CPPI Strategy Analysis for Insurance Project

Comprehensive Analysis of Portfolio Insurance Strategies

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Date: February 2025

Motivation

In today's increasingly volatile financial environment, robust risk management techniques are essential, particularly for institutions in the insurance sector. Portfolio insurance strategies, such as CPPI and OBPI, have emerged as vital tools for protecting capital while still allowing participation in market gains. This research explores advanced mathematical models and simulation techniques to evaluate and optimize these strategies under varying market conditions. By integrating continuous-time stochastic models with real-world data, the study aims to enhance understanding of how to balance capital preservation with growth, ultimately contributing to more resilient financial practices in the insurance industry.

Abstract

This report presents a comprehensive analysis of portfolio insurance strategies, focusing primarily on Constant Proportion Portfolio Insurance (CPPI) and Option-Based Portfolio Insurance (OBPI). We develop a rigorous mathematical framework for simulating asset price dynamics using both Geometric Brownian Motion (GBM) and Jump Diffusion models. The study examines the naïve CPPI strategy where the portfolio's exposure to the risky asset is dynamically adjusted based on the cushion $C_t = V_t - Ne^{-r(T-t)}$ and further extends it with a dynamic floor that adapts to market conditions via an adjustment term. A detailed parameter sensitivity analysis is conducted to investigate the effects of the guarantee N, the multiplier m, and, for OBPI, the strike price K on the final portfolio value. Simulation results under both continuous and jump diffusion settings, as well as applications on real-world data, highlight the trade-offs between capital preservation and market participation. The findings underscore the critical importance of careful parameter calibration in achieving robust portfolio performance in volatile markets.

Keywords: Portfolio Insurance, CPPI, OBPI, Geometric Brownian Motion, Jump Diffusion, Dynamic Floor, Parameter Sensitivity, Risk Management, Simulation

Contents

1 Introduction **1**

2 Simulation of Stochastic Processes **2**
2.1 Simulating Geometric Brownian Motion (GBM) . . . 2
2.2 Simulating a Jump Diffusion Process . . . 3

3 Preliminary Investment Strategies **4**
3.1 Risk-Free Investment Only . . . 4
3.2 Naïve CPPI Strategy and Parameter Analysis . . . 5
3.2.1 Sensitivity Analysis on the Guarantee N . . . 6
3.2.2 Limitations and Potential Enhancements . . . 7

4 Constant proportion portfolio insurance (CPPI) Strategy **8**
4.1 CPPI Strategy with GBM . . . 8
4.2 CPPI Strategy with Jump Diffusion . . . 9
4.3 Sensitivity Analysis on the Guarantee N and the Multiplier m . . . 9
4.4 Limitations and Potential Enhancements . . . 10

5 Option-Based Portfolio Insurance (OBPI) **11**
5.1 OBPI under Geometric Brownian Motion . . . 11
5.2 OBPI under Jump Diffusion . . . 12
5.3 Sensitivity Analysis on the Guarantee N and the Strike K . . . 13
5.4 Limitations and Potential Enhancements . . . 14

6 CPPI with a Dynamic Floor **15**
6.1 Simulation under GBM . . . 16
6.2 Simulation under Jump Diffusion . . . 16
6.3 Parameter Sensitivity Analysis: Multiplier m and Guarantee N . . . 17
6.4 Limitations and Potential Enhancements . . . 18

7 Application on Real-World Data **19**
7.1 Data Presentation . . . 19
7.2 Application of Naïve CPPI on Real Data . . . 19
7.3 Application of CPPI on Real Data . . . 20
7.4 Application of OBPI on Real Data . . . 21
7.5 Application of CPPI with Dynamic Floor on Real Data . . . 22

8 Conclusion **23**

1 Introduction

Innovative risk management strategies have become increasingly important in portfolio management, particularly within the insurance sector. One widely used method is Constant Proportion Portfolio Insurance (CPPI), designed to protect capital while allowing participation in market gains. The core idea of CPPI is to adjust exposure to risky assets based on the cushion, defined as:

$$C_t = V_t - P_t,$$

where V_t is the portfolio value and P_t is the protective floor. The exposure to the risky asset is then determined by:

$$E_t = m\, C_t,$$

with $m > 0$ being the leverage multiplier.

In our framework, a client begins at time $t = 0$ with an initial portfolio value V_0 and a target value N to be achieved at maturity T. The portfolio comprises a risk-free asset, modeled as a zero-coupon bond, and a risky asset. The risk-free asset's price is given by:

$$P_t = Ne^{-r(T-t)},$$

where $r \geq 0$ is the constant, continuously compounded interest rate.

We make the following simplifying assumptions:

- The risk-free rate r is constant.
- $V_0 \geq Ne^{-rT}$ (otherwise, all funds would be allocated to the risk-free asset).
- For simplicity, we assume $V_0 \geq N$.
- Transaction costs are ignored.

Under continuous rebalancing, the risk-free component will grow to exactly N by maturity, while any excess is exposed to market gains through the leveraged risky asset. To analyze our strategy, we consider two models for the evolution of the risky asset's price: Geometric Brownian Motion (GBM) and Jump Diffusion. The GBM model captures the continuous dynamics of asset prices, while the Jump Diffusion model accounts for sudden, abrupt changes. These two versions are chosen to provide a comprehensive analysis of the CPPI strategy under different market conditions.

2 Simulation of Stochastic Processes

In this section, we describe the theoretical framework used to simulate the evolution of asset prices under two different stochastic models : the Geometric Brownian Motion (GBM) and a Jump Diffusion process. These simulations form the backbone of our numerical analysis and allow us to generate sample paths that illustrate the dynamic behavior of the underlying asset.

