2020 AMC 10A Solution

Problem1

What value of x satisfies $x - \frac{3}{4} = \frac{5}{12} - \frac{1}{3}$?

$\textbf{(A)}\ -\frac{2}{3} \qquad \textbf{(B)}\ \frac{7}{36} \qquad \textbf{(C)}\ \frac{7}{12} \qquad \textbf{(D)}\ \frac{2}{3} \qquad \textbf{(E)}\ \frac{5}{6}$

Solution

Adding $\frac{3}{4}$ to both sides, $x = \frac{5}{12} - \frac{1}{3} + \frac{3}{4} = \frac{5}{12} - \frac{4}{12} + \frac{9}{12} = \boxed{\textbf{(E)}\ \frac{5}{6}}$.

Problem2

The numbers $3, 5, 7, a,$ and b have an average (arithmetic mean) of 15. What is the average of a and b?

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 15 \qquad \textbf{(C)}\ 30 \qquad \textbf{(D)}\ 45 \qquad \textbf{(E)}\ 60$

Solution

The arithmetic mean of the numbers $3, 5, 7, a,$ and b is equal to $\frac{3+5+7+a+b}{5} = \frac{15+a+b}{5} = 15$. Solving for $a+b$, we get $a+b=60$. Dividing by 2 to find the average of the two numbers a and b gives $\frac{60}{2} = \boxed{\textbf{(C)}\ 30}$.

Problem3

Assuming $a \neq 3$, $b \neq 4$, and $c \neq 5$, what is the value in simplest form of the following expression? $\frac{a-3}{5-c} \cdot \frac{b-4}{3-a} \cdot \frac{c-5}{4-b}$

$\textbf{(A) } -1 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } \frac{abc}{60} \qquad \textbf{(D) } \frac{1}{abc} - \frac{1}{60} \qquad \textbf{(E) } \frac{1}{60} - \frac{1}{abc}$

Solution

Note that $a-3$ is -1 times $3-a$.
Likewise, $b-4$ is -1 times $4-b$ and $c-5$ is -1 times $5-c$.
Therefore, the product of the given fraction equals $(-1)(-1)(-1) = \boxed{\textbf{(A) } -1}$.

Problem4

A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid $\$0.50$ per mile, and her only expense is gasoline at $\$2.00$ per gallon. What is her net rate of pay, in dollars per hour, after this expense?

$\textbf{(A) } 20 \qquad \textbf{(B) } 22 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 26$

Solution

Since the driver travels 60 miles per hour and each hour she uses 2 gallons of gasoline, she spends $4 per hour on gas. If she gets $0.50 per mile, then she gets $30 per hour of driving. Subtracting the gas cost, her net rate of pay per hour is $\boxed{\textbf{(E) } 26}$.

Problem5

What is the sum of all real numbers x for which $|x^2 - 12x + 34| = 2?$

$\textbf{(A) } 12 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 21 \qquad \textbf{(E) } 25$

Solution 1

Split the equation into two cases, where the value inside the absolute value is positive and nonpositive.

Case 1:

The equation yields $x^2 - 12x + 34 = 2$, which is equal to $(x-4)(x-8) = 0$. Therefore, the two values for the positive case is 4 and 8.

Case 2:

Similarly, taking the nonpositive case for the value inside the absolute value notation yields $-x^2 + 12x - 34 = 2$. Factoring and simplifying gives $(x-6)^2 = 0$, so the only value for this case is 6.

Summing all the values results in $4+8+6 = \boxed{\textbf{(C) } 18}$.

Solution 2

We have the equations $x^2 - 12x + 32 = 0$ and $x^2 - 12x + 36 = 0$.

Notice that the second is a perfect square with a double root at $x = 6$, and the first has real roots. By Vieta's, the sum of the roots of the first equation is 12. $12 + 6 = \boxed{\textbf{(C) } 18}$

Problem6

How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by $5?$

$\textbf{(A) } 80 \qquad \textbf{(B) } 100 \qquad \textbf{(C) } 125 \qquad \textbf{(D) } 200 \qquad \textbf{(E) } 500$

Solution

The ones digit, for all numbers divisible by 5, must be either 0 or 5. However, from the restriction in the problem, it must be even, giving us exactly one choice (0) for this digit. For the middle two digits, we may choose any even integer from $[0, 8]$, meaning that we have 5 total options. For the first digit, we follow similar intuition but realize that it cannot be 0, hence giving us 4 possibilities. Therefore, using the multiplication rule, we get $4 \times 5 \times 5 \times 1 = \boxed{\textbf{(B) } 100}$. ~ciceronii

Problem7

The 25 integers from -10 to $14,$ inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

$\textbf{(A) } 2 \qquad \textbf{(B) } 5 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 50$

Solution

Without loss of generality, consider the five rows in the square. Each row must have the same sum of numbers, meaning that the sum of all the numbers in the square divided by 5 is the total value per row. The sum of the 25 integers is $-10+9+...+14 = 11+12+13+14 = 50$, and the common sum is $\frac{50}{5}=\boxed{\text{(C) } 10}$.

Solution 2

Take the sum of the middle 5 values of the set (they will turn out to be the mean of each row). We get $0 + 1 + 2 + 3 + 4 = \boxed{\textbf{(C) } 10}$ as our answer. ~Baolan

Problem8

What is the value of

$$1+2+3-4+5+6+7-8+\cdots+197+198+199-200?$$

$\textbf{(A) } 9,800 \qquad \textbf{(B) } 9,900 \qquad \textbf{(C) } 10,000 \qquad \textbf{(D) } 10,100 \qquad \textbf{(E) } 10,200$

Solution 1

Split the even numbers and the odd numbers apart. If we group every 2 even numbers together and add them, we get a total of $50 \cdot (-2) = -100$. Summing the odd numbers is equivalent to summing the first 100 odd numbers, which is equal to $100^2 = 10000$. Adding these two, we obtain the answer of $\boxed{\text{(B) } 9900}$.

Solution 2 (bashy)

We can break this entire sum down into 4 integer bits, in which the sum is $2x$, where x is the first integer in this bit. We can find that the first sum of every sequence is $4x - 3$, which we plug in for the 50 bits in the entire sequence is $1 + 2 + 3 + \cdots + 50 = 1275$, so then we can plug it into the first term of every sequence equation we got above $4(1275) - 3(50) = 4950$, and so the sum of every bit is $2x$, and we only found the value of x, the sum of the sequence is $4950 \cdot 2 = \boxed{(B) 9900}$.-middletonkids

Solution 3

Another solution involves adding everything and subtracting out what is not needed. The first step involves solving $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \cdots + 197 + 198 + 199 + 200$. To do this, we can simply multiply 200 and 201 and divide by 2 to get us 20100. The next step involves subtracting out the numbers with minus signs. We actually have to do this twice, because we need to take out the numbers we weren't supposed to add and then subtract them from the problem. Then, we can see that from 4 to 200, incrementing by 4, there are 50 numbers that we have to subtract. To do this we can do 50 times 51divided by 2, and then we can multiply by 4, because we are counting by fours, not ones. Our answer will be 5100, but remember, we have to do this twice. Once we do that,

we will get 10200. Finally, we just have to do $20100-10200$, and our answer is $\boxed{\text{(B) }9900}$.
$- Phineas1500$

Solution 4

In this solution, we group every 4 terms. Our groups should be: $1+2+3-4=2$, $5+6+7-8=10$,

$9+10+11-12=18$, ...

$197+198+199-200=394$. We add them together to get this expression: $2+10+18+...+394$. This can be rewritten as $8*(0+1+2+...+49)+100$. We add this to get $\boxed{\textbf{(B)}\ 9900}$. ~Baolan

Solution 5

We can split up this long sum into groups of four integers. Finding the first few sums, we have that $1+2+3-4=2$, $5+6+7-8=10$, and $9+10+11-12=18$. Notice that this is an increasing arithmetic sequence, with a common difference of 8. We can find the sum of the arithmetic sequence by finding the average of the first and last terms, and then multiplying by the number of terms in the sequence. The first term is $1+2+3-4$, or 2, the last term is $197+198+199-200$, or 394, and there are $200 \div 4$ or 50 terms. So, we have that the sum of the sequence is $\frac{(394+2)\cdot 50}{2}$, or $\boxed{\text{(B) }9900}$. ~Arctic_Bunny

Solution 3

Taking the average of the first and last terms, -10 and 14, we have that the mean of the set is 2. There are 5 values in each row, column or diagonal, so the value of the common sum is $5 \cdot 2$, or $\boxed{\textbf{(C) } 10}$. ~Arctic_Bunny, edited by KINGLOGIC

Problem9

A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of $N?$

$\textbf{(A) } 9 \qquad \textbf{(B) } 18 \qquad \textbf{(C) } 27 \qquad \textbf{(D) } 36 \qquad \textbf{(E) } 77$

Solution

The least common multiple of 7 and 11 is 77. Therefore, there must be 77 adults and 77 children. The total number of benches is $\frac{77}{7}+\frac{77}{11}=11+7=\boxed{\text{(B) }18}$.

Solution 2

This is similar to Solution 1, with the same basic idea, but we don't need to calculate the LCM. Since both $7$and 11 are prime, their LCM must be their product. So the answer would be $7+11=\boxed{\text{(B) }18}$. ~Baolan

Problem10

Seven cubes, whose volumes are 1, 8, 27, 64, 125, 216, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

$\textbf{(A) } 644 \qquad \textbf{(B) } 658 \qquad \textbf{(C) } 664 \qquad \textbf{(D) } 720 \qquad \textbf{(E) } 749$

Solution 1

The volume of each cube follows the pattern of n^3 ascending, for n is between 1 and 7.

We see that the total surface area can be comprised of three parts: the sides of the cubes, the tops of the cubes, and the bottom of the $7 \times 7 \times 7$ cube (which is just $7 \times 7 = 49$). The sides areas can be measured as the sum $4\sum_{n=0}^{7} n^2$, giving us 560. Structurally, if we examine the tower from the top, we see that it really just forms a 7×7 square of area 49. Therefore, we can say that the total surface area is $560 + 49 + 49 = \boxed{\textbf{(B)}\ 658}$. Alternatively, for the area of the tops, we could have found the sum $\sum_{n=0}^{6}((n+1)^2 - n^2)$, giving us 49 as well.

~ciceronii

Solution 2

It can quickly be seen that the side lengths of the cubes are the integers from 1 to 7, inclusive.

First, we will calculate the total surface area of the cubes, ignoring overlap. This value
is

$$6(1^2+2^2+\cdots+7^2) = 6\sum_{n=1}^{7} n^2 = 6\left(\frac{7(7+1)(2\cdot 7+1)}{6}\right) = 7\cdot 8\cdot 15 = 840$$

. Then, we need to subtract out the overlapped parts of the cubes. Between each consecutive pair of cubes, one of the smaller cube's faces is completely covered, along with an equal area of one of the larger cube's faces. The total area of the overlapped parts of the cubes is thus equal to $2\sum_{n=1}^{6} n^2 = 182$. Subtracting the overlapped surface area from the total surface area, we get $840 - 182 = \boxed{\textbf{(B)}\ 658}$. ~emerald_block

Solution 3 (a bit more tedious than others)

It can be seen that the side lengths of the cubes using cube roots are all integers from 1 to 7, inclusive.

Only the cubes with side length 1 and 7 have 5 faces in the surface area and the rest have 3. Also, since the

cubes are stacked, we have to find the difference between

each n^2 and $(n-1)^2$ side length as n ranges from 7 to

2.

We then come up with
this:
$5(49)+13+4(36)+11+4(25)+9+4(16)+7+4(9)+5+4(4)+3+5(1)$
.

