1 Black-Scholes Equation Check Solution Correctness

1.1

Cumulative normal distribution function, $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{-\infty} exp(\frac{-y^2}{2}) dy$. $\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})$.

1.2

$$d_2 = d_1 - \sigma\sqrt{T-t}$$
$$\implies \frac{d_2^2}{2} = \frac{d_1^2}{2} + \frac{\sigma^2(T-t)}{2} - d_1\sigma\sqrt{T-t}$$
$$\implies \frac{d_1^2}{2} - \frac{d_2^2}{2} = d_1\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2} = log(\frac{S}{K}) + r(T-t)$$
$$\implies log(\frac{S}{K}) = \frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t)$$
$$\implies \frac{S}{K} = exp(\frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t)) = \frac{exp(\frac{-d_2^2}{2})}{exp(\frac{-d_1^2}{2} - r(T-t))} = \frac{\mathcal{N}'(d_2)}{\mathcal{N}'(d_1)exp(r(T-t))}$$
$$\implies S\mathcal{N}'(d_1) = Kexp(-r(T-t))\mathcal{N}'(d_2)$$

1.3

$$d_1 = \frac{log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}$$
$$\implies \frac{\partial d_1}{\partial S} = \frac{1}{\sigma\sqrt{(T-t)}} \frac{\partial}{\partial S}(logS - logK + (r + \frac{\sigma^2}{2})(T-t)) = \frac{1}{\sigma S\sqrt{(T-t)}}$$
$$Similarly \quad d_2 = d_1 - \sigma\sqrt{T-t} \implies \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{\sigma S\sqrt{(T-t)}}$$

1.4

$c = S\mathcal{N}(d_1) - Kexp(-r(T-t))\mathcal{N}(d_2)$

$$
\begin{aligned}
\frac{\partial c}{\partial t} &= \frac{\partial}{\partial t}[S\mathcal{N}(d_1) - Kexp(-r(T-t))\mathcal{N}(d_2)] \\
&= S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} - Kexp(-r(T-t))[r\mathcal{N}(d_2) + \mathcal{N}'(d_2)\frac{\partial d_2}{\partial t}] \\
&= S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} - Kexp(-r(T-t))[r\mathcal{N}(d_2) + \mathcal{N}'(d_2)(\frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}})] \\
&= [S\mathcal{N}'(d_1) - Kexp(-r(T-t))\mathcal{N}'(d_2)]\frac{\partial d_1}{\partial t} - Kexp(-r(T-t))[r\mathcal{N}(d_2) + \frac{\sigma}{2\sqrt{T-t}}\mathcal{N}'(d_2)] \\
&= -rKexp(-r(T-t))\mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}}S\mathcal{N}'(d_1)
\end{aligned}
$$

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) + \frac{1}{\sigma S\sqrt{(T-t)}}[S\mathcal{N}'(d_1) - Kexp(-r(T-t))\mathcal{N}'(d_2)] = \mathcal{N}(d_1)$$

1.5

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) \implies \frac{\partial^2 c}{\partial S^2} = \mathcal{N}'(d_1)\frac{\partial d_1}{\partial S} = \mathcal{N}'(d_1)\frac{1}{\sigma S \sqrt{(T-t)}}$$

$$
\begin{aligned}
\frac{\partial c}{\partial t} &= -rK exp(-r(T-t))\mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}}S\mathcal{N}'(d_1) \\
\frac{\sigma^2 S^2}{2}\frac{\partial^2 c}{\partial S^2} &= \mathcal{N}'(d_1)\frac{1}{\sigma S\sqrt{T-t}} \times \frac{\sigma^2 S^2}{2} = \frac{\sigma}{2\sqrt{T-t}}S\mathcal{N}'(d_1) \\
rS\frac{\partial c}{\partial S} &= \mathcal{N}(d_1)rS \\
-rc &= -r(S\mathcal{N}(d_1) - Kexp(-r(T-t))\mathcal{N}(d_2)) = -rS\mathcal{N}(d_1) + rKexp(-r(T-t))\mathcal{N}(d_2)
\end{aligned}
$$

Adding the above - $\frac{\partial c}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0$. So, Black-Scholes PDE is correct for European call option.