2.1 Simulating Geometric Brownian Motion (GBM)

The Geometric Brownian Motion is a widely used model for asset prices in continuous time. It is defined by the stochastic differential equation (SDE):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t,$$

where:

- μ is the drift rate,
- σ is the volatility,
- B_t represents a standard Brownian motion.

The analytical solution to this SDE is given by:

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}\, z\right],$$

with z being a standard normal random variable. In our simulation, the time interval $[0, T]$ is discretized using a small time step dt (e.g., $dt = \frac{1}{252}$ for daily steps over one year, $T = 1$ year). At each step, the asset price is updated by generating an independent normally distributed random variable and applying the Euler-Maruyama discretization scheme.

For instance, with parameters $S_0 = 100$, $\mu = 0.08$, and $\sigma = 0.15$, multiple sample paths are generated to illustrate the evolution of the asset price over one year.

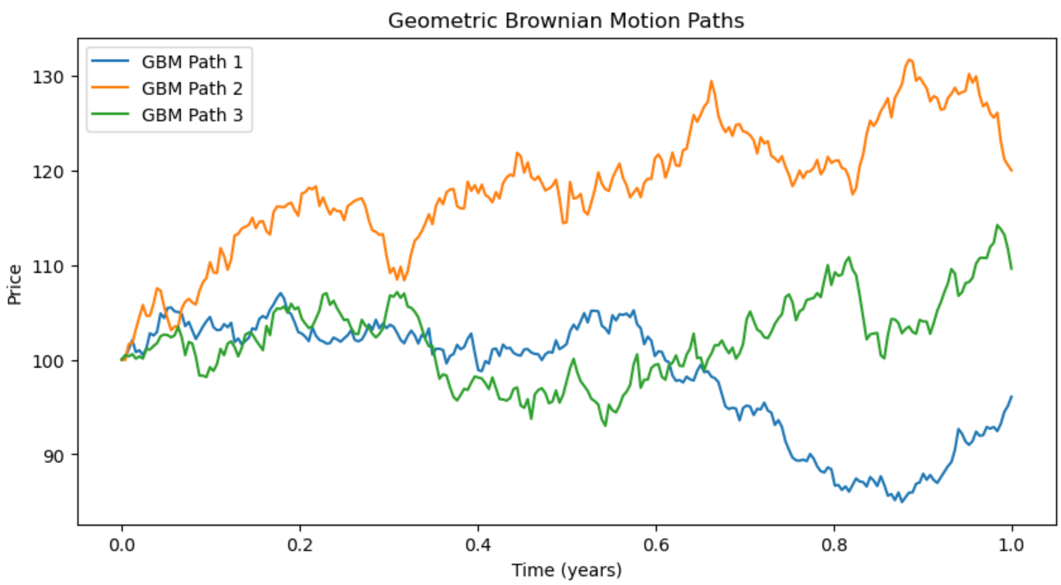

Figure 1: Geometric Brownian Motion Paths

2.2 Simulating a Jump Diffusion Process

The Jump Diffusion model extends the GBM by incorporating sudden, discontinuous changes (jumps) in the asset price. Its dynamics are described by the SDE:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t + (J-1)S_t\, dN_t,$$

where:

- dN_t is an increment of a Poisson process with intensity λ, representing the number of jumps in a small time interval dt,
- J is a random variable modeling the jump multiplier, typically assumed to be log-normally distributed (i.e., $\ln(J) \sim \mathcal{N}(\text{Jump}_\mu, \text{Jump}_{\sigma^2})$).

With the parameters $S_0 = 100$, $\mu = 0.08$, and $\sigma = 0.15$, the continuous part of the model follows a standard Geometric Brownian Motion, which simulates the natural drift and volatility of the asset's price over time. In addition, the jump component is governed by:

- $\lambda = 10$: This is the average jump intensity per year, meaning that on average, there are 10 jump events occurring each year.
- $\text{Jump}_\mu = -0.1$: This parameter represents the mean of the logarithm of the jump multiplier. A negative value indicates that the jumps are, on average, downward, thereby reducing the asset's price.
- $\text{Jump}_{\sigma^2} = 0.3$: This parameter controls the volatility or dispersion of the jump sizes. A higher value implies a wider range of possible jump magnitudes.

Thus, the jump diffusion process combines the smooth evolution of prices from the GBM with occasional abrupt changes, which can cause significant price shifts from one day to the next. This allows the model to capture both regular market dynamics and the impact of sudden market shocks.

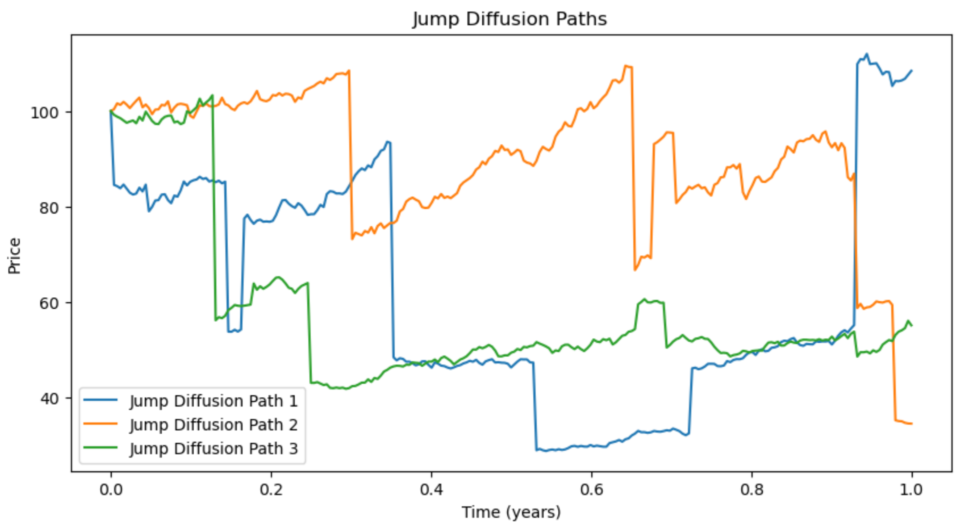

Figure 2: Jump Diffusion Paths

3 Preliminary Investment Strategies

Before introducing the CPPI technique, we review some basic strategies that serve as the foundation for more advanced portfolio management approaches.

