MICT-5101: Probability and Stochastic Process[1]

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Lecture Outline I

Introduction

1 Introduction

1.1 Text & Reference Book List

Text Book

1. Ross, S. (2010): *Introduction to Probability Models*, 10th edition, Pearson, Prentice Hall.
2. Anthony J. Hayter (2012): *Probability and Statistics for Engineers and Scientists* 4th Edition, Duxbury Press.

Reference Book List

1. Mehdhi, J. (2009): *Stochastic Processes* , 3rd Revised Edition, New Age Science.
2. Beichelt F. (2016): *Applied probability and stochastic processes*, 7th edition, CRC Press.
3. Ross, S. (2020): *Introduction to Probability and Statistics for Engineers and Scientists*, 6th Edition, Pearson Education Inc.

Fundamentals of Probability Models

1. **Part I:** Probability Theory
   - Basic Concepts of Probability
   - Random Variable
   - Expectation
   - Some Probability Distributions
     - Bernoulli
     - Binomial
     - Poisson
     - Uniform and
     - Normal
     - exponential
     - ...
2. **Part II:** Stochastic Processes
   - Basics of Stochastic Processes
   - Random Point Processes
   - Discrete-Time Markov Chains
   - ...

Chapter 3: The Poisson Process

2 Chapter 3: The Poisson Process

Counting Processes

A **counting process** is a stochastic process $N(t)$ that counts the number of events that have occurred by time t, where:

- State Space: $N(t)$ takes non-negative integer values (0, 1, 2, ...).
- Non-decreasing: $N(t_1) \leq N(t_2)$ for all $t_1 < t_2$.
- Initial Condition: Typically, $N(0) = 0$ (indicating no events have occurred at time zero).
- For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval $(s, t]$

Counting processes are used to model random events over time, such as arrivals of customers, failures of machines, or occurrences of specific events.

Common Types

- Poisson Process
- Renewal Process

These processes have specific properties regarding the distribution and timing of events.

Examples of Counting Processes

1. Customer Arrivals at a Store:
   - Let $N(t)$ be the number of customers entering by time t (in hours).
   - Modeled as a Poisson process if arrivals are independent and at a constant rate.
2. Calls Received at a Call Center:
   - Let $N(t)$ be the total number of calls received by time t (in minutes).
   - Also modeled as a Poisson process.
3. Machine Failures in a Factory:
   - Let $N(t)$ be the number of machine failures by time t (in days).
   - Can be modeled using a renewal process.
4. Earthquake Occurrences:
   - Let $N(t)$ denote the number of earthquakes by time t (in years).
   - May be modeled as a non-homogeneous Poisson process.

Two Important Assumptions

1. **Independent Increments:**
   - The increments are independent if the number of events in disjoint intervals is independent.
   - For $0 \leq t_1 < t_2 < t_3 < t_4$:

$$N(t_2) - N(t_1) \quad \text{and} \quad N(t_4) - N(t_3) \text{ are independent.}$$

2. **Stationary Increments**
   - A counting process has stationary increments if the distribution of the number of events in any time interval depends only on the length of the interval, not on its position.
   - For any h and t:

$$N(t+h) - N(t) \sim \text{Distribution depending only on } h.$$

i.e., $[N(t_2+h) - N(t_1+h)]$ has the same distribution as the number of events in the interval (t_1, t_2) i.e., $[N(t_2) - N(t_1)]$ for $t_1 < t_2$ and $h > 0$.

Poisson Process

A stochastic process $N(t)$ is called a Poisson process with rate $\lambda > 0$ if it satisfies the following conditions:

- Initial Condition:

$$N(0) = 0$$

- Independent Increments: For any $0 \leq t_1 < t_2$, the increment $N(t_2) - N(t_1)$ is independent of $N(t_1)$.
- Stationary Increments: For any $s, t \geq 0$:

$$N(t+s) - N(t) \sim \text{Poisson}(\lambda s)$$

- Distribution of Increments:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots$$

where $P(N(t) = k)$ is the probability of observing k events in the interval $[0, t]$.