2 Black-Scholes Option Price vs Actual Option Price

Black-Scholes equation has a closed form solution, $c = S\mathcal{N}(d_1) - Kexp(-r(T-t))\mathcal{N}(d_2)$, given that the volatility, i.e. Standard Deviation of log returns, $u_i = log(a_i)$ is constant for an asset [1]. In this formula, interest r, time to maturity, T, current stock price of the underlying asset, S are constants of the domain. Only, volatility, σ needs to be derived from past data. As long as the volatility does not change over time until maturity, the option price will be fair.
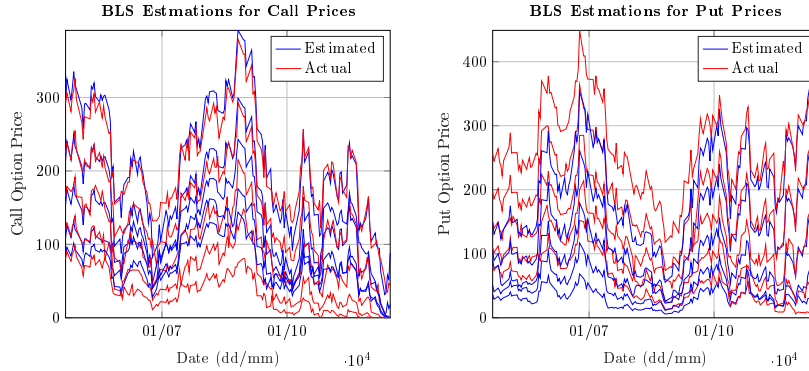

Figure 1: Black-Scholes vs Actual Option Prices

Algorithm 1 Calculating Black-Scholes Option Price

$r = 0.06$ // interest rate
for $i \in \{1,...,3T/4\}$ **do** // T = total days = 222
    $idx = T/4 + i$ // current day
    $S = stocks(idx)$ // today's stock price
    for $j \in \{1,...,N\}$ **do** // N = number of options = 10
        $K = strikePrices(j)$ // strike price of the option
        $\tau = (dates(T,j) + 1 - dates(idx,j))/365$ // maturity time in years
        $\sigma = std(log(returns(i:T/4+i-1,j)))$ // volatility as SD of log returns
        $[p,c] = blsprice(S,K,r,\tau,\sigma)$ // matlab function for Black-Scholes price
        if $isCallOption(j)$ **then**
            $estPrices(i,j) = c$
        else
            $estPrices(i,j) = p$

Algorithm 1 describes how I calculated the option prices for the 10 options from $T/4 + 1$ to T days. Then I plotted the estimated option prices against the actual traded prices in Figure 1.
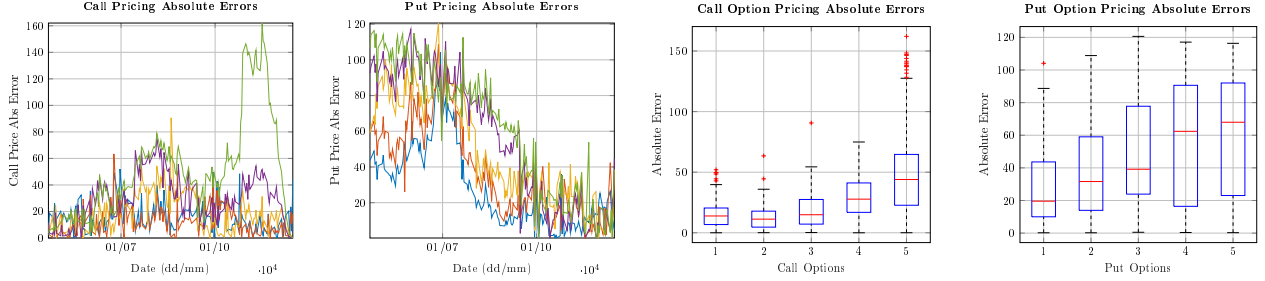

Figure 2: Absolute Error between Black-Scholes and Actual Option Prices

Then I calculated the absolute error between the Black-Scholes option price and the actual traded price. They were plotted in Figure 2. Finally, using Matlab's Financial Toolbox, I calculated the greeks for each option over that period and plotted them in Figure 3.

