2020 AMC 10B Solution

Problem1

What is the value of $1-(-2)-3-(-4)-5-(-6)?$

$\textbf{(A)}\ -20 \qquad \textbf{(B)}\ -3 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 21$

Solution

We know that when we subtract negative numbers, $a-(-b)=a+b$.

The equation becomes $1+2-3+4-5+6=\boxed{\textbf{(D)}\ 5}$

Problem2

Carl has 5 cubes each having side length 1, and Kate has 5 cubes each having side length 2. What is the total volume of these 10 cubes?

$\textbf{(A)}\ 24 \qquad \textbf{(B)}\ 25 \qquad \textbf{(C)}\ 28 \qquad \textbf{(D)}\ 40 \qquad \textbf{(E)}\ 45$

Solution

A cube with side length 1 has volume $1^3=1$, so 5 of these will have a total volume of $5\cdot1=5$.

A cube with side length 2 has volume $2^3=8$, so 5 of these will have a total volume of $5\cdot8=40$.

$5+40=\boxed{\textbf{(E)}}$ ~quacker88

Problem 3

The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of $w$to $y?$

$\textbf{(A)}\ 4:3 \qquad \textbf{(B)}\ 3:2 \qquad \textbf{(C)}\ 8:3 \qquad \textbf{(D)}\ 4:1 \qquad \textbf{(E)}\ 16:3$

Solution 1

WLOG, let $w = 4$ and $x = 3$.

Since the ratio of z to x is $1:6$, we can substitute in the value of x to get $\frac{z}{3} = \frac{1}{6} \implies z = \frac{1}{2}$.

The ratio of y to z is $3:2$, so $\frac{y}{\frac{1}{2}} = \frac{3}{2} \implies y = \frac{3}{4}$.

The ratio of w to y is then $\frac{4}{\frac{3}{4}} = \frac{16}{3}$ so our answer is $\boxed{\textbf{(E) }16:3}$ ~quacker88

Solution 2

We need to somehow link all three of the ratios together. We can start by connecting the last two ratios together by multiplying the last ratio by two.

$z:x = 1:6 = 2:12$, and since $y:z = 3:2$, we can link them together to get $y:z:x = 3:2:12$.

Finally, since $x:w = 3:4 = 12:16$, we can link this again to get: $y:z:x:w = 3:2:12:16$,

so $w:y = \boxed{\textbf{(E) }16:3}$ ~quacker88

Problem4

The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

$\textbf{(A) }2 \qquad \textbf{(B) }3 \qquad \textbf{(C) }5 \qquad \textbf{(D) }7 \qquad \textbf{(E) }11$

Solution

Since the three angles of a triangle add up to 180° and one of the angles is 90° because it's a right triangle, $a^\circ + b^\circ = 90^\circ$.

The greatest prime number less than 90 is 89. If $a = 89^\circ$, then $b = 90^\circ - 89^\circ = 1^\circ$, which is not prime.

The next greatest prime number less than 90 is 83. If $a = 83^\circ$, then $b = 7^\circ$, which IS prime, so we have our answer $\boxed{\textbf{(D) } 7}$ ~quacker88

Solution 2

Looking at the answer choices, only 7 and 11 are coprime to 90. Testing 7, the smaller angle, makes the other angle 83 which is prime, therefore our answer is $\boxed{\textbf{(D) } 7}$

Problem5

How many distinguishable arrangements are there of 1 brown tile, 1 purple tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)

$\textbf{(A) } 210 \qquad \textbf{(B) } 420 \qquad \textbf{(C) } 630 \qquad \textbf{(D) } 840 \qquad \textbf{(E) } 1050$

Solution

Let's first find how many possibilities there would be if they were all distinguishable, then divide out the ones we overcounted.

There are $7!$ ways to order 7 objects. However, since there's $3! = 6$ ways to switch the yellow tiles around without changing anything (since they're indistinguishable) and $2! = 2$ ways to order the green tiles, we have to divide out these possibilities.

$\frac{7!}{6 \cdot 2} = \boxed{\textbf{(B) } 420}$ ~quacker88

Solution

We can repeat chooses extensively to find the answer. There are 7 choose 3 ways to arrange the brown tiles which is 35. Then from the remaining tiles there are 4 choose $2 = 6$ ways to arrange the red tiles. And now from the remaining two tiles and two slots we can see there are two ways to arrange the purple and brown tiles, giving us an answer of $35 * 6 * 2 = 420$ $\frac{7!}{6 \cdot 2} = \boxed{\textbf{(B) } 420}$

Problem6

Driving along a highway, Megan noticed that her odometer showed 15951 (miles). This number is a palindrome-it reads the same forward and backward. Then 2 hours later, the odometer displayed the next higher palindrome. What was her average speed, in miles per hour, during this 2-hour period?

$\textbf{(A)}\ 50 \qquad \textbf{(B)}\ 55 \qquad \textbf{(C)}\ 60 \qquad \textbf{(D)}\ 65 \qquad \textbf{(E)}\ 70$

Solution

In order to get the smallest palindrome greater than 15951, we need to raise the middle digit. If we were to raise any of the digits after the middle, we would be forced to also raise a digit before the middle to keep it a palindrome, making it unnecessarily larger.

So we raise 9 to the next largest value, 10, but obviously, that's not how place value works, so we're in the 16000s now. To keep this a palindrome, our number is now 16061.

So Megan drove $16061 - 15951 = 110$ miles. Since this happened over 2 hours, she drove at $\frac{110}{2} = \boxed{\textbf{(B)}\ 55}$ mph. ~quacker88

Problem7

How many positive even multiples of 3 less than 2020 are perfect squares?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 9 \qquad \textbf{(D)}\ 10 \qquad \textbf{(E)}\ 12$

Solution

Any even multiple of 3 is a multiple of 6, so we need to find multiples of 6 that are perfect squares and less than 2020. Any solution that we want will be in the form $(6n)^2$, where n is a positive integer. The smallest possible value is at $n=1$, and the largest is at $n=7$ (where the expression equals 1764). Therefore, there are a total of $\boxed{\textbf{(A)}\ 7}$ possible numbers.-PCChess

Problem8

Points P and Q lie in a plane with $PQ = 8$. How many locations for point R in this plane are there such that the triangle with vertices P, Q, and R is a right triangle with area 12 square units?

$\textbf{(A)}\ 2 \qquad\textbf{(B)}\ 4 \qquad\textbf{(C)}\ 6 \qquad\textbf{(D)}\ 8 \qquad\textbf{(E)}\ 12$

Solution 1

There are 3 options here:

1. $\textbf{P}$ is the right angle.

It's clear that there are 2 points that fit this, one that's directly to the right of P and one that's directly to the left. We don't need to find the length, we just need to know that it is possible, which it is.

2. $\textbf{Q}$ is the right angle.

Using the exact same reasoning, there are also 2 solutions for this one.

3. The new point is the right angle.

(Diagram temporarily removed due to asymptote error)

The diagram looks something like this. We know that the altitude to base $\overline{AB}$ must be 3 since the area is 12. From here, we must see if there are valid triangles that satisfy the necessary requirements.

First of all, $\dfrac{\overline{BC}\cdot\overline{AC}}{2} = 12 \implies \overline{BC}\cdot\overline{AC} = 24$ because of the area.

Next, $\overline{BC}^2 + \overline{AC}^2 = 64$ from the Pythagorean Theorem.

From here, we must look to see if there are valid solutions. There are multiple ways to do this:

$\textbf{Recognizing min \& max:}$

We know that the minimum value of $\overline{BC}^2 + \overline{AC}^2 = 64$ is when $\overline{BC} = \overline{AC} = \sqrt{24}$. In this case, the equation becomes $24 + 24 = 48$, which is LESS than 64. $\overline{BC} = 1, \overline{AC} = 24$. The equation becomes $1 + 576 = 577$, which is obviously greater than 64. We can conclude that there are values for $\overline{BC}$ and $\overline{AC}$ in between that satisfy the Pythagorean Theorem.

And since $\overline{BC} \neq \overline{AC}$, the triangle is not isoceles, meaning we could reflect it over $\overline{AB}$ and/or the line perpendicular to $\overline{AB}$ for a total of 4 triangles this case.

Solution 2

Note that line segment $\overline{PQ}$ can either be the shorter leg, longer leg or the hypotenuse. If it is the shorter leg, there are two possible points for Q that can satisfy the requirements - that being above or below $\overline{PQ}$. As such, there are 2 ways for this case. Similarly, one can find that there are also 2 ways for point Q to lie if $\overline{PQ}$ is the longer leg. If it is a hypotenuse, then there are 4 possible points because the arrangement of the shorter and longer legs can be switched, and can be either above or below the line segment. Therefore, the answer is $2 + 2 + 4 = \boxed{\textbf{(D)}\ 8}$.

