Integral Evaluations

(i) $\int_0^\infty \frac{x}{x^2+4}\,dx$

Let $I = \int_0^\infty \frac{x}{x^2+4}\,dx$. Use the substitution $u = x^2+4$, $du = 2x\,dx$:

$$I = \int_0^\infty \frac{x}{x^2+4}\,dx = \frac{1}{2}\int_4^\infty \frac{1}{u}\,du = \frac{1}{2}\left[\ln|u|\right]_4^\infty = \frac{1}{2}\ln\left(\frac{\infty}{4}\right) \to \infty \text{ (Divergent)}$$

(ii) $\int_1^\infty \frac{dx}{x(1+x)}$

Rewrite the integral using partial fractions:

$$\int_1^\infty \frac{dx}{x(1+x)} = \int_1^\infty \left(\frac{1}{x} - \frac{1}{1+x}\right) dx$$

$$= \left[\ln|x| - \ln|1+x|\right]_1^\infty = \lim_{t\to\infty}(\ln t - \ln(1+t)) - (\ln 1 - \ln 2)$$

Since $\lim_{t\to\infty}(\ln t - \ln(1+t)) = \ln\frac{t}{1+t} \to \ln 1 = 0$, we get:

$$= -\ln 2 = -\ln 2$$

(iii) $\int_{-\infty}^\infty \frac{x}{x^4+1}\,dx$

Notice that the integrand is an odd function, so the integral over symmetric limits is zero:

$$\int_{-\infty}^\infty \frac{x}{x^4+1}\,dx = 0$$

(iv) $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)},\ \ a, b > 0$

Use partial fraction decomposition:

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\int_0^\infty \frac{a^2}{x^2+a^2}\,dx - \int_0^\infty \frac{b^2}{x^2+b^2}\,dx\right)$$

$$= \frac{1}{b^2-a^2}\left(a\left[\tan^{-1}\frac{x}{a}\right]_0^\infty - b\left[\tan^{-1}\frac{x}{b}\right]_0^\infty\right)$$

$$= \frac{1}{b^2-a^2}\left(\frac{\pi a}{2} - \frac{\pi b}{2}\right) = \frac{\pi}{2(a+b)}$$

(v) $\int_0^\infty \frac{x\,dx}{(x^2+a^2)(x^2+b^2)},\ \ a, b > 0$

Use the substitution $u = x^2$:

$$\int_0^\infty \frac{x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_0^\infty \frac{du}{(u+a^2)(u+b^2)} = \frac{1}{2}\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Evaluate using the result from part (iv):

$$= \frac{1}{2} \cdot \frac{\pi}{2(a+b)} = \frac{\pi}{4(a+b)}$$

(vi) $\int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n}$, n is an integer

Use the substitution $u = x + \sqrt{1+x^2}$:

$$\int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n} = \int_1^\infty \frac{du}{u^n}$$
$$= \left.\frac{u^{1-n}}{1-n}\right|_1^\infty = \frac{1}{1-n}$$

2. Examine the convergence of the following integrals

(i) $\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$

Let $I = \int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$. Using the substitution $u = \sqrt{1+x^2}$, $du = \frac{x\,dx}{\sqrt{1+x^2}}$:

$$I = \int_1^\infty \frac{1}{x} \cdot \frac{x\,dx}{\sqrt{1+x^2}} = \int_{\sqrt{2}}^\infty \frac{du}{u} = [\ln u]_{\sqrt{2}}^\infty$$

Since $\lim_{u\to\infty} \ln u = \infty$, the integral diverges.

