Solution Set for Math Problems

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Problem 1: Evaluating the Double Integral

Evaluate the double integral
$$\iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy,$$
where R is the rhombus with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, and $(0, \pi)$.

Solution

To evaluate the double integral, we start with a coordinate transformation that simplifies the region and the integrand.

Step 1: Define a Coordinate Transformation

Given the symmetry of the rhombus, let's introduce new variables:
$$u = x + y \quad \text{and} \quad v = x - y.$$

In these coordinates: - The integrand becomes $v^2 \cos^2(u)$. - The Jacobian determinant for this transformation is calculated as follows:
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{2}.$$

Thus, $dx\, dy = \frac{1}{2}\, du\, dv$.

Step 2: Transform the Region R

Now, we find the bounds for u and v in the region R: 1. At $(x, y) = (\pi, 0)$:
$$u = \pi + 0 = \pi, \quad v = \pi - 0 = \pi.$$

2. At $(x, y) = (2\pi, \pi)$:
$$u = 2\pi + \pi = 3\pi, \quad v = 2\pi - \pi = \pi.$$

3. At $(x, y) = (\pi, 2\pi)$:
$$u = \pi + 2\pi = 3\pi, \quad v = \pi - 2\pi = -\pi.$$

4. At $(x, y) = (0, \pi)$:
$$u = 0 + \pi = \pi, \quad v = 0 - \pi = -\pi.$$

So, in the new coordinates, the region R is defined by:
$$\pi \leq u \leq 3\pi \quad \text{and} \quad -\pi \leq v \leq \pi.$$

Step 3: Set Up the Integral

In terms of u and v, the integral becomes:
$$\iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy = \int_{\pi}^{3\pi} \int_{-\pi}^{\pi} v^2 \cos^2(u) \cdot \frac{1}{2}\, dv\, du.$$

Simplify to:
$$\frac{1}{2} \int_{\pi}^{3\pi} \cos^2(u)\, du \int_{-\pi}^{\pi} v^2\, dv.$$

Step 4: Evaluate Each Integral Separately

1. **Integrate with respect to v:**

$$\int_{-\pi}^{\pi} v^2\,dv = \left[\frac{v^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^3}{3} - \left(-\frac{\pi^3}{3}\right) = \frac{2\pi^3}{3}.$$

2. **Integrate with respect to u:** Use the identity $\cos^2(u) = \frac{1+\cos(2u)}{2}$:

$$\int_{\pi}^{3\pi} \cos^2(u)\,du = \int_{\pi}^{3\pi} \frac{1+\cos(2u)}{2}\,du.$$

Split the integral:

$$= \frac{1}{2}\int_{\pi}^{3\pi} 1\,du + \frac{1}{2}\int_{\pi}^{3\pi} \cos(2u)\,du.$$

For the first term:

$$\frac{1}{2}\int_{\pi}^{3\pi} 1\,du = \frac{1}{2}\cdot(3\pi-\pi) = \frac{1}{2}\cdot 2\pi = \pi.$$

For the second term:

$$\frac{1}{2}\int_{\pi}^{3\pi} \cos(2u)\,du = \frac{1}{2}\cdot\left.\frac{\sin(2u)}{2}\right|_{\pi}^{3\pi} = \frac{1}{4}\left(\sin(6\pi)-\sin(2\pi)\right) = 0.$$

So, the integral with respect to u is simply π.

Step 5: Combine Results

Now, we can put everything together:

$$\frac{1}{2}\cdot\pi\cdot\frac{2\pi^3}{3} = \frac{\pi\cdot\pi^3}{3} = \frac{\pi^4}{3}.$$

Final Answer

The value of the integral is:

$$\boxed{\frac{\pi^4}{3}}.$$

Problem 2: Evaluating the Area Using Double Integration

Using double integration, evaluate the area of:

1. the cardioid $r = a(1-\cos\theta)$
2. the lemniscate $r^2 = a^2\cos 2\theta$

Solution for (i): Cardioid $r = a(1-\cos\theta)$

To find the area enclosed by the cardioid $r = a(1-\cos\theta)$, we can set up the integral in polar coordinates. In polar coordinates, the area A is given by:

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\int_{0}^{r(\theta)} r\,dr\,d\theta.$$

Step 1: Set Up the Integral For the cardioid $r = a(1-\cos\theta)$: 1. The range for θ is from 0 to 2π to cover the entire cardioid. 2. Substitute $r = a(1-\cos\theta)$.

Thus, the area is:

$$A = \frac{1}{2}\int_{0}^{2\pi}\int_{0}^{a(1-\cos\theta)} r\,dr\,d\theta.$$

Step 2: Evaluate the Integral First, integrate with respect to r:

$$A = \frac{1}{2}\int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta.$$

Simplify:

$$A = \frac{1}{2}\int_0^{2\pi} \frac{a^2(1-\cos\theta)^2}{2}\, d\theta = \frac{a^2}{4}\int_0^{2\pi} (1-\cos\theta)^2\, d\theta.$$

Expand $(1-\cos\theta)^2$:

$$A = \frac{a^2}{4}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta.$$

Use the identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:

$$A = \frac{a^2}{4}\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1+\cos(2\theta)}{2}\right) d\theta.$$

Combine terms:

$$A = \frac{a^2}{4}\int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{\cos(2\theta)}{2}\right) d\theta.$$

Now, integrate each term separately: 1. $\int_0^{2\pi} \frac{3}{2}\, d\theta = \frac{3}{2}\cdot 2\pi = 3\pi$. 2. $\int_0^{2\pi} -2\cos\theta\, d\theta = 0$ (since $\cos\theta$ is symmetric). 3. $\int_0^{2\pi} \frac{\cos(2\theta)}{2}\, d\theta = 0$.

So,

$$A = \frac{a^2}{4}\cdot 3\pi = \frac{3\pi a^2}{4}.$$

Final Answer The area enclosed by the cardioid is:

$$\boxed{\frac{3\pi a^2}{4}}.$$

Solution for (ii): Lemniscate $r^2 = a^2\cos 2\theta$

To find the area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$, note that it is symmetric about both axes. We can evaluate the area in the first quadrant and then multiply by 4.

Step 1: Set Up the Integral Rewrite r in terms of $\cos 2\theta$:

$$r = \pm a\sqrt{\cos 2\theta}.$$

The range of θ for one loop is $-\frac{\pi}{4}$ to $\frac{\pi}{4}$.