Examples of Poisson Processes

- Customer Arrivals: The number of customers arriving at a store in an hour can be modeled as a Poisson process if arrivals are random and independent.
- Call Center Calls: The number of incoming calls at a call center during a specific time period can be modeled as a Poisson process.
- Traffic Accidents: The number of accidents occurring at a particular intersection in a day can be modeled as a Poisson process.
- Email Arrivals: The number of emails received in an inbox over a fixed time interval can be treated as a Poisson process if they arrive randomly.

Derivation of Poisson Process

Assumption:

In order to derive Poisson process, we assume

1. The number of events in different disjoint time intervals are independent
2. The distribution of number of events depends only on the length of the interval
3. The number of occurrences at time 0 is zero. i.e., $N(0) = 0$
4. $P_0(0) = 1$
5. $P\{N(h) = 1\} = \lambda h + O(h)$
6. $P\{N(h) \geq 2\} = O(h)$

In order to derive Poisson process, Let

$$
\begin{aligned}
P_0(t+h) &= \Pr\{N(t+h)=0\} \\
&= \Pr\{N(t)=0,\ N(t+h)-N(t)=0\} \\
&= \Pr\{N(t)=0\}\cdot\Pr\{N(h)=0\} \\
&= P_0(t)\cdot[1-\Pr\{N(h)\geq 1\}] \\
&= P_0(t)\cdot[1-\lambda h+O(h)] \qquad [\because\ O(h)+O(h)=O(h)\ \ and\ \ -O(h)=O(h)]
\end{aligned}
$$

$$\Rightarrow P_0(t+h)-P_0(t)=-\lambda hP_0(t)+O(h) \qquad [\because\ O(h)\times any\ Value=O(h)]$$

$$\Rightarrow \lim_{h\to 0}\frac{P_0(t+h)-P_0(t)}{h}=-\lambda P_0(t)+\lim_{h\to 0}\frac{O(h)}{h}$$

$$\Rightarrow P_0'(t)=-\lambda P_0(t)$$

$$\Rightarrow \frac{P_0'(t)}{P_0(t)}=-\lambda$$

$$\Rightarrow \ln P_0(t)=-\lambda t+C \qquad [\text{Integrating both sides w. r. to } t]$$

$$\Rightarrow P_0(t)=ke^{-\lambda t} \qquad [Here,\ k=e^c] \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

From (1)

$$P_0(0)=ke^{-\lambda 0}=1 \qquad \Rightarrow k=1$$

$$Now,\quad P_0(t)=e^{-\lambda t} \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

$$
\begin{aligned}
P_n(t+h) &= \Pr\{N(t+h)=n\} \\
&= \Pr\{N(t)=n, N(h)=0\} + \Pr\{N(t)=n-1, N(h)=1\} + \Pr\{N(t)=n-2, N(h)=2\} + \cdots \\
&= P_n(t)\Pr\{N(h)=0\} + P_{n-1}(t)\Pr\{N(h)=1\} + P_{n-2}(t)\Pr\{N(h)=2\} + \cdots \\
&= P_n(t)[1-\lambda h + O(h)] + P_{n-1}(t)[\lambda t + O(h)] + O(h) \\
&= P_n(t) - \lambda h P_n(t) + \lambda h P_{n-1}(t) + O(h)
\end{aligned}
$$

$$\Rightarrow P_n(t+h) - P_n(t) = -\lambda h P_n(t) + \lambda h P_{n-1}(t) + O(h)$$

$$\Rightarrow \lim_{h\to 0} \frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \lim_{h\to 0}\frac{O(h)}{h}$$

$$\Rightarrow P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

$$\Rightarrow P_n'(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

$$\Rightarrow e^{\lambda t}[P_n'(t) + \lambda P_n(t)] = \lambda e^{\lambda t} P_{n-1}(t)$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t}P_n(t)\right] = \lambda e^{\lambda t} P_{n-1}(t) \quad \cdots \quad \cdots \quad \cdots \quad (3) \quad \left[\because e^{mx}\{f'(x)+mf(x)\} = \frac{d}{dx}\left(e^{mx}f(x)\right)\right]$$