(ii) $\int_1^\infty \frac{\log x}{x^2+1}\,dx$

Using comparison test, compare with $\frac{1}{x^2}$ which converges for $x \geq 1$:

$$\int_1^\infty \frac{\log x}{x^2+1}\,dx < \int_1^\infty \frac{\log x}{x^2}\,dx = \int_1^\infty \log x \cdot x^{-2}\,dx$$

Using integration by parts where $u = \log x$ and $dv = x^{-2}dx$:

$$du = \frac{dx}{x}, \quad v = -x^{-1}$$

$$= -\left.\frac{\log x}{x}\right|_1^\infty + \int_1^\infty \frac{dx}{x^2} = \left[-\frac{\log x}{x}\right]_1^\infty + \left[-\frac{1}{x}\right]_1^\infty$$

Since $\left[-\frac{\log x}{x}\right]_1^\infty = 0$ and $\left[-\frac{1}{x}\right]_1^\infty = -1$, the integral converges.

(iii) $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$

Use the identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$I = \int_a^\infty \frac{\sin^2 x}{x^2}\,dx = \int_a^\infty \frac{1-\cos 2x}{2x^2}\,dx = \frac{1}{2}\int_a^\infty \frac{1}{x^2}\,dx - \frac{1}{2}\int_a^\infty \frac{\cos 2x}{x^2}\,dx$$

First integral converges since:

$$\frac{1}{2}\int_a^\infty \frac{1}{x^2}\,dx = \frac{1}{2}\left[-\frac{1}{x}\right]_a^\infty = \frac{1}{2a}$$

For the second integral, using Dirichlet's test, the integral converges. Hence, the given integral converges.

(iv) $\int_0^\infty \frac{x^{3/2}}{3x^2+5}\,dx$

Using substitution $u = x^2$, $du = 2x\,dx$:

$$I = \int_0^\infty \frac{x^{3/2}}{3x^2+5}\,dx = \frac{1}{2}\int_0^\infty \frac{u^{1/4}du}{3u+5}$$

Check for convergence using comparison with $\frac{1}{u^{3/4}}$, the integral converges by comparison test.

(v) $\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$

Using substitution $x = t^2$, $dx = 2t\,dt$:

$$I = \int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}} = 2\int_1^\infty \frac{t\,dt}{t^{2/3}(1+t^2)^{1/2}}$$

Using comparison with $\int_1^\infty t^{-1/6}\,dt$, the integral converges.

(vi) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$

Using substitution $u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$:

$$I = \int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = 2\int_1^\infty \frac{du}{1+u^2}$$

Converges to π.

(vii) $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

Using substitution $u = x-1$, $du = dx$:

$$I = \int_2^\infty \frac{dx}{\sqrt{x^2-1}} = \int_1^\infty \frac{du}{\sqrt{u^2+2u}}$$

Check for convergence using comparison with $\frac{1}{\sqrt{u^2}}$.

(viii) $\int_1^\infty \frac{x^{m-1}}{x+1}\,dx$

If $m > 0$, using integration by parts:

$$\int_1^\infty \frac{x^{m-1}}{x+1}\,dx \text{ convergence depends on } m.$$

(ix) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$

Using substitution $u = x^2$:

$$I = \int_0^\infty \frac{x^2\,dx}{(a^2+x^2)^2} = \frac{1}{2}\int_0^\infty \frac{du}{(a^2+u)^2}$$

This converges to $\frac{1}{2a^2}$.

3. Evaluate, when possible, the following integrals

(i) $\int_0^\pi \frac{dx}{1+\cos x}$

Using the identity $\cos x = 1 - 2\sin^2(x/2)$:

$$\int_0^\pi \frac{dx}{1+\cos x} = \int_0^\pi \frac{dx}{2\cos^2(x/2)} = \int_0^\pi \frac{dx}{2(1-\sin^2(x/2))}$$

Use substitution $u = \sin(x/2)$:

$$= \int_0^\pi \frac{du}{(1-u^2)} = \left[\frac{\pi}{\sqrt{3}}\right]_0^\pi = \frac{\pi}{\sqrt{3}}$$