We then add all of this and get $\boxed{\textbf{(B) } 658}$.

Problem 11

What is the median of the following list of 4040 numbers$?$

$$1, 2, 3, ..., 2020, 1^2, 2^2, 3^2, ..., 2020^2$$

$\textbf{(A) } 1974.5 \qquad \textbf{(B) } 1975.5 \qquad \textbf{(C) } 1976.5 \qquad \textbf{(D) } 1977.5 \qquad \textbf{(E) } 1978.5$

Solution 1

We can see that 44^2 is less than 2020. Therefore, there are 1976 of the 4040 numbers after 2020. Also, there are 2064 numbers that are under and equal to 2020. Since 44^2 is equal to 1936, it, with the other squares, will shift our median's placement up 44. We can find that the median of the whole set is 2020.5, and $2020.5-44$ gives us 1976.5. Our answer is $\boxed{\textbf{(C) } 1976.5}$.

~aryam

Solution 2

As we are trying to find the median of a 4040-term set, we must find the average of the 2020th and 2021st terms.

Since $45^2=2025$ is slightly greater than 2020, we know that the 44 perfect squares 1^2 through 44^2 are less than 2020, and the rest are greater. Thus, from the number 1 to the number 2020, there are $2020 + 44 = 2064$ terms. Since 44^2 is $44 + 45 = 89$ less than $45^2 = 2025$ and 84 less than 2020, we will only need to consider the perfect square terms going down from the 2064th term, 2020, after going down 84 terms. Since the 2020th and 2021st terms are only 44 and 43 terms away from the 2064th term, we can simply subtract 44 from 2020 and 43 from 2020 to get the two terms, which are 1976 and 1977. Averaging the two, we get $\boxed{\textbf{(C) } 1976.5}$. ~emerald_block

Solution 3

We want to know the 2020th term and the 2021th term to get the median.

We know that $44^2 = 1936$

So numbers $1^2, 2^2, \ldots, 44^2$ are in between 1 to 1936.

So the sum of 44 and 1936 will result in 1980, which means that 1936 is the 1980th number.

Also, notice that $45^2 = 2025$, which is larger than 2021.

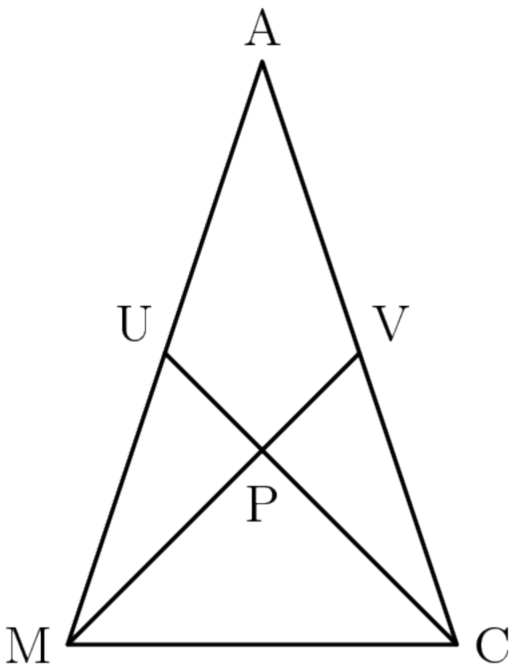
Then the 2020th term will be $1936 + 40 = 1976$, and similarly the 2021th term will be 1977.

Solving for the median of the two numbers, we get $\boxed{\textbf{(C) } 1976.5}$

Problem12

Triangle AMC is isoceles with $AM = AC$.

Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



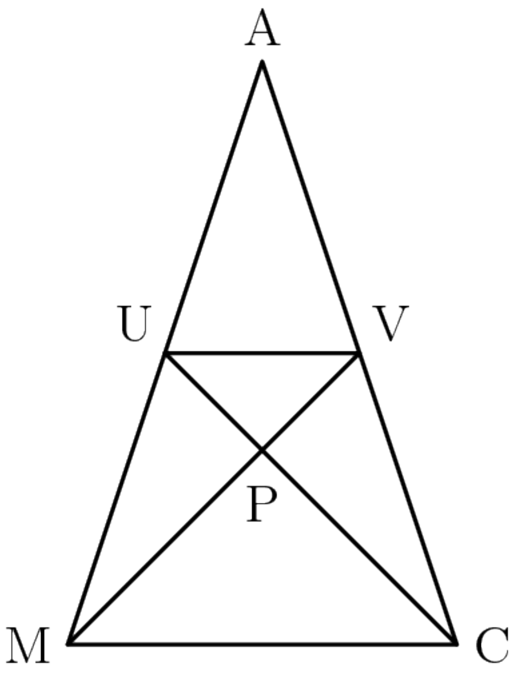
$\textbf{(A)}\ 48 \qquad \textbf{(B)}\ 72 \qquad \textbf{(C)}\ 96 \qquad \textbf{(D)}\ 144 \qquad \textbf{(E)}\ 192$

Solution 1

Since quadrilateral $UVCM$ has perpendicular diagonals, its area can be found as half of the product of the length of the diagonals. Also note that $\triangle AUV$ has $\frac{1}{4}$ the area of triangle AMC by similarity, so $[UVCM] = \frac{3}{4}\cdot [AMC].$ Thus, $\frac{1}{2} \cdot 12 \cdot 12 = \frac{3}{4}\cdot [AMC]$

$$72 = \frac{3}{4}\cdot [AMC] [AMC] = 96 \rightarrow \boxed{\textbf{(C)}}.$$

Solution 2 (Trapezoid)



We know that $\triangle AUV \sim \triangle AMC$, and since the ratios of its sides are $\frac{1}{2}$, the ratio of of their areas is $(\frac{1}{2})^2 = \frac{1}{4}$.

If $\triangle AUV$ is $\frac{1}{4}$ the area of $\triangle AMC$, then trapezoid $MUVC$ is $\frac{3}{4}$ the area of $\triangle AMC$.

Let's call the intersection of $\overline{UC}$ and $\overline{MV}$ P. Let $\overline{UP} = x$.

Then $\overline{PC} = 12 - x$. Since $\overline{UC} \perp \overline{MV}$, $\overline{UP}$ and $\overline{CP}$ are heights of triangles $\triangle MUV$ and $\triangle MCV$, respectively. Both of these triangles have base 12.

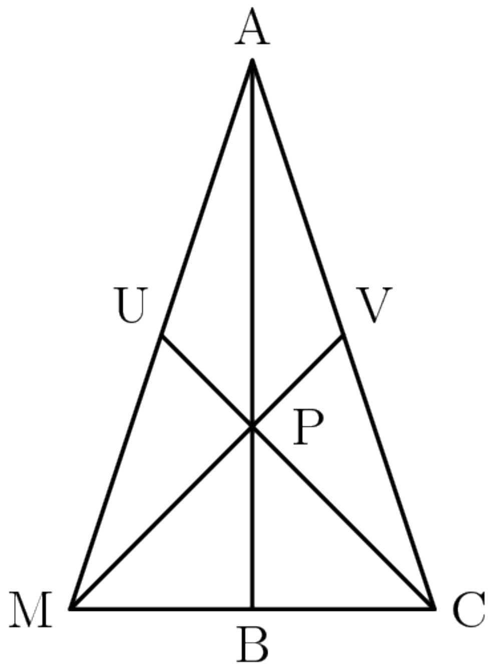
Area of $\triangle MUV = \dfrac{x \cdot 12}{2} = 6x$

Area of $\triangle MCV = \dfrac{(12 - x) \cdot 12}{2} = 72 - 6x$

Adding these two gives us the area of trapezoid $MUVC$, which is $6x + (72 - 6x) = 72$.

This is $\frac{3}{4}$ of the triangle, so the area of the triangle is $\frac{4}{3} \cdot 72 = \boxed{\textbf{(C) } 96}$ ~quacker88, diagram by programjames1

Solution 3 (Medians)



Draw median $\overline{AB}$.

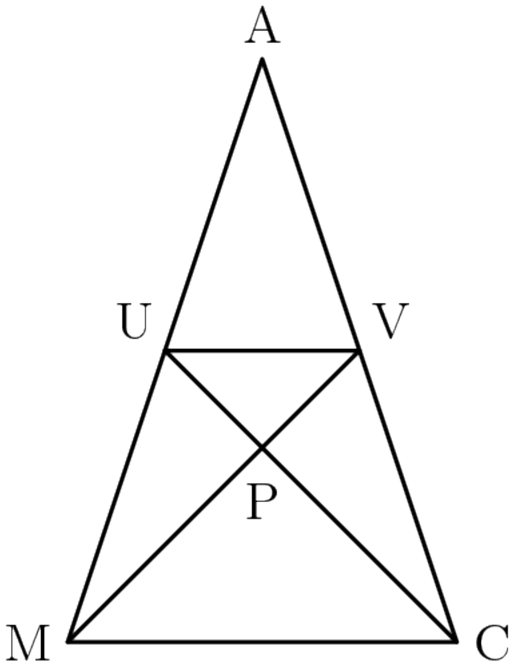
Since we know that all medians of a triangle intersect at the incenter, we know that $\overline{AB}$ passes through point P. We also know that medians of a triangle divide each other into segments of ratio $2:1$. Knowing this, we can see that $\overline{PC}:\overline{UP} = 2:1$, and since the two segments sum to 12, $\overline{PC}$ and $\overline{UP}$ are 8 and 4, respectively.

Finally knowing that the medians divide the triangle into 6 sections of equal area, finding the area of $\triangle PUM$ is enough. $\overline{PC} = \overline{MP} = 8$.

The area of $\triangle PUM = \frac{4 \cdot 8}{2} = 16$. Multiplying this by 6 gives us $6 \cdot 16 = \boxed{\textbf{(C) } 96}$

~quacker88

Solution 4 (Triangles)



We know that $AU = UM, AV = VC$, so $UV = \frac{1}{2}MC$.

As $\angle UPM = \angle VPC = 90$, we can see that $\triangle UPM \cong \triangle VPC$ and $\triangle UVP \sim \triangle MPC$ with a side ratio of $1 : 2$.

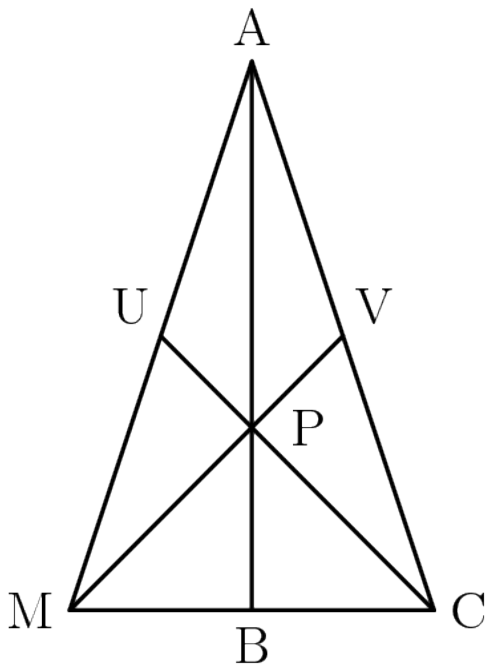
So $UP = VP = 4, MP = PC = 8$.

With that, we can see that $[\triangle UPM] = 16$, and the area of trapezoid $MUVC$ is 72.

As said in solution 1, $[\triangle AMC] = 72/\frac{3}{4} = \boxed{\textbf{(C) } 96}$.

-QuadraticFunctions, solution 1 by ???

