2021 AMC 10A Solution

Problem1

What is the value of $(2^2-2)-(3^2-3)+(4^2-4)$

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 8 \qquad \textbf{(E) } 12$

Solution 1

$(4-2)-(9-3)+(16-4)=2-6+12=8.$ This corresponds to $\boxed{\textbf{(D) } 8}$

Solution 2

$$\begin{aligned}(2^2-2)-(3^2-3)+(4^2-4) &= 2(2-1)-3(3-1)+4(4-1)\\ &= 2(1)-3(2)+4(3)\\ &= 2-6+12\\ &= \boxed{\textbf{(D) } 8}.\end{aligned}$$

Problem2

Portia's high school has 3 times as many students as Lara's high school. The two high schools have a total of 2600 students. How many students does Portia's high school have?

$\textbf{(A) } 600 \qquad \textbf{(B) } 650 \qquad \textbf{(C) } 1950 \qquad \textbf{(D) } 2000 \qquad \textbf{(E) } 2050$

Solution 1

The following system of equations can be formed with p representing the number of students in Portia's high school and l representing the number of students in Lara's high school. $p=3lp+l=2600$

Substituting p with $3l$ we get $4l=2600$. Solving for l, we get $l=650$.

Since we need to find p we multiply 650 by 3 to get $p=1950$, which is $\boxed{\text{C}}$

Solution 2 (One Variable)

Suppose Lara's high school has x students. It follows that Portia's high school has $3x$ students. We know that $x+3x=2600$, or $4x=2600$. Our answer is $3x=2600\left(\frac{3}{4}\right)=650(3)=\boxed{\textbf{(C) }1950}$.

Solution 3 (Arithmetics)

Clearly, 2600 students is 4 times as many students as Lara's high school. Therefore, Lara's high school has $2600\div4=650$ students, and Portia's high school has $650\cdot3=\boxed{\textbf{(C) }1950}$ students.

Solution 4 (Answer Choices)

Solution 4.1 (Quick Inspection)

The number of students in Portia's high school must be a multiple of $3.$ This eliminates $\textbf{(B)}$, $\textbf{(D)}$, and $\textbf{(E)}$. Since $\textbf{(A)}$ is too small (as $600+600/3<2600$ is clearly true), we are left with $\boxed{\textbf{(C) }1950}$.

Solution 4.2 (Plug in the Answer Choices)

For $\textbf{(A)}$, we have $600+\frac{600}{3}=800\neq2600.$ So, $\textbf{(A)}$ is incorrect.

For $\textbf{(B)}$, we have $650+\frac{650}{3}=866\frac{2}{3}\neq2600.$ So, $\textbf{(B)}$ is incorrect.

For $\textbf{(C)}$, we have $1950+\frac{1950}{3}=2600.$ So, $\boxed{\textbf{(C) }1950}$ is correct. For completeness, we will check choices $\textbf{(D)}$ and $\textbf{(E)}$.

For $\textbf{(D)}$, we have $2000+\frac{2000}{3}=2666\frac{2}{3}\neq 2600.$ So, $\textbf{(D)}$ is incorrect.

For $\textbf{(E)}$, we have $2050+\frac{2050}{3}=2733\frac{1}{3}\neq 2600.$ So, $\textbf{(E)}$ is incorrect.

Problem3

The sum of two natural numbers is $17{,}402$. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

$\textbf{(A)}\ 10{,}272\qquad\textbf{(B)}\ 11{,}700\qquad\textbf{(C)}\ 13{,}362\qquad\textbf{(D)}\ 14{,}238\qquad\textbf{(E)}\ 15{,}426$

Solution 1

The units digit of a multiple of 10 will always be 0. We add a 0 whenever we multiply by 10. So, removing the units digit is equal to dividing by 10.

Let the smaller number (the one we get after removing the units digit) be a. This means the bigger number would be $10a$.

We know the sum is $10a+a=11a$ so $11a=17402$.

So $a=1582$. The difference is $10a-a=9a$. So, the answer is $9(1582)=14238=\boxed{\textbf{(D)}}$.

Solution 2 (Lazy Speed)

Since the ones place of a multiple of 10 is 0, this implies the other integer has to end with a 2 since both integers sum up to a number that ends with a 2. Thus, the ones place of the difference has to be $10-2=8$, and the only answer choice that ends with an 8 is $\boxed{\textbf{(D)}\ 14238}$

~CoolJupiter 2021

Another quick solution is to realize that the sum is represents a number n added to $10n$. The difference is $9n$, which is $\frac{9}{11}$ of the given sum.

Solution 3 (Vertical Addition and Logic)

Let the larger number be $\overline{AB,CD0}.$ It follows that the smaller number is $\overline{A,BCD}.$ Adding vertically, we have

$$\begin{array}{cccccc} & A & B & C & D & 0 \\ + & & A & B & C & D \\ \hline & 1 & 7 & 4 & 0 & 2 \end{array}$$

Working from right to left, we have $D=2\Rightarrow C=8\Rightarrow B=5\Rightarrow A=1.$The larger number is $15,820$ and the smaller number is $1,582.$ Their difference is $15,820-1,582=\boxed{\textbf{(D)}\ 14{,}238}.$

Problem4

A cart rolls down a hill, travelling 5 inches the first second and accelerating so that during each successive 1-second time interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?

$\textbf{(A)}\ 215 \qquad \textbf{(B)}\ 360 \qquad \textbf{(C)}\ 2992 \qquad \textbf{(D)}\ 3195 \qquad \textbf{(E)}\ 3242$

Solution 1 (Arithmetic Series)

Since $\text{Distance} = \text{Speed} \times \text{Time},$we seek the sum

$$5(1)+12(1)+19(1)+26(1)+\cdots=5+12+19+26+\cdots,$$

in which there are 30 addends. The last addend is $5+7(30-1)=208.$ Therefore, the requested sum is

$$5+12+19+26+\cdots+208=\frac{(5+208)(30)}{2}=\boxed{\textbf{(D)}\ 3195}.$$

Recall that to find the sum of an arithmetic series, we take the average of the first and last terms, then multiply by the number of terms, namely $\dfrac{\text{First}+\text{Last}}{2}\cdot\text{Count}.$

Solution 2 (Answer Choices and Modular Arithmetic)

From the 30-term sum $5+12+19+26+\cdots$ in the previous solution, taking modulo 10 gives $5+12+19+26+\cdots \equiv 3(0+1+2+3+4+5+6+7+8+9) = 3(45) \equiv 5 \pmod{10}.$

The only answer choices that are $5 \mod 10$ are $\textbf{(A)}$ and $\textbf{(D)}$. By a quick estimation, $\textbf{(A)}$ is too small, leaving us with $\boxed{\textbf{(D)}\ 3195}$.

Problem5

The quiz scores of a class with $k > 12$ students have a mean of 8. The mean of a collection of 12 of these quiz scores is 14. What is the mean of the remaining quiz scores of terms of k?

$\textbf{(A)}\ \frac{14-8}{k-12} \qquad \textbf{(B)}\ \frac{8k-168}{k-12} \qquad \textbf{(C)}\ \frac{14}{12}-\frac{8}{k} \qquad \textbf{(D)}\ \frac{14(k-12)}{k^2} \qquad \textbf{(E)}\ \frac{14(k-12)}{8k}$

Solution 1 (Generalized)

The total score in the class is $8k.$ The total score on the 12 quizzes is $12\cdot14=168.$ Therefore, for the remaining quizzes ($k-12$ of them), the total score is $8k-168.$ Their mean score is $\boxed{\textbf{(B)}\ \frac{8k-168}{k-12}}.$

~MRENTHUSIASM

Solution 2 (Convenient Values and Observations)

Set $k=13.$ The answer is the same as the last student's quiz score, which is $8\cdot13-14\cdot12<0.$ From the answer choices, only $\boxed{\textbf{(B)}\ \frac{8k-168}{k-12}}$ yields a negative value for $k=13.$

Problem6

Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to 2 miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at 3 miles per hour. She meets Jean at the halfway point. What was Jean's average speed, in miles per hour, until they meet?

$\textbf{(A)}\ \frac{12}{13} \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ \frac{13}{12} \qquad \textbf{(D)}\ \frac{24}{13} \qquad \textbf{(E)}\ 2$

Solution 1 (Generalized Distance)

Let $2d$ miles be the distance from the start to the fire tower. When Chantal meets Jean, she has traveled
for

$$\frac{d}{4}+\frac{d}{2}+\frac{d}{3}=d\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{3}\right)=d\left(\frac{3}{12}+\frac{6}{12}+\frac{4}{12}\right)=\frac{13}{12}d$$

hours. Jean also has traveled for $\frac{13}{12}d$ hours, and he travels for d miles. So, his average speed is $\frac{d}{\left(\frac{13}{12}d\right)}=\boxed{\textbf{(A)}\ \frac{12}{13}}$miles per hour.

Solution 2 (Convenient Distance)

We use the same template as shown in Solution 1, except that we replace d with a concrete number.

Let 24 miles be the distance from the start to the fire tower. When Chantal meets Jean, she travels for $\frac{12}{4}+\frac{12}{2}+\frac{12}{3}=3+6+4=13$ hours. Jean also has traveled for 13 hours, and he travels for 12 miles. So, his average speed is $\boxed{\textbf{(A)}\ \frac{12}{13}}$miles per hour.

Problem7

Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that all of his happy snakes can add, none of his purple snakes can subtract, and all of his snakes that can't subtract also can't add. Which of these conclusions can be drawn about Tom's snakes?

$\textbf{(A)}$ Purple snakes can add.

$\textbf{(B)}$ Purple snakes are happy.

$\textbf{(C)}$ Snakes that can add are purple.

$\textbf{(D)}$ Happy snakes are not purple.

$\textbf{(E)}$ Happy snakes can't subtract.

