2021 Fall AMC 10A Solution

Problem1

What is the value of $\frac{(2112-2021)^2}{169}$?

$\textbf{(A) } 7 \qquad \textbf{(B) } 21 \qquad \textbf{(C) } 49 \qquad \textbf{(D) } 64 \qquad \textbf{(E) } 91$

Solution 1 (Laws of Exponents)

We have $\frac{(2112-2021)^2}{169}=\frac{91^2}{169}=\frac{91^2}{13^2}=\left(\frac{91}{13}\right)^2=7^2=\boxed{\textbf{(C) } 49}$.

Solution 2 (Difference of Squares)

We have $\frac{(2112-2021)^2}{169}=\frac{91^2}{169}=\frac{(10^2-3^2)^2}{169}=\frac{(13\cdot7)^2}{169}=\frac{13^2\cdot7^2}{13^2}=7^2=\boxed{\textbf{(C) } 49}$.

Problem2

Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

$\textbf{(A) } 16 \qquad \textbf{(B) } 17 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 19 \qquad \textbf{(E) } 20$

Solution

We construct the following table:

Scenario	Length	Width	Area
Initial	4	6	24
Menkara shortens one side.	3	6	18
Menkara shortens other side instead.	4	5	20

Therefore, the answer is $\boxed{\textbf{(E) } 20}$.

Problem3

What is the maximum number of balls of clay of radius 2 that can completely fit inside a cube of side length 6 assuming the balls can be reshaped but not compressed before they are packed in the cube?

$\textbf{(A) } 3 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 6 \qquad \textbf{(E) } 7$

Solution 1 (Inequality)

The volume of the cube is $V_{\text{cube}} = 6^3 = 216,$ and the volume of a clay ball is $V_{\text{ball}} = \dfrac{4}{3}\cdot\pi\cdot 2^3 = \dfrac{32}{3}\pi.$

Since the balls can be reshaped but not compressed, the maximum number of balls that can completely fit inside a cube is $\left\lfloor \dfrac{V_{\text{cube}}}{V_{\text{ball}}} \right\rfloor = \left\lfloor \dfrac{81}{4\pi} \right\rfloor.$

Approximating with $\pi \approx 3.14,$ we have $12 < 4\pi < 13,$ or $\left\lfloor \dfrac{81}{13} \right\rfloor \leq \left\lfloor \dfrac{81}{4\pi} \right\rfloor \leq \left\lfloor \dfrac{81}{12} \right\rfloor.$ We simplify to get $6 \leq \left\lfloor \dfrac{81}{4\pi} \right\rfloor \leq 6,$ from which $\left\lfloor \dfrac{81}{4\pi} \right\rfloor = \boxed{\textbf{(D) } 6}.$

Solution 2 (Inequality)

As shown in Solution 1, we conclude that the maximum number of balls that can completely fit inside a cube is $\left\lfloor\frac{81}{4\pi}\right\rfloor.$

By an underestimation $\pi \approx 3,$ we have $4\pi > 12,$ or $\frac{81}{4\pi} < 6\frac{3}{4}.$

By an overestimation $\pi \approx \frac{22}{7},$ we have $4\pi < \frac{88}{7},$ or $\frac{81}{4\pi} > 6\frac{39}{88}.$

Together, we get $6 < 6\frac{39}{88} < \frac{81}{4\pi} < 6\frac{3}{4} < 7,$ from which $\left\lfloor\frac{81}{4\pi}\right\rfloor = \boxed{\textbf{(D) } 6}.$

Solution 3 (Approximation)

As shown in Solution 1, we conclude that the maximum number of balls that can completely fit inside a cube is $\left\lfloor\frac{81}{4\pi}\right\rfloor.$

Approximating with $\pi \approx 3,$ we have $\frac{81}{4\pi} \approx 6\frac{3}{4}.$ Since π is about 5% greater than $3,$ it is safe to claim that $\left\lfloor\frac{81}{4\pi}\right\rfloor = \boxed{\textbf{(D) } 6}.$

Problem4

Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

$\textbf{(A)}\ 2\frac{3}{4} \qquad\textbf{(B)}\ 3\frac{3}{4} \qquad\textbf{(C)}\ 4\frac{1}{2} \qquad\textbf{(D)}\ 5\frac{1}{2} \qquad\textbf{(E)}\ 6\frac{3}{4}$

Solution 1

If Mr. Lopez chooses Route A, then he will spend $\frac{6}{30}=\frac{1}{5}$ hour, or $\frac{1}{5}\cdot60=12$ minutes.

If Mr. Lopez chooses Route B, then he will spend $\frac{9/2}{40}+\frac{1/2}{20}=\frac{11}{80}$ hour, or $\frac{11}{80}\cdot60=8\frac{1}{4}$ minutes.

Therefore, Route B is quicker than Route A by $12-8\frac{1}{4}=\boxed{\textbf{(B)}\ 3\frac{3}{4}}$ minutes.

Solution 2

We use the equation $d = st$ to solve this problem. Recall that 1 mile per hour is equal to $\frac{1}{60}$ mile per minute.

For Route A, the distance is 6 miles and the speed to travel this distance is $\frac{1}{2}$ mile per minute. Thus, the time it takes on Route A is 12 minutes.

For Route B, we have to use the equation twice: once for the distance of $5-\frac{1}{2}=\frac{9}{2}$ miles with a speed of $\frac{2}{3}$ mile per minute and a distance of $\frac{1}{2}$ miles at a speed of $\frac{1}{3}$ mile per minute. Thus, the time it takes to go on Route B is $\frac{9}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot 3 = \frac{27}{4} + \frac{3}{2} = \frac{33}{4}$ minutes. Thus, Route B is $12 - \frac{33}{4} = \frac{15}{4} = 3\frac{3}{4}$ faster than Route A. Thus, the answer is $\boxed{\textbf{(B)}\ 3\frac{3}{4}}$.

Problem5

The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit $A.$ What is $A?$

$\textbf{(A)}\ 1 \qquad\textbf{(B)}\ 3 \qquad\textbf{(C)}\ 5 \qquad\textbf{(D)}\ 7 \qquad\textbf{(E)}\ 9$

Solution

First, modulo 2 or 5, $\underline{20210A} \equiv A$. Hence, $A \neq 0, 2, 4, 5, 6, 8$.

Second modulo 3, $\underline{20210A} \equiv 2 + 0 + 2 + 1 + 0 + A \equiv 5 + A$.

Hence, $A \neq 1, 4, 7$.

Third, modulo 11, $\underline{20210A} \equiv A + 1 + 0 - 0 - 2 - 2 \equiv A - 3$.

Hence, $A \neq 3$.

Therefore, the answer is $\boxed{\textbf{(E)}\ 9}$.

Solution 2

$202100 \implies$ divisible by 2.

$202101 \implies$ divisible by 3.

$202102 \implies$ divisible by 2.

$202103 \implies$ divisible by 11.

$202104 \implies$ divisible by 2.

$202105 \implies$ divisible by 5.

$202106 \implies$ divisible by 2.

$202107 \implies$ divisible by 3.

$202108 \implies$ divisible by 2.

This leaves only $A = \boxed{\textbf{(E) } 9}$.

Problem6

Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

$\textbf{(A) } 6 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 15$

Solution

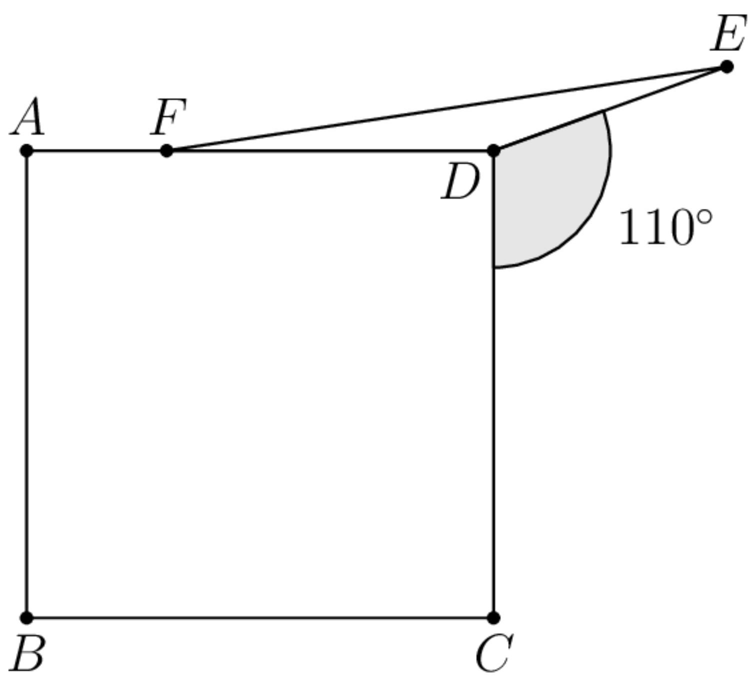
There are $41 - 1 = 40$ gaps between the 41 telephone poles, so the distance of each gap is $5280 \div 40 = 132$ feet.

Each of Oscar's leaps covers $132 \div 12 = 11$ feet, and each of Elmer's strides covers $132 \div 44 = 3$feet.

Therefore, Oscar's leap is $11 - 3 = \boxed{\textbf{(B) } 8}$ feet longer than Elmer's stride.

Problem7

As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



$\textbf{(A) } 160 \qquad \textbf{(B) } 164 \qquad \textbf{(C) } 166 \qquad \textbf{(D) } 170 \qquad \textbf{(E) } 174$

Solution

By angle subtraction, we have $\angle ADE = 360^\circ - \angle ADC - \angle CDE = 160^\circ$. Note that $\triangle DEF$ is isosceles, so $\angle DFE = \dfrac{180^\circ - \angle ADE}{2} = 10^\circ.$ Finally, we get $\angle AFE = 180^\circ - \angle DFE = \boxed{\textbf{(D) } 170}$ degrees.

Problem8

A two-digit positive integer is said to be $cuddly$ if it is equal to the sum of its nonzero tens digit and the square of its units digit. How many two-digit positive integers are cuddly?

$\textbf{(A) }0\qquad\textbf{(B) }1\qquad\textbf{(C) }2\qquad\textbf{(D) }3\qquad\textbf{(E) }4$

Solution 1

Note that the number $\underline{xy} = 10x + y.$ By the problem statement,

$$10x + y = x + y^2 \implies 9x = y^2 - y \implies 9x = y(y-1).$$

From this we see that $y(y-1)$ must be divisible by $9.$ This only happens when $y = 9.$ Then, $x = 8.$ Thus, there is only $\boxed{\textbf{(B) } 1}$ cuddly number, which is $89.$

Solution 2

If the tens digit is a and the ones digit is b then the number is $10 + b$ so we have the equation $10a + b = a + b^2$. We can guess and check after narrowing the possible cuddly numbers down to $13, 14, 24, 25, 35, 36, 46, 47, 57, 68, 78, 89,$ and 99. (We can narrow it down to these by just thinking about how a's value affects b's value and then check all the possiblities.) Checking all of these we get that there is only $\boxed{\textbf{(B) } 1}$ 2-digit cuddly number, and it is 89.

Problem9

When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?

$\textbf{(A)}\ \frac{3}{8} \qquad \textbf{(B)}\ \frac{4}{9} \qquad \textbf{(C)}\ \frac{5}{9} \qquad \textbf{(D)}\ \frac{9}{16} \qquad \textbf{(E)}\ \frac{5}{8}$

Solution

Since an even number is 3 times more likely to appear than an odd number, the probability of an even number appearing is $\frac{3}{4}$. Since the problem states that the sum of the two die must be even, the numbers must both be even or both be odd. We either have EE or OO, so we have $\frac{3}{4}\cdot\frac{3}{4}+\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}+\frac{9}{16}=\frac{10}{16}=\boxed{\textbf{(E)}\ \frac{5}{8}}$.

Problem10

A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50, 20, 20, 5,$ and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

$\textbf{(A)}\ {-}18.5 \qquad \textbf{(B)}\ {-}13.5 \qquad \textbf{(C)}\ 0 \qquad \textbf{(D)}\ 13.5 \qquad \textbf{(E)}\ 18.5$

Solution

The formula for expected values is $\text{Expected Value}=\sum(\text{Outcome}\cdot\text{Probability})$. We have

$$\begin{aligned}
t &= 50 \cdot \frac{1}{5} + 20 \cdot \frac{1}{5} + 20 \cdot \frac{1}{5} + 5 \cdot \frac{1}{5} + 5 \cdot \frac{1}{5} \\
&= (50 + 20 + 20 + 5 + 5) \cdot \frac{1}{5} \\
&= 100 \cdot \frac{1}{5} \\
&= 20, \\
s &= 50 \cdot \frac{50}{100} + 20 \cdot \frac{20}{100} + 20 \cdot \frac{20}{100} + 5 \cdot \frac{5}{100} + 5 \cdot \frac{5}{100} \\
&= 25 + 4 + 4 + 0.25 + 0.25 \\
&= 33.5.
\end{aligned}$$

Therefore, the answer is $t - s = \boxed{\textbf{(B) } -13.5}$.

Problem11

Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

$\textbf{(A) }70\qquad\textbf{(B) }84\qquad\textbf{(C) }98\qquad\textbf{(D) }105\qquad\textbf{(E) }126$

Solution 1 (One Variable)

Let x be the length of the ship. Then, in the time that Emily walks 210 steps, the ship moves $210 - x$steps. Also, in the time that Emily walks 42 steps, the ship moves $x - 42$ steps. Since the ship and Emily both travel at some constant rate, $\frac{210}{210-x} = \frac{42}{x-42}$. Dividing both sides by 42 and cross multiplying, we get $5(x-42)=210-x$, so $6x=420$, and $x=\boxed{\textbf{(A) }70}$.

Solution 2 (Two Variables)

Let the speed at which Emily walks be 42 steps per hour. Let the speed at which the ship is moving be s. Walking in the direction of the ship, it takes her 210 steps, or $\frac{210}{42}=5$ hours, to travel. We can create an equation: $d=5(42-s)$, where d is the length of the ship. Walking in the opposite direction of the ship, it takes her 42 steps, or $42/42=1$ hour. We can create a similar equation: $d=1(42+s)$. Now we have two variables and two equations. We can equate the expressions for d and solve for s:

$$210-5s=42+s$$
$$s=28.$$

Therefore, we have $d=42+s=\boxed{\textbf{(A) }70}$.

Solution 3 (Three Variables)

Suppose that Emily and the ship take steps simultaneously such that Emily's steps cover a greater length than the ship's steps.

Let L be the length of the ship, E be Emily's step length, and S be the ship's step length. We wish to find $\frac{L}{E}$.

When Emily walks from the back of the ship to the front, she walks a distance of $210E$ and the front of the ship moves a distance of $210S$. We have $210E=L+210S$ for this scenario, which rearranges to

$$210E-210S=L. \qquad (1)$$

When Emily walks in the opposite direction, she walks a distance of $42E$ and the back of the ship moves a distance of $42S$. We have $42E = L - 42S$ for this scenario, which rearranges to $42E + 42S = L. \qquad (2)$ We multiply (2) by 5 and then add (1) to get $420E = 6L$, from which $\frac{L}{E} = \boxed{\textbf{(A) } 70}$.

Solution 4 (Relative Speeds)

Call the speed of the boat v_s and the speed of Emily v_e.

Consider the scenario when Emily is walking along with the boat. Relative to an observer on the boat, her speed is $v_e - v_s$.

Consider the scenario when Emily is walking in the opposite direction. Relative to an observer on the boat, her speed is $v_e + v_s$

Since Emily takes 210 steps to walk along with the boat and 42 steps to walk opposite the boat, that means it takes her 5 times longer to walk the length of a stationary boat at $v_e - v_s$ compared to $v_e + v_s$.

This means that $5(v_e - v_s) = v_e + v_s \rightarrow v_s = \frac{2v_e}{3}$.

As Emily takes 210 steps to walk the length of the boat at a speed of $v_e - \frac{2v_e}{3} = \frac{v_e}{3}$, she must take $\frac{1}{3}$ of the time to walk the length of the boat at a speed of v_e, so our answer is $210/3 \rightarrow \boxed{\textbf{(A) } 70}$

Problem12

The base-nine representation of the number N is $27{,}006{,}000{,}052_{\rm nine}.$ What is the remainder when N is divided by $5?$

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 4$

Solution 1 (Modular Arithmetic)

Recall that $9\equiv -1\pmod{5}.$ We expand N by the definition of bases:

$$\begin{aligned} N&=27{,}006{,}000{,}052_9 \\ &= 2\cdot9^{10} + 7\cdot9^9 + 6\cdot9^6 + 5\cdot9 + 2 \\ &\equiv 2\cdot(-1)^{10} + 7\cdot(-1)^9 + 6\cdot(-1)^6 + 5\cdot(-1) + 2 &&\pmod{5} \\ &\equiv 2-7+6-5+2 &&\pmod{5} \\ &\equiv -2 &&\pmod{5} \\ &\equiv \boxed{\textbf{(D) } 3} &&\pmod{5}. \end{aligned}$$

Solution 2 (Powers of 9)

We need to first convert N into a regular base-10 number:

$$N=27{,}006{,}000{,}052_9 = 2\cdot9^{10} + 7\cdot9^9 + 6\cdot9^6 + 5\cdot9 + 2.$$

Now, consider how the last digit of 9 changes with changes of the power of 9 :

$9^0 = 1$

$9^1 = 9$

$9^2 = 81$

$9^3 = 729$

$9^4 = 6561$

$\vdots$ Note that if x is odd, then $9^x\equiv 4\pmod{5}.$ On the other hand, if x is even, then $9^x\equiv 1\pmod{5}.$

Therefore, we have

$$\begin{aligned} N &\equiv 2\cdot(1)+7\cdot(4)+6\cdot(1)+5\cdot(4)+2\cdot(1) && \pmod 5 \\ &\equiv 2+28+6+20+2 && \pmod 5 \\ &\equiv 58 && \pmod 5 \\ &\equiv \boxed{\textbf{(D) } 3} && \pmod 5. \end{aligned}$$

Note that for the odd case, $9^x \equiv -1 \pmod 5$ may simplify the process further, as given by Solution 1.

Problem13

Each of 6 balls is randomly and independently painted either black or white with equal probability. What is the probability that every ball is different in color from more than half of the other 5 balls?

$\textbf{(A) } \frac{1}{64} \qquad \textbf{(B) } \frac{1}{6} \qquad \textbf{(C) } \frac{1}{4} \qquad \textbf{(D) } \frac{5}{16} \qquad \textbf{(E) } \frac{1}{2}$

Solution 1

Note that for this restriction to be true, there must be 3 balls of each color. There are a total of $2^6 = 64$ways to color the balls, and there are $\binom{6}{3} = 20$ ways for three balls chosen to be painted white. Thus, the answer is $\frac{20}{64} = \boxed{\textbf{(D) } \frac{5}{16}}$.

Solution 2

For this restriction to be upheld, there must be three black and three white balls. One such way for this to occur is the arrangement $BBBWWW$, which has a $\frac{1}{2^6}$ probability of occuring. However, there are $\frac{6!}{3!\cdot 3!}$ ways to arrange the three black and three white balls, meaning that the answer is, $\frac{1}{64}\cdot\frac{6!}{3!\cdot 3!}=\boxed{\textbf{(D) } \frac{5}{16}}$.