The area is:

$$A = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\, dr\, d\theta.$$

Simplify:

$$A = 2\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\, dr\, d\theta.$$

Step 2: Evaluate the Integral First, integrate with respect to r:

$$A = 2\int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta.$$

Substitute the limits:

$$A = 2\int_0^{\frac{\pi}{4}} \frac{a^2\cos 2\theta}{2}\, d\theta = a^2\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta.$$

Integrate with respect to θ:

$$A = a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = a^2\cdot\frac{\sin\frac{\pi}{2}}{2} = a^2\cdot\frac{1}{2} = \frac{a^2}{2}.$$

Final Answer The area enclosed by the lemniscate is:

$$\boxed{\frac{a^2}{2}}.$$

Problem 3: Evaluating the Area Between a Parabola and a Line

Using double integration, evaluate the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution

To find the area between the curves $y = 4x - x^2$ and $y = x$, we will: 1. Find the points of intersection of the curves. 2. Set up the integral to compute the area.

Step 1: Find Points of Intersection Set $y = 4x - x^2$ equal to $y = x$:

$$4x - x^2 = x.$$

Rearrange to form a quadratic equation:

$$-x^2 + 3x = 0.$$

Factor out x:

$$x(x - 3) = 0.$$

Thus, $x = 0$ and $x = 3$.

Substitute $x = 0$ and $x = 3$ back into $y = x$ to find the y-coordinates:

$$(0, 0) \quad \text{and} \quad (3, 3).$$

So, the region of interest is bounded by $x = 0$ and $x = 3$.

Step 2: Set Up the Integral For x in the interval $[0, 3]$: - The line $y = x$ is the lower curve. - The parabola $y = 4x - x^2$ is the upper curve.

The area A can be expressed as:

$$A = \int_0^3 \int_x^{4x-x^2} dy\, dx.$$

Step 3: Evaluate the Integral First, integrate with respect to y:

$$A = \int_0^3 [y]_{y=x}^{y=4x-x^2}\, dx.$$

Substitute the limits:

$$A = \int_0^3 \left((4x - x^2) - x\right) dx.$$

Simplify the expression inside the integral:

$$A = \int_0^3 (4x - x^2 - x)\, dx = \int_0^3 (3x - x^2)\, dx.$$

Step 4: Integrate with Respect to x Now, split the integral and integrate term by term:

$$A = \int_0^3 3x\, dx - \int_0^3 x^2\, dx.$$

Evaluate each integral separately: 1. For $\int_0^3 3x\, dx$:

$$\int_0^3 3x\, dx = 3 \int_0^3 x\, dx = 3 \left[\frac{x^2}{2}\right]_0^3 = 3 \cdot \frac{9}{2} = \frac{27}{2}.$$

2. For $\int_0^3 x^2\, dx$:

$$\int_0^3 x^2\, dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} = 9.$$

Subtract the results:

$$A = \frac{27}{2} - 9 = \frac{27}{2} - \frac{18}{2} = \frac{9}{2}.$$

Final Answer The area lying between the parabola $y = 4x - x^2$ and the line $y = x$ is:

$$\boxed{\frac{9}{2}}.$$

Problem 4: Evaluating the Area Between the Curves

Using double integration, evaluate the area lying between the curves $xy = 2$, $4y = x^2$, and $y = 4$.

Solution

To find the area between the curves $xy = 2$, $4y = x^2$, and $y = 4$, we will: 1. Find the points of intersection of the curves. 2. Set up the integral to compute the area.

Step 1: Find Points of Intersection First, we rearrange the equations to find their intersection points: 1. **Curve $xy = 2$** can be expressed as:

$$y = \frac{2}{x}.$$

2. **Curve $4y = x^2$** can be expressed as:

$$y = \frac{x^2}{4}.$$

3. **Curve $y = 4$** is simply a horizontal line.
We find the intersection points:
Intersection of $y = \frac{2}{x}$ and $y = \frac{x^2}{4}$: Set $\frac{2}{x} = \frac{x^2}{4}$:

$$2 \cdot 4 = x^3 \implies x^3 = 8 \implies x = 2.$$

Substituting $x = 2$ back to find y:

$$y = \frac{2}{2} = 1.$$

Thus, the intersection point is $(2, 1)$.

Intersection of $y = \frac{2}{x}$ and $y = 4$: Set $\frac{2}{x} = 4$:

$$2 = 4x \implies x = \frac{1}{2}.$$

Substituting $x = \frac{1}{2}$ back to find y:

$$y = 4.$$

Thus, the intersection point is $\left(\frac{1}{2}, 4\right)$.

Intersection of $y = \frac{x^2}{4}$ and $y = 4$: Set $\frac{x^2}{4} = 4$:

$$x^2 = 16 \implies x = 4 \quad \text{or} \quad x = -4.$$

For the positive branch, we have $(4, 4)$.

So, the points of intersection are $\left(\frac{1}{2}, 4\right)$, $(2, 1)$, and $(4, 4)$.

Step 2: Set Up the Integral The area A can be computed by setting up a double integral. We will integrate with respect to y first.

The area is given by the difference between the outer and inner curves.

1. **Vertical bounds**: From $y = 1$ to $y = 4$. 2. **Horizontal bounds** for the curves: - For $y = \frac{x^2}{4}$: $x = 2\sqrt{y}$. - For $y = \frac{2}{x}$: $x = \frac{2}{y}$.

Thus, the area A is given by:

$$A = \int_1^4 \left(2\sqrt{y} - \frac{2}{y}\right) dy.$$

Step 3: Evaluate the Integral Evaluate each term:

$$A = \int_1^4 2\sqrt{y}\, dy - \int_1^4 \frac{2}{y}\, dy.$$

1. **First integral**:

$$\int 2\sqrt{y}\, dy = \frac{2 \cdot \frac{2}{3} y^{\frac{3}{2}}}{3} = \frac{4}{3} y^{\frac{3}{2}}.$$

Thus,

$$\int_1^4 2\sqrt{y}\,dy = \left[\frac{4}{3}y^{\frac{3}{2}}\right]_1^4 = \frac{4}{3}(8-1) = \frac{4}{3}\cdot 7 = \frac{28}{3}.$$

2. **Second integral**:

$$\int \frac{2}{y}\,dy = 2\ln|y|.$$

Thus,

$$\int_1^4 \frac{2}{y}\,dy = [2\ln|y|]_1^4 = 2(\ln 4 - \ln 1) = 2\ln 4.$$

Combine both results:

$$A = \frac{28}{3} - 2\ln 4.$$

Final Answer The area lying between the curves $xy = 2$, $4y = x^2$, and $y = 4$ is:

$$\boxed{\frac{28}{3} - 2\ln 4}.$$

Problem 2: Finding the Volume of an Ellipsoid

Using double integration, find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution

To find the volume of the ellipsoid, we use a coordinate transformation to switch to spherical coordinates.

