MICT-5101: Probability and Stochastic Process[1]

Dr. Md. Rezaul Karim
PhD(KULeuven & UHasselt), MS(Biostatistics), MS(Statistics)
Professor, Department of Statistics and Data Science
Jahangirnagar University (JU), Savar, Dhaka - 1342, Bangladesh

MS-2024

[1]These course slides should not be reproduced nor used by others (without permission).

Lecture Outline I

Chapter 5. Convergence and Limit Theorems

1 Chapter 5. Convergence and Limit Theorems

What is Convergence?

In probability theory, convergence describes how a sequence of random variables behaves as the sample size n increases. There are several types of convergence, each with specific definitions and properties. We will focus on the most common types:

- Types of Convergence:
  - Convergence in Distribution
  - Convergence in Probability
  - Almost Sure Convergence
  - Convergence in Mean
- Key Theorems:
  - Law of Large Numbers (LLN)
  - Central Limit Theorem (CLT)
  - Other limit theorems
- Asymptotic Behavior: Understanding how random variables behave as the number of observations increases.

Convergence in Distribution (Weak Convergence):

A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x),$$

for all x at which $F_X(x)$ is continuous.

Example: If X_n are random variables representing the sum of n i.i.d. Bernoulli trials with probability p, then $X_n \xrightarrow{d} \text{Binomial}(n, p)$ as $n \to \infty$.

- **Example 1: Central Limit Theorem (CLT)**
  If $X_1, X_2, \ldots$ are **i.i.d.** random variables with mean μ and variance σ^2, then the sample mean

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

  converges in distribution to a **normal distribution** as $n \to \infty$:

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1).$$

- **Example 2: Convergence to a Poisson Distribution**
  Let $X_n \sim \text{Poisson}(\lambda_n)$ where $\lambda_n \to \lambda$. Then,

$$X_n \xrightarrow{d} \text{Poisson}(\lambda)$$

  as $n \to \infty$, i.e., a Poisson random variable with parameter λ.

Exercise - 1: Let $X_2, X_3, X_4, \ldots$ be a sequence of random variables such that

$$F_{X_n}(x) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nx} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that X_n converges in distribution to Exponential(1).

Solution

Let $X \sim \text{Exponential}(1)$. For $x \leq 0$, we have

$$F_{X_n}(x) = F_X(x) = 0, \quad \text{for } n = 2, 3, 4, \ldots.$$

For $x \geq 0$, we have

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} \left(1 - \left(1 - \frac{1}{n}\right)^{nx}\right) = 1 - \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{nx}.$$

Using the known limit $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$, we get

$$= 1 - e^{-x} = F_X(x), \quad \text{for all } x \geq 0.$$

Thus, we conclude that $X_n \xrightarrow{d} X$.

Exercise - 2: Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables such that

$$X_n \sim \text{Binomial}(n, \lambda_n), \quad \text{for } n \in \mathbb{N}, n > \lambda,$$

where $\lambda > 0$ is a constant. Show that X_n converges in distribution to Poisson(λ).

Solution

We have

$$
\begin{aligned}
\lim_{n\to\infty} P_{X_n}(k) &= \lim_{n\to\infty} \binom{n}{k} (\lambda_n)^k (1-\lambda_n)^{n-k} \\
&= \lambda^k \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{1}{n}\right)^k (1-\lambda_n)^{n-k}. \\
&= \lambda^k \frac{1}{k!} \lim_{n\to\infty} \left[\frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} \right] \\
&\qquad \times \left((1-\lambda_n)^n\right) \left((1-\lambda_n)^{-k}\right).
\end{aligned}
$$

Note that for a fixed k, we have

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k} = 1,$$
$$\lim_{n\to\infty} (1-\lambda_n)^{-k} = 1,$$
$$\lim_{n\to\infty} (1-\lambda_n)^{n} = e^{-\lambda}.$$

Thus, we conclude

$$\lim_{n\to\infty} P_{X_n}(k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Convergence in Probability

A sequence of random variables X_n converges to X in probability (denoted $X_n \xrightarrow{P} X$) if for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0.$$