Solution 3

To get every ball different in color from more than half of the other 5 balls, we must have 3 black balls and 3 white balls.

Following from the binomial theorem, this happens with probability $\binom{6}{3}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{6-3}=\frac{5}{16}.$

Therefore, the answer is $\boxed{\textbf{(D) } \frac{5}{16}}$.

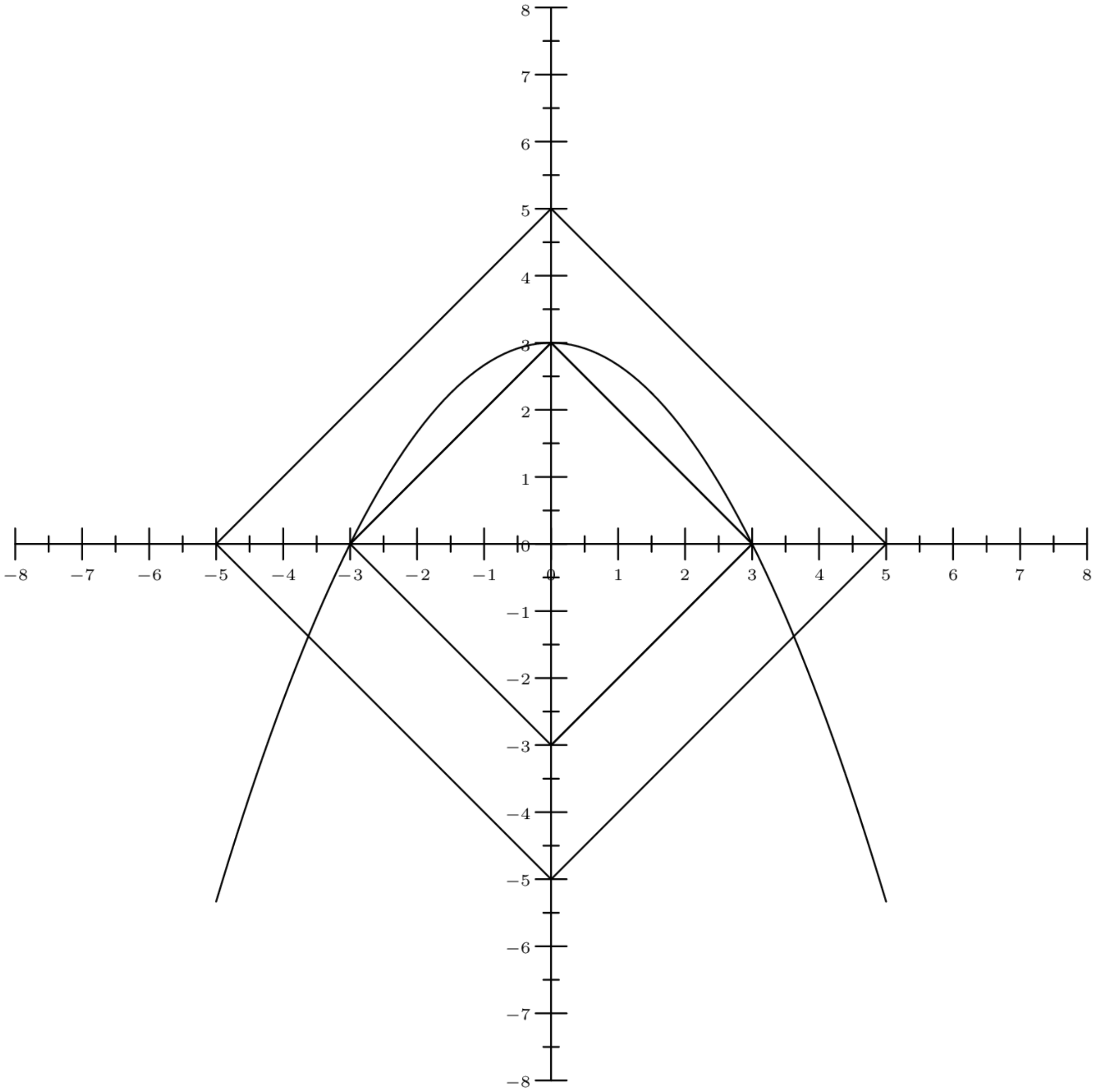
Problem14

How many ordered pairs (x,y) of real numbers satisfy the following system of equations?
$$x^2+3y=9$$
$$(|x|+|y|-4)^2=1$$

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 5 \qquad \textbf{(E) } 7$

Solution 1 (Graphing)

The second equation is $(|x|+|y|-4)^2=1$. We know that the graph of $|x|+|y|$ is a very simple diamond shape, so let's see if we can reduce this equation to that form:
$$(|x|+|y|-4)^2=1 \implies |x|+|y|-4=\pm1 \implies |x|+|y|=\{3,5\}.$$
We now have two separate graphs for this equation and one graph for the first equation, so let's put it on the coordinate plane:



We see from the graph that there are 5 intersections, so the answer is $\boxed{\textbf{(D) }5}$.

Solution 2 (Unrigorous but Feasible)

We can manipulate the first equation to get $y=-\frac{x^2}{3}+3$. From the second equation, we have that $|x|+|y|-4=1$ or $|x|+|y|-4=-1$. We will consider each case separately.

If $|x|+|y|-4=1$, then $|x|+|y|=5$. The graph of this is a square with vertices $(5,0)$, $(-5,0)$, $(0,5)$ and $(0,-5)$. The parabola from the first equation is downwards facing, and its vertex is inside this square; the parabola will clearly intersect the square twice. Therefore, this case gives us $\underline{2}$ solutions.

If $|x|+|y|-4=-1$, then $|x|+|y|=3$. The graph of this is a square with vertices $(3,0)$, $(-3,0)$, $(0,3)$ and $(0,-3)$. The vertex of the parabola from the first equation is on one of the corners of this square (in particular, $(0,3)$). Also, at $y=0$, the parabola has x intercepts of ± 3; the square passes through both of those points. If we continue to move down, the square narrows in, while the parabola continues to expand. Therefore, these are our only 3 intersection points in this case: $(0,3)$, $(3,0)$ and $(-3,0)$. This case gives us $\underline{3}$ solutions.

Adding these two cases together, we get our final answer of $\boxed{\textbf{(D) } 5}$.

Problem15

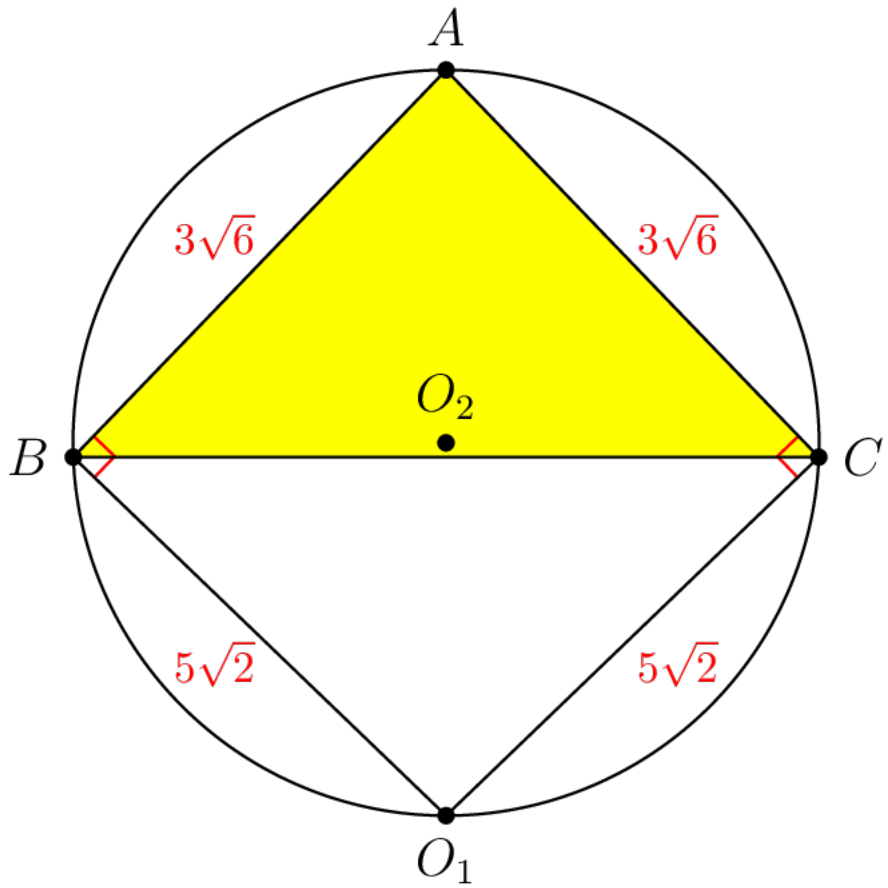
Isosceles triangle ABC has $AB=AC=3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C?

$\textbf{(A) } 24\pi \qquad \textbf{(B) } 25\pi \qquad \textbf{(C) } 26\pi \qquad \textbf{(D) } 27\pi \qquad \textbf{(E) } 28\pi$

Solution 1 (Cyclic Quadrilateral)

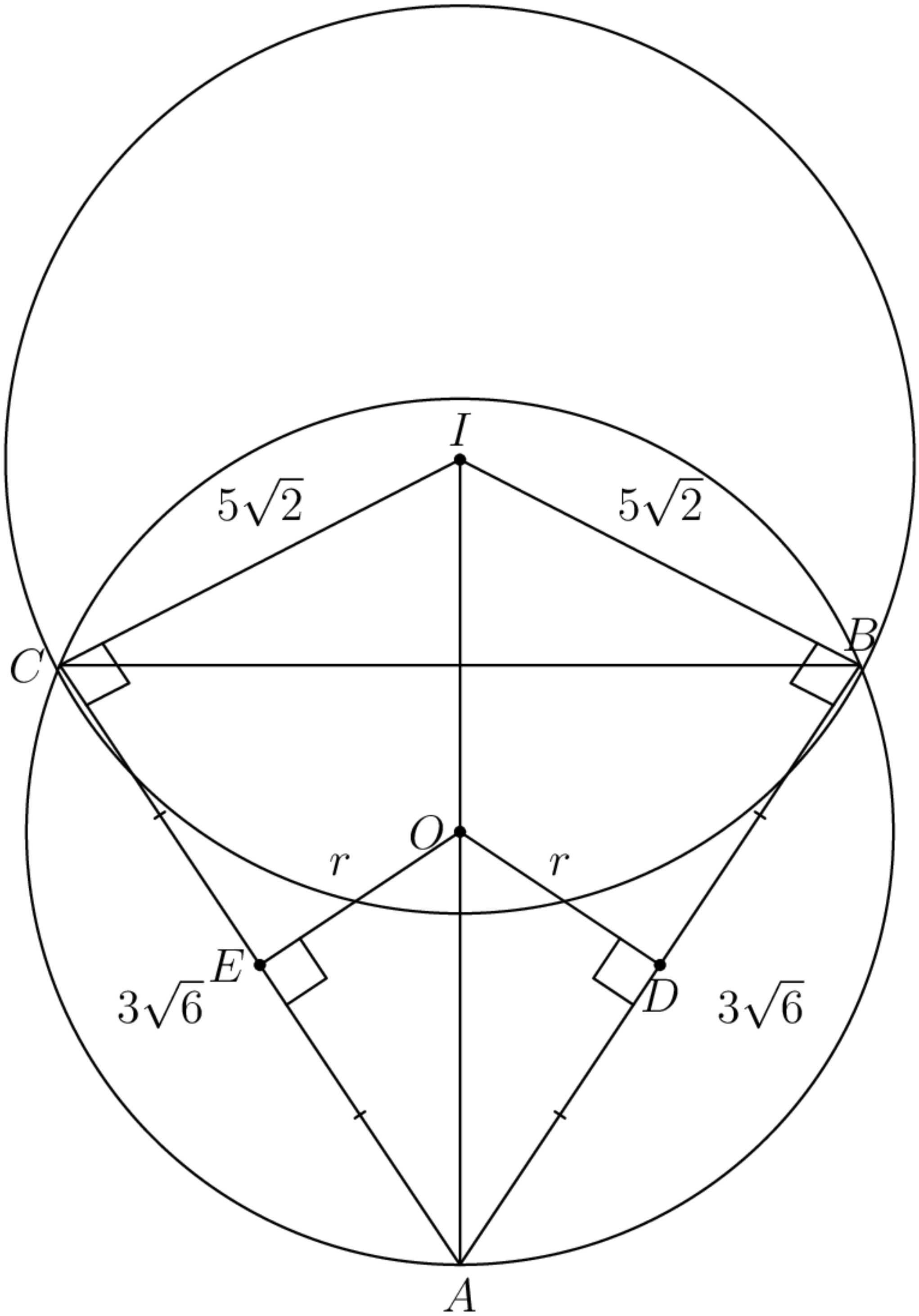
Let $\odot O_1$ be the circle with radius $5\sqrt{2}$ that is tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C. Note that $\angle ABO_1 = \angle ACO_1 = 90^\circ$. Since the opposite angles of quadrilateral ABO_1C are supplementary, quadrilateral ABO_1C is cyclic.

Let $\odot O_2$ be the circumcircle of quadrilateral $ABO_1C.$ It follows that $\odot O_2$ is also the circumcircle of $\triangle ABC,$ as shown below:



By the Inscribed Angle Theorem, we conclude that $\overline{AO_1}$ is the diameter of $\odot O_2.$ By the Pythagorean Theorem on right $\triangle ABO_1,$ we have $AO_1 = \sqrt{AB^2 + BO_1^2} = 2\sqrt{26}.$ Therefore, the area of $\odot O_2$ is $\pi \cdot \left(\frac{AO_1}{2}\right)^2 = \boxed{\textbf{(C) } 26\pi}.$

Solution 2 (Similar Triangles)



Because circle I is tangent to $\overline{AB}$ at $B, \angle ABI \cong 90^\circ$. Because O is the circumcenter of $\triangle ABC, \overline{OD}$ is the perpendicular bisector of $\overline{AB}$, and $\angle BAI \cong \angle DAO$, so therefore $\triangle ADO \sim \triangle ABI$ by AA similarity. Then we have $\frac{AD}{AB} = \frac{DO}{BI} \implies \frac{1}{2} = \frac{r}{5\sqrt{2}} \implies r = \frac{5\sqrt{2}}{2}$. We also know that $\overline{AD} = \frac{3\sqrt{6}}{2}$ because of the perpendicular bisector, so the hypotenuse of $\triangle ADO$ is

$$\sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{6}}{2}\right)^2} = \sqrt{\frac{25}{2} + \frac{27}{2}} = \sqrt{26}.$$

This is the radius of the circumcircle of $\triangle ABC$, so the area of this circle is $\boxed{\textbf{(C) } 26\pi}$.

Solution 3 (Trigonometry)

Denote by O the center of the circle that is tangent to line AB at B and to line AC at C.

Because this circle is tangent to line AB at B, we have $OB \perp AB$ and $OB = 5\sqrt{2}$.

Because this circle is tangent to line AC at C, we have $OC \perp AC$ and $OC = 5\sqrt{2}$.

Because $AB = AC$, $OB = OC$, $AO = AO$, we get $\triangle ABO \cong \triangle ACO$. Hence, $\angle BAO = \angle CAO$.

Let AO and BC meet at point D.

Because $AB = AC, \angle BAO = \angle CAO, AD = AD$, we get $\triangle ABD \cong \triangle ACD$.

Hence, $BD = CD$ and $\angle ADB = \angle ADC = 90^\circ$.

Denote $\theta = \angle BAO$. Hence, $\angle BAC = 2\theta$.

Denote by R the circumradius of $\triangle ABC$. In $\triangle ABC$, following from the law of sines, $2R = \frac{BC}{\sin \angle BAC}$.

Problem16

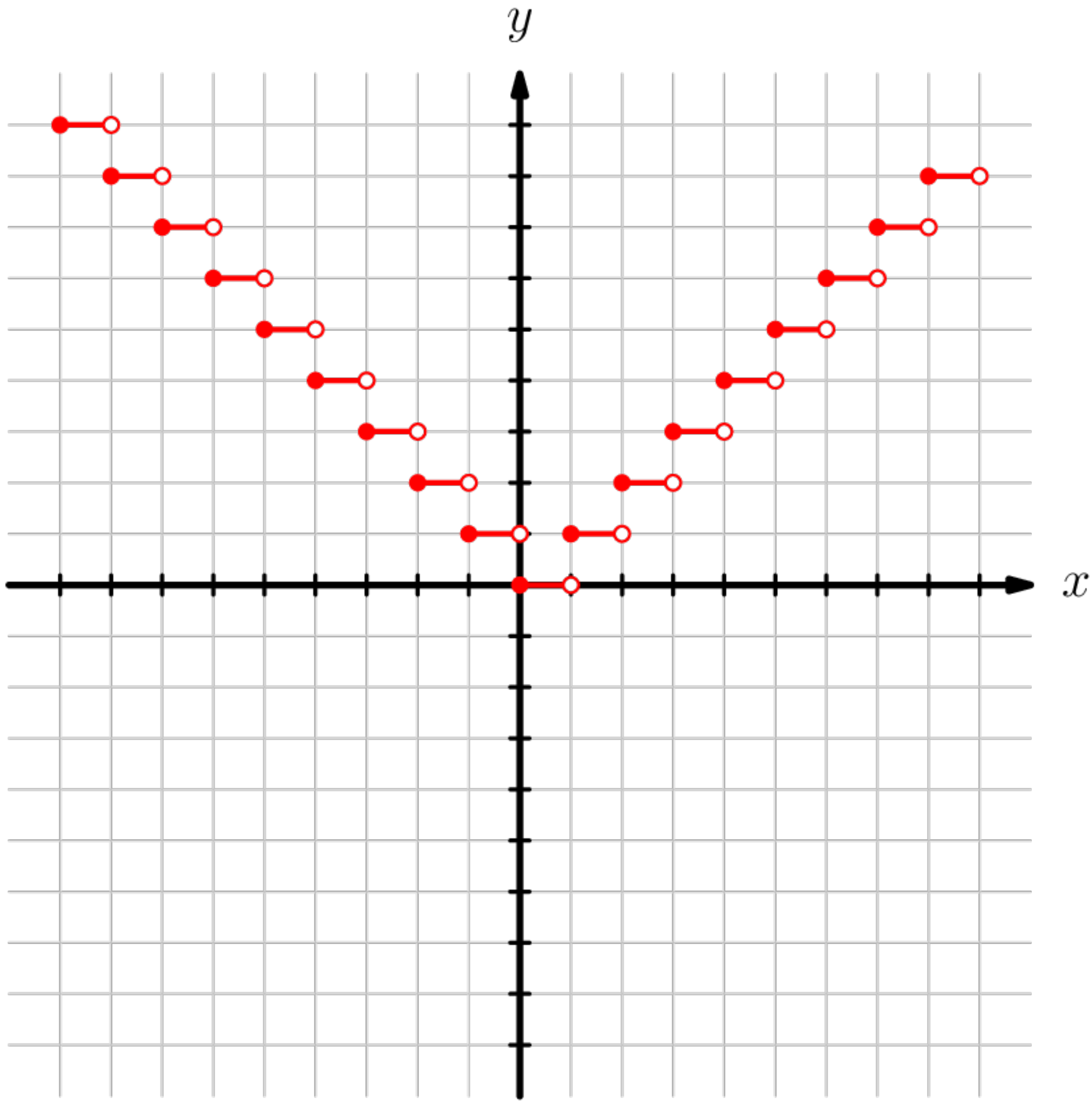
The graph of $f(x) = |\lfloor x \rfloor| - |\lfloor 1 - x \rfloor|$ is symmetric about which of the following? (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

