COMP6212 Assignment 2 : Shakib-Bin Hamid 25250094 sh3g12

# 1 Black Scholes Equation Check Solution Correctness

## 1.1

Cumulative normal distribution function, $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{-\infty} exp(\frac{-y^2}{2})dy$. $\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})$.

## 1.2

$$
\begin{aligned}
& d_2 = d_1 - \sigma\sqrt{T-t} \\
\implies & \frac{d_2^2}{2} = \frac{d_1^2}{2} + \frac{\sigma^2(T-t)}{2} - d_1\sigma\sqrt{T-t} \\
\implies & \frac{d_1^2}{2} - \frac{d_2^2}{2} = d_1\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2} = log(\frac{S}{K}) + r(T-t) \\
\implies & log(\frac{S}{K}) = \frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t) \\
\implies & \frac{S}{K} = exp(\frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t)) = \frac{exp(\frac{-d_2^2}{2})}{exp(\frac{-d_1^2}{2} - r(T-t))} = \frac{\mathcal{N}'(d_2)}{\mathcal{N}'(d_1)exp(r(T-t))} \\
\implies & S\mathcal{N}'(d_1) = Kexp(-r(T-t))\mathcal{N}'(d_2)
\end{aligned}
$$

## 1.3

$$
\begin{aligned}
& d_1 = \frac{log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \\
\implies & \frac{\partial d_1}{\partial S} = \frac{1}{\sigma\sqrt{(T-t)}} \frac{\partial}{\partial S}(logS - logK + (r + \frac{\sigma^2}{2})(T-t)) = \frac{1}{\sigma S\sqrt{(T-t)}} \\
& Similarly \quad d_2 = d_1 - \sigma\sqrt{T-t} \implies \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{\sigma S\sqrt{(T-t)}}
\end{aligned}
$$

## 1.4

$c = S\mathcal{N}(d_1) - Kexp(-r(T-t))\mathcal{N}(d_2)$

$$
\begin{aligned}
-\frac{\partial c}{\partial t} &= -\frac{\partial}{\partial t}[S\mathcal{N}(d_1) - Kexp(-r(T-t))\mathcal{N}(d_2)] \\
&= -S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} + Kexp(-r(T-t))[-r\mathcal{N}(d_2) + \mathcal{N}'(d_2)\frac{\partial d_2}{\partial t}] \\
&= -S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} + Kexp(-r(T-t))[-r\mathcal{N}(d_2) + \mathcal{N}'(d_2)(\frac{\partial d_1}{\partial t} - \frac{\sigma}{2\sqrt{T-t}})] \\
&= [-S\mathcal{N}'(d_1) + Kexp(-r(T-t))\mathcal{N}'(d_2)]\frac{\partial d_1}{\partial t} + Kexp(-r(T-t))[-r\mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}}\mathcal{N}'(d_2)] \\
&= -rKexp(-r(T-t))\mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}}S\mathcal{N}'(d_1)
\end{aligned}
$$

$$
\frac{\partial c}{\partial S} = \mathcal{N}(d_1) + \frac{1}{\sigma S\sqrt{(T-t)}}[S\mathcal{N}'(d_1) - Kexp(-r(T-t))\mathcal{N}'(d_2)] = \mathcal{N}(d_1)
$$

The above follows because -

$$
\begin{aligned}
Kexp(-r(T-t))\mathcal{N}'(d_2) &= Kexp(-r(T-t))\mathcal{N}'(d_1-\sigma\sqrt{T-t}) \\
&= Kexp(-r(T-t))\frac{1}{\sqrt{2\pi}}exp(\frac{-(d_1-\sigma\sqrt{T-t})^2}{2}) \\
&= Kexp(-r(T-t))\frac{1}{\sqrt{2\pi}}exp(\frac{-d_1^2}{2})exp(\frac{2d_1\sigma\sqrt{T-t}-\sigma^2(T-t)}{2}) \\
&= Kexp(-r(T-t))\mathcal{N}'(d_1)exp(d_1\sigma\sqrt{T-t})exp(-\sigma^2(T-t)) \\
&= K\mathcal{N}'(d_1)exp(-r(T-t)-\frac{\sigma^2(T-t)}{2}+log(\frac{S}{K})+r(T-t)+\frac{\sigma^2(T-t)}{2}) \\
&= K\mathcal{N}'(d_1)\frac{S}{K} = S\mathcal{N}'(d_1)
\end{aligned}
$$

**1.5**

$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) \implies \frac{\partial^2 c}{\partial S^2} = \mathcal{N}'(d_1)\frac{\partial d_1}{\partial S} = \mathcal{N}'(d_1)\frac{1}{\sigma S\sqrt{(T-t)}}$

$$
\begin{aligned}
\frac{\partial c}{\partial t} &= -rKexp(-r(T-t))\mathcal{N}(d_2) - \frac{\sigma}{2\sqrt{T-t}}S\mathcal{N}'(d_1) \\
\frac{\sigma^2 S^2}{2}\frac{\partial^2 c}{\partial S^2} &= \mathcal{N}'(d_1)\frac{1}{\sigma\sqrt{T-t}} \times \frac{\sigma^2 S^2}{2} = \frac{\sigma}{2\sqrt{T-t}}S\mathcal{N}'(d_1) \\
rS\frac{\partial c}{\partial S} &= \mathcal{N}(d_1)rS \\
-rc &= -r(S\mathcal{N}(d_1) - Kexp(-r(T-t))\mathcal{N}(d_2)) = -rS\mathcal{N}(d_1) + rKexp(-r(T-t))\mathcal{N}(d_2)
\end{aligned}
$$

Adding the above - $\frac{\partial c}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0$. So, black scholes PDE is correct for European call option.