Solution 5 (Only Pythagorean Theorem)



Let AB be the height. Since medians divide each other into a $2:1$ ratio, and the medians have length 12, we have $PC = MP = 8$ and $UP = UV = 4$. From right triangle $\triangle MUP$, $MU^2 = MP^2 + UP^2 = 8^2 + 4^2 = 80,$ so $MU = \sqrt{80} = 4\sqrt{5}$. Since CU is a median, $AM = 8\sqrt{5}$. From right triangle $\triangle MPC$, $MC^2 = MP^2 + PC^2 = 8^2 + 8^2 = 128,$ which implies $MC = \sqrt{128} = 8\sqrt{2}$. By symmetry $MB = \dfrac{8\sqrt{2}}{2} = 4\sqrt{2}$.

Applying the Pythagorean Theorem to right triangle $\triangle MAB$ gives

$$AB^2 = AM^2 - MB^2 = 8\sqrt{5}^2 - 4\sqrt{2}^2 = 288,$$

so $AB = \sqrt{288} = 12\sqrt{2}$. Then the area of $\triangle AMC$ is

$$\frac{AB \cdot MC}{2} = \frac{8\sqrt{2} \cdot 12\sqrt{2}}{2} = \frac{96 \cdot 2}{2} = \boxed{\textbf{(C) } 96}$$

Solution 6 (Drawing)

(NOT recommended) Transfer the given diagram, which happens to be to scale, onto a piece of a graph paper. Counting the boxes should give a reliable result since the answer choices are relatively far apart. -Lingjun

Solution 7

Given a triangle with perpendicular medians with lengths x and y, the area will be $\frac{2xy}{3} = \boxed{\textbf{(C)}\ 96}$.

Solution 8 (Fastest)

Connect the line segment UV and it's easy to see quadrilateral $UVMC$ has an area of the product of its diagonals divided by 2 which is 72. Now, solving for triangle AUV could be an option, but the drawing shows the area of AUV will be less than the quadrilateral meaning the the area of AMC is less than $72 * 2$ but greater than 72, leaving only one possible answer choice, $\boxed{\textbf{(C)}\ 96}$.

Problem 13

A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0,0), (0,4), (4,4),$ and $(4,0)$. What is the probability that the sequence of jumps ends on a vertical side of the square$?$

$\textbf{(A)}\ \frac{1}{2}\qquad\textbf{(B)}\ \frac{5}{8}\qquad\textbf{(C)}\ \frac{2}{3}\qquad\textbf{(D)}\ \frac{3}{4}\qquad\textbf{(E)}\ \frac{7}{8}$

Solution

Drawing out the square, it's easy to see that if the frog goes to the left, it will immediately hit a vertical end of the square. Therefore, the probability of this happening is $\frac{1}{4} * 1 = \frac{1}{4}$. If the frog goes to the right, it will be in the center of the square at $(2,2)$, and by symmetry (since the frog is equidistant from all sides of the square), the chance it will hit a vertical side of a square is $\frac{1}{2}$. The probability of this happening is $\frac{1}{4} * \frac{1}{2} = \frac{1}{8}$.

If the frog goes either up or down, it will hit a line of symmetry along the corner it is closest to and furthest to, and again, is equidistant relating to the two closer sides and also equidistant relating the two further sides. The probability for it to hit a vertical wall is $\frac{1}{2}$. Because there's a $\frac{1}{2}$ chance of the frog going up and down, the total probability for this case is $\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$ and summing up all the cases, $\frac{1}{4} + \frac{1}{8} + \frac{1}{4} = \frac{5}{8} \implies \boxed{\textbf{(B) } \frac{5}{8}}$.

Solution 2

Let's say we have our four by four grid and we work this out by casework. A is where the frog is, while B and C are possible locations for his second jump, while O is everything else. If we land on a C, we have reached the vertical side. However, if we land on a B, we can see that there is an equal chance of reaching the horizontal or vertical side, since we are symmetrically between them. So we have the probability of landing on a C is 1/4, while B is 3/4. Since C means that we have "succeeded", while B means that we have a half chance, we compute $1 \cdot C + \frac{1}{2} \cdot B$.

$1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{31}{44} + \frac{3}{8}$We get $\frac{5}{8}$, or BO O O O OO B O O O C A B O OO B O O OO O O O O-yeskay

Solution 3

If the frog is on one of the 2 diagonals, the chance of landing on vertical or horizontal each becomes $\frac{1}{2}$. Since it starts on $(1,2)$, there is a $\frac{3}{4}$ chance (up, down, or right) it will reach a diagonal on the first jump and $\frac{1}{4}$ chance (left) it will reach the vertical side. The probablity of landing on a vertical is $\frac{1}{4}+\frac{3}{4}*\frac{1}{2}=\boxed{\textbf{(B)}\frac{5}{8}.}$ - Lingjun.

Solution 4 (Complete States)

Let $P_{(x,y)}$ denote the probability of the frog's sequence of jumps ends with it hitting a vertical edge when it is at (x,y). Note that $P_{(1,2)}=P_{(3,2)}$ by reflective symmetry over the line $x=2$.

Similarly, $P_{(1,1)}=P_{(1,3)}=P_{(3,1)}=P_{(3,3)}$, and $P_{(2,1)}=P_{(2,3)}$.

Now we create equations for the probabilities at each of these points/states by considering the probability of going either up, down, left, or right from that point: $P_{(1,2)}=\frac{1}{4}+\frac{1}{2}P_{(1,1)}+\frac{1}{4}P_{(2,2)}$

$$P_{(2,2)}=\frac{1}{2}P_{(1,2)}+\frac{1}{2}P_{(2,1)}$$

$$P_{(1,1)}=\frac{1}{4}+\frac{1}{4}P_{(1,2)}+\frac{1}{4}P_{(2,1)}$$

$P_{(2,1)}=\frac{1}{2}P_{(1,1)}+\frac{1}{4}P_{(2,2)}$We have a system of 4 equations in 4 variables, so we can solve for each of these probabilities. Plugging the second equation into the fourth equation

gives $P_{(2,1)}=\frac{1}{2}P_{(1,1)}+\frac{1}{4}\left(\frac{1}{2}P_{(1,2)}+\frac{1}{2}P_{(2,1)}\right)$

$$P_{(2,1)}=\frac{8}{7}\left(\frac{1}{2}P_{(1,1)}+\frac{1}{8}P_{(1,2)}\right)=\frac{4}{7}P_{(1,1)}+\frac{1}{7}P_{(1,2)}$$

Plugging in the third equation into this gives

$$P_{(2,1)} = \frac{4}{7}\left(\frac{1}{4} + \frac{1}{4}P_{(1,2)} + \frac{1}{4}P_{(2,1)}\right) + \frac{1}{7}P_{(1,2)}$$

$$P_{(2,1)} = \frac{7}{6}\left(\frac{1}{7} + \frac{2}{7}P_{(1,2)}\right) = \frac{1}{6} + \frac{1}{3}P_{(1,2)} \ (*)$$

Next, plugging in the second and third equation into the first equation yields

$$P_{(1,2)} = \frac{1}{4}+\frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}P_{(1,2)} + \frac{1}{4}P_{(2,1)}\right)+\frac{1}{4}\left(\frac{1}{2}P_{(1,2)} + \frac{1}{2}P_{(2,1)}\right)$$

$$P_{(1,2)} = \frac{3}{8} + \frac{1}{4}P_{(1,2)} + \frac{1}{4}P_{(2,1)}$$

Now plugging in (*) into this, we get

$$P_{(1,2)} = \frac{3}{8} + \frac{1}{4}P_{(1,2)} + \frac{1}{4}\left(\frac{1}{6} + \frac{1}{3}P_{(1,2)}\right)$$

$$P_{(1,2)} = \frac{3}{2} \cdot \frac{5}{12} = \boxed{\textbf{(B) } \frac{5}{8}}$$

Problem14

Real numbers x and y satisfy $x + y = 4$ and $x \cdot y = -2$. What is the value of

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y?$$

$\textbf{(A)}\ 360 \qquad \textbf{(B)}\ 400 \qquad \textbf{(C)}\ 420 \qquad \textbf{(D)}\ 440 \qquad \textbf{(E)}\ 480$

Solution

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y = x + \frac{x^3}{y^2} + y + \frac{y^3}{x^2} = \frac{x^3}{x^2} + \frac{y^3}{x^2} + \frac{y^3}{y^2} + \frac{x^3}{y^2}$$

Continuing to combine

$$\frac{x^3+y^3}{x^2}+\frac{x^3+y^3}{y^2} = \frac{(x^2+y^2)(x^3+y^3)}{x^2y^2} = \frac{(x^2+y^2)(x+y)(x^2-xy+y^2)}{x^2y^2}$$

From the givens, it can be concluded that $x^2y^2 = 4$.

Also, $(x+y)^2 = x^2 + 2xy + y^2 = 16$This means that $x^2+y^2=20$. Substituting this information into $\frac{(x^2+y^2)(x+y)(x^2-xy+y^2)}{x^2y^2}$, we have $\frac{(20)(4)(22)}{4} = 20 \cdot 22 = \boxed{\textbf{(D) } 440}$. ~PCChess

Solution 2

As above, we need to calculate $\frac{(x^2+y^2)(x^3+y^3)}{x^2y^2}$. Note that $x, y,$ are the roots of x^2-4x-2 and so $x^3 = 4x^2+2x$ and $y^3 = 4y^2+2y$. Thus

$x^3+y^3 = 4(x^2+y^2)+2(x+y) = 4(20)+2(4) = 88$

where $x^2+y^2=20$ and $x^2y^2=4$ as in the previous solution. Thus the answer is $\frac{(20)(88)}{4} = \boxed{\textbf{(D) } 440}$.

- $\textbf{Emathmaster}$

Solution 3

Note that $(x^3+y^3)(\frac{1}{y^2}+\frac{1}{x^2}) = x+\frac{x^3}{y^2}+\frac{y^3}{x^2}+y.$ Now, we only need to find the values of $x^3+y^3$and $\frac{1}{y^2}+\frac{1}{x^2}.$

Recall that $x^3+y^3=(x+y)(x^2-xy+y^2),$ and that $x^2-xy+y^2=(x+y)^2-3xy.$ We are able to solve the second equation, and doing so gets us $4^2-3(-2)=22.$ Plugging this into the first equation, we get $x^3+y^3=4(22)=88.$

In order to find the value of $\frac{1}{y^2}+\frac{1}{x^2},$ we find a common denominator so that we can add them together. This gets us $\frac{x^2}{x^2y^2}+\frac{y^2}{x^2y^2}=\frac{x^2+y^2}{(xy)^2}.$ Recalling that $x^2+y^2=(x+y)^2-2xy$ and solving this equation, we get $4^2-2(-2)=20.$ Plugging this into the first equation, we get $\frac{1}{y^2}+\frac{1}{x^2}=\frac{20}{(-2)^2}=5.$

Solving the original equation, we get $x+\frac{x^3}{y^2}+\frac{y^3}{x^2}+y=(88)(5)=\boxed{\textbf{(D) }440}.$ ~emerald_block

Solution 4 (Bashing)

This is basically bashing using Vieta's formulas to find x and y (which I highly do not recommend, I only wrote this solution for fun).

We use Vieta's to find a quadratic relating x and y. We set x and y to be the roots of the quadratic $Q(n)=n^2-4n-2$ (because $x+y=4$, and $xy=-2$). We can solve the quadratic to get the roots $2+\sqrt{6}$ and $2-\sqrt{6}$. x and y are "interchangeable", meaning that it doesn't matter which solution x or $y$is, because it'll return the same result when plugged in. So we plug in $2+\sqrt{6}$ for x and $2-\sqrt{6}$ and get $\boxed{\textbf{(D) }440}$ as our answer.