Problem9

How many ordered pairs of integers (x, y) satisfy the equation $x^{2020} + y^2 = 2y$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ \text{infinitely many}$

Solution

Rearranging the terms and and completing the square for y yields the result $x^{2020} + (y-1)^2 = 1$. Then, notice that x can only be 0, 1 and -1 because any value of x^{2020} that is greater than 1 will cause the term $(y-1)^2$ to be less than 0, which is impossible as y must be real. Therefore, plugging in the above values for x gives the ordered pairs $(0,0)$, $(1,1)$, $(-1,1)$, and $(0,2)$ gives a total of $\boxed{\textbf{(D) }4}$ ordered pairs.

Solution 2

Bringing all of the terms to the LHS, we see a quadratic equation $y^2-2y+x^{2020}=0$ in terms of y. Applying the quadratic formula, we get $y=\frac{2\pm\sqrt{4-4\cdot1\cdot x^{2020}}}{2}=\frac{2\pm\sqrt{4(1-x^{2020})}}{2}$. In order for y to be real, which it must be given the stipulation that we are seeking integral answers, we know that the discriminant, $4(1-x^{2020})$ must be nonnegative. Therefore, $4(1-x^{2020})\geq 0 \implies x^{2020}\leq 1$. Here, we see that we must split the inequality into a compound, resulting in $-1\leq x\leq 1$.

The only integers that satisfy this are $x\in\{-1,0,1\}$. Plugging these values back into the quadratic equation, we see that $x=\{-1,1\}$ both produce a discriminant of 0, meaning that there is only 1 solution for y. If $x=\{0\}$, then the discriminant is nonzero, therefore resulting in two solutions for y.

Thus, the answer is $2\cdot1+1\cdot2=\boxed{\textbf{(D) }4}$.

~Tiblis

Solution 3, x first

Set it up as a quadratic in terms of y: $y^2 - 2y + x^{2020} = 0$ Then the discriminant is $\Delta = 4 - 4x^{2020}$ This will clearly only yield real solutions when $x^{2020} \leq 1$, because it is always positive. Then $x = -1, 0, 1$. Checking each one: -1 and 1 are the same when raised to the 2020th power: $y^2 - 2y + 1 = (y-1)^2 = 0$ This has only has solutions 1, so $(\pm 1, 1)$ are solutions. Next, if $x = 0$: $y^2 - 2y = 0$ Which has 2 solutions, so $(0, 2)$ and $(0, 0)$

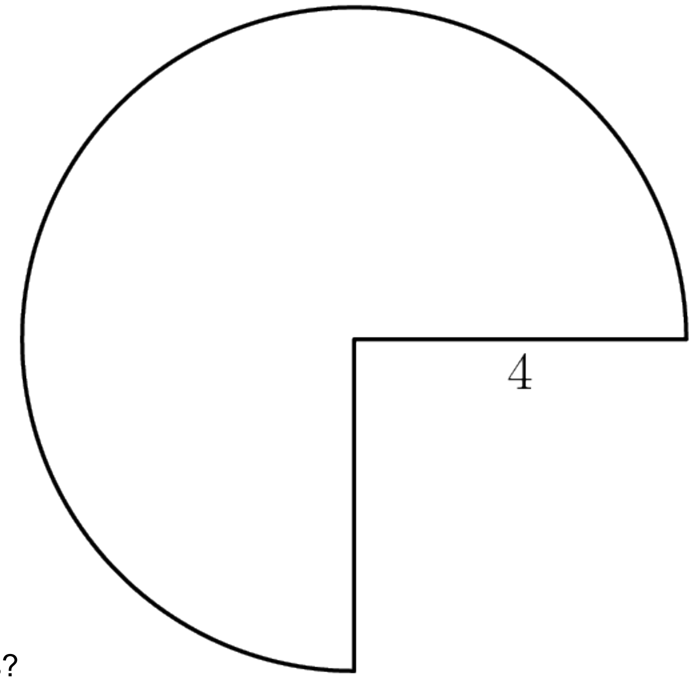
These are the only 4 solutions, so $\boxed{\textbf{(D) } 4}$

Solution 4, y first

Move the y^2 term to the other side to get $x^{2020} = 2y - y^2 = y(2-y)$. Because $x^{2020} \geq 0$ for all x, then $y(2-y) \geq 0 \Rightarrow y = 0, 1, 2$. If $y = 0$ or $y = 2$, the right side is 0 and therefore $x = 0$. When $y = 1$, the right side become 1, therefore $x = 1, -1$. Our solutions are $(0, 2)$, $(0, 0)$, $(1, 1)$, $(-1, 1)$. There are 4 solutions, so the answer is $\boxed{\textbf{(D) } 4}$ - wwt7535

Problem 10

A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic



inches?

$\textbf{(A) } 3\pi\sqrt{5} \qquad \textbf{(B) } 4\pi\sqrt{3} \qquad \textbf{(C) } 3\pi\sqrt{7} \qquad \textbf{(D) } 6\pi\sqrt{3} \qquad \textbf{(E) } 6\pi\sqrt{7}$

Solution

Notice that when the cone is created, the radius of the circle will become the slant height of the cone and the intact circumference of the circle will become the circumference of the base of the cone.

We can calculate that the intact circumference of the circle is $8\pi \cdot \frac{3}{4} = 6\pi$. Since that is also equal to the circumference of the cone, the radius of the cone is 3. We also have that the slant height of the cone is 4. Therefore, we use the Pythagorean Theorem to calculate that the height of the cone is $\sqrt{4^2 - 3^2} = \sqrt{7}$. The volume of the cone is $\frac{1}{3} \cdot \pi \cdot 3^2 \cdot \sqrt{7} = \boxed{\textbf{(C) } 3\pi\sqrt{7}}$ -PCChess

Solution 2 (Last Resort/Cheap)

Using a ruler, measure a circle of radius 4 and cut out the circle and then the quarter missing. Then, fold it into a cone and measure the diameter to be 6 cm $\implies r = 3$. You can form a right triangle with sides 3, 4, and then through the Pythagorean theorem the height h is found to be $h^2 = 4^2 - 3^2 \implies h = \sqrt{7}$. The volume of a cone is $\frac{1}{3}\pi r^2 h$.

Plugging in we find $V = 3\pi\sqrt{7} \implies \boxed{\textbf{(C)}}$

Problem11

Ms. Carr asks her students to read any 5 of the 10 books on a reading list. Harold randomly selects 5 books from this list, and Betty does the same. What is the probability that there are exactly 2 books that they both select?

$\textbf{(A)}\ \frac{1}{8} \qquad \textbf{(B)}\ \frac{5}{36} \qquad \textbf{(C)}\ \frac{14}{45} \qquad \textbf{(D)}\ \frac{25}{63} \qquad \textbf{(E)}\ \frac{1}{2}$

Solution

We don't care about which books Harold selects. We just care that Betty picks 2 books from Harold's list and 3 that aren't on Harold's list.

The total amount of combinations of books that Betty can select is $\binom{10}{5} = 252$.

There are $\binom{5}{2} = 10$ ways for Betty to choose 2 of the books that are on Harold's list.

From the remaining 5 books that aren't on Harold's list, there are $\binom{5}{3} = 10$ ways to choose 3 of them.

$\frac{10 \cdot 10}{252} = \boxed{\textbf{(D)}\ \frac{25}{63}}$ ~quacker88

Problem12

The decimal representation of $\frac{1}{20^{20}}$ consists of a string of zeros after the decimal point, followed by a 9 and then several more digits. How many zeros are in that initial string of zeros after the decimal point?

$\textbf{(A) } 23 \qquad\textbf{(B) } 24 \qquad\textbf{(C) } 25 \qquad\textbf{(D) } 26 \qquad\textbf{(E) } 27$

Solution 1

$$\frac{1}{20^{20}} = \frac{1}{(10\cdot 2)^{20}} = \frac{1}{10^{20}\cdot 2^{20}}$$

Now we do some estimation. Notice that $2^{20} = 1024^2$, which means that 2^{20} is a little more than $1000^2 = 1,000,000$. Multiplying it with 10^{20}, we get that the denominator is about $1\underbrace{00\dots0}_{26 \text{ zeros}}$. Notice that when we divide 1 by an n digit number, there are $n-1$ zeros before the first nonzero digit. This means that when we divide 1 by the 27 digit integer $1\underbrace{00\dots0}_{26 \text{ zeros}}$, there are $\boxed{\textbf{(D) } 26}$ zeros in the initial string after the decimal point. -PCChess

Solution 2

First rewrite $\frac{1}{20^{20}}$ as $\frac{5^{20}}{10^{40}}$. Then, we know that when we write this in decimal form, there will be 40 digits after the decimal point. Therefore, we just have to find the number of digits in 5^{20}.

$\log 5^{20} = 20\log 5$ and memming $\log 5 \approx 0.69$ (alternatively use the fact that $\log 5 = 1 - \log 2$),

$\lfloor 20\log 5\rfloor + 1 = \lfloor 20\cdot 0.69\rfloor + 1 = 13 + 1 = 14$ digits.

Our answer is $\boxed{\textbf{(D) } 26}$.