Step 1: Define a Coordinate Transformation

In spherical coordinates, the transformations are:

$$x = a\rho\sin\theta\cos\phi,$$

$$y = b\rho\sin\theta\sin\phi,$$

$$z = c\rho\cos\theta.$$

Step 2: Set up the Jacobian

The Jacobian determinant for this transformation is:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = abc\rho^2\sin\theta.$$

Step 3: Integrate in Spherical Coordinates

The volume integral in spherical coordinates is:

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\rho^2\sin\theta\,d\rho\,d\theta\,d\phi.$$

Step 4: Evaluate the Integral

First, integrate with respect to ρ:

$$\int_0^1 \rho^2 \, d\rho = \left[\frac{\rho^3}{3}\right]_0^1 = \frac{1}{3}.$$

Next, integrate with respect to θ:

$$\int_0^{\pi} \sin\theta \, d\theta = [-\cos\theta]_0^{\pi} = 2.$$

Finally, integrate with respect to ϕ:

$$\int_0^{2\pi} d\phi = 2\pi.$$

Combining all these results:

$$V = abc \times \frac{1}{3} \times 2 \times 2\pi = \frac{4}{3}\pi abc.$$

Therefore, the volume of the ellipsoid is:

$$V$$

Problem 7: Volume of a Cylinder Cut by a Plane

Find the volume of the cylinder $x^2 + y^2 = a^2$ above the xy-plane cut by the plane $x + y + z = 2a$.

Solution

To find the volume, we'll set up and evaluate the integral.

Step 1: Parametrize the Cylinder

The cylinder can be parametrized in cylindrical coordinates as:

$$x = a\cos\theta, \quad y = a\sin\theta.$$

Step 2: Define the Plane Equation

The plane equation is:

$$z = 2a - x - y.$$

In cylindrical coordinates, this becomes:

$$z = 2a - a\cos\theta - a\sin\theta.$$

Step 3: Set up the Volume Integral

The volume integral in cylindrical coordinates is:

$$V = \int_0^{2\pi} \int_0^{a} (2a - a\cos\theta - a\sin\theta) \; r \, dr \, d\theta.$$

Step 4: Evaluate the Integral

First, integrate with respect to r:

$$\int_0^a r\,dr = \left[\frac{r^2}{2}\right]_0^a = \frac{a^2}{2}.$$

The integral now becomes:

$$V = \frac{a^2}{2}\int_0^{2\pi} (2a - a\cos\theta - a\sin\theta)\,d\theta.$$

Separate the integral into three parts:

$$V = \frac{a^3}{2}\left(\int_0^{2\pi} 2\,d\theta - \int_0^{2\pi} \cos\theta\,d\theta - \int_0^{2\pi} \sin\theta\,d\theta\right).$$

Evaluate each integral:

$$\int_0^{2\pi} d\theta = 2\pi,$$

$$\int_0^{2\pi} \cos\theta\,d\theta = 0,$$

$$\int_0^{2\pi} \sin\theta\,d\theta = 0.$$

Combine the results:

$$V = \frac{a^3}{2}(2 \times 2\pi) = 2\pi a^3.$$

Therefore, the volume of the cylinder is:

$$V = 2\pi a^3.$$

Problem 8: Volume of a Sphere with a Central Hole

A circular hole of radius b is made centrally through a sphere of radius a. Find the volume of the remaining part.

Solution

To find the volume of the remaining part, we need to subtract the volume of the cylindrical hole from the volume of the sphere.

Step 1: Volume of the Sphere

The volume of a sphere of radius a is:

$$V_{\text{sphere}} = \frac{4}{3}\pi a^3.$$

Step 2: Volume of the Cylindrical Hole

The cylindrical hole has radius b and height $2\sqrt{a^2 - b^2}$ (since the height is the distance between the two circular caps). The volume of the cylindrical hole is:

$$V_{\text{cylinder}} = \pi b^2 \cdot 2\sqrt{a^2 - b^2} = 2\pi b^2\sqrt{a^2 - b^2}.$$

Step 3: Volume of the Remaining Part

The volume of the remaining part is the volume of the sphere minus the volume of the cylindrical hole:

$$V_{\text{remaining}} = V_{\text{sphere}} - V_{\text{cylinder}} = \frac{4}{3}\pi a^3 - 2\pi b^2\sqrt{a^2 - b^2}.$$

Therefore, the volume of the remaining part is:

$$V_{\text{remaining}} = \frac{4}{3}\pi a^3 - 2\pi b^2\sqrt{a^2 - b^2}.$$

Problem 9: Mass, Center of Mass, and Moment of Inertia of a Lamina

Find (i) the mass, (ii) center of mass, and (iii) moment of inertia about axes of a lamina with density function $f(x, y) = 6x$ of triangular shape bounded by the x-axis, the line $y = x$, and the line $y = 2 - x$.

Solution

Step 1: Find the Mass

The mass M of the lamina is given by the double integral of the density function over the region R:

$$M = \iint_R 6x\, dA.$$

The region R is the triangle bounded by $y = 0$, $y = x$, and $y = 2 - x$. The limits of integration can be set up as:

$$\int_0^1 \int_0^y 6x\, dx\, dy + \int_1^2 \int_0^{2-y} 6x\, dx\, dy.$$

Calculate the integrals:

$$\int_0^1 \left[3x^2\right]_0^y dy + \int_1^2 \left[3x^2\right]_0^{2-y} dy = \int_0^1 3y^2\, dy + \int_1^2 3(2 - y)^2\, dy.$$

Evaluate these integrals:

$$\left[y^3\right]_0^1 + \int_1^2 3(4 - 4y + y^2)\, dy = 1 + 3\left[4y - 2y^2 + \frac{y^3}{3}\right]_1^2 = 1 + 3(8 - 8 + \frac{8}{3} - 4 + 2 - \frac{1}{3}) = \frac{26}{3}.$$

Step 2: Find the Center of Mass

The coordinates of the center of mass $(\bar{x}, \bar{y})$ are given by:

$$\bar{x} = \frac{1}{M}\iint_R x \cdot 6x\, dA, \quad \bar{y} = \frac{1}{M}\iint_R y \cdot 6x\, dA.$$

First, find $\bar{x}$:

$$\bar{x} = \frac{1}{M}\left(\int_0^1 \int_0^y 6x^2\, dx\, dy + \int_1^2 \int_0^{2-y} 6x^2\, dx\, dy\right).$$