~Baolan

Solution 5 (Bashing Part 2)

This usually wouldn't work for most problems like this, but we're lucky that we can quickly expand and factor this expression in this question.

We first change the original expression to $4+\frac{x^5+y^5}{x^2y^2}$, because $x+y=4$. This is equal to $4+\frac{(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)}{4}=x^4+y^4-x^3y-xy^3+8$. We can factor and reduce x^4+y^4 to $(x^2+y^2)^2-2x^2y^2=((x+y)^2-2xy)^2-8=400-8=392$. Now our expression is just $400-(x^3y+xy^3)$. We factor x^3y+xy^3 to get $(xy)(x^2+y^2)=-40$. So the answer would be $400-(-40)=\boxed{(\textbf{D})440}$.

Solution 6 (Complete Binomial Theorem)

We first simplify the expression to $x+y+\frac{x^5+y^5}{x^2y^2}$. Then, we can solve for x and y given the system of equations in the problem.

Since $xy=-2$, we can substitute $\frac{-2}{x}$ for y. Thus, this becomes the equation $x-\frac{2}{x}=4.$ Multiplying both sides by x, we obtain $x^2-2=4x$, or $x^2-4x-2=0$. By the quadratic formula we obtain $x=2\pm\sqrt{6}$. We also easily find that given $x = 2 \pm \sqrt{6}$, y equals the conjugate of x. Thus, plugging our values in for x and y, our expression equals $4 + \frac{(2-\sqrt{6})^5 + (2+\sqrt{6})^5}{(2-\sqrt{6})^2(2+\sqrt{6})^2}$

By the binomial theorem, we observe that every second terms of the expansions x^5 and y^5 will cancel out (since a positive plus a negative of the same absolute value equals zero). We also observe that the other terms not canceling out are doubled when summing the expansions of $x^5 + y^5$. Thus, our expression equals $4 + \frac{2(2^5 + \binom{5}{2}2^3 \times 6 + \binom{5}{4}2 \times 36)}{(2-\sqrt{6})^2(2+\sqrt{6})^2}$. which equals $4 + \frac{2(872)}{4}$ which equals $\boxed{\textbf{(D)} 440}$.

Problem15

A positive integer divisor of $12!$ is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 12 \qquad \textbf{(D)}\ 18 \qquad \textbf{(E)}\ 23$

Solution

The prime factorization of $12!$ is $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$. This yields a total of $11 \cdot 6 \cdot 3 \cdot 2 \cdot 2$ divisors of $12!$.In order to produce a perfect square divisor, there must be an even exponent for each number in the prime factorization. Note that 7 and 11 can not be in the prime factorization of a perfect square because there is only one of each in $12!$. Thus, there are $6 \cdot 3 \cdot 2$ perfect squares. (For 2, you can have 0, 2, 4, 6, 8, or $1$0 2s, etc.) The probability that the divisor chosen is a perfect square is

$$\frac{6 \cdot 3 \cdot 2}{11 \cdot 6 \cdot 3 \cdot 2 \cdot 2} = \frac{1}{22} \implies \frac{m}{n} = \frac{1}{22} \implies m + n = 1 + 22 = \boxed{\textbf{(E)}\ 23}$$

Problem16

A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0), (2020,0), (2020,2020)$, and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

$\textbf{(A) } 0.3 \qquad \textbf{(B) } 0.4 \qquad \textbf{(C) } 0.5 \qquad \textbf{(D) } 0.6 \qquad \textbf{(E) } 0.7$

Solution 1

Diagram

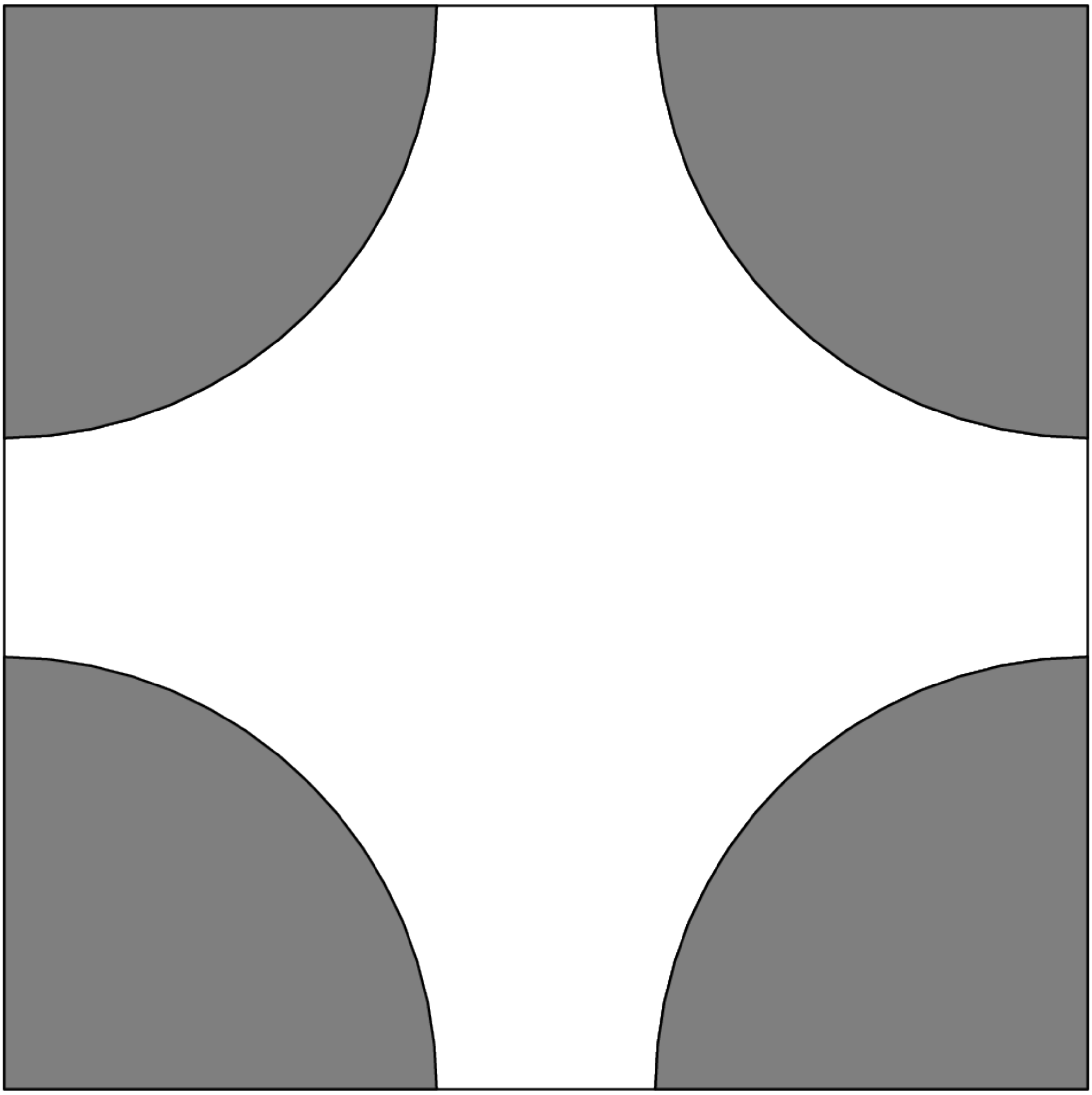


Diagram by MathandSki Using Asymptote

Note: The diagram represents each unit square of the given $2020 * 2020$ square.

Solution

We consider an individual one-by-one block.

If we draw a quarter of a circle from each corner (where the lattice points are located), each with radius d, the area covered by the circles should be 0.5. Because of this, and the fact that there are four circles, we write

$$4 * \frac{1}{4} * \pi d^2 = \frac{1}{2}$$

Solving for d, we obtain $d = \frac{1}{\sqrt{2\pi}}$, where with $\pi \approx 3$, we get $d = \frac{1}{\sqrt{6}}$, and from here, we simplify and see that $d \approx 0.4 \implies \boxed{\textbf{(B) } 0.4.}$ ~Crypthes

$\textbf{Note:}$ To be more rigorous, note that $d < 0.5$ since if $d \geq 0.5$ then clearly the probability is greater than $\frac{1}{2}$. This would make sure the above solution works, as if $d \geq 0.5$ there is overlap with the quartercircles. - $\textbf{Emathmaster}$

Solution 2

As in the previous solution, we obtain the equation $4 * \frac{1}{4} * \pi d^2 = \frac{1}{2}$, which simplifies to $\pi d^2 = \frac{1}{2} = 0.5$. Since π is slightly more than 3, d^2 is slightly less than $\frac{0.5}{3} = 0.1\overline{6}$. We notice that $0.1\overline{6}$ is slightly more than $0.4^2 = 0.16$, so d is roughly $\boxed{\textbf{(B) } 0.4}$. ~[emerald_block]

Solution 3 (Estimating)

As above, we find that we need to estimate $d = \frac{1}{\sqrt{2\pi}}$.

Note that we can approximate $2\pi \approx 6.28 \approx 6.25$ and so $\frac{1}{\sqrt{2\pi}} \approx \frac{1}{\sqrt{6.25}} = \frac{1}{2.5} = 0.4$.

And so our answer is $\boxed{\textbf{(B) } 0.4}$.

Problem 17

Define $P(x) = (x - 1^2)(x - 2^2) \cdots (x - 100^2)$. How many integers n are there such that $P(n) \leq 0$?

$\textbf{(A) } 4900 \qquad \textbf{(B) } 4950 \qquad \textbf{(C) } 5000 \qquad \textbf{(D) } 5050 \qquad \textbf{(E) } 5100$

Solution 1

Notice that $P(x)$ is a product of many integers. We either need one factor to be 0 or an odd number of negative factors.

Case 1: There are 100 integers n for which $P(x) = 0$

Case 2: For there to be an odd number of negative factors, n must be between an odd number squared and an even number squared. This means that there are $2 + 6 + \cdots + 198$ total possible values of n. Simplifying, there are 5000 possible numbers.

Summing, there are $\boxed{\textbf{(E) } 5100}$ total possible values of n. ~PCChess

Solution 2

Notice that $P(x)$ is nonpositive when x is between 100^2 and 99^2, 98^2 and 97^2 $\ldots$, 2^2 and 1^2 (inclusive), which means that the amount of values equals $((100+99)(100-99)+1)+((98+97)(98-97)+1)+\ldots+((2+1)(2-1)+1)$.

This reduces to

$$200+196+192+\ldots+4=4(1+2+\ldots+50)=4\frac{50\cdot 51}{2}=\boxed{\textbf{(E) }5100}$$

~Zeric

Solution 3 (end behavior)

We know that $P(x)$ is a 100-degree function with a positive leading coefficient. That is, $P(x)=x^{100}+ax^{99}+bx^{98}+...+(\text{constant})$.

Since the degree of $P(x)$ is even, its end behaviors match. And since the leading coefficient is positive, we know that both ends approach ∞ as x goes in either direction.

$$\lim_{x\to-\infty}P(x)=\lim_{x\to\infty}P(x)=\infty$$

So the first time $P(x)$ is going to be negative is when it intersects the x-axis at an x-intercept and it's going to dip below. This happens at 1^2, which is the smallest intercept.

However, when it hits the next intercept, it's going to go back up again into positive territory, we know this happens at 2^2. And when it hits 3^2, it's going to dip back into negative territory. Clearly, this is going to continue to snake around the intercepts until 100^2.

To get the amount of integers below and/or on the x-axis, we simply need to count the integers. For example, the amount of integers in between the $[1^2,2^2]$ interval we got earlier, we subtract and add one. $(2^2-1^2+1)=4$ integers, so there are four integers in this interval that produce a negative result.

