1 Efficient Frontier

Let a portfolio of N assets be $\boldsymbol{\pi}$, whose expected return is $\boldsymbol{\mu}$ and the co-variance is $\boldsymbol{\Sigma}$.

1.1 Efficient Frontier with 3 Assets

According to the paper, the expected return of the portfolio, $E = \sum_{i=1}^{N} \pi_i \mu_i = \boldsymbol{\pi}^t \boldsymbol{\mu}$. The risk is analogous to the variance of the returns, i.e. $V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} \pi_i \pi_j = \boldsymbol{\pi}^t \boldsymbol{\Sigma} \boldsymbol{\pi}$.

Given $\boldsymbol{\mu} = \boldsymbol{m}$ and $\boldsymbol{\Sigma} = \boldsymbol{C}$ for a 3 assets, we can generate 100 random portfolios, where each portfolio $\boldsymbol{\pi} = (\pi_1 \pi_2 \pi_3)^t$ s.t. $\mathbf{1}^t \boldsymbol{\pi} = 1$ by `y=randn(3,1); y=y/norm(y,1)`. Then we can calculate $E - V$ for each of the portfolios by `E=y'*m; V=sqrt(y'*C*y)`.

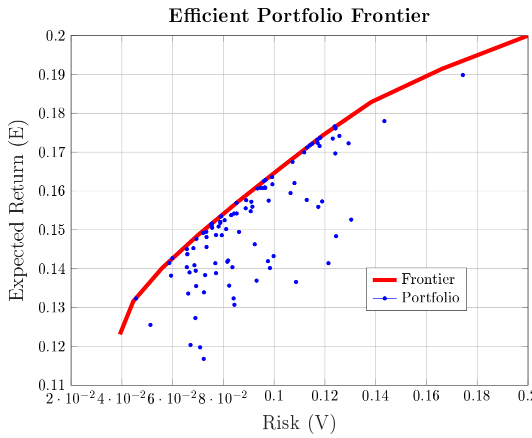


Figure 1: Efficient Portfolio

Finally I make the scatter plot and on the same figure I plot the efficient frontier using `estimateFrontier` function. As expected all the random portfolios were on the correct one side of the frontier.

1.2 Efficient Frontier with 2 Assets

To prepare three 2 asset porfolios, we remove the data points that are not necessary, i.e. that has the third asset. First I plotted random returns generated by the 2 asset mean and variance using `mvnrnd`. As can be noticed from Figure 2, asset 2 and 3 are almost uncorrelated, asset 1 and 2 are negatively correlated, asset 1 and 3 are positively correlated.

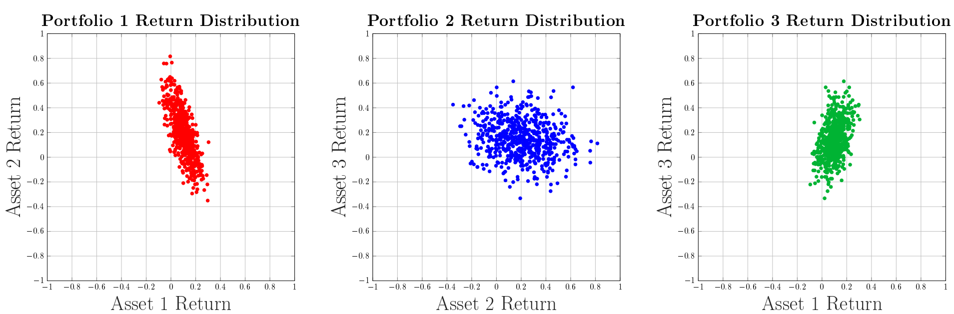


Figure 2: Distribution of 2 Asset Returns

As done previously with all three assets, I generate 100 random portfolios for each of the three 2 asset combinations and plot the $E - V$ scatters along with the efficient frontiers. Notice that in case of 2 asset portfolios, every portfolio construction is efficient and the frontier has a bend, i.e. risk increases for the lowest returns.