Example 1: Convergence of Sample Mean
Let $X_n = \overline{X}_n$ be the sample mean of n **i.i.d.** random variables with mean μ. By the **Weak Law of Large Numbers**,

$$\overline{X}_n \xrightarrow{P} \mu \text{ as } n \to \infty.$$

This means that for any $\epsilon > 0$,

$$P(|\overline{X}_n - \mu| \geq \epsilon) \to 0.$$

Example 2: Convergence of Binomial Distribution to Normal
Let $X_n \sim \text{Binomial}(n, p)$, and assume $n \to \infty$ while p remains constant. The **Law of Large Numbers** tells us that as $n \to \infty$,

$$\frac{X_n}{n} \xrightarrow{P} p.$$

Thus, X_n/n converges in probability to p.

Almost Sure Convergence (**Strong Convergence**)

A sequence of random variables X_n converges to X almost surely (denoted $X_n \xrightarrow{a.s.} X$) if

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

Example 1: Strong Law of Large Numbers

Let $X_1, X_2, \ldots$ be **i.i.d.** random variables with mean μ and variance σ^2. The **Strong Law of Large Numbers** states that

$$\overline{X}_n \xrightarrow{a.s.} \mu \text{ as } n \to \infty.$$

This means that the sample mean $\overline{X}_n$ converges to μ with probability **1**.

Example 2: Convergence of Random Walks

Consider a simple random walk where $X_n = S_n = \sum_{i=1}^{n} X_i$ and X_i are i.i.d. random variables taking values ± 1 with probability 1/2. The **Strong Law of Large Numbers** guarantees that

$$\frac{S_n}{n} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

This shows that the average step size of the random walk tends to 0 almost surely as the number of steps increases.

Convergence in Mean (or L^p Convergence)

A sequence of random variables X_n converges to X in L^p (denoted $X_n \xrightarrow{L^p} X$) if

$$\lim_{n\to\infty} E\left[|X_n - X|^p\right] = 0.$$

Example 1: Convergence in Mean Square
Let X_n be the sequence of random variables such that $E[X_n^2] \to 0$ as $n \to \infty$. Then, X_n converges to 0 in **mean square**, i.e.,

$$X_n \xrightarrow{L^2} 0.$$

This implies

$$E[X_n^2] \to 0 \quad \text{as} \quad n \to \infty.$$

This also implies **convergence in probability**.

Example 2: Convergence of the Sequence of Random Variables
Let $X_n = \frac{1}{n}$ with probability 1, so that $E[X_n^2] = \frac{1}{n^2}$. As $n \to \infty$,

$$X_n \xrightarrow{L^2} 0 \quad \text{since} \quad E[X_n^2] = \frac{1}{n^2} \to 0.$$

Relationships Between Types of Convergence

- **Almost Sure Convergence** ($X_n \xrightarrow{a.s.} X$) $\Rightarrow$ **Convergence in Probability** ($X_n \xrightarrow{P} X$).
- **Convergence in Probability** ($X_n \xrightarrow{P} X$) $\Rightarrow$ **Convergence in Distribution** ($X_n \xrightarrow{d} X$).
- **Convergence in Distribution** ($X_n \xrightarrow{d} X$) does not necessarily imply **Convergence in Probability** or **Almost Sure Convergence**.
- **Convergence in Mean Square** ($X_n \xrightarrow{L^2} X$) $\Rightarrow$ **Convergence in Probability** ($X_n \xrightarrow{P} X$).

Law of Large Numbers (LLN)

The Law of Large Numbers (LLN) states that sample averages converge to the expected value as the sample size increases.

- **Weak Law of Large Numbers (WLLN):** If $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with mean $\mu = \mathbb{E}[X_i]$, then the sample average $\bar{X}_n$ converges in probability to μ as $n \to \infty$:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu \quad \text{as} \quad n \to \infty.$$

Example: If $X_1, X_2, \ldots$ are i.i.d. random variables with $\mathbb{E}[X_i] = 5$, then as n increases, the sample mean $\bar{X}_n$ will converge in probability to 5.

- **Strong Law of Large Numbers (SLLN):** The Strong Law of Large Numbers states that with probability 1, the sample average $\bar{X}_n$ converges almost surely to the population mean μ as $n \to \infty$:

$$\bar{X}_n \xrightarrow{a.s.} \mu \quad \text{as} \quad n \to \infty.$$

Example: Consider rolling a fair die. The sample mean of the outcomes of n rolls, $\bar{X}_n$, will almost surely converge to the expected value of the die, which is 3.5, as $n \to \infty$.