3.1 Risk-Free Investment Only

The simplest and most naive strategy is to invest the entire initial capital exclusively in the risk-free asset. This approach guarantees a fixed return but does not allow any participation in the potentially higher gains of the risky asset.

Let:

- V_0 be the initial portfolio value provided by the client,
- $r \geq 0$ be the continuously compounded risk-free rate, and
- T be the investment horizon.

Under this strategy, the portfolio evolves deterministically. Its value at maturity is given by

$$V_T = V_0 e^{rT} > N.$$

If the client's objective is to secure a target amount N at time T, then one must choose the initial investment as

$$V_0 = N e^{-rT},$$

which ensures that at maturity, $V_T = N$. Although this strategy guarantees the target amount, it is conservative because it excludes any exposure to the risky asset, and hence, forfeits any additional market-driven gains.

Figure 3 illustrates the evolution of the portfolio over time under this strategy using the parameters: $r = 2\%$, $V_0 = 1000$, and $T = 1$ year.

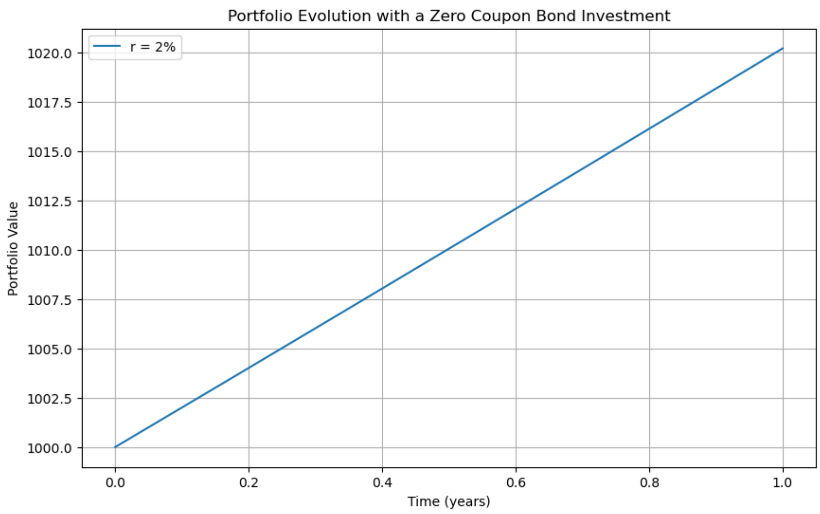

Figure 3: Evolution of the risk-free portfolio

3.2 Naïve CPPI Strategy and Parameter Analysis

In the naïve CPPI strategy (i.e., a CPPI without a multiplier), the portfolio is dynamically managed by allocating funds between a risk-free asset and a risky asset. At each time step, a protective floor typically ensuring a minimum guaranteed value is set. While this floor is generally referred to as the "protective floor," it is often determined as the price of a zero-coupon bond maturing at the investment horizon. A protective floor is set as

$$P_t = Ne^{-r(T-t)},$$

where N is the guarantee level and r is the constant risk-free rate. The available cushion is then

$$C_t = V_t - P_t,$$

and the exposure to the risky asset is determined by the multiplier m:

$$E_t = m\,C_t.$$

If the current portfolio value V_t falls below the floor P_t, the strategy allocates all funds to the risk-free asset.

We apply this naïve CPPI strategy in three different settings:

1. Application with GBM: Using parameters $S_0 = 1000$, $\mu = 0.08$, $\sigma = 0.15$, $T = 1$ year, and $dt = \frac{1}{252}$, $r = 0.02$, the naïve CPPI strategy is implemented on these simulated GBM paths. Figure 4 illustrates the portfolio evolution under this setting, comparing two types of investors : an aggressive investor with $N = 700$ and a conservative investor with $N = 900$, both starting with an initial capital of $V_0 = 1000$.