1.3 Use of `linprog` in NaiveMV

In order to calculate the efficient frontier, we need two extreme points - maximum return for a portfolio regardless of the risk and minimum risk regardless of the return. For the first case, we have $E = \underset{w}{max}(\boldsymbol{\pi}^t \boldsymbol{\mu})$

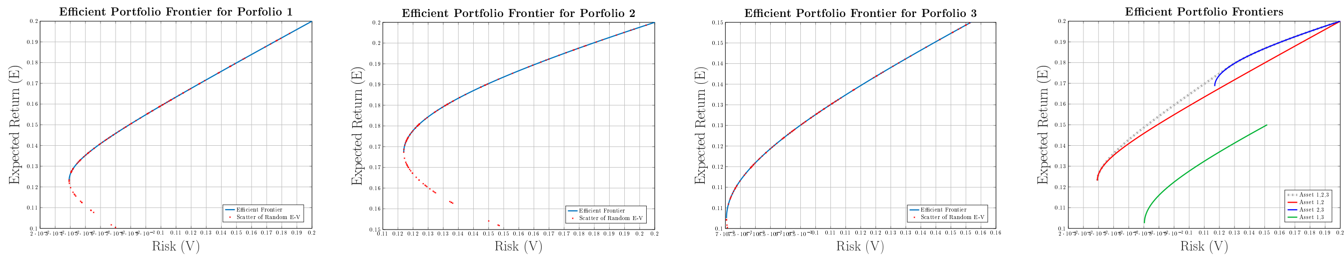


Figure 3: Efficient Frontier for 2 Asset Portfolios

s.t. $\mathbf{1}^t\boldsymbol{\pi} = 1$ which gives the portfolio that maximises the return regardless of the risk. We can then calculate $E - V$ for this portfolio, thus giving us the top corner of the $E - V$ graphs here. This is a linear equation of $\boldsymbol{\pi}$. Matlab's `linprog` function can solve linear equation can solve such equations of the following form -

$$\min_x (f^t x) s.t. \begin{cases} A.x \leq b \\ A_{eq}.x = b_{eq} \\ lb \leq x \leq ub \end{cases}$$

```
f=-ERet;A=[];b=[];Aeq=ones(1,N);beq=1;lb=0;ub=1
```

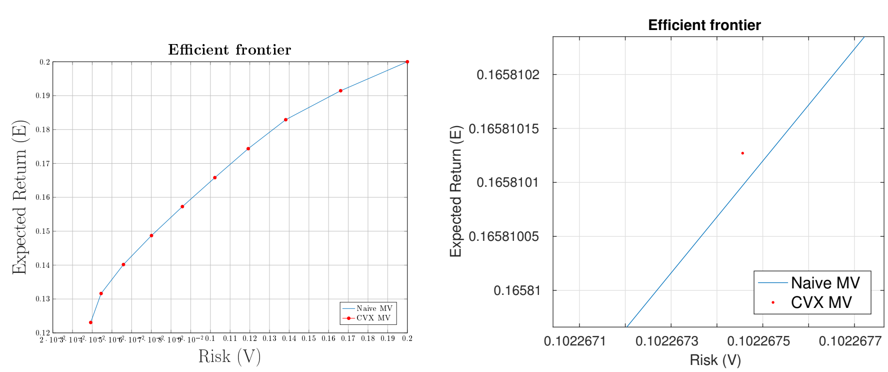
However, to calculate the portfolio that minimises the risk we need to solve a quadratic equation of $\boldsymbol{\pi}$, $\underset{w}{mim}(\boldsymbol{\pi}^t\boldsymbol{\Sigma}\boldsymbol{\pi})$ s.t. $\mathbf{1}^t\boldsymbol{\pi} = 1$. In this case we use the `quadprog` function. Finally, we choose N expected returns between the two extreme points and calculate the porfolio that minimises the risk while achieving the chosen expected returns. Thus the efficient portfolio is created.