As can be seen from Figure 1, the call option prices closely followed the Black-Scholes solution. One exception here is a spike which can also be seen in Figure 2 as well where the absolute error was quite substantial compared to other call options.

On the other hand, there was a systematic difference between the Black-Scholes formula and actual prices for the put options, seen in both Figure 1 and 2. All put options were traded at a higher price than that derived from Black-Scholes formula. Thus the absolute errors were also quite large and dispersed.

In Figure 3, one of the volatilities has a spike near the end. This is precisely the call option whose price spiked at the same time. This leads me to believe that the price is correlated to the volatility of the underlying asset. Vega, $\nu = \frac{\partial c}{\partial \sigma}$ demonstrates exactly this. Clearly, as time to maturity draws near, the effect steadily goes down to 0.

Figure 3: Black-Scholes Parameters

Delta $\Delta = \frac{\partial c}{\partial S}$ is also depicted in the same Figure over the same time period. Note that the horizontal scale is the time from the beginning of observation to the maturity date.

3 Implied Volatility

In implementing Black-Scholes solution, I calculated the volatility from the previous $T/4$ day's historical data. However, actual traders will use other factors, such as their experience or recent market movements etc. to negotiate or write the prices for their options. So, the prices observed for the options will almost always be different from those derived from the Black-Scholes equation, as seen in the previous section.
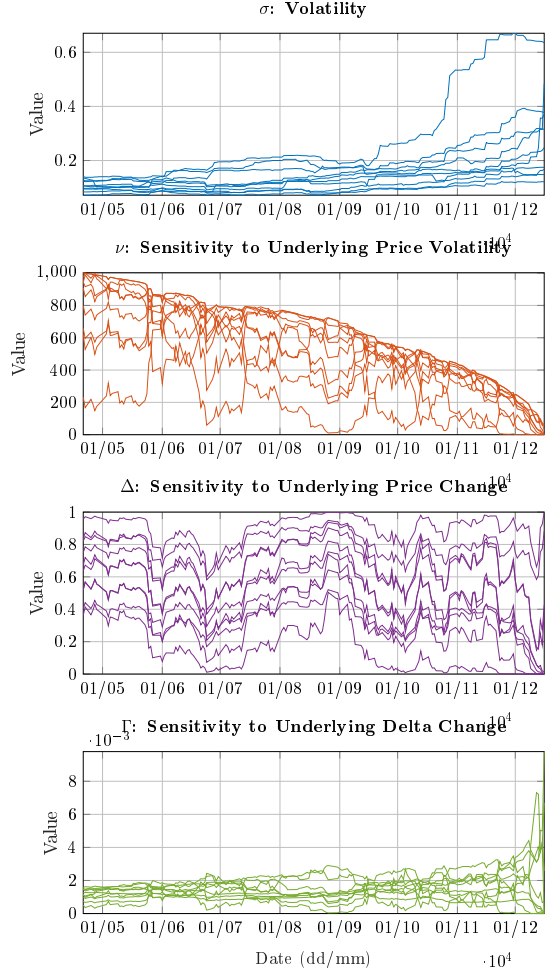
However, this gives a chance to calculate what the traders or market believe the volatility should be by working backwards. This is called the implied volatility.
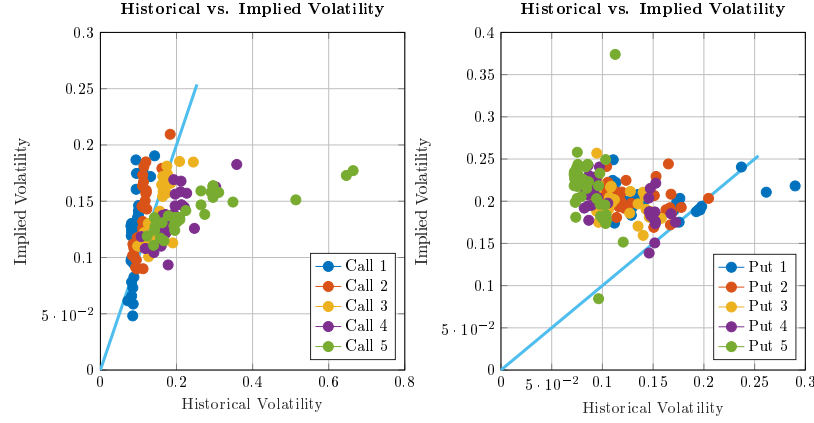