$\textbf{(A)}$ the y-axis $\qquad$ $\textbf{(B)}$ the line $x = 1$ $\qquad$ $\textbf{(C)}$ the origin $\qquad$ $\textbf{(D)}$ the point $\left(\frac{1}{2}, 0\right)$ $\qquad$ $\textbf{(E)}$ the point $(1, 0)$

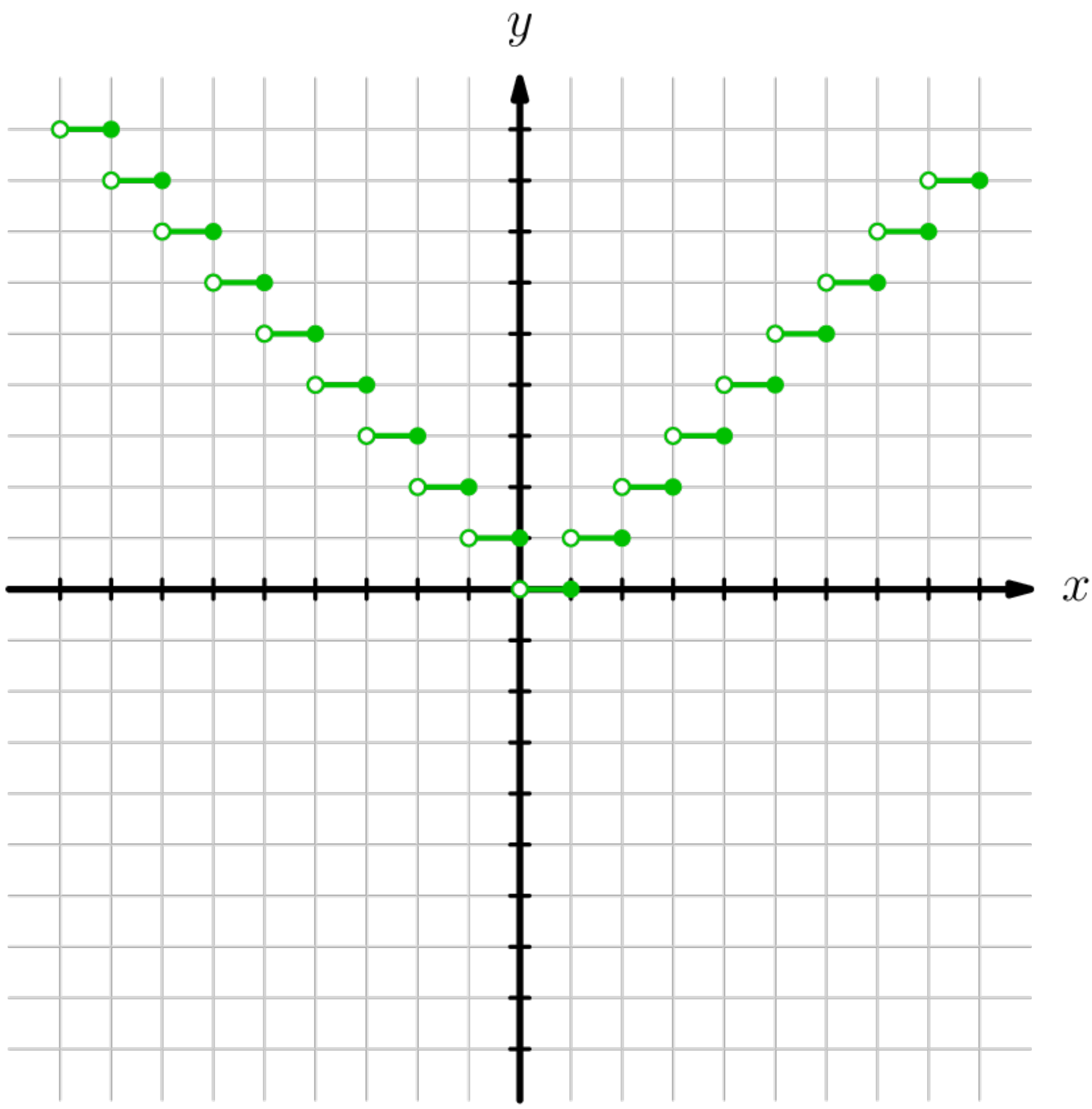
Solution 1 (Graphing)

Let $y_1 = |\lfloor x \rfloor|$ and $y_2 = |\lfloor 1 - x \rfloor| = |\lfloor -(x-1) \rfloor|$. Note that the graph of y_2 is a reflection of the graph of y_1 about the y-axis, followed by a translation 1 unit to the right.

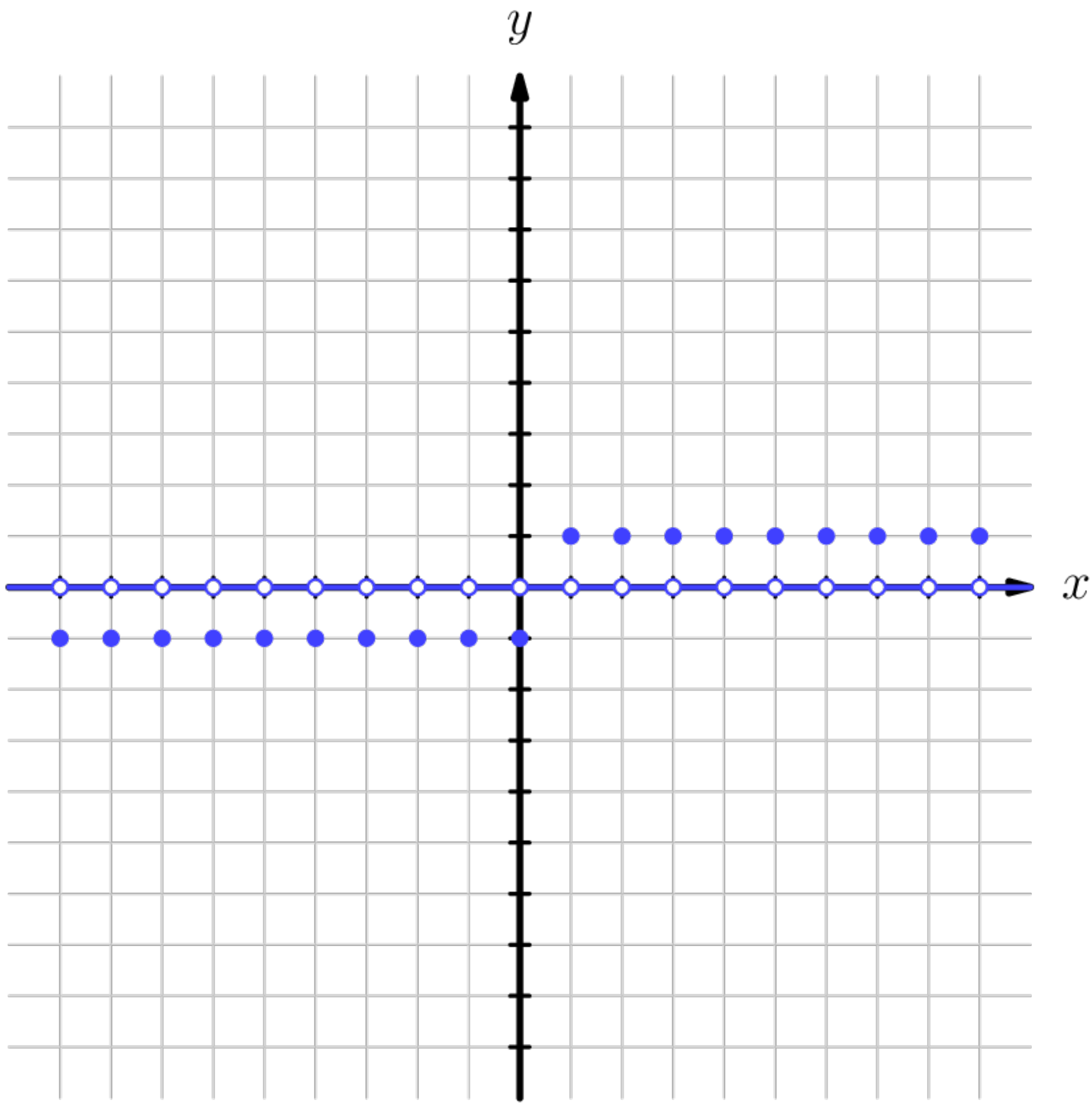
The graph of y_1 is shown below:



The graph of y_2 is shown below:



The graph of $f(x) = y_1 - y_2$ is shown below:



Therefore, the graph of $f(x)$ is symmetric about $\boxed{\textbf{(D)}\ \text{the point}\ \left(\frac{1}{2},0\right)}$.

Solution 2 (Casework)

For all $x \in \mathbb{R}$ and $n \in \mathbb{Z},$ note that:

1. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ and $\lceil x+n \rceil = \lceil x \rceil + n$
2. $\lfloor -x \rfloor = -\lceil x \rceil$

3. $$\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ 1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

We rewrite $f(x)$ as
$$\begin{aligned} f(x) &= |\lfloor x \rfloor| - |\lfloor 1 - x \rfloor| \\ &= |\lfloor x \rfloor| - | - \lceil x - 1 \rceil| \\ &= |\lfloor x \rfloor| - | - \lceil x \rceil + 1|. \end{aligned}$$
We apply casework to the value of x :

1. $x \in \mathbb{Z}^-$

   It follows that $f(x) = -x - (-x + 1) = -1.$

2. $x = 0$

   It follows that $f(x) = 0 - 1 = -1.$

3. $x \in \mathbb{Z}^+$

   It follows that $f(x) = x - (x - 1) = 1.$

4. $x \notin \mathbb{Z}$ and $x < 0$

   It follows that
   $f(x) = -\lfloor x \rfloor - (-\lceil x \rceil + 1) = (\lceil x \rceil - \lfloor x \rfloor) - 1 = 0.$

5. $x \notin \mathbb{Z}$ and $0 < x < 1$

   It follows that $f(x) = 0 - 0 = 0.$

6. $x \notin \mathbb{Z}$ and $x > 1$

It follows that

$$f(x) = \lfloor x \rfloor - (\lceil x \rceil - 1) = (\lfloor x \rfloor - \lceil x \rceil) + 1 = 0.$$

Together, we have $f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Z}^- \cup \{0\} \\ 1 & \text{if } x \in \mathbb{Z}^+ \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases},$ so the graph of $f(x)$ is symmetric about $\boxed{\textbf{(D)} \ \text{the point} \ \left(\frac{1}{2}, 0\right)}.$

Alternatively, we can eliminate $\textbf{(A)}, \textbf{(B)}, \textbf{(C)},$ and $\textbf{(E)}$ once we finish with Case 3. This leaves us with $\textbf{(D)}.$

Solution 3 (Casework)

Denote $x = a + b$, where $a \in \mathbb{Z}$ and $b \in [0, 1)$. Hence, a is the integer part of x and b is the decimal part of x.

$\textbf{Case 1}$: $b = 0$.

We have

$$\begin{aligned} f(x) &= |\lfloor x \rfloor| - |\lfloor 1 - x \rfloor| \\ &= |a| - |1 - a| \\ &= \begin{cases} a - (a-1) & \text{if } a \in \mathbb{Z} \text{ and } a \geq 1 \\ -1 & \text{if } a = 0 \\ -a - (1-a) & \text{if } a \in \mathbb{Z} \text{ and } a \leq -1 \end{cases} \\ &= \begin{cases} 1 & \text{if } a \in \mathbb{Z} \text{ and } a \geq 1 \\ -1 & \text{if } a = 0 \\ -1 & \text{if } a \in \mathbb{Z} \text{ and } a \leq -1 \end{cases} \\ &= \begin{cases} 1 & \text{if } a \in \mathbb{Z} \text{ and } a \geq 1 \\ -1 & \text{if } a \in \mathbb{Z} \text{ and } a \leq 0 \end{cases} \end{aligned}$$

$\textbf{Case 2}$: $b \neq 0$.

We have
$$\begin{aligned} f(x) &= |\lfloor x \rfloor| - |\lfloor 1-x \rfloor| \\ &= |a| - |\lfloor 1-a-b \rfloor| \\ &= |a| - |\lfloor -a+(1-b) \rfloor| \\ &= |a| - |-a| \\ &= 0. \end{aligned}$$

Therefore, the graph of $f(x)$ is symmetric through the point $\left(\frac{1}{2}, 0\right)$.

Therefore, the answer is $\boxed{\textbf{(D)} \quad \text{the point } \left(\frac{1}{2}, 0\right)}$.

Therefore, the area of the circumcircle of $\triangle ABC$ is

$$\begin{aligned}\pi R^2 &= \pi\left(\frac{BC}{2\sin\angle BAC}\right)^2\\ &= \pi\left(\frac{2BD}{2\sin\angle BAC}\right)^2\\ &= \pi\left(\frac{BD}{\sin 2\theta}\right)^2\\ &= \pi\left(\frac{AB\sin\theta}{\sin 2\theta}\right)^2\\ &= \pi\left(\frac{AB\sin\theta}{2\sin\theta\cos\theta}\right)^2\\ &= \pi\left(\frac{AB}{2\cos\theta}\right)^2\\ &= \pi\left(\frac{AO}{2}\right)^2\\ &= \frac{\pi}{4}\left(AB^2+OB^2\right)\\ &= \boxed{\textbf{(C) } 26\pi}.\end{aligned}$$

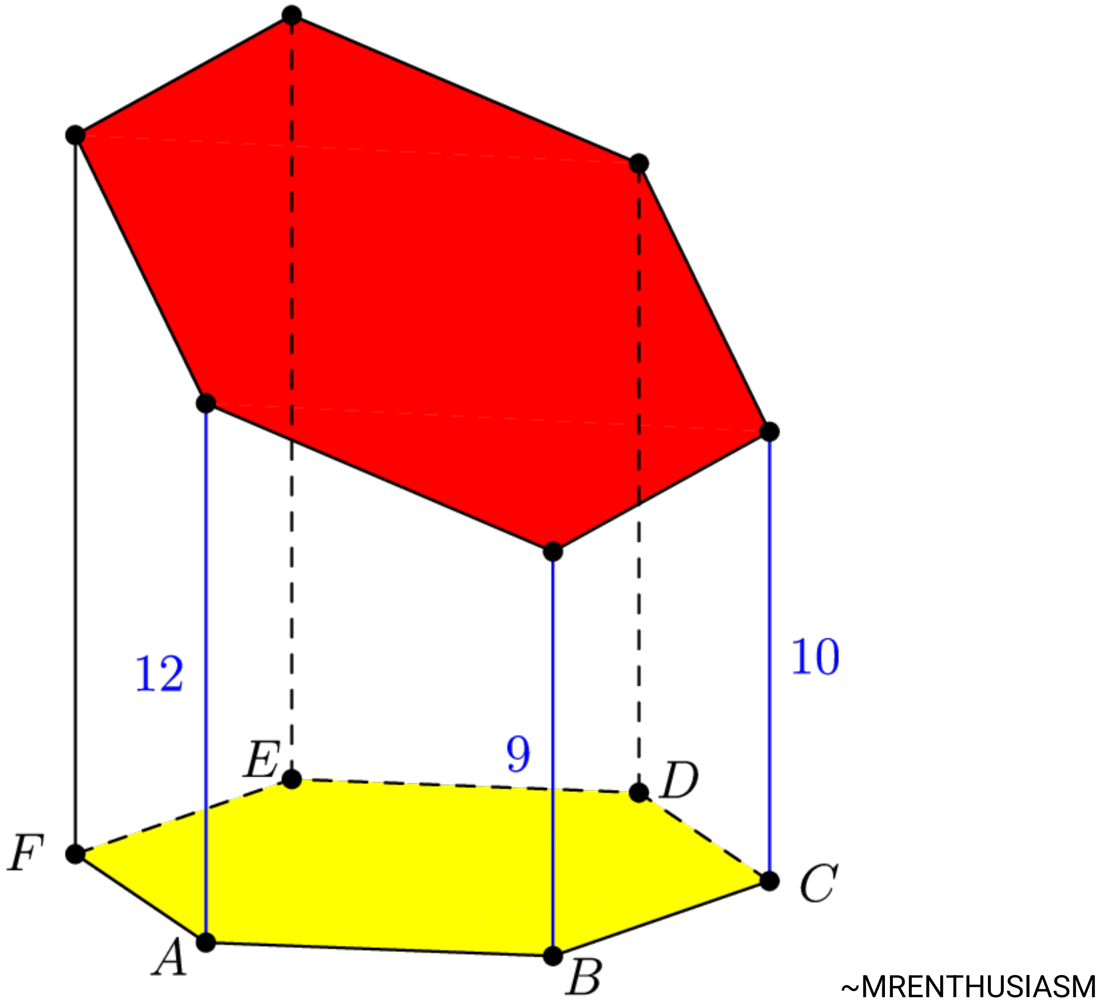
Problem17

An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $ABCDEF$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of pillars at A, B, and C are 12, 9, and 10 meters, respectively. What is the height, in meters, of the pillar at E?