1.4 Efficient Frontier : NaiveMV vs CVX

Using the CVX tool we can declarively perform the convex optimisations we performed earlier with `linprog` and `quadprog`, as follows -

```
cvx_begin quiet
        variable w(N,1)
        minimize( -ERet'*w )
        subject to
                ones(1,N)*w == 1;
                w >= zeros(N,1);
cvx_end
```

```
cvx_begin quiet
        variable w(N,1)
        minimize( 0.5*w'*ECov*w + zeros(N,1)'*w)
        subject to
                ones(1,N) * x == 1;
                x >= zeros(N,1);
cvx_end
```



(a) Similarity (b) Difference

Figure 4: NaiveMV vs Using CVX

The results are extremely similar, since differences only show up in 10^{-7} scale. However, the CVX tool was noticeably slower.

2 Markowitz vs Naive $1/N$ Strategy on FTSE 100

2.1 Getting FTSE 100 Data

I downloaded the FTSE 100 index and the top 30 most traded companies' data over the last 3 years using a `bash` script. Then I used a `ruby` script to fill in the missing days (which may be weekends) with the previous day's data. As a result all data had the same number of rows. The returns will be calculated based on the adjusted close values. Figure 5 plots the data.

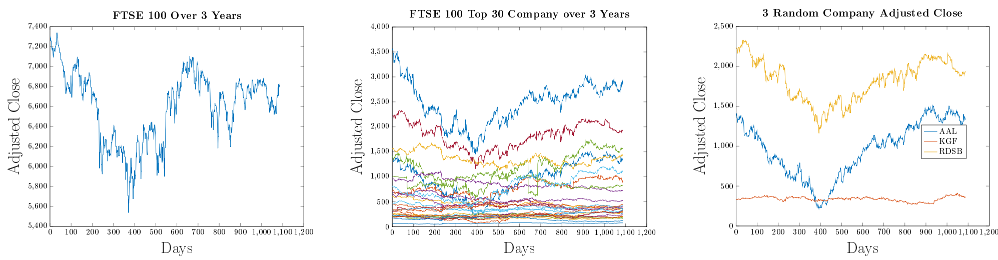


Figure 5: FTSE 100, Top 30 Most Traded Company and Selected 3 Company Adjusted Close over 3 Years

2.2 Returns and Efficient Portfolio of 3 Random Assets

I started by calculating the returns as percentage. I defined the return in two ways -

- $return(i+1) = (val(i+1) - \boldsymbol{val(i)})/val(i+1)$, return based on daily investment
- $return(i+1) = (val(i+1) - \boldsymbol{val(1)})/val(i+1)$, return based on first investment

I expected portfolio returns in first definition to be jittery in a small window since daily returns go up and down. Whereas the second definition would be smoother, although it implicitly imposes a correlation. Figure 6 and 8 build a portfolio out of **all** 30 assets, whereas Figure 7 and 11 select **3** random assets, which were controlled using `rng(1)`. The resultant assets were AAL, KGF and RDSB. Their adjusted closing values were plotted in Figure 5.

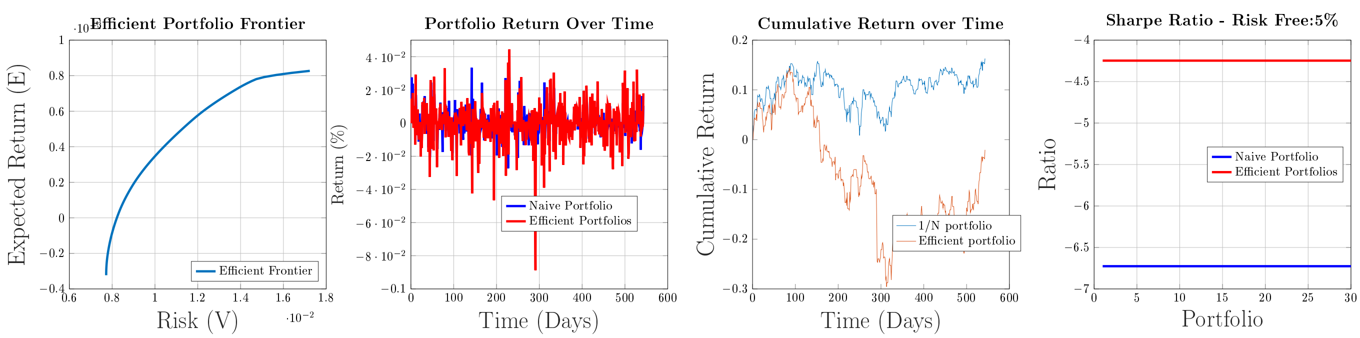


Figure 6: Portfolio of **all** 30 assets analysis where returns are based on **daily** change