Solution 1

We know that purple snakes cannot subtract, thus they cannot add either. Since happy snakes must be able to add, the purple snakes cannot be happy. Therefore, we know that the happy snakes are not purple and the answer is $\boxed{\textbf{(D)}}$.

Solution 2 (Explains Solution 1 Using Arrows)

We are given that

$$(1)\ \text{Happy} \Rightarrow \text{can add}$$

$$(2)\ \text{Purple} \Rightarrow \text{cannot subtract}$$

$$(3)\ \text{Cannot subtract} \Rightarrow \text{cannot add}$$

Combining (2) and (3) into $(*)$ below, we have

$$(1)\ \text{Happy} \Rightarrow \text{can add}$$

$$(*)\ \text{Purple} \Rightarrow \text{cannot subtract} \Rightarrow \text{cannot add}$$

Clearly, the answer is $\boxed{\textbf{(D)}}$.

Problem8

When a student multiplied the number 66 by the repeating decimal $\underline{1}.\underline{abab}... = \underline{1}.\overline{\underline{ab}}$,where a and b are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{ab}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit number $\underline{ab}$?

$\textbf{(A) }15 \qquad \textbf{(B) }30 \qquad \textbf{(C) }45 \qquad \textbf{(D) }60 \qquad \textbf{(E) }75$

Solution

It is known that $0.\overline{ab} = \frac{ab}{99}$ and $0.ab = \frac{ab}{100}$. Let $\overline{ab} = x$. We have that $66(1+\frac{x}{100})+0.5 = 66(1+\frac{x}{99})$. Solving gives that $100x - 75 = 99x$ so $x = \boxed{\text{(E)} 75}$.

Problem9

What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

$\textbf{(A) }0 \qquad \textbf{(B) }\frac{1}{4} \qquad \textbf{(C) }\frac{1}{2} \qquad \textbf{(D) }1 \qquad \textbf{(E) }2$

Solution 1

Expanding, we get that the expression is $x^2+2xy+y^2+x^2y^2-2xy+1$ or $x^2+y^2+x^2y^2+1$. By the trivial inequality(all squares are nonnegative) the minimum value for this is $\boxed{\text{(D)} 1}$, which can be achieved at $x=y=0$.
~aop2014

Solution 2 (Beyond Overkill)

Like solution 1, expand and simplify the original equation to $x^2+y^2+x^2y^2+1$ and let $f(x,y)=x^2+y^2+x^2y^2+1$. To find local extrema, find where $\nabla f(x,y)=\mathbf{0}$. First, find the first partial derivative with respect to x and y and find where they are 0: $\frac{\partial f}{\partial x}=2x+2xy^2=2x(1+y^2)=0 \implies x=0$

$$\frac{\partial f}{\partial y}=2y+2yx^2=2y(1+x^2)=0 \implies y=0$$

Thus, there is a local extreme at $(0,0)$. Because this is the only extreme, we can assume that this is a minimum because the problem asks for the minimum (though this can also be proven using the partial second derivative test) and the global minimum since it's the only minimum, meaning $f(0,0)$ is the minimum of $f(x,y)$. Plugging $(0,0)$ into $f(x,y)$, we find 1 $\implies \boxed{\textbf{(D) } 1}$

Problem10

Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

$\textbf{(A) } 3^{127}+2^{127} \qquad \textbf{(B) } 3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63} \qquad \textbf{(C) } 3^{128}-2^{128} \qquad \textbf{(D) } 3^{128}+2^{128} \qquad \textbf{(E) } 5^{127}$

Solution 1

All you need to do is multiply the entire equation by $(3-2)$. Then all the terms will easily simplify by difference of squares and you will get $3^{128}-2^{128}$ or $\boxed{C}$ as your final answer. Notice you don't need to worry about $3-2$ because that's equal to 1.

-Lemonie

Solution 2

If you weren't able to come up with the $(3-2)$ insight, then you could just notice that the answer is divisible by $(2+3)=5$, and $(2^2+3^2)=13$. We can then use Fermat's Little Theorem for $p=5,13$ on the answer choices to determine which of the answer choices are divisible by both 5 and 13. This is $\boxed{C}$.

Solution 3

After expanding the first few terms, the result after each term appears to be

$$2^{2^n-1}+2^{2^n-2}\cdot 3^1+2^{2^n-3}\cdot 3^2+...+2^1\cdot 3^{2^n-2}+3^{2^n-1}$$

where n is the number of terms expanded. We can prove this using mathematical induction. The base step is trivial. When expanding another term, all of the previous terms multiplied by $2^{2^{n-1}}$ would give

$$2^{2^n-1}+2^{2^n-2}\cdot 3^1+2^{2^n-3}\cdot 3^2+...+2^{2^{n-1}+1}\cdot 3^{2^{n-1}-1}+2^{2^{n-1}}\cdot 3^{2^{n-1}}$$

, and all the previous terms multiplied by $3^{2^{n-1}}$ would give

$$3^{2^n-1}+3^{2^n-2}\cdot 2^1+3^{2^n-3}\cdot 2^2+...+3^{2^{n-1}+1}\cdot 2^{2^{n-1}-1}+3^{2^{n-1}}\cdot 2^{2^{n-1}}$$

. Their sum is equal to

$$2^{2^n-1}+2^{2^n-2}\cdot 3^1+2^{2^n-3}\cdot 3^2+...+2^1\cdot 3^{2^n-2}+3^{2^n-1}$$

, so the proof is complete. Since $\frac{3^{2^n}-2^{2^n}}{3-2}$ is equal to

$$2^{2^n-1}+2^{2^n-2}\cdot 3^1+2^{2^n-3}\cdot 3^2+...+2^1\cdot 3^{2^n-2}+3^{2^n-1}$$

, the answer is $\frac{3^{2^7}-2^{2^7}}{3-2}=\boxed{C}$.

-SmileKat32

Solution 4 (Engineer's Induction)

We can compute some of the first few partial products, and notice that $\prod_{k=0}^{2^n}(2^{2^n}+3^{2^n})=3^{2^{n+1}}-2^{2^{n+1}}$. As we don't have to prove this, we get the product is $3^{2^7}-2^{2^7}=3^{128}-2^{128}$, and smugly click $\boxed{\textbf{(C)}\ 3^{128}-2^{128}}$. ~rocketsri

Problem11

For which of the following integers b is the base-b number 2021_b-221_b not divisible by 3?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

Solution

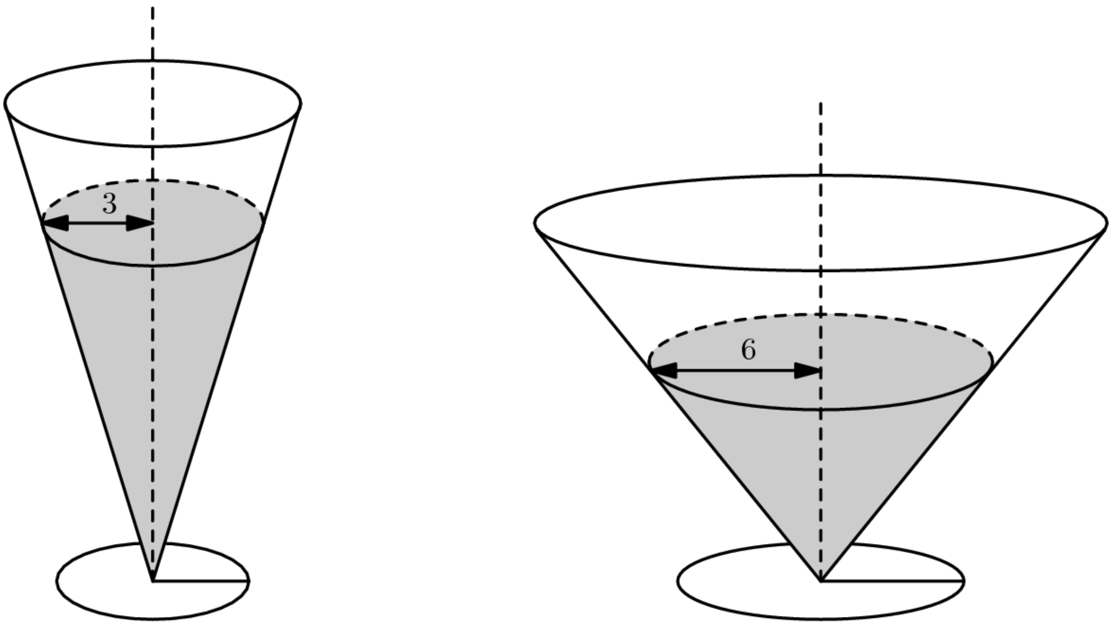
We have

$$2021_b-221_b=2000_b-200_b=2b^3-2b^2=2b^2(b-1).$$

This expression is divisible by 3 **unless** $b\equiv2\pmod{3}.$ The only choice congruent to 2 modulo 3 is $\boxed{\textbf{(E)}\ 8}$.

Problem12

Two right circular cones with vertices facing down as shown in the figure below contains the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?



(A) $1:1$ **(B)** $47:43$ **(C)** $2:1$ **(D)** $40:13$ **(E)** $4:1$

Solution 1 (Use Tables to Organize Information)

Initial Scenario

	Base	Height	Volume
Narrow Cone	3	h_1	$\frac{1}{3}\pi(3)^2h_1 = 3\pi h_1$
Wide Cone	6	h_2	$\frac{1}{3}\pi(6)^2h_2 = 12\pi h_2$

By similar triangles:

For the narrow cone, the ratio of base radius to height is $\frac{3}{h_1}$, which remains constant.

For the wide cone, the ratio of base radius to height is $\frac{6}{h_2}$, which remains constant.

Equating the initial volumes gives $3\pi h_1 = 12\pi h_2$, which simplifies to $\frac{h_1}{h_2} = 4.$

Final Scenario (Two solutions follow from here.)