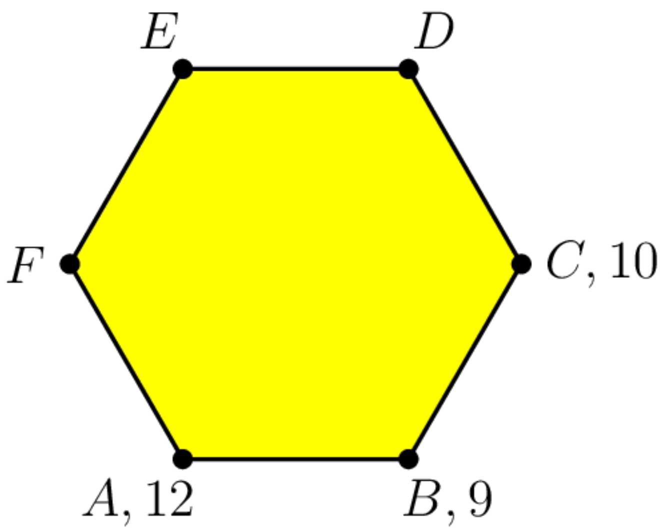
$\textbf{(A) } 9 \qquad \textbf{(B) } 6\sqrt{3} \qquad \textbf{(C) } 8\sqrt{3} \qquad \textbf{(D) } 17 \qquad \textbf{(E) } 12\sqrt{3}$

Diagrams

Three-Dimensional Diagram



Two-Dimensional Diagram



Solution 1 (Height From the Center)

The pillar at B has height 9 and the pillar at A has height $12.$ Since the solar panel is flat, the inclination from pillar B to pillar A is $3.$ Call the center of the hexagon $G.$ Since $\overrightarrow{CG} \parallel \overrightarrow{BA},$ it follows that the solar panel has height 13 at $G.$ Since the solar panel is flat, the heights of the solar panel at $B, G,$ and E are collinear. Therefore, the pillar at E has height $9+4+4=\boxed{\textbf{(D) } 17}.$

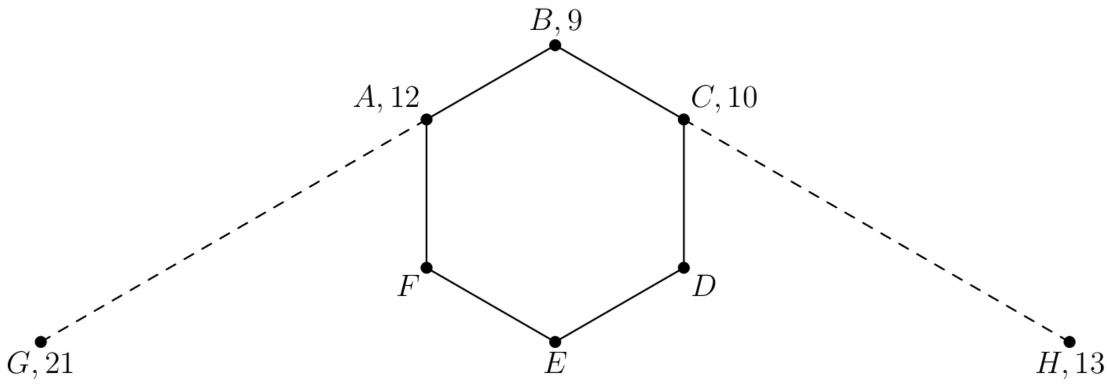
Solution 2 (Height From Each Vertex)

Let the height of the pillar at D be $x.$ Notice that the difference between the heights of pillar C and pillar D is equal to the difference between the heights of pillar A and pillar $F.$ So, the height at F is $x+2.$ Now, doing the same thing for pillar E we get the height is $x+3.$ Therefore, we can see the difference between the heights at pillar C and pillar D is half the difference between the

heights at B and E, so
$$\begin{aligned} x+3-9 &= 2\cdot(x-10) \\ x-6 &= 2\cdot(x-10) \\ x &= 14. \end{aligned}$$
The answer is $x+3=\boxed{\textbf{(D) } 17}$.

Solution 3 (Extend the Sides)

We can extend BA and BC to G and H, respectively, such that $AG=CH$ and E lies on $\overline{GH}$:



Because of hexagon proportions, $\dfrac{BA}{AG}=\dfrac{1}{3}$ and $\dfrac{BC}{CH}=\dfrac{1}{3}$. Let g be the height of G. Because A, B and G lie on the same line, $\dfrac{12-9}{g-12}=\dfrac{1}{3}$, so $g-12=9$ and $g=21$. Similarly, the height of H is 13. E is the midpoint of GH, so we can take the average of these heights to get our answer, $\boxed{\textbf{(D) } 17}$.

Solution 4 (Averages of Heights)

Denote by h_X the height of any point X.

Denote by M the midpoint of A and C.

Hence, $h_M = \frac{h_A + h_C}{2} = 11.$ Denote by O the center of $ABCDEF$. Because $ABCDEF$ is a regular hexagon, O is the midpoint of B and E. Hence, $h_O = \frac{h_E + h_B}{2} = \frac{h_E + 9}{2}.$

Because $ABCDEF$ is a regular hexagon, M is the midpoint of B and O.

Hence, $h_M = \frac{h_B + h_O}{2} = \frac{9 + h_O}{2}.$ Solving these equations, we get $h_E = \boxed{\textbf{(D) } 17}$.

Solution 5 (Vectors)

In this solution, we define **rise** as the change of height (in meters) from the solar panel to the ground. It follows that the rise from B to A is $12 - 9 = 3,$ and the rise from B to C is $10 - 9 = 1.$ Note that $\vec{BE} = 2\vec{BA} + 2\vec{BC},$ so the rise from B to E is $2 \cdot 3 + 2 \cdot 1 = 8.$

Together, the height of the pillar at E is $9 + 8 = \boxed{\textbf{(D) } 17}$ meters.

Solution 6 (Vectors)

WLOG, let the side length of the hexagon be 6.

Establish a 3D coordinate system, in which $A = (0,0,0)$. Let the coordinates of B and C be $(6,0,0)$, $\left(9, -3\sqrt{3}, 0\right)$, respectively. Then, the solar panel passes through $P=(0,0,12), Q=(6,0,9), R=\left(9,-3\sqrt{3},10\right)$

.

The vector $\vec{PQ}=\langle 6,0,-3\rangle$ and $\vec{PR}=\left\langle 9,-3\sqrt{3},-2\right\rangle$.

Computing $\vec{PQ}\times\vec{PR}$ by the matrix $\begin{bmatrix} i & j & k \\ 6 & 0 & -3 \\ 9 & -3\sqrt{3} & -2 \end{bmatrix}$ gives the result $-9\sqrt{3}i-15j-18\sqrt{3}k$. Therefore, a normal vector of the plane of the solar panel is $\left\langle -9\sqrt{3},-15,-18\sqrt{3}\right\rangle$, and the equation of the plane is $-9\sqrt{3}x-15y-18\sqrt{3}z=k$.

Substituting $(x,y,z)=(0,0,12)$, we find that $k=-216\sqrt{3}$.

Since $E=\left(0,-6\sqrt{3}\right)$, we substitute $(x,y)=\left(0,-6\sqrt{3}\right)$ into $-9\sqrt{3}x-15y-18\sqrt{3}z=-216\sqrt{3}$, which gives $z=\boxed{\textbf{(D)}\ 17}$.

Problem18

A farmer's rectangular field is partitioned into 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field? 

$\textbf{(A) }12\qquad\textbf{(B) }64\qquad\textbf{(C) }84\qquad\textbf{(D) }90\qquad\textbf{(E) }144$

Solution 1 (Casework)

There are 4 possibilities for the top-left section. It follows that the top-right and bottom-left sections each have 3 possibilities, so they have $3^2=9$ combinations. We have two cases:

1. **The top-right and bottom-left sections have the same crop.**

   Note that 3 of the 9 combinations of the top-right and bottom-left sections satisfy this case, from which the bottom-right section has 3 possibilities. Therefore, there are $4\cdot3\cdot3=36$ ways in this case.

2. **The top-right and bottom-left sections have different crops.**

   Note that 6 of the 9 combinations of the top-right and bottom-left sections satisfy this case, from which the bottom-right section has 2 possibilities. Therefore, there are $4\cdot6\cdot2=48$ ways in this case.

Together, the answer is $36+48=\boxed{\textbf{(C) }84}$.

Solution 2 (Casework)

We will do casework on the type of crops in the field.

Case 1: all of a kind.

If all four sections have the same type of crop, there are simply $\underline{4}$ ways to choose crops for the sections.

Case 2: $\mathbf{3}$ of a kind, $\mathbf{1}$ of another kind.

Since the one of another kind must be adjacent to two of the other crops, when choosing the type of crops in this case, we cannot choose soybeans and potatoes, or corn and wheat. Therefore, there are $4 \cdot 3 - 2 \cdot 2 = 8$ choices for the two crops we choose for the section (notice we did not choose by 2, since the crop we pick first will be the unique one), and 4 ways to choose which section the unique crop is planted on. This gives us a total of $4 \cdot 8 = \underline{32}$ ways to choose crops for the sections.

Case 3: $\mathbf{2}$ of a kind, $\mathbf{2}$ of another kind.

We cannot choose corn and wheat, or soybeans and potatoes, once again, because if we do, the two would have to be adjacent in some way, which the problem disallows. So, there are $\binom{4}{2} - 2 = 4$ ways to choose our two crops (notice that we did choose by 2, since there are two of both crops). There are $\binom{4}{2} = 6$ ways to choose where one of the crops go, so there are $4 \cdot 6 = \underline{24}$ ways to choose crops for the sections.

Case 4: $\mathbf{2}$ of a kind, $\mathbf{1}$ of another kind, $\mathbf{1}$ of another kind.

In cases 2 and 3, we excluded the possibility of choosing bad pairs for our crops (i.e. soybeans and potatoes, or corn and wheat). In this case, it is inevitable that we choose a bad pair, because we are choosing 3 crops this time. The two sections of the same kind must contain the crop that is not part of the bad pair in the trio: for example, if we choose corn, soybeans and potatoes as our three crop types, nor soybeans and potatoes can be the type which occupies two sections in this case; corn must be the one to do so. There are 4 ways to choose the crop that is not part of the bad pair, and then 1 way to choose the bad pair, giving us $4 \cdot 1 = 4$ ways to choose the crops. To separate the bad pair of crops, the two of a kind must be diagonally placed. There are 2 ways to choose where the two of a kind go, and 2 ways to choose which of the bad pair goes where, giving us $2 \cdot 2 = 4$ ways to choose the positions for the crops. In total, there are $4 \cdot 4 = \underline{16}$ ways to choose crops for the sections.

Case 5: every single crop.

Bad pairs must be on the same diagonal, so there are 2 ways to choose which pair gets which diagonal, and $2 \cdot 2 = 4$ ways to choose which of each pair goes where on the diagonal, giving us $2 \cdot 4 = \underline{8}$ ways to choose crops for the sections.

Adding up all our values, we get our final answer of $4 + 32 + 24 + 16 + 8 = \boxed{\textbf{(C) } 84}$.

Solution 3 (Casework)

To lighten notation, we use C, W, S, P to denote corn, wheat, soybeans, and potatoes, respectively.

We use I, II, III, IV to denote four quadrants, respectively.

We determine an arrangement in the following steps.

Step 1: Determine the crop planted in I.

The number of ways is 4.

Step 2: Determine the crops planed in II, III, IV.

To find the number of arrangements in this step, without loss of generality, we assume that we plant C in I.

We do the following casework analysis.

Case 1: Both II and IV are planted with C.

In this case, the number of ways to plant in III is 3.

Case 2: In II and IV, only one quadrant is planted with C, and another quadrant is planted with either S or P.

In this case, we determine an arrangement in the following steps.

Step 2.1: Determine whether C is planted in II or IV.

The number of ways is 2.

Step 2.2: In II or IV not planted with C, determine whether it is planted with S or P.

The number of ways is 2.

Step 2.3: Determine the crop planted in III.

The number of ways is 2.

Following from the rule of product, the number of ways in this case is $2 \cdot 2 \cdot 2 = 8$.

Case 3: II and IV are both planted with S or W.

In this case, we determine an arrangement in the following steps.

Step 2.1: Determine whether S or W is planted in II and IV.

The number of ways is 2.

Step 2.2: Determine the crop planted in III.

The number of ways is 3.

Following from the rule of product, the number of ways in this case is $2 \cdot 3 = 6$.

Case 4: In II and IV, exactly one quadrant is planted with S and another one is planted with W.

Step 2.1: Determine which quadrant in II and IV is planted with S.

The number of ways is 2.

Step 2.2: Determine the crop planted in III.

The number of ways is 2.

Following from the rule of product, the number of ways in this case is $2 \cdot 2 = 4$.

Putting all cases together, the total number of arrangements in Step 2 is $3 + 8 + 6 + 4 = 21$.

Following from the rule of product, the total number of arrangements is $4 \cdot 21 = 84$.

Therefore, the answer is $\boxed{\textbf{(C)}\ 84}$.

Solution 4 (Educated Guess)

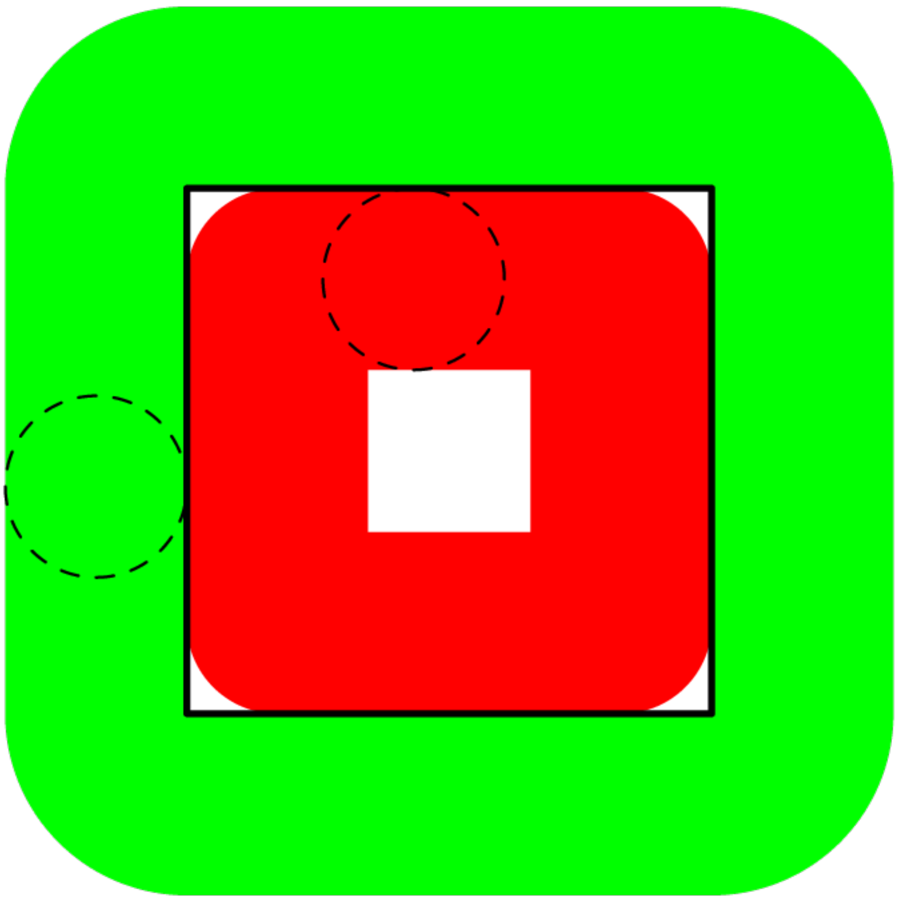
The top right box has 4 choices and the top left box has 3 choices. Thus, it is reasonable to assume that the answer is a multiple of 12. We know that the answer will not be too small or too large, so the answer is $\boxed{\textbf{(C)}\ 84}$.

Problem19

A disk of radius 1 rolls all the way around the inside of a square of side length $s > 4$ and sweeps out a region of area A. A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area $2A$. The value of s can be written as $a + \frac{b\pi}{c}$, where a, b, and c are positive integers and b and c are relatively prime. What is $a + b + c$?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 11 \qquad \textbf{(C)}\ 12 \qquad \textbf{(D)}\ 13 \qquad \textbf{(E)}\ 14$

Diagram



Solution

The side length of the inner square traced out by the disk with radius 1 is $s-4.$ However, there is a piece at each corner (bounded by two line segments and one 90° arc) where the disk never sweeps out. The combined area of these four pieces is $(1+1)^2-\pi\cdot1^2=4-\pi.$ As a result, we have $A=s^2-(s-4)^2-(4-\pi)=8s-20+\pi$.Now, we consider the second disk. The part it sweeps is comprised of four quarter circles with radius $2$and four rectangles with side lengths of 2 and s. When we add it all together, we have $2A=8s+4\pi,$ or $A=4s+2\pi$.We equate the expressions for $A,$ and then solve for s :

$8s-20+\pi=4s+2\pi$.We get $s=5+\frac{\pi}{4},$ so the answer is $5+1+4=\boxed{\textbf{(A) } 10}$.

Problem20

How many ordered pairs of positive integers (b,c) exist where both $x^2+bx+c=0$ and $x^2+cx+b=0$ do not have distinct, real solutions?

$\textbf{(A) } 4 \qquad \textbf{(B) } 6 \qquad \textbf{(C) } 8 \qquad \textbf{(D) } 10 \qquad \textbf{(E) } 12$

Solution 1 (Casework)

A quadratic equation does not have real solutions if and only if the discriminant is nonpositive. We conclude that:

1. Since $x^2+bx+c=0$ does not have real solutions, we have $b^2\leq 4c.$

2. Since $x^2 + cx + b = 0$ does not have real solutions, we have $c^2 \leq 4b.$

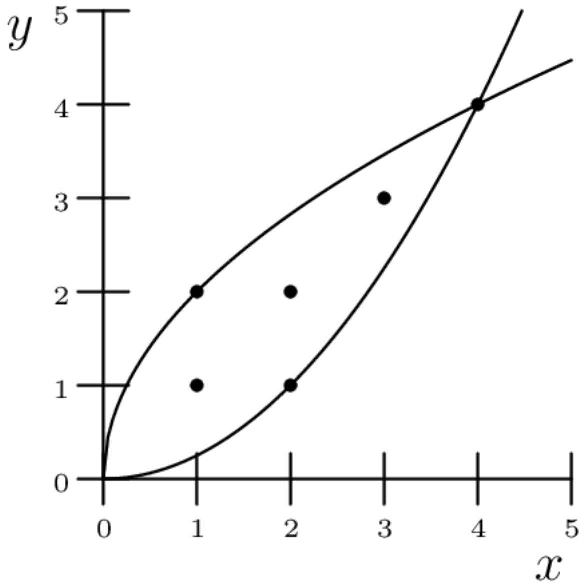
Squaring the first inequality, we get $b^4 \leq 16c^2.$ Multiplying the second inequality by $16,$ we get $16c^2 \leq 64b.$ Combining these results, we get $b^4 \leq 16c^2 \leq 64b.$ We apply casework to the value of $b:$

- If $b = 1,$ then $1 \leq 16c^2 \leq 64,$ from which $c = 1, 2.$
- If $b = 2,$ then $16 \leq 16c^2 \leq 128,$ from which $c = 1, 2.$
- If $b = 3,$ then $81 \leq 16c^2 \leq 192,$ from which $c = 3.$
- If $b = 4,$ then $256 \leq 16c^2 \leq 256,$ from which $c = 4.$

Together, there are $\boxed{\textbf{(B) } 6}$ ordered pairs $(b, c),$ namely $(1,1), (1,2), (2,1), (2,2), (3,3),$ and $(4,4).$

Solution 2 (Graphing)

Similar to Solution 1, use the discriminant to get $b^2 \leq 4c$ and $c^2 \leq 4b$.

These can be rearranged to $c \geq \frac{1}{4}b^2$ and $b \geq \frac{1}{4}c^2$. Now, we can roughly graph these two inequalities, letting one of them be the x axis and the other be y. The graph of solutions should be above the parabola and under its inverse, meaning we want points on the graph or in the first area enclosed by the two graphs: 

We are looking for lattice points (since b and c are positive integers), of which we can count $\boxed{\textbf{(B) } 6}$.

Solution 3 (Graphing)

We need to solve the following system of inequalities: $\left\{ \begin{array}{l} b^2 - 4c \leq 0 \\ c^2 - 4b \leq 0 \end{array} \right.$. Feasible solutions are in the region formed between two parabolas $b^2 - 4c = 0$ and $c^2 - 4b = 0$.

