

## 1 Black Scholes Equation Check Solution Correctness

### 1.1

Cumulative normal distribution function,  $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{-\infty} \exp(-\frac{y^2}{2}) dy$ .  $\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ .

### 1.2

$$\begin{aligned}
 d_2 &= d_1 - \sigma\sqrt{T-t} \\
 \Rightarrow \frac{d_2^2}{2} &= \frac{d_1^2}{2} + \frac{\sigma^2(T-t)}{2} - d_1\sigma\sqrt{T-t} \\
 \Rightarrow \frac{d_1^2}{2} - \frac{d_2^2}{2} &= d_1\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2} = \log\left(\frac{S}{K}\right) + r(T-t) \\
 \Rightarrow \log\left(\frac{S}{K}\right) &= \frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t) \\
 \Rightarrow \frac{S}{K} &= \exp\left(\frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t)\right) = \frac{\exp(-\frac{d_2^2}{2})}{\exp(-\frac{d_1^2}{2} - r(T-t))} = \frac{\mathcal{N}'(d_2)}{\mathcal{N}'(d_1)\exp(r(T-t))} \\
 \Rightarrow SN'(d_1) &= K\exp(-r(T-t))\mathcal{N}'(d_2)
 \end{aligned}$$

### 1.3

$$\begin{aligned}
 d_1 &= \frac{\log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \\
 \Rightarrow \frac{\partial d_1}{\partial S} &= \frac{1}{\sigma\sqrt{(T-t)}} \frac{\partial}{\partial S} (\log S - \log K + (r + \frac{\sigma^2}{2})(T-t)) = \frac{1}{\sigma S\sqrt{(T-t)}} \\
 \text{Similarly } d_2 &= d_1 - \sigma\sqrt{T-t} \Rightarrow \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{\sigma S\sqrt{(T-t)}}
 \end{aligned}$$

### 1.4

$$c = SN(d_1) - K\exp(-r(T-t))\mathcal{N}(d_2)$$

$$\begin{aligned}
 -\frac{\partial c}{\partial t} &= -\frac{\partial}{\partial t} [SN(d_1) - K\exp(-r(T-t))\mathcal{N}(d_2)] \\
 &= -SN'(d_1)\frac{\partial d_1}{\partial t} + K\exp(-r(T-t))[-r\mathcal{N}(d_2) + \mathcal{N}'(d_2)\frac{\partial d_2}{\partial t}] \\
 &= -SN'(d_1)\frac{\partial d_1}{\partial t} + K\exp(-r(T-t))[-r\mathcal{N}(d_2) + \mathcal{N}'(d_2)(\frac{\partial d_1}{\partial t} - \frac{\sigma}{2\sqrt{T-t}})] \\
 &= [-SN'(d_1) + K\exp(-r(T-t))\mathcal{N}'(d_2)]\frac{\partial d_1}{\partial t} + K\exp(-r(T-t))[-r\mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}}\mathcal{N}'(d_2)] \\
 &= -rK\exp(-r(T-t))\mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}}SN'(d_1)
 \end{aligned}$$

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) + \frac{1}{\sigma S\sqrt{(T-t)}} [SN'(d_1) - K\exp(-r(T-t))\mathcal{N}'(d_2)] = \mathcal{N}(d_1)$$

## 1.5

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) \implies \frac{\partial^2 c}{\partial S^2} = \mathcal{N}'(d_1) \frac{\partial d_1}{\partial S} = \mathcal{N}'(d_1) \frac{1}{\sigma S \sqrt{T-t}}$$

$$\begin{aligned} \frac{\partial c}{\partial t} &= -r \text{Exp}(-r(T-t)) \mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}} S \mathcal{N}'(d_1) \\ \frac{\sigma^2 S^2}{2} \frac{\partial^2 c}{\partial S^2} &= \mathcal{N}'(d_1) \frac{1}{\sigma S \sqrt{T-t}} \times \frac{\sigma^2 S^2}{2} = \frac{\sigma}{2\sqrt{T-t}} S \mathcal{N}'(d_1) \\ rS \frac{\partial c}{\partial S} &= \mathcal{N}(d_1) rS \\ -rc &= -r(S \mathcal{N}(d_1) - \text{Exp}(-r(T-t)) \mathcal{N}(d_2)) = -rS \mathcal{N}(d_1) + r \text{Exp}(-r(T-t)) \mathcal{N}(d_2) \end{aligned}$$

Adding the above -  $\frac{\partial c}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc = 0$ . So, black scholes PDE is correct for European call option.