

1 Black Scholes Equation Check Solution Correctness

Cumulative normal distribution function, $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{-\infty} \exp(-\frac{y^2}{2}) dy$. $\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$.

$$\begin{aligned}
 d_2 &= d_1 - \sigma\sqrt{T-t} \\
 \Rightarrow \frac{d_2^2}{2} &= \frac{d_1^2}{2} + \frac{\sigma^2(T-t)}{2} - d_1\sigma\sqrt{T-t} \\
 \Rightarrow \frac{d_1^2}{2} - \frac{d_2^2}{2} &= d_1\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2} = \log\left(\frac{S}{K}\right) + r(T-t) \\
 \Rightarrow \log\left(\frac{S}{K}\right) &= \frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t) \\
 \Rightarrow \frac{S}{K} &= \exp\left(\frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t)\right) = \frac{\exp(-\frac{d_2^2}{2})}{\exp(-\frac{d_1^2}{2} - r(T-t))} = \frac{\mathcal{N}'(d_2)}{\mathcal{N}'(d_1)\exp(r(T-t))} \\
 \Rightarrow SN'(d_1) &= K\exp(-r(T-t))\mathcal{N}'(d_2)
 \end{aligned}$$

$$\begin{aligned}
 d_1 &= \frac{\log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \\
 \Rightarrow \frac{\partial d_1}{\partial S} &= \frac{1}{\sigma\sqrt{(T-t)}} \frac{\partial}{\partial S} (\log S - \log K + (r + \frac{\sigma^2}{2})(T-t)) = \frac{1}{\sigma S\sqrt{(T-t)}} \\
 \text{Similarly } d_2 &= d_1 - \sigma\sqrt{T-t} \Rightarrow \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{\sigma S\sqrt{(T-t)}}
 \end{aligned}$$

$$\begin{aligned}
 c &= SN(d_1) - K\exp(-r(T-t))\mathcal{N}(d_2) \\
 \Rightarrow \frac{\partial c}{\partial S} &= \mathcal{N}(d_1) + \frac{1}{\sigma S\sqrt{(T-t)}} [SN'(d_1) - K\exp(-r(T-t))\mathcal{N}'(d_2)] = \mathcal{N}(d_1)
 \end{aligned}$$

$$\begin{aligned}
 K\exp(-r(T-t))\mathcal{N}'(d_2) &= K\exp(-r(T-t))\mathcal{N}'(d_1 - \sigma\sqrt{T-t}) \\
 &= K\exp(-r(T-t)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_1 - \sigma\sqrt{T-t})^2}{2}\right) \\
 &= K\exp(-r(T-t)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_1^2}{2}\right) \exp\left(\frac{2d_1\sigma\sqrt{T-t} - \sigma^2(T-t)}{2}\right) \\
 &= K\exp(-r(T-t))\mathcal{N}'(d_1) \exp(d_1\sigma\sqrt{T-t}) \exp(-\sigma^2(T-t)) \\
 &= K\mathcal{N}'(d_1) \exp(-r(T-t) - \frac{\sigma^2(T-t)}{2} + \log\left(\frac{S}{K}\right) + r(T-t) + \frac{\sigma^2(T-t)}{2}) \\
 &= K\mathcal{N}'(d_1) \frac{S}{K} = SN'(d_1)
 \end{aligned}$$

$$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) \Rightarrow \frac{\partial^2 c}{\partial S^2} = \mathcal{N}(d_1) \frac{\partial d_1}{\partial S} = \mathcal{N}(d_1) \frac{1}{\sigma S\sqrt{(T-t)}}$$