Solution 3 (Brute Force)

Just as in Solution $2,$ we rewrite $\frac{1}{20^{20}}$ as $\frac{5^{20}}{10^{40}}.$ We then calculate 5^{20} entirely by hand, first doing $5^5 \cdot 5^5,$then multiplying that product by itself, resulting in $95,367,431,640,625.$ Because this is 14 digits, after dividing this number by 10 fourteen times, the decimal point is before the $9.$ Dividing the number again by 10 twenty-six more times allows a string of $\boxed{\textbf{(D) } 26}$ zeroes to be formed. -OreoChocolate

Solution 4 (Smarter Brute Force)

Just as in Solutions 2 and $3,$ we rewrite $\frac{1}{20^{20}}$ as $\frac{5^{20}}{10^{40}}.$ We can then look at the number of digits in powers of 5. $5^1 = 5$, $5^2 = 25$, $5^3 = 125$, $5^4 = 625$, $5^5 = 3125$, $5^6 = 15625$, $5^7 = 78125$ and so on. We notice after a few iterations that every power of five with an exponent of $1 \mod 3$, the number of digits doesn't increase. This means 5^{20} should have $20-6$ digits since there are 6 numbers which are $1 \mod 3$ from 0 to 20, or 14 digits total. This means our expression can be written as $\frac{k \cdot 10^{14}}{10^{40}}$, where k is in the range $[1,10)$. Canceling gives $\frac{k}{10^{26}}$, or 26 zeroes before the k since the number k should start on where the one would be in 10^{26}. ~aop2014

Solution 5 (Logarithms)

$$|\lceil\log \frac{1}{20^{20}}\rceil| = |\lceil\log 20^{-20}\rceil| = |\lceil-20\log(20)\rceil| = |\lceil-20(\log 10+\log 2)\rceil| = |\lceil-20(1+0.301)\rceil| = |\lceil-26.02\rceil| = |-26| = \boxed{\textbf{(D) } 26}$$

Problem13

Andy the Ant lives on a coordinate plane and is currently at $(-20, 20)$ facing east (that is, in the positive x-direction). Andy moves 1 unit and then turns 90° degrees left. From there, Andy moves 2 units (north) and then turns 90° degrees left. He then moves 3 units (west) and again turns 90° degrees left. Andy continues his progress, increasing his distance each time by 1 unit and always turning left. What is the location of the point at which Andy makes the 2020th left turn?

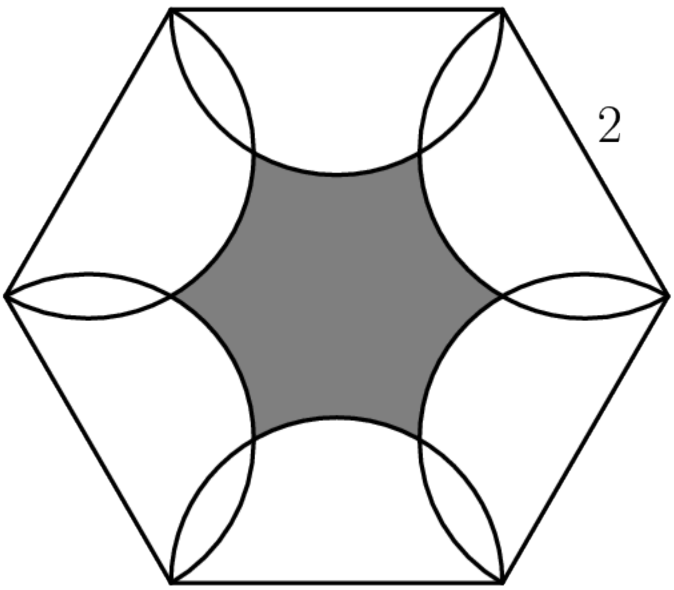
$\textbf{(A)}\ (-1030, -994) \qquad \textbf{(B)}\ (-1030, -990) \qquad \textbf{(C)}\ (-1026, -994) \qquad \textbf{(D)}\ (-1026, -990) \qquad \textbf{(E)}\ (-1022, -994)$

Solution 1

You can find that every four moves both coordinates decrease by 2. Therefore, both coordinates need to decrease by two 505 times. You subtract, giving you the answer of $\boxed{\textbf{(B)}\ \text{(-1030,-990)}}$ ~happykeeper

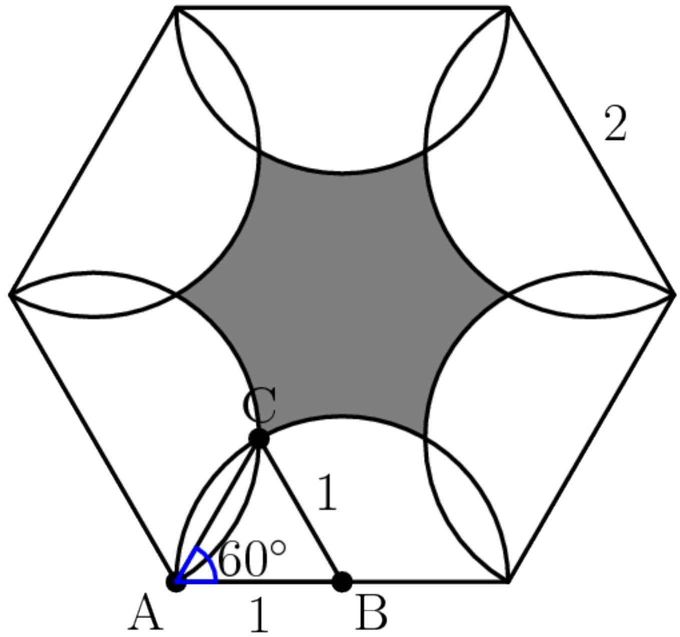
Problem14

As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded region — inside the hexagon but outside all of the semicircles?

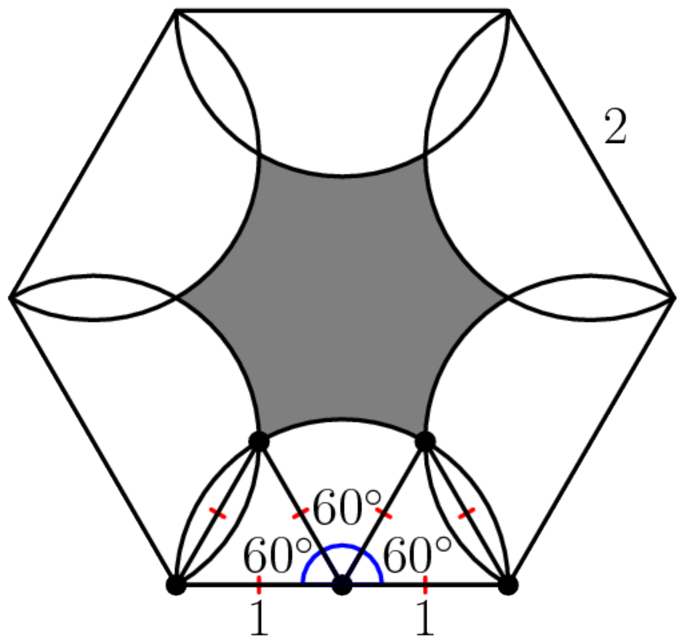


$\textbf{(A)}\ 6\sqrt{3}-3\pi \qquad \textbf{(B)}\ \frac{9\sqrt{3}}{2}-2\pi \qquad \textbf{(C)}\ \frac{3\sqrt{3}}{2}-\frac{\pi}{3} \qquad \textbf{(D)}\ 3\sqrt{3}-\pi \qquad \textbf{(E)}\ \frac{9\sqrt{3}}{2}-\pi$

Solution 1



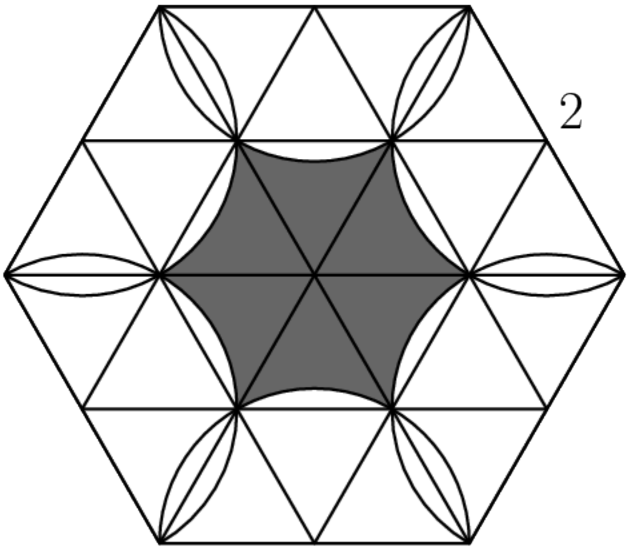
Let point A be a vertex of the regular hexagon, let point B be the midpoint of the line connecting point A and a neighboring vertex, and let point C be the second intersection of the two semicircles that pass through point A. Then, $BC = 1$, since B is the center of the semicircle with radius 1 that C lies on, $AB = 1$, since B is the center of the semicircle with radius 1 that A lies on, and $\angle BAC = 60^\circ$, as a regular hexagon has angles of 120○, and $\angle BAC$ is half of any angle in this hexagon. Now, using the sine law, $\frac{1}{\sin \angle ACB} = \frac{1}{\sin 60^\circ}$, so $\angle ACB = 60^\circ$. Since the angles in a triangle sum to 180○, $\angle ABC$ is also 60○. Therefore, $\triangle ABC$ is an equilateral triangle with side lengths of 1.