Define $f\left(b \right) = \frac{b^2}{4}$ and $g\left(b \right) = 2 \sqrt{b}$. Therefore, all feasible solutions are in the region formed between the graphs of these two functions.

For $b = 1$, we have $f(b) = \frac{1}{4}$ and $g(b) = 2$. Hence, the feasible c are $1, 2$.

For $b = 2$, we have $f(b) = 1$ and $g(b) = 2\sqrt{2}$. Hence, the feasible c are $1, 2$.

For $b = 3$, we have $f(b) = \frac{9}{4}$ and $g(b) = 2\sqrt{3}$. Hence, the feasible c is 3.

For $b = 4$, we have $f(b) = 4$ and $g(b) = 4$. Hence, the feasible c is 4.

For $b > 4$, we have $f(b) > g(b)$. Hence, there is no feasible c.

Putting all cases together, the correct answer is $\boxed{\textbf{(B) } 6}$.

Solution 4 (Oversimplified but Risky)

A quadratic equation $Ax^2 + Bx + C = 0$ has one real solution if and only if $\sqrt{B^2 - 4AC} = 0.$ Similarly, it has imaginary solutions if and only if $\sqrt{B^2 - 4AC} < 0.$ We proceed as following:

We want both $x^2 + bx + c$ to be 1 value or imaginary and $x^2 + cx + b$ to be 1 value or imaginary. $x^2 + 4x + 4$ is one such case since $\sqrt{b^2 - 4ac}$ is $0.$ Also, $x^2 + 3x + 3, x^2 + 2x + 2, x^2 + x + 1$ are always imaginary for both b and $c.$ We also have $x^2 + x + 2$ along with $x^2 + 2x + 1$ since the latter has one solution, while the first one is imaginary. Therefore, we have $\boxed{\textbf{(B) } 6}$ total ordered pairs of integers.

Problem21

Each of the 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

$\textbf{(A)}\ 1 \qquad\textbf{(B)}\ 4 \qquad\textbf{(C)}\ 8 \qquad\textbf{(D)}\ 12 \qquad\textbf{(E)}\ 16$

Solution 1 (Multinomial Coefficients)

For simplicity purposes, we assume that the balls are indistinguishable and the bins are distinguishable.

Let d be the number of ways to distribute 20 balls into 5 bins. We have $p=\frac{5\cdot4\cdot\binom{20}{3,5,4,4,4}}{d}$ and $q=\frac{\binom{20}{4,4,4,4,4}}{d}$. Therefore, the answer is

$$\frac pq=\frac{5\cdot4\cdot\binom{20}{3,5,4,4,4}}{\binom{20}{4,4,4,4,4}}=\frac{5\cdot4\cdot\frac{20!}{3!\cdot5!\cdot4!\cdot4!\cdot4!}}{\frac{20!}{4!\cdot4!\cdot4!\cdot4!\cdot4!}}=\frac{5\cdot4\cdot(4!\cdot4!\cdot4!\cdot4!\cdot4!)}{3!\cdot5!\cdot4!\cdot4!\cdot4!}=\frac{5\cdot4\cdot4}{5}=\boxed{\textbf{(E)}\ 16}.$$

Remark

By the stars and bars argument, we get $d=\binom{20+5-1}{5-1}=\binom{24}{4}$.

~MRENTHUSIASM

Solution 2 (Binomial Coefficients)

For simplicity purposes, the balls are indistinguishable and the bins are distinguishable.

Let q be equal to $\frac{x}{a}$ where a is the total number of combinations and x is the number of cases where every bin ends up with 4 balls.

Notice that we can take 1 ball from one bin and place it in another bin so that some bin ends up with 3balls, another with 5 balls, and the other three with 4 balls each. We have $x\cdot\frac{\binom{5}{1}\cdot\binom{4}{1}\cdot\binom{4}{1}}{5}=16x.$ Therefore, we get $p=\frac{16x}{a},$ from which $\frac{p}{q}=\boxed{\textbf{(E) } 16}.$

~Hoju

Solution 3 (Binomial Coefficients)

Since both of the boxes will have 3 boxes with 4 balls in them, we can leave those out. There are $\binom{6}{3}=20$ ways to choose where to place the 3 and the 5. After that, there are $\binom{8}{3}=56$ ways to put the 3 and 5 balls being put into the boxes. For the $4,4,4,4,4$ case, after we canceled the $4,4,4$ out, we have $\binom{8}{4}=70$ ways to put the 4 balls inside the boxes. Therefore, we have $\frac{56\cdot 20}{70}$ which is equal to $8\cdot 2=\boxed{\textbf{(E) } 16}.$

~Arcticturn

Solution 4 (Set Theory)

Construct the set A consisting of all possible $3-5-4-4-4$ bin configurations, and construct set B consisting of all possible $4-4-4-4-4$ configurations. If we let N be the total number of configurations possible, it's clear we want to solve for $\frac{p}{q}=\frac{\frac{|A|}{N}}{\frac{|B|}{N}}=\frac{|A|}{|B|}.$

Consider drawing an edge between an element in A and an element in B if it is possible to reach one configuration from the other by moving a single ball (note this process is reversible). Let us consider the total number of edges drawn.

From any element in A, we may take one of the 5 balls in the 5-bin and move it to the 3-bin to get a valid element in B. This implies the number of edges is $5|A|$.

On the other hand for any element in B, we may choose one of the 20 balls and move it to one of the other 4 bins to get a valid element in A. This implies the number of edges is $80|B|$.

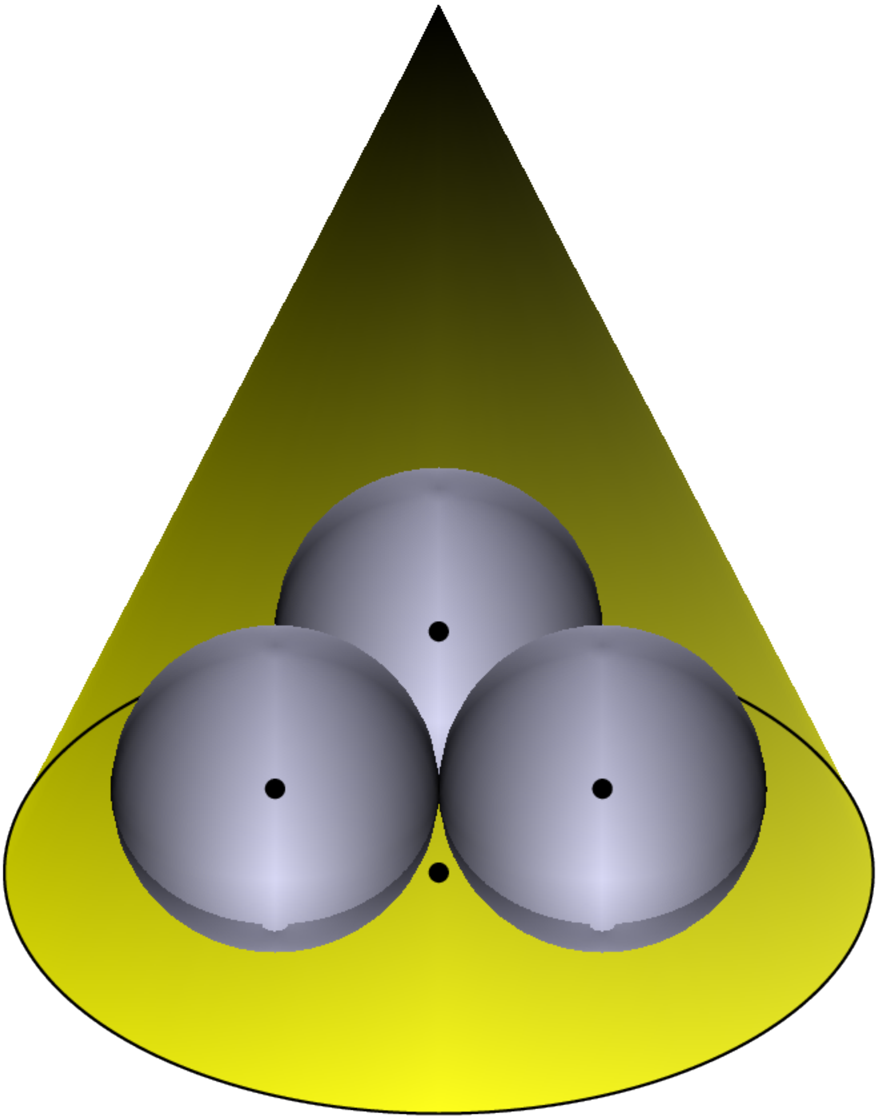
Since they must be equal, then $$5|A| = 80|B| \rightarrow \frac{|A|}{|B|} = \frac{80}{5} = \boxed{\textbf{(E) } 16}.$$

Problem22

Inside a right circular cone with base radius 5 and height 12 are three congruent spheres with radius r. Each sphere is tangent to the other two spheres and also tangent to the base and side of the cone. What is r?

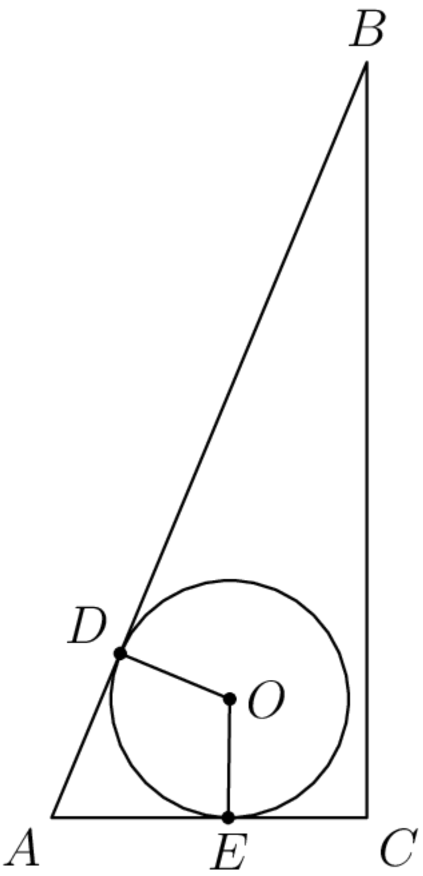
$\textbf{(A)}\ \frac{3}{2} \qquad \textbf{(B)}\ \frac{90-40\sqrt{3}}{11} \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ \frac{144-25\sqrt{3}}{44} \qquad \textbf{(E)}\ \frac{5}{2}$

Diagram



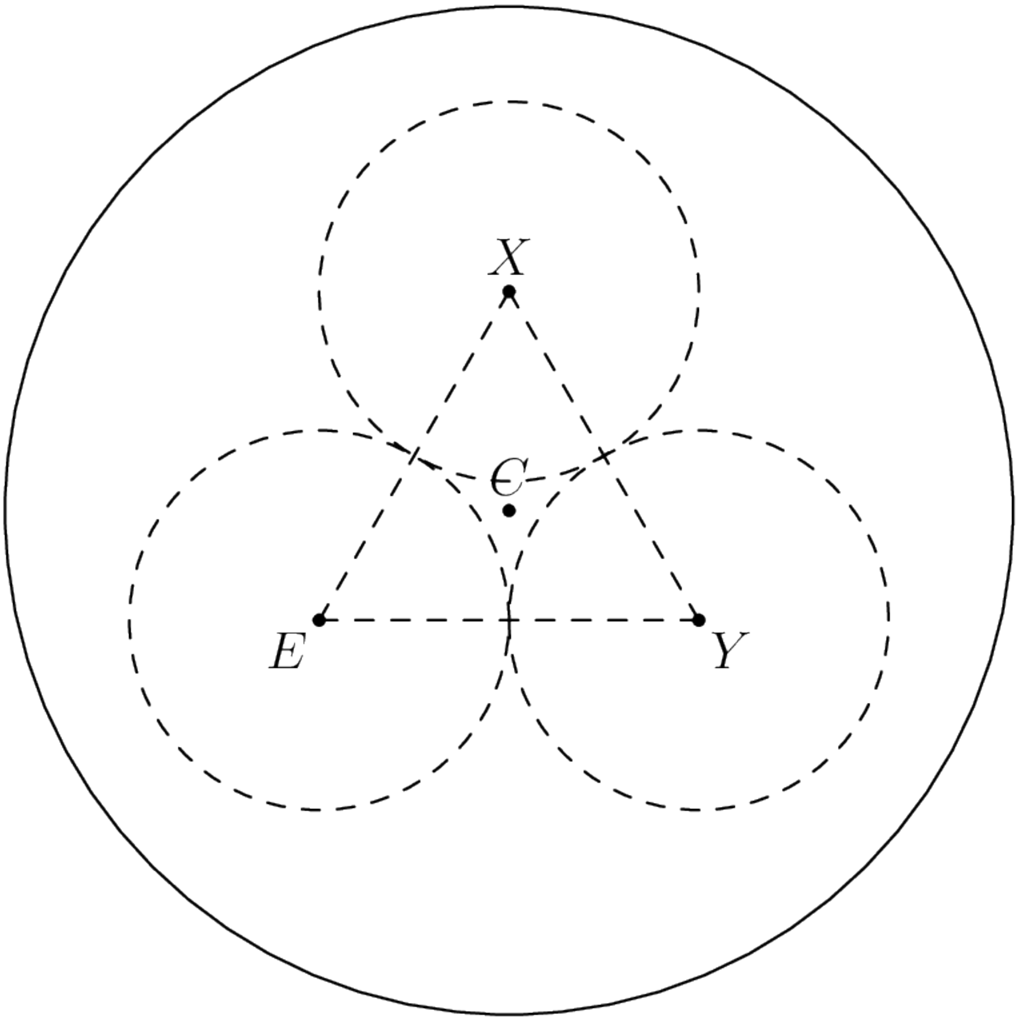
Solution 1 (Cross Sections and Angle Bisectors)

We can take half of a cross section of the sphere, as

such: 

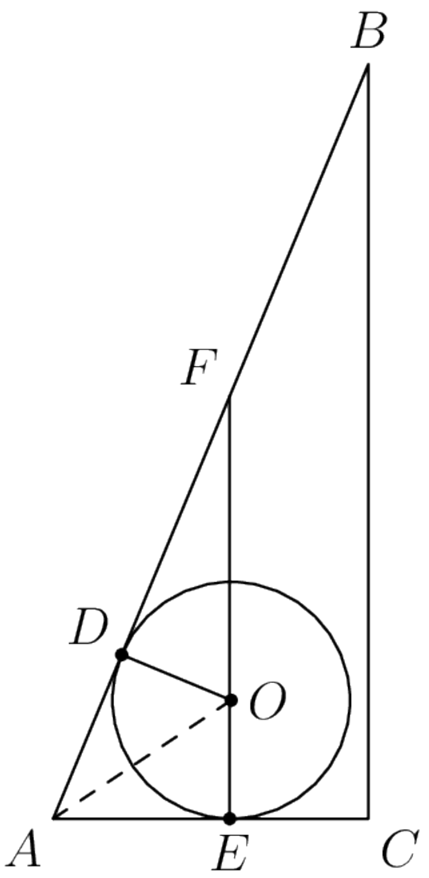
Notice that we chose a cross section where one of the spheres was tangent to the lateral surface of the cone at D.

To evaluate r, we will find AE and EC in terms of r; we also know that $AE + EC = 5$, so with this, we can solve r. Firstly, to find EC, we can take a bird's eye view of the



cone:

Note that C is the centroid of equilateral triangle EXY. Also, since all of the medians of an equilateral triangle are also altitudes, we want to find two-thirds of the altitude from E to XY; this is because medians cut each other into a 2 to 1 ratio. This equilateral triangle has a side length of $2r$, therefore it has an altitude of length $r\sqrt{3}$; two thirds of this is $\frac{2r\sqrt{3}}{3}$, so $EC = \frac{2r\sqrt{3}}{3}$.



To evaluate AE in terms of r, we will extend $\overline{OE}$ past point O to $\overline{AB}$ at point F. $\triangle AEF$ is similar to $\triangle ACB$. Also, AO is the angle bisector of $\angle EAB$. Therefore, by the angle bisector theorem, $\frac{OE}{OF} = \frac{AE}{AF} = \frac{5}{13}$. Also, $OE = r$, so $\frac{r}{OF} = \frac{5}{13}$, so $OF = \frac{13r}{5}$. This means that

$$AE = \frac{5 \cdot EF}{12} = \frac{5 \cdot (OE + OF)}{12} = \frac{5 \cdot (r + \frac{13r}{5})}{12} = \frac{18r}{12} = \frac{3r}{2}.$$

We have that $EC = \frac{2r\sqrt{3}}{3}$ and that $AE = \frac{3r}{2}$, so $AC = EC + AE = \frac{2r\sqrt{3}}{3} + \frac{3r}{2} = \frac{4r\sqrt{3} + 9r}{6}$. We also were given that $AC = 5$. Therefore, we have $\frac{4r\sqrt{3}+9r}{6} = 5$. This is a simple linear equation in terms of r. We can solve for r to get $r = \boxed{\textbf{(B) } \frac{90-40\sqrt{3}}{11}}$.

Solution 2 (Cross Sections and Areas)

Denote by O_1, O_2, O_3 the centers of three spheres.

Because three congruent spheres are tangent to the base of the cone, the plane formed by O_1, O_2, O_3(denoted as α) is parallel to the base, with the distance r.

Let D be the point that the sphere with center O_1 meets the base of the cone at. Hence, $O_1 D = r$.

Because three congruent spheres are mutually externally tangent to each other, $\triangle O_1 O_2 O_3$ is equilateral, with side length $2r$.

Let O be the center of the base, V be the vertex of the base. Let line OV and plane α intersect at point D. By symmetry, E is the center $\triangle O_1 O_2 O_3$.

Hence, $O_1 E = \frac{\sqrt{3}}{3} O_1 O_2 = \frac{2\sqrt{3}}{3}$.

Let F be the point that the sphere with center O_1 meets the side of the cone at. Hence, $O_1 F = r$.

Let line VF and the base intersect at point A.

Hence, we only need to analyze the following 2-d geometry problem:

In $\triangle VOA$ with $\angle O = 90^\circ$, $VO = 12$, $OA = 5$, there is an interior point O_1 whose distances to OA, OV, VA, are r, $\frac{2\sqrt{3}}{3}$, and r, respectively. What is r?

Now, we solve this problem.

We compute the area of $\triangle VOA$ in two ways.