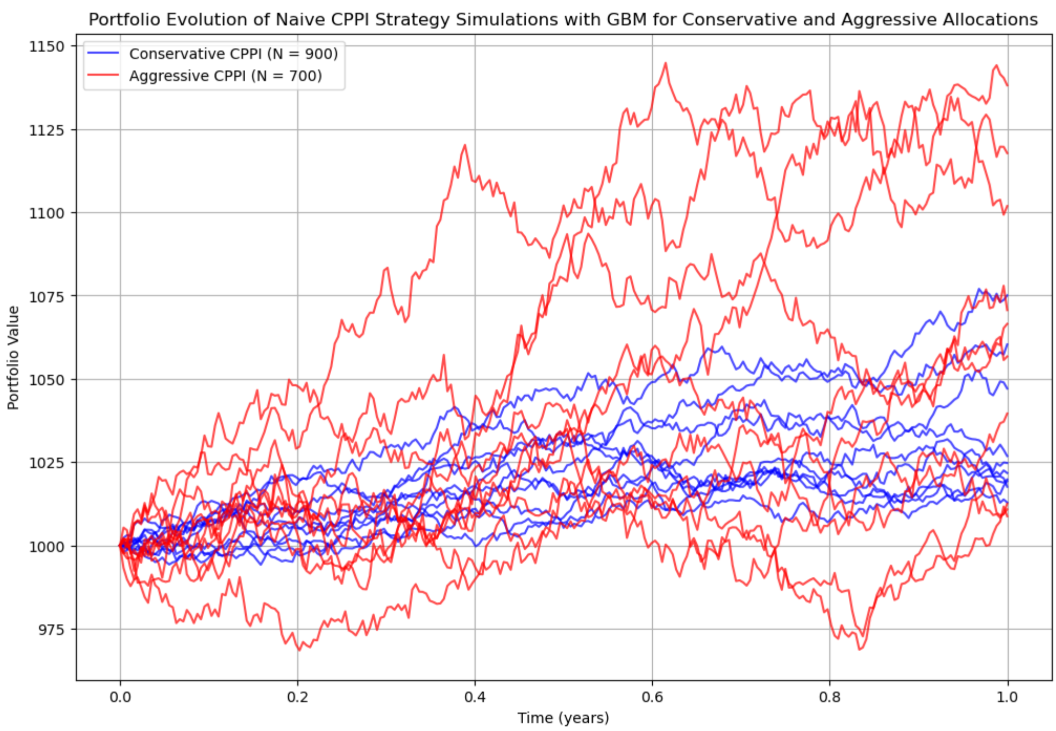

Figure 4: Naïve CPPI portfolio evolution with GBM

2. Application with Jump Diffusion: Using the same parameters as in Figure 4, along with $\lambda = 10$, $\text{Jump}_{\mu} = -0.1$, and $\text{Jump}_{\sigma^2} = 0.3$, the naïve CPPI strategy is applied to asset paths simulated under the jump diffusion model. Figure 5 illustrates the evolution of several portfolio paths under this model, comparing the same investor types than in the GBM application.

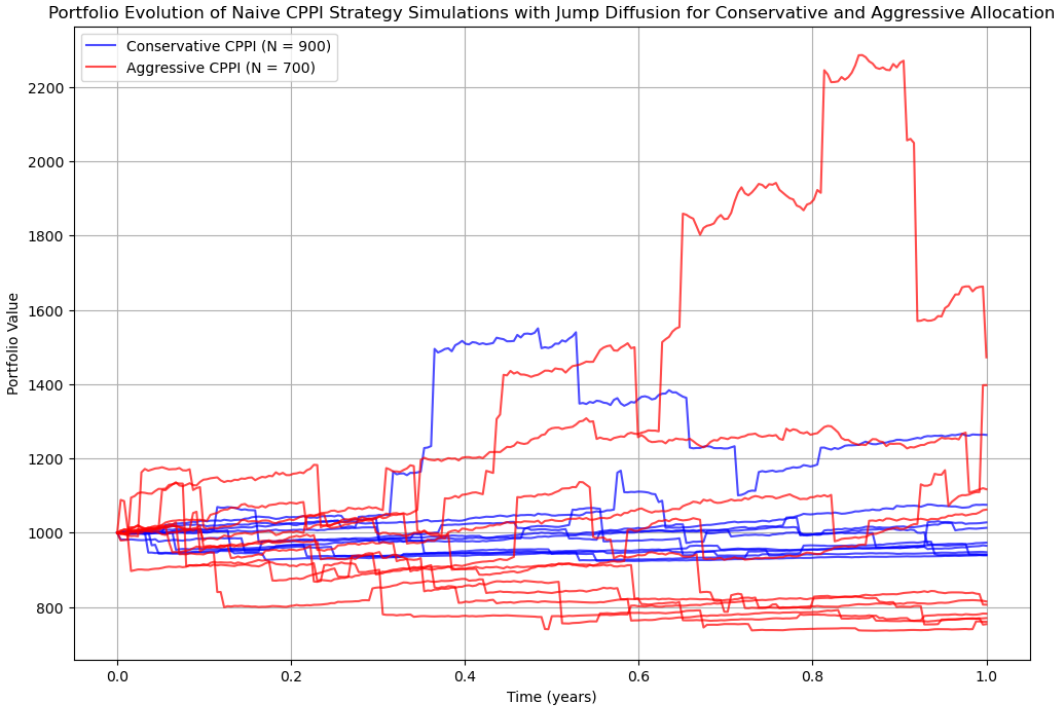

Figure 5: Naïve CPPI portfolio evolution with Jump Diffusion.

3.2.1 Sensitivity Analysis on the Guarantee N

The guarantee parameter N directly determines the protective floor $P_t = Ne^{-r(T-t)}$. A lower N results in a lower floor, which increases the cushion $C_t = V_t - P_t$ and hence, leads to a higher allocation to the risky asset via $E_t = m\,C_t$. Mathematically, when N approaches 0, the floor becomes negligible (i.e., $P_t \approx 0$), and the entire portfolio is effectively available for investment in the risky asset, leading to a higher average final portfolio value. Conversely, as N approaches 100, a larger portion of the portfolio is reserved to secure the guarantee, reducing the cushion and the risky allocation, and therefore yielding a lower final portfolio value. Furthermore, if $N > 100$, the algorithm will invest entirely in the risk-free asset because the portfolio value falls below the floor, ensuring the guarantee but limiting any potential upside.

Figure 6 illustrates the impact of varying N on the average final portfolio value. The results confirm that smaller N values tend to yield higher portfolio returns, while larger N values lead to more conservative outcomes with lower returns. Due to the limited number of simulations (100), the curve appears less smooth, reflecting the inherent variability in the simulated paths.