Solution 1.1 (Fraction Trick)

Let the base radii of the narrow cone and the wide cone be $3x$ and $6y,$ respectively, where $x,y>1.$ We have the following table:

	Base	Height	Volume
Narrow Cone	$3x$	h_1x	$\frac13\pi(3x)^2h_1=3\pi h_1x^3$
Wide Cone	$6y$	h_2y	$\frac13\pi(6y)^2h_2=12\pi h_2y^3$

Equating the final volumes gives $3\pi h_1x^3=12\pi h_2y^3,$ which simplifies to $x^3=y^3,$ or $x=y.$

Lastly, the requested ratio is $\dfrac{h_1x-h_1}{h_2y-h_2}=\dfrac{h_1(x-1)}{h_2(y-1)}=\dfrac{h_1}{h_2}=\boxed{\textbf{(E) }4}.$

PS:

1. This problem uses the following fraction trick:

For unequal positive numbers a,b,c and $d,$ if $\dfrac ab=\dfrac cd=k,$ then $\dfrac{a\pm c}{b\pm d}=k.$

<u>Quick Proof</u>

From $\dfrac ab=\dfrac cd=k,$ we know that $a=bk$ and $c=dk.$

Therefore, $\dfrac{a\pm c}{b\pm d}=\dfrac{bk\pm dk}{b\pm d}=\dfrac{(b\pm d)k}{b\pm d}=k.$

2. The work above shows that, regardless of the shape or the volume of the solid dropped in, as long as the solid sinks to the bottom and is completely submerged without spilling any liquid, the answer will remain unchanged.

~MRENTHUSIASM

Solution 1.2 (Bash)

Let the base radii of the narrow cone and the wide cone be r_1 and r_2, respectively.

Let the rises of the liquid levels of the narrow cone and the wide cone be Δh_1 and Δh_2, respectively. We have the following table:

	Base	**Height**	**Volume**
Narrow Cone	r_1	$h_1+\Delta h_1$	$\frac{1}{3}\pi r_1^2(h_1+\Delta h_1)$
Wide Cone	r_2	$h_2+\Delta h_2$	$\frac{1}{3}\pi r_2^2(h_2+\Delta h_2)$

By similar triangles discussed above, we have

$$\frac{3}{h_1}=\frac{r_1}{h_1+\Delta h_1} \quad \Rightarrow \quad r_1=\frac{3}{h_1}(h_1+\Delta h_1) \qquad (1)$$

$$\frac{6}{h_2}=\frac{r_2}{h_2+\Delta h_2} \quad \Rightarrow \quad r_2=\frac{6}{h_2}(h_2+\Delta h_2) \qquad (2)$$

The volume of the marble dropped in is $\frac{4}{3}\pi(1)^3=\frac{4}{3}\pi.$

Now, we set up an equation for the volume of the narrow cone and solve for Δh_1 :

$$\begin{aligned}
\frac{1}{3}\pi r_1^2(h_1+\Delta h_1) &= 3\pi h_1+\frac{4}{3}\pi \\
\frac{1}{3}\pi\underbrace{\left(\frac{3}{h_1}(h_1+\Delta h_1)\right)^2}_{\text{by }(1)}(h_1+\Delta h_1) &= 3\pi h_1+\frac{4}{3}\pi \\
\frac{3}{h_1^2}(h_1+\Delta h_1)^3 &= 3h_1+\frac{4}{3} \\
(h_1+\Delta h_1)^3 &= h_1^3+\frac{4h_1^2}{9} \\
\Delta h_1 &= \sqrt[3]{h_1^3+\frac{4h_1^2}{9}}-h_1.
\end{aligned}$$

Next, we set up an equation for the volume of the wide cone Δh_2 :

$\frac{1}{3}\pi r_2^2(h_2+\Delta h_2)=12\pi h_2+\frac{4}{3}\pi.$ Using the exact same process from above (but with different numbers), we get $\Delta h_2=\sqrt[3]{h_2^3+\frac{h_2^2}{9}}-h_2.$ Recall that $\frac{h_1}{h_2}=4.$ Therefore, the requested ratio is

$$\begin{aligned}\frac{\Delta h_1}{\Delta h_2}&=\frac{\sqrt[3]{h_1^3+\frac{4h_1^2}{9}}-h_1}{\sqrt[3]{h_2^3+\frac{h_2^2}{9}}-h_2}\\&=\frac{\sqrt[3]{(4h_2)^3+\frac{4(4h_2)^2}{9}}-4h_2}{\sqrt[3]{h_2^3+\frac{h_2^2}{9}}-h_2}\\&=\frac{\sqrt[3]{4^3\left(h_2^3+\frac{h_2^2}{9}\right)}-4h_2}{\sqrt[3]{h_2^3+\frac{h_2^2}{9}}-h_2}\\&=\frac{4\sqrt[3]{h_2^3+\frac{h_2^2}{9}}-4h_2}{\sqrt[3]{h_2^3+\frac{h_2^2}{9}}-h_2}\\&=\boxed{\textbf{(E) }4}.\end{aligned}$$

~MRENTHUSIASM

Solution 2 (Quick and dirty)

The heights of the cones are not given, so suppose the heights are very large (i.e. tending towards infinity) in order to approximate the cones as cylinders with base radii 3 and 6 and infinitely large height. Then the base area of the wide cylinder is 4 times that of the narrow cylinder. Since we are dropping a ball of the same volume into each cylinder, the water level in the narrow cone/cylinder should rise $\boxed{\textbf{(E) }4}$ times as much.

Problem13

What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 2\sqrt{3} \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 3\sqrt{3} \qquad \textbf{(E)}\ 6$

Solution

Drawing the tetrahedron out and testing side lengths, we realize that the triangles ABD and ABC are right triangles. It is now easy to calculate the volume of the tetrahedron using the formula for the volume of a pyramid: $\frac{3 \cdot 4 \cdot 2}{3 \cdot 2} = 4$, so we have an answer of $\boxed{C}$.

Problem14

All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B?

$\textbf{(A)}\ -88 \qquad \textbf{(B)}\ -80 \qquad \textbf{(C)}\ -64 \qquad \textbf{(D)}\ -41 \qquad \textbf{(E)}\ -40$

Solution 1:

By Vieta's formulae, the sum of the 6 roots is 10 and the product of the 6 roots is 16. By inspection, we see the roots are 1, 1, 2, 2, 2, and 2, so the function is

$(z-1)^2(z-2)^4 = (z^2-2z+1)(z^4-8z^3+24z^2-32z+16)$

. Therefore, $B = -32 - 48 - 8 = \boxed{\textbf{(A)} -88}$. ~JHawk0224

Solution 2:

Using the same method as Solution 1, we find that the roots are $2, 2, 2, 2, 1,$ and 1. Note that B is the negation of the 3rd symmetric sum of the roots. Using casework on the number of 1's in each of the $\binom{6}{3} = 20$ products $r_a \cdot r_b \cdot r_c$, we obtain $B = -\left(\binom{4}{3}\binom{2}{0} \cdot 2^3 + \binom{4}{2}\binom{2}{1} \cdot 2^2 \cdot 1 + \binom{4}{1}\binom{2}{2} \cdot 2\right) = -(32 + 48 + 8) = \boxed{\textbf{(A)} -88}$.

Problem15

Values for $A, B, C,$ and D are to be selected from $\{1, 2, 3, 4, 5, 6\}$ without replacement (i.e. no two letters have the same value). How many ways are there to make such choices so that the two curves $y = Ax^2 + B$ and $y = Cx^2 + D$ intersect? (The order in which the curves are listed does not matter; for example, the choices $A = 3, B = 2, C = 4, D = 1$ is considered the same as the choices $A = 4, B = 1, C = 3, D = 2.$)

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 60 \qquad \textbf{(C)}\ 90 \qquad \textbf{(D)}\ 180 \qquad \textbf{(E)}\ 360$

Solution 1 (Intuition):

Visualizing the two curves, we realize they are both parabolas with the same axis of symmetry. Now assume that the first equation is above the second, since order doesn't matter. Then $C > A$ and $B > D$. Therefore the number of ways to choose the four integers is $\tbinom{6}{2}\tbinom{4}{2} = 90$, and the answer is $\boxed{C}$. ~IceWolf10

Solution 2 (Algebra):

Setting $y = Ax^2 + B = Cx^2 + D$, we find that $Ax^2 - Cx^2 = x^2(A - C) = D - B$, so $x^2 = \dfrac{D - B}{A - C} \geq 0$ by the trivial inequality. This implies that $D - B$ and $A - C$ must both be positive or negative. If two distinct values are chosen for (A, C) and (B, D) respectively, there are 2 ways to order them so that both the numerator and denominator are positive/negative (increasing and decreasing). We must divide by 2 at the end, however, since the 2 curves aren't considered distinct. Calculating, we get $\frac{1}{2} \cdot \binom{6}{2}\binom{4}{2} \cdot 2 = \boxed{\textbf{(C) } 90}$.

Problem16

In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$ What is the median of the numbers in this list?

$\textbf{(A) } 100.5 \qquad \textbf{(B) } 134 \qquad \textbf{(C) } 142 \qquad \textbf{(D) } 150.5 \qquad \textbf{(E) } 167$

Solution 1

There are $1 + 2 + .. + 199 + 200 = \frac{(200)(201)}{2} = 20100$ numbers in total. Let the median be k. We want to find the median k such that $\frac{k(k+1)}{2} = 20100/2,$ or $k(k+1) = 20100.$ Note that $\sqrt{20100} \approx 142$. Plugging this value in as k gives $\frac{1}{2}(142)(143) = 10153.$ $10153 - 142 < 10050,$ so 142 is the 152nd and 153rd numbers, and hence, our desired answer. $\boxed{\text{(C) } 142}$.

Note that we can derive $\sqrt{20100} \approx 142$ through the formula $\sqrt{n} = \sqrt{a+b} \approx \sqrt{a} + \frac{b}{2\sqrt{a}+1},$ where a is a perfect square less than or equal to n. We set a to 19600, so $\sqrt{a} = 140$, and $b = 500$. We then have $n \approx 140 + \dfrac{500}{2(140)+1} \approx 142$.

