COMP6212 Assignment 2 : Shakib-Bin Hamid 25250094 sh3g12

1 Black Scholes Equation Check Solution Correctness

Cumulative normal distribution function, $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{-\infty} e^{\frac{-y^2}{2}} dy$. $\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.

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\begin{aligned}
& d_2 = d_1 - \sigma\sqrt{T-t} \\
\implies \quad & \frac{d_2^2}{2} = \frac{d_1^2}{2} + \frac{\sigma^2(T-t)}{2} - d_1\sigma\sqrt{T-t} \\
\implies \quad & \frac{d_1^2}{2} - \frac{d_2^2}{2} = d_1\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2} = log(\frac{S}{K}) + r(T-t) \\
\implies \quad & log(\frac{S}{K}) = \frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t) \\
\implies \quad & \frac{S}{K} = exp(\frac{d_1^2}{2} - \frac{d_2^2}{2} - r(T-t)) = \frac{exp(\frac{-d_2^2}{2})}{exp(\frac{-d_1^2}{2} - r(T-t))} = \frac{\mathcal{N}'(d_2)}{\mathcal{N}'(d_1)e^{r(T-t)}} \\
\implies \quad & S\mathcal{N}'(d_1) = Ke^{-r(T-t)}\mathcal{N}'(d_2)
\end{aligned}
$$

$$
\begin{aligned}
& d_1 = \frac{log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \\
\implies \quad & \frac{\partial d_1}{\partial S} = \frac{1}{\sigma\sqrt{(T-t)}} \frac{\partial}{\partial S}(logS - logK + (r + \frac{\sigma^2}{2})(T-t)) = \frac{1}{\sigma S\sqrt{(T-t)}} \\
Similarly \quad & d_2 = d_1 - \sigma\sqrt{T-t} \implies \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{\sigma S\sqrt{(T-t)}}
\end{aligned}
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\begin{aligned}
& c = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \\
\implies \quad & \frac{\partial c}{\partial S} = \mathcal{N}(d_1) + \frac{1}{\sigma S\sqrt{(T-t)}}[S\mathcal{N}'(d_1) - Ke^{-r(T-t)}\mathcal{N}'(d_2)] = \mathcal{N}(d_1)
\end{aligned}
$$

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\begin{aligned}
Ke^{-r(T-t)}\mathcal{N}'(d_2) &= Ke^{-r(T-t)}\mathcal{N}'(d_1 - \sigma\sqrt{T-t}) \\
&= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}} e^{\frac{-(d_1-\sigma\sqrt{T-t})^2}{2}} \\
&= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} e^{\frac{2d_1\sigma\sqrt{T-t}-\sigma^2(T-t)}{2}} \\
&= Ke^{-r(T-t)}\mathcal{N}'(d_1)e^{d_1\sigma\sqrt{T-t}}e^{-\sigma^2(T-t)} \\
&= K\mathcal{N}'(d_1)e^{-r(T-t)-\frac{\sigma^2(T-t)}{2}+log(\frac{S}{K})+r(T-t)+\frac{\sigma^2(T-t)}{2}} \\
&= K\mathcal{N}'(d_1)\frac{S}{K} = S\mathcal{N}'(d_1)
\end{aligned}
$$

$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) \implies \frac{\partial^2 c}{\partial S^2} = \mathcal{N}(d_1)\frac{\partial d_1}{\partial S} = \mathcal{N}(d_1)\frac{1}{\sigma S\sqrt{(T-t)}}$