Doing this with all of the other intervals, we have

$$(2^2-1^2+1)+(4^2-3^2+1)+...+(100^2-99^2+1).$$

Proceed with Solution 2. ~quacker88

Problem18

Let (a,b,c,d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $0,1,2,3.$ For how many such quadruples is it true that $a\cdot d-b\cdot c$ is odd? (For example, $(0,3,1,1)$ is one such quadruple, because $0\cdot 1-3\cdot 1=-3$ is odd.)

$$\textbf{(A) } 48 \qquad \textbf{(B) } 64 \qquad \textbf{(C) } 96 \qquad \textbf{(D) } 128 \qquad \textbf{(E) } 192$$

Solution

Solution 1 (Parity)

In order for $a\cdot d-b\cdot c$ to be odd, consider parity. We must have (even)-(odd) or (odd)-(even). There are $2\cdot 4+2\cdot 2=12$ ways to pick numbers to obtain an even product. There are $2\cdot 2=4$ ways to obtain an odd product. Therefore, the total amount of ways to make $a\cdot d-b\cdot c$ odd is $2\cdot(12\cdot 4)=\boxed{\textbf{(C) } 96}$.

-Midnight

Solution 2 (Basically Solution 1 but more in depth)

Consider parity. We need exactly one term to be odd, one term to be even. Because of symmetry, we can set ad to be odd and bc to be even, then multiply by $2.$ If ad is odd, both a and d must be odd, therefore there are $2\cdot 2=4$ possibilities for $ad.$ Consider $bc.$ Let us say that b is even. Then there are $2\cdot 4=8$possibilities for $bc.$ However, b can be odd, in which case we have $2\cdot 2=4$ more possibilities for $bc.$ Thus there are 12 ways for us to choose bc and 4 ways for us to choose $ad.$ Therefore, also considering symmetry, we have $2*4*12=96$ total values of $ad-bc.$ (C)

Solution 3 (Complementary Counting)

There are 4 ways to choose any number independently and 2 ways to choose any odd number independently. To get an even products, we count: $\mathrm{P(any\ number)} \cdot \mathrm{P(any\ number)} - \mathrm{P(odd)} \cdot \mathrm{P(odd)}$, which is $4 \cdot 4 - 2 \cdot 2 = 12$. The number of ways to get an odd product can be counted like so: $\mathrm{P(odd)} \cdot \mathrm{P(odd)}$, which is $2 \cdot 2$, or 4. So, for one product to be odd the other to be even: $2 \cdot 4 \cdot 12 = \boxed{(C) 96}$(order matters). ~ Anonymous and Arctic_Bunny

Solution 4 (Solution 3 but more in depth)

We use complementary counting: If the difference is even, then we can subtract those cases. There are a total of $4^4=256$ cases.

For an even difference, we have (even)-(even) or (odd-odd).

From Solution 3:

"There are 4 ways to choose any number independently and 2 ways to choose any odd number independently. even products:(number)*(number)-(odd)*(odd): $4 \cdot 4 - 2 \cdot 2 = 12$. odd products: (odd)*(odd): $2 \cdot 2 = 4$."

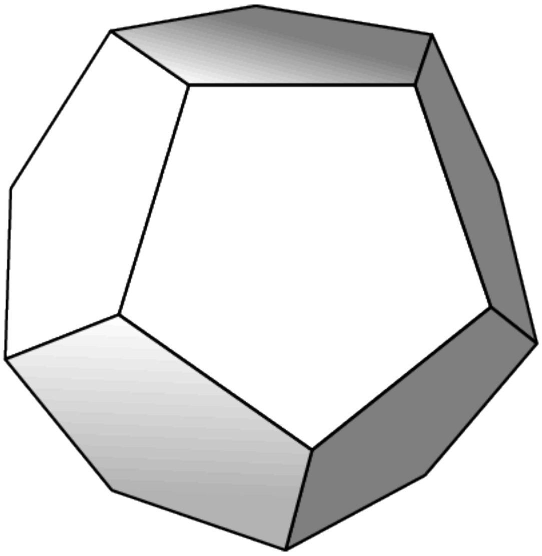
With this, we easily calculate $256-12^2-4^2=\textbf{(C) 96}$.

Problem19

As shown in the figure below, a regular dodecahedron (the polyhedron consisting of 12 congruent regular pentagonal faces) floats in space with two horizontal faces. Note that there is a ring of five slanted faces adjacent to the top face, and a ring of five slanted faces adjacent to the bottom face. How many ways are there to move from the top face to the bottom face via a sequence of adjacent faces so that each face is visited at most once and moves are not permitted from the bottom ring to the top ring?

$\textbf{(A)}\ 125 \qquad \textbf{(B)}\ 250 \qquad \textbf{(C)}\ 405 \qquad \textbf{(D)}\ 640 \qquad \textbf{(E)}\ 810$

Diagram



Solution 1

Since we start at the top face and end at the bottom face without moving from the lower ring to the upper ring or revisiting a face, our journey must consist of the top face, a series of faces in the upper ring, a series of faces in the lower ring, and the bottom face, in that order.

We have 5 choices for which face we visit first on the top ring. From there, we have 9 choices for how far around the top ring we go before moving down: $1, 2, 3,$ or 4 faces around clockwise, $1, 2, 3,$ or 4 faces around counterclockwise, or immediately going down to the lower ring without visiting any other faces in the upper ring.

We then have 2 choices for which lower ring face to visit first (since every upper-ring face is adjacent to exactly 2 lower-ring faces) and then once again 9 choices for how to travel around the lower ring. We then proceed to the bottom face, completing the trip.

Multiplying together all the numbers of choices we have, we get $5 \cdot 9 \cdot 2 \cdot 9 = \boxed{\textbf{(E) } 810}$.

Solution 2

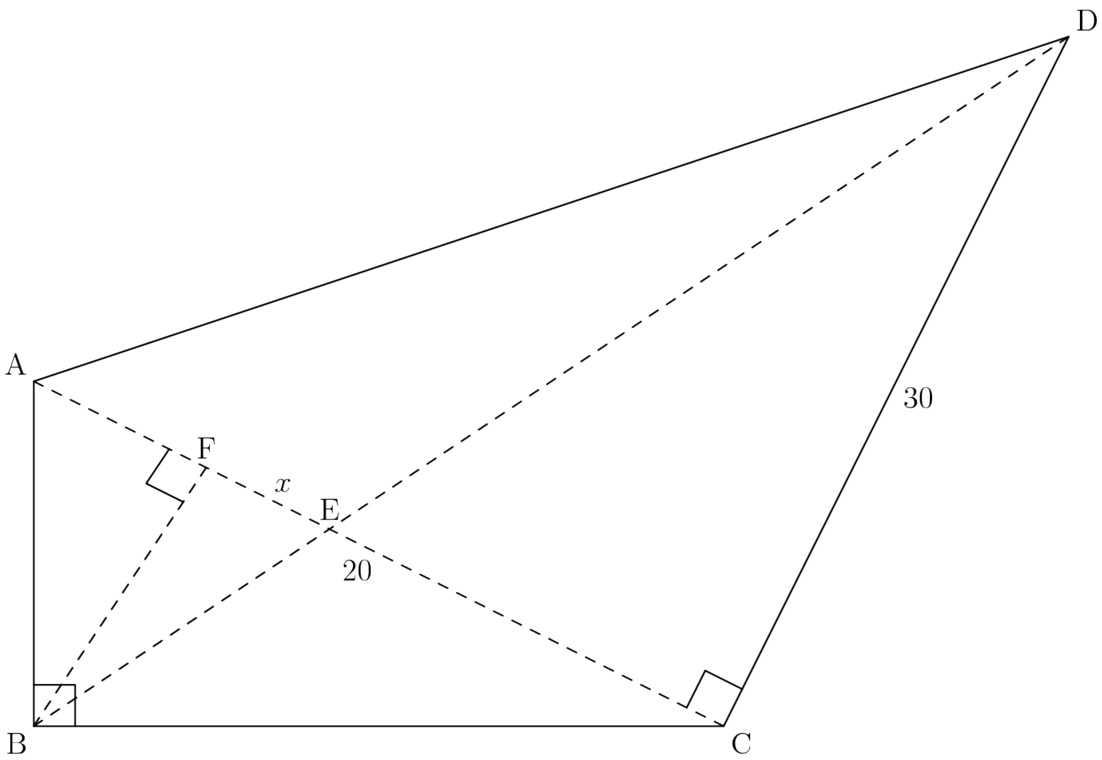
Swap the faces as vertices and the vertices as faces. Then, this problem is the same as 2016 AIME I #3which had an answer of $\boxed{\textbf{(E) } 810}$. - $\textbf{Emathmaster}$

Problem20

Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ, AC = 20,$ and $CD = 30.$ Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point $E,$ and $AE = 5.$ What is the area of quadrilateral $ABCD?$

$\textbf{(A) } 330 \qquad\textbf{(B) } 340 \qquad\textbf{(C) } 350 \qquad\textbf{(D) } 360 \qquad\textbf{(E) } 370$

Solution 1 (Just Drop An Altitude)



It's crucial to draw a good diagram for this one.

Since $AC = 20$ and $CD = 30$, we get $[ACD] = 300$. Now we need to find $[ABC]$ to get the area of the whole quadrilateral. Drop an altitude from B to AC and call the point of intersection F. Let $FE = x$. Since $AE = 5$, then $AF = 5 - x$. By dropping this altitude, we can also see two similar triangles, BFE and DCE. Since EC is $20 - 5 = 15$, and $DC = 30$, we get that $BF = 2x$. Now, if we redraw another diagram just of ABC, we get that $(2x)^2 = (5-x)(15+x)$. Now expanding, simplifying, and dividing by the GCF, we get $x^2+2x-15=0$. This factors to $(x+5)(x-3)$. Since lengths cannot be negative, $x=3$.

Since $x=3$, $[ABC]=60$.

So

$$[ABCD] = [ACD] + [ABC] = 300 + 60 = \boxed{\textbf{(D) } 360}$$

(I'm very sorry if you're a visual learner but now you have a diagram by ciceronii)