Solution 2

The xth number of this sequence is $\left\lceil \dfrac{-1 \pm \sqrt{1+8x}}{2} \right\rceil$ via the quadratic formula. We can see that if we halve x we end up getting $\left\lceil \dfrac{-1 \pm \sqrt{1+4x}}{2} \right\rceil$. This is approximately the number divided by $\sqrt{2}$. $\dfrac{200}{\sqrt{2}} = 141.4$ and since 142 looks like the only number close to it, it is answer $\boxed{(C) 142}$ ~Lopkiloinm

Solution 3 (answer choices)

We can look at answer choice C, which is 142 first. That means that the number of numbers from 1 to $142$is roughly the number of numbers from 143 to 200.

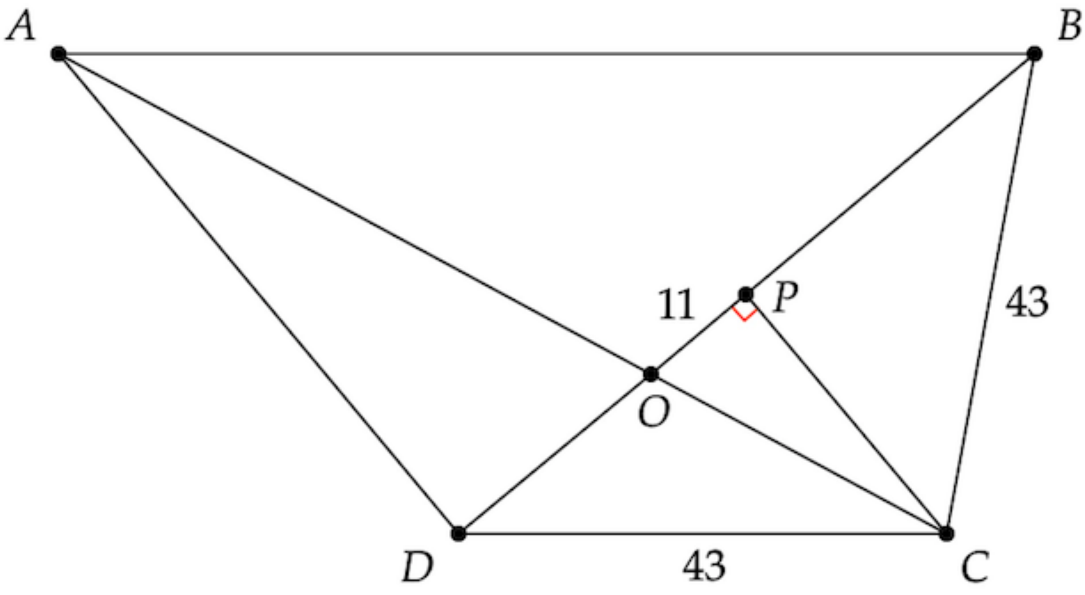
The number of numbers from 1 to 142 is $\dfrac{142(142+1)}{2}$ which is approximately $10000.$ The number of numbers from 143 to 200 is $\dfrac{200(200+1)}{2} - \dfrac{142(142+1)}{2}$ which is approximately 10000 as well. Therefore, we can be relatively sure the answer choice is $\boxed{(C)\ 142}$.

Problem17

Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}, BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

$\textbf{(A) } 65 \qquad \textbf{(B) } 132 \qquad \textbf{(C) } 157 \qquad \textbf{(D) } 194 \qquad \textbf{(E) } 215$

Diagram



~MRENTHUSIASM (by Geometry Expressions)

Solution 1

Angle chasing reveals that $\triangle BPC \sim \triangle BDA$, therefore $2 = \frac{BD}{BP} = \frac{AB}{BC} = \frac{AB}{43}$ $AB = 86$Additional angle chasing shows that $\triangle ABO \sim \triangle CDO$, therefore $2 = \frac{AB}{CD} = \frac{BP}{PD} = \frac{\frac{BD}{2}+11}{\frac{BD}{2}-11} BD = 66$

Since $\triangle ADB$ is right, the Pythagorean theorem implies that $AD = \sqrt{86^2 - 66^2} AD = 4\sqrt{190}$

$4\sqrt{190} \implies 4 + 190 = \boxed{\textbf{D) } 194}$

Solution 2 (One Pair of Similar Triangles, then Areas)

Since $\triangle BCD$ is isosceles with legs $\overline{CB}$ and $\overline{CD}$, it follows that the median $\overline{CP}$ is also an altitude of $\triangle BCD.$ Let $DO = x$ and $CP = h.$ We have $PB = x + 11.$

Since $\triangle ADO \sim \triangle CPO$ by AA, we have $AD = CP \cdot \frac{DO}{PO} = h \cdot \frac{x}{11}.$

Let the brackets denote areas. Notice that $[ADO] = [BCO]$ (By the same base/height, $[ADC] = [BCD].$ Subtracting $[OCD]$ from both sides gives $[ADO] = [BCO].$). Doubling both sides, we have

$$2[ADO] = 2[BCO]$$
$$\frac{x^2 h}{11} = (x+22)h$$
$$x^2 = 11x + 11 \cdot 22$$
$$(x-22)(x+11) = 0$$
$$x = 22.$$

In $\triangle CPB,$ we have $h = \sqrt{43^2 - 33^2} = \sqrt{76 \cdot 10} = 2\sqrt{190}$ and $AD = h \cdot \frac{x}{11} = 4\sqrt{190}.$ Finally, $4 + 190 = \boxed{\textbf{(D) } 194}.$

Solution 3 (short)

Let $CP = y$ and CP is perpendicular bisector of DB. Let $DO = x$, so $DP = PB = 11 + x$.

(1) $\triangle CPO \sim \triangle ADO$, so we get $\frac{AD}{x} = \frac{y}{11}$, or $AD = \frac{xy}{11}$.

(2) pythag on $\triangle CDP$ gives $(11 + x)^2 + y^2 = 43^2$.

(3) $\triangle BPC \sim \triangle BDA$ with ratio $1:2$, so $AD = 2y$.

Thus, $xy/11 = 2y$, or $x = 22$. And $y = \sqrt{43^2 - 33^2} = 2\sqrt{190}$, so $AD = 4\sqrt{190}$ and the answer is $\boxed{194}$.

Solution 4 - Extending the line

Observe that $\triangle BPC$ is congruent to $\triangle DPC$; both are similar to $\triangle BDA$. Let's extend $\overline{AD}$ and $\overline{BC}$ past points D and C respectively, such that they intersect at a point E. Observe that $\angle BDE$ is 90 degrees, and that $\angle DBE \cong \angle PBC \cong \angle DBA \implies \angle DBE \cong \angle DBA$. Thus, by ASA, we know that $\triangle ABD \cong \triangle EBD$, thus, $AD = ED$, meaning D is the midpoint of AE. Let M be the midpoint of $\overline{DE}$. Note that $\triangle CME$ is congruent to $\triangle BPC$, thus $BC = CE$, meaning C is the midpoint of $\overline{BE}$.

Therefore, $\overline{AC}$ and $\overline{BD}$ are both medians of $\triangle ABE$. This means that O is the centroid of $\triangle ABE$; therefore, because the centroid divides the median in a 2:1 ratio, $\frac{BO}{2} = DO = \frac{BD}{3}$. Recall that P is the midpoint of BD; $DP = \frac{BD}{2}$. The question tells us that $OP = 11$; $DP - DO = 11$; we can write this in terms of DB; $\frac{DB}{2} - \frac{DB}{3} = \frac{DB}{6} = 11 \implies DB = 66$.

We are almost finished. Each side length of $\triangle ABD$ is twice as long as the corresponding side length $\triangle CBP$ or $\triangle CPD$, since those triangles are similar; this means that $AB = 2 \cdot 43 = 86$. Now, by Pythagorean theorem on $\triangle ABD$,
$AB^2 - BD^2 = AD^2 \implies 86^2 - 66^2 = AD^2 \implies AD = \sqrt{3040} \implies AD = 4\sqrt{190}$
$4 + 190 = \boxed{194, \textbf{D}}$.

Problem18

Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Furthermore, suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

$$\textbf{(A)}\ \frac{17}{32} \qquad \textbf{(B)}\ \frac{11}{16} \qquad \textbf{(C)}\ \frac{7}{9} \qquad \textbf{(D)}\ \frac{7}{6} \qquad \textbf{(E)}\ \frac{25}{11}$$

Solution 1

Looking through the solutions we can see that $f(\frac{25}{11})$ can be expressed as $f(\frac{25}{11} \cdot 11) = f(11) + f(\frac{25}{11})$ so using the prime numbers to piece together what we have we can get $10=11+f(\frac{25}{11})$,

so $f(\frac{25}{11})=-1$ or $\boxed{E}$.

$$f(\frac{25}{11}\cdot 11)=f(25)=f(5)+f(5)=10$$

Solution 2

We know that $f(p)=f(p\cdot 1)=f(p)+f(1)$. By transitive, we have $f(p)=f(p)+f(1)$. Subtracting $f(p)$ from both sides gives $0=f(1)$. Also

$$f(2)+f\left(\frac{1}{2}\right)=f(1)=0 \implies 2+f\left(\frac{1}{2}\right)=0 \implies f\left(\frac{1}{2}\right)=-2$$

$$f(3)+f\left(\frac{1}{3}\right)=f(1)=0 \implies 3+f\left(\frac{1}{3}\right)=0 \implies f\left(\frac{1}{3}\right)=-3$$

$$f(11)+f\left(\frac{1}{11}\right)=f(1)=0 \implies 11+f\left(\frac{1}{11}\right)=0 \implies f\left(\frac{1}{11}\right)=-11$$

In $\textbf{(A)}$ we have $f\left(\frac{17}{32}\right)=17+5f\left(\frac{1}{2}\right)=17-5(2)=7$.