Central Limit Theorem (CLT)

The Central Limit Theorem explains the behavior of the sample mean (or sum) of i.i.d. random variables.

- **Classical Central Limit Theorem:** If $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with mean μ and variance σ^2, then as $n \to \infty$,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

  This means that the distribution of the sample mean $\bar{X}_n$ approaches a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$. **Example:** If you flip a fair coin n times, each flip is a Bernoulli trial with $p = 0.5$. The average of these n flips will approximate a normal distribution with mean 0.5 and variance $\frac{0.25}{n}$ as n increases.

- **Implications of the CLT:**
  - The sample mean $\bar{X}_n$ follows a normal distribution for large n, regardless of the distribution of the individual X_i's.
  - This is why the normal distribution is often used as an approximation for sample statistics in hypothesis testing and confidence intervals, especially with large sample sizes.

Applications of Limit Theorems

Limit theorems are widely used in statistics and data analysis:

- **Estimating Population Parameters:** The LLN guarantees that sample averages get closer to the population mean as sample size increases.
- **Hypothesis Testing and Confidence Intervals:** The CLT justifies the use of normal approximations for statistical inference methods, even for non-normal data, when the sample size is sufficiently large.
- **Risk Management and Portfolio Theory:** The LLN and CLT are used to model the risk and returns of investments, ensuring that average returns converge to expected values as the sample size increases.

Chapter 6. Continuous-Time Markov Chains

Continuous Markov Chain (CTMC)

A Continuous-Time Markov Chain (CTMC) is a type of stochastic process where the system transitions between states over continuous time, and the future evolution of the system depends only on its present state, not on the past states. It satisfies the **Markov property**, and transitions occur at continuous time points governed by transition rates. A CTMC is a family of random variables $\{X(t), t \geq 0\}$ with the following properties:

1. The state space $S = \{s_1, s_2, \dots\}$ is a countable set of states.
2. The process satisfies the Markov property:

$$P(X(t+s) = j \mid X(t) = i) = P(X(s) = j \mid X(0) = i).$$

3. Transitions occur at any point in time (as opposed to discrete time steps). That is, continuous time here.

4. The process evolves according to an infinitesimal generator Q, where the rate matrix $Q = [Q_{ij}]$ determines the rates of transition between states, where:

$$Q_{ij} = \lim_{\Delta t \to 0} \frac{P(X(t + \Delta t) = j \mid X(t) = i)}{\Delta t}, \quad \text{for } i \neq j.$$

For $i \neq j$, Q_{ij} is the rate of transitioning from state i to state j, and for each state i, diagonal elements Q_{ii} are determined by:

$$Q_{ii} = -\sum_{j \neq i} Q_{ij}.$$

The Q is called a transition rate matrix.

5. The transition probabilities $P(t)$, where $P_{ij}(t)$ denotes the probability of transitioning from state i to state j at time t, satisfy the Kolmogorov Forward Equation (or master equation):

$$\frac{d}{dt}P(t) = P(t)Q$$

with the initial condition:

$$P(0) = I,$$

where I is the identity matrix.

This defines the behavior of the continuous-time Markov chain, including its transitions and probabilities over time.

Example of CTMC: Disease Progression Model

A Continuous-Time Markov Chain (CTMC) can model systems where the state changes over continuous time. In this example, we model the progression of a disease in a patient.

States:

- **State 0: Healthy** - The patient is in a healthy state.
- **State 1: Sick** - The patient has contracted a disease.
- **State 2: Recovered** - The patient has recovered from the disease.

The transitions between the states are governed by the following rates:

- From **Healthy (State 0)** to **Sick (State 1)** at rate λ,
- From **Sick (State 1)** to **Recovered (State 2)** at rate μ.

The corresponding transition rate matrix Q is:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\mu & \mu \\ 0 & 0 & 0 \end{bmatrix}$$

Transition Rate Matrix with Death State

Consider a system with the following states:

- **State 0: Healthy** - The patient is in a healthy state.
- **State 1: Sick** - The patient has contracted a disease.
- **State 2: Recovered** - The patient has recovered from the disease.
- **State 3: Death** - The patient has passed away due to the disease.