~ Solution by Ultraman

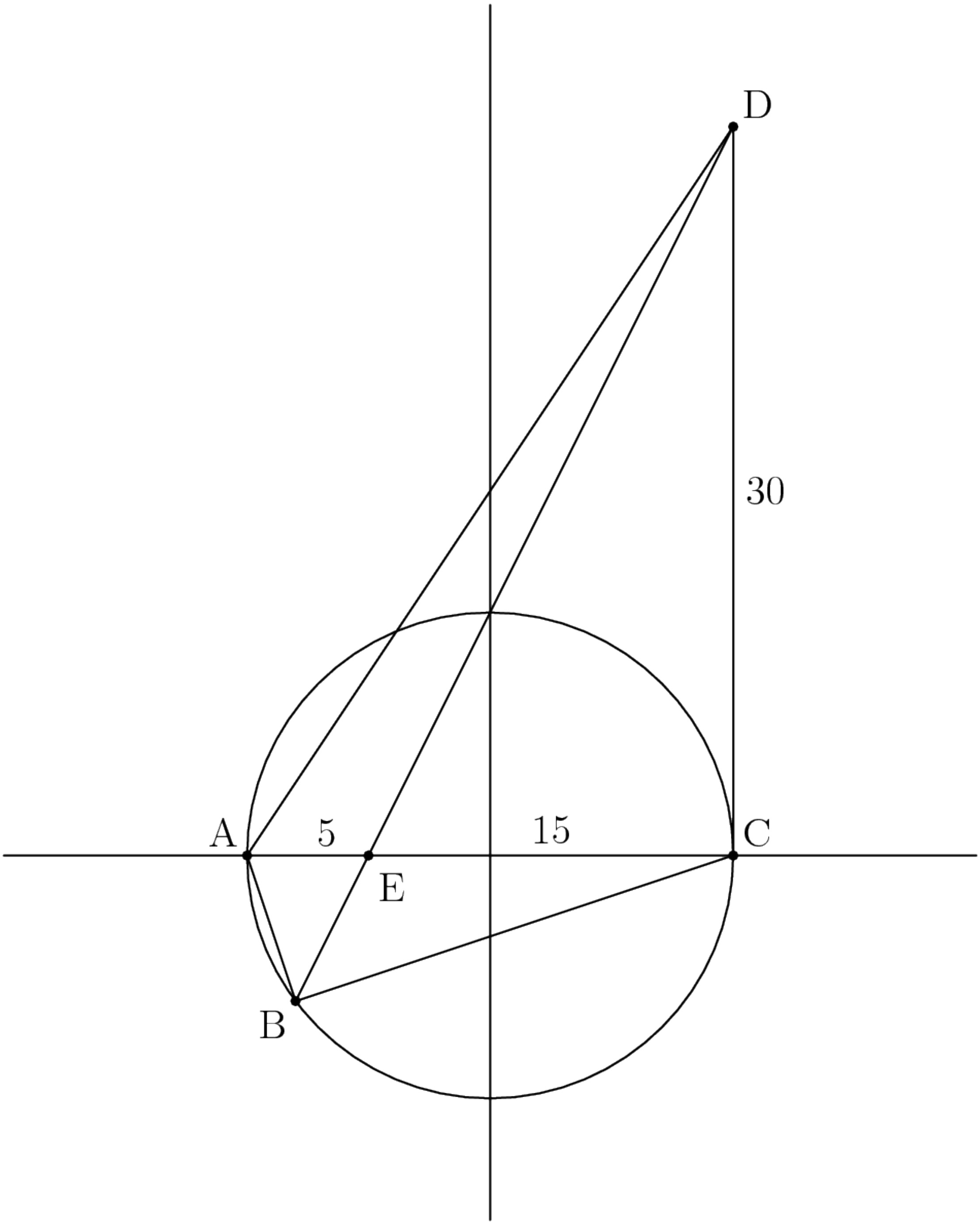
~ Diagram by ciceronii

Solution 2 (Pro Guessing Strats)

We know that the big triangle has area 300. Use the answer choices which would mean that the area of the little triangle is a multiple of 10. Thus the product of the legs is a multiple of 20. Guess that the legs are equal to $\sqrt{20a}$ and $\sqrt{20b}$, and because the hypotenuse is 20 we get $a+b=20$. Testing small numbers, we get that when $a=2$ and $b=18$, ab is indeed a square. The area of the triangle is thus 60, so the answer is $\boxed{\textbf{(D) } 360}$.

~tigershark22 ~(edited by HappyHuman)

Solution 3 (coordinates)



Let the points be $A(-10, 0)$, $B(x, y)$, $C(10, 0)$, $D(10, 30)$, and $E(-5, 0)$, respectively. Since B lies on line DE, we know that $y = 2x + 10$. Furthermore, since $\angle ABC = 90°$, B lies on the circle with diameter AC, so $x^2 + y^2 = 100$. Solving for x and y with these equations, we get the solutions $(0, 10)$ and $(-8, -6)$. We immediately discard the $(0, 10)$ solution as y should be negative. Thus, we conclude that

$$[ABCD] = [ACD] + [ABC] = \frac{20 \cdot 30}{2} + \frac{20 \cdot 6}{2} = \boxed{\textbf{(D)}\ 360}$$

.

Solution 4 (Trigonometry)

Let $\angle C = \angle ACB$ and $\angle B = \angle CBE$. Using Law of Sines on $\triangle BCE$ we get $\frac{BE}{\sin C} = \frac{CE}{\sin B} = \frac{15}{\sin B}$ and LoS on $\triangle ABE$ yields $\frac{BE}{\sin{(90-C)}} = \frac{5}{\sin{(90-B)}} = \frac{BE}{\cos C} = \frac{5}{\cos B}$. Divide the two to get $\tan B = 3\tan C$. Now, $\tan{\angle CED} = 2 = \tan{\angle B + \angle C} = \frac{4\tan C}{1 - 3\tan^2 C}$ and solve the quadratic, taking the positive solution (C is acute) to get $\tan C = \frac{1}{3}$. So if $AB = a$, then $BC = 3a$ and $[ABC] = \frac{3a^2}{2}$. By Pythagorean Theorem, $10a^2 = 400 \iff \frac{3a^2}{2} = 60$ and the answer is $300 + 60 \iff \boxed{\textbf{(D)}}$.

(This solution is incomplete, can someone complete it please-Lingjun) ok Latex edited by kc5170

We could use the famous m-n rule in trigonometry in triangle ABC with Point E [Unable to write it here.Could anybody write the expression] We will find that BD is angle bisector of triangle ABC(because we will get tan (x)=1) Therefore by converse of angle bisector theorem AB:BC = 1:3. By using phythagorean theorem we have values of AB and AC. AB.AC = 120. Adding area of ABC and ACD Answer••360

Problem21

There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that $\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}$. What is k?

$\textbf{(A) } 117 \qquad \textbf{(B) } 136 \qquad \textbf{(C) } 137 \qquad \textbf{(D) } 273 \qquad \textbf{(E) } 306$

Solution 1

First, substitute 2^{17} with a. Then, the given equation becomes $\frac{a^{17}+1}{a+1} = a^{16} - a^{15} + a^{14} ... - a^1 + a^0$. Now consider only $a^{16} - a^{15}$. This equals $a^{15}(a-1) = a^{15} * (2^{17} - 1)$. Note that $2^{17} - 1$ equals $2^{16} + 2^{15} + ... + 1$, since the sum of a geometric sequence is $\frac{a^n - 1}{a - 1}$. Thus, we can see that $a^{16} - a^{15}$ forms the sum of 17 different powers of 2. Applying the same method to each of $a^{14} - a^{13}$, $a^{12} - a^{11}$, ... , $a^2 - a^1$, we can see that each of the pairs forms the sum of 17 different powers of 2. This gives us $17 * 8 = 136$. But we must count also the a^0 term. Thus, Our answer is $136 + 1 = \boxed{\textbf{(C) } 137}$.

~seanyoon777

Solution 2

(This is similar to solution 1) Let $x = 2^{17}$. Then, $2^{289} = x^{17}$. The LHS can be rewritten as

$$\frac{x^{17}+1}{x+1}=x^{16}-x^{15}+\cdots+x^2-x+1=(x-1)(x^{15}+x^{13}+\cdots+x^1)+1$$

. Plugging 2^{17} back in for x, we have $(2^{17}-1)(2^{15\cdot17}+2^{13\cdot17}+\cdots+2^{1\cdot17})+1=(2^{16}+2^{15}+\cdots+2^0)(2^{15\cdot17}+2^{13\cdot17}+\cdots+2^{1\cdot17})+1$. When expanded, this will have $17\cdot8+1=137$ terms. Therefore, our answer is $\boxed{\textbf{(C) } 137}$.

Solution 3 (Intuitive)

Multiply both sides by $2^{17}+1$ to get $2^{289}+1=2^{a_1}+2^{a_2}+\ldots+2^{a_k}+2^{a_1+17}+2^{a_2+17}+\ldots+2^{a_k+17}$.

Notice that $a_1=0$, since there is a 1 on the LHS. However, now we have an extra term of 2^{18} on the right from 2^{a_1+17}. To cancel it, we let $a_2=18$. The two 2^{18}'s now combine into a term of 2^{19}, so we let $a_3=19$. And so on, until we get to $a_{18}=34$. Now everything we don't want telescopes into 2^{35}. We already have that term since we let $a_2=18\implies a_2+17=35$.

Everything from now on will automatically telescope to 2^{52}. So we let a_{19} be 52.

As you can see, we will have to add 17 a_n's at a time, then "wait" for the sum to automatically telescope for the next 17 numbers, etc, until we get to 2^{289}. We only need to add a_n's between odd multiples of 17 and even multiples. The largest even multiple of 17 below 289 is $17\cdot16$, so we will have to add a total of $17\cdot8$ a_n's. However, we must not forget we let $a_1=0$ at the beginning, so our answer is $17\cdot8+1=\boxed{\textbf{(C) } 137}$.

Solution 4

Note that the expression is equal to something slightly lower than 2^{272}. Clearly, answer choices (D) and (E) make no sense because the lowest sum for 273 terms is $2^{273}-1$. (A) just makes no sense. (B) and (C) are 1 apart, but because the expression is odd, it will have to contain $2^0=1$, and because (C) is 1bigger, the answer is $\boxed{\textbf{(C) } 137}$.
~Lcz

Solution 5

In order to shorten expressions, $\#$ will represent 16 consecutive 0s when expressing numbers.

Think of the problem in binary. We have

$$\frac{1\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#1_2}{1\#1_2}$$

Note that

$$(2^{17}+1)(2^0+2^{34}+2^{68}+\cdots+2^{272})=2^0(2^{17}+1)+2^{34}(2^{17}+1)+2^{68}(2^{17}+1)+\cdots 2^{272}(2^{17}+1)$$
$$=1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1_2$$

and

$$(2^{17}+1)(2^{17}+2^{51}+2^{85}+\cdots+2^{255})=2^{17}(2^{17}+1)+2^{51}(2^{17}+1)+2^{85}(2^{17}+1)+\cdots 2^{255}(2^{17}+1)$$
$$=1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#0_2$$

Since

$$\begin{aligned} &1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1_2 \\ - &\quad 1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#1\#0_2 \\ = &1\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#0\#1_2 \end{aligned}$$

this means that

$$(2^{17}+1)(2^0+2^{34}+2^{68}+\cdots+2^{272})-(2^{17}+1)(2^{17}+2^{51}+2^{85}+\cdots+2^{255})=2^{289}$$

so

$$\frac{2^{289}+1}{2^{17}+1}=(2^0+2^{34}+2^{68}+\cdots+2^{272})-(2^{17}+2^{51}+2^{85}+\cdots+2^{255})$$

$$=2^0+(2^{34}-2^{17})+(2^{68}-2^{51})+\cdots+(2^{272}-2^{255})$$

Expressing each of the pairs of the form $2^{n+17}-2^n$ in binary, we have

$$\begin{array}{rr} & 100000000000000000\cdots0_2 \\ - & 10\cdots0_2 \\ = & 11111111111111110\cdots0_2 \end{array}$$

or

$$2^{n+17}-2^n=2^{n+16}+2^{n+15}+2^{n+14}+\cdots+2^n$$

This means that each pair has 17 terms of the form 2^n.

Since there are 8 of these pairs, there are a total of $8\cdot 17=136$ terms.

Accounting for the 2^0 term, which was not in the pair, we have a total of $136+1=\boxed{\textbf{(C) } 137}$ terms.

Problem22

For how many positive integers $n\leq 1000$ is $\left\lfloor \frac{998}{n}\right\rfloor+\left\lfloor \frac{999}{n}\right\rfloor+\left\lfloor \frac{1000}{n}\right\rfloor$ not divisible by 3? (Recall that $\lfloor x\rfloor$ is the greatest integer less than or equal to x.)

$\textbf{(A) } 22 \qquad \textbf{(B) } 23 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 26$

Solution 1 (Casework)

Expression: $\left\lfloor \frac{998}{n}\right\rfloor+\left\lfloor \frac{999}{n}\right\rfloor+\left\lfloor \frac{1000}{n}\right\rfloor$

Solution:

Let $a = \left\lfloor \frac{998}{n} \right\rfloor$

Since $\frac{1000}{n} - \frac{998}{n} = \frac{2}{n}$, for any integer $n \geq 2$, the difference between the largest and smallest terms before the $\lfloor x \rfloor$ function is applied is less than or equal to 1, and thus the terms must have a range of 1 or less after the function is applied.