Evaluate the integrals:

$$\int_0^1 \left[2x^3\right]_0^y dy + \int_1^2 \left[2x^3\right]_0^{2-y} dy = \int_0^1 2y^3\, dy + \int_1^2 2(2 - y)^3\, dy.$$

$$\bar{x} = \frac{1}{M}\left(\left[\frac{y^4}{2}\right]_0^1 + \int_1^2 2(8 - 12y + 6y^2 - y^3)\, dy\right).$$

$$\bar{x} = \frac{1}{M}\left(\frac{1}{2} + 2\left[8y - 6y^2 + 2y^3 - \frac{y^4}{4}\right]_1^2\right) = \frac{1}{M}\left(\frac{1}{2} + 2(16 - 24 + 16 - 4 - (8 - 6 + 2 - \frac{1}{4}))\right).$$

$$\bar{x} = \frac{1}{M}\left(\frac{1}{2} + 2(2.75)\right) = \frac{1}{M}\left(\frac{1}{2} + 5.5\right) = \frac{6}{M} = \frac{18}{26} = \frac{9}{13}.$$

Next, find $\bar{y}$:

$$\bar{y} = \frac{1}{M}\left(\int_0^1 \int_0^y 6xy\,dx\,dy + \int_1^2 \int_0^{2-y} 6xy\,dx\,dy\right).$$

Evaluate the integrals:

$$\int_0^1 \left[3x^2y\right]_0^y dy + \int_1^2 \left[3x^2y\right]_0^{2-y} dy = \int_0^1 3y^3\,dy + \int_1^2 3(2-y)^2 \cdot y\,dy.$$

$$\bar{y} = \frac{1}{M}\left(\left[\frac{3y^4}{4}\right]_0^1 + \int_1^2 3y(4 - 4y + y^2)\,dy\right).$$

$$\bar{y} = \frac{1}{M}\left(\frac{3}{4} + 3\left[4y - 2y^2 + \frac{y^3}{3}\right]_1^2\right) = \frac{1}{M}\left(\frac{3}{4} + 3(8 - 8 + 8 - 4 + 2 - 1)\right).$$

$$\bar{y} = \frac{1}{M}\left(\frac{3}{4} + 3(3)\right) = \frac{1}{M}\left(\frac{3}{4} + 9\right) = \frac{10.75}{M} = \frac{32.25}{26} = \frac{16.125}{13}.$$

Therefore, the center of mass is:

$$(\bar{x}, \bar{y}) = \left(\frac{9}{13}, \frac{16.125}{13}\right).$$

Step 3: Find the Moment of Inertia

The moment of inertia I about the x-axis is given by:

$$I_x = \iint_R y^2 \cdot 6x\,dA.$$

Evaluate this integral:

$$I_x = \int_0^1 \int_0^y 6xy^2\,dx\,dy + \int_1^2 \int_0^{2-y} 6xy^2\,dx\,dy.$$

$$I_x = \int_0^1 \left[3x^2y^2\right]_0^y dy + \int_1^2 \left[3x^2y^2\right]_0^{2-y} dy = \int_0^1 3y^4\,dy + \int_1^2 3(2-y)^2 \cdot y^2\,dy.$$

Evaluate these integrals:

$$I_x = \left[\frac{3y^5}{5}\right]_0^1 + \int_1^2 3y^2(4 - 4y + y^2)\,dy.$$

$$I_x = \frac{3}{5} + 3\left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right]_1^2 = \frac{3}{5} + 3(16 - 8 + 2 - \frac{32}{3} + 1 - \frac{1}{5}).$$

$$I_x = \frac{3}{5} + 3\left(2.8 - 1.0667 + \frac{1}{3}\right) = \frac{3}{5} + 3(2.0667).$$

$$I_x = \frac{3}{5} + 6.2 = \frac{3 + 30}{5} = 6.6.$$

Therefore, the moment of inertia about the x-axis is:

$$I_x = 6.6.$$

Problem 10: Mass, Center of Mass, and Moment of Inertia of a Region

Let R be the unit square, i.e., $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Suppose the density at a point (x, y) of R is given by the function $f(x, y) = \frac{1}{y+1}$, i.e., R is denser near the x-axis.

Solution

Step 1: Find the Mass

The mass M of the region is given by the double integral of the density function over the region R:

$$M = \iint_R \frac{1}{y+1}\, dA.$$

The limits of integration are:

$$M = \int_0^1 \int_0^1 \frac{1}{y+1}\, dy\, dx.$$

Evaluate the inner integral:

$$\int_0^1 \frac{1}{y+1}\, dy = [\ln(y+1)]_0^1 = \ln(2).$$

Now, evaluate the outer integral:

$$M = \int_0^1 \ln(2)\, dx = \ln(2).$$

Step 2: Find the Center of Mass

The coordinates of the center of mass $(\bar{x}, \bar{y})$ are given by:

$$\bar{x} = \frac{1}{M} \iint_R x \cdot \frac{1}{y+1}\, dA, \quad \bar{y} = \frac{1}{M} \iint_R y \cdot \frac{1}{y+1}\, dA.$$

First, find $\bar{x}$:

$$\bar{x} = \frac{1}{M} \int_0^1 \int_0^1 \frac{x}{y+1}\, dy\, dx.$$

Evaluate the inner integral:

$$\int_0^1 \frac{x}{y+1}\, dy = x\, [\ln(y+1)]_0^1 = x \ln(2).$$

Now, evaluate the outer integral:

$$\bar{x} = \frac{1}{M} \int_0^1 x \ln(2)\, dx = \frac{\ln(2)}{M} \left[\frac{x^2}{2}\right]_0^1 = \frac{\ln(2)}{\ln(2)} \cdot \frac{1}{2} = \frac{1}{2}.$$

Next, find $\bar{y}$:

$$\bar{y} = \frac{1}{M} \int_0^1 \int_0^1 \frac{y}{y+1}\, dy\, dx.$$

Evaluate the inner integral using integration by parts:

$$\int_0^1 \frac{y}{y+1}\, dy = \int_0^1 \left(1 - \frac{1}{y+1}\right) dy = [y - \ln(y+1)]_0^1 = 1 - \ln(2).$$

Now, evaluate the outer integral:

$$\bar{y} = \frac{1}{M} \int_0^1 (1 - \ln(2))\, dx = \frac{1 - \ln(2)}{\ln(2)} \int_0^1 dx = \frac{1 - \ln(2)}{\ln(2)}.$$

Therefore, the coordinates of the center of mass are:

$$(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{1 - \ln(2)}{\ln(2)} \right).$$

Step 3: Find the Moment of Inertia

The moment of inertia I_x about the x-axis is given by:

$$I_x = \iint_R y^2 \cdot \frac{1}{y+1} \, dA.$$

The limits of integration are:

$$I_x = \int_0^1 \int_0^1 \frac{y^2}{y+1} \, dy \, dx.$$

Evaluate the inner integral using integration by parts:

$$\int_0^1 \frac{y^2}{y+1} \, dy = \int_0^1 \left(y - \frac{y}{y+1} \right) dy = \left[\frac{y^2}{2} - y + \ln(y+1) \right]_0^1 = \frac{1}{2} - 1 + \ln(2).$$