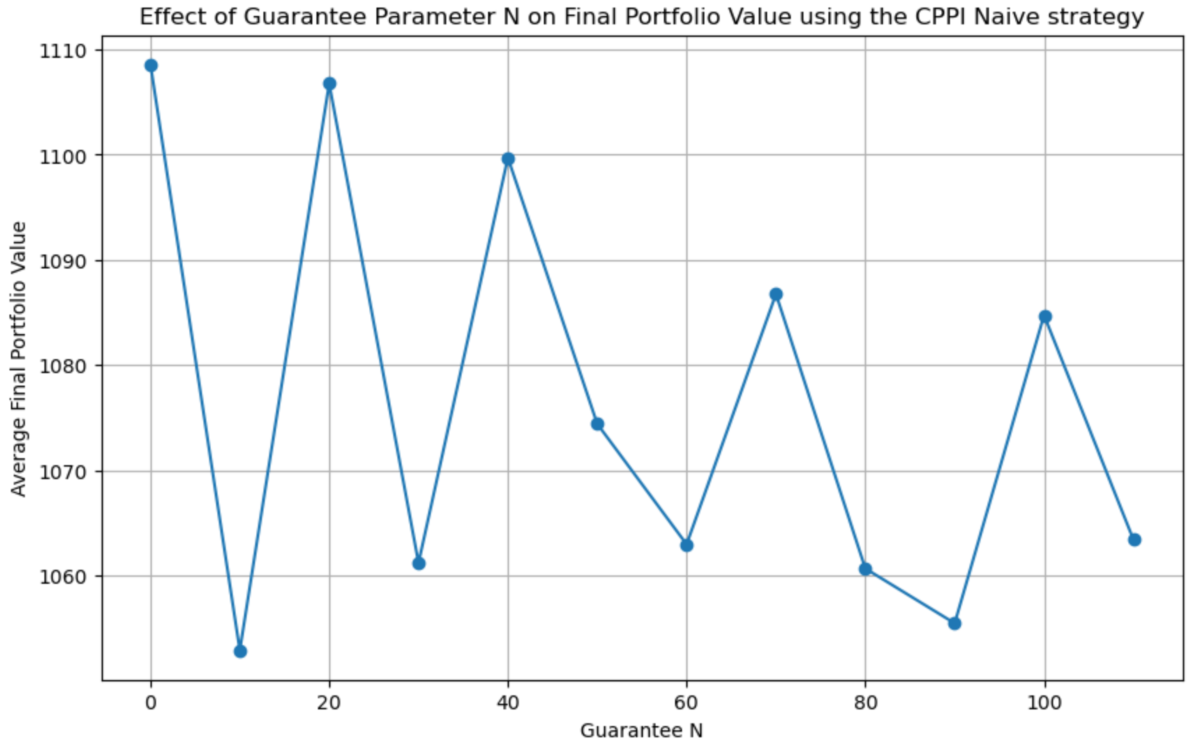

Figure 6: Evolution of the average portfolio value as a function of N under the naïve CPPI strategy

3.2.2 Limitations and Potential Enhancements

While the naïve CPPI strategy effectively splits investments between a risk-free asset and a risky asset ensuring that the portfolio value remains above a predefined guarantee it has certain limitations. This strategy relies solely on a static guarantee N to determine the protective floor and invests any excess (i.e. the cushion $C_t = V_t - P_t$) in the risky asset. Although this simple mechanism provides a baseline level of capital protection, it is inherently rigid.

In particular, without an adjustable leverage factor, the strategy does not dynamically adjust its exposure to the risky asset in response to changing market conditions. An adaptive mechanism that introduces a variable multiplier could enhance performance by increasing exposure during favorable conditions and reducing it when market risks escalate. Additionally, the guarantee parameter N plays a critical role : a very high N significantly raises the protective floor, reducing the available cushion and thereby limiting potential gains from the risky asset. Conversely, a very low N enlarges the cushion and increases exposure to the risky asset, but may also compromise capital protection. Moreover, if N is set above a critical threshold (e.g., $N > 100$ when $V_0 = 100$), the algorithm may allocate the entire portfolio to the risk-free asset, ensuring the guarantee but sacrificing any participation in market upswings.

Thus, while the naïve CPPI strategy provides a basic framework for balancing investments between risk-free and risky assets, its performance could be significantly improved by simply incorporating an adaptive multiplier to allow for more flexible risk exposure.

4 Constant proportion portfolio insurance (CPPI) Strategy

In the CPPI strategy, the portfolio is dynamically managed by allocating funds between a risk-free asset and a risky asset, with the addition of a multiplier to leverage the exposure to the risky asset. The strategy follows the same principles as the naïve CPPI, with the key difference being the incorporation of the multiplier, which amplifies the allocation to the risky asset based on the cushion. If $V_t < P_t$, the entire portfolio is shifted into the risk-free asset to preserve capital.

The following sections explore the application of this CPPI strategy under two different asset price models first using Geometric Brownian Motion (GBM) and then incorporating a Jump Diffusion process followed by a detailed sensitivity analysis on the parameters m and N.

4.1 CPPI Strategy with GBM

In our simulation, with parameters $S_0 = 1000$, $\mu = 0.08$, $\sigma = 0.15$, $T = 1$ year, and $dt = \frac{1}{252}$, $V_0 = 1000$, $r = 0.02$, $N = 900$, the CPPI strategy is applied to GBM-generated paths. For each simulation, the portfolio is rebalanced continuously according to the rule :

$$\text{If } V_t < P_t,\ V_{t+1} = V_t\, e^{r\,dt}; \quad \text{otherwise, } V_{t+1} = (V_t - m\,(V_t - P_t))e^{r\,dt} + m\,(V_t - P_t)\frac{S_{t+1}}{S_t}.$$

Figure 7 illustrates the evolution of sample portfolio paths under the CPPI strategy using GBM. Two cases are presented : one with $m = 1$, representing the naïve CPPI strategy and another with $m = 10$, highlighting the impact of leverage on portfolio performance. The comparison demonstrates how the introduction of a multiplier enhances the CPPI strategy by increasing exposure to the risky asset, potentially leading to higher returns compared to the naïve CPPI approach.

