2021 AMC 10B Solution

Problem1

How many integer values of x satisfy $|x| < 3\pi$?

$\textbf{(A)}\ 9 \qquad\textbf{(B)}\ 10 \qquad\textbf{(C)}\ 18 \qquad\textbf{(D)}\ 19 \qquad\textbf{(E)}\ 20$

Solution 1

Since 3π is about 9.42, we multiply 9 by 2 for the numbers from 1 to 9 and the numbers from -1 to $-9$and add 1 to account for the zero to get $\boxed{\textbf{(D)}\ 19}$
~smarty101 and edited by Tony_Li2007

Solution 2

$3\pi \approx 9.4.$ There are two cases here.

When $x > 0, |x| > 0,$ and $x = |x|.$ So then $x < 9.4$

When $x < 0, |x| > 0,$ and $x = -|x|.$ So then $-x < 9.4$. Dividing by -1 and flipping the sign, we get $x > -9.4.$

From case 1 and 2, we know that $-9.4 < x < 9.4$. Since x is an integer, we must have x between $-9$and 9. There are a total of $9 - (-9) + 1 = \boxed{\textbf{(D)}\ 19}$ integers.

Problem2

What is the value of $\sqrt{(3-2\sqrt{3})^2}+\sqrt{(3+2\sqrt{3})^2}$?

$\textbf{(A)}\ 0 \qquad\textbf{(B)}\ 4\sqrt{3}-6 \qquad\textbf{(C)}\ 6 \qquad\textbf{(D)}\ 4\sqrt{3} \qquad\textbf{(E)}\ 4\sqrt{3}+6$

Solution

Note that the square root of any square is always the absolute value of the squared number because the square root function will only return a positive number. By squaring both 3 and $2\sqrt{3}$, we see that $2\sqrt{3} > 3$, thus $3 - 2\sqrt{3}$ is negative, so we must take the absolute value of $3 - 2\sqrt{3}$, which is just $2\sqrt{3} - 3$. Knowing this, the first term in the expression equals $2\sqrt{3} - 3$ and the second term is $3 + 2\sqrt{3}$, and summing the two gives $\boxed{\textbf{(D) } 4\sqrt{3}}$.

~bjc, abhinavg0627 and JackBocresion

Solution 2

Let $x = \sqrt{(3-2\sqrt{3})^2} + \sqrt{(3+2\sqrt{3})^2}$,

then $x^2 = (3-2\sqrt{3})^2 + 2\sqrt{(-3)^2} + (3+2\sqrt{3})^2$.

The $2\sqrt{(-3)^2}$ term is there due to difference of squares. Simplifying the expression gives us $x^2 = 48$, so $x = \boxed{\textbf{(D) } 4\sqrt{3}}$ ~ shrungpatel

Problem3

In an after-school program for juniors and seniors, there is a debate team with an equal number of students from each class on the team. Among the 28 students in the program, 25% of the juniors and 10% of the seniors are on the debate team. How many juniors are in the program?

$\textbf{(A) } 5 \qquad \textbf{(B) } 6 \qquad \textbf{(C) } 8 \qquad \textbf{(D) } 11 \qquad \textbf{(E) } 20$

Solution 1

Say there are j juniors and s seniors in the program. Converting percentages to fractions, $\frac{j}{4}$ and $\frac{s}{10}$ are on the debate team, and since an equal number of juniors and seniors are on the debate team, $\frac{j}{4} = \frac{s}{10}.$

Cross-multiplying and simplifying we get $5j = 2s.$ Additionally, since there are 28 students in the program, $j + s = 28.$ It is now a matter of solving the system of equations $5j = 2sj + s = 28,$and the solution is $j = 8, s = 20.$ Since we want the number of juniors, the answer is $\boxed{(C)\ 8}$.

Solution 2 (Fast and not rigorous)

We immediately see that E is the only possible amount of seniors, as 10% can only correspond with an answer choice ending with 0. Thus the number of seniors is 20 and the number of juniors is $28 - 20 = 8 \rightarrow \boxed{C}$.

Solution 3

Since there are an equal number of juniors and seniors on the debate team, suppose there are x juniors and x seniors. This number represents $25\% = \frac{1}{4}$ of the juniors and $10\% = \frac{1}{10}$ of the seniors, which tells us that there are $4x$ juniors and $10x$ seniors. There are 28 juniors and seniors in the program altogether, so we get $10x + 4x = 28,$

$14x=28$,$x=2$.Which means there are $4x=8$ juniors on the debate team, $\boxed{\text{(C) } 8}$.

Problem4

At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

$\textbf{(A)}\ 23 \qquad \textbf{(B)}\ 32 \qquad \textbf{(C)}\ 37 \qquad \textbf{(D)}\ 41 \qquad \textbf{(E)}\ 64$

Solution

There are 46 students paired with a blue partner. The other 11 students wearing blue shirts must each be paired with a partner wearing a shirt of the opposite color. There are 64 students remaining. Therefore the requested number of pairs is $\frac{64}{2}=\boxed{\textbf{(B)}\ 32}$

Problem5

The ages of Jonie's four cousins are distinct single-digit positive integers. Two of the cousins' ages multiplied together give 24, while the other two multiply to 30. What is the sum of the ages of Jonie's four cousins?

$\textbf{(A)}\ 21 \qquad \textbf{(B)}\ 22 \qquad \textbf{(C)}\ 23 \qquad \textbf{(D)}\ 24 \qquad \textbf{(E)}\ 25$

Solution

First look at the two cousins' ages that multiply to 24. Since the ages must be single-digit, the ages must either be $3 \text{ and } 8$ or $4 \text{ and } 6.$

Next, look at the two cousins' ages that multiply to 30. Since the ages must be single-digit, the only ages that work are $5 \text{ and } 6.$ Remembering that all the ages must all be distinct, the only solution that works is when the ages are $3, 8$ and $5, 6$.

We are required to find the sum of the ages, which is $3 + 8 + 5 + 6 = \boxed{(B) \ 22}$.

Problem6

Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

$\textbf{(A)}\ 74 \qquad \textbf{(B)}\ 75 \qquad \textbf{(C)}\ 76 \qquad \textbf{(D)}\ 77 \qquad \textbf{(E)}\ 78$

Solution 1

WLOG, assume there are 3 students in the morning class and 4 in the afternoon class. Then the average is $\frac{3 \cdot 84 + 4 \cdot 70}{7} = \boxed{\textbf{(C)}\ 76}$

Solution 2

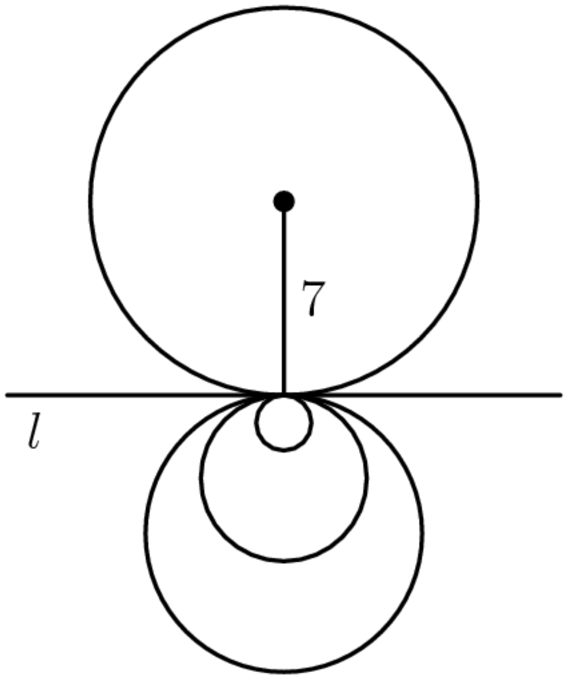
Let there be $3x$ students in the morning class and $4x$ students in the afternoon class. The total number of students is $3x + 4x = 7x$. The average is $\frac{3x \cdot 84 + 4x \cdot 70}{7x} = 76$. Therefore, the answer is $\boxed{\textbf{(C)} 76}$.

Problem7

In a plane, four circles with radii $1, 3, 5,$ and 7 are tangent to line l at the same point $A,$ but they may be on either side of l. Region S consists of all the points that lie inside exactly one of the four circles. What is the maximum possible area of region S?