When, $n=1$

$$
\begin{aligned}
&\frac{d}{dt}\left[e^{\lambda t}P_1(t)\right] = \lambda e^{\lambda t}P_0(t) \\
&\Rightarrow \frac{d}{dt}\left[e^{\lambda t}P_1(t)\right] = \lambda e^{\lambda t}e^{-\lambda t} \\
&\Rightarrow \frac{d}{dt}\left[e^{\lambda t}P_1(t)\right] = \lambda \\
&\Rightarrow e^{\lambda t}P_1(t) = \lambda t + c \qquad [\text{Integrating both sides w.r.t. } t] \\
&\Rightarrow P_1(t) = e^{-\lambda t}(\lambda t + c) \quad \cdots \quad \cdots \quad \cdots \quad (4)
\end{aligned}
$$

$$\text{For } t = 0$$

$$P_1(0) = e^{-\lambda \cdot 0}(\lambda \cdot 0 + c) \qquad [\because \; P_1(0) = 0]$$

$$\Rightarrow 0 = c$$

$$\Rightarrow c = 0$$

$$\text{Now} \quad P_1(t) = \lambda t e^{\lambda t} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (5)$$

From equation (3) when $n = 2$

$$\frac{d}{dt}\left[e^{\lambda t} P_2(t)\right] = \lambda e^{\lambda t} P_1(t)$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t} P_2(t)\right] = \lambda e^{\lambda t} e^{-\lambda t} \lambda t$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t} P_1(t)\right] = \lambda^2 t$$

$$\Rightarrow e^{\lambda t} P_2(t) = \frac{\lambda^2 t^2}{2} + c \qquad [\text{Integrating both sides w.r.t. } t]$$

$$\Rightarrow P_2(t) = \frac{e^{-\lambda t}(\lambda t)^2}{2!} \qquad [\because \; P_2(0) = 0 \text{ i.e., } c = 0]$$

$$\text{Similarly,} \quad P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad ; \; n = 0, \; 1, \; 2, \cdots$$

Properties of Poisson Process:

1. $Mean = \lambda t$, $Variance = \lambda t$

2. Probability generating function $P(s) = e^{\lambda(s-1)t}$

3. Sum of independent Poisson process is also a Poisson process.

4. The difference of two independent Poisson process is not a Poisson process.

5. Random selection from a Poisson process is also a Poisson process.

6. Suppose $[N(t), t \geq 0]$ be a Poisson process such that $s < t$ then

$$\Pr\{N(s) = k \mid N(t) = n\} = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k} \qquad ; \qquad k = 0, 1, 2, \cdots.$$

7. Suppose $[N(t), t \geq 0]$ be a Poisson process then the autocorrelation coefficient between $N(t)$ and $N(t+s)$ is $\sqrt{\dfrac{t}{t+s}}$.

Example:

Suppose that customers arrive at a Bank according to a Poisson process with a mean rate λ per minute. Then the number of customers $N(t)$ arriving in an interval of duration t minutes follows Poisson distribution with mean λt. If the rate of arrival is 3 per minute, then in an arrival of 2 minutes, find the probability that the number of customer arriving is:
i) Exactly 4. ii) Greater than 4. iii) Less than 4.

Solution:

i) The Probability that the number of customers arriving exactly 4 is:

$$P_4(t) = \frac{e^{-\lambda t}(\lambda t)^4}{4!} \qquad ; \; \textit{Where } \lambda = 3 \textit{ and } t = 2$$

$$= \frac{e^{-6}(6)^4}{4!} = 0.233$$

ii) Greater than 4 is:

$$\sum_{k=5}^{\infty} P_k(2) = 1 - \sum_{k=0}^{4} P_k(2) = 1 - \sum_{k=0}^{4} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad ; \; \textit{Where } \lambda = 3 \textit{ and } t = 2$$

$$= 1 - \sum_{k=0}^{4} \frac{e^{-6}(6)^k}{k!} = 1 - e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right)$$

$$= 1 - 0.285 = 0.715$$

iii) Less than 4 is:

$$\sum_{k=0}^{3} P_k(2) = \sum_{k=0}^{3} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad ; \; \textit{Where } \lambda = 3 \textit{ and } t = 2$$

$$= \sum_{k=0}^{3} \frac{e^{-6}(6)^k}{k!} = e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!}\right) = 0.152$$

Interarrival Time:

The intervals between successive occurrences in a Poisson process are called interarrival times. Let T_i denote the time between two successive occurrences E_i and E_{i+1}. The sequence $\{T_i, i = 1, 2, \ldots\}$ represents the interarrival times. The distribution of T is given by:

$$f(T) = \lambda e^{-\lambda t} \quad \text{for } T > 0, \quad \lambda = \text{rate of arrivals.}$$

Theorem: Interarrival Time.