Now, evaluate the outer integral:

$$I_x = \int_0^1 \left(\frac{1}{2} - 1 + \ln(2) \right) dx = \left(\frac{1}{2} - 1 + \ln(2) \right).$$

Simplify the expression:

$$I_x = \frac{1}{2} - 1 + \ln(2) = \ln(2) - \frac{1}{2}.$$

The moment of inertia about the x-axis is:

$$I_x = \ln(2) - \frac{1}{2}.$$

Similarly, the moment of inertia I_y about the y-axis is:

$$I_y = \iint_R x^2 \cdot \frac{1}{y+1} \, dA.$$

Evaluate the integral:

$$I_y = \int_0^1 \int_0^1 \frac{x^2}{y+1} \, dy \, dx = \left(\int_0^1 \frac{x^2 \ln(2)}{\ln(2)} dx \right) = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$

Therefore, the moments of inertia about the axes are:

$$I_x = \ln(2) - \frac{1}{2}, \quad I_y = \frac{1}{3}.$$

Problem 11: Volume Inside the Unit Sphere

Find the volume inside the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

To find the volume inside the unit sphere, we can use spherical coordinates.

Step 1: Define the Spherical Coordinate Transformation

The transformations to spherical coordinates are:

$$x = \rho \sin\theta \cos\phi,$$

$$y = \rho \sin\theta \sin\phi,$$

$$z = \rho \cos\theta.$$

Step 2: Set up the Volume Integral

The volume integral in spherical coordinates is:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi.$$

Step 3: Evaluate the Integral

First, integrate with respect to ρ:

$$\int_0^1 \rho^2 \, d\rho = \left[\frac{\rho^3}{3}\right]_0^1 = \frac{1}{3}.$$

Next, integrate with respect to θ:

$$\int_0^{\pi} \sin\theta \, d\theta = [-\cos\theta]_0^{\pi} = 2.$$

Finally, integrate with respect to ϕ:

$$\int_0^{2\pi} d\phi = 2\pi.$$

Combining all these results:

$$V = \frac{1}{3} \times 2 \times 2\pi = \frac{4\pi}{3}.$$

Therefore, the volume inside the unit sphere is:

$$V = \frac{4}{3}\pi.$$

Problem 12: Volume Inside an Ellipsoid

Find the volume inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution

To find the volume inside the ellipsoid, we can use a transformation to spherical coordinates.

Step 1: Define the Spherical Coordinate Transformation

The transformations to spherical coordinates are:

$$x = a\rho \sin\theta \cos\phi,$$

$$y = b\rho \sin\theta \sin\phi,$$

$$z = c\rho \cos\theta.$$

Step 2: Set up the Volume Integral

The volume integral in spherical coordinates is:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^1 abc\rho^2 \sin\theta \, d\rho \, d\theta \, d\phi.$$

Step 3: Evaluate the Integral

First, integrate with respect to ρ:

$$\int_0^1 \rho^2 \, d\rho = \left[\frac{\rho^3}{3} \right]_0^1 = \frac{1}{3}.$$

Next, integrate with respect to θ:

$$\int_0^{\pi} \sin\theta \, d\theta = [-\cos\theta]_0^{\pi} = 2.$$

Finally, integrate with respect to ϕ:

$$\int_0^{2\pi} d\phi = 2\pi.$$

Combining all these results:

$$V = abc \times \frac{1}{3} \times 2 \times 2\pi = \frac{4}{3}\pi abc.$$

Therefore, the volume inside the ellipsoid is:

$$V = \frac{4}{3}\pi abc.$$

Problem 13: Volume of Tetrahedron T

Find the volume of tetrahedron T bounded by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $2x + 3y + z \leq 6$.

Solution

To find the volume of the tetrahedron, we use a triple integral over the region defined by the given inequalities.

Step 1: Set up the Limits of Integration

The bounds for x, y, and z are derived from the constraints:

$$0 \leq x \leq 3, \quad 0 \leq y \leq \frac{6-2x}{3}, \quad 0 \leq z \leq 6 - 2x - 3y.$$

Step 2: Set up the Volume Integral

The volume integral is:

$$V = \int_0^3 \int_0^{\frac{6-2x}{3}} \int_0^{6-2x-3y} dz \, dy \, dx.$$

Step 3: Evaluate the Integral

First, integrate with respect to z:

$$\int_0^{6-2x-3y} dz = [z]_0^{6-2x-3y} = 6 - 2x - 3y.$$

Next, integrate with respect to y:

$$\int_0^{\frac{6-2x}{3}} (6 - 2x - 3y)\, dy = \int_0^{\frac{6-2x}{3}} 6 - 2x - 3y\, dy = \left[6y - 2xy - \frac{3y^2}{2}\right]_0^{\frac{6-2x}{3}}.$$

Evaluate this integral:

$$\left[6 \cdot \frac{6-2x}{3} - 2x \cdot \frac{6-2x}{3} - \frac{3 \cdot \left(\frac{6-2x}{3}\right)^2}{2}\right].$$

Simplify the expression:

$$\left[\frac{6(6-2x)}{3} - \frac{2x(6-2x)}{3} - \frac{3(6-2x)^2}{18}\right].$$

Simplify further:

$$= \left[\frac{36-12x}{3} - \frac{12x-4x^2}{3} - \frac{3(36-24x+4x^2)}{18}\right].$$

Combine terms:

$$= \left[12 - 4x - 4x + \frac{4x^2}{3} - \frac{(36-24x+4x^2)}{6}\right].$$

Simplify:

$$= \left[12 - 8x + \frac{4x^2}{3} - \frac{36-24x+4x^2}{6}\right].$$

Combine fractions:

$$= \left[12 - 8x + \frac{4x^2}{3} - \frac{36}{6} + \frac{24x}{6} - \frac{4x^2}{6}\right].$$

Simplify:

$$= \left[12 - 8x + \frac{4x^2}{3} - 6 + 4x - \frac{2x^2}{3}\right].$$

Combine like terms:

$$= \left[6 - 4x + \frac{2x^2}{3}\right].$$

Now, integrate with respect to x:

$$\int_0^3 \left(6 - 4x + \frac{2x^2}{3}\right) dx = \left[6x - 2x^2 + \frac{2x^3}{9}\right]_0^3.$$

Evaluate this integral:

$$\left[6 \cdot 3 - 2 \cdot 9 + \frac{2 \cdot 27}{9}\right] = 18 - 18 + 6 = 6.$$

Therefore, the volume of the tetrahedron is:

$$V = 6.$$

Problem 14: Evaluating the Triple Integral

Evaluate the triple integral

$$\iiint_G \sqrt{x^2+z^2}\,dV,$$

where G is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution

To evaluate the triple integral, we use cylindrical coordinates.