In $\textbf{(B)}$ we have $f\left(\frac{11}{16}\right)=11+4f\left(\frac{1}{2}\right)=11-4(2)=3$.

In $\textbf{(C)}$ we have $f\left(\frac{7}{9}\right)=7+2f\left(\frac{1}{3}\right)=7-2(3)=1$.

In $\textbf{(D)}$ we have $f\left(\frac{7}{6}\right)=7+f\left(\frac{1}{2}\right)+f\left(\frac{1}{3}\right)=7-2-3=2$.

In $\textbf{(E)}$ we have $f\left(\frac{25}{11}\right) = 10 + f\left(\frac{1}{11}\right) = 10 - 11 = -1$.

Thus, our answer is $\boxed{\textbf{(E)} \frac{25}{11}}$

~JHawk0224 ~awesomediabrine

Solution 3 (Deeper)

Consider the rational $\frac{a}{b}$, for a, b integers. We have $f(a) = f\left(\frac{a}{b} \cdot b\right) = f\left(\frac{a}{b}\right) + f(b)$.

So $f\left(\frac{a}{b}\right) = f(a) - f(b)$. Let p be a prime. Notice that $f(p^k) = kf(p)$. And $f(p) = p$. So if $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $f(a) = a_1p_1 + a_2p_2 + ... + a_kp_k$.

We simply need this to be greater than what we have for $f(b)$. Notice that for answer choices A, B, C, and D, the numerator (a) has less prime factors than the denominator, and so they are less likely to work. We check E first, and it works, therefore the answer is $\boxed{\textbf{(E)}}$.

~yofro

Solution 4 (Most Comprehensive, Similar to Solution 3)

We have the following important results:

$$(1) \ f\left(\prod_{k=1}^{n} a_k\right) = \sum_{k=1}^{n} f(a_k)$$ for all positive rational numbers a_k and positive integers n

$(2)\ f\left(a^n\right)=nf(a)$ for all positive rational numbers a and positive integers n

$(3)\ f(1)=0$

$(4)\ f\left(\frac{1}{a}\right)=-f(a)$ for all positive rational numbers a

Proofs

Result (1) can be shown by induction.

Result (2) : Since positive powers are just repeated multiplication of the base, we will use result (1) to prove result (2) :

$$f\left(a^n\right)=f\left(\prod_{k=1}^{n} a\right)=\sum_{k=1}^{n} f(a)=nf(a).$$

Result (3) : For all positive rational numbers $a,$ we have $f(a)=f(a\cdot 1)=f(a)+f(1).$ Therefore, we get $f(1)=0.$ So, result (3) is true.

Result (4) : For all positive rational numbers $a,$ we have $f(a)+f\left(\frac{1}{a}\right)=f\left(a\cdot\frac{1}{a}\right)=f(1)=0.$ It follows that $f\left(\frac{1}{a}\right)=-f(a),$ and result (4) is true.

For all positive integers x and y, suppose $\prod_{k=1}^{m} p_k^{e_k}$ and $\prod_{k=1}^{n} q_k^{d_k}$ are their prime factorizations, respectively, we have

$$\begin{aligned} f\left(\frac{x}{y}\right) &= f(x) + f\left(\frac{1}{y}\right) \\ &= f(x) - f(y) \\ &= f\left(\prod_{k=1}^{m} p_k^{e_k}\right) - f\left(\prod_{k=1}^{n} q_k^{d_k}\right) \\ &= \left[\sum_{k=1}^{m} f\left(p_k^{e_k}\right)\right] - \left[\sum_{k=1}^{n} f\left(q_k^{d_k}\right)\right] \\ &= \left[\sum_{k=1}^{m} e_k f\left(p_k\right)\right] - \left[\sum_{k=1}^{n} d_k f\left(q_k\right)\right] \\ &= \left[\sum_{k=1}^{m} e_k p_k\right] - \left[\sum_{k=1}^{n} d_k q_k\right]. \end{aligned}$$

We apply function f on each fraction in the choices:

$$\begin{array}{lcccccccc} \textbf{(A)} & f\left(\frac{17}{32}\right) & = & f\left(\frac{17^1}{2^5}\right) & = & [1(17)] - [5(2)] & = & 7 \\ \textbf{(B)} & f\left(\frac{11}{16}\right) & = & f\left(\frac{11^1}{2^4}\right) & = & [1(11)] - [4(2)] & = & 3 \\ \textbf{(C)} & f\left(\frac{7}{9}\right) & = & f\left(\frac{7^1}{3^2}\right) & = & [1(7)] - [2(3)] & = & 1 \\ \textbf{(D)} & f\left(\frac{7}{6}\right) & = & f\left(\frac{7^1}{2^1 \cdot 3^1}\right) & = & [1(7)] - [1(2) + 1(3)] & = & 2 \\ \textbf{(E)} & f\left(\frac{25}{11}\right) & = & f\left(\frac{5^2}{11^1}\right) & = & [2(5)] - [1(11)] & = & -1. \end{array}$$

Therefore, the answer is $\boxed{\textbf{(E) } \frac{25}{11}}$.

Solution 5

The problem gives us that f(p)=p. If we let a=p and b=1, we get f(p)=f(p)+f(1), which implies f(1)=0. Notice that the answer choices are all fractions, which means we will have to multiply an integer by a fraction to be able to solve it. Therefore, let's try plugging in fractions and try to solve them. Note that if we plug in a=p and b=1/p, we get f(1)=f(p)+f(1/p). We can solve for f(1/p) as -f(p)! This gives us the information we need to solve the problem. Testing out the answer choices gives us the answer of E.

Problem19

The area of the region bounded by the graph of $x^2 + y^2 = 3|x - y| + 3|x + y|$ is $m + n\pi$, where m and n are integers. What is $m + n$?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 27 \qquad \textbf{(C)}\ 36 \qquad \textbf{(D)}\ 45 \qquad \textbf{(E)}\ 54$

Solution 1

In order to attack this problem, we need to consider casework:

Case 1: $|x - y| = x - y, |x + y| = x + y$

Substituting and simplifying, we have $x^2 - 6x + y^2 = 0$,

i.e. $(x - 3)^2 + y^2 = 3^2$, which gives us a circle of radius 3 centered at $(3, 0)$.

Case 2: $|x - y| = y - x, |x + y| = x + y$

Substituting and simplifying again, we have $x^2 + y^2 - 6y = 0$,

i.e. $x^2 + (y - 3)^2 = 3^2$. This gives us a circle of radius 3 centered at $(0, 3)$.

Case 3: $|x - y| = x - y, |x + y| = -x - y$

Doing the same process as before, we have $x^2+y^2+6y=0$,

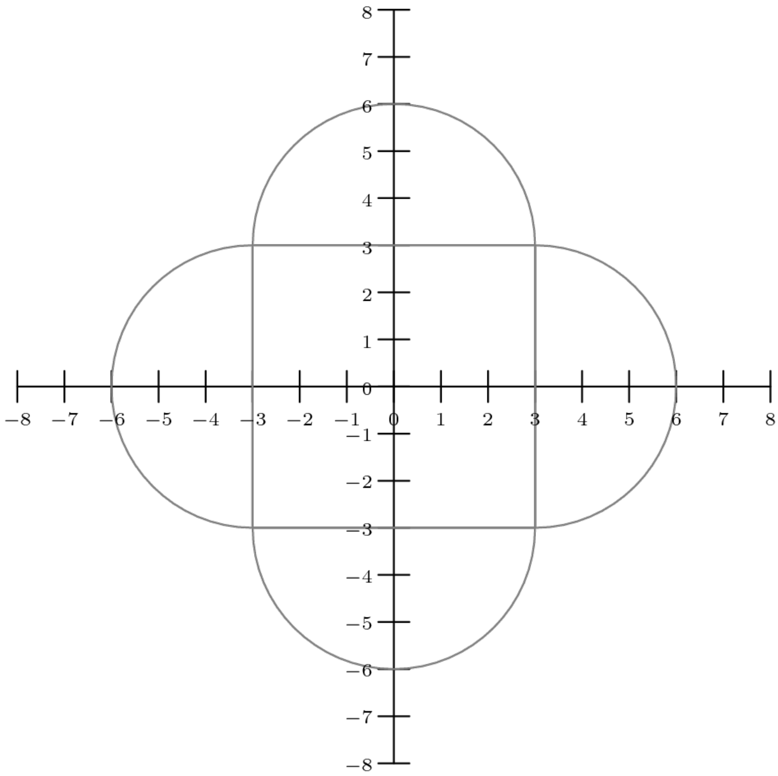
i.e. $x^2+(y+3)^2=3^2$. This gives us a circle of radius 3 centered at $(0,-3)$.

Case 4: $|x-y|=y-x, |x+y|=-x-y$

One last time: we have $x^2+y^2+6x=0$,

i.e. $(x+3)^2+y^2=3^2$. This gives us a circle of radius 3centered at $(-3,0)$.

After combining all the cases and drawing them on the Cartesian Plane, this is what the diagram looks like:



Now, the area of the shaded region is just a square with side length 6 with four semicircles of radius 3. The area is $6\cdot 6+4\cdot\dfrac{9\pi}{2}=36+18\pi$. The answer is $36+18$ which is $\boxed{\textbf{(E) } 54}$

Problem20

In how many ways can the sequence $1, 2, 3, 4, 5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

$\textbf{(A)}\ 10 \qquad\textbf{(B)}\ 18 \qquad\textbf{(C)}\ 24 \qquad\textbf{(D)}\ 32 \qquad\textbf{(E)}\ 44$

Solution 1 (Bashing)

We write out the 120 cases. These cases are the ones that work:

$13254, 14253, 14352, 15243, 15342, 21435, 21534, 23154, 24153, 24351, 25143, 25341, 31425, 31524, 32415, 32451, 34152, 34251, 35142, 35241, 41325, 41523, 42315, 42513, 43512, 45132, 45231, 51324, 51423, 52314, 52413, 53412.$

We count these out and get $\boxed{\text{D: } 32}$ permutations that work.