The time between two occurrences in a Poisson process $[N(t), t \geq 0]$ with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$.

Proof:

Let X be the random variable representing the interval between two successive occurrences of $[N(t), t \geq 0]$ and let $P[X \leq x] = F(x)$ be its distribution function.

Let us denote two successive events by E_i & E_{i+1} and suppose that E_i occurred at the instant t_i. Then,

$$\begin{aligned}
P[X > x] &= P[E_{i+1} \text{ did not occur in } (t_i, t_1 + x) \text{ given that } E_i \text{ occurred at the instant } t_i] \\
&= P[E_{i+1} \text{ did not occur in } (t_i, t_1 + x) \mid N(t_i) = i] \\
&= P[\text{no occurrence take place in an interval } (t_i, t_1 + x) \text{ of length } X \mid N(t_i) = i] \\
&= P[N(x) = 0 \mid N(t_i) = i] \\
&= P[N(x) = 0] \\
&= e^{-\lambda x}
\end{aligned}$$

Now,

$$\begin{aligned}
F(x) &= P[X \leq x] \\
&= 1 - P[X > x] \\
&= 1 - e^{-\lambda x} \qquad ; x > 0
\end{aligned}$$

Then, the density function is

$$f(x) = F'(x) = \lambda e^{-\lambda x} \quad ; x > 0$$

Waiting Time:

The quantity of interest S_n, the arrival time of n^{th} event, is known as waiting time until the n^{th} event occurs. Let $\{T_i\}$ be a sequence of interarrival time then it is denoted by

$$S_n = \sum_{i=1}^{n} T_i \qquad ; n \geq 1$$

Here, T_i follows exponential distribution with parameter λ and S_n follows gamma distribution with parameters n, λ. That is, the probability density of S_n is given by

$$f_{S_n}(t) = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} \qquad ; \qquad t \geq 0$$

Example:

Suppose car passes Prantic Gate at a Poisson rate of 1 per minute. If 5% of the cars are Toyota then

a) What is the probability that at least one Toyota passes by during an hour.

b) Given that 10 Toyotas had passed by an hour, what is the expected number of cars has passed by at that time.

c) If 50 cars have passed by an hour, what is the probability that 5 of them are Toyotas.

Solution:

Let $N(t)$ $\rightarrow$ number of cars passed by t

$N_1(t)$ $\rightarrow$ number of Toyotas passed by t

$N_2(t)$ $\rightarrow$ number of non-Toyotas passed by t

Then $N(t)$ is a Poisson process with rate $\lambda = 1$ per minute. $N_1(t)$ is a Poisson process with rate $\lambda p = 0.05 \times 1 = 0.05$ per minute. $N_2(t)$ is a Poisson process with rate $\lambda q = 0.95 \times 1 = 0.95$ per minute.

a) $$P\{N_1(60) \geq 1\} = 1 - P\{N_1(60) = 0\} = 1 - e^{-0.05 \times 60} = 1 - e^{-3}$$

Comment: The probability that at least one Toyota passes by during an hour is $1 - e^{-3}$.

b) $$E\{N(60) = n \mid N_1(60) = 10\} = E\{10 + N_2(60)\} = 10 + 0.95 \times 60 = 67$$

Comment: Given that 10 Toyotas had passed by an hour, the expected number of cars has passed by at that time is 67.

c) $$P\{N_1(60) = 5 \mid N(60) = 50\} = \binom{50}{5} p^5 (1-p)^{50-5} = \binom{50}{5} (0.05)^5 (0.95)^{50-5} = 0.07$$

Comment: 50 cars have passed by an hour, the probability that 5 of them are Toyotas is 0.07.

Example:

Suppose that customers arrive at a counter in accordance with a Poisson process with mean rate of 2 per minute $(\lambda = 2/\text{minute})$. Then the interval between any two successive arrivals follows exponential distribution with mean $\frac{1}{\lambda} = \frac{1}{2}$ minute. What is the probability that the interval between two successive arrivals is

a) More that 1 minute

b) 4 minutes or less

c) Between 1 and 2 minutes

d) Find the expected time until the 9th customer.