First, we have
$$\text{Area } \triangle VOA = \frac{1}{2} OA \cdot OV = 30.$$

Second, we have
$$\begin{aligned}
\text{Area } \triangle VOA &= \text{Area } \triangle O_1OA + \text{Area } \triangle O_1OV + \text{Area } \triangle O_1VA \\
&= \frac{1}{2} \cdot OA \cdot O_1D + \frac{1}{2} \cdot OV \cdot O_1E + \frac{1}{2} \cdot VA \cdot O_1F \\
&= \frac{1}{2} \cdot 5 \cdot r + \frac{1}{2} \cdot 12 \cdot \frac{2\sqrt{3}}{3} + \frac{1}{2} \cdot 13 \cdot r \\
&= \left(9 + 4\sqrt{3}\right) r.
\end{aligned}$$

These two approaches to compute $\text{Area } \triangle VOA$ should give me the same number. Hence,
$$r = \frac{30}{9 + 4\sqrt{3}} = \boxed{\textbf{(B) } \frac{90 - 40\sqrt{3}}{11}}.$$

Solution 3 (Coordinate Geometry)

We will use coordinates. WLOG, let the coordinates of the center of the base of the cone be the origin. Then, let the center of one of the spheres be $\left(0, \frac{2r}{\sqrt{3}}, r\right)$. Note that the distance between this point and the plane given by $12y + 5z = 60$ is r. Thus, by the point-to-plane distance formula, we have $$\frac{\left|12 \cdot \frac{2r}{\sqrt{3}} + 5r - 60\right|}{\sqrt{0^2 + 5^2 + 12^2}} = r.$$ Solving for r yields $r = \boxed{\textbf{(B) } \frac{90-40\sqrt{3}}{11}}$.

Problem23

For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \ge 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \le 50$ is $f_{50}(n) = 12?$

$\textbf{(A) } 7 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 10 \qquad \textbf{(E) } 11$

Solution 1

First, we can test values that would make $f(x) = 12$ true. For this to happen x must have 6 divisors, which means its prime factorization is in the form pq^2 or p^5, where p and q are prime numbers. Listing out values less than 50 which have these prime factorizations, we find $12, 20, 28, 44, 18, 45, 50$ for pq^2, and just 32 for p^5.

Here 12 especially catches our eyes, as this means if one of $f_i(n) = 12$, each of $f_{i+1}(n), f_{i+2}(n), \dots$ will all be 12. This is because $f_{i+1}(n) = f(f_i(n))$ (as given in the problem statement), so were $f_i(n) = 12$, plugging this in we get $f_{i+1}(n) = f(12) = 12$, and thus the pattern repeats. Hence, as long as for a i, such that $i \le 50$ and $f_i(n) = 12$, $f_{50}(n) = 12$ must be true, which also immediately makes all our previously listed numbers, where $f(x)=12$, possible values of n.

We also know that if $f(x)$ were to be any of these numbers, x would satisfy $f_{50}(n)$ as well. Looking through each of the possibilities aside from 12, we see that $f(x)$ could only possibly be equal to 20 and 18, and still have x less than or equal to 50. This would mean x must have 10, or 9 divisors, and testing out, we see that x will then be of the form p^4q, or p^2q^2. The only two values less than or equal to 50 would be 48 and 36 respectively. From here there are no more possible values, so tallying our possibilities we count $\boxed{\textbf{(D) } 10}$ values (Namely $12, 20, 28, 44, 18, 45, 50, 32, 36, 48$).

Solution 2

$\textbf{Observation 1}$: $f_1(12)=12$.

Hence, if n has the property that $f_j(n)=12$ for some j, then $f_k(n)=12$ for all $k>j$.

$\textbf{Observation 2}$: $f_1(8)=8$.

Hence, if n has the property that $f_j(n)=8$ for some j, then $f_k(n)=8$ for all $k>j$.

$\textbf{Case 1}$: $n=1$.

We have $f_1(n)=2$, $f_2(n)=f_1(2)=4$, $f_3(n)=f_1(4)=6$, $f_4(n)=f_1(6)=8$. Hence, Observation 2 implies $f_{50}(n)=8$.

$\mathbf{Case\ 2}$: n is prime.

We have $f_1(n) = 4$, $f_2(n) = f_1(4) = 6$, $f_3(n) = f_1(6) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

$\mathbf{Case\ 3}$: The prime factorization of n takes the form p_1^2.

We have $f_1(n) = 6$, $f_2(n) = f_1(6) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

$\mathbf{Case\ 4}$: The prime factorization of n takes the form p_1^3.

We have $f_1(n) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

$\mathbf{Case\ 5}$: The prime factorization of n takes the form p_1^4.

We have $f_1(n) = 10$, $f_2(n) = f_1(10) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

$\mathbf{Case\ 6}$: The prime factorization of n takes the form p_1^5.

We have $f_1(n) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$.

In this case the only n is $2^5 = 32$.

$\mathbf{Case\ 7}$: The prime factorization of n takes the form $p_1 p_2$.

We have $f_1(n) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

$\mathbf{Case\ 8}$: The prime factorization of n takes the form $p_1 p_2^2$.

We have $f_1(n) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$.

In this case, all n are $18, 50, 12, 20, 45, 28, 44$.

$\mathbf{Case\ 9}$: The prime factorization of n takes the form $p_1p_2^3$.

We have $f_1(n) = 16$, $f_2(n) = f_1(16) = 10$, $f_3(n) = f_1(10) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

$\mathbf{Case\ 10}$: The prime factorization of n takes the form $p_1p_2^4$.

We have $f_1(n) = 20$, $f_2(n) = f_1(20) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$.

In this case, the only n is 48.

$\mathbf{Case\ 11}$: The prime factorization of n takes the form $p_1^2p_2^2$.

We have $f_1(n) = 18$, $f_2(n) = f_1(18) = 12$. Hence, Observation 1 implies $f_{50}(n) = 12$.

In this case, the only n is 36.

$\mathbf{Case\ 12}$: The prime factorization of n takes the form $p_1p_2p_3$.

We have $f_1(n) = 16$, $f_2(n) = f_1(16) = 10$, $f_3(n) = f_2(10) = 8$. Hence, Observation 2 implies $f_{50}(n) = 8$.

Putting all cases together, the number of feasible $n \leq 50$ is $\boxed{\textbf{(D) } 10}$.

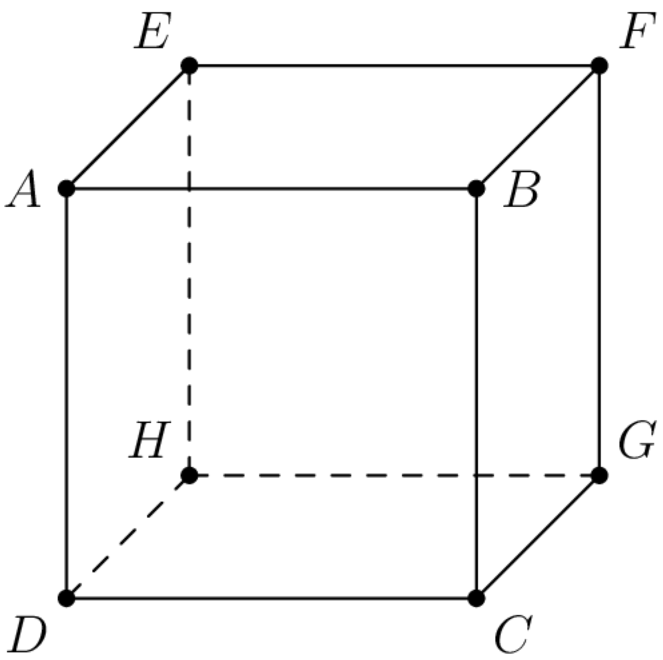
Problem24

Each of the 12 edges of a cube is labeled 0 or 1. Two labelings are considered different even if one can be obtained from the other by a sequence of one or more rotations and/or reflections. For how many such labelings is the sum of the labels on the edges of each of the 6 faces of the cube equal to 2?

$\textbf{(A) } 8 \qquad \textbf{(B) } 10 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 16 \qquad \textbf{(E) } 20$

Solution 1

For simplicity purposes, we name this cube $ABCDEFGH$ by vertices,



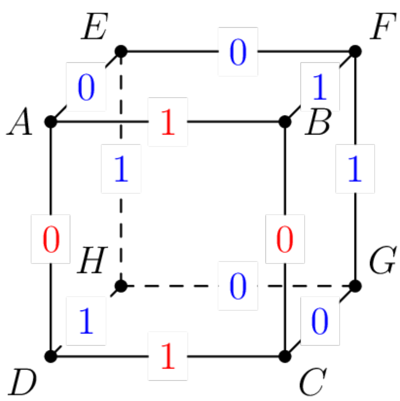
as shown below. Note that for each face of this cube, two edges are labeled 0 and two edges are labeled $1.$ For all twelve edges of this cube, we conclude that six edges are labeled $0,$ and six edges are labeled $1.$

We apply casework to face $ABCD.$ Recall that there are $\binom{4}{2}=6$ ways to label its edges:

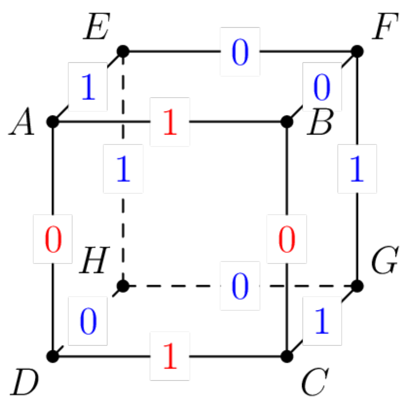
1. **Opposite edges have the same label.**

   There are 2 ways to label the edges of $ABCD.$ We will consider one of the ways, then multiply the count by $2.$ Without the loss of generality, we assume that $\overline{AB},\overline{BC},\overline{CD},\overline{DA}$ are labeled $1,0,1,0,$ respectively:

   We apply casework to the label of $\overline{AE},$ as shown below.



The label of $\overline{AE}$ is 0.



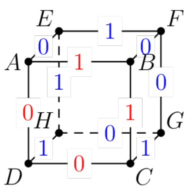
The label of $\overline{AE}$ is 1.

We have $2 \cdot 2 = 4$ such labelings for this case.

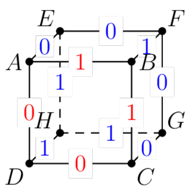
2. **Opposite edges have different labels.**

   There are 4 ways to label the edges of $ABCD$. We will consider one of the ways, then multiply the count by 4. Without the loss of generality, we assume that $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are labeled $1, 1, 0, 0,$ respectively:

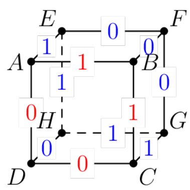
   We apply casework to the labels of $\overline{AE}$ and $\overline{BF}$, as shown below.

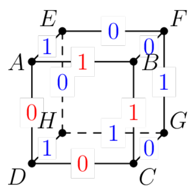


The label of $\overline{AE}$ is 0.
The label of $\overline{BF}$ is 0.



The label of $\overline{AE}$ is 0.
The label of $\overline{BF}$ is 1.





The label of $\overline{AE}$ is 1.
The label of $\overline{BF}$ is 0.

   We have $4 \cdot 4 = 16$ such labelings for this case.

Therefore, we have $4 + 16 = \boxed{\textbf{(E) } 20}$ such labelings in total.

Solution 2

Since we want the sum of the edges of each face to be 2, we need there to be two 1s and two 0s on each face. Through experimentation, we find that either $2, 4,$ or all of them have 1s adjacent to 1s and 0s adjacent to 0 on each face. WLOG, let the first face (counterclockwise) be $0, 0, 1, 1$. In this case we are trying to have all of them be adjacent to each other. First face: $0, 0, 1, 1$. Second face: 2 choices: $1, 0, 0, 1$ or $0, 0, 1, 1$. After that, it is basically forced and everything will fall in to place. Since we assumed WLOG, we need to multiply 2 by 4 to get a total of 8 different arrangements.

Secondly, 4 of the faces have all of them adjacent and 2 of the faces do not: WLOG counting counterclockwise, we have $0, 0, 1, 1$. Then, we choose the other face next to it. There are two cases, which are $0, 1, 0, 1$ and $1, 0, 1, 0$. Therefore, this subcase has 4 different arrangements. Then, we can choose the face at front to be $1, 0, 1, 0$. This has 4 cases. The sides can either be $0, 1, 1, 0$ or $1, 1, 0, 0$. Therefore, we have another 8 cases.

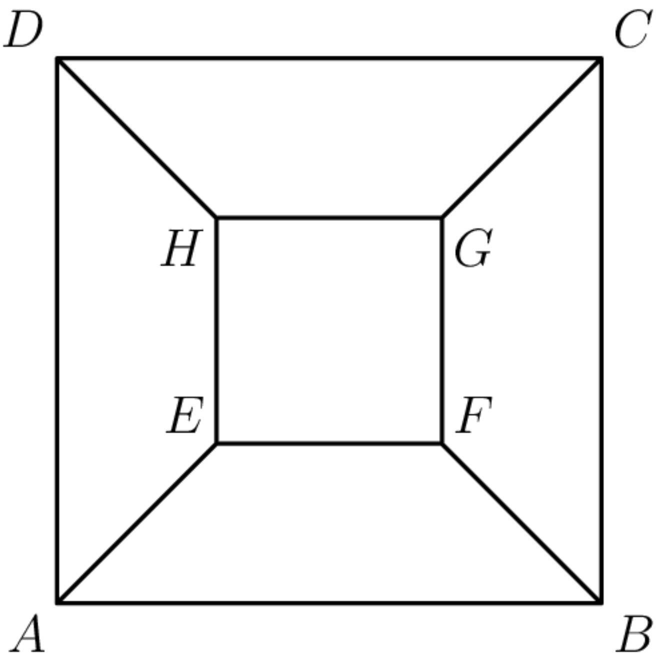
Summing these up, we have $8 + 4 + 8 = 20$. Therefore, our answer is $\boxed{\textbf{(E) } 20}$.

Remark

It is very easy to get disorganized when counting, so when doing this problem, make sure to draw a diagram of the cube. Labeling is a bit harder, since we often confuse one side with another. Try doing the problem by labeling sides on the lines (literally letting the lines pass through your 0s and 1s.) I found that to be very helpful when solving this problem.

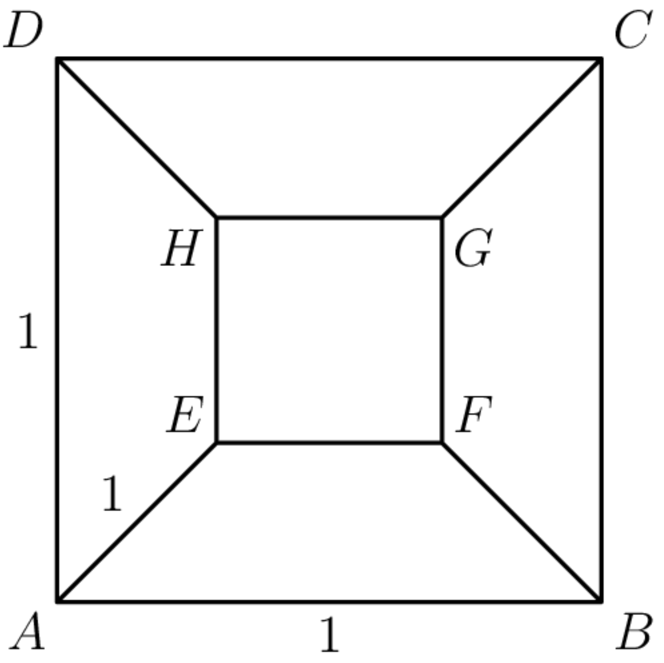
~Arcticturn

Solution 3

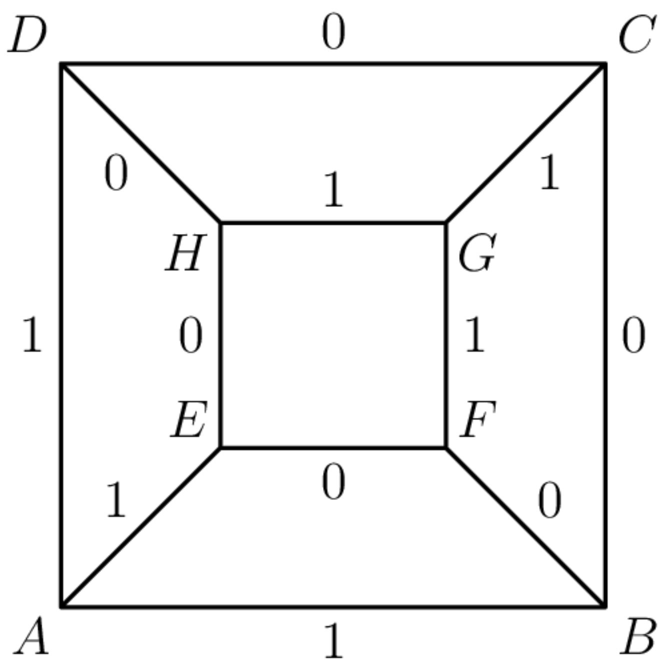


We see that each face has to have 2 1's and 2 0's. We can start with edges connecting to A.

Case 1

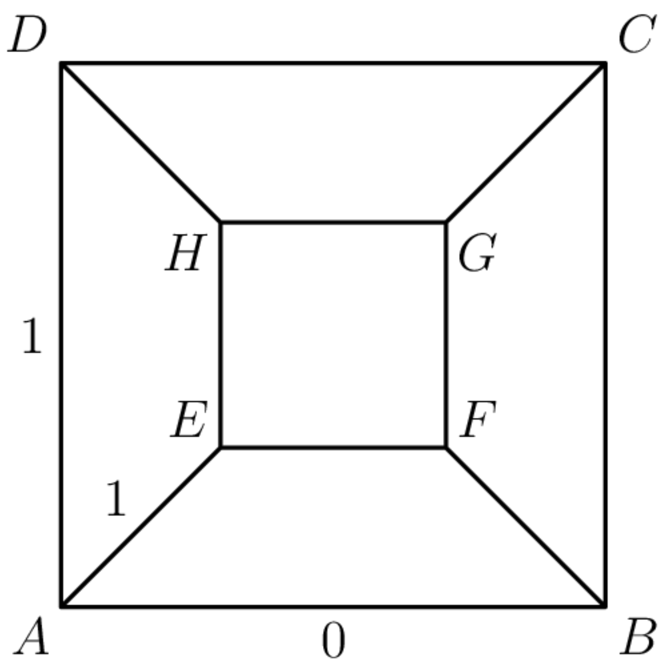


This goes to:

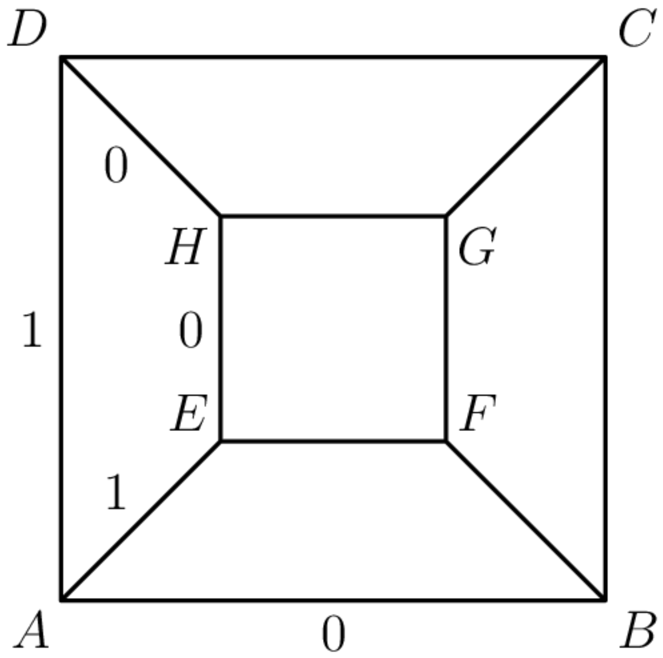


We can see that we choose 2 diametrically opposite vertices to put 3 1's on the connecting edges. As a result, this case has $\frac{8}{2} = 4$ orientations.