$\textbf{(A)}\ 24\pi \qquad \textbf{(B)}\ 32\pi \qquad \textbf{(C)}\ 64\pi \qquad \textbf{(D)}\ 65\pi \qquad \textbf{(E)}\ 84\pi$

Solution



After a bit of wishful thinking and inspection, we find that the above configuration maximizes our area, which is $49\pi + (25-9)\pi = 65\pi \rightarrow \boxed{\textbf{(D)}}$

Problem8

Mr. Zhou places all the integers from 1 to 225 into a 15 by 15 grid. He places 1 in the middle square (eighth row and eighth column) and places other numbers one by one clockwise, as shown in part in the diagram below. What is the sum of the greatest number and the least number that appear in the second row from the top?

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	21	22	23	24	25	$\cdots$
$\cdots$	20	7	8	9	10	$\cdots$
$\cdots$	19	6	1	2	11	$\cdots$
$\cdots$	18	5	4	3	12	$\cdots$
$\cdots$	17	16	15	14	13	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$\textbf{(A)}\ 367 \qquad \textbf{(B)}\ 368 \qquad \textbf{(C)}\ 369 \qquad \textbf{(D)}\ 379 \qquad \textbf{(E)}\ 380$

Solution 1

By observing that the right-top corner of a square will always be a square, we know that the top right corner of the 15x15 grid is 225. We can subtract 14 to get the value of the top-left corner; 211. We can then find the value of the bottom left and right corners similarly. From there, we can find the value of the box on the far right in the 2nd row from the top by subtracting 13, since the length of the side will be one box shorter. Similarly, we find the value for the box 2nd from the left and 2nd from the top, which is 157. We know that the least number in the 2nd row will be 157, and the greatest will be the number to its left, which is 1 less than 211. We then sum 157 and 210 to get $\boxed{\textbf{(A) } 367}$.

-Dynosol

Solution 2: Draw It Out

Drawing out the diagram, we get $\boxed{\textbf{(A) } 367}$. Note that this should mainly be used just to check your answer.

Problem9

The point $P(a,b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1,5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6,3)$. What is $b - a?$

$\textbf{(A) } 1 \qquad \textbf{(B) } 3 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 7 \qquad \textbf{(E) } 9$

Solution

The final image of P is $(-6,3)$. We know the reflection rule for reflecting over $y=-x$ is $(x,y) --> (-y, -x)$. So before the reflection and after rotation the point is $(-3,6)$.

By definition of rotation, the slope between $(-3,6)$ and $(1,5)$ must be perpendicular to the slope between (a,b) and $(1,5)$. The first slope is $\frac{5-6}{1-(-3)}=\frac{-1}{4}$. This means the slope of P and $(1,5)$ is 4. Rotations also preserve distance to the center of rotation, and since we only "travelled" up and down by the slope once to get from $(3,-6)$ to $(1,5)$ it follows we shall only use the slope once to travel from $(1,5)$ to P.

Therefore point P is located at $(1+1,5+4)=(2,9)$. The answer is $9-2=7=\boxed{\textbf{(D)}}$.

Problem10

An inverted cone with base radius $12\mathrm{cm}$ and height $18\mathrm{cm}$ is full of water. The water is poured into a tall cylinder whose horizontal base has radius of $24\mathrm{cm}$. What is the height in centimeters of the water in the cylinder?

$\textbf{(A)}\ 1.5 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 4.5 \qquad \textbf{(E)}\ 6$

Solution 1

The volume of a cone is $\frac{1}{3}\cdot\pi\cdot r^2\cdot h$ where r is the base radius and h is the height. The water completely fills up the cone so the volume of the water is $\frac{1}{3}\cdot 18\cdot 144\pi=6\cdot 144\pi$.

The volume of a cylinder is $\pi\cdot r^2\cdot h$ so the volume of the water in the cylinder would be $24\cdot 24\cdot\pi\cdot h$.

We can equate these two expressions because the water volume stays the same like this $24\cdot 24\cdot\pi\cdot h=6\cdot 144\pi$. We get $4h=6$ and $h=\frac{6}{4}$.

So the answer is $1.5 = \boxed{\textbf{(A)}}$.

Solution 2 (ratios)

The water completely fills up the cone. For now, assume the radius of both cone and cylinder are the same. Then the cone has $\frac{1}{3}$ of the volume of the cylinder, and so the height is divided by 3. Then, from the problem statement, the radius is doubled, meaning the area of the base is quadrupled (since $2^2 = 4$).

Therefore, the height is divided by 3 and divided by 4, which is $18 \div 3 \div 4 = 1.5 = \boxed{\textbf{(A)}}$.

Problem11

Grandma has just finished baking a large rectangular pan of brownies. She is planning to make rectangular pieces of equal size and shape, with straight cuts parallel to the sides of the pan. Each cut must be made entirely across the pan. Grandma wants to make the same number of interior pieces as pieces along the perimeter of the pan. What is the greatest possible number of brownies she can produce?

$\textbf{(A) }24\qquad\textbf{(B) }30\qquad\textbf{(C) }48\qquad\textbf{(D) }60\qquad\textbf{(E) }64$

Solution 1

Let the side lengths of the rectangular pan be m and n. It follows that $(m-2)(n-2) = \frac{mn}{2}$, since half of the brownie pieces are in the interior. This gives

$$2(m-2)(n-2) = mn \iff mn - 4m - 4n + 8 = 0.$$

Adding 8 to both sides and applying Simon's Favorite Factoring Trick, we obtain $(m-4)(n-4) = 8$. Since m and n are both positive, we obtain $(m,n) = (5,12), (6,8)$ (up to ordering). By inspection, $5 \cdot 12 = \boxed{\textbf{(D) }60}$ maximizes the number of brownies.

Solution 2

Obviously, no side of the rectangular pan can have less than 5 brownies beside it. We let one side of the pan have 5 brownies, and let the number of brownies on its adjacent side be x. Therefore, $5x = 2 \cdot 3(x-2)$, and solving yields $x = 12$ and there are $5 \cdot 12 = 60$ brownies in the pan. 64 is the only choice larger than 60, but it cannot be the answer since the only way to fit 64 brownies in a pan without letting a side of it have less than 5 brownies beside it is by forming a square of 8 brownies on each side, which does not meet the requirement. Thus the answer is $\boxed{\textbf{(D) } 60}$.

Problem12

Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

$\textbf{(A) } 1 : 16 \qquad \textbf{(B) } 1 : 15 \qquad \textbf{(C) } 1 : 14 \qquad \textbf{(D) } 1 : 8 \qquad \textbf{(E) } 1 : 3$

Solution 1

Prime factorize N to get $N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2$. For each odd divisor n of N, there exist even divisors $2n, 4n, 8n$ of N, therefore the ratio is $1 : (2+4+8) \rightarrow \boxed{\textbf{(C)}}$

Solution 2

Prime factorizing N, we see $N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2$. The sum of N's odd divisors are the sum of the factors of N without 2, and the sum of the even divisors is the sum of the odds subtracted by the total sum of divisors. The sum of odd divisors is given by

$$a = (1+3+3^2+3^3+3^4+3^5)(1+5)(1+7)(1+17+17^2)$$

and the total sum of divisors is

$(1+2+4+8)(1+3+3^2+3^3+3^4+3^5)(1+5)(1+7)(1+17+17^2) = 15a$. Thus, our ratio is $\frac{a}{15a-a} = \frac{a}{14a} = \frac{1}{14}\boxed{C}$

Problem13

Let n be a positive integer and d be a digit such that the value of the numeral $\underline{32d}$ in base n equals 263, and the value of the numeral $\underline{324}$ in base n equals the value of the numeral $\underline{11d1}$ in base six. What is $n + d?$

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 11 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 15 \qquad \textbf{(E)}\ 16$

Solution

We can start by setting up an equation to convert $\underline{32d}$ base n to base 10. To convert this to base 10, it would be $3n^2 + 2n + d.$ Because it is equal to 263, we can set this equation to 263. Finally, subtract $d$from both sides to get $3n^2 + 2n = 263 - d$.