Solution:

Suppose x represents time between two successive arrivals then

$$f(x) = \lambda e^{-\lambda x} \quad ; \quad x > 0$$

a) $P\{x > 1\} = 2\int_1^\infty e^{-2x}\,dx = e^{-2} = 0.135$

b) $P\{x \le 4\} = 1 - 2\int_4^\infty e^{-2x}\,dx = 1 - e^{-2\times 4} = 1 - e^{-8} = 0.99967$

c) $P\{1 \le x \le 2\} = 2\int_1^2 e^{-2x}\,dx = e^{-2} - e^{-4} = 0.0179$

d) We know expected waiting time is $\frac{n}{\lambda}$. Thus, the expected time until the 9th customer is $\frac{n}{\lambda} = \frac{9}{2}$.

Conditional Distribution of Arrival Times

Arrival Times:
Let $A_1, A_2, \ldots, A_n$ be the arrival times of the first n events in a Poisson process. The arrival times are defined as:

$$A_i = T_1 + T_2 + \cdots + T_i,$$

where T_i are the interarrival times.

Objective:
We want to find the joint distribution of $A_1, A_2, \ldots, A_n$ given that there are exactly n arrivals by time t:

$$P(A_1 \leq a_1, A_2 \leq a_2, \ldots, A_n \leq a_n | N(t) = n) \quad \text{for } 0 < a_1 < a_2 < \ldots < a_n < t.$$

Understanding the Setup

Properties of the Poisson Process:
- Given $N(t) = n$, the n arrival times are uniformly distributed in the interval $[0, t]$. - The arrival times are independent and identically distributed (i.i.d.).

Order Statistics:
The arrival times $A_1, A_2, \ldots, A_n$ can be thought of as the order statistics of n uniformly distributed random variables in $[0, t]$.

Implication:
For $0 < a_1 < a_2 < \ldots < a_n < t$, the joint distribution can be derived based on the properties of uniform distributions.

Derivation of Joint Distribution

Step 1: Conditioning on $N(t) = n$

We express the joint probability as:

$$P(A_1 \leq a_1, A_2 \leq a_2, \ldots, A_n \leq a_n | N(t) = n) = \frac{P(A_1 \leq a_1, A_2 \leq a_2, \ldots, A_n \leq a_n, N(t) = n)}{P(N(t) = n)}.$$

Step 2: Joint Probability Numerator

Given $N(t) = n$, the arrival times are uniformly distributed. The joint distribution of the order statistics is:

$$P(A_1 \leq a_1, A_2 \leq a_2, \ldots, A_n \leq a_n | N(t) = n) = \frac{a_1 a_2 \cdots a_n}{t^n} \cdot \frac{1}{n!},$$

where $\frac{1}{n!}$ accounts for the ordering of the arrival times.

Step 3: Denominator

For the Poisson process:

$$P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

Combining Results:

Substituting the numerator and denominator, we have:

$$P(A_1 \leq a_1, A_2 \leq a_2, \ldots, A_n \leq a_n | N(t) = n) = \frac{\frac{a_1 a_2 \cdots a_n}{t^n} \cdot \frac{1}{n!}}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}}.$$

Simplifying:

The $n!$ terms cancel out, leading to:

$$P(A_1 \leq a_1, A_2 \leq a_2, \ldots, A_n \leq a_n | N(t) = n) = \frac{a_1 a_2 \cdots a_n}{(\lambda t)^n} e^{-\lambda t}.$$

Problems on Poisson Process

Problem 1:

A call center receives an average of 10 calls per hour. What is the probability that exactly 5 calls are received in a 30-minute interval?

Solution:

Let $\lambda = 10$ calls/hour. In a 30-minute interval, $\lambda = 5$ calls (since $\frac{1}{2}$ hour).

$$\lambda t = \frac{10 \text{ calls}}{60 \text{ minutes}} \times 30 \text{ minutes} = 5 \text{ calls}$$

The number of calls follows a Poisson distribution:

$$P(N(0.5) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

For $k = 5$:

$$P(N(0.5) = 5) = \frac{(5)^5 e^{-5}}{5!} = \frac{3125 e^{-5}}{120} \approx 0.175.$$

Problems on Poisson Process (cont.)