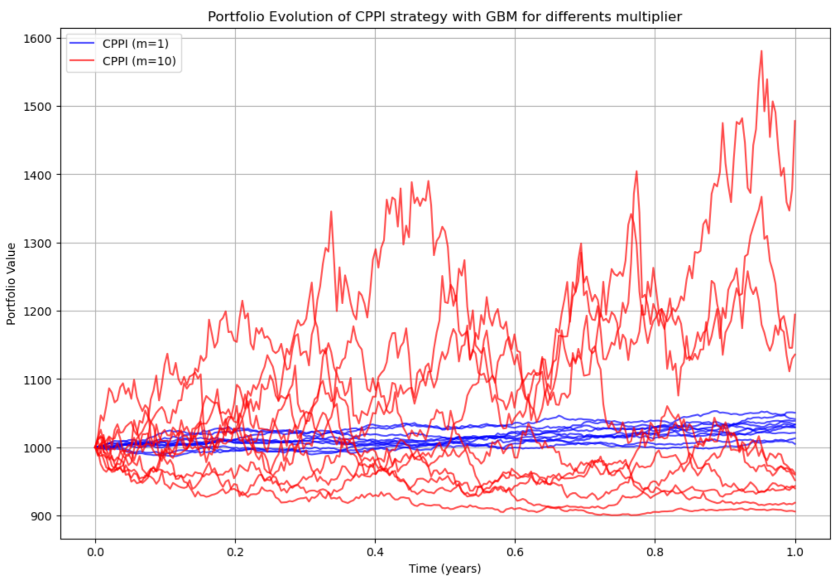

Figure 7: Portfolio evolution of the CPPI strategy with GBM for different multiplier values

4.2 CPPI Strategy with Jump Diffusion

Using the same parameters as in Figure 7, along with parameters $\lambda = 10$, $\text{Jump}_\mu = -0.1$, and $\text{Jump}_{\sigma^2} = 0.3$, the simulation reflects both continuous price evolution and abrupt changes. The CPPI rebalancing rule remains the same, but the risky asset returns now incorporate jump effects. Figure 8 shows sample portfolio evolution paths under the CPPI strategy using the Jump Diffusion model.

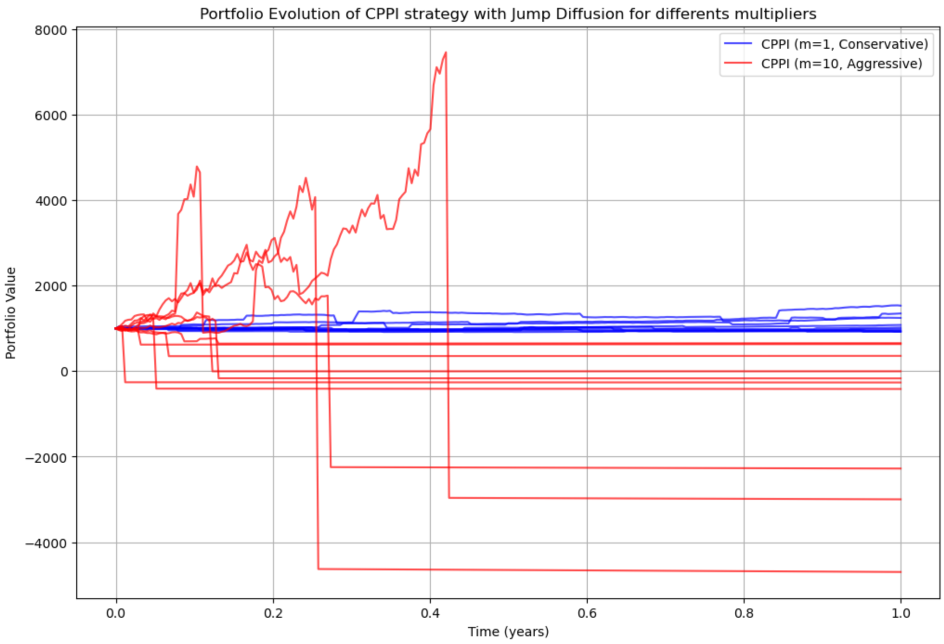

Figure 8: Portfolio evolution of the CPPI strategy with Jump Diffusion

4.3 Sensitivity Analysis on the Guarantee N and the Multiplier m

The performance of the CPPI strategy is highly sensitive to the choice of the guarantee N (defining the floor) and the multiplier m. A lower N increases the cushion, allowing for greater exposure to the risky asset and potentially higher portfolio values. Conversely, a higher N restricts the cushion, reducing the risky allocation and leading to a more conservative portfolio. In extreme cases, when N exceeds a critical threshold (e.g., $N > 100$ when $V_0 = 100$), the strategy fully shifts to the risk-free asset to ensure the guarantee.

The multiplier m directly scales the risky asset allocation. A higher m increases exposure and potential returns, but also amplifies losses. If m is small (e.g., close to 1), the portfolio follows a more conservative path with limited upside.

Figure 9 presents a heatmap of the average final portfolio value as a function of m and N. The results confirm that larger m and lower N lead to higher portfolio values, while smaller m and higher N result in more conservative outcomes. To generate this heatmap, we conducted 50 simulations per parameter pair. The limited number of simulations explains the slight irregularities in the heatmap; a higher number of simulations would yield a smoother representation. Nonetheless, the overall trends remain clear: increasing m enhances portfolio performance, while a high N constrains market participation.