We can also set up equations to convert $\underline{324}$ base n and $\underline{11d1}$ base 6 to base 10. The equation to covert $\underline{324}$ base n to base 10 is $3n^2 + 2n + 4.$ The equation to convert $\underline{11d1}$ base 6 to base 10 is $6^3 + 6^2 + 6d + 1.$

Simplify $6^3 + 6^2 + 6d + 1$ so it becomes $6d + 253.$ Setting the above equations equal to each other, we have $3n^2 + 2n + 4 = 6d + 253.$ Subtracting 4 from both sides gets $3n^2 + 2n = 6d + 249.$

We can then use equations $3n^2+2n=263-d$

$3n^2+2n=6d+249$ to solve for d. Set $263-d$ equal to $6d+249$ and solve to find that $d=2$.

Plug $d=2$ back into the equation $3n^2+2n=263-d$. Subtract 261 from both sides to get your final equation of $3n^2+2n-261=0$. Solve using the quadratic formula to find that the solutions are 9 and -10. Because the base must be positive, $n=9$.

Adding 2 to 9 gets $\boxed{\textbf{(B)}11}$

-Zeusthemoose (edited for readability)

Solution 2

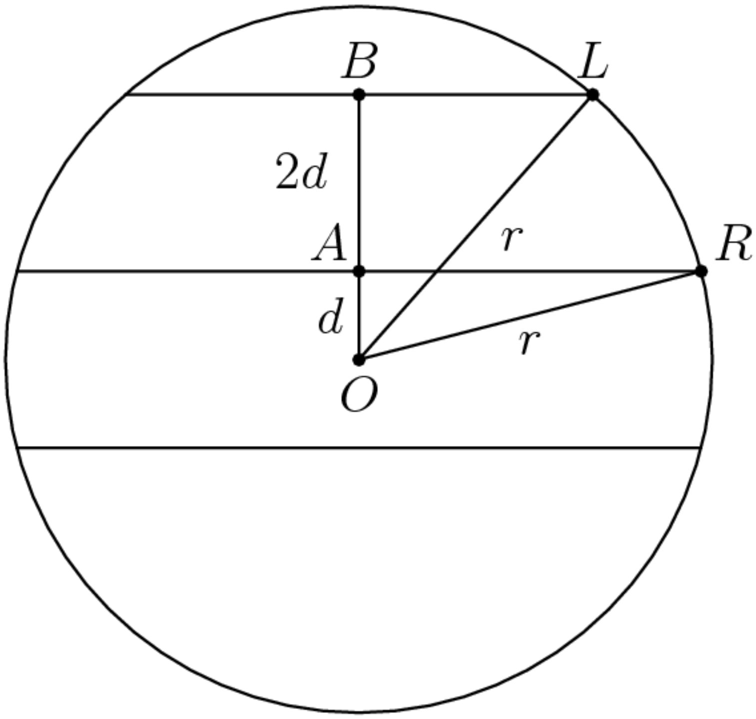
$32d$ is greater than 263 when both are interpreted in base 10, so n is less than 10. Some trial and error gives $n=9$. 263 in base 9 is 322, so the answer is $9+2=\boxed{\textbf{(B)}11}$.

Problem14

Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38,$ and 34. What is the distance between two adjacent parallel lines?

$\textbf{(A) }5\frac12 \qquad \textbf{(B) }6 \qquad \textbf{(C) }6\frac12 \qquad \textbf{(D) }7 \qquad \textbf{(E) }7\frac12$

Solution 1



Since two parallel chords have the same length (38), they must be equidistant from the center of the circle. Let the perpendicular distance of each chord from the center of the circle be d. Thus, the distance from the center of the circle to the chord of length 34 is

$$2d + d = 3d$$

and the distance between each of the chords is just $2d$. Let the radius of the circle be r. Drawing radii to the points where the lines intersect the circle, we create two different right triangles:

- One with base $\frac{38}{2} = 19$, height d, and hypotenuse r ($\triangle RAO$ on the diagram)

- Another with base $\frac{34}{2} = 17$, height $3d$, and hypotenuse r ($\triangle LBO$ on the diagram)

By the Pythagorean theorem, we can create the following system of equations:

$$19^2 + d^2 = r^2$$

$17^2 + (3d)^2 = r^2$

Solving, we find $d = 3$, so $2d = \boxed{\textbf{(B) } 6}$.

Solution 2 (Coordinates)

Because we know that the equation of a circle is $(x-a)^2 + (y-b)^2 = r^2$ where the center of the circle is (a, b) and the radius is r, we can find the equation of this circle by centering it on the origin. Doing this, we get that the equation is $x^2 + y^2 = r^2$. Now, we can set the distance between the chords as $2d$ so the distance from the chord with length 38 to the diameter is d.

Therefore, the following points are on the circle as the y-axis splits the chord in half, that is where we get our x value:

$(19, d)$

$(19, -d)$

$(17, -3d)$

Now, we can plug one of the first two value in as well as the last one to get the following equations:

$19^2 + d^2 = r^2$

$17^2 + (3d)^2 = r^2$

Subtracting these two equations, we get $19^2 - 17^2 = 8d^2$ - therefore, we get $72 = 8d^2 \rightarrow d^2 = 9 \rightarrow d = 3$. We want to find $2d = 6$ because that's the distance between two chords. So, our answer is $\boxed{B}$.

Problem15

The real number x satisfies the equation $x+\frac{1}{x}=\sqrt{5}$. What is the value of $x^{11}-7x^{7}+x^{3}$?

$\textbf{(A)}\ -1 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ \sqrt{5}$

Solution 1

We square $x+\frac{1}{x}=\sqrt{5}$ to get $x^2+2+\frac{1}{x^2}=5$. We subtract 2 on both sides for $x^2+\frac{1}{x^2}=3$ and square again, and see that $x^4+2+\frac{1}{x^4}=9$ so $x^4+\frac{1}{x^4}=7$. We can divide our original expression of $x^{11}-7x^{7}+x^{3}$ by x^7 to get that it is equal to $x^7(x^4-7+\frac{1}{x^4})$. Therefore because $x^4+\frac{1}{x^4}$ is 7, it is equal to $x^7(0)=\boxed{(B) 0}$.

Solution 2

Multiplying both sides by x and using the quadratic formula, we get $\frac{\sqrt{5}\pm 1}{2}$. We can assume that it is $\frac{\sqrt{5}+1}{2}$, and notice that this is also a solution the equation $x^2-x-1=0$, i.e. we have $x^2=x+1$. Repeatedly using this on the given (you can also just note Fibonacci numbers),

$$\begin{aligned}(x^{11})-7x^7+x^3&=(x^{10}+x^9)-7x^7+x^3\\&=(2x^9+x^8)-7x^7+x^3\\&=(3x^8+2x^7)-7x^7+x^3\\&=(3x^8-5x^7)+x^3\\&=(-2x^7+3x^6)+x^3\\&=(x^6-2x^5)+x^3\\&=(-x^5+x^4+x^3)\\&=-x^3(x^2-x-1)=\boxed{\textbf{(B)}0}\end{aligned}$$

~Lcz

Solution 3

We can immediately note that the exponents of $x^{11}-7x^7+x^3$ are an arithmetic sequence, so they are symmetric around the middle term.

So, $x^{11}-7x^7+x^3=x^7(x^4-7+\frac{1}{x^4})$. We can see that since $x+\frac{1}{x}=\sqrt{5}$, $x^2+2+\frac{1}{x^2}=5$ and therefore $x^2+\frac{1}{x^2}=3$. Continuing from here, we get $x^4+2+\frac{1}{x^4}=9$, so $x^4-7+\frac{1}{x^4}=0$. We don't even need to find what x^3 is! This is since $x^3\cdot 0$ is evidently $\boxed{(B)0}$, which is our answer.

Problem16

Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, $1357, 89,$ and 5 are all uphill integers, but $32, 1240,$ and 466 are not. How many uphill integers are divisible by 15?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

Solution 1

The divisibility rule of 15 is that the number must be congruent to 0 mod 3 and congruent to 0 mod 5. Being divisible by 5 means that it must end with a 5 or a 0. We can rule out the case when the number ends with a 0 immediately because the only integer that is uphill and ends with a 0 is 0 which is not positive. So now we know that the number ends with a 5. Looking at the answer choices, the answer choices are all pretty small, so we can generate all of the numbers that are uphill and are divisible by 3. These numbers are $15, 45, 135, 345, 1245, 12345$ which are 6 numbers C.