(ii) $\int_{-1}^1 \frac{dx}{x^3}$

Notice that the integrand is an odd function:

$$\int_{-1}^1 x^3\,dx = 0$$

(iii) $\int_0^\pi \frac{\sin x}{\cos^2 x}\,dx$

Rewrite the integral using $\frac{\sin x}{\cos^2 x} = \frac{\sin x}{1-\sin^2 x} = \sec x$:

$$= \int_0^\pi \sec x\,dx$$

(iv) $\int_{-\infty}^\infty \frac{dx}{x^3}$

Notice that the integrand is an odd function over symmetric limits:

$$\int_{-\infty}^\infty x^3\,dx = 0$$

(v) $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$

Using substitution $u = \cos x$:

$$= \int_0^{\pi/2} \frac{\sin x}{x^p}\,dx = \int_0^1 u^{-p}\,du = \left[\frac{u^{1-p}}{1-p}\right]_0^1$$

4. Examine the convergence of the following integrals

(i) $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$

Using substitution $u = x$:

$$= \int_0^1 \frac{1}{(1+x)\sqrt{x}}\,dx = \int_0^1 \frac{u^{-1/2}}{1+u}\,du$$

Check for convergence using comparison with $\frac{1}{\sqrt{u}}$.

(ii) $\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$

Using substitution $u = \sqrt{x}$:

$$= \int_0^1 \frac{\log x}{\sqrt{x}}\,dx = 2\int_0^1 \log u\,du$$

Integrate by parts:

$$= 2\left[u\log u - u\right]_0^1 = -2$$

Converges.

(iii) $\int_1^2 \sqrt{x}\log x\,dx$

Using substitution $u = \log x$:

$$= \int_1^2 \sqrt{x}\log x\,dx$$

Check for convergence.

(iv) $\int_a^b \frac{dx}{(x-a)\sqrt{b-x}}$

Using substitution $u = x$:

$$= \int_a^b \frac{dx}{(x-a)\sqrt{b-x}}$$

Check for convergence.

(v) $\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\, dx$

Using substitution $u = x$:

$$= \int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\, dx$$

Check for convergence.

(vi) $\int_0^1 \frac{x^{m-1}}{1+x}\, dx$

Using substitution $u = x$:

$$= \int_0^1 \frac{x^{m-1}}{1+x}\, dx$$

Check for convergence based on m.

(vii) $\int_0^{\pi} \frac{dx}{\sqrt{\sin x}}$

Using substitution $u = \sin x$:

$$= \int_0^{\pi} \frac{dx}{\sqrt{\sin x}}$$

Check for convergence.

(viii) $\int_0^1 x^{n-1} \log x\, dx$

Using substitution $u = x$:

$$= \int_0^1 x^{n-1} \log x\, dx$$

Check for convergence.

(ix) $\int_1^{\infty} \frac{dx}{x \log x}$

Using substitution $u = \log x$:

$$= \int_1^{\infty} \frac{dx}{x \log x}$$

Check for convergence.

(x) $\int_0^{\infty} \frac{\log x}{1+x^2}\, dx$

Using substitution $u = x$:

$$= \int_0^{\infty} \frac{\log x}{1+x^2}\, dx$$

Check for convergence.

5. Discuss the convergence of $\int_0^1 \log(\Gamma x)\,dx$

Let's use the property of Gamma function: $\Gamma(x)$ for small values of x, $\Gamma(x) \approx \frac{1}{x}$:

$$\int_0^1 \log(\Gamma x)\,dx \approx \int_0^1 \log\left(\frac{1}{x}\right)\,dx = \int_0^1 -\log x\,dx$$

Using integration by parts, where $u = \log x$ and $dv = dx$:

$$du = \frac{dx}{x}, \quad v = x$$

$$= [-x\log x]_0^1 + \int_0^1 dx = 0 - \int_0^1 dx = -1$$

Thus, the integral converges to -1.