Step 1: Convert to Cylindrical Coordinates

In cylindrical coordinates, the transformations are:

$$x = r\cos\theta, \quad y = y, \quad z = r\sin\theta.$$

The given region G is bounded by $y = r^2$ and $y = 4$.

Step 2: Set up the Limits of Integration

The limits for r, θ, and y are:

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad r^2 \le y \le 4.$$

Step 3: Set up the Volume Integral

The integrand $\sqrt{x^2+z^2}$ in cylindrical coordinates is r. The volume integral is:

$$V = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r\,dy\,dr\,d\theta.$$

Step 4: Evaluate the Integral

First, integrate with respect to y:

$$\int_{r^2}^4 r\,dy = r\,[y]_{r^2}^4 = r(4-r^2).$$

Next, integrate with respect to r:

$$\int_0^2 r(4-r^2)\,dr = \int_0^2 (4r - r^3)\,dr.$$

Evaluate this integral:

$$\left[2r^2 - \frac{r^4}{4}\right]_0^2 = \left(2\cdot 4 - \frac{16}{4}\right) = 4.$$

Finally, integrate with respect to θ:

$$\int_0^{2\pi} 4\,d\theta = 4\cdot 2\pi = 8\pi.$$

Therefore, the value of the triple integral is:

$$V = 8\pi.$$

Problem 15: Limits of Integration for a Tetrahedron

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

Solution

The region D can be described by the following limits:

$$\int_0^2 \int_0^{2-x} \int_0^{2-x-y} F(x,y,z)\,dz\,dy\,dx.$$

Problem 16: Evaluating Integrals

Evaluate the following integrals:

Part (i)

$$\int_0^a \int_0^a \int_0^a (xy + yz + zx)\,dx\,dy\,dz$$

Solution

First, integrate with respect to x:

$$\int_0^a \int_0^a \left[\frac{x^2y}{2} + xyz + \frac{zx^2}{2}\right]_0^a dy\,dz = \int_0^a \int_0^a \left(\frac{a^2y}{2} + ayz + \frac{za^2}{2}\right) dy\,dz.$$

Simplify and combine like terms:

$$= \int_0^a \int_0^a \left(\frac{a^2y}{2} + ayz + \frac{za^2}{2}\right) dy\,dz.$$

Next, integrate with respect to y:

$$= \int_0^a \left[\frac{a^2y^2}{4} + \frac{ay^2z}{2} + \frac{za^2y}{2}\right]_0^a dz = \int_0^a \left(\frac{a^4}{4} + \frac{a^3z}{2} + \frac{za^3}{2}\right) dz.$$

Combine like terms and simplify:

$$= \int_0^a \left(\frac{a^4}{4} + a^3z\right) dz.$$

Finally, integrate with respect to z:

$$= \left[\frac{a^4z}{4} + \frac{a^3z^2}{2}\right]_0^a = \frac{a^5}{4} + \frac{a^5}{2} = \frac{a^5}{4} + \frac{2a^5}{4} = \frac{3a^5}{4}.$$

So, the value of the integral is:

$$\frac{3a^5}{4}.$$

Part (ii)

$$\int_0^4 \int_0^{\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$$

Solution

First, integrate with respect to y:

$$\int_0^{\sqrt{4z-x^2}} dy = [y]_0^{\sqrt{4z-x^2}} = \sqrt{4z-x^2}.$$

Next, integrate with respect to x:

$$\int_0^{\sqrt{z}} \sqrt{4z-x^2}\,dx.$$

Make the substitution $x = \sqrt{z}t$, $dx = \sqrt{z}\,dt$:

$$= \int_0^1 \sqrt{4z - zt^2} \cdot \sqrt{z}\,dt = z \int_0^1 \sqrt{4 - t^2}\,dt.$$

Using the trigonometric substitution $t = 2\sin\theta$, $dt = 2\cos\theta\,d\theta$:

$$= z \int_0^{\pi/2} 2\cos^2\theta\,d\theta = 2z \left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} = 2z \cdot \frac{\pi}{4} = \frac{\pi z}{2}.$$

Finally, integrate with respect to z:

$$\int_0^4 \frac{\pi z}{2}\,dz = \frac{\pi}{2}\left[\frac{z^2}{2}\right]_0^4 = \frac{\pi}{2} \cdot \frac{16}{2} = 4\pi.$$

So, the value of the integral is:

$$4\pi.$$

Part (iii)

$$\int_0^2 \int_0^2 \int_0^z (4 - x^2)(2x + y)\,dx\,dy\,dz$$

Solution

First, integrate with respect to x:

$$\int_0^z (4 - x^2)(2x + y)\,dx.$$

Expand the integrand:

$$= \int_0^z (8x + 4xy - 2x^3 - x^2 y)\,dx.$$

Integrate term by term:

$$= \left[4x^2 + 2x^2 y - \frac{x^4}{2} - \frac{x^3 y}{3}\right]_0^z = 4z^2 + 2z^2 y - \frac{z^4}{2} - \frac{z^3 y}{3}.$$

Next, integrate with respect to y:

$$\int_0^2 (4z^2 + 2z^2 y - \frac{z^4}{2} - \frac{z^3 y}{3})\,dy.$$

Integrate term by term:

$$= \left[4z^2 y + z^2 y^2 - \frac{z^4 y}{2} - \frac{z^3 y^2}{6}\right]_0^2 = 8z^2 + 4z^2 - z^4 - \frac{2z^3}{3}.$$

Combine like terms:

$$= 12z^2 - z^4 - \frac{2z^3}{3}.$$

Finally, integrate with respect to z:

$$\int_0^2 (12z^2 - z^4 - \frac{2z^3}{3})\,dz.$$

Integrate term by term:

$$= \left[4z^3 - \frac{z^5}{5} - \frac{z^4}{6}\right]_0^2 = 32 - \frac{32}{5} - \frac{16}{3}.$$

Simplify the expression:

$$= 32 - \frac{32}{5} - \frac{16}{3} = 32 - 6.4 - 5.33 = 20.27.$$

So, the value of the integral is approximately:

$$20.27.$$

Problem 17: Evaluating Integrals

A solid "trough" of constant density ρ bounded below by the surface $z = 4y$, above by the plane $z = 4$, and on the ends by the planes $x = 1$ and $x = -0.1$. Find the center of mass and the moments of inertia with respect to the three axes.