Solution 2 (Casework)

Reading the terms from left to right, we have two cases:

$\text{Case } \#1\text{: } +, -, +, -$

$\text{Case } \#2\text{: } -, +, -, +$

($+$ stands for increase and $-$ stands for decrease.)

For $\text{Case } \#1,$ note that for the second and fourth terms, one of which must be a $5,$ and the other one must be a 3 or $4.$ We have four sub-cases:

$(1)\ \_3\_5\_$

$(2)\ \_5\_3\_$

$(3)\ \_4\_5\_$

$(4)\ \_5\_4\_$

For (1), the first two blanks must be 1 and 2 in some order, and the last blank must be a 4, for a total of 2possibilities. Similarly, (2) also has 2 possibilities.

For (3), there are no restrictions for the numbers $1, 2,$ and $3.$ So, we have $3! = 6$ possibilities. Similarly, (4) also has 6 possibilities.

Together, $\text{Case } \#1$ has $2+2+6+6=16$ possibilities. By symmetry, $\text{Case } \#2$ also has 16possibilities. Together, the answer is $16+16=\boxed{\textbf{(D) } 32}$.

Solution 3 (similar to solution 2)

Like Solution 2, we have two cases. Due to symmetry, we just need to count one of the cases. For the purpose of this solution, we will be doing $-,+,-,+$. Instead of starting with 5, we start with 1.

There are two ways to place it:

_1_ _ _

_ _ _1_

Now we place 2, it can either be next to 1 and on the outside, or is place in where 1 would go in the other case. So now we have another two "sub case":

_1_2_(case 1)

21_ _ _(case 2)

There are 3! ways to arrange the rest for case 1, since there is no restriction.

For case 2, we need to consider how many ways to arrange 3,4,5 in a $a>b<c$ fashion. It should seem pretty obvious that b has to be 3, so there will be 2! way to put 4 and 5.

Now we find our result, times 2 for symmetry, times 2 for placement of 1 and times (3!+2!) for the two different cases for placement of 2. This give us $2*2*(3!+2!) = 4*(6+2) = 32$.

Solution 4: Symmetry

We only need to find the # of rearrangements when 5 is the 4th digit and 5th digit. Find the total, and multiply by 2. Then we can get the answer by adding the case when 5 is the third digit.

Case 1: 5 is the 5th digit. __ __ __ __ 5

Then 4 can only be either 1st digit or the 3rd digit.

4 __ __ __ 5, then the only way is that 3 is the 3rd digit, so it can be either 231 or 132, give us 2 results.

__ __ 4 __ 5, then the 1st digit must be 2 or 3, 2 gives us 1 way, and 3 gives us 2 ways. (Can't be 1 because the first digit would increasing). Therefore, 4 in the middle and 5 in the last would result in 3 ways.

Case 2: 5 is the fourth digit. __ __ __ 5 __

Then the last digit can be all of the 4 numbers 1, 2, 3, and 4. Let's say if the last digit is 4, then the 2nd digit would be the largest for the remaining digits to prevent increasing order or decreasing order. Then the remaining two are interchangeable, give us $2!$ ways. All of the 4 can work, so case 2 would result in $2! + 2! + 2! + 2! = 8$ ways.

Case 3: 5 is in the middle. __ __ 5 __ __

Then there are only two cases: 1. 42513, then 4 and 3 are interchangeable, which results in $2! * 2!$. Or it can be 43512, then 4 and 2 are interchangeable, but it can not be 23514, so there can only be 2 possible ways: $43512, 21534$.

Therefore, case 3 would result in $4 + 2 = 6$ ways.

$8 + 3 + 2 = 13$, so the total ways for case 1 and case 2 with both increasing and decreasing would be $13 * 2 = 26.$

$26 + 6 = \boxed{\textbf{(D) }32}$.

Problem21

Let $ABCDEF$ be an equiangular hexagon. The lines $AB, CD,$ and EF determine a triangle with area $192\sqrt{3}$, and the lines $BC, DE,$ and FA determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m+n\sqrt{p}$, where $m, n,$ and p are positive integers and p is not divisible by the square of any prime. What is $m+n+p$?

$\textbf{(A)}\ 47 \qquad \textbf{(B)}\ 52 \qquad \textbf{(C)}\ 55 \qquad \textbf{(D)}\ 58 \qquad \textbf{(E)}\ 63$

Solution (Misplaced problem?)

Note that the extensions of the given lines will determine an equilateral triangle because the hexagon is equiangular. The area of the first triangle is $192\sqrt{3}$, so the side length is $\sqrt{192 \cdot 4} = 16\sqrt{3}$. The area of the second triangle is $324\sqrt{3}$, so the side length is $\sqrt{4 \cdot 324} = 36$. We can set the first value equal to $AB+CD+EF$ and the second equal to $BC+DE+FA$ by substituting some lengths in with different sides of the same equilateral triangle. The perimeter of the hexagon is just the sum of these two, which is $16\sqrt{3}+36$ and $16+3+36=\boxed{55\ \textbf{(C)}}$

Problem22

Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 13 \qquad \textbf{(C)}\ 15 \qquad \textbf{(D)}\ 17 \qquad \textbf{(E)}\ 20$

Solution

Suppose the roommate took pages a through b, or equivalently, page numbers $2a-1$ through $2b$. Because there are $(2b-2a+2)$ numbers taken,

$$\frac{(2a-1+2b)(2b-2a+2)}{2}+19(50-(2b-2a+2))=\frac{50*51}{2} \implies (2a+2b-39)(b-a+1)=\frac{50*13}{2}=25*13.$$

The first possible solution that comes to mind is if

$$2a+2b-39=25, b-a+1=13 \implies a+b=32, b-a=12$$

, which indeed works, giving $b=22$ and $a=10$. The answer is $22-10+1=\boxed{(\mathbf{B})13}$

Solution 2 (Different Variable Choice, Similar Logic)

Suppose the smallest page number removed is $k,$ and n pages are removed. It follows that the largest page number removed is $k+n-1.$

Remarks:

1. n pages are removed means that $\frac{n}{2}$ sheets are removed, from which n must be even.
2. k must be odd, as the smallest page number removed is on the right side (odd-numbered).
3. $$1+2+3+\cdots+50=\frac{51(50)}{2}=1275.$$
4. The sum of the page numbers removed is $\frac{(2k+n-1)n}{2}.$

$$\frac{1275 - \frac{(2k+n-1)n}{2}}{50-n} = 19$$
$$1275 - \frac{(2k+n-1)n}{2} = 19(50-n)$$
$$2550 - (2k+n-1)n = 38(50-n)$$
$$2550 - (2k+n-1)n = 1900 - 38n$$

Together, we have $650 = (2k+n-39)n.$

The factors of 650 are $1, 2, 5, 10, 13, 25, 26, 50, 65, 130, 325, 650.$

Since n is even, we only have a few cases to consider:

$\boldsymbol{n}$	$\boldsymbol{2k+n-39}$	$\boldsymbol{k}$
2	325	181
10	65	47
26	25	19
50	13	1
130	5	negative
650	1	negative

Since $1 \leq k \leq 50,$ only $k = 47, 19, 1$ are possible:

If $k = 47,$ then the note pages will run out if we take 10 pages starting from page $47.$

If $k = 1,$ then the average page number of the remaining pages will be undefined, as there is no page remaining (after taking 50 pages starting from page 1).

So, the only possibility is $k = 19,$ from which $n = 26$ pages are taken out, which is $\frac{n}{2} = \boxed{\textbf{(B) } 13}$ sheets.

Solution 3

Let n be the number of sheets borrowed, with an average page number $k+25.5$. The remaining $25-n$sheets have an average page number of 19 which is less than 25.5, the average page number of all 50 pages, therefore $k>0$. Since the borrowed sheets start with an odd page number and end with an even page number we have $k\in\mathbb{N}$. We notice that $n<25$ and $k\leq(49+50)/2-25.5=24<25$.

The weighted increase of average page number from 25.5 to $k+25.5$ should be equal to the weighted decrease of average page number from 25.5 to 19, where the weights are the page number in each group (borrowed vs. remained), therefore

$$2nk=2(25-n)(25.5-19)=13(25-n)\implies 13|n\text{ or }13|k$$

Since $n,k<25$ we have either $n=13$ or $k=13$.
If $n=13$ then $k=6$. If $k=13$ then $2n=25-n$which is impossible. Therefore the answer should be $n=\boxed{\textbf{(B) } 13}$

Problem23

Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

$$\textbf{(A) }\frac{9}{16}\qquad\textbf{(B) }\frac{5}{8}\qquad\textbf{(C) }\frac{3}{4}\qquad\textbf{(D) }\frac{25}{32}\qquad\textbf{(E) }\frac{13}{16}$$

Solution 1 (complementary counting)

We will use complementary counting. First, the frog can go left with probability $\frac{1}{4}$. We observe symmetry, so our final answer will be multiplied by 4 for the 4 directions, and since $4 \cdot \frac{1}{4} = 1$, we will ignore the leading probability.

From the left, she either goes left to another edge ($\frac{1}{4}$) or back to the center ($\frac{1}{4}$). Time for some casework.

$\textbf{Case 1:}$ She goes back to the center.

Now, she can go in any 4 directions, and then has 2 options from that edge. This gives $\frac{1}{2}$. --End case 1

$\textbf{Case 2:}$ She goes to another edge (rightmost).

Subcase 1: She goes back to the left edge. She now has 2 places to go, giving $\frac{1}{2}$

Subcase 2: She goes to the center. Now any move works.