Solution 2

First, note how the number must end in either 5 or 0 in order to satisfying being divisible by 15. However, the number can't end in 0 because it's not strictly greater than the previous digits. Thus, our number must end in 5. We do casework on the number of digits.

Case 1 = 1 digit. No numbers work, so 0

Case 2 = 2 digits. We have the numbers $15, 45,$ and 75, but 75 isn't an uphill number, so 2 numbers.

Case 3 = 3 digits. We have the numbers $135, 345$. So 2 numbers.

Case 4 = 4 digits. We have the numbers $1235, 1245$ and 2345, but only 1245 satisfies this condition, so 1number.

Case 5= 5 digits. We have only 12345, so 1 number.

Adding these up, we have $2+2+1+1=6$. $\boxed{C}$

~JustinLee2017

Solution 3

Like solution 2, we can proceed by using casework. A number is divisible by 15 if is divisible by 3 and $5.$ In this case, the units digit must be $5,$ otherwise no number can be formed.

Case 1: sum of digits = 6

There is only one number, $15.$

Case 2: sum of digits = 9

There are two numbers: 45 and $135.$

Case 3: sum of digits = 12

There are two numbers: 345 and $1245.$

Case 4: sum of digits = 15

There is only one number, $12345.$

We can see that we have exhausted all cases, because in order to have a larger sum of digits, then a number greater than 5 needs to be used, breaking the conditions of the problem. The answer is $\textbf{(C)}.$

Problem17

Ravon, Oscar, Aditi, Tyrone, and Kim play a card game. Each person is given 2 cards out of a set of 10 cards numbered $1, 2, 3, \dots, 10.$ The score of a player is the sum of the numbers of their cards. The scores of the players are as follows: Ravon--11, Oscar--4, Aditi--7, Tyrone--16, Kim--17. Which of the following statements is true?

$\textbf{(A) }\text{Ravon was given card 3.}\qquad \textbf{(B) }\text{Aditi was given card 3.}\qquad \textbf{(C) }\text{Ravon was given card 4.}\qquad \textbf{(D) }\text{Aditi was given card 4}$

$\textbf{(E) }\text{Tyrone was given card 7}$

Solution

Oscar must be given 3 and 1, so we rule out $\textbf{(A)}$ and $\textbf{(B)}$. If Tyrone had card 7, then he would also have card 9, and then Kim must have 10 and 7 so we rule out $\textbf{(E)}$. If Aditi was given card 4, then she would have card 3, which Oscar already had. So the answer is $\boxed{\textbf{(C) }\text{Ravon was given card 4.}}$

~smarty101 and smartypantsno_3

Solution 2

Oscar must be given 3 and 1. Aditi cannot be given 3 or 1, so she must have 2 and 5. Similarly, Ravon cannot be given 1, 2, 3, or 5, so he must have 4 and 7, and the answer is $\boxed{\textbf{(C)}\ \text{Ravon was given card 4.}}$.

Problem18

A fair 6-sided die is repeatedly rolled until an odd number appears. What is the probability that every even number appears at least once before the first occurrence of an odd number?

$\textbf{(A)}\ \frac{1}{120} \qquad \textbf{(B)}\ \frac{1}{32} \qquad \textbf{(C)}\ \frac{1}{20} \qquad \textbf{(D)}\ \frac{3}{20} \qquad \textbf{(E)}\ \frac{1}{6}$

Solution 1

There is a $\frac{3}{6}$ chance that the first number we choose is even.

There is a $\frac{2}{5}$ chance that the next number that is distinct from the first is even.

There is a $\frac{1}{4}$ chance that the next number distinct from the first two is even.

$\frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20}$, so the answer is $\boxed{\textbf{(C)}\ \frac{1}{20}}$.

~Tucker

Solution 2

Every set of three numbers chosen from $\{1, 2, 3, 4, 5, 6\}$ has an equal chance of being the first 3 distinct numbers rolled.

Therefore, the probability that the first 3 distinct numbers are $\{2,4,6\}$ is $\frac{1}{\binom{6}{3}}=\boxed{(C)\ \frac{1}{20}}$

~kingofpineapplz

Solution 3 (Quicksolve)

Note that the problem is basically asking us to find the probability that in some permutation of $1,2,3,4,5,6$ that we get the three even numbers in the first three spots.

There are $6!$ ways to order the 6 numbers and $3!(3!)$ ways to order the evens in the first three spots and the odds in the next three spots.

Therefore the probability is $\frac{3!(3!)}{6!}=\frac{1}{20}=\boxed{\textbf{(C)}}$.

--abhinavg0627

Solution 4

Let P_n denote the probability that the first odd number appears on roll n and all our conditions are met. We now proceed with complementary counting.

For $n\le 3$, it's impossible to have all 3 evens appear before an odd. Note that for $n\geq 4,$

$$P_n=\frac{1}{2^n}-\frac{1}{2^n}\left(\frac{\binom{3}{2}(2^{n-1}-2)+\binom{3}{2}}{3^{n-1}}\right)=\frac{1}{2^n}-\left(\frac{3(2^{n-1})-3}{2^n\cdot 3^{n-1}}\right)=\frac{1}{2^n}-\left(\frac{1}{2\cdot 3^{n-2}}-\frac{1}{2^n\cdot 3^{n-2}}\right).$$

Summing for all n, we get our answer of

$$\left(\frac{1}{2^4}+\frac{1}{2^5}+...\right)-\left(\frac{1}{2\cdot 3^2}+\frac{1}{2\cdot 3^3}+...\right)+\left(\frac{1}{2^4\cdot 3^2}+\frac{1}{2^5\cdot 3^3}+...\right)=\left(\frac{1}{8}\right)-\left(\frac{\frac{1}{18}}{\frac{2}{3}}\right)+\left(\frac{\frac{1}{144}}{\frac{5}{6}}\right)=\left(\frac{1}{8}\right)-\left(\frac{1}{12}\right)+\left(\frac{1}{120}\right)=\boxed{\textbf{(C)}\ \frac{1}{20}}.$$

Problem19

Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to $40.$ The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

$\textbf{(A) }36.2 \qquad \textbf{(B) }36.4 \qquad \textbf{(C) }36.6\qquad \textbf{(D) }36.8 \qquad \textbf{(E) }37$

Solution 1

Let x be the greatest integer, y be the smallest, z be the sum of the numbers in S excluding x and y, and $k$be the number of elements in S.

Then, $S = x+y+z$

Firstly, when the greatest integer is removed, $\frac{S-x}{k-1} = 32$

When the smallest integer is also removed, $\frac{S-x-y}{k-2} = 35$

When the greatest integer is added back, $\frac{S-y}{k-1} = 40$

We are given that $x = y+72$

After you substitute $x = y+72$, you have 3 equations with 3 unknowns $S,$, y and k.

$S - y - 72 = 32k - 32$

$S - 2y - 72 = 35k - 70$

$S - y = 40k - 40$

This can be easily solved to yield $k = 10$, $y = 8$, $S = 368$.

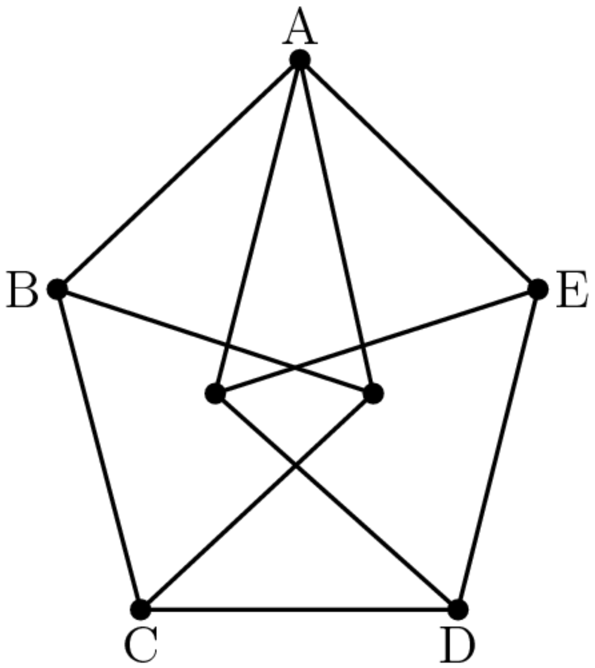
$\therefore$ average value of all integers in the set $= S/k = 368/10 = 36.8$, D)

Solution 2

We should plug in 36.2 and assume everything is true except the 35 part. We then calculate that part and end up with 35.75. We also see with the formulas we used with the plug in that when you increase by $0.2$the 35.75 part decreases by 0.25. The answer is then $\boxed{(D) 36.8}$. You can work backwards because it is multiple choice and you don't have to do critical thinking.