The transitions between the states are governed by the following rates:

- From **Healthy (State 0)** to **Sick (State 1)** at rate λ,
- From **Sick (State 1)** to **Recovered (State 2)** at rate μ,
- From **Sick (State 1)** to **Death (State 3)** at rate ν.

The corresponding transition rate matrix Q is:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -(\mu+\nu) & \mu & \nu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Time Spent in Each State

The time spent in each state follows an exponential distribution. For state i, the expected time spent in the state before transitioning is given by:

$$\text{Expected time in state } i = \frac{1}{|q_{ii}|}$$

For example:

- In State 0 (Healthy), the expected time before transitioning to Sick is $\frac{1}{\lambda}$.
- In State 1 (Sick), the expected time before transitioning to Recovered or More Severe is $\frac{1}{\mu+\nu}$.
- In State 2 (Recovered), the system stays in this state indefinitely (since $q_{22} = 0$).

Example: Simple Queueing System

Consider a simple queueing system where customers arrive and leave the queue. We will model this as a continuous-time Markov chain.

States:

- State 0: No customers in the queue.
- State 1: One customer in the queue.
- State 2: Two customers in the queue.

Transition Rates:

- From State 0 to State 1: Customer arrives at rate λ.
- From State 1 to State 0: A customer leaves at rate μ.
- From State 1 to State 2: Another customer arrives at rate λ.
- From State 2 to State 1: A customer leaves at rate μ.

Transition Rate Matrix

The transition rates can be represented in a generator matrix Q as follows:

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{pmatrix}$$

Where:

- $q_{01} = \lambda$: Rate of transition from State 0 to State 1 (customer arrival).
- $q_{10} = \mu$: Rate of transition from State 1 to State 0 (customer leaves).
- $q_{12} = \lambda$: Rate of transition from State 1 to State 2 (another customer arrives).
- $q_{21} = \mu$: Rate of transition from State 2 to State 1 (customer leaves).

Transition Rates Explanation

- The diagonal elements of the generator matrix are negative and represent the total rate of leaving a given state:

$$q_{00} = -\lambda, \quad q_{11} = -(\lambda + \mu), \quad q_{22} = -\lambda.$$

- The off-diagonal elements are positive and represent the rate of transitioning from one state to another:

$$q_{01} = \lambda, \quad q_{10} = \mu, \quad q_{12} = \lambda, \quad q_{21} = \mu.$$

These transition rates determine how the system evolves over time.

Exponential Waiting Times

Given the process is in state i, the waiting time until the next transition follows an exponential distribution with rate $-Q_{ii}$. Specifically, for a transition from state i to state j:

$$P(X(t) = j \mid X(0) = i) = \begin{cases} 1 - e^{-Q_{ii}t}, & \text{if } i = j, \\ e^{-Q_{ii}t} Q_{ij}, & \text{if } i \neq j. \end{cases}$$

This defines the transition probabilities based on exponential waiting times between transitions.

Applications of Continuous Markov Chains

Continuous Markov Chains are widely used in several fields:

- **Queuing Theory**: To model customer arrivals and service times.
- **Population Dynamics**: To describe biological processes like birth, death, or migration.
- **Reliability Engineering**: To model system failures and repairs.
- **Finance**: For modeling transitions in asset prices or credit ratings.

Birth-Death Process

A birth-death process is a type of stochastic process that models systems where entities (e.g., individuals, particles, etc.) can increase or decrease in number over time. The transitions between states happen randomly, and the system evolves based on the rates of birth and death events. In birth-death process, the system is described by a set of discrete states, typically $\{0, 1, 2, \dots\}$, where each state represents the number of individuals or entities in the system. For example:

- State 0: No individuals (empty system),
- State 1: One individual in the system,
- State 2: Two individuals in the system,
- $\dots$

Transitions between States

The system can transition between states based on two types of events:

- Birth: An entity enters the system, increasing the population by one.
- Death: An entity leaves the system, decreasing the population by one.

The rates at which these transitions occur are:

- Birth rate λ_i: The rate at which an individual is born when the system is in state i.
- Death rate μ_i: The rate at which an individual dies when the system is in state i.