Problem 2:
An airport experiences an average of 3 flight arrivals every 10 minutes. What is the probability that there are no arrivals in a 5-minute interval?

Solution:
First, calculate the rate λ for a 5-minute interval:

$$\lambda t = \frac{3}{10} \times 5 = 1.5.$$

The number of arrivals follows a Poisson distribution:

$$P(N(5) = 0) = \frac{(1.5)^0 e^{-1.5}}{0!} = e^{-1.5} \approx 0.223.$$

Problems on Poisson Process (cont.)

Problem 3:

A factory produces an average of 2 defective items per hour. What is the probability that in a 2-hour period, there are at most 3 defective items?

Solution:

For a 2-hour period, $\lambda t = 2 \times 2 = 4$. We need to find:

$$P(N(2) \leq 3) = P(N(2) = 0) + P(N(2) = 1) + P(N(2) = 2) + P(N(2) = 3).$$

Calculating each term:

$$P(N(2) = k) = \frac{4^k e^{-4}}{k!} \quad \text{for } k = 0, 1, 2, 3.$$

Thus,

$$P(N(2) = 0) = \frac{4^0 e^{-4}}{0!} = e^{-4},$$

$$P(N(2)=1)=\frac{4^1e^{-4}}{1!}=4e^{-4},$$

$$P(N(2)=2)=\frac{4^2e^{-4}}{2!}=\frac{16e^{-4}}{2}=8e^{-4},$$

$$P(N(2)=3)=\frac{4^3e^{-4}}{3!}=\frac{64e^{-4}}{6}\approx 10.67e^{-4}.$$

Finally,

$$P(N(2)\leq 3)\approx e^{-4}(1+4+8+10.67)\approx e^{-4}\times 23.67.$$

Nonhomogeneous Poisson Process (NHPP)

A Nonhomogeneous Poisson Process is a stochastic process $N(t)$ characterized by the following properties:

1. **Time-Varying Intensity:** The rate of events $\lambda(t)$ is a non-negative function that varies with time.
2. **Poisson Distribution for Event Counts:** The number of events $N(t)$ that occur in the interval $(0, t]$ is distributed as:

$$N(t) \sim \text{Poisson}\left(\Lambda(0,t)\right) \quad \text{where } \Lambda(0,t) = \int_0^t \lambda(u)\, du$$

   This parameter represents the expected number of events from time 0 to t.

3. **Independent Increments:** Counts of events in non-overlapping intervals are independent. For non-overlapping intervals $(a, b]$ and $(c, d]$:

$$N(b) - N(a) \text{ and } N(d) - N(c) \text{ are independent.}$$

4. **Initial State:** The process starts at zero: $N(0) = 0$, indicating that no events have occurred at the start of observation.

Examples of Nonhomogeneous Poisson Processes

1. **Call Arrivals in a Call Center**
   - $\lambda(t) = 5 + 3\sin\left(\frac{\pi t}{12}\right)$
   - The call rate varies throughout the day, peaking during lunch and evening hours.
2. **Customer Arrivals at a Retail Store**
   - $\lambda(t) = 10 + 2t \quad (0 \leq t \leq 10)$
   - Customer arrivals increase as closing time approaches, reflecting higher foot traffic.
3. **Earthquake Occurrences**
   - $\lambda(t) = \frac{1}{1+e^{-0.1(t-50)}}$
   - The likelihood of earthquakes increases over time due to geological pressures.
4. **Webpage Visits**
   - $\lambda(t) = 20 + 10\cos\left(\frac{2\pi t}{24}\right)$
   - Visits vary throughout the day, peaking during business hours and dropping overnight.