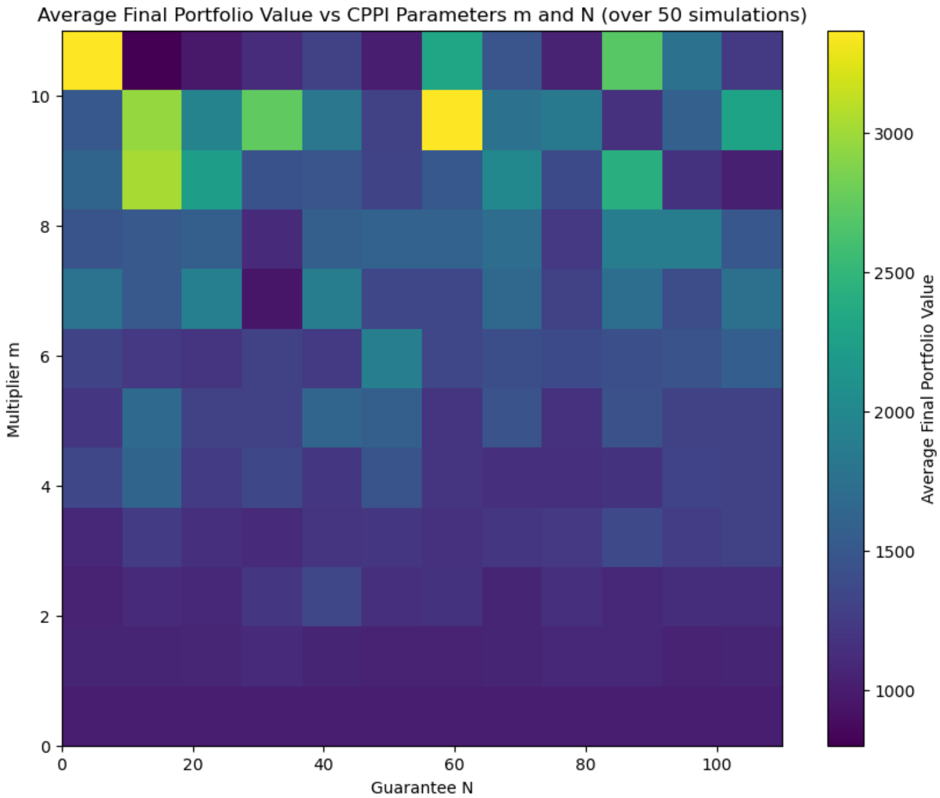

Figure 9: Average final portfolio value as a function of the multiplier m and guarantee N in the CPPI strategy

4.4 Limitations and Potential Enhancements

The basic CPPI strategy, with its fixed multiplier and static floor, ensures that if $V_t < P_t$ the portfolio is fully shifted to the risk-free asset. However, its static parameters determine the cushion $C_t = V_t - P_t$ solely based on the preset guarantee N and do not adapt to changing market conditions.

If N is too high, the cushion is small, limiting exposure to the risky asset and reducing potential gains. Conversely, if N is too low, the increased cushion boosts risky asset allocation, which may enhance returns but at the cost of capital protection. Additionally, a fixed multiplier m scales the risky allocation linearly, so a higher m amplifies both gains and losses, making the strategy highly sensitive to market fluctuations.

Enhanced strategies, such as CPPI with a dynamic floor and OBPI, address these issues. In CPPI with a dynamic floor, an additional adjustment term Δ_t allows the floor to adapt to market conditions, moderating the impact of N and maintaining a more optimal balance between risk-free and risky exposures. Similarly, OBPI uses an option-based approach to secure a minimum payoff while capturing upside potential.

In summary, while the basic CPPI strategy lays a solid foundation for portfolio insurance, its static parameters can restrict performance. Incorporating an adaptive multiplier or a dynamic floor can offer a more flexible and responsive risk management framework.

5 Option-Based Portfolio Insurance (OBPI)

The Option-Based Portfolio Insurance (OBPI) strategy provides capital protection by statically allocating funds between a risk-free asset and a European call option on a risky asset. At inception ($t = 0$), an amount

$$P_0 = Ne^{-rT}$$

is invested in a zero-coupon bond that guarantees a payoff of N at maturity T. The remaining funds, $V_0 - P_0$, are used to purchase a European call option with a strike price K. This leads to a terminal portfolio value of

$$V_T = N + \max(S_T - K, 0),$$

ensuring that the investor receives at least N at maturity, while capturing additional upside if the risky asset's price exceeds K.

Central to the OBPI strategy is the pricing of the European call option. To determine its cost, the Black-Scholes model is employed, providing a closed-form solution under the assumption that the underlying asset follows a Geometric Brownian Motion. The Black-Scholes call option price is given by:

$$C_0 = S_0\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

This pricing function is essential as it quantifies the cost and potential payoff of the call option component. By determining the fair value of the option based on parameters such as S_0, K, r, σ, and T, the OBPI strategy can effectively balance the allocation between the protective bond and the call option, ensuring capital preservation while also participating in market upside.

5.1 OBPI under Geometric Brownian Motion

Using parameters $S_0 = 1000$, $\mu = 0.08$, $\sigma = 0.15$, $T = 1$ year, and $dt = \frac{1}{252}$, $V_0 = 1000$, $r = 0.02$, $K = 100$, $N = 900$, the OBPI strategy is applied on simulated GBM paths. Figure 10 illustrates the evolution of an OBPI portfolio under GBM, showing how the risk-free component grows to N while the call option contributes additional value when the asset price exceeds the strike K.