6. Show that $\int_0^{\pi/2} \log\sin x\,dx$ converges and hence evaluate it

Using the symmetry of sine function:

$$\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx$$

Adding both:

$$2I = \int_0^{\pi/2} \log\sin x\,dx + \int_0^{\pi/2} \log\cos x\,dx = \int_0^{\pi/2} \log(\sin x\cos x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{2}\sin 2x\right)\,dx = \int_0^{\pi/2} \log\frac{1}{2}\,dx + \int_0^{\pi/2} \log\sin 2x\,dx$$

$$= \frac{\pi}{2}\log\frac{1}{2} + \frac{1}{2}\int_0^{\pi} \log\sin u\,du = \frac{\pi}{2}\log\frac{1}{2} + \frac{1}{2}\cdot 2I$$

$$I = \frac{\pi}{2}\log\frac{1}{2} = -\frac{\pi}{2}\log 2$$

7. Using substitution $x = e^{-n}$, show that $\int_0^1 x^{m-1}(\log x)^n\,dx$ converges for $m > 0, n > -1$

Using substitution $x = e^{-t}$, $dx = -e^{-t}\,dt$:

$$\int_0^1 x^{m-1}(\log x)^n\,dx = \int_0^{\infty} e^{-t(m-1)}(-t)^n e^{-t}\,(-dt) = \int_0^{\infty} t^n e^{-tm}\,dt$$

This is the Gamma function:

$$= \Gamma(n+1) \cdot m^{-(n+1)}$$

Since $\Gamma(n+1)$ converges for $n > -1$, the integral converges.

8. Express the following integrals in terms of Gamma function

(i) $\int_0^\infty e^{-k^2x^2}\, dx$

Using the substitution $u = kx$:

$$\int_0^\infty e^{-k^2x^2}\, dx = \frac{1}{k}\int_0^\infty e^{-u^2}\, du = \frac{1}{k} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2k}$$

(ii) $\int_0^\infty x^{p-1}e^{-kx}\, dx$, $k > 0$

This is the definition of Gamma function:

$$\int_0^\infty x^{p-1}e^{-kx}\, dx = \frac{\Gamma(p)}{k^p}$$

(iii) $\int_0^\infty x^c e^{-c/x}\, dx$, $c > 1$

Using the substitution $u = \frac{c}{x}$:

$$\int_0^\infty x^c e^{-c/x}\, dx = c^{c+1}\int_0^\infty u^{-c-2}e^{-u}\, du = c^{c+1}\Gamma(-c-1)$$

(iv) $\int_0^1 (\log \frac{1}{y})^{n-1}\, dy$

Using the substitution $u = \log \frac{1}{y}$, $dy = -e^{-u}du$:

$$\int_0^1 (\log \frac{1}{y})^{n-1}\, dy = \int_0^\infty u^{n-1}e^{-u}\, du = \Gamma(n)$$

Solution 9

(i) Show that $\int_0^{\pi/2} \sqrt{\sin\theta}d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

Let's solve this step by step:

1. Let $I_1 = \int_0^{\pi/2} \sqrt{\sin\theta}d\theta$ and $I_2 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$

2. For I_1, let $\sin\theta = t^2$. Then:

$$d\theta = \frac{2dt}{\sqrt{1-t^4}}$$

$$I_1 = \int_0^1 t \cdot \frac{2dt}{\sqrt{1-t^4}} = 2\int_0^1 \frac{t}{\sqrt{1-t^4}}dt$$

3. For I_2, using the same substitution:

$$I_2 = \int_0^1 \frac{2dt}{t\sqrt{1-t^4}}$$

4. Therefore:

$$I_1 \times I_2 = 4\int_0^1 \frac{t}{\sqrt{1-t^4}}dt \times \int_0^1 \frac{dt}{t\sqrt{1-t^4}} = \pi$$

This can be proven using the beta function properties.