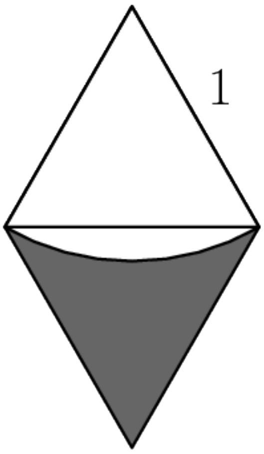
Since the area of a regular hexagon can be found with the formula $\frac{3\sqrt{3}s^2}{2}$, where S is the side length of the hexagon, the area of this hexagon is $\frac{3\sqrt{3}(2^2)}{2} = 6\sqrt{3}$. Since the area of an equilateral triangle can be found with the formula $\frac{\sqrt{3}}{4}s^2$, where S is the side length of the equilateral triangle, the area of an equilateral triangle with side lengths of 1 is $\frac{\sqrt{3}}{4}(1^2) = \frac{\sqrt{3}}{4}$. Since the area of a circle can be found with the formula πr^2, the area of a sixth of a circle with radius 1 is $\frac{\pi(1^2)}{6} = \frac{\pi}{6}$. In each sixth of the hexagon, there are two equilateral triangles colored white, each with an area of $\frac{\sqrt{3}}{4}$, and one sixth of a circle with radius 1 colored white, with an area of $\frac{\pi}{6}$. The rest of the sixth is colored gray. Therefore, the total area that is colored white in each sixth of the hexagon is $2(\frac{\sqrt{3}}{4}) + \frac{\pi}{6}$, which equals $\frac{\sqrt{3}}{2} + \frac{\pi}{6}$, and the total area colored white is $6(\frac{\sqrt{3}}{2}+\frac{\pi}{6})$, which equals $3\sqrt{3}+\pi$. Since the area colored gray equals the total area of the hexagon minus the area colored white, the area colored gray is $6\sqrt{3}-(3\sqrt{3}+\pi)$, which equals $\boxed{\textbf{(D)}\ 3\sqrt{3}-\pi}$.

Solution 2

First, subdivide the hexagon into 24 equilateral triangles with side length 1:



Now note that the entire shaded region is just 6 times this part:



The entire rhombus is just 2 equilatrial triangles with side lengths of 1, so it has an area of: $2\cdot\frac{\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$ The arc that is not included has an area of: $\frac{1}{6}\cdot\pi\cdot1^2=\frac{\pi}{6}$ Hence, the area of the shaded region in that section is $\frac{\sqrt{3}}{2}-\frac{\pi}{6}$ For a final area of: $6\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)=3\sqrt{3}-\pi \Rightarrow \boxed{(\mathrm{D})}$

Problem15

Steve wrote the digits 1, 2, 3, 4, and 5 in order repeatedly from left to right, forming a list of $10,000$ digits, beginning $123451234512\ldots$. He then erased every third digit from his list (that is, the 3rd, 6th, 9th, $\ldots$ digits from the left), then erased every fourth digit from the resulting list (that is, the 4th, 8th, 12th, $\ldots$ digits from the left in what remained), and then erased every fifth digit from what remained at that point. What is the sum of the three digits that were then in the positions $2019, 2020, 2021$?

$\textbf{(A)}\ 7 \qquad \textbf{(B)}\ 9 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 11 \qquad \textbf{(E)}\ 12$

Solution 1

After erasing every third digit, the list becomes $1245235134\ldots$ repeated. After erasing every fourth digit from this list, the list becomes $124235341452513\ldots$ repeated. Finally, after erasing every fifth digit from this list, the list becomes $124253415251\ldots$ repeated. Since this list repeats every 12 digits and since $2019, 2020, 2020$ are $3, 4, 5$ respectively in $\pmod{12}$, we have that the 2019th, 2020th, and 2021st digits are the 3rd, 4th, and 5th digits respectively. It follows that the answer is $4+2+5=\boxed{\textbf{(D)}\ 11}$.
~dolphin7

Problem16

Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

$\textbf{(A)}$ Bela will always win. $\textbf{(B)}$ Jenn will always win. $\textbf{(C)}$ Bela will win if and only if n is odd. $\textbf{(D)}$ Jenn will win if and only if n is odd. $\textbf{(E)}$ Jenn will win if and only if $n > 8$.

Solution

Notice that to use the optimal strategy to win the game, Bela must select the middle number in the range $[0, n]$ and then mirror whatever number Jenn selects. Therefore, if Jenn can select a number within the range, so can Bela. Jenn will always be the first person to run out of a number to choose, so the answer is $\boxed{\textbf{(A)} \text{ Bela will always win}}$.

Solution 2 (Guessing)

First of all, realize that the value of n should have no effect on the strategy at all. This is because they can choose real numbers, not integers, so even if n is odd, for example, they can still go halfway. Similarly, there is no reason the strategy would change when $n > 8$.

So we are left with (A) and (B). From here it is best to try out random numbers and try to find the strategy that will let Bela win, but if you can't find it, realize that it is more likely the answer is $\boxed{\textbf{(A)} \text{ Bela will always win}}$ since Bela has the first move and thus has more control.

Problem17

There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

$\textbf{(A)}\ 11 \qquad\textbf{(B)}\ 12 \qquad\textbf{(C)}\ 13 \qquad\textbf{(D)}\ 14 \qquad\textbf{(E)}\ 15$

Solution

Let us use casework on the number of diagonals.

Case 1: 0 diagonals There are 2 ways: either 1 pairs with 2, 3 pairs with 4, and so on or 10 pairs with 1, 2pairs with 3, etc.

Case 2: 1 diagonal There are 5 possible diagonals to draw (everyone else pairs with the person next to them.

Note that there cannot be 2 diagonals.

Case 3: 3 diagonals

Note that there cannot be a case with 4 diagonals because then there would have to be 5 diagonals for the two remaining people, thus a contradiction.

Case 4: 5 diagonals There is 1 way to do this.

Thus, in total there are $2+5+5+1=\boxed{13}$ possible ways.

Problem18

An urn contains one red ball and one blue ball. A box of extra red and blue balls lie nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?

$\textbf{(A)}\ \frac{1}{6} \qquad \textbf{(B)}\ \frac{1}{5} \qquad \textbf{(C)}\ \frac{1}{4} \qquad \textbf{(D)}\ \frac{1}{3} \qquad \textbf{(E)}\ \frac{1}{2}$

Solution

Let R denote that George selects a red ball and B that he selects a blue one. Now, in order to get 3 balls of each color, he needs 2 more of both R and B.

There are 6 cases: $RRBB, RBRB, RBBR, BBRR, BRBR, BRRB$ (we can confirm that there are only 6 since $\binom{4}{2}=6$). However we can clump $RRBB+BBRR$, $RBRB+BRBR$, and $RBBR+BRRB$ together since they are equivalent by symmetry.

$\textbf{CASE 1:}$ $RRBB$ and $BBRR$

Let's find the probability that he picks the balls in the order of $RRBB$.

The probability that the first ball he picks is red is $\frac{1}{2}$.

Now there are 2 reds and 1 blue in the urn. The probability that he picks red again is now $\frac{2}{3}$.

There are 3 reds and 1 blue now. The probability that he picks a blue is $\frac{1}{4}$.

Finally, there are 3 reds and 2 blues. The probability that he picks a blue is $\frac{2}{5}$.

So the probability that the $RRBB$ case happens is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{2}{5} = \frac{1}{30}$. However, since the $BBRR$case is the exact same by symmetry, case 1 has a probability of $\frac{1}{30} \cdot 2 = \frac{1}{15}$ chance of happening.

$\textbf{CASE 2:}$ $RBRB$ and $BRBR$

Let's find the probability that he picks the balls in the order of $RBRB$.

The probability that the first ball he picks is red is $\frac{1}{2}$.

Now there are 2 reds and 1 blue in the urn. The probability that he picks blue is $\frac{1}{3}$.

There are 2 reds and 2 blues now. The probability that he picks a red is $\frac{1}{2}$.

Finally, there are 3 reds and 2 blues. The probability that he picks a blue is $\frac{2}{5}$.

So the probability that the $RBRB$ case happens is $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{30}$. However, since the $BRBR$case is the exact same by symmetry, case 2 has a probability of $\frac{1}{30} \cdot 2 = \frac{1}{15}$ chance of happening.