Problem20

The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written is $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n ?$



$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 22 \qquad \textbf{(D)}\ 23 \qquad \textbf{(E)}\ 24$

Solution 1

Let M be the midpoint of CD. Noting that AED and ABC are $120-30-30$ triangles because of the equilateral triangles,

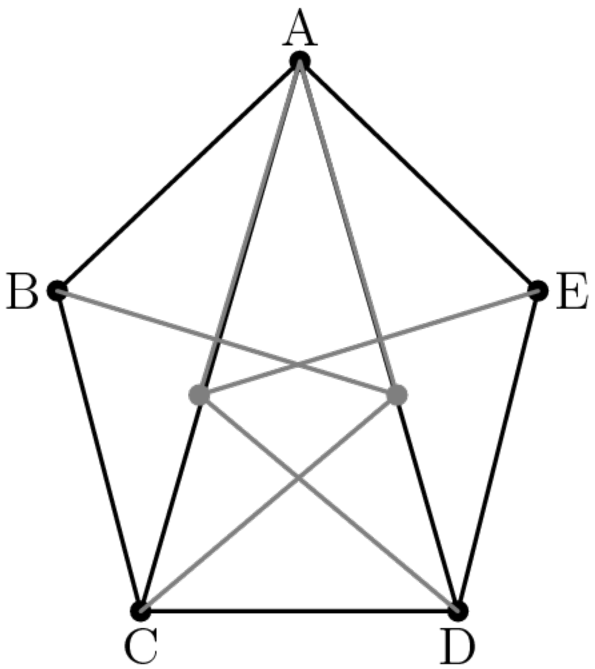
$$AM = \sqrt{AD^2 - MD^2} = \sqrt{12-1} = \sqrt{11} \implies [ACD] = \sqrt{11}$$

. Also, $[AED] = 2 \cdot 2 \cdot \frac{1}{2} \cdot \sin 120^o = \sqrt{3}$ and so

$$[ABCDE] = [ACD] + 2[AED] = \sqrt{11} + 2\sqrt{3} = \sqrt{11} + \sqrt{12} \implies \boxed{\textbf{(D) } 23}$$

.

Solution 2



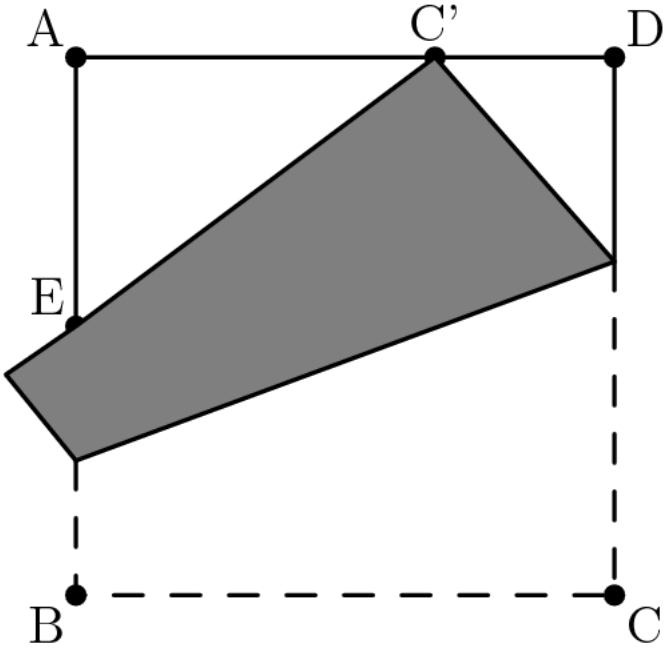
Draw diagonals AC and AD to split the pentagon into three parts. We can compute the area for each triangle and sum them up at the end. For triangles ABC and ADE, they each have area $2 \cdot \frac{1}{2} \cdot \frac{4\sqrt{3}}{4} = \sqrt{3}$. For triangle ACD, we can see that $AC = AD = 2\sqrt{3}$ and $CD = 2$. Using Pythagorean Theorem, the altitude for this triangle is $\sqrt{11}$, so the area is $\sqrt{11}$. Adding each part up, we get $2\sqrt{3} + \sqrt{11} = \sqrt{12} + \sqrt{11} \implies \boxed{\textbf{(D) } 23}$.

Problem21

A square piece of paper has side length 1 and vertices $A, B, C,$ and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C', and edge $\overline{BC}$ intersects edge $\overline{AB}$ at point E.

Suppose that $C'D = \frac{1}{3}$. What is the perimeter of triangle $\triangle AEC'$?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 1+\frac{2}{3}\sqrt{3} \qquad \textbf{(C)}\ \sqrt{136} \qquad \textbf{(D)}\ 1+\frac{3}{4}\sqrt{3} \qquad \textbf{(E)}\ \frac{7}{3}$



Solution 1

We can set the point on CD where the fold occurs as point F. Then, we can set FD as x, and CF as $1-x$ because of symmetry due to the fold. It can be recognized that this is a right triangle, and solving for x, we get,

$$x^2 + \left(\frac{1}{3}\right)^2 = (1-x)^2 \rightarrow x^2 + \frac{1}{9} = x^2 - 2x + 1 \rightarrow x = \frac{4}{9}$$

We know this is a 3-4-5 triangle because the side lengths are $\frac{3}{9}, \frac{4}{9}, \frac{5}{9}$. We also know that EAC' is similar to $C'DF$ because angle C' is a right angle. Now, we can use similarity to find out that the perimeter is just the perimeter of $C'DF * \frac{AC'}{DF}$. Thats just $\frac{4}{3} * \frac{\frac{2}{3}}{\frac{4}{9}} = \frac{4}{3} * \frac{3}{2} = 2$. Therefore, the final answer is $\boxed{A}$

~Tony_Li2007

Solution 2

Let line we're reflecting over be ℓ, and let the points where it hits AB and CD, be M and N, respectively. Notice, to reflect over a line we find the perpendicular passing through the midpoint of the two points (which are the reflected and the original). So, we first find the equation of the line ℓ. The segment CC' has slope $\frac{0-1}{1-2/3} = -3$, implying line ℓ has a slope of $\frac{1}{3}$. Also, the midpoint of segment CC' is $\left(\frac{5}{6}, \frac{1}{2}\right)$, so line ℓ passes through this point. Then, we get the equation of line ℓ is simply $y = \frac{1}{3}x + \frac{2}{9}$. Then, if the point where B is reflected over line ℓ is B', then we get BB' is the line $y = -3x$. The intersection of ℓ and segment BB' is $\left(-\frac{1}{15}, \frac{1}{5}\right)$. So, we get $B' = \left(-\frac{2}{15}, \frac{2}{5}\right)$. Then, line segment $B'C'$ has equation $y = \frac{3}{4}x + \frac{1}{2}$, so the point E is the y-intercept, or $\left(0, \frac{1}{2}\right)$. This implies that $AE = \frac{1}{2}, AC' = \frac{2}{3}$, and by the Pythagorean Theorem, $EC' = \frac{5}{6}$ (or you could notice $\triangle AEC'$ is a $3-4-5$ right triangle). Then, the perimeter is $\frac{1}{2} + \frac{2}{3} + \frac{5}{6} = 2$, so our answer is $\boxed{\textbf{(A) } 2}$.

Solution 3 (Fakesolve):

Assume that E is the midpoint of $\overline{AB}$. Then, $\overline{AE} = \frac{1}{2}$ and since $C'D = \frac{1}{3}$, $\overline{AC'} = \frac{2}{3}$. By the Pythagorean Theorem, $\overline{EC'} = \frac{5}{6}$.

It easily follows that our desired perimeter is $2 \rightarrow \boxed{A}$ ~samrocksnature

Problem22

Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n?$

$\textbf{(A)}\ 47 \qquad \textbf{(B)}\ 94 \qquad \textbf{(C)}\ 227 \qquad \textbf{(D)}\ 471 \qquad \textbf{(E)}\ 542$

Solution 1

Let our denominator be $(5!)^3$, so we consider all possible distributions.