This means that for every integer $n \geq 2$,

- if $\frac{998}{n}$ is an integer and $n \neq 2$, then the three terms in the expression above must be (a, a, a),
- if $\frac{998}{n}$ is an integer because $n = 2$, then $\frac{1000}{n}$ will be an integer and will be 1 greater than $\frac{998}{n}$; thus the three terms in the expression must be $(a, a, a+1)$,
- if $\frac{999}{n}$ is an integer, then the three terms in the expression above must be $(a, a+1, a+1)$,
- if $\frac{1000}{n}$ is an integer, then the three terms in the expression above must be $(a, a, a+1)$, and
- if none of $\left\{ \frac{998}{n}, \frac{999}{n}, \frac{1000}{n} \right\}$ are integral, then the three terms in the expression above must be (a, a, a).

The last statement is true because in order for the terms to be different, there must be some integer in the interval $\left(\frac{998}{n}, \frac{999}{n}\right)$ or the interval $\left(\frac{999}{n}, \frac{1000}{n}\right)$. However, this means that multiplying the integer by $n$should produce a new integer between 998 and 999 or 999 and 1000, exclusive, but because no such integers exist, the terms cannot be different, and thus, must be equal.

● Note that $n = 1$ does not work; to prove this, we just have to substitute 1 for n in the expression. This gives us

$$\left\lfloor \frac{998}{1} \right\rfloor + \left\lfloor \frac{999}{1} \right\rfloor + \left\lfloor \frac{1000}{1} \right\rfloor = 998 + 999 + 1000 = 2997 = 999 \cdot 3$$

which is divisible by 3.

Now, we test the five cases listed above (where $n \geq 2$)

Case 1: n divides 998 and $n \neq 2$

As mentioned above, the three terms in the expression are (a, a, a), so the sum is $3a$, which is divisible by 3. Therefore, the first case does not work (**0** cases).

Case 2: n divides 998 and $n = 2$

As mentioned above, in this case the terms must be $(a, a, a + 1)$, which means the sum is $3a + 1$, so the expression is not divisible by 3. Therefore, this is **1** case that works.

Case 3: n divides 999

Because n divides 999, the number of possibilities for n is the same as the number of factors of 999.

999 = $3^3 \cdot 37^1$. So, the total number of factors of 999 is $4 \cdot 2 = 8$.

However, we have to subtract 1, because the case $n=1$ does not work, as mentioned previously. This leaves $8-1=$ **7** cases.

Case 4: n divides 1000

Because n divides 1000, the number of possibilities for n is the same as the number of factors of 1000.

1000 = $5^3 \cdot 2^3$. So, the total number of factors of 1000 is $4 \cdot 4 = 16$.

Again, we have to subtract 1, so this leaves $16-1=15$ cases. We have also overcounted the factor 2, as it has been counted as a factor of 1000 and as a separate case (Case 2). $15-1=14$, so there are actually **14** valid cases.

Case 5: n divides none of $\{998, 999, 1000\}$

Similar to Case 1, the value of the terms of the expression are (a, a, a). The sum is $3a$, which is divisible by 3, so this case does not work (**0** cases).

Now that we have counted all of the cases, we add them.

$0+1+7+14+0=22$, so the answer is $\boxed{\textbf{(A)} 22}$.

~dragonchomper, additional edits by emerald_block

Solution 2 (Solution 1 but simpler)

$\ast$ Note that this solution does not count a majority of cases that are important to consider in similar problems, though they are not needed for this problem, and therefore it may not work with other, similar problems.

Notice that you only need to count the number of factors of 1000 and 999, excluding 1. 1000 has 16 factors, and 999 has 8. Adding them gives you 24, but you need to subtract 2 since 1 does not work.

Therefore, the answer is 24 - 2 = $\boxed{\textbf{(A)} 22}$.

-happykeeper, additional edits by dragonchomper

Solution 3

NOTE: For this problem, whenever I say $*\text{factors}*$, I will be referring to all the factors of the number except for 1.

Now, quickly observe that if $n > 2$ divides 998, then $\left\lfloor \frac{999}{n} \right\rfloor$ and $\left\lfloor \frac{1000}{n} \right\rfloor$ will also round down to $\frac{998}{n}$, giving us a sum of $3 \cdot \frac{998}{n}$, which does not work for the question. However, if $n > 2$ divides 999, we see that $\left\lfloor \frac{998}{n} \right\rfloor = \frac{999}{n} - 1$ and $\left\lfloor \frac{1000}{n} \right\rfloor = \left\lfloor \frac{999}{n} \right\rfloor$. This gives us a sum of $3 \cdot \left\lfloor \frac{999}{n} \right\rfloor - 1$, which is clearly not divisible by 3. Using the same logic, we can deduce that $(n > 2)|1000$ too works (for our problem). Thus, we need the factors of 999 and 1000 and we don't have to eliminate any because the $\text{gcf}(999, 1000) = 1$. But we have to be careful! See that when $n|998, 999, 1000$, then our problem doesn't get fulfilled. The only n that satisfies that is $n = 1$. So, we have:

$$999 = 3^3 \cdot 37 \implies (3+1)(1+1) - 1 *\text{factors}* \implies 7$$

;

$$1000 = 2^3 \cdot 5^3 \implies (3+1)(3+1) - 1 *\text{factors}* \implies 15$$

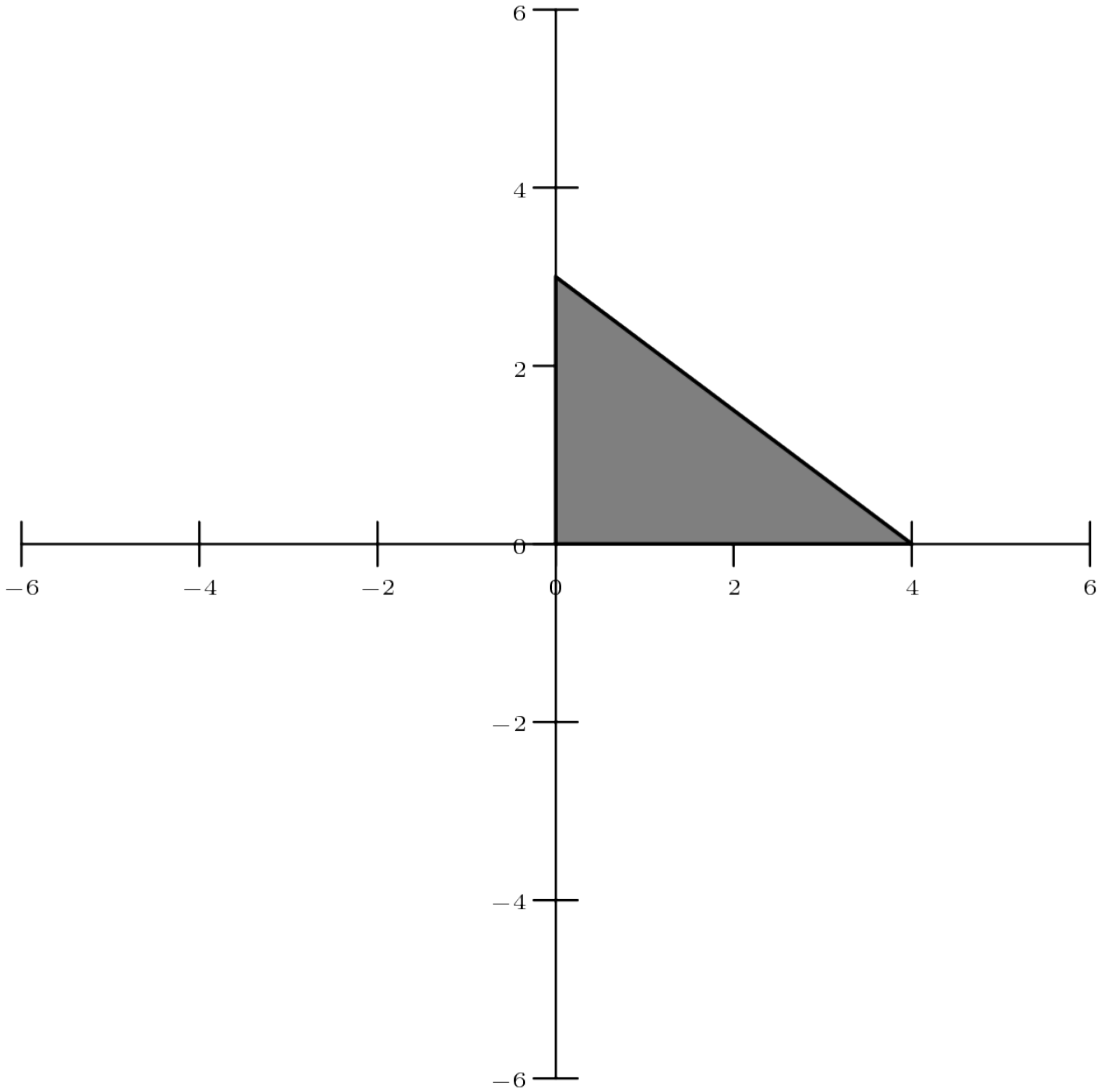
. Adding them up gives a total of $7 + 15 = \boxed{\textbf{(A)} 22}$ workable n's.

Problem23

Let T be the triangle in the coordinate plane with vertices $(0,0), (4,0)$, and $(0,3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90°, 180°, and 270° counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a 90° rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

$\textbf{(A) } 12 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 17 \qquad \textbf{(D) } 20 \qquad \textbf{(E) } 25$

Solution



First, any combination of motions we can make must reflect T an even number of times. This is because every time we reflect T, it changes orientation. Once T has been flipped once, no combination of rotations will put it back in place because it is the mirror image; however, flipping it again changes it back to the original orientation. Since we are only allowed 3 transformations and an even number of them must be reflections, we either reflect T 0 times or 2 times.

Case 1: 0 reflections on T

In this case, we must use 3 rotations to return T to its original position. Notice that our set of rotations, $\{90^\circ, 180^\circ, 270^\circ\}$, contains every multiple of 90° except for 0°. We can start with any two rotations a, b in $\{90^\circ, 180^\circ, 270^\circ\}$ and there must be exactly one $c \equiv -a - b \pmod{360^\circ}$ such that we can use the three rotations (a, b, c) which ensures that $a + b + c \equiv 0^\circ \pmod{360^\circ}$.

That way, the composition of rotations a, b, c yields a full rotation. For example, if $a = b = 90^\circ$,
then $c \equiv -90^\circ - 90^\circ = -180^\circ \pmod{360^\circ}$,

so $c = 180^\circ$ and the rotations $(90^\circ, 90^\circ, 180^\circ)$ yields a full rotation.

The only case in which this fails is when c would have to equal 0°. This happens when (a, b) is already a full rotation,

namely, $(a, b) = (90^\circ, 270^\circ), (180^\circ, 180^\circ)$, or $(270^\circ, 90^\circ)$.
However, we can simply subtract these three cases from the total.
Selecting (a, b) from $\{90^\circ, 180^\circ, 270^\circ\}$ yields $3 \cdot 3 = 9$choices, and with 3 that fail, we are left with 6 combinations for case 1.

Case 2: 2 reflections on T

In this case, we first eliminate the possibility of having two of the same reflection. Since two reflections across the x-axis maps T back to itself, inserting a rotation before, between, or after these two reflections would change T's final location, meaning that any combination involving two reflections across the x-axis would not map T back to itself. The same applies to two reflections across the y-axis.