$\textbf{CASE 3:}$ $RBBR$ and $BRRB$

Let's find the probability that he picks the balls in the order of $RBBR$.

The probability that the first ball he picks is red is $\frac{1}{2}$.

Now there are 2 reds and 1 blue in the urn. The probability that he picks blue is $\frac{1}{3}$.

There are 2 reds and 2 blues now. The probability that he picks a blue is $\frac{1}{2}$.

Finally, there are 2 reds and 3 blues. The probability that he picks a red is $\frac{2}{5}$.

So the probability that the $RBBR$ case happens is $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{30}$. However, since the $BRBR$case is the exact same by symmetry, case 3 has a probability of $\frac{1}{30} \cdot 2 = \frac{1}{15}$ chance of happening.

Adding up the cases, we have $\frac{1}{15} + \frac{1}{15} + \frac{1}{15} = \boxed{\textbf{(B) } \frac{1}{5}}$ ~quacker88

Solution 2

We know that we need to find the probability of adding 2 red and 2 blue balls in some order. There are 6 ways to do this, since there are $\binom{4}{2} = 6$ ways to arrange $RRBB$ in some order. We will show that the probability for each of these 6 ways is the same.

We first note that the denominators should be counted by the same number. This number is $2 \cdot 3 \cdot 4 \cdot 5 = 120$. This is because 2, 3, 4, and 5 represent how many choices there are for the four steps. No matter what the $k-th$ step involves $k+1$ numbers to choose from.

The numerators are the number of successful operations. No matter the order, the first time a red is added will come from 1 choice and the second time will come from 2 choices, since that is how many reds are in the urn originally. The same goes for the blue ones. The numerator must equal $(1 \cdot 2)^2$.

Therefore, the probability for each of the orderings of $RRBB$ is $\frac{4}{120} = \frac{1}{30}$. There are 6 of these, so the total probability is $\boxed{\textbf{(B) }\frac{1}{5}}$.

Solution 3

First, notice that when George chooses a ball he just adds another ball of the same color. On George's first move, he either chooses the red or the blue with a $\frac{1}{2}$ chance each. We can assume he chooses Red(chance $\frac{1}{2}$), and then multiply the final answer by two for symmetry. Now, there are two red balls and one blue ball in the urn. Then, he can either choose another Red(chance $\frac{2}{3}$), in which case he must choose two blues to get three of each, with probability $\frac{1}{4} \cdot \frac{2}{5} = \frac{1}{10}$ or a blue for two blue and two red in the urn, with chance $\frac{1}{3}$. If he chooses blue next, he can either choose a red then a blue, or a blue then a red. Each of these has a $\frac{1}{2} \cdot \frac{2}{5}$ for total of $2 \cdot \frac{1}{5} = \frac{2}{5}$. The total probability that he ends up with three red and three blue is $2 \cdot \frac{1}{2}(\frac{2}{3} \cdot \frac{1}{10} + \frac{1}{3} \cdot \frac{2}{5}) = \frac{1}{15} + \frac{2}{15} = \boxed{\textbf{(B)} \frac{1}{5}}$. ~aop2014

Solution 4

Let the probability that the urn ends up with more red balls be denoted $P(R)$. Since this is equal to the probability there are more blue balls, the probability there are equal amounts is $1 - 2P(R)$. $P(R) =$ the probability no more blues are chosen plus the probability only 1 more blue is chosen. The first case, $P(\text{no more blues}) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$.

The second case, $P(\text{1 more blue}) = 4 \cdot \frac{1 \cdot 1 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{5}$. Thus, the answer is $1 - 2\left(\frac{1}{5} + \frac{1}{5}\right) = 1 - \frac{4}{5} = \boxed{\textbf{(B) } \frac{1}{5}}$.

~JHawk0224

Solution 5

By conditional probability after 4 rounds we have 5 cases: RRRBBB, RRRRBB, RRBBBB, RRRRRB and RBBBBB. Thus the probability is $\frac{1}{5}$. Put $\boxed{B}$.

~FANYUCHEN20020715

Edited by Kinglogic

Solution 6

Here X stands for R or B, and Y for the remaining color. After 3 rounds one can either have a 4+1 configuration (XXXXY), or 3+2 configuration (XXXYY). The probability of getting to XXXYYY from XXXYY is $\frac{2}{5}$. Observe that the probability of arriving to 4+1 configuration is $\frac{2}{3}\cdot\frac{3}{4}=\frac{1}{2}$($\frac{2}{3}$ to get from XXY to XXXY, $\frac{3}{4}$ to get from XXXY to XXXXY). Thus the probability of arriving to 3+2 configuration is also $\frac{1}{2}$, and the answer is $\frac{1}{2}\cdot\frac{2}{5}=\boxed{\textbf{(B) }\frac{1}{5}}$.

Solution 7

We can try to use dynamic programming to solve this problem. (Informatics Olympiad hahaha)

We let $dp[i][j]$ be the probability that we end up with i red balls and j blue balls. Notice that there are only two ways that we can end up with i red balls and j blue balls: one is by fetching a red ball from the urn when we have $i-1$ red balls and j blue balls and the other is by fetching a blue ball from the urn when we have i red balls and $j-1$ blue balls.

Then we have

$$dp[i][j]=\frac{i-1}{i-1+j}dp[i-1][j]+\frac{j-1}{i-1+j}dp[i][j-1]$$

Then we start can with $dp[1][1]=1$ and try to compute $dp[3][3]$.

$i \setminus j$	1	2	3
1	1	$1/2$	$1/3$
2	$1/2$	$1/3$	$1/4$
3	$1/3$	$1/4$	$1/5$

The answer is $\boxed{\textbf{(B) }\frac{1}{5}}$.

Problem19

In a certain card game, a player is dealt a hand of 10 cards from a deck of 52 distinct cards. The number of distinct (unordered) hands that can be dealt to the player can be written as $158A00A4AA0$. What is the digit A?

$\textbf{(A) } 2 \qquad\textbf{(B) } 3 \qquad\textbf{(C) } 4 \qquad\textbf{(D) } 6 \qquad\textbf{(E) } 7$

Solution 1

$158A00A4AA0 \equiv 1+5+8+A+0+0+A+4+A+A+0 \equiv 4A \pmod{9}$

We're looking for the amount of ways we can get 10 cards from a deck of 52, which is represented by $\binom{52}{10}$.

$$\binom{52}{10} = \frac{52\cdot51\cdot50\cdot49\cdot48\cdot47\cdot46\cdot45\cdot44\cdot43}{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

We need to get rid of the multiples of 3, which will subsequently get rid of the multiples of 9 (if we didn't, the zeroes would mess with the equation since you can't divide by 0)

$9\cdot5=45$, $8\cdot6=48$, $\frac{51}{3}$ leaves us with 17.

$$\frac{52\cdot\cancel{51}^{17}\cdot50\cdot49\cdot\cancel{48}\cdot47\cdot46\cdot\cancel{45}\cdot44\cdot43}{10\cdot\cancel{9}\cdot\cancel{8}\cdot7\cdot\cancel{6}\cdot\cancel{5}\cdot4\cdot\cancel{3}\cdot2\cdot1}$$

Converting these into $\pmod{9}$, we have

$$\binom{52}{10} \equiv \frac{(-2)\cdot(-1)\cdot(-4)\cdot4\cdot2\cdot1\cdot(-1)\cdot(-2)}{1\cdot(-2)\cdot4\cdot2\cdot1} \equiv (-1)\cdot(-4)\cdot(-1)\cdot(-2) \equiv 8 \pmod{9}$$

$4A \equiv 8 \pmod{9} \implies A = \boxed{\textbf{(A) } 2}$ ~quacker88

Solution 2

$$\binom{52}{10} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 5 \cdot 7 \cdot 47 \cdot 46 \cdot 11 \cdot 43$$

Since this number is divisible by 4 but not 8, the last 2 digits must be divisible by 4 but the last 3 digits cannot be divisible by 8. This narrows the options down to 2 and 6.

Also, the number cannot be divisible by 3. Adding up the digits, we get $18 + 4A$. If $A = 6$, then the expression equals 42, a multiple of 3. This would mean that the entire number would be divisible by 3, which is not what we want. Therefore, the only option is $\boxed{\textbf{(A) } 2}$-PCChess

Solution 3

It is not hard to check that 13 divides the number,

$$\binom{52}{10} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 {\cdot} 17 {\cdot} 5 {\cdot} 7 {\cdot} 47 {\cdot} 46 {\cdot} 11 {\cdot} 43.$$

As $10^3 \equiv -1 \pmod{13}$, using $\pmod{13}$ we have $13 | \overline{AA0} - \overline{0A4} + \overline{8A0} - \overline{15} = 110A + 781$.

Thus $6A + 1 \equiv 0 \pmod{13}$, implying $A \equiv 2 \pmod{13}$ so the answer is $\boxed{\textbf{(A) } 2}$.