Propositon: Sum of Two Processes

- Let's denote two non-homogeneous Poisson processes as $N_1(t)$ and $N_2(t)$ with respective intensity functions $\lambda_1(t)$ and $\lambda_2(t)$. Then the total process

$$N(t) = N_1(t) + N_2(t)$$

is also a nonhomogeneous Poisson process with intensity function

$$\lambda(t) = \lambda_1(t) + \lambda_2(t)$$

- This means that for any time interval $[a, b]$, the number of events in that interval follows a Poisson distribution with parameter equal to the integral of the total intensity over that interval:

$$N(b) - N(a) \sim \text{Poisson}\left(\int_a^b \lambda(t)\,dt\right)$$

Example of a Non-Homogeneous Poisson Process

Consider a restaurant that experiences varying customer arrival rates throughout the day. Let $N(t)$ be the number of customers arriving at the restaurant by time t.

- The arrival rate $\lambda(t)$ varies over time:

$$\lambda(t) = \begin{cases} 10 & \text{for } 0 \leq t < 12 \quad \text{(morning)} \\ 20 & \text{for } 12 \leq t < 14 \quad \text{(lunch)} \\ 30 & \text{for } 14 \leq t < 17 \quad \text{(afternoon)} \\ 25 & \text{for } 17 \leq t < 21 \quad \text{(dinner)} \\ 5 & \text{for } t \geq 21 \quad \text{(late night)} \end{cases}$$

Calculating the Probability of Arrivals

To find the number of arrivals from $t = 12$ to $t = 14$:

$$\mathbb{E}[N(14) - N(12)] = \int_{12}^{14} \lambda(t)\, dt = \int_{12}^{14} 20\, dt = 20 \times (14 - 12) = 40$$

Therefore:

$$N(14) - N(12) \sim \text{Poisson}(40)$$

Another Example of a Non-Homogeneous Poisson Process

Consider a retail store that experiences varying customer arrival rates throughout the day, modeled as a non-homogeneous Poisson process. Let $N(t)$ be the number of purchases made at the store by time t, where t is measured in hours from 9 AM:

$$\lambda(t) = \begin{cases} 2 + 2t & \text{for } 0 \leq t < 4 \quad \text{(9 AM to 1 PM)} \\ 10 + 1.5(t-4) & \text{for } 4 \leq t < 8 \quad \text{(1 PM to 5 PM)} \\ 16 + 0.5(t-8) & \text{for } 8 \leq t \leq 12 \quad \text{(5 PM to 9 PM)} \end{cases}$$

Expected Purchases: 9 AM to 1 PM

$$\mathbb{E}[N(4)] = \int_0^4 (2 + 2t)\, dt = \left[2t + t^2\right]_0^4 = 8 + 16 = 24$$

Expected Purchases: 1 PM to 5 PM

$$\mathbb{E}[N(8) - N(4)] = \int_4^8 (10 + 1.5(t - 4))\, dt = \int_4^8 (4 + 1.5t)\, dt$$

$$= \left[4t + 0.75t^2\right]_4^8 = (32 + 48) - (16 + 12) = 80 - 28 = 52$$

Expected Purchases: 5 PM to 9 PM

$$\mathbb{E}[N(12) - N(8)] = \int_8^{12} (16 + 0.5(t-8))\, dt = \int_8^{12} (12 + 0.5t)\, dt$$

$$= \left[12t + 0.25t^2\right]_8^{12} = (144 + 36) - (96 + 16) = 180 - 112 = 68$$

Compound Poisson Processes

Let us define a process

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

where $N(t)$ is a Poisson process with intensity λ and Y_i are *independent and identically distributed* (i.i.d.) random variables. Then the process $X(t)$ is called Compound Poisson Process in which $X(t)$ is defined as 0 if $N(t) = 0$.

Example:

i) Suppose customers leave a supermarket in accordance with a Poisson process. If Y_i the amount spent by the i^{th} customer, $i = 1, 2, \quad \ldots,$ are independent and identically distributed, then $[X(t), t \geq 0]$ is a compound Poisson process where $X(t)$ denotes the total amount of money spent by time t.

ii) Suppose customers leave a supermarket in accordance with a Poisson process. If Y_i be the amount spent by the i^{th} customer $i = 1, 2, \cdots$ are independent and identically distributed then $[X(t), t \geq 0]$ is a compound Poisson process, where $X(t)$ denotes the total amount of money spent by time t.