$\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot 1 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$ for this case. --End case 2

She goes back to the center in Case 1 with probability $\frac{1}{4}$, and to the right edge with probability $\frac{1}{4}$

So, our answer is $\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{8} = \frac{1}{4}(\frac{1}{2} + \frac{3}{8}) = \frac{1}{4} \cdot \frac{7}{8} = \frac{7}{32}$

But, don't forget complementary counting. So, we get $1 - \frac{7}{32} = \frac{25}{32} \implies \boxed{D}$. ~ firebolt360

Video Solution for those who prefer: https://youtu.be/ude2rzO1cmk ~ firebolt360

Solution 2 (direct counting and probability states)

We can draw a state diagram with three states: center, edge, and corner. Denote center by M, edge by E, and corner by C. There are a few ways Frieda can reach a corner in four or less moves: EC, EEC, EEEC, EMEC. Then, calculating the probabilities of each of these cases happening, we have $1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{1}{2} = \frac{25}{32}$, so the answer is $\boxed{D}$. ~IceWolf10

Solution 3 (Similar to Solution 2, but Finds the Numerator and Denominator Separately)

Denominator

There are $4^4 = 256$ ways to make 4 hops without restrictions.

Numerator (Casework)

Suppose Frieda makes 4 hops without stopping. We perform casework on which hop reaches a corner for the first time.

(1) Hop $\#2$ (Hops $\#3$ and $\#4$ have no restrictions)

The 4 independent hops have $4, 2, 4, 4$ options, respectively. So, this case has $4 \cdot 2 \cdot 4 \cdot 4 = 128$ ways.

(2) Hop $\#3$ (Hop $\#4$ has no restriction)

No matter which direction the first hop takes, the second hop must "wrap around".

The 4 independent hops have $4, 1, 2, 4$ options, respectively. So, this case has $4 \cdot 1 \cdot 2 \cdot 4 = 32$ ways.

(3) Hop $\#4$

Two sub-cases:

(3.1) The second hop "wraps around". It follows that the third hop also "wraps around".

The 4 independent hops have $4, 1, 1, 2$ options, respectively. So, this sub-case has $4 \cdot 1 \cdot 1 \cdot 2 = 8$ ways.

(3.2) The second hop backs to the center.

The 4 independent hops have $4, 1, 4, 2$ options, respectively. So, this sub-case has $4 \cdot 1 \cdot 4 \cdot 2 = 32$ways.

Together, Case (3) has $8 + 32 = 40$ ways.

The numerator is $128 + 32 + 40 = 200.$

Probability

$$\frac{200}{256} = \boxed{\textbf{(D)}\ \frac{25}{32}}.$$

Solution 4

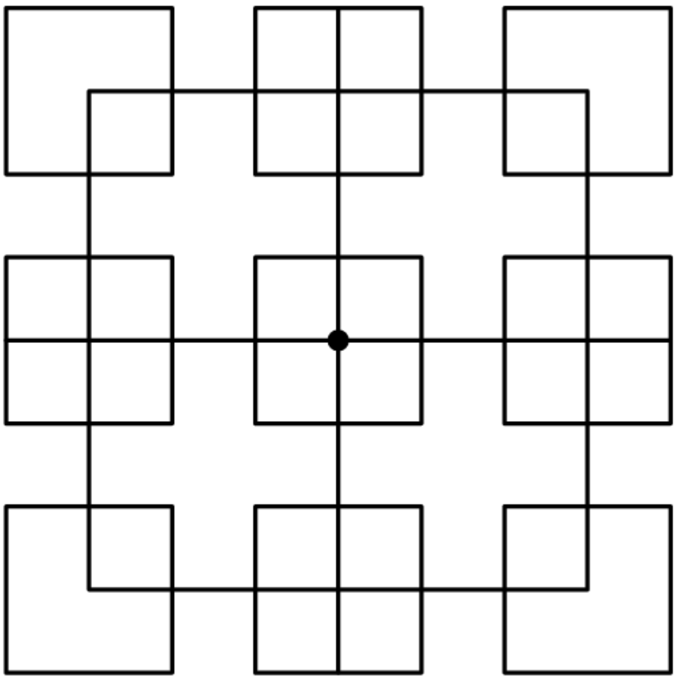
Let C_n be the probability that Frieda is on the central square after n moves, E_n be the probability that Frieda is on one of the four squares on the middle of the edges after n moves, and V_n (V for vertex) be the probability that Frieda is on a corner after n moves. The only way to reach the center is by moving in 1specific direction out of 4 total directions from the middle of an edge, so $C_{n+1} = \frac{E_n}{4}$. The ways to reach the middle of an edge are by moving in any direction from the center or by moving in 1 specific direction from the middle of an edge, so $E_{n+1} = C_n + \frac{E_n}{4}$. The ways to reach a corner are by simply staying there after reaching there in a previous move or by moving in 2 specific directions from the middle of an edge, so $V_{n+1} = V_n + \frac{E_n}{2}$.

Since Frieda always start from the center, $C_0 = 1$, $E_0 = 0$, and $V_0 = 0$.

We use the previous formulas to work out V_4 and find it to be $\boxed{\textbf{(D)}\ \frac{25}{32}}$.

Solution 5

Imagine an infinite grid of 2 by 2 squares such that there is a 2 by 2 square centered at $(3x, 3y)$ for all ordered pairs of integers (x, y).



It is easy to see that the problem is equivalent to Frieda moving left, right, up, or down on this infinite grid starting at $(0, 0)$. (minus the teleportations) Since counting the complement set is easier, we'll count the number of 4-step paths such that Frieda never reaches a corner point.

In other words, since the reachable corner points are $(\pm1, \pm1), (\pm1, \pm2), (\pm2, \pm1)$, and $(\pm2, \pm2)$, Frieda can only travel along the collection of points included in S, where S is all points on $x = 0$ and $y = 0$ such that $|y| < 4$ and $|x| < 4$, respectively, plus all points on the big square with side length 6centered at $(0,0)$. We then can proceed with casework:

Case 1: Frieda never reaches $(0,\pm 3)$ nor $(\pm 3,0)$.

When Frieda only moves horizontally or vertically for her four moves, she can do so in $2^4-4=12$ ways for each case . Thus, $12\cdot 2$ total paths for the subcase of staying in one direction. (For instance, all length 4 combinations of F and B except $FFFF$, $BBBB$, $FFFB$, and $BBBF$ for the horizontal direction.)

There is another subcase where she changes directions during her path. There are four symmetric cases for this subcase depending on which of the four quadrants Frieda hugs. For the first quadrant, the possible paths are $FBUD$, $FBUU$, $UDFB$, and $UDFF$. Thus, a total of $4\cdot 4=16$ ways for this subcase.

Total for Case 1: $24+16=40$

Case 2: Frieda reaches $(0,\pm 3)$ or $(\pm 3,0)$.

Once Frieda reaches one of the points listed above (by using three moves), she has four choices for her last move. Thus, a total of $4\cdot 4=16$ paths for this case.

Our total number of paths never reaching coroners is

thus $16+40=56$, making for an answer of $\frac{4^4-56}{4^4}=\boxed{\textbf{(D)}\ \frac{25}{32}}$.

Solution 6 (Casework)

We take cases on the number of hops needed to reach a corner. For simplicity, denote E as a move that takes Frieda to an edge, W as wrap-around move and C as a corner move. Also, denote O as a move that takes us to the center.

2 Hops

Then, Frieda will have to (E,C) as her set of moves. There are 4 ways to move to an edge, and 2 corners to move to, for a total of $4\cdot 2=8$ cases here. Then, there are 4 choices for each move, for a probability of $\frac{8}{4 \cdot 4} = \frac{1}{2}$.

3 Hops

In this case, Frieda must wrap-around. There's only one possible combination, just (E, W, C). There are 4ways to move to an edge, 1 way to wrap-around (you must continue in the same direction) and 2 corners, for a total of $4 \cdot 1 \cdot 2 = 8$ cases here. Then, there are 4 choices for each move, for a probability of $\frac{8}{4 \cdot 4 \cdot 4} = \frac{1}{8}$.

4 Hops

Lastly, there are two cases we must consider here. The first case is (E, O, E, C), and the second is (E, W, W, C). For the first case, there are 4 ways to move to an edge, 1 way to return to the center, 4ways to move to an edge once again, and 2 ways to move to a corner. Hence, there is a total of $4 \cdot 1 \cdot 4 \cdot 2 = 32$ cases here. Then, for the second case, there are 4 ways to move to a corner, 1 way to wrap-around, 1 way to wrap-around again, and 2 ways to move to a corner. This implies there are $4 \cdot 1 \cdot 1 \cdot 2 = 8$ cases here. Then, there is a total of $8 + 32 = 40$ cases, out of a total of $4^4 = 256$ cases, for a probability of $\frac{40}{256} = \frac{5}{32}$.

Then, the total probability that Frieda ends up on a corner is $\frac{1}{2} + \frac{1}{8} + \frac{5}{32} = \frac{25}{32}$, corresponding to choice $\boxed{\textbf{(D)}\ \frac{25}{32}}$. ~rocketsri

Solution 7

I denote 3x3 grid by

- HOME square (x1)

- CORN squares (x4)

- SIDE squares (x4)

Transitions:

- HOME always move to SIDE

- CORN is DONE

- SIDE move to CORN with $p=1/4,$ move to SIDE with $p=1/4,$ and move to CORN with $p=1/2.$

After one move, will be on SIDE square

After two moves, will be $1/2+1/4(\text{HOME})+1/4(\text{SIDE})$

After three moves, will be

$$1/2+1/4(\text{SIDE})+1/4(1/2+1/4(\text{SIDE})+1/4(\text{HOME}))$$

After four moves, probability on CORN will be $1/2+1/4(1/2)+1/4(1/2+1/4(1/2))=1/2+1/8+5/32=25/32.$

Problem24

The interior of a quadrilateral is bounded by the graphs of $(x+ay)^2=4a^2$ and $(ax-y)^2=a^2$, where a a positive real number. What is the area of this region in terms of a, valid for all $a>0$?