- Emathmaster

Solution 4

As mentioned above, $\binom{52}{10} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 17 \cdot 13 \cdot 7 \cdot 47 \cdot 46 \cdot 11 \cdot 43 = 158A00A4AA0.$

We can divide both sides of $10 \cdot 17 \cdot 13 \cdot 7 \cdot 47 \cdot 46 \cdot 11 \cdot 43 = 158A00A4AA0$ by 10 to obtain $17 \cdot 13 \cdot 7 \cdot 47 \cdot 46 \cdot 11 \cdot 43 = 158A00A4AA$, which means A is simply the units digit of the left-hand side. This value is $7 \cdot 3 \cdot 7 \cdot 7 \cdot 6 \cdot 1 \cdot 3 \equiv \boxed{\textbf{(A) } 2} \pmod{10}.$

Problem20

Let B be a right rectangular prism (box) with edges lengths $1, 3,$ and 4, together with its interior. For real $r \geq 0$, let $S(r)$ be the set of points in 3-dimensional space that lie within a distance r of some point B. The volume of $S(r)$ can be expressed as $ar^3 + br^2 + cr + d$, where $a, b, c,$ and d are positive real numbers. What is $\frac{bc}{ad}$?

$\textbf{(A)}\ 6 \qquad \textbf{(B)}\ 19 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 26 \qquad \textbf{(E)}\ 38$

Solution

Split $S(r)$ into 4 regions:

1. The rectangular prism itself
2. The extensions of the faces of B
3. The quarter cylinders at each edge of B
4. The one-eighth spheres at each corner of B

Region 1: The volume of B is 12, so $d = 12$

Region 2: The volume is equal to the surface area of B times r. The surface area can be computed to be $2(4*3+3*1+4*1) = 38$, so $c = 38$.

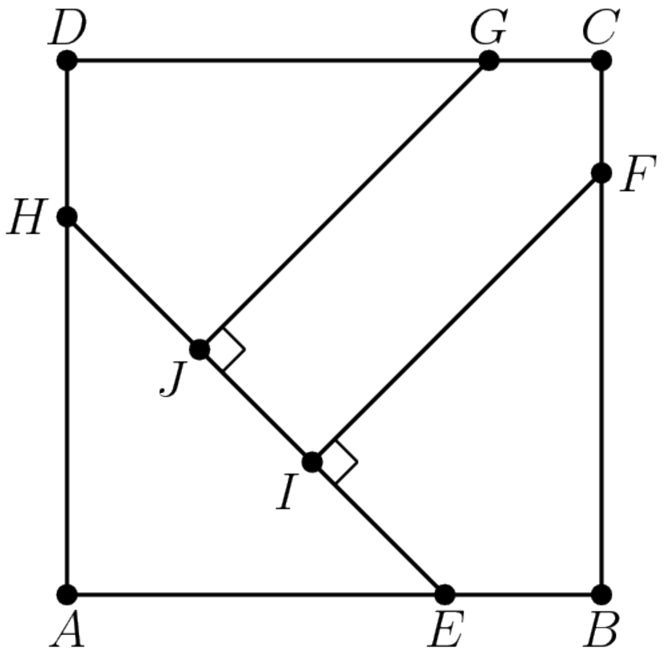
Region 3: The volume of each quarter cylinder is equal to $(\pi * r^2 * h)/4$. The sum of all such cylinders must equal $(\pi * r^2)/4$ times the sum of the edge lengths. This can be computed as $4(4+3+1) = 32$, so the sum of the volumes of the quarter cylinders is $8\pi * r^2$, so $b = 8\pi$

Region 4: There is an eighth of a sphere of radius r at each corner. Since there are 8 corners, these add up to one full sphere of radius r. The volume of this sphere is $\frac{4}{3}\pi * r^3$, so $a = \frac{4\pi}{3}$.

Using these values, $\dfrac{(8\pi)(38)}{(4\pi/3)(12)} = \boxed{\textbf{(B) } 19}$

Problem21

In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH.$ Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area $1.$ What



is FI^2?

$\textbf{(A) } \frac{7}{3} \qquad \textbf{(B) } 8-4\sqrt2 \qquad \textbf{(C) } 1+\sqrt2 \qquad \textbf{(D) } \frac{7}{4}\sqrt2 \qquad \textbf{(E) } 2\sqrt2$

Solution

Since the total area is 4, the side length of square $ABCD$ is 2. We see that since triangle HAE is a right isosceles triangle with area 1, we can determine sides HA and AE both to be $\sqrt{2}$. Now, consider extending FB and IE until they intersect. Let the point of intersection be K. We note that EBK is also a right isosceles triangle with side $2-\sqrt{2}$ and find it's area to be $3-2\sqrt{2}$. Now, we notice that FIK is also a right isosceles triangle and find it's area to be $\frac{1}{2}FI^2$. This is also equal to $1+3-2\sqrt{2}$ or $4-2\sqrt{2}$. Since we are looking for FI^2, we want two times this. That gives $\boxed{\textbf{(B)}\ 8-4\sqrt{2}}$.~TLiu

Solution 2

Since this is a geometry problem involving sides, and we know that HE is 2, we can use our ruler and find the ratio between FI and HE. Measuring(on the booklet), we get that HE is about 1.8 inches and FI is about 1.4 inches. Thus, we can then multiply the length of HE by the ratio of $\frac{1.4}{1.8}$, of which we then get $FI=\frac{14}{9}$. We take the square of that and get $\frac{196}{81}$, and the closest answer to that is $\boxed{\textbf{(B)}\ 8-4\sqrt{2}}$. ~Celloboy (Note that this is just a strategy I happened to use that worked. Do not press your luck with this strategy, for it was a lucky guess)

Solution 3

Draw the auxiliary line AC. Denote by M the point it intersects with HE, and by N the point it intersects with GF. Last, denote by x the segment FN, and by y the segment FI. We will find two equations for $x$and y, and then solve for y^2.

Since the overall area of $ABCD$ is $4 \implies AB=2$, and $AC=2\sqrt{2}$. In addition, the area of $\triangle AME=\frac{1}{2} \implies AM=1$.

The two equations for x and y are then:

- Length of AC: $1+y+x=2\sqrt{2} \implies x=(2\sqrt{2}-1)-y$

- Area of CMIF: $\frac{1}{2}x^2 + xy = \frac{1}{2} \quad \Longrightarrow \quad x(x+2y) = 1$.

Substituting the first into the second,

yields $\left[\left(2\sqrt{2}-1\right)-y\right] \cdot \left[\left(2\sqrt{2}-1\right)+y\right] = 1$

Solving for y^2 gives $\boxed{\textbf{(B) } 8-4\sqrt{2}}$ ~DrB

Solution 4

Plot a point F' such that F' and I are collinear and extend line FB to point B' such that $FIB'F'$ forms a square. Extend line AE to meet line $F'B'$ and point E' is the intersection of the two. The area of this square is equivalent to FI^2. We see that the area of square $ABCD$ is 4, meaning each side is of length 2. The area of the pentagon $EIFF'E'$ is 2. Length $AE = \sqrt{2}$, thus $EB = 2-\sqrt{2}$. Triangle $EB'E'$ is isosceles, and the area of this triangle is $\frac{1}{2}(4-2\sqrt{2})(2-\sqrt{2}) = 6-4\sqrt{2}$. Adding these two areas, we get $2+6-4\sqrt{2} = 8-4\sqrt{2} \rightarrow \boxed{\text{(B)}}$. --OGBooger

Solution 5 (HARD Calculation)

We can easily observe that the area of square $ABCD$ is 4 and its side length is 2 since all four regions that build up the square has area 1. Extend FI and let the intersection with AB be K. Connect AC, and let the intersection of AC and HE be L. Notice that since the area of triangle AEH is 1 and $AE = AH$, $AE = AH = \sqrt{2}$, therefore $BE = HD = 2-\sqrt{2}$. Let $CG = CF = m$, then $BF = DG = 2 - m$. Also notice that $KB = 2 - m$, thus $KE = KB - BE = 2 - m - (2 - \sqrt{2}) = \sqrt{2} - m$. Now use the condition that the area of quadrilateral $BFIE$ is 1, we can set up the following equation: $\frac{1}{2}(2-m)^2 - \frac{1}{4}(\sqrt{2}-m)^2 = 1$ We solve the equation and yield $m = \frac{8 - 2\sqrt{2} - \sqrt{64 - 32\sqrt{2}}}{2}$. Now notice that

$$FI = AC - AL = 2\sqrt{2} - 1 - \frac{\sqrt{2}}{2} * \frac{8 - 2\sqrt{2} - \sqrt{64 - 32\sqrt{2}}}{2}$$

$$= 2\sqrt{2} - 1 - \frac{8\sqrt{2} - 4 - \sqrt{128 - 64\sqrt{2}}}{4}$$

$$= \frac{\sqrt{128 - 64\sqrt{2}}}{4}.$$

Hence $FI^2 = \frac{128 - 64\sqrt{2}}{16} = 8 - 4\sqrt{2}$. -HarryW

Problem22

What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

$\textbf{(A) } 100 \qquad \textbf{(B) } 101 \qquad \textbf{(C) } 200 \qquad \textbf{(D) } 201 \qquad \textbf{(E) } 202$

Solution

Let $x = 2^{50}$. We are now looking for the remainder of $\frac{4x^4 + 202}{2x^2 + 2x + 1}$.