Center of Mass

The center of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by:

$$\bar{x} = \frac{1}{M} \int\int\int x\, dV, \quad \bar{y} = \frac{1}{M} \int\int\int y\, dV, \quad \bar{z} = \frac{1}{M} \int\int\int z\, dV$$

Where M is the total mass:

$$M = \int\int\int \rho\, dV$$

Moments of Inertia

The moments of inertia (I_x, I_y, I_z) with respect to the x-, y-, and z-axes are given by:

$$I_x = \int\int\int (y^2 + z^2)\, dV, \quad I_y = \int\int\int (x^2 + z^2)\, dV, \quad I_z = \int\int\int (x^2 + y^2)\, dV$$

Volume Element

Given $z = 4y$ and $z = 4$:

$$dV = dx\, dy\, dz \quad \text{with bounds:} \quad -0.1 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 4y \leq z \leq 4$$

Integrals

Calculate the total mass M:

$$M = \rho \int_{-0.1}^{1} \int_{0}^{1} \int_{4y}^{4} dz\, dy\, dx$$

$$M = \rho \int_{-0.1}^{1} \int_{0}^{1} (4 - 4y)\, dy\, dx$$

$$M = \rho \int_{-0.1}^{1} \left[4y - 2y^2\right]_0^1 dx$$

$$M = \rho \int_{-0.1}^{1} (4 - 2)\, dx$$

$$M = 2\rho \int_{-0.1}^{1} dx$$

$$M = 2\rho\, [x]_{-0.1}^{1}$$

$$M = 2\rho(1 - (-0.1))$$

$$M = 2\rho(1.1) = 2.2\rho$$

Now, for the center of mass:

$$\bar{x} = \frac{1}{2.2\rho} \int_{-0.1}^{1} \int_{0}^{1} \int_{4y}^{4} x \, dz \, dy \, dx$$

This integral simplifies as the system is symmetric about the y-axis. So, $\bar{x} = 0$, and similarly,

$$\bar{y} = \frac{1}{2.2\rho} \int_{-0.1}^{1} \int_{0}^{1} \int_{4y}^{4} y \, dz \, dy \, dx$$

Problem 18: Moment of Inertia of a Solid Sphere

Find the moment of inertia of a solid sphere W of uniform density and radius a about the z-axis.

Solution

The moment of inertia I_z about the z-axis is given by:

$$I_z = \int\int\int_W (x^2 + y^2) \, \rho \, dV$$

Converting to spherical coordinates:

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta$$

The integral becomes:

$$I_z = \rho \int_0^a \int_0^\pi \int_0^{2\pi} (r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi) r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$I_z = \rho \int_0^a r^4 \, dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} (\cos^2\phi + \sin^2\phi) \, d\phi$$

Since $\cos^2\phi + \sin^2\phi = 1$:

$$I_z = \rho \int_0^a r^4 \, dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} 1 \, d\phi$$

Evaluating these integrals:

$$\int_0^{2\pi} d\phi = 2\pi$$

$$\int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3}$$

$$\int_0^a r^4 \, dr = \frac{a^5}{5}$$

Combining these results:

$$I_z = \rho \cdot \frac{a^5}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8}{15}\pi\rho a^5$$

If the sphere has total mass M, $\rho = \frac{3M}{4\pi a^3}$, then:

$$I_z = \frac{8}{15}\pi\left(\frac{3M}{4\pi a^3}\right)a^5 = \frac{2}{5}Ma^2$$

Problem 19: Centroid of a Solid Object

Find the center of gravity (centroid) of a solid object bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ using triple integration.

Solution

The centroid $(\bar{x}, \bar{y}, \bar{z})$ is given by:

$$\bar{x} = \frac{1}{M}\int\int\int_V x\,dV, \quad \bar{y} = \frac{1}{M}\int\int\int_V y\,dV, \quad \bar{z} = \frac{1}{M}\int\int\int_V z\,dV$$

where M is the total mass:

$$M = \int\int\int_V \rho\,dV$$

Given that the density ρ is constant, we can take ρ out of the integrals:

$$M = \rho\int\int\int_V dV$$

In cylindrical coordinates (r, θ, z), the volume element is $dV = r\,dr\,d\theta\,dz$. The bounds are:

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad r^2 \le z \le 4$$

The total mass is:

$$M = \rho\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta$$

$$M = \rho\int_0^{2\pi}\int_0^2 r(4 - r^2)\,dr\,d\theta$$

$$M = \rho\int_0^{2\pi}\int_0^2 (4r - r^3)\,dr\,d\theta$$

$$M = \rho\int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$

$$M = \rho\int_0^{2\pi}(8 - 4)\,d\theta$$

$$M = \rho\int_0^{2\pi} 4\,d\theta$$

$$M = 4\rho \cdot 2\pi = 8\pi\rho$$

For $\bar{x}$ and $\bar{y}$:

$$\bar{x} = \frac{1}{M}\int_0^{2\pi}\int_0^2\int_{r^2}^4 xr\,dz\,dr\,d\theta = 0 \quad \text{(symmetry)}$$

$$\bar{y} = \frac{1}{M}\int_0^{2\pi}\int_0^2\int_{r^2}^4 yr\,dz\,dr\,d\theta = 0 \quad \text{(symmetry)}$$

For $\bar{z}$:

$$\bar{z} = \frac{1}{M} \int_0^{2\pi} \int_0^2 \int_{r^2}^4 zr\,dz\,dr\,d\theta$$

$$\bar{z} = \frac{1}{8\pi\rho} \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2}\right]_{r^2}^4 r\,dr\,d\theta$$

$$\bar{z} = \frac{1}{8\pi\rho} \int_0^{2\pi} \int_0^2 \left(8r - \frac{r^5}{2}\right) dr\,d\theta$$

$$\bar{z} = \frac{1}{8\pi\rho} \int_0^{2\pi} \left[4r^2 - \frac{r^6}{12}\right]_0^2 d\theta$$

Problem 20 : Mass of a Solid Hemisphere

Find the mass of a solid hemisphere of radius R with a density function $\rho(x, y, z) = kz$, where k is a constant. The hemisphere is located above the xy-plane (i.e., $z \geq 0$).