As expected, the portfolio returns over time were moving between (± 0.02) if we calculated returns based on daily investment and were smoother compared to first investment. The efficient frontier in both cases were of expected shape.

2.3 Comparison with Naive Strategy

After chronologically sorting the data, I picked the first half of the data as the training set and the rest as the validation set. Using the first half, I calculated the Expected Return as the mean and the Risk as the covariance using `m = mean(returnsTrain);C = cov(returnsTrain);`. Then Financial Toolbox calculated the efficient frontier. Then I picked $E_{max}/2$ to be the target return and calculated the weights, which I

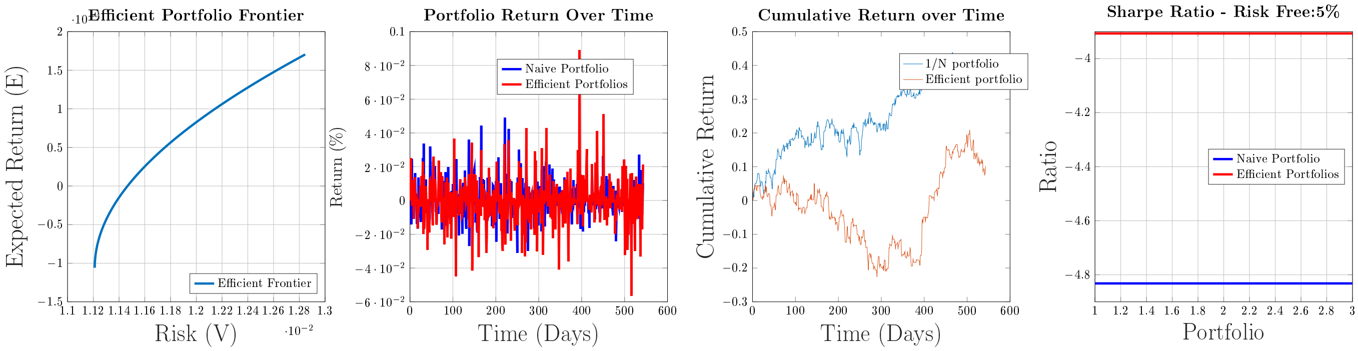


Figure 7: Portfolio of **3 random** assets analysis where returns are based on **daily** change

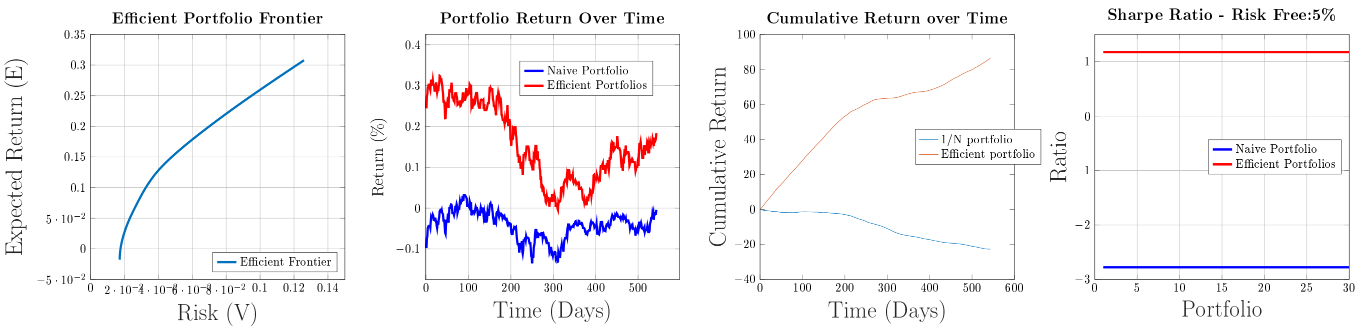


Figure 8: Portfolio of **all** 30 assets analysis where returns are based on **first return**