$\textbf{(A)}\ \frac{8a^2}{(a+1)^2}\qquad\textbf{(B)}\ \frac{4a}{a+1}\qquad\textbf{(C)}\ \frac{8a}{a+1}\qquad\textbf{(D)}\ \frac{8a^2}{a^2+1}\qquad\textbf{(E)}\ \frac{8a}{a^2+1}$

Diagram

Graph in Desmos: https://www.desmos.com/calculator/nagjmnkywx

~MRENTHUSIASM

Solution 1

The conditions $(x+ay)^2=4a^2$ and $(ax-y)^2=a^2$ give $|x+ay|=|2a|$ and $|ax-y|=|a|$ or $x+ay=\pm 2a$ and

$ax - y = \pm a$. The slopes here are perpendicular, so the quadrilateral is a rectangle. Plug in $a = 1$ and graph it. We quickly see that the area is $2\sqrt{2} \cdot \sqrt{2} = 4$, so the answer can't be A or $B$by testing the values they give (test it!). Now plug in $a = 2$. We see using a ruler that the sides of the rectangle are about $\frac{7}{4}$ and $\frac{7}{2}$. So the area is about $\frac{49}{8} = 6.125$.

Testing C we get $\frac{16}{3}$ which is clearly less than 6, so it is out. Testing D we get $\frac{32}{5}$ which is near our answer, so we leave it. Testing E we get $\frac{16}{5}$, way less than 6, so it is out. So, the only plausible answer is $\boxed{D}$ ~firebolt360

Solution 2 (Casework)

For the equation $(x + ay)^2 = 4a^2$, the cases are

$(1)\ x + ay = 2a$. This is a line with x-intercept $2a$, y-intercept 2, and slope $-\frac{1}{a}$.

$(2)\ x + ay = -2a$. This is a line with x-intercept $-2a$, y-intercept -2, and slope $-\frac{1}{a}$.

For the equation $(ax - y)^2 = a^2$, the cases are

$(1')\ ax - y = a$. This is a line with x-intercept 1, y-intercept $-a$, and slope a.

$(2')\ ax - y = -a$. This is a line with x-intercept -1, y-intercept a, and slope a.

Plugging $a = 2$ into the choices gives

$\textbf{(A)}\ \frac{32}{9} \qquad \textbf{(B)}\ \frac{8}{3} \qquad \textbf{(C)}\ \frac{16}{3} \qquad \textbf{(D)}\ \frac{32}{5} \qquad \textbf{(E)}\ \frac{16}{5}$

Plugging $a=2$ into the four above equations and solving systems of equations for intersecting lines [(1) and $(1')$, (1) and $(2')$, (2) and $(1')$, (2) and $(2')$], we get the respective solutions $(x,y)=\left(\frac{8}{5},\frac{6}{5}\right),(0,2),\left(-\frac{8}{5},-\frac{6}{5}\right),(0,-2).$

Solution 2.1 (Rectangle)

Since the slopes of the intersecting lines (from the four above equations) are negative reciprocals, the quadrilateral is a rectangle. Finally, by the Distance Formula, the length and width of the rectangle are $\frac{8\sqrt5}{5}$ and $\frac{4\sqrt5}{5}.$ The area we seek is $\left(\frac{8\sqrt5}{5}\right)\left(\frac{4\sqrt5}{5}\right)=\frac{32}{5}.$

The answer is $\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}.$

~MRENTHUSIASM

Solution 2.2 (Shoelace Formula)

Even if we do not recognize that the solutions form the vertices of a rectangle, we can apply the Shoelace Formula on **consecutive** vertices

$$\begin{aligned}(x_1,y_1)&=\left(\frac{8}{5},\frac{6}{5}\right), \\ (x_2,y_2)&=(0,2), \\ (x_3,y_3)&=\left(-\frac{8}{5},-\frac{6}{5}\right), \\ (x_4,y_4)&=(0,-2).\end{aligned}$$

The area formula is

$$\begin{aligned} A &= \frac{1}{2}\left|(x_1y_2+x_2y_3+x_3y_4+x_4y_1)-(y_1x_2+y_2x_3+y_3x_4+y_4x_1)\right| \\ &= \frac{1}{2}\left|\left[\frac{8}{5}\cdot 2+0\cdot\left(-\frac{6}{5}\right)+\left(-\frac{8}{5}\right)\cdot(-2)+0\cdot\frac{6}{5}\right]-\left[\frac{6}{5}\cdot 0+2\cdot\left(-\frac{8}{5}\right)+\left(-\frac{6}{5}\right)\cdot 0+(-2)\cdot\frac{8}{5}\right]\right| \\ &= \frac{1}{2}\left|\left[\frac{16}{5}+\frac{16}{5}\right]-\left[-\frac{16}{5}-\frac{16}{5}\right]\right| \\ &= \frac{1}{2}\left|\frac{64}{5}\right| \\ &= \frac{32}{5}. \end{aligned}$$

Therefore, the answer is $\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}$.

Suggested Reading for the Shoelace Formula: https://artofproblemsolving.com/wiki/index.php/Shoelace_Theorem

~MRENTHUSIASM

Solution 3 (Geometry)

Similar to Solution 2, we will use the equations of the four cases:

(1) $x+ay=2a.$ This is a line with x-intercept $2a$, y-intercept 2, and slope $-\frac{1}{a}$.

(2) $x+ay=-2a.$ This is a line with x-intercept $-2a$, y-intercept -2, and slope $-\frac{1}{a}$.

(3)* $ax-y=a.$ This is a line with x-intercept 1, y-intercept $-a$, and slope $a.$

(4)* $ax-y=-a.$ This is a line with x-intercept -1, y-intercept a, and slope $a.$

The area of the rectangle created by the four equations can be written as $2a\cdot\cos A\cdot 4\sin A$

= $8a\cos A\cdot\sin A$

= $8a \cdot \dfrac{1}{\sqrt{a^2+1}} \cdot \dfrac{a}{\sqrt{a^2+1}}$

= $\boxed{\textbf{(D) } \dfrac{8a^2}{a^2+1}}$.

(Note: $\tan A =$ slope a)

-fnothing4994

Solution 4 (bruh moment solution)

Trying $a=1$ narrows down the choices to options $\textbf{(C)}$, $\textbf{(D)}$ and $\textbf{(E)}$.

Trying $a=2$ and $a=3$ eliminates $\textbf{(C)}$ and $\textbf{(E)}$, to obtain $\boxed{\textbf{(D) } \dfrac{8a^2}{a^2+1}}$.

Problem25

How many ways are there to place 3 indistinguishable red chips, 3 indistinguishable blue chips, and 3indistinguishable green chips in the squares of a 3×3 grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally?

$\textbf{(A) } 12 \qquad \textbf{(B) } 18 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 30 \qquad \textbf{(E) } 36$

Solution 1

Call the different colors A,B,C. There are $3! = 6$ ways to rearrange these colors to these three letters, so 6must be multiplied after the letters are permuted in the grid. WLOG assume that A is in the center. $\begin{array}{ccc} ? & ? & ? \\ ? & \text{A} & ? \\ ? & ? & ? \end{array}$In this configuration, there are two cases, either all the A's lie on the same diagonal:$\begin{array}{ccc} ? & ? & \text{A} \\ ? & \text{A} & ? \\ \text{A} & ? & ? \end{array}$or all the other two A's are on adjacent corners: $\begin{matrix} A & ? & A \\ ? & A & ? \\ ? & ? & ? \end{matrix}$ In the first case there are two ways to order them since there are two diagonals, and in the second case there are four ways to order them since there are four pairs of adjacent corners.

In each case there is only one way to put the three B's and the three C's as shown in the diagrams. $\begin{matrix} C & B & AA & B & A \\ B & A & CC & A & C \\ A & C & BB & C & B \end{matrix}$ This means that there are $4+2=6$ ways to arrange A,B, and C in the grid, and there are 6 ways to rearrange the colors. Therefore, there are $6 \cdot 6 = 36$ ways in total, which is $\boxed{\text{E}}$.

-happykeeper

Solution 2 (Casework)

Without the loss of generality, we place a red ball in the top-left square. There are two cases:

Case (1): The two balls adjacent to the top-left red ball have different colors.

R	B	
G	R	

Each placement has 6 permutations, as there are $3! = 6$ ways to permute RBG.

There are three sub-cases for Case (1):

R	B	R
G	R	G
B	G	B

R	B	G
G	R	B
R	B	G

R	B	G
G	R	B
B	G	R

So, Case (1) has $3 \cdot 6 = 18$ ways.

Case (2): The two balls adjacent to the top-left red ball have the same

R	B	
B		

color. Each placement has 6 permutations, as there are $\binom{3}{2}\binom{2}{1} = 6$ ways to choose three balls consisting of exactly two colors (RBB, RGG, BRR, BGG, GRR, GBB). There are three sub-cases for Case (2):

R	B	R
B	G	B
G	R	G

R	B	G
B	G	R
R	B	G

R	B	G
B	G	R
G	R	B

So, Case (2) has $3 \cdot 6 = 18$ ways.

Together, the answer is $18 + 18 = \boxed{\textbf{(E) } 36}$.

~MRENTHUSIASM

Solution 3 (Casework and Derangements)

Case (1): We have a permutation of R, B, and G as all of the rows. There are $3!$ ways to rearrange these three colors. After finishing the first row, we move onto the second. Notice how the second row must be a derangement of the first one. By the derangement formula, $\frac{3!}{e} \approx 2$, so there are two possible permutations of the second row. (Note: You could have also found the number of derangements of PIE). Finally, there are 2 possible permutations for the last row. Thus, there are $3! \cdot 2 \cdot 2 = 24$ possibilities.

Case (2): All of the rows have two balls that are the same color and one that is different. There are obviously 3 possible configurations for the first row, 2 for the second, and 2 for the third. $3 \cdot 2 \cdot 2 = 12$.

Therefore, our answer is $24 + 12 = \boxed{\textbf{(E) } 36}$.