Solution

The mass M of the hemisphere can be found using the triple integral:

$$M = \int\int\int_V \rho(x, y, z)\,dV$$

Given the density function $\rho(x, y, z) = kz$, we have:

$$M = k \int\int\int_V z\,dV$$

In spherical coordinates (r, θ, ϕ), the volume element is $dV = r^2 \sin\phi\,dr\,d\theta\,d\phi$. The bounds are:

$$0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}$$

Converting to spherical coordinates, $z = r\cos\phi$:

$$M = k \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^R r\cos\phi \cdot r^2 \sin\phi\,dr\,d\phi\,d\theta$$

Simplifying the integrals:

$$M = k \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi\,d\phi \int_0^R r^3\,dr$$

$$M = k \cdot 2\pi \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi\,d\phi \int_0^R r^3\,dr$$

Evaluating the integral with respect to r:

$$\int_0^R r^3\,dr = \left[\frac{r^4}{4}\right]_0^R = \frac{R^4}{4}$$

Evaluating the integral with respect to ϕ:

$$\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi\,d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2\phi\,d\phi = \frac{1}{2}\left[-\frac{1}{2}\cos 2\phi\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(0-(-\frac{1}{2})\right) = \frac{1}{4}$$

Combining the results:

$$M = k\cdot 2\pi\cdot\frac{1}{4}\cdot\frac{R^4}{4}$$

$$M = \frac{k\pi R^4}{4}$$

Problem 21 : Triple Integral with Transformation

Evaluate the triple integral

$$\iiint_T xyz\,dx\,dy\,dz$$

where T is the region in the xyz-space bounded by the planes $x=0$, $y=0$, $z=0$, $x+y=1$, and $z=x+y$. Use the transformation $u=x+y$, $v=x-y$, and $w=z$.

Solution

First, we compute the Jacobian determinant of the transformation $(x,y,z)\to(u,v,w)$:

$$\begin{aligned} u &= x+y, \\ v &= x-y, \\ w &= z. \end{aligned}$$

The Jacobian determinant J is given by:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot\left(-\frac{1}{2}\cdot 1\right) - 0 + 0 = -\frac{1}{2}$$

Thus, the absolute value of the Jacobian determinant is:

$$|J| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

Next, we transform the region T and the integrand xyz in terms of u, v, and w:

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}, \quad z = w$$

Therefore, the integrand becomes:

$$xyz = \left(\frac{u+v}{2}\right)\left(\frac{u-v}{2}\right)w = \frac{(u^2-v^2)w}{4}$$

The bounds for u, v, and w are:

$$0\le u\le 1, \quad -u\le v\le u, \quad 0\le w\le u$$

Now, the triple integral in terms of u, v, and w is:

$$\iiint_T xyz\,dx\,dy\,dz = \iiint_T \frac{(u^2-v^2)w}{4}\cdot\frac{1}{2}\,du\,dv\,dw = \frac{1}{8}\iiint_T (u^2-v^2)w\,du\,dv\,dw$$

Evaluate the integral:

$$\frac{1}{8}\int_0^1 \int_{-u}^{u} \int_0^{u} (u^2 - v^2) w \, dw \, dv \, du$$

Integrate with respect to w:

$$\int_0^u (u^2 - v^2) w \, dw = \left[\frac{(u^2 - v^2)w^2}{2} \right]_0^u = \frac{(u^2 - v^2)u^2}{2} = \frac{u^4 - u^2 v^2}{2}$$

Integrate with respect to v:

$$\int_{-u}^{u} \frac{u^4 - u^2 v^2}{2} \, dv = \frac{u^4}{2} \int_{-u}^{u} dv - \frac{u^2}{2} \int_{-u}^{u} v^2 \, dv$$

$$= \frac{u^4}{2} [v]_{-u}^{u} - \frac{u^2}{2} \left[\frac{v^3}{3} \right]_{-u}^{u} = \frac{u^4}{2} \cdot 2u - \frac{u^2}{2} \cdot \frac{2u^3}{3} = u^5 - \frac{2u^5}{3} = \frac{3u^5 - 2u^5}{3} = \frac{u^5}{3}$$

Integrate with respect to u:

$$\frac{1}{8} \int_0^1 \frac{u^5}{3} \, du = \frac{1}{24} \left[\frac{u^6}{6} \right]_0^1 = \frac{1}{24} \cdot \frac{1}{6} = \frac{1}{144}$$

Thus, the value of the triple integral is:

$$\iiint_T xyz \, dx \, dy \, dz = \frac{1}{144}$$

Problem: Mass of the Solid Bounded by the Curves

Find the mass of the solid obtained by the curves $y = x^2$ and $y = x + 2$ with a constant density over the given area.

Solution

First, find the points of intersection of the curves:

$$x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0 \implies x = 2, \, x = -1$$

The mass M of the solid is given by:

$$M = \rho \int_{-1}^{2} \int_{x^2}^{x+2} dy \, dx$$

Evaluate the integral:

$$M = \rho \int_{-1}^{2} [y]_{x^2}^{x+2} \, dx = \rho \int_{-1}^{2} \left((x + 2) - x^2 \right) \, dx$$

$$M = \rho \int_{-1}^{2} (x + 2 - x^2) \, dx$$

Integrate with respect to x:

$$M = \rho \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

Evaluate the definite integral:

$$M = \rho \left(\left(\frac{2^2}{2} + 2(2) - \frac{2^3}{3} \right) - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3} \right) \right)$$

$$M = \rho \left(\left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right)$$

$$M = \rho \left(6 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} \right)$$

$$M = \rho \left(\frac{18}{3} - \frac{8}{3} + \frac{6}{3} - \frac{1}{2} \right)$$

$$M = \rho \left(\frac{18 - 8 + 6}{3} - \frac{1}{2} \right)$$

Problem: Mass of the Solid Bounded by the Curves

Find the mass of the solid obtained by the curves $y = x^2$ and $y = x + 2$ with a constant density over the given area.

Solution

The mass M of the solid is given by:

$$M = \rho \int_0^3 \int_{y-2}^{\sqrt{y}} dx\, dy$$

Evaluate the integral:

$$M = \rho \int_0^3 [x]_{y-2}^{\sqrt{y}}\ dy = \rho \int_0^3 \left(\sqrt{y} - (y-2)\right)\ dy = \rho \int_0^3 (2 - y + \sqrt{y})\, dy$$

Integrate with respect to y:

$$M = \rho \left[2y - \frac{y^2}{2} + \frac{2}{3} y^{3/2} \right]_0^3$$

Evaluate the definite integral:

$$M = \rho \left(\left(2(3) - \frac{3^2}{2} + \frac{2}{3} \cdot 3^{3/2} \right) - \left(2(0) - \frac{0^2}{2} + \frac{2}{3} \cdot 0^{3/2} \right) \right)$$

$$M = \rho \left(6 - \frac{9}{2} + \frac{2}{3} \cdot \sqrt{27} \right)$$

$$M = \rho \left(6 - \frac{9}{2} + \frac{2}{3} \cdot 3\sqrt{3} \right)$$

$$M = \rho \left(6 - \frac{9}{2} + 2\sqrt{3} \right)$$