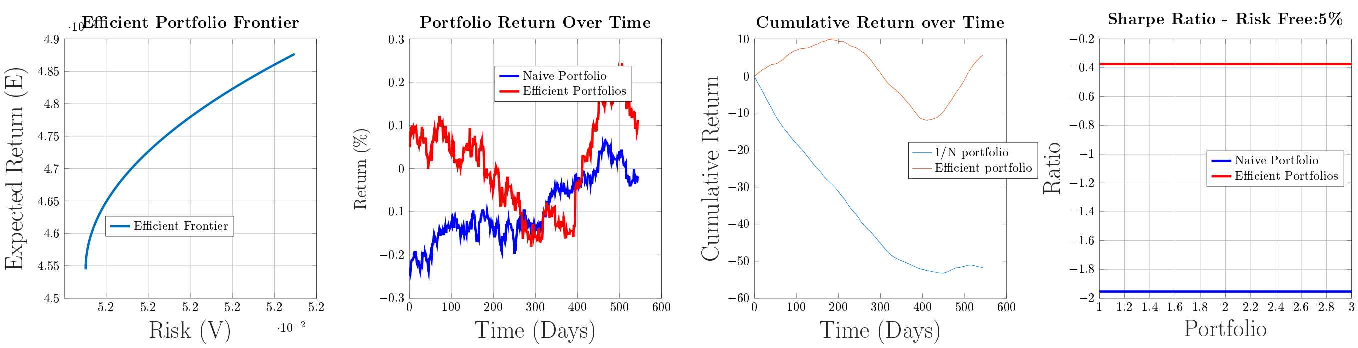


Figure 9: Portfolio of **3 random** assets analysis where returns are based on **first return**

picked as the Markowitz portfolio for comparison. Similarly `1/N*ones(N,1)` was the naive 1/N strategy portfolio.

I started the comparison by calculating the returns over the validation set. I used the Markowitz portfolio selected from the training data and the 1/N portfolio stated above -

```
effReturn = returnsTest * efficientWeights';
naiveReturn = returnsTest * naiveWeights';
```

I plotted them in the second subfigures in the figures in this section. In this particular selection of assets, the markowitz portfolio beat the naive one. However, it was not the case every time. On some runs of the experiment, the naive strategy was giving better returns on the graph.

Then I calculated the sharpe ratio, defined as - $r = (m - r_0)/\sigma$, where r_0 is the risk free return. I set `riskFree = 0.05` and calculated the following -

```
naiveSharpe = (mean(naiveReturn) - riskFree)/std(naiveReturn);
effSharpe = (mean(effReturn) - riskFree)/std(effReturn);
```

The sharpe ratio is plotted in the 4th subfigures in the figures in this section. Surprisingly the sharpe ratios in both cases were often negative, meaning that risk free assets, if it returned 5%, would be better. The reason for this is that the assets actually performed poorly in adjusted close values during this time, as seen in Figure 5. Unsurprisingly, if we set $r_0 = 0$, i.e. no returns are risk free, then the sharpe ratios are positive. Also, a larger sharpe ratio can suggest a better performing portfolio. So, whichever portfolio showed better returns on the 2nd subfigures had the larger sharpe ratio on the 4th subfigures. I think the sharpe ratio is a reasonable measure of performance, but it depends largely on the estimate of the risk free assets like - government bonds etc.

3 Index Tracking

I first selected the first half of the data as training data and the rest as the validation data.