We use PIE (Principle of Inclusion and Exclusion) to count the successful ones.

When we have at 1 box with all 3 balls the same color in that box, there are ${}_5C_1 \cdot {}_5P_1 \cdot (4!)^3$ ways for the distributions to occur (${}_5C_1$ for selecting one of the five boxes for a uniform color, ${}_5P_1$ for choosing the color for that box, $4!$ for each of the three people to place their remaining items).

However, we overcounted those distributions where two boxes had uniform color, and there are ${}_5C_2 \cdot {}_5P_2 \cdot (3!)^3$ ways for the distributions to occur (${}_5C_2$ for selecting two of the five boxes for a uniform color, ${}_5P_2$ for choosing the color for those boxes, $3!$ for each of the three people to place their remaining items).

Again, we need to re-add back in the distributions with three boxes of uniform color... and so on so forth.

Our success by PIE is $_5C_1 \cdot {_5P_1} \cdot (4!)^3 - {_5C_2} \cdot {_5P_2} \cdot (3!)^3 + {_5C_3} \cdot {_5P_3} \cdot (2!)^3 - {_5C_4} \cdot {_5P_4} \cdot (1!)^3 + {_5C_5} \cdot {_5P_5} \cdot (0!)^3 = 120 \cdot 2556.$

$\frac{120 \cdot 2556}{120^3} = \frac{71}{400}$, yielding an answer of $\boxed{471 \textbf{(D)}}$.

Solution 2

As In Solution 1, the probability is

$$\frac{\binom{5}{1} \cdot 5 \cdot (4!)^3 - \binom{5}{2} \cdot 5 \cdot 4 \cdot (3!)^3 + \binom{5}{3} \cdot 5 \cdot 4 \cdot 3 \cdot (2!)^3 - \binom{5}{4} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot (1!)^3 + \binom{5}{5} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(5!)^3}$$

$$= \frac{5 \cdot 5 \cdot (4!)^3 - 10 \cdot 5 \cdot 4 \cdot (3!)^3 + 10 \cdot 5 \cdot 4 \cdot 3 \cdot (2!)^3 - 5 \cdot 5! + 5!}{(5!)^3}.$$

Dividing by $5!$, we get $\frac{5 \cdot (4!)^2 - 10 \cdot (3!)^2 + 10 \cdot (2!)^2 - 5 + 1}{(5!)^2}$. Dividing by 4, we get $\frac{5 \cdot 6 \cdot 24 - 10 \cdot 9 + 10 - 1}{30 \cdot 120}$. Dividing by 9, we get $\frac{5 \cdot 2 \cdot 8 - 10 + 1}{10 \cdot 40} = \frac{71}{400} \implies \boxed{\textbf{(D) } 471}$.

Solution 3

Use complementary counting. Denote T_n as the total number of ways to put n colors to n boxes by 3 people out of which f_n ways are such that no box has uniform color. Notice $T_n = (n!)^3$. From this setup we see the question is asking for $1 - \frac{f_5}{(5!)^3}$. To find f_5 we want to exclude the cases of a) one box of the same color, b) 2 boxes of the same color, c) 3 boxes of same color, d) 4 boxes of the same color, and e) 5 boxes of the same color. Cases d) and e) coincide. From this, we have

$$f_5 = T_5 - \binom{5}{1}\binom{5}{1} \cdot f_4 - \binom{5}{2}\binom{5}{2} \cdot 2! \cdot f_3 - \binom{5}{3}\binom{5}{3} \cdot 3! \cdot f_2 - 5!$$

In case b), there are $\binom{5}{2}$ ways to choose 2 boxes that have the same color, $\binom{5}{2}$ ways to choose the two colors, 2! ways to permute the 2 chosen colors, and f_3 ways so that the remaining 3 boxes don't have the same color. The same goes for cases a) and c). In case e), the total number of ways to permute 5 colors is $5!$. Now, we just need to calculate f_2, f_3 and f_4.

We have $f_2 = T_2 - 2 = (2!)^3 - 2 = 6$, since we subtract the number of cases where the boxes contain uniform colors, which is 2.

In the same way, $f_3 = T_3 - \left[3! + \binom{3}{1}\binom{3}{1} \cdot f_2\right] = 156$ - again, we must subtract the number of ways at least 1 box has uniform color. There are $3!$ ways if 3 boxes each contains uniform color. Two boxes each contains uniform color is the same as previous. If one box has the same color, there are $\binom{3}{1}$ ways to pick a box, and $\binom{3}{1}$ ways to pick a color for that box, 1! ways to permute the chosen color, and f_2 ways for the remaining 2 boxes to have non-uniform colors.

Similarly,

$$f_4 = (4!)^3 - \left[4! + \binom{4}{2}\binom{4}{2} \cdot 2! \cdot f_2 + \binom{4}{1}\binom{4}{1} \cdot f_3\right] = 10,872$$

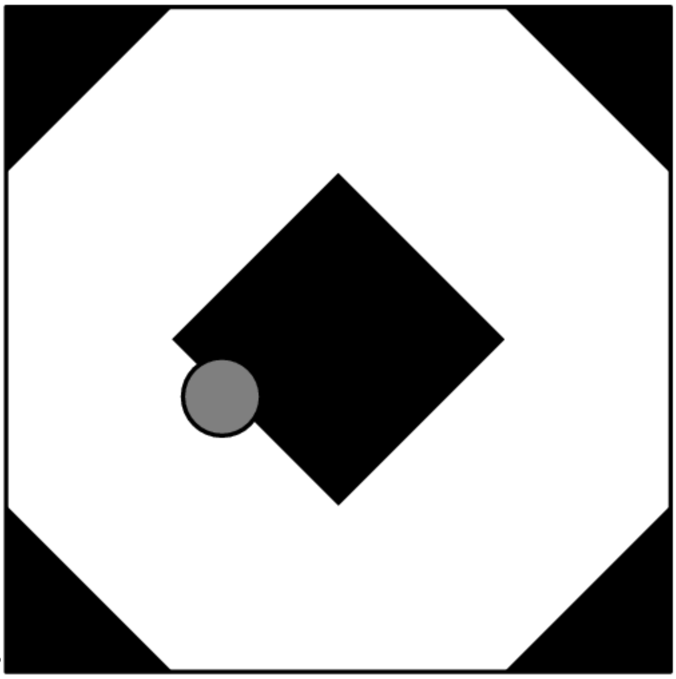
Thus,

$$f_5 = f_5 = (5!)^3 - \left[\binom{5}{1}\binom{5}{1} \cdot f_4 + \binom{5}{2}\binom{5}{2} \cdot 2! \cdot f_3 + \binom{5}{3}\binom{5}{3} \cdot 3! \cdot f_2 + 5!\right] = (5!)^3 - 306,720$$

Thus, the probability is $\frac{306,720}{(5!)^3} = 71/400$ and the answer is $\boxed{\textbf{(D) } 471}$

Problem23

A square with side length 8 is colored white except for 4 black isosceles right triangular regions with legs of length 2 in each corner of the square and a black diamond with side length $2\sqrt{2}$ in the center of the square, as shown in the diagram. A circular coin with diameter 1 is dropped onto the square and lands in a random location where the coin is completely contained within the square. The probability that the coin will cover part of the black region of the square can be written as $\frac{1}{196}\left(a+b\sqrt{2}+\pi\right)$, where a and b are positive integers.



What is $a+b$?

$\textbf{(A) } 64 \qquad \textbf{(B) } 66 \qquad \textbf{(C) } 68 \qquad \textbf{(D) } 70 \qquad \textbf{(E) } 72$

Solution

To find the probability, we look at the $\frac{\text{success region}}{\text{total possible region}}$. For the coin to be completely contained within the square, we must have the distance from the center of the coin to a side of the square to be at least $\frac{1}{2}$, as it's the radius of the coin. This implies the $\text{total possible region}$ is a square with side length $8-\frac{1}{2}-\frac{1}{2}=7$, with an area of 49. Now, we consider cases on where needs to land to partially cover a black region.