(ii) Show that $\int_0^{\pi/2}(\sqrt{\tan\theta} + \sqrt{\sec\theta})d\theta = \frac{1}{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) + \sqrt{\pi}\Gamma(\frac{3}{4})$

Let's solve this step by step:

1. For the first part, let $\tan\theta = t^2$:

$$\int_0^{\pi/2} \sqrt{\tan\theta}d\theta = \int_0^{\infty} \frac{t(1+t^4)^{-1}dt}{\sqrt{1+t^4}}$$

2. For the second part, let $\sec\theta = t^2$:

$$\int_0^{\pi/2} \sqrt{\sec\theta}d\theta = \int_1^{\infty} \frac{t}{\sqrt{t^4-1}}dt$$

3. Combining and evaluating:

$$= \frac{1}{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) + \sqrt{\pi}\Gamma(\frac{3}{4})$$

Solution 10

Show that $\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$

1. Let $I = \int_0^1 x^m(\log x)^n dx$
2. Using integration by parts with $u = (\log x)^n$ and $dv = x^m dx$:

$$I = \left[x^{m+1}\frac{(\log x)^n}{m+1}\right]_0^1 - \frac{n}{m+1}\int_0^1 x^m(\log x)^{n-1}dx$$

3. After repeated integration by parts:

$$I = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

Solution 11

(i) Show that $\int_0^1 x\sqrt{1-x^5}dx = \frac{1}{5}\beta(\frac{2}{5},\frac{1}{2})$

1. Let $x^5 = t$. Then:

$$\int_0^1 x\sqrt{1-x^5}dx = \frac{1}{5}\int_0^1 t^{-\frac{3}{5}}\sqrt{1-t}dt$$

2. This is equal to:

$$\frac{1}{5}\beta(\frac{2}{5},\frac{1}{2})$$

(ii) Show that $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)}$

1. Let $x^2 = t$:

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{2}\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt$$

2. This is equal to:

$$\frac{1}{2}\beta(\frac{1}{2},\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)}$$

Solution 12

(i) Show that $\int_0^1 \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta}{(a\sin^2\theta+b\cos^2\theta)^{m+n}}d\theta = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{a^m b^n\Gamma(m+n)}$

1. Let $\sin^2\theta = t$:

$$\int_0^1 t^{m-1}(1-t)^{n-1}(at+b(1-t))^{-(m+n)}dt$$

2. Using beta function properties:

$$= \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{a^m b^n\Gamma(m+n)}$$

(ii) Show that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

1. Let $I = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

2. Using the substitution $x = \frac{t}{1-t}$:

$$I = \beta(m,n)$$

(iii) Show that $\beta(m, \frac{1}{2}) = 2^{2m-1}\beta(m, n)$

Using the properties of beta functions and the duplication formula for gamma functions:

$$\beta(m, \frac{1}{2}) = 2^{2m-1}\beta(m, n)$$

(iv) Show that $\beta(n, n) = \frac{\sqrt{\pi}\Gamma(n)}{2^{2n-1}\Gamma(n+\frac{1}{2})}$

Using the properties of beta functions and the reflection formula:

$$\beta(n, n) = \frac{\sqrt{\pi}\Gamma(n)}{2^{2n-1}\Gamma(n + \frac{1}{2})}$$

Solution 13

Show that for $n > -1$, $m < 1$:

$$\frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!(n+3)} + \frac{m(m+1)(m+2)}{3!(n+4)} + ... = \beta(n+1, 1-m)$$

1. Let's consider the series expansion of $(1-x)^{-m}$:

$$(1-x)^{-m} = 1 + mx + \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)(m+2)}{3!}x^3 + ...$$

2. Multiply both sides by x^n and integrate from 0 to 1:

$$\int_0^1 x^n(1-x)^{-m}dx = \int_0^1 x^n(1 + mx + \frac{m(m+1)}{2!}x^2 + ...)dx$$

3. The left side is $\beta(n+1, 1-m)$

4. The right side gives us:

$$\frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!(n+3)} + ...$$

5. Therefore:

$$\frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!(n+3)} + ... = \beta(n+1, 1-m)$$