3.1 Greedy Forward Search Asset Selection

Algorithm 1 Greedy Asset Selection for Index Tracking

S = {}
for $i \in \{1, ..., N/5\}$ **do**
  $e = \{\}$
  for $j \in \{1, ..., N\}$ **do**
    if $j \in \{S\}$ **then**
      $e(j) = \infty$
    else
      $s = S \bigcup \{j\}$
      $\boldsymbol{r_t} = R(:, s)$
      $\min_{\boldsymbol{w}}(||\boldsymbol{r_t} * \boldsymbol{w'} - \boldsymbol{y_t}||_2) s.t. \mathbf{1}^t \boldsymbol{w} = 1, \boldsymbol{w} \geq \mathbf{0}$
      $e(j) = \sqrt{mean(||\boldsymbol{r_t} * \boldsymbol{w'} - \boldsymbol{y_t}||_2^2)}$
  $idx = minIdx(e)$
  $S = S \bigcup \{idx\}$

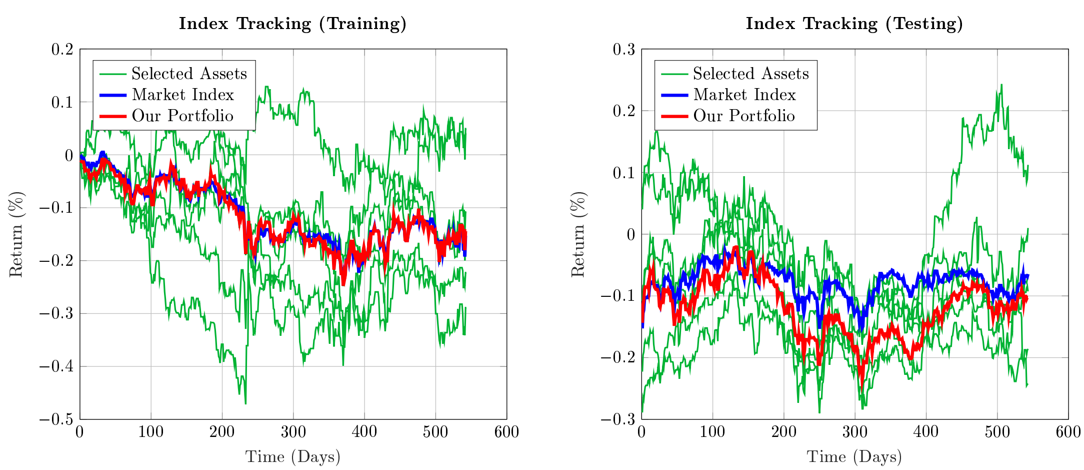


Figure 10: Greedy Index Tracking

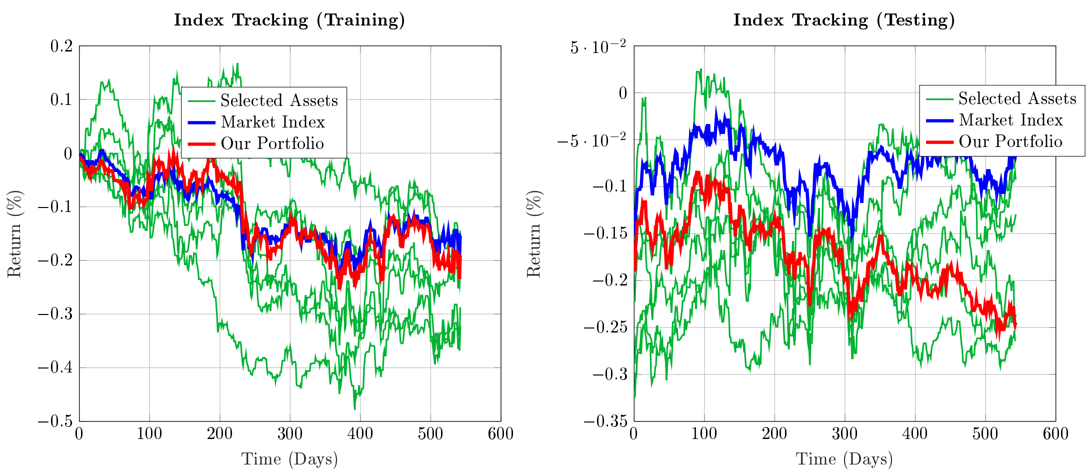


Figure 11: Sparse Index Tracking