Case 2

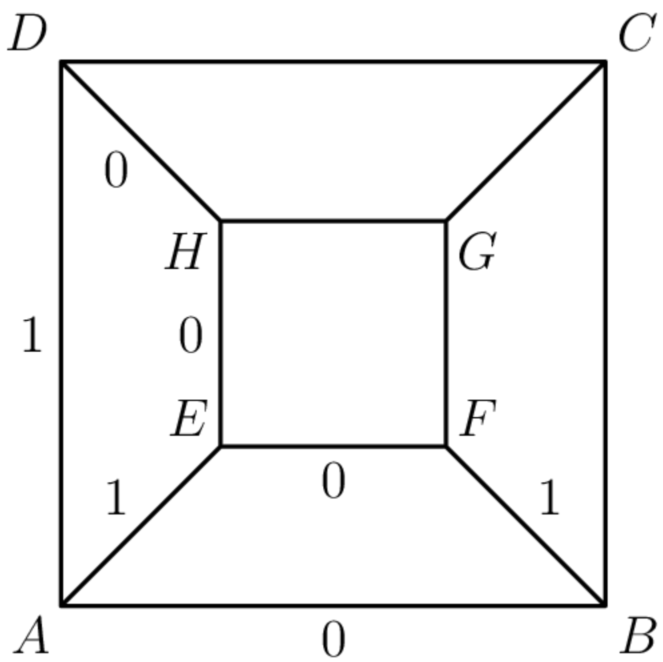


Filling out a bit more, we have:

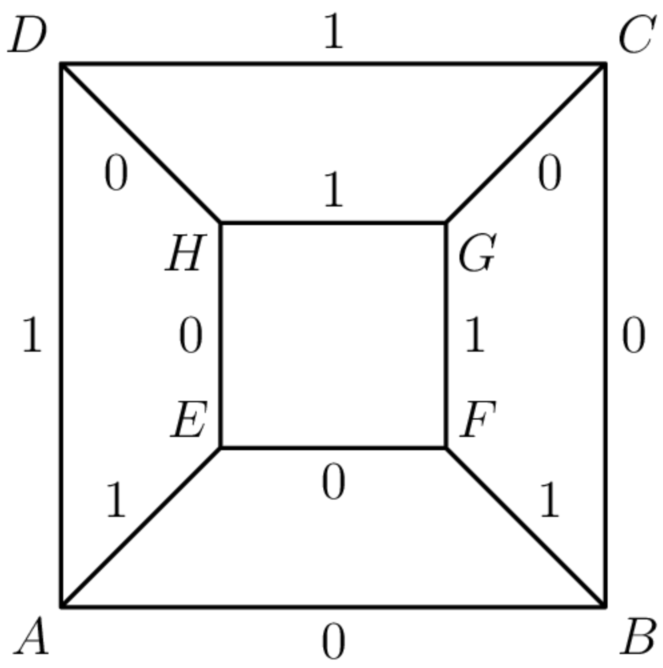


Let's try filling out BC and CD first.

Case 2.1

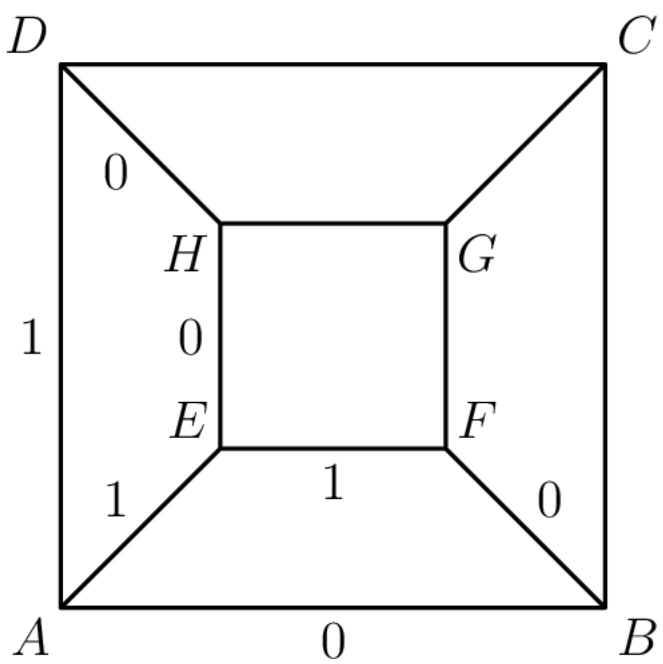


This goes to:



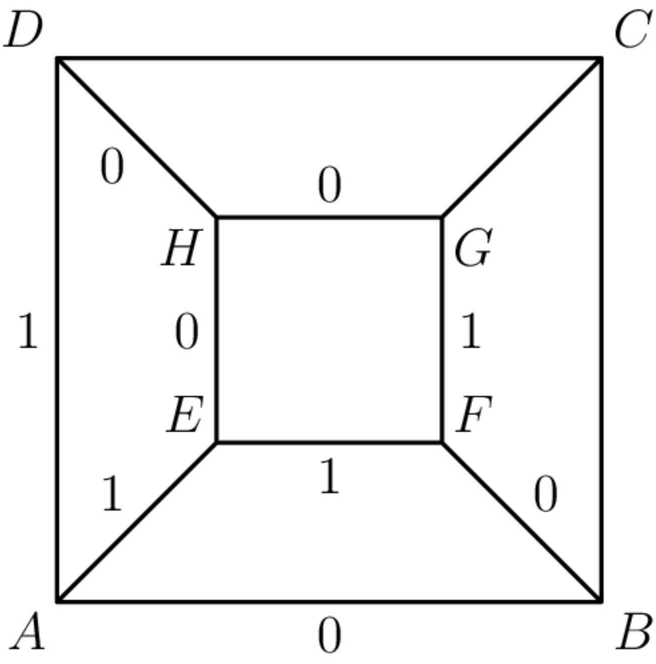
We can see that it consists of chains of three 1's, with the middle of each chain being opposite edges. As a result, this case has $\frac{12}{2} = 6$ orientations.

Case 2.2



Oh no... We have different ways of filling out FG and GH. More casework!

Case 2.2.1



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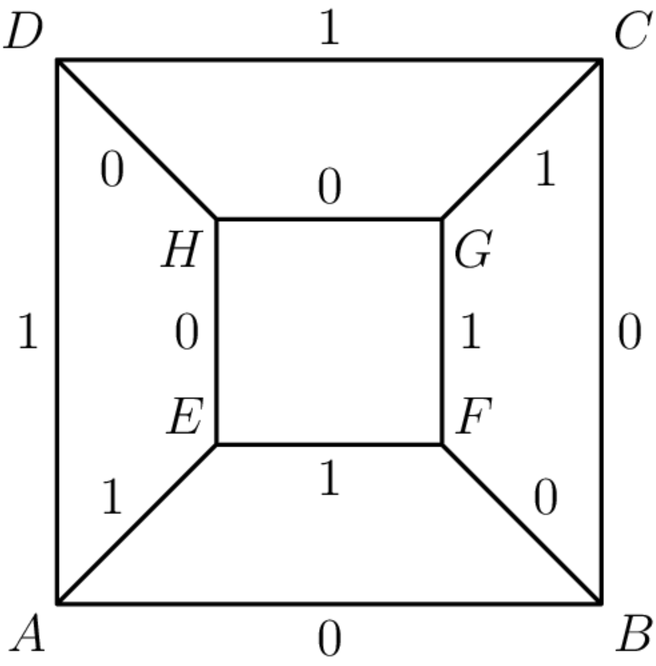
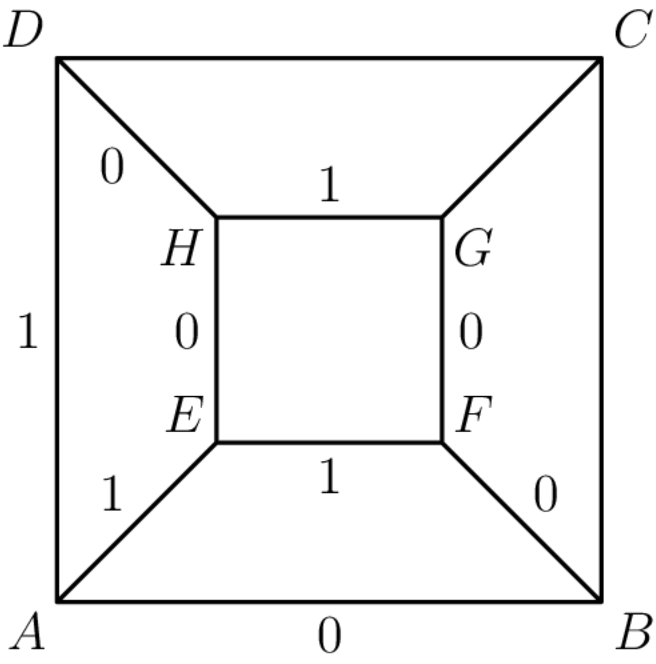
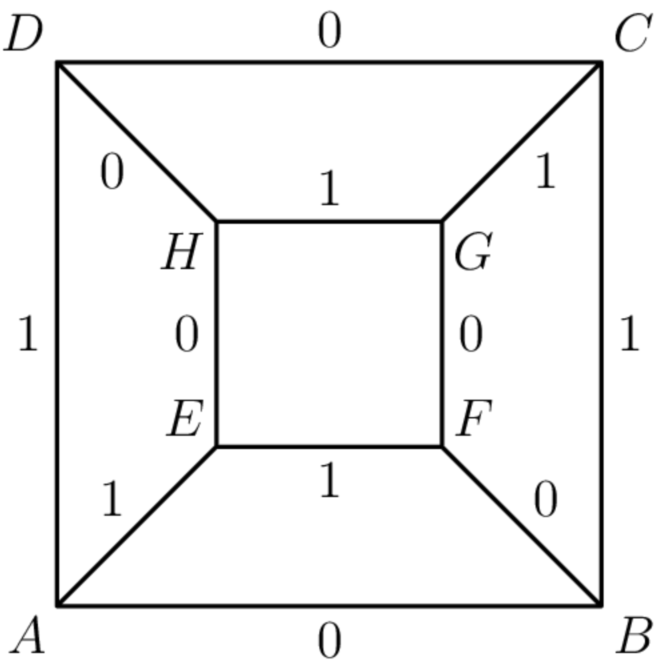


We can see that this is the inverse of case 1 (Define inverse to mean swapping 1's for 0's and 0's for 1's). Therefore, this should also have 4 orientations.

Case 2.2.2



This goes to:



This is the inverse of case 2.1, so this will also have 6 orientations.

Putting Them All Together

We see that if the 3 edges connecting to A has two 0's, and one 1, it would have the same solutions as if it had two 1's, and one 0. The solutions would just be inverted. As case 2.1 and case 2.2.2 are inverses, and case 2.2.1 has case 1 as an inverse, there would not be any additional solutions.

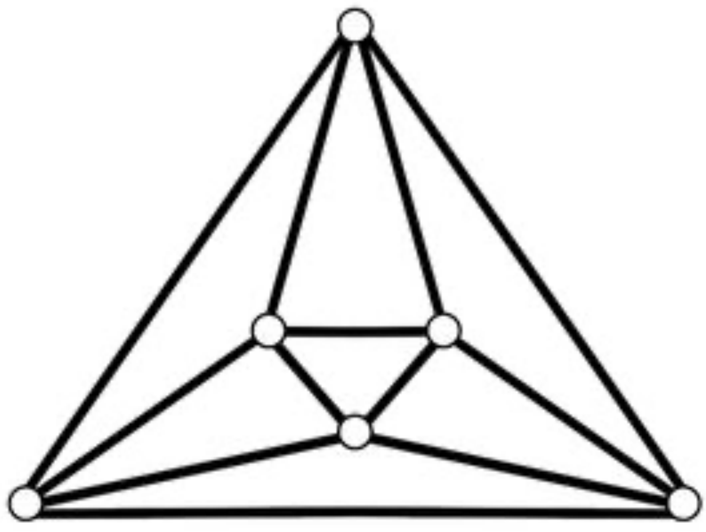
Similarly, if the 3 edges connecting to A has three 0's, it would be the same as the inverse of case 1, or case 2.2.1, resulting in no new solutions.

Putting all the cases together, we have $4+6+4+6=\boxed{\textbf{(E) } 20}$ solutions.

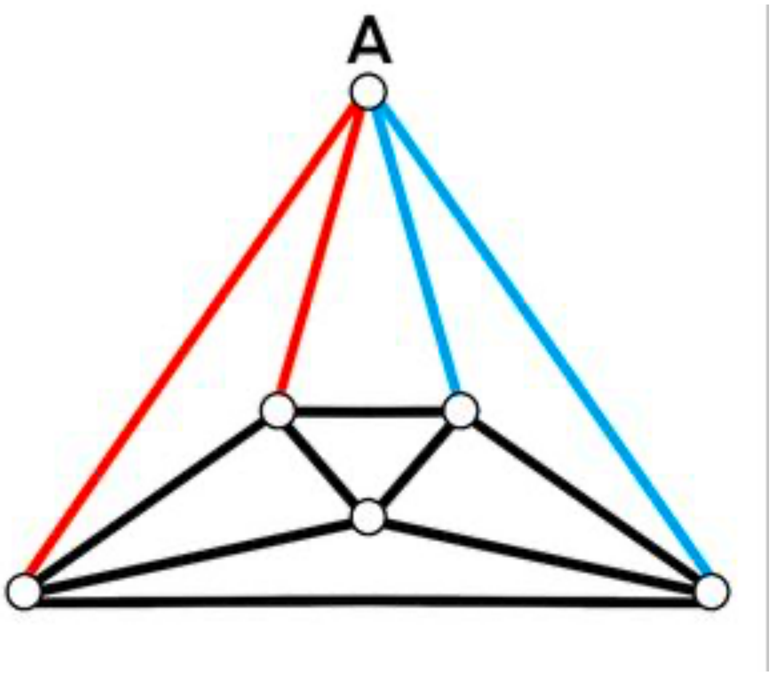
Solution 4

The problem states the sum of the labels on the edges of each of the 6 faces of the cube equal to 2. That is, the sum of the labels on the 4 edges of a face is equal to 2. The labels can only be 0 or 1, meaning 2edges are labeled 1, the other 2 are labeled 0.

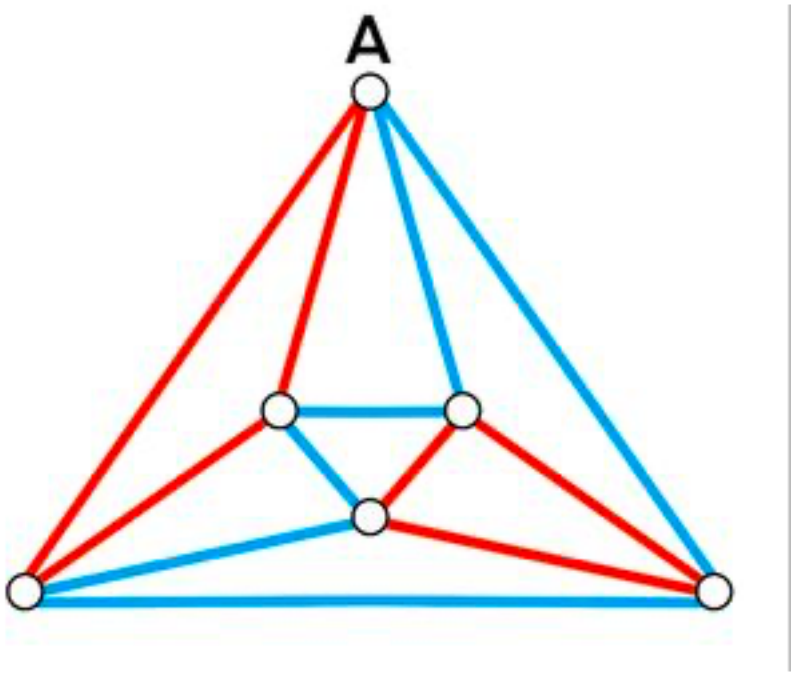
This problem can be approached by Graph Coloring of Graph Theory. Note that each face of the cube connects to 4 other faces, each with a shared edge. We use the following graph to represent the problem. Each vertex represents a face, each edge represent the cube's edge. Each vertex has 4 edges connecting to 4 other vertices. The edges can be colored red or blue, with red as label 0, and blue as label 1. Each vertex must have 2 red edges and 2 blue edges.



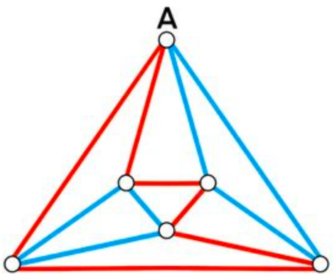
$\textbf{Case 1}$: 2 adjacent red edges from vertex A. There are 4 ways to choose 2 red edges adjacent to each other and connect to 2 vertices with an edge between them as shown below.

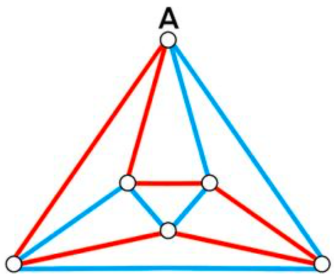


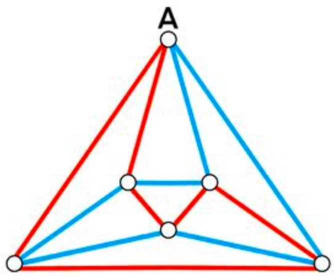
Case 1.1: 2 adjacent red edges from vertex A form a closed loop with a third red edge. There is only 1 case as shown below.



Case 1.2: 2 adjacent red edges from vertex A does not form a closed loop with a third red edge. There are 3 cases as shown below.