Thus, the system will transition from state i to:

- State $i + 1$ (birth) with rate λ_i,
- State $i - 1$ (death) with rate μ_i.

Transition Rate Matrix

The transition rate matrix Q describes the rates at which the system transitions between states. For a simple birth-death process with 3 states (State 0, State 1, State 2), the transition rate matrix is:

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\ 0 & \mu_2 & -\mu_2 \end{bmatrix}$$

Where:

- $Q_{01} = \lambda_0$: The rate of transitioning from State 0 to State 1 (birth).
- $Q_{10} = \mu_1$: The rate of transitioning from State 1 to State 0 (death).
- $Q_{12} = \lambda_1$: The rate of transitioning from State 1 to State 2 (birth).
- $Q_{21} = \mu_2$: The rate of transitioning from State 2 to State 1 (death).
- $Q_{11} = -(\lambda_1 + \mu_1)$: The rate of leaving State 1 (the sum of birth and death rates).
- $Q_{22} = -\mu_2$: The rate of leaving State 2 (death rate).

Example: Population of a Town

Consider a town with a population of n people:

- Birth (Arrival): A new person moves into the town at a rate λ (birth rate).
- Death (Departure): A person leaves the town at a rate μ (death rate).

The system can transition from:

- State n to $n+1$ (birth),
- State n to $n-1$ (death).

The transitions between these states are governed by the birth and death rates λ and μ, respectively.

Transition Rate Matrix

The transition rate matrix Q for a Birth-Death process with states $0, 1, 2, \ldots$ is given by:

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \ldots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \ldots \\ 0 & \mu & -(\lambda+\mu) & \lambda & \ldots \\ 0 & 0 & \mu & -(\lambda+\mu) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Where:

- The diagonal elements represent the total rate of leaving a state (negative).
- The off-diagonal elements represent the rates of transitioning between states.

Transition Diagram

The transition diagram for a simple Birth-Death Process can be visualized as follows:

$$\text{State } 0 \xrightarrow{\lambda} \text{State } 1 \xrightarrow{\lambda} \text{State } 2 \xrightarrow{\lambda} \dots$$

$$\text{State } 1 \xleftarrow{\mu} \text{State } 0, \quad \text{State } 2 \xleftarrow{\mu} \text{State } 1, \dots$$

- λ is the rate of transition to a higher state (birth).
- μ is the rate of transition to a lower state (death).

Applications of Birth-Death Processes

Birth-death processes are used to model a wide range of systems:

- Population Dynamics: Modeling the growth and decline of populations due to birth and death events.
- Queuing Systems: Modeling customer arrival (birth) and departure (death) in server-client systems.
- Epidemiology: Modeling the spread of diseases where individuals can either recover or die (death) or become infected (birth).
- Chemical Reactions: Describing systems where molecules are created (birth) or decay (death).
- Reliability Theory: Modeling the failure (death) and repair (birth) of components in systems.

Equilibrium Distribution in Birth-Death Process

In a birth-death process, the stationary (equilibrium) distribution $\pi = [\pi_0, \pi_1, \pi_2, \dots]$ satisfies the equation:

$$\pi Q = 0$$

where Q is the transition rate matrix. The general form of the transition rate matrix for a birth-death process is:

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where:

- λ_i is the birth rate (transition from state i to $i+1$),
- μ_i is the death rate (transition from state i to $i-1$),
- $-(\lambda_i + \mu_i)$ is the total rate of leaving state i.

Equilibrium Distribution and Balance Equations

To find the equilibrium distribution, we need to solve the system of linear equations given by:

$$\pi Q = 0.$$

The condition $\pi Q = 0$ gives a set of balance equations for each state i.
Balance Equation for State i: For state i, the flow into state i equals the flow out of state i. Specifically:

- The flow into state i comes from state $i-1$ with rate λ_{i-1}.
- The flow out of state i goes to state $i+1$ with rate λ_i, and to state $i-1$ with rate μ_i.