Near The Center Square

We can have the center of the coin land within $\frac{1}{2}$ of the center square, or inside of the center square. We have that the center lands either outside of the square, or inside. So, we have a region with $\frac{1}{2}$ emanating from every point on the exterior of the square, forming 4 quarter circles and 4 rectangles. The 4 quarter circles combine to make a full circle, with radius of $\frac{1}{2}$, so that has an area of $\frac{\pi}{4}$. The area of a rectangle is $2\sqrt{2}\cdot\frac{1}{2}=\sqrt{2}$, so 4 of them combine to an area of $4\sqrt{2}$. The area of the black square is simply $(2\sqrt{2})^2=8$. So, for this case, we have a combined total of $8+4\sqrt{2}+\frac{\pi}{4}$. Onto the second (and last) case.

Near A Triangle

We can also have the coin land within $\frac{1}{2}$ of one of the triangles. By symmetry, we can just find the successful region for one of them, then multiply by 4. Consider this diagram. We can draw in an altitude from the bottom corner of the square to hit the hypotenuse of the blue triangle. The length of this when passing through the black region is $\sqrt{2}$, and when passing through the white region (while being contained in the blue triangle) is $\frac{1}{2}$. However, we have to subtract off when it doesn't pass through the red square. Then, it's the hypotenuse of a small isosceles right triangle with side lengths of $\frac{\frac{1}{2}}{\sqrt{2}}$, or $\frac{\sqrt{2}}{2}$. So, our altitude of the blue triangle is $\sqrt{2}+\frac{1}{2}-\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+1}{2}$. Then, recall, the area of an isosceles right triangle is h^2, where h is the altitude from the right angle. So, squaring this, we get $\frac{3+2\sqrt{2}}{4}$. Now, we have to multiply this by 4 to account for all of the black triangles, to get $3+2\sqrt{2}$ as the final area for this case.

Finishing

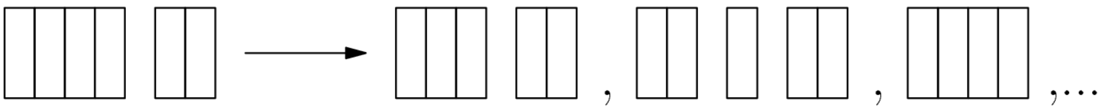
Then, to have the coin touching a black region, we add up the area of our successful regions,
or

$$8+4\sqrt{2}+\frac{\pi}{4}+3+2\sqrt{2}=11+6\sqrt{2}+\frac{\pi}{4}=\frac{44+24\sqrt{2}+\pi}{4}$$

. The total region is 49, so our probability is $\frac{\frac{44+24\sqrt{2}+\pi}{4}}{49}=\frac{44+24\sqrt{2}+\pi}{196}$, which implies $a+b=44+24=68$. This corresponds to answer choice $\boxed{\textbf{(C) }68}$

Problem24

Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3,2),(2,1,2),(4),(4,1),(2,2)$, or $(1,1,2)$.



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

$\textbf{(A) }(6,1,1) \qquad \textbf{(B) }(6,2,1) \qquad \textbf{(C) }(6,2,2)\qquad \textbf{(D) }(6,3,1) \qquad \textbf{(E) }(6,3,2)$

Solution

First we note that symmetrical positions are losing for the player to move. Then we start checking small positions. (n) is always winning for the first player. Furthermore, $(3,2,1)$ is losing and so is $(4,1)$. We look at all the positions created from $(6,2,1)$, as $(6,1,1)$ is obviously winning by playing $(2,2,1,1)$.There are several different positions that can be played by the first player from $(6,2,1)$. They are $(2,2,2,1),(1,3,2,1),(4,2,1),(6,1),(5,2,1),(4,1,2,1),(3,2,2,1)$. Now we list refutations for each of these moves:

$$(2,2,2,1) - (2,1,2,1)$$

$$(1,3,2,1) - (3,2,1)$$

$$(4,2,1) - (4,1)$$

$$(6,1) - (4,1)$$

$$(5,2,1) - (3,2,1)$$

$$(4,1,2,1)-(2,1,2,1)$$

$$(3,2,2,1)-(1,2,2,1)$$

This proves that $(6,2,1)$ is losing for the first player.

-Note: In general, this game is very complicated. For example $(8,7,5,3,2)$ is winning for the first player but good luck showing that.

Solution 2 (Process of Elimination)

$(6,1,1)$ can be turned into $(2,2,1,1)$ by Arjun, which is symmetric, so Beth will lose.

$(6,3,1)$ can be turned into $(3,1,3,1)$ by Arjun, which is symmetric, so Beth will lose.

$(6,2,2)$ can be turned into $(2,2,2,2)$ by Arjun, which is symmetric, so Beth will lose.

$(6,3,2)$ can be turned into $(3,2,3,2)$ by Arjun, which is symmetric, so Beth will lose.

That leaves $(6,2,1)$ or $\boxed{\textbf{(B)}}$.

Solution 3 (Nim-values)

Let the nim-value of the ending game state, where someone has just removed the final brick, be 0. Then, any game state with a nim-value of 0 is losing. It is well-known that the nim-value of a supergame (a combination of two or more individual games) is the binary xor function on the nim-values of the individual games that compose the supergame. Therefore, we calculate the nim-values of the states with a single wall up to 6 bricks long (since the answer choices only go up to 6).

First, the game with 1 brick has a nim-value of 1.

Similarly, the game with 2 bricks has a nim-value of 2.

Next, we consider a 3 brick wall. After the next move, the possible resulting game states are 1 brick, a 2brick wall, or 2 separate bricks. The first two options have nim-values of 1 and 2. The final option has a nim-value of $1 \oplus 1 = 0$, so the nim-value of this game state is 3.

Next, the 4 brick wall. The possible states are a 2 brick wall, a 3 brick wall, a 2 brick wall and a 1 brick wall, or a 1 brick wall and a 1 brick wall. The nim-values of these states are 2, 3, 3, and 0, respectively, and hence the nim-value of this game state is 1.

[Why is the nim-value of it 1? - awesomediabrine

https://en.wikipedia.org/wiki/Mex_(mathematics)]

The possible game states after the 5 brick wall are the following: a 3 brick wall, a 4 brick wall, a 3 brick wall and a 1 brick wall, a two 2 brick walls, and a 2 brick wall plus a 1 brick wall. The nim-values of these are 3, 1, 2, 0, and 3, respectively, meaning the nim-value of a 5 brick wall is 4.

Finally, we find the nim-value of a 6 brick wall. The possible states are a 5 brick wall, a 4 brick wall and a 1brick wall, a 3 brick wall and a 2 brick wall, a 4 brick wall, a 3 brick wall and a 1 brick wall, and finally two 2brick walls. The nim-values of these game states are 4, 0, 1, 1, 2, and 0, respectively. This means the 6 brick wall has a nim-value of 3.

The problem is asking which of the answer choices is losing, or has a nim-value of 0. We see that option $\textbf{(A)}$ has a nim-value of $3 \oplus 1 \oplus 1 = 3$, option $\textbf{(B)}$ has a nim-value of $3 \oplus 2 \oplus 1 = 0$, option $\textbf{(C)}$ has a nim-value of $3 \oplus 2 \oplus 2 = 3$, option $\textbf{(D)}$ has a nim-value of $3 \oplus 3 \oplus 1 = 1$, and option $\textbf{(E)}$ has a nim-value of $3 \oplus 3 \oplus 2 = 2$, so the answer is $\boxed{\textbf{(B)}\ (6, 2, 1)}$.

This method can also be extended to solve the note after the first solution. The nim-values of the 7 brick wall and the 8 brick wall are 2 and 1, using the same method as above. The nim-value of $(8, 7, 5, 3, 2)$ is therefore $1 \oplus 2 \oplus 4 \oplus 3 \oplus 2 = 6$, which is winning.