Therefore, we must use one reflection about the x-axis, one reflection about the y-axis, and one rotation. Since a reflection about the x-axis changes the sign of the y component, a reflection about the y-axis changes the sign of the x component, and a 180° rotation changes both signs, these three transformation composed (in any order) will suffice. It is therefore only a question of arranging the three, giving us $3! = 6$combinations for case 2.

Combining both cases we get $6+6=\boxed{\textbf{(A) } 12}$

Solution 2(Rewording solution 1)

As in the previous solution, note that we must have either 0 or 2 reflections because of orientation since reflection changes orientation that is impossible to fix by rotation. We also know we can't have the same reflection twice, since that would give a net of no change and would require an identity rotation.

Suppose there are no reflections. Denote 90° as 1, 180° as 2, and 270° as 3, just for simplification purposes. We want a combination of 3 of these that will sum to either 4 or 8(0 and 12 is impossible since the minimum is 3 and the max is 9). 4 can be achieved with any permutation of $(1-1-2)$ and 8 can be achieved with any permutation of $(2-3-3)$. This case can be done in $3+3=6$ ways.

Suppose there are two reflections. As noted already, they must be different, and as a result will take the triangle to the opposite side of the origin if we don't do any rotation. We have 1 rotation left that we can do though, and the only one that will return to the original position is 2, which is 180° AKA reflection across origin. Therefore, since all 3 transformations are distinct. The three transformations can be applied anywhere since they are commutative(think quadrants). This gives 6 ways.

$$6+6=\boxed{(A)12}$$

Problem24

Let n be the least positive integer greater than 1000 for which

$$\gcd(63, n+120) = 21 \quad \text{and} \quad \gcd(n+63, 120) = 60.$$

What is the sum of the digits of n?

$\textbf{(A) } 12 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 21 \qquad \textbf{(E) } 24$

Solution 1

We know that $gcd(63, n+120) = 21$, so we can write $n + 120 \equiv 0 \pmod{21}$. Simplifying, we get $n \equiv 6 \pmod{21}$. Similarly, we can write $n + 63 \equiv 0 \pmod{60}$, or $n \equiv -3 \pmod{60}$. Solving these two modular congruences, $n \equiv 237 \pmod{420}$ which we know is the only solution by CRT (Chinese Remainder Theorem). Now, since the problem is asking for the least positive integer greater than 1000, we find the least solution is $n = 1077$. However, we are have not considered cases where $gcd(63, n+120) = 63$ or $gcd(n+63, 120) = 120$. $1077 + 120 \equiv 0 \pmod{63}$ so we try $n = 1077 + 420 = 1497$. $1497 + 63 \equiv 0 \pmod{120}$ so again we add 420 to n. It turns out that $n = 1497 + 420 = 1917$ does indeed satisfy the original conditions, so our answer is $1 + 9 + 1 + 7 = \boxed{\textbf{(C) } 18}$.

Solution 2 (bashing)

We are given that $\gcd(63, n+120) = 21$ and $\gcd(n+63, 120) = 60$.

This tells us that $n+120$ is divisible by 21 but not 63. It also tells us that $n+63$ is divisible by 60 but not 120. Starting, we find the least value of $n+120$ which is divisible by 21 which satisfies the conditions for n, which is 1134, making $n = 1014$. We then now keep on adding 21 until we get a number which satisfies the second equation. This number turns out to be 1917, whose digits add up to $\boxed{\textbf{(C) } 18}$.

-Midnight

Solution 3 (bashing but worse)

Assume that n has 4 digits. Then $n = abcd$, where a, b, c, d represent digits of the number (not to get confused with $a * b * c * d$). As given the problem, $gcd(63, n + 120) = 21$ and $gcd(n + 63, 120) = 60$. So we know that $d = 7$ (last digit of n). That means that $12 + abc \equiv 0 \pmod 7$ and $7 + abc \equiv 0 \pmod 6$. We can bash this after this. We just want to find all pairs of numbers (x, y) such that x is a multiple of 7 that is 5 greater than a multiple of 6. Our equation for $12 + abc$ would be $42 * j + 35 = x$ and our equation for $7 + abc$ would be $42 * j + 30 = y$, where j is any integer. We plug this value in until we get a value of abc that makes $n = abc7$ satisfy the original problem statement (remember, $abc > 100$). After bashing for hopefully a couple minutes, we find that $abc = 191$ works.

So $n = 1917$ which means that the sum of its digits is $\boxed{\textbf{(C) } 18}$.

~ Baolan

Solution 4

The conditions of the problem reduce to the following. $n + 120 = 21k$ where $gcd(k, 3) = 1$ and $n + 63 = 60l$ where $gcd(l, 2) = 1$. From these equations, we see that $21k - 60l = 57$. Solving this diophantine equation gives us that $k = 20a + 57, l = 7a + 19$ form. Since, n is greater than 1000, we can do some bounding and get that $k > 53$ and $l > 17$. Now we start the bash by plugging in numbers that satisfy these conditions. We get $l = 53$, $k = 97$. So the answer is $\boxed{1917}$.

Solution 5

You can first find that n must be congruent to $6 \equiv 0 \pmod{21}$ and $57 \equiv 0 \pmod{60}$. The we can find that $n = 21x + 6$ and $n = 60y + 57$, where x and y are integers. Then we can find that y must be odd, since if it was even the gcd will be 120, not 60. Also, the unit digit of n has to be 7, since the unit digit of 60y is always 0 and the unit digit of 57 is 7. Therefore, you can find that x must end in 1 to satisfy n having a unit digit of 7. Also, you can find that x must not be a multiple of three or else the gcd will be 63. Therefore, you can test values for x and you can find that x=91 satisfies all these conditions.Therefore, n is 1917 and $1 + 9 + 1 + 7$ $= \boxed{(C) 18}$.-happykeeper

Solution 6 (Reverse Euclidean Algorithm)

We are given that $\gcd(63, n + 120) = 21$ and $\gcd(n + 63, 120) = 60.$ By applying the Euclidean algorithm, but in reverse, we have

$$\gcd(63, n+120) = \gcd(63, n+120+63) = \gcd(63, n+183) = 21$$

and

$$\gcd(n+63, 120) = \gcd(n+63+120, 120) = \gcd(n+183, 120) = 60.$$

We now know that $n + 183$ must be divisible by 21 and $60,$ so it is divisible by $\operatorname{lcm}(21, 60) = 420.$ Therefore, $n + 183 = 420k$ for some integer $k.$ We know that $3 \nmid k,$ or else the first condition won't hold ($\gcd$ will be 63) and $2 \nmid k,$ or else the second condition won't hold ($\gcd$ will be 120). Since $k = 1$ gives us too small of an answer, then $k = 5 \implies n = 1917,$ so the answer is $1 + 9 + 1 + 7 = \boxed{(C) 18}$.

Problem25

Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

$\textbf{(A) } \frac{7}{36} \qquad \textbf{(B) } \frac{5}{24} \qquad \textbf{(C) } \frac{2}{9} \qquad \textbf{(D) } \frac{17}{72} \qquad \textbf{(E) } \frac{1}{4}$

Solution 1

Consider the probability that rolling two dice gives a sum of s, where $s \leq 7$. There are $s-1$ pairs that satisfy this, namely $(1, s-1), (2, s-2), \ldots, (s-1, 1)$, out of $6^2 = 36$ possible pairs. The probability is $\frac{s-1}{36}$.

Therefore, if one die has a value of a and Jason rerolls the other two dice, then the probability of winning is $\frac{7-a-1}{36} = \frac{6-a}{36}$.

In order to maximize the probability of winning, a must be minimized. This means that if Jason rerolls two dice, he must choose the two dice with the maximum values.

Thus, we can let $a \leq b \leq c$ be the values of the three dice, which we will call A, B, and C respectively. Consider the case when $a+b<7$. If $a+b+c=7$, then we do not need to reroll any dice. Otherwise, if we reroll one die, we can roll dice C in the hope that we get the value that makes the sum of the three dice 7. This happens with probability $\frac{1}{6}$. If we reroll two dice, we will roll B and C, and the probability of winning is $\frac{6-a}{36}$, as stated above.

However, $\frac{1}{6} > \frac{6-a}{36}$, so rolling one die is always better than rolling two dice if $a+b<7$.

Now consider the case where $a+b\geq 7$. Rerolling one die will not help us win since the sum of the three dice will always be greater than 7. If we reroll two dice, the probability of winning is, once again, $\frac{6-a}{36}$. To find the probability of winning if we reroll all three dice, we can let each dice have 1 dot and find the number of ways to distribute the remaining 4 dots. By Stars and Bars, there are $\binom{6}{2}=15$ ways to do this, making the probability of winning $\frac{15}{6^3}=\frac{5}{72}$.

In order for rolling two dice to be more favorable than rolling three dice, $\frac{6-a}{36}>\frac{5}{72}\rightarrow a\leq 3$.

Thus, rerolling two dice is optimal if and only if $a\leq 3$ and $a+b\geq 7$. The possible triplets (a,b,c) that satisfy these conditions, and the number of ways they can be permuted, are $(3,4,4)\rightarrow 3$ ways. $(3,4,5)\rightarrow 6$ ways. $(3,4,6)\rightarrow 6$ ways. $(3,5,5)\rightarrow 3$ ways. $(3,5,6)\rightarrow 6$ ways. $(3,6,6)\rightarrow 3$ ways. $(2,5,5)\rightarrow 3$ ways. $(2,5,6)\rightarrow 6$ ways. $(2,6,6)\rightarrow 3$ ways. $(1,6,6)\rightarrow 3$ ways.

There are $3+6+6+3+6+3+3+6+3+3=42$ ways in which rerolling two dice is optimal, out of $6^3=216$ possibilities, Therefore, the probability that Jason will reroll two dice is $\frac{42}{216}=\boxed{\textbf{(A)}\ \frac{7}{36}}$

Solution 2

We count the numerator. Jason will pick up no dice if he already has a 7 as a sum. We need to assume he does not have a 7 to begin with. If Jason decides to pick up all the dice to re-roll, by Stars and Bars(or whatever...), there will be 2 bars and 4 stars(3 of them need to be guaranteed because a roll is at least 1) for a probability of $\frac{15}{216} = \frac{2.5}{36}$. If Jason picks up 2 dice and leaves a die showing k, he will need the other two to sum to $7 - k$. This happens with probability $\frac{6-k}{36}$ for integers $1 \leq k \leq 6$. If the roll is not 7, Jason will pick up exactly one die to re-roll if there can remain two other dice with sum less than 7, since this will give him a $\frac{1}{6}$ chance which is a larger probability than all the cases unless he has a 7 to begin with. We have $\frac{1}{6} > \underline{\frac{5,4,3}{36}} > \frac{2.5}{36} > \frac{2,1,0}{36}$. We count the underlined part's frequency for the numerator without upsetting the probability greater than it.

Let a be the roll we keep. We know a is at most 3 since 4 would cause Jason to pick up all the dice. When $a = 1$, there are 3 choices for whether it is rolled 1st, 2nd, or 3rd, and in this case the other two rolls have to be at least 6(or he would have only picked up 1). This give $3 \cdot 1^2 = 3$ ways.

Similarly, $a = 2$ gives $3 \cdot 2^2 = 12$ because the 2 can be rolled in 3 places and the other two rolls are at least 5. $a = 3$ gives $3 \cdot 3^2 = 27$. Summing together gives the numerator of 42. The denominator is $6^3 = 216$, so we have $\frac{42}{216} = \boxed{(A)\frac{7}{36}}$