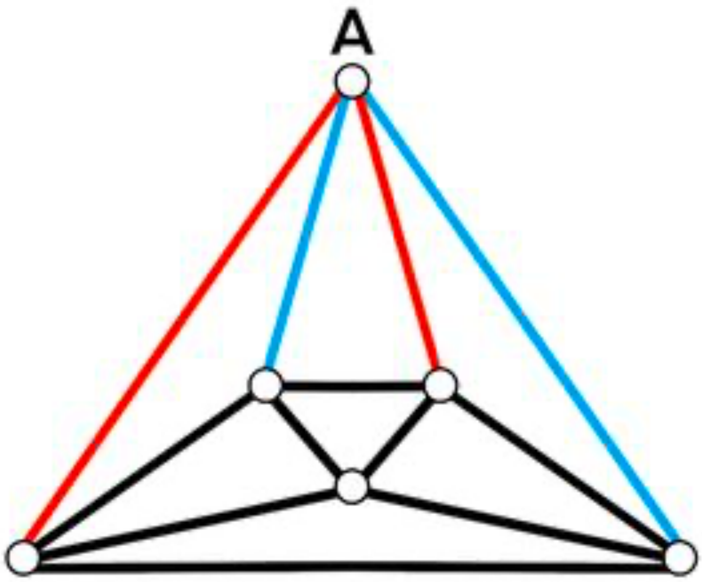




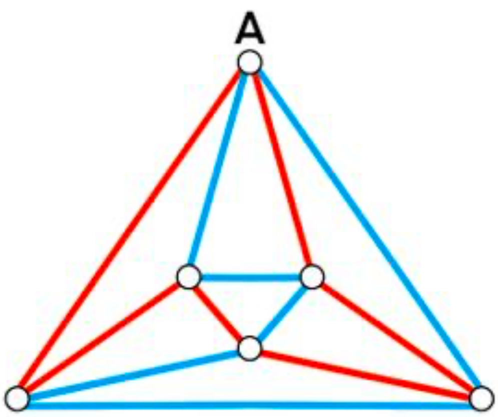


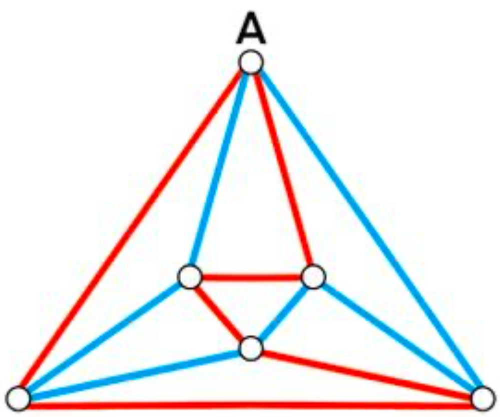
In case 1, there are total $4 \cdot (1+3) = 16$ ways.

$\mathbf{Case\ 2}$: 2 red edges from vertex A with 1 blue edge in between. There are 2 ways to choose 2 red edges with 1 blue edge in between.



There are only 2 cases as shown below.





In case 2, there are total $2 \cdot 2 = 4$ ways.

From both case 1 and case 2, there are $16+4=\boxed{\textbf{(E) } 20}$ ways in total.

Problem25

A quadratic polynomial with real coefficients and leading coefficient 1 is called $\textit{disrespectful}$ if the equation $p(p(x))=0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

$\textbf{(A) } \frac{5}{16} \qquad\textbf{(B) } \frac{1}{2} \qquad\textbf{(C) } \frac{5}{8} \qquad\textbf{(D) } 1 \qquad\textbf{(E) } \frac{9}{8}$

Solution 1 (Vieta's Formulas)

Let r_1 and r_2 be the roots of $\tilde{p}(x)$.

Then, $\tilde{p}(x)=(x-r_1)(x-r_2)=x^2-(r_1+r_2)x+r_1r_2$.

The solutions to $\tilde{p}(\tilde{p}(x))=0$ is the union of the solutions to $\tilde{p}(x)-r_1=x^2-(r_1+r_2)x+(r_1r_2-r_1)=0$ and $\tilde{p}(x)-r_2=x^2-(r_1+r_2)x+(r_1r_2-r_2)=0$. Note that one of these two quadratics has one solution (a double root) and the other has two as there are exactly three solutions. WLOG, assume that the quadratic with one root is $x^2-(r_1+r_2)x+(r_1r_2-r_1)=0$. Then, the discriminant is 0, so $(r_1+r_2)^2 = 4r_1r_2 - 4r_1$.

Thus, $r_1-r_2=\pm 2\sqrt{-r_1}$, but for $x^2-(r_1+r_2)x+(r_1r_2-r_2)=0$ to have two solutions, it must be the case that $r_1-r_2=-2\sqrt{-r_1}$ *. It follows that the sum of the roots of $\tilde{p}(x)$ is $2r_1+2\sqrt{-r_1}$, whose maximum value occurs when $r_1 = -\frac{1}{4}$. Solving for r_2 yields $r_2 = \frac{3}{4}$.

Therefore, $\tilde{p}(x) = x^2 - \frac{1}{2}x - \frac{3}{16}$, so $\tilde{p}(1) = \boxed{\textbf{(A) } \frac{5}{16}}$.

Remark

$*$ For $x^2 - (r_1 + r_2)x + (r_1 r_2 - r_2) = 0$ to have two solutions, the discriminant $(r_1 + r_2)^2 - 4r_1 r_2 + 4r_2$ must be positive. From here, we get that $(r_1 - r_2)^2 > -4r_2$,

so $-4r_1 > -4r_2 \implies r_1 < r_2$. Hence, $r_1 - r_2$ is negative,

so $r_1 - r_2 = -2\sqrt{-r_1}$.

Solution 2 (Vertex Form)

Let $p(x) = (x-h)^2 + k$ for some real constants h and k. Suppose that $p(x)$ has real roots r and s.

Since $p(p(x)) = 0$, we conclude that $p(x) = r$ or $p(x) = s$. Without the loss of generality, we assume that $p(x) = r$ has two real solutions and $p(x) = s$ has one real solution. Therefore, we have $k = s$, from which $p(x) = (x-h)^2 + s$.

As $p(s) = 0$, we expand the left side to obtain $(s-h)^2 + s = 0$, or

$$s^2 - (2h-1)s + h^2 = 0. \qquad (\bigstar)$$

Since $(\bigstar)$ has real solutions for s, the discriminant is nonnegative: $(2h-1)^2-4h^2\geq 0.$ We solve this inequality to get $h\leq\frac{1}{4}.$

Either by the axis of symmetry or Vieta's Formulas, note that $r+s=2h.$ As we wish to maximize $2h,$ we maximize $h.$ Substituting $h=\frac{1}{4}$ into $(\bigstar),$ we obtain $s^2+\frac{1}{2}s+\frac{1}{16}=0.$ We factor the left side to get $\left(s+\frac{1}{4}\right)^2=0,$ or $s=-\frac{1}{4}.$

Finally, the unique such polynomial is $\tilde{p}(x)=\left(x-\frac{1}{4}\right)^2-\frac{1}{4},$ from which $\tilde{p}(1)=\boxed{\textbf{(A) } \frac{5}{16}}.$

Solution 3 (Symmetry)

Let $\tilde{p}(x)=(x-h)^2+k$. We seek to maximize the sum of the roots $2h$, so we maximize h.

Let $P(x)=\tilde{p}(\tilde{p}(x))=((x-h)^2+k-h)^2+k$. Note that $P(x)$ is symmetric about $x=h$.

$P(x)=0$ has 3 real solutions, Due to the complex conjugate theorem, $P(x)$ must have 4 real roots. Therefore, $P(x)$ must have exactly 1 double root.

This root cannot be to the left or to the right of $x = h$, as the symmetry of the function would mean that there would be another double root reflected across the $x = h$. It follows that the double root could only be situated at $x = h$.

$0 = P(h) = ((h-h)^2 + k - h)^2 + k = (k-h)^2 + k$.

Expanding and writing this out in terms of k, $k^2 + (1-2h)k + h^2 = 0$.

In order for this to have a solution, the discriminant has to be non-negative. In other words, $(1-2h)^2 - 4h^2 \geq 0$.

This simplifies to $1 - 4h \geq 0$, or $h \leq \frac{1}{4}$.

As we seek to maximize h, we set $h = \frac{1}{4}$ and see that $k = -\frac{1}{4}$.

Therefore, $\tilde{p}(x) = \left(x - \frac{1}{4}\right)^2 - \frac{1}{4}$,

and $\tilde{p}(1) = \left(1 - \frac{1}{4}\right)^2 - \frac{1}{4} = \frac{9}{16} - \frac{1}{4} = \boxed{\textbf{(A) } \frac{5}{16}}$.

Solution 4 (Discriminant)

Equation $p\left(p\left(x\right)\right) = 0$ is equivalent to the following system of equations:

$$\begin{aligned} p\left(u\right) = 0, \\ p\left(x\right) - u = 0. \end{aligned}$$

Denote $p\left(x\right) = x^2 + px + q$.

Denote by r_1 and r_2 two roots of $p\left(x\right) = 0$.

Because $p\left(p\left(x\right)\right)=0$ has three real roots, we must have the properties that r_1 and 2 are real with $r_1 \neq r_2$. Without loss of generality, we assume $r_1 < r_2$.

We notice that all roots of $p\left(p\left(x\right)\right)=0$ are the collection of all roots of $p\left(x\right)-r_1=0$ and all roots of $p\left(x\right)-r_2=0$.

Because each of these two equations is quadratic, it has two roots (may be identical). To get a total number of three roots, one equation must have two identical roots.

Because $r_1 < r_2$, equation $p\left(x\right)-r_1=0$ has two identical roots. Hence, the discriminant of this equation satisfies $p^2-4\left(q-r_1\right)=0. \qquad (1)$

Because $r_1 < r_2$, $r_1=\dfrac{-p-\sqrt{p^2-4q}}{2}$.

Hence, Equation (1) can be written as $p^2-4\left(q-\dfrac{-p-\sqrt{p^2-4q}}{2}\right)=0.$

This can be reorganized as $\left(p^2-4q\right)-2\sqrt{p^2-4q}-2p=0. \qquad (2)$

Define $x=\sqrt{p^2-4q}$. Hence, the value of p should ensure that equation $x^2-2x-2p=0$ has at least one real nonnegative root.

This condition can be satisfied if the discriminant of this equation is nonnegative: $\left(-2\right)^2-4\cdot 1\cdot\left(-2p\right)\geq 0$. Hence, $p\geq -\dfrac{1}{2}$.

Now, we are ready to find $\tilde{p}\left(x\right)$.

Following from Vieta's formula, $r_1 + r_2 = -p$. Hence, to get $r_1 + r_2$ maximized, we need to find smallest p.

Because $p \geq -\frac{1}{2}$, the smallest p is $-\frac{1}{2}$. Plugging this value into Equation (2), we get $\sqrt{p^2 - 4q} = 1$. Hence, $q = -\frac{3}{16}$.

Thus, $\tilde{p}(x) = x^2 - \frac{1}{2}x - \frac{3}{16}$. Therefore, $\tilde{p}(1) = \boxed{\textbf{(A)}\ \frac{5}{16}}$.

Solution 5 (Factored Form)

The disrespectful function $p(x)$ has leading coefficient 1, so it can be written in factored form as $(x-r)(x-s)$. Now the problem states that all $p(x)$ must satisfy $p(p(x)) = 0$. Plugging our form in, we get $((x-r)(x-s)-r)((x-r)(x-s)-s) = 0$. The roots of this equation are $(x-r)(x-s) = r$ and $(x-r)(x-s) = s$. By the fundamental theorem of algebra, each root must have two roots for a total of four possible values of x yet the problem states that this equation is satisfied by three values of x. Therefore one equation must give a double root. Without loss of generality, let the equation $(x-r)(x-s) = r$ be the equation that produces the double root. Expanding gives $x^2 - (r+s)x + rs - r = 0$. We know that if there is a double root to this equation, the discriminant must be equal to zero, so $(r+s)^2 - 4(rs-r) = 0 \implies r^2 + 2rs + s^2 - 4rs + 4r = 0 \implies r^2 - 2rs + s^2 + 4r = 0$.

From here two solutions can progress.

Solution 5.1 (Quadratic Formula)

We can rewrite $r^2 - 2rs + s^2 + 4r = 0$ as $(r-s)^2 + 4r = 0$. Let's keep our eyes on the ball; we want to find the disrespectful quadratic that maximizes the sum of the roots, which is $r + s$. Let this be equal to a new variable, m, so that our problem is reduced to maximizing this variable. We can rewrite our equation in terms of m as

$$(2r-m)^2 + 4r = 0 \implies m^2 - 4rm + 4r^2 + 4r = 0.$$

This is a quadratic in m, so we can use the quadratic formula:

$$m = \frac{4r \pm \sqrt{16r^2 - 4(4r^2+4r)}}{2} = 2r \pm \sqrt{-4r} = 2\left(r \pm \sqrt{-r}\right).$$

It will be easier to think without the square root, so let $q = \sqrt{-r}$. We can rewrite the equation as $m = 2(-q^2 \pm q)$. We want to maximize m, so we take the plus value of the right-hand-side of the equation.

Then, $m = 2(-q^2 + q) \implies m = -2q(q-1)$. To maximize m, we find the vertex of the right-hand side of the equation. The vertex of $-2q(q-1)$ is the average of the roots of the equation which is $\frac{0+1}{2} = \frac{1}{2}$. This means that $r = -q^2 \implies \boldsymbol{r = -\frac{1}{4}}$

and $m = -2q(q-1) \implies m = \frac{1}{2}$.

Therefore, $m - r = s \implies \boldsymbol{s = \frac{3}{4}}$.

Solution 5.2 (Derivation-Rotated Conics)

We see that the equation $r^2 - 2rs + s^2 + 4r = 0$ is in the form of the general equation of a rotated conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$

Because $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$, this rotated conic is a parabola.

The definition of a parabola is the locus of all points that are equidistant from a point (focus) and line (directrix). Let the focus and directrix of this particular parabola be (a, b) and $y = mx + c$. Then we can try to find the general form of a rotated parabola in terms of $a, b, m,$ and c.

The distance between two points (x, y) and (a, b) is $\sqrt{(x-a)^2 + (y-b)^2}$. Therefore this is the distance from any point on the parabola to the focus.

The distance from a point (x, y) to a line $y = mx + c \implies mx - y + c = 0$ is $\dfrac{|mx - y + c|}{\sqrt{m^2+1}}$.

We can set these two equal to each other and we get $\sqrt{(x-a)^2 + (y-b)^2} = \dfrac{|mx - y + c|}{\sqrt{m^2+1}}$. Squaring both sides of the equation, we get $(x-a)^2 + (y-b)^2 = \dfrac{(mx - y + c)^2}{m^2+1}$.

Expanding both sides of the equation gives

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = \frac{m^2x^2 + y^2 + c^2 + 2mxc - 2yc - 2mxy}{m^2+1}.$$

Multiplying both sides of the equation by $m^2 + 1$ and rearranging gives

$$x^2 + 2mx + m^2y^2 - 2((m^2+1)a + mc)x - 2((m^2+1)b - c)y + (m^2+1)(a^2+b^2) - c^2.$$

Now we can compare to our rotated parabola, $r^2 - 2rs + s^2 + 4r = 0$. From this, $-2 = 2m$ or $m = -1$. From here we have a system of three equations:

$$\begin{aligned} -2((m^2+1)a+mc)&=4,\\ -2((m^2+1)b-c)&=0,\\ (m^2+1)(a^2+b^2)-c^2&=0. \end{aligned}$$

Plugging in $m=-1$ we get

$$\begin{aligned} -2(2a-c)&=4,\\ -2(2b-c)&=0,\\ 2(a^2+b^2)-c^2&=0. \end{aligned}$$

Solving for the first equation, $c=2+2a$.

Subtracting the first two equations, $-4a+4b=4 \implies b=a+1$.

Plugging into the third equation,

$2a^2+2a^2+4a+2=c^2 \implies 4a^2+4a+2=c^2$.

Substituting c in, we get

$4a^2+4a+2=4a^2+8a+4 \implies 4a+2=0 \implies a=-\frac{1}{2}$

.

Now $b=a+1=\frac{1}{2}$ and $c=2+2a=1$.

This means that the focus of the parabola is $\left(-\frac{1}{2},\frac{1}{2}\right)$ and the directrix is $y=-x+1$. The maximum value of $r+s$ would lie at the vertex of the parabola, which is the midpoint of the focus and the foot of the focus at the directrix. The line that the vertex and focus lie on is perpendicular to the directrix, so it has slope 1. It can be written as $y=x+d$ and must go through $\left(-\frac{1}{2},\frac{1}{2}\right)$ so $d=1$. This perpendicular line intersects the directrix, so to find the point at which this foot occurs, we set the equation of the

$$y = x + 1,$$

lines equal to each other: $y = -x + 1$. Adding, we get $2y = 2$ or $y = 1$ and $x = 0$. The vertex of the parabola is now at the midpoint of $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(0, 1)$ which is $\left(-\frac{1}{4}, \frac{3}{4}\right)$.

Therefore, we have $\boldsymbol{r = -\frac{1}{4}}$ and $\boldsymbol{s = \frac{3}{4}}$.

Solutions 5.1 and 5.2 Rejoined

Now that we know the roots of $\tilde{p}(1)$, we can plug in our equation:

$$(x-r)(x-s) = \left(1 - \left(-\frac{1}{4}\right)\right)\left(1 - \frac{3}{4}\right) = \frac{5}{4} \cdot \frac{1}{4} = \frac{5}{16} = \boxed{\textbf{(A) } \frac{5}{16}}.$$

~KingRavi

Solution 6 (Factored Form)

Let $p(x) = (x-p)(x-q)$. Then,

$$p(p(x)) = ((x-p)(x-q)-p)((x-p)(x-q)-q) = 0$$

, which means that either $(x-p)(x-q)-p = 0$ or $(x-p)(x-q)-q = 0$.

Both of these equations are quadratics, so $p(p(x))$ has four roots, unless there's a double root.

Without loss of generality, let $(x-p)(x-q)-p$ be the expression that produces the double root, so its discriminant is zero. When expanded,

$$(x-p)(x-q)-p = x^2 - (p+q)x + pq - p = 0.$$

The value of x is irrelevant to the discriminant, which is $(-(p+q))^2 - 4(pq-p)$. Setting this equal to zero and simplifing, this equation becomes $p^2 - 2pq + q^2 + 4p = 0,$ which is a quadratic in p.

Now, we seek the value of p. The previous quadratic is equivalent to $p^2 - (2q-4)p + q^2 = 0$. Using the quadratic formula by having $a = 1, b = -(2q-4)$, and $c = q^2,$ we have

$$p = \frac{2q-4 \pm \sqrt{(-(2q-4))^2 - 4(1(q^2))}}{2(1)} = \frac{2q-4 \pm 4\sqrt{1-q}}{2} = q-2+2\sqrt{1-q}.$$

Our focus is on maximizing $p+q$, so we need the maximum values of p and q respectively (by taking the positive square root). Adding q, we see that $p+q = 2\left(q-1+\sqrt{1-q}\right).$

Let $k = \sqrt{1-q}.$ Then, $-k^2 = q-1,$ so

$$p+q = 2(-k^2+k).$$

The graph of $-k^2+k$ is a parabola that opens up downwards, and has its maximum value at y-value of the vertex. The x coordinate of the vertex is the average of the two roots, which in this case, are 0 and 1, so the average of these two is $\frac{1}{2}$. This means that the maximum value of $-k^2+k$ occurs at $k = \frac{1}{2}.$

Substituting $k = \sqrt{1-q},$ we see that $q = \frac{3}{4}.$

Since $p = q-2+2\sqrt{1-q},$ we can plug $q = \frac{3}{4}$ in, to see that $p = -\frac{1}{4}.$

Because the roots of this quadratic are $\frac{3}{4}$ and $-\frac{1}{4}$, our quadratic is $\left(x-\frac{3}{4}\right)\left(x+\frac{1}{4}\right)$. Letting 1 be x, $\tilde{p}(1)=\frac{5}{16}$, which is $\boxed{\textbf{(A) } \frac{5}{16}}$.