Video Solution by OmegaLearn (Game Theory)

https://youtu.be/zkSBMVAfYLo

Problem25

Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between $1$and $30,$ inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx.$ The possible values of m lie in an interval of length $\frac{a}{b},$ where a and b are relatively prime positive integers. What is $a + b?$

$\textbf{(A)}\ 31 \qquad \textbf{(B)}\ 47 \qquad \textbf{(C)}\ 62 \qquad \textbf{(D)}\ 72 \qquad \textbf{(E)}\ 85$

Solution 1

First, we find a numerical representation for the number of lattice points in S that are under the line $y = mx.$ For any value of $x,$ the highest lattice point under $y = mx$ is $\lfloor mx \rfloor.$ Because every lattice point from $(x, 1)$ to $(x, \lfloor mx \rfloor)$ is under the line, the total number of lattice points under the line is $\sum_{x=1}^{30}(\lfloor mx \rfloor).$

Now, we proceed by finding lower and upper bounds for $m.$ To find the lower bound, we start with an approximation. If 300 lattice points are below the line, then around $\frac{1}{3}$ of the area formed by S is under the line. By using the formula for a triangle's area, we find that when $x = 30, y \approx 20.$ Solving for m assuming that $(30, 20)$ is a point on the line, we get $m = \frac{2}{3}.$ Plugging in m to $\sum_{x=1}^{30}(\lfloor mx \rfloor),$ we get:

$$\sum_{x=1}^{30}(\lfloor\frac{2}{3}x\rfloor) = 0+1+2+2+3+\cdots+18+18+19+20$$

We have a repeat every 3 values (every time $y=\frac{2}{3}x$ goes through a lattice point). Thus, we can use arithmetic sequences to calculate the value above:

$$\sum_{x=1}^{30}(\lfloor\frac{2}{3}x\rfloor) = 0+1+2+2+3+\cdots+18+18+19+20$$

$$= \frac{20(21)}{2}+2+4+6+\cdots+18 = 210+\frac{20}{2}\cdot 9 = 300$$

This means that $\frac{2}{3}$ is a possible value of $m.$ Furthermore, it is the lower bound for $m.$ This is because $y=\frac{2}{3}x$ goes through many points (such as $(21, 14)$). If m was lower, $y=\frac{2}{3}x$ would no longer go through some of these points, and there would be less than 300 lattice points under it.

Now, we find an upper bound for $m.$ Imagine increasing m slowly and rotating the line $y=mx,$ starting from the lower bound of $m=\frac{2}{3}.$ The upper bound for m occurs when $y=mx$ intersects a lattice point again (look at this problem to get a better idea of what's happening: https://artofproblemsolving.com/wiki/index.php/2011_AMC_10B_Problems/Problem_24).

In other words, we are looking for the first $m>\frac{2}{3}$ that is expressible as a ratio of positive integers $\frac{p}{q}$ with $q\le 30.$ For each $q=1,\ldots,30$, the smallest multiple of $\frac{1}{q}$ which exceeds $\frac{2}{3}$ is $1, \frac{2}{2}, \frac{3}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{19}{27}, \frac{19}{28}, \frac{20}{29}, \frac{21}{30}$ respectively, and the smallest of these is $\frac{19}{28}$. Note: start listing the multiples of $\frac{1}{q}$ from $\frac{21}{30}$ and observe that they get further and further away from $\frac{2}{3}$. Alternatively, see the method of finding upper bounds in solution 2.

The lower bound is $\frac{2}{3}$ and the upper bound is $\frac{19}{28}$. Their difference is $\frac{1}{84}$, so the answer is $1 + 84 = \boxed{85}$.

~JimY

Solution 2

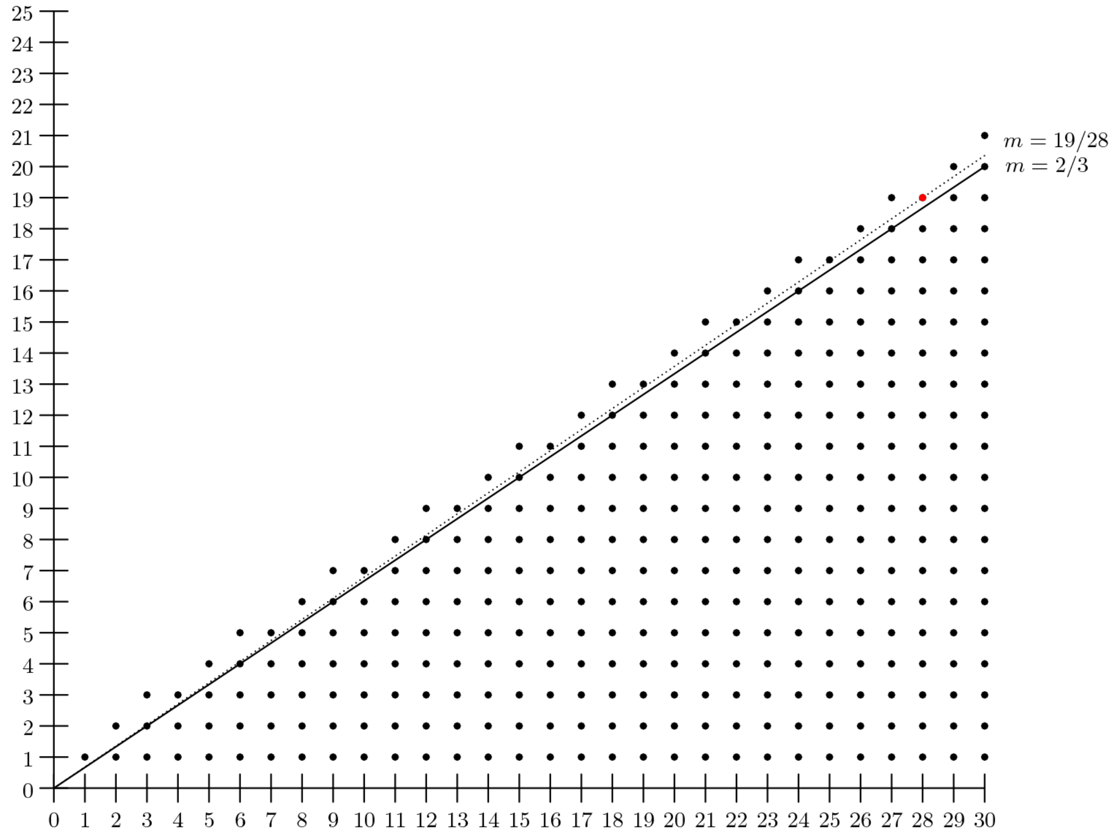
I know that I want about $\frac{2}{3}$ of the box of integer coordinates above my line. There are a total of 30 integer coordinates in the desired range for each axis which gives a total of 900 lattice points. I estimate that the slope, m, is $\frac{2}{3}$. Now, although there is probably an easier solution, I would try to count the number of points above the line to see if there are 600 points above the line. The line $y = \frac{2}{3}x$ separates the area inside the box so that $\frac{2}{3}$ of the are is above the line.

I find that the number of coordinates with $x = 1$ above the line is 30, and the number of coordinates with $x = 2$ above the line is 29. Every time the line $y = \frac{2}{3}x$ hits a y-value with an integer coordinate, the number of points above the line decreases by one. I wrote out the sum of 30 terms in hopes of finding a pattern. I graphed the first couple positive integer x-coordinates, and found that the sum of the integers above the line is $30+29+28+28+27+26+26\ldots+10$. The even integer repeats itself every third term in the sum. I found that the average of each of the terms is 20, and there are 30 of them which means that exactly 600 above the line as desired. This give a lower bound because if the slope decreases a little bit, then the points that the line goes through will be above the line.

To find the upper bound, notice that each point with an integer-valued x-coordinate is either $\frac{1}{3}$ or $\frac{2}{3}$ above the line. Since the slope through a point is the y-coordinate divided by the x-coordinate, a shift in the slope will increase the y-value of the higher x-coordinates. We turn our attention to $x=28,29,30$ which the line $y=\frac{2}{3}x$ intersects at $y=\frac{56}{3},\frac{58}{3},20$. The point (30,20) is already counted below the line, and we can clearly see that if we slowly increase the slope of the line, we will hit the point (28,19) since (28, $\frac{56}{3}$) is closer to the lattice point. The slope of the line which goes through both the origin and (28,19) is $y=\frac{19}{28}x$. This gives an upper bound of $m=\frac{19}{28}$.

Taking the upper bound of m and subtracting the lower bound yields $\frac{19}{28}-\frac{2}{3}=\frac{1}{84}$. This is answer $1+84=\boxed{\textbf{(E) } 85}$.

Diagram



Video Solution , Very Easy

https://youtu.be/PC8flZzICFg ~hippopotamus1