Theta\}$. Let δ be an estimator of $g(\theta)$. Define $\eta(T) = \mathrm{E}[\delta(X) \mid T]$. If $R(g(\theta), \delta) < \infty$, and $L(\theta, \cdot)$ is convex for all θ, then $R(g(\theta), \eta(T)) \leq R(g(\theta), \delta)$. *Hint*: You may need Jensen's inequality.

7. (2p) Prove the following Lehmann-Scheffé Theorem: Let T be a complete and sufficient statistic for a parameter θ. Let $\delta(X)$ be any unbiased estimator of $g(\theta)$. Then $\eta(T) = \mathrm{E}[\delta(X) \mid T]$ is the unique unbiased estimator of $g(\theta)$ that minimizes the frequentist risk $R(g(\theta), d)$, if $L(\theta, \cdot)$ is convex for all θ.

8. (2p) Suppose that δ is an admissible decision rule and has constant risk. Show that δ is minimax.

9. (3p) In this task, we consider the maximum likelihood estimator. During the lecture, we implicitly assume that the density $p(\cdot \mid \theta)$ we assumed is the true density that generates the data. Now suppose that our assumption is $p(\cdot \mid \theta)$ but the truth is $g(x)$. Prove heuristically that the maximum likelihood $\hat{\theta}$ converges in probability to θ^* that minimizes the Kullback-Leibler divergence. Also prove heuristically that $\hat{\theta}$ is asymptotically normal.

10. (2p) Suppose that we observe iid data X_1, ..., X_n with distribution function F. We would like to use bootstrap to approximate the distribution of $T_n(\{X_i, i = 1, ..., n\}, F)$. Suppose that there exists a function $F_A(x)$ such that

$$\sup_t |P[T_n(\{X_i\}, F) \leq t] - F_A(t)| \to 0,$$
$$\sup_t \left| P\left(T\left(X^*, \hat{F}_n\right) \leq t \mid \hat{F}_n\right) - F_A(t) \right| \xrightarrow{P} 0,$$

where X^* is the bootstrap sample. Show that bootstrap is consistent.