We could proceed with polynomial division, but the denominator looks awfully similar to the Sophie Germain Identity, which states that $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$

Let's use the identity, with $a = 1$ and $b = x$, so we have

$$1+4x^4=(1+2x^2+2x)(1+2x^2-2x)$$

Rearranging, we can see that this is exactly what we need:

$$\frac{4x^4+1}{2x^2+2x+1}=2x^2-2x+1$$

So $\dfrac{4x^4+202}{2x^2+2x+1}=\dfrac{4x^4+1}{2x^2+2x+1}+\dfrac{201}{2x^2+2x+1}$

Since the first half divides cleanly as shown earlier, the remainder must be $\boxed{\textbf{(D) } 201}$ ~quacker88

Solution 2

Similar to Solution 1, let $x=2^{50}$. It suffices to find remainder of $\dfrac{4x^4+202}{2x^2+2x+1}$. Dividing polynomials results in a remainder of $\boxed{\textbf{(D) } 201}$.

MAA Original Solution

$$2^{202}+202=(2^{101})^2+2\cdot 2^{101}+1-2\cdot 2^{101}+201$$

$$=(2^{101}+1)^2-2^{102}+201$$

$$=(2^{101}-2^{51}+1)(2^{101}+2^{51}+1)+201.$$

Thus, we see that the remainder is surely $\boxed{\textbf{(D) } 201}$

Problem23

Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1),$ and $D(1,-1)$. Consider the following four transformations: $L,$ a rotation of 90° counterclockwise around the origin; $R,$ a rotation of 90° clockwise around the origin; $H,$ a reflection across the x-axis; and $V,$ a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1, 1)$ to $(-1, -1)$ and would send the vertex B at $(-1, 1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

$\textbf{(A)}\ 2^{37} \qquad\textbf{(B)}\ 3\cdot 2^{36} \qquad\textbf{(C)}\ 2^{38} \qquad\textbf{(D)}\ 3\cdot 2^{37} \qquad\textbf{(E)}\ 2^{39}$

Solution

Let (+) denote counterclockwise/starting orientation and (-) denote clockwise orientation. Let 1,2,3, and 4 denote which quadrant A is in.

Realize that from any odd quadrant and any orientation, the 4 transformations result in some permutation of $(2+, 2-, 4+, 4-)$.

The same goes that from any even quadrant and any orientation, the 4 transformations result in some permutation of $(1+, 1-, 3+, 3-)$.

We start our first 19 moves by doing whatever we want, 4 choices each time. Since 19 is odd, we must end up on an even quadrant.

As said above, we know that exactly one of the four transformations will give us $(1+)$, and we must use that transformation.

Thus $4^{19} = \boxed{(C) 2^{38}}$

Solution 2

Hopefully, someone will think of a better one, but here is an indirect answer, use only if you are really desperate. 20 moves can be made, and each move have 4 choices, so a total of $4^{20} = 2^{40}$ moves. First, after the 20 moves, Point A can only be in first quadrant $(1,1)$ or third quadrant $(-1,-1)$. Only the one in the first quadrant works, so divide by 2. Now, C must be in the opposite quadrant as A. B can be either in the second ($(-1,1)$) or fourth quadrant ($(1,-1)$) , but we want it to be in the second quadrant, so divide by 2 again. Now as A and B satisfy the conditions, C and D will also be at their original spot. $\frac{2^{40}}{2\cdot 2} = 2^{38}$. The answer is $\boxed{C}$ ~Kinglogic

Solution 3

The total number of sequence is $4^{20} = 2^{40}$.

Note that there can only be even number of reflections since they result in the same anti-clockwise orientation of the verices A, B, C, D. Therefore, the probability of having the same anti-clockwise orientation with the original arrangement after the transformation is $\frac{1}{2}$.

Next, even number of reflections mean that there must be even number of rotations since their sum is even. Even rotations result only in the original position or 180° rotation of it.

Since rotation R and rotation L cancels each other out, the difference between the numbers of them define the final position. The probability of the transformation returning the vertices to the orginal position given that there are even number of rotations is equivalent to the probability that

$$|n(R) - n(L)| \equiv 0 \pmod 4$$ when

$$|n(H) - n(V)| \equiv 0 \pmod 4$$

or

$$|n(R) - n(L)| \equiv 2 \pmod 4$$ when

$$|n(H) - n(V)| \equiv 2 \pmod 4$$

which is again, $\frac{1}{2}$.

Therefore, $2^{40} \cdot \frac{1}{2} \cdot \frac{1}{2} = \boxed{\textbf{(C) } 2^{38}}$ ~joshuamh111

Solution 4

Notice that any pair of two of these transformations either swaps the x and y-coordinates, negates the x and y-coordinates, swaps and negates the x and y-coordinates, or leaves the original unchanged. Furthermore, notice that for each of these results, if we apply another pair of transformations, one of these results will happen again, and with equal probability. Therefore, no matter what state after we apply the first 19 pairs of transformations, there is a $\frac{1}{4}$ chance the last pair of transformations will return the figure to its original position. Therefore, the answer is $\frac{4^{20}}{4} = 4^{19} = \boxed{\textbf{(C) } 2^{38}}$

Problem24

How many positive integers n satisfy $$\dfrac{n+1000}{70} = \lfloor \sqrt{n} \rfloor?$$ (Recall that $\lfloor x\rfloor$ is the greatest integer not exceeding x.)

$\textbf{(A) } 2 \qquad\textbf{(B) } 4 \qquad\textbf{(C) } 6 \qquad\textbf{(D) } 30 \qquad\textbf{(E) } 32$

Solution

First notice that the graphs of $(x+1000)/70$ and $\sqrt{n}$ intersect at 2 points. Then, notice that $(n+1000)/70$ must be an integer. This means that n is congruent to $50 \pmod{70}$.

For the first intersection, testing the first few values of n (adding 70 to n each time and noticing the left side increases by 1 each time) yields $n=20$ and $n=21$. timating from the graph can narrow down the other cases, being $n = 47$, d $n = 50$. This results in a total of 6 cases, for an answer of $\boxed{\textbf{(C) } 6}$.

~DrJoyo

Solution 2 (Graphing)

One intuitive approach to the question is graphing. Obviously, you should know what the graph of the square root function is, and if any function is floored (meaning it is taken to the greatest integer less than a value), a stair-like figure should appear. The other function is simply a line with a slope of $1/70$. If you precisely draw out the two regions of the graph where the derivative of the square function nears the derivative of the linear function, you can now deduce that 3 values of intersection lay closer to the left side of the stair, and 3 values lay closer to the right side of the stair.

With meticulous graphing, you can realize that the answer is $\boxed{\textbf{(C) } 6}$.

A in-depth graph with intersection points is linked below.https://www.desmos.com/calculator/e5wk9adbuk

Solution 3

- Not a reliable or in-depth solution (for the guess and check students)

We can first consider the equation without a floor function:

$$\frac{n+1000}{70} = \sqrt{n}$$

Multiplying both sides by 70 and then squaring:

$$n^2 + 2000n + 1000000 = 4900n$$

Moving all terms to the left:

$$n^2 - 2900n + 1000000 = 0$$

Now we can use wishful thinking to determine the factors:

$$(n-400)(n-2500) = 0$$

This means that for $n = 400$ and $n = 2500$, the equation will hold without the floor function.

Now we can simply check the multiples of 70 around 400 and 2500 in the original equation:

For $n = 330$, left hand side $= 19$ but $18^2 < 330 < 19^2$ so right hand side $= 18$

For $n = 400$, left hand side $= 20$ and right hand side $= 20$

For $n = 470$, left hand side $= 21$ and right hand side $= 21$

For $n = 540$, left hand side $= 22$ but $540 > 23^2$ so right hand side $= 23$

Now we move to $n = 2500$

For $n = 2430$, left hand side $= 49$ and $49^2 < 2430 < 50^2$ so right hand side $= 49$

For $n = 2360$, left hand side $= 48$ and $48^2 < 2360 < 49^2$ so right hand side $= 48$

For $n = 2290$, left hand side $= 47$ and $47^2 < 2360 < 48^2$ so right hand side $= 47$

For $n = 2220$, left hand side $= 46$ but $47^2 < 2220$ so right hand side $= 47$

For $n = 2500$, left hand side $= 50$ and right hand side $= 50$

For $n = 2570$, left hand side $= 51$ but $2570 < 51^2$ so right hand side $= 50$

Therefore we have 6 total solutions, $n = 400, 470, 2290, 2360, 2430, 2500 = \boxed{\textbf{(C) } 6}$

Solution 4

This is my first solution here, so please forgive me for any errors.