Mean of Compound Process:

$$E[X(t)] = E[E\{X(t) \mid N(t)\}] \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now,

$$E\left\{X(t) \middle/ N(t)\right\} = E\left[\sum_{i=1}^{N(t)} Y_i \mid N(t)\right]$$
$$= E\left[\sum_{i=1}^{N(t)} Y_i\right]$$
$$= \sum_{i=1}^{N(t)} E(Y_i)$$
$$= N(t)E(Y_i)$$

From equation (1) we get,

$$E[X(t)] = E[N(t)E(Y_i)]$$
$$= \lambda t \, E(Y_i)$$

Variance of Compound Process:

$$V[X(t)] = V[E\{X(t) \mid N(t)\}] + E[V\{X(t) \mid N(t)\}] \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$V\{X(t) \mid N(t)\} = V\left[\sum_{i=1}^{N(t)} Y_i \mid N(t)\right]$$

$$= V\left[\sum_{i=1}^{N(t)} Y_i\right]$$

$$= \sum_{i=1}^{N(t)} V(Y_i)$$

$$= N(t)V(Y_i)$$

From equation (2)

$$V[X(t)] = V[N(t)E(Y_i)] + E[N(t)V(Y_i)]$$

$$= \lambda t[E(Y_i)]^2 + \lambda t V(Y_i)$$

$$= \lambda t[\{E(Y_i)\}^2 + V(Y_i)]$$

$$= \lambda t E(Y_i^2)$$

Scenario

Consider a supermarket that receives customers throughout the day. Each customer makes a purchase, which varies in amount.

- Let $N(t)$ represent the number of customers who enter the supermarket by time t.
- Assume $N(t)$ follows a Poisson distribution with intensity λ:

$$\lambda = 5 \text{ customers per hour}$$

- Each customer's spending amount Y_i is modeled as an i.i.d. random variable.
- Assume the spending amounts are normally distributed: $Y_i \sim \mathcal{N}(50, 15^2)$.

The total spending amount $X(t)$ in the supermarket by all customers up to time t can be expressed as:

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

Expected Number of Customers

The expected number of customers in one hour is:

$$\mathbb{E}[N(t)] = \lambda t$$

For $t = 1$ hour:

$$\mathbb{E}[N(1)] = 5$$

The expected spending amount is:

$$\mathbb{E}[Y] = 50$$

The expected total spending in one hour is given by:

$$\mathbb{E}[X(1)] = \mathbb{E}[N(1)] \cdot \mathbb{E}[Y] = 5 \cdot 50 = 250$$

Calculating Variance

The variance of the spending can be computed as:

$$\mathrm{Var}(X(1)) = \mathbb{E}[N(1)] \cdot \mathrm{Var}(Y) + \mathrm{Var}(N(1)) \cdot (\mathbb{E}[Y])^2$$

- Assuming the variance of spending is:

$$\mathrm{Var}(Y) = 15^2 = 225$$

- For $N(1)$:

$$\mathrm{Var}(N(1)) = 5$$

Thus,

$$\mathrm{Var}(X(1)) = 5 \cdot 225 + 5 \cdot (50)^2 = 1125 + 12500 = 13625$$

Example 0.5. Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the number of people in each family is independent and takes on the values $1, 2, 3, 4, 5$ with respective probabilities $1/4, 1/4, 1/3, 1/12, 1/12$, then what is the expected value and variance of the number of individuals migrating to this area during a fixed six-week period?

Answer. Let $N(t)$ be the number of families that migrate to the area over t weeks, and Y_i the size of each family. Then the number of individuals migrating to this area over t weeks is $X(t) = \sum_{i=1}^{N(t)} Y_i$.

Since

$$\mathrm{E}(Y_1) = \frac{5}{2} \qquad \text{and} \quad \mathrm{E}(Y_1^2) = \frac{1}{4} + 1 + 3 + \frac{41}{12} = \frac{23}{3},$$

we have

$$\mathrm{E}(X(6)) = 2 \cdot 6 \cdot \frac{5}{2} = 30, \quad \mathrm{Var}(X(6)) = 2 \cdot 6 \cdot \frac{23}{3} = 92$$

CLT of a Compound Poisson Process

As $t \to \infty$, the distribution of

$$\frac{X(t) - \mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}} = \frac{X(t) - \lambda t \mu_Y}{\sqrt{\lambda t(\sigma_Y^2 + \mu_Y^2)}}$$

converges to a standard normal distribution $N(0, 1)$.