Figure 4: Historical vs Implied Volatility Scatter Plot

I started by randomly selecting $T/5$ days between day $T/4 + 1$ and T and sorting them so that they fall in chronological order. Then I used Matlab's Financial Toolbox to calculate the implied volatility for those days. In some cases, it could not be computed and returned `NaN`. Getting rid of such days' data, I was left with ≈ 30 days' worth of implied volatility. I then also calculated the historical volatility of those days from previous $T/4$ days' data. Finally, I plotted a scatter diagram in Figure 4. Since the selected data was sorted chronologicaly, I also plotted both the implied and historical volatility over time in Figure 5.
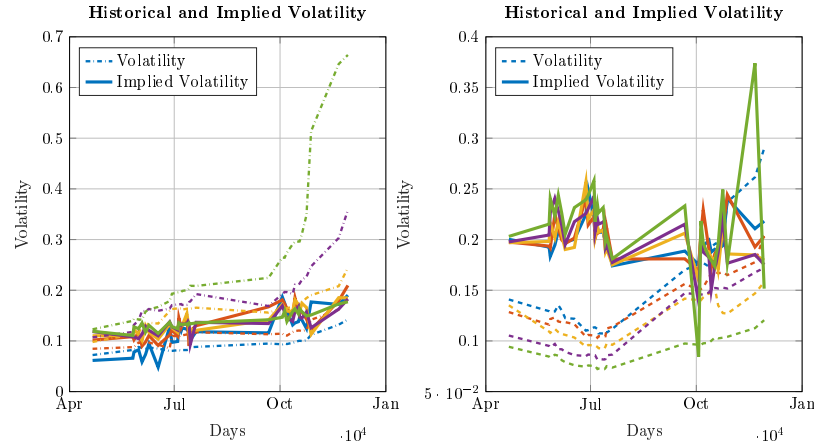

Figure 5: Historical and Implied Volatility over Time

An assumption of the Black-Scholes formula is that volatility of an asset is constant on any day, leading up to the maturity day. As, we have seen this is not the case. What is of interest then is how it changes depending on the strike price. A graph of implied volatility vs strike price on a day will have a valley or 'smile' shape, giving the name 'Volatility Smile'. So, I picked 10 random days and for each day calculated the implied volatility on that day for each of the 5 strike prices. Then I plotted the volatility smile in Figure 6.
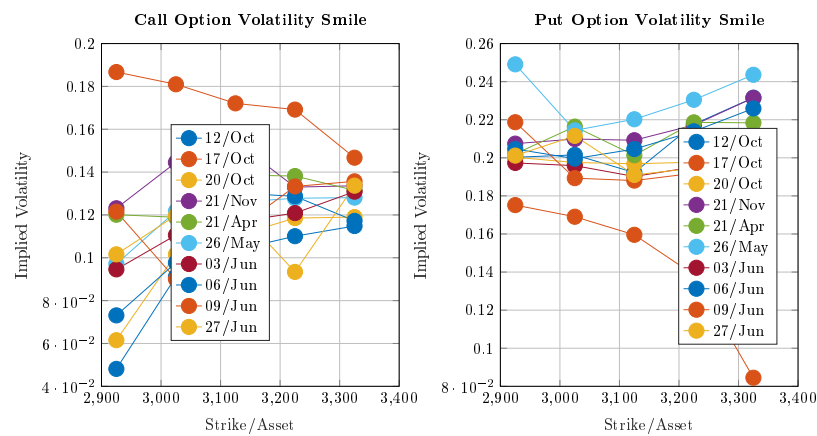
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Figure 6: Volatility Smile

References

[1] J. C. Hull *Options, Futures and Other Derivatives*. Prentice Hall, 2009