This gives the balance equation:

$$\pi_{i-1}\lambda_{i-1} = \pi_i\mu_i + \pi_i\lambda_i.$$

Equilibrium Distribution of a Birth-Death Process

Consider a simple **birth-death process** with states $0, 1, 2, \ldots$.
The transition rates are:

- Birth rate: λ (rate from state i to state $i+1$),
- Death rate: μ (rate from state i to state $i-1$).

For state i, the balance equation is:

$$\pi_{i-1}\lambda = \pi_i(\mu + \lambda)$$

This gives the recursive relationship:

$$\pi_i = \frac{\lambda}{\mu + \lambda}\pi_{i-1}.$$

Solving recursively for $\pi_1, \pi_2, \ldots$:

$$\pi_1 = \frac{\lambda}{\mu + \lambda}\pi_0,$$

$$\pi_2 = \left(\frac{\lambda}{\mu + \lambda}\right)^2 \pi_0,$$

$$\pi_3 = \left(\frac{\lambda}{\mu + \lambda}\right)^3 \pi_0,$$

In general, we have:

$$\pi_i = \left(\frac{\lambda}{\mu + \lambda}\right)^i \pi_0.$$

Now, using the **normalization condition** $\sum_{i=0}^{\infty} \pi_i = 1$, we get:

$$\sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu + \lambda} \right)^i \pi_0 = 1.$$

This is a geometric series with first term 1 and common ratio $\frac{\lambda}{\mu+\lambda}$, so the sum is:

$$\frac{1}{1 - \frac{\lambda}{\mu+\lambda}} = \frac{\mu + \lambda}{\mu}.$$

The normalization condition becomes:

$$\pi_0 \cdot \frac{\mu + \lambda}{\mu} = 1,$$

so we solve for π_0:

$$\pi_0 = \frac{\mu}{\mu + \lambda}.$$

Finally, the equilibrium distribution is:

$$\pi_i = \left(\frac{\lambda}{\mu + \lambda}\right)^i \pi_0 = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i.$$

Example: Birth-Death Process with 3 States

Consider a birth-death process with the following states:

- **State 0**: No individuals.
- **State 1**: One individual.
- **State 2**: Two individuals.

The rates are:

- $\lambda_0 = 1$ (rate of birth from state 0 to state 1),
- $\mu_1 = 0.5$ (rate of death from state 1 to state 0),
- $\lambda_1 = 1.5$ (rate of birth from state 1 to state 2),
- $\mu_2 = 1$ (rate of death from state 2 to state 1).

The transition rate matrix Q for this process is:

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\ 0 & \mu_2 & -\mu_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -2 & 1.5 \\ 0 & 1 & -1 \end{bmatrix}$$

The equilibrium distribution $\pi = [\pi_0, \pi_1, \pi_2]$ must satisfy the condition $\pi Q = 0$.

This gives the following balance equations for each state:

- For state 0: $\pi_0 \lambda_0 = \pi_1 \mu_1$,
- For state 1: $\pi_1 \lambda_0 = \pi_0 \mu_1 + \pi_2 \mu_2$,
- For state 2: $\pi_2 \lambda_1 = \pi_1 \mu_2$.

Substituting the given values of λ_i and μ_i, we get:

- $\pi_0 \cdot 1 = \pi_1 \cdot 0.5 \quad \text{(from state 0)},$
- $\pi_1 \cdot 1 = \pi_0 \cdot 0.5 + \pi_2 \cdot 1 \quad \text{(from state 1)},$
- $\pi_2 \cdot 1.5 = \pi_1 \cdot 1 \quad \text{(from state 2)}.$

Solving these equations, we first find:

$$\pi_1 = 2\pi_0 \quad \text{(from the first equation).}$$

Substituting $\pi_1 = 2\pi_0$ into the third equation:

$$\pi_2 = \frac{2}{3}\pi_0.$$

Finally, using the normalization condition $\pi_0 + \pi_1 + \pi_2 = 1$, we get:

$$\pi_0 + 2\pi_0 + \frac{2}{3}\pi_0 = 1.$$

Solving this, we get:

$$\pi_0 = \frac{3}{7}, \quad \pi_1 = \frac{6}{7}, \quad \pi_2 = \frac{2}{7}.$$

Hence, the equilibrium distribution is:

$$\pi = \left[\frac{3}{7}, \frac{6}{7}, \frac{2}{7}\right].$$