We are given that $\dfrac{n+1000}{70} = \lfloor \sqrt{n} \rfloor$

$\lfloor\sqrt{n}\rfloor$ must be an integer, which means that $n+1000$ is divisible by 70. As $1000 \equiv 20 \pmod{70}$, this means that $n \equiv 50 \pmod{70}$, so we can write $n = 70k+50$ for $k \in \mathbb{Z}$.

Therefore,

$$\frac{n+1000}{70} = \frac{70k+1050}{70} = k+15 = \lfloor\sqrt{70k+50}\rfloor$$

Also, we can say that $\sqrt{70k+50}-1 \le k+15$ and $k+15 \le \sqrt{70k+50}$

Squaring the second inequality, we get $k^2+30k+225 \le 70k+50 \implies k^2-40k+175 \le 0 \implies (k-5)(k-35) \le 0 \implies 5 \le k \le 35$.

Similarly solving the first inequality gives us $k \le 19-\sqrt{155}$ or $k \ge 19+\sqrt{155}$

$\sqrt{155}$ is slightly larger than 12, so instead, we can say $k \le 6$ or $k \ge 32$.

Combining this with $5 \le k \le 35$, we get $k = 5, 6, 32, 33, 34, 35$ are all solutions for k that give a valid solution for n, meaning that our answer is $\boxed{\textbf{(C)} 6}$.

-Solution By Qqqwerw

Solution 5

We start with the given equation $\frac{n+1000}{70} = \lfloor\sqrt{n}\rfloor$ From there, we can start with the general inequality that $\lfloor\sqrt{n}\rfloor \le \sqrt{n} < \lfloor\sqrt{n}\rfloor + 1$. This means that $\frac{n+1000}{70} \le \sqrt{n} < \frac{n+1070}{70}$ Solving each inequality separately gives us two inequalities:

$n-70\sqrt{n}+1000 \leq 0 \rightarrow (\sqrt{n}-50)(\sqrt{n}-20) \leq 0 \rightarrow 20 \leq \sqrt{n} \leq 50$

$n - 70\sqrt{n} + 1070 > 0 \rightarrow \sqrt{n} < 35 - \sqrt{155}, \sqrt{n} > 35 + \sqrt{155}$

Simplifying and approximating decimals yields 2 solutions for one inequality and 4 for the other. Hence $2 + 4 = \boxed{\textbf{(C)} 6}$.

Problem25

Let $D(n)$ denote the number of ways of writing the positive integer n as a product $n = f_1 \cdot f_2 \cdots f_k$, where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

$\textbf{(A) } 112 \qquad \textbf{(B) } 128 \qquad \textbf{(C) } 144 \qquad \textbf{(D) } 172 \qquad \textbf{(E) } 184$

Solution

Note that $96 = 2^5 \cdot 3$. Since there are at most six not necessarily distinct factors > 1 multiplying to 96, we have six cases: $k = 1, 2, ..., 6.$ Now we look at each of the six cases.

$k = 1$: We see that there is 1 way, merely 96.
$k = 2$: This way, we have the 3 in one slot and 2 in another, and symmetry. The four other 2's leave us with 5 ways and symmetry doubles us so we have 10.

$k = 3$: We have $3, 2, 2$ as our baseline. We need to multiply by 2 in 3 places, and see that we can split the remaining three powers of 2 in a manner that is $3 - 0 - 0$, $2 - 1 - 0$ or $1 - 1 - 1$.
A $3 - 0 - 0$ split has $6 + 3 = 9$ ways of happening ($24 - 2 - 2$ and symmetry; $2 - 3 - 16$ and symmetry), a $2 - 1 - 0$ split has $6 \cdot 3 = 18$ ways of happening (due to all being distinct) and

a $1-1-1$ split has 3 ways of happening ($6-4-4$ and symmetry) so in this case we have $9+18+3=30$ ways.

$k=4$: We have $3,2,2,2$ as our baseline, and for the two other 2's, we have a $2-0-0-0$ or $1-1-0-0$ split. The former grants us $4+12=16$ ways ($12-2-2-2$ and symmetry and $3-8-2-2$ and symmetry) and the latter grants us also $12+12=24$ ways ($6-4-2-2$ and symmetry and $3-4-4-2$ and symmetry) for a total of $16+24=40$ ways.

$k=5$: We have $3,2,2,2,2$ as our baseline and one place to put the last two: on another two or on the three. On the three gives us 5 ways due to symmetry and on another two gives us $5\cdot 4=20$ ways due to symmetry. Thus, we have $5+20=25$ ways.

$k=6$: We have $3,2,2,2,2,2$ and symmetry and no more twos to multiply, so by symmetry, we have 6ways.

Thus, adding, we have $1+10+30+40+25+6=\boxed{\textbf{(A) }112}$.

~kevinmathz

Solution 2

As before, note that $96=2^5\cdot 3$, and we need to consider 6 different cases, one for each possible value of k, the number of factors in our factorization. However, instead of looking at each individually, find a general form for the number of possible factorizations with k factors. First, the factorization needs to contain one factor that is itself a multiple of 3, and there are k to choose from, and the rest must contain at least one factor of 2. Next, consider the remaining $6-n$ factors of 2 left to assign to the k factors. Using stars and bars, the number of ways to do this is $\binom{(6-k)+k-1}{6-k} = \binom{5}{6-k}$ This

makes $k\binom{5}{6-k}$ possibilities for each k.

To obtain the total number of factorizations, add all possible values for k:

$$\sum_{k=1}^{6} k\binom{5}{6-k} = 1+10+30+40+25+6 = \boxed{\textbf{(A) } 112}$$

.

Solution 3

Begin by examining f_1. f_1 can take on any value that is a factor of 96 except 1.

For each choice of f_1, the resulting $f_2 \dots f_k$ must have a product of $96/f_1$.

This means the number of ways the rest f_a, $1 < a <= k$ can be written by the scheme stated in the problem for each f_1 is equal to $D(96/f_1)$, since the product of $f_2 \cdot f_3 \dots \cdot f_k = x$ is counted as one valid product if and only if $f_1 \cdot x = 96$, the product x has the properties that factors are greater than 1, and differently ordered products are counted separately.

For example, say the first factor is 2. Then, the remaining numbers must multiply to 48, so the number of ways the product can be written beginning with 2 is $D(48)$. To add up all the number of solutions for every possible starting factor, $D(96/f_1)$ must be calculated and summed for all possible f_1, except 96 and 1, since a single 1 is not counted according to the problem statement. The 96 however, is counted, but only results in 1 possibility, the first and only factor being 96. This means

$D(96) = D(48) + D(32) + D(24) + D(16) + D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1$

.

Instead of calculating D for the larger factors first, reduce $D(48)$, $D(32)$, and $D(24)$ into sums of $D(m)$where $m <= 16$ to ease calculation.

Following the recursive definition $D(n) = ($sums of $D(c)) + 1$where c takes on every divisor of n except for 1 and itself, the sum simplifies to
$D(96) = (D(24) + D(16) + D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1)$
+
$(D(16) + D(8) + D(4) + D(2) + 1) + D(24) + D(16) + D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1.$

$D(24) = D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + 1$
, so the sum further simplifies to
$D(96) = 3D(16) + 4D(12) + 5D(8) + 4D(6) + 5D(4) + 4D(3) + 5D(2) + 5$
, after combining terms. From quick casework,
$D(16) = 8, D(12) = 8, D(8) = 4, D(6) = 3, D(4) = 2, D(3) = 1$
and $D(2) = 1$. Substituting these values into the expression above,

$D(96) = 3 \cdot 8 + 4 \cdot 8 + 5 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 4 \cdot 1 + 5 \cdot 1 + 5 = \boxed{\textbf{(A) } 112}$
.

~monmath a.k.a Fmirza

Solution 4

Note that $96 = 3 \cdot 2^5$, and that D of a perfect power of a prime is relatively easy to calculate. Also note that you can find $D(96)$ from $D(32)$ by simply totaling the number of ways there are to insert a 3 into a set of numbers that multiply to 32.

First, calculate $D(32)$. Since $32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, all you have to do was find the number of ways to divide the 2's into groups, such that each group has at least one 2. By stars and bars, this results in 1 way with five terms, 4 ways with four terms, 6 ways with three terms, 4 ways with two terms, and 1 way with one term. (The total, 16, is not needed for the remaining calculations.)

Then, to get $D(96)$, in each possible $D(32)$ sequence, insert a 3 somewhere in it, either by placing it somewhere next to the original numbers (in one of $n+1$ ways, where n is the number of terms in the $D(32)$ sequence), or by multiplying one of the numbers by 3 (in one of n ways). There are $2+1=3$ways to do this with one term, $3+2=5$ with two, 7 with three, 9 with four, and 11 with five.

The resulting number of possible sequences is $3\cdot 1+5\cdot 4+7\cdot 6+9\cdot 4+11\cdot 1=\boxed{\textbf{(A) }112}$.

~emerald_block