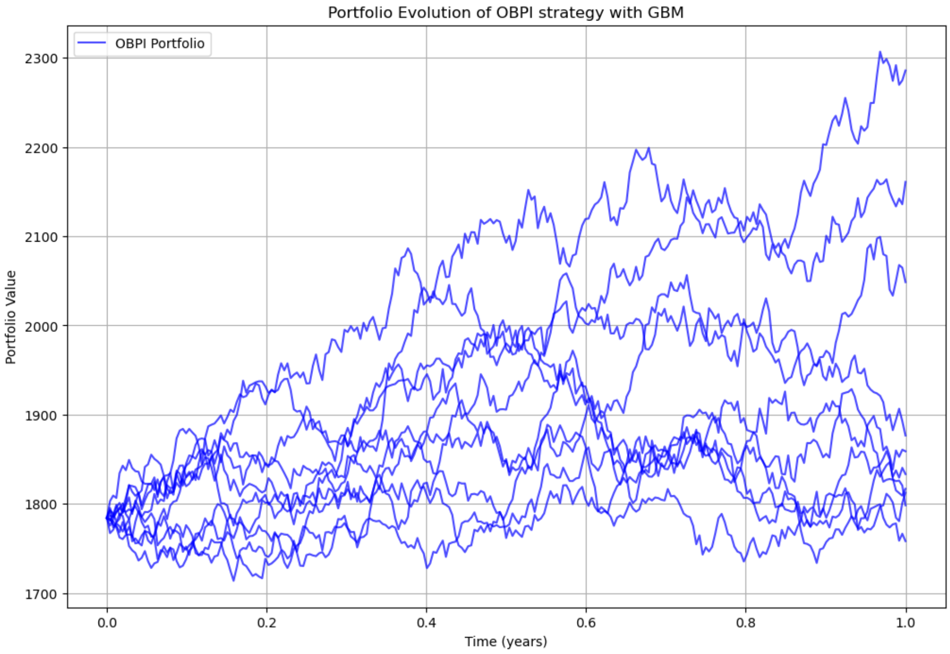

Figure 10: Portfolio evolution of OBPI strategy under GBM

5.2 OBPI under Jump Diffusion

Using the same parameters as in Figure 10, along with parameters $\lambda = 10$, $\text{Jump}_\mu = -0.1$, and $\text{Jump}_{\sigma^2} = 0.3$, the Jump Diffusion model captures both the continuous price evolution and sudden market shocks. The OBPI strategy is then applied on these simulated paths. Figure 11 shows sample portfolio evolutions under the OBPI strategy using Jump Diffusion.

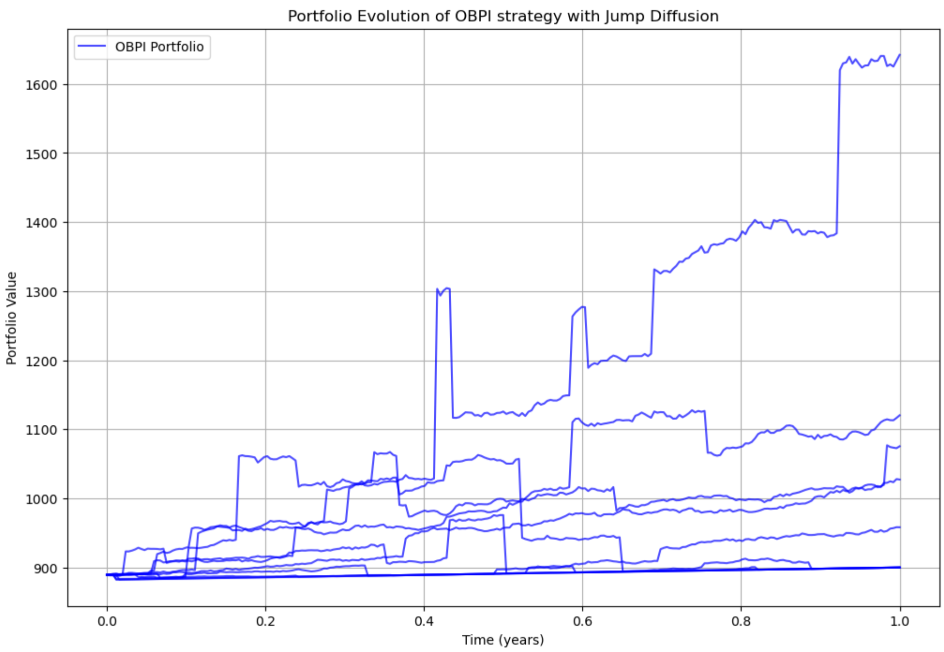

Figure 11: Portfolio evolution of OBPI strategy under Jump Diffusion

5.3 Sensitivity Analysis on the Guarantee N and the Strike K

The performance of the OBPI strategy is highly dependent on two key parameters: the guarantee N and the strike price K. The risk-free component is calculated as

$$P_0 = Ne^{-rT},$$

which guarantees that the investor receives at least N at maturity. The remaining funds are used to purchase a European call option with strike K, leading to a terminal portfolio value of

$$V_T = N + \max(S_T - K, 0).$$

A lower guarantee N reduces the amount invested in the risk-free asset, thereby increasing the capital available for the call option. Similarly, a lower strike price K raises the probability that the option will be exercised (i.e., that $S_T > K$). Consequently, when both N and K are low, the OBPI portfolio tends to achieve a higher average final value. Conversely, higher values of N and K constrain the funds available for the option and reduce the likelihood of a favorable option payoff, resulting in a lower final portfolio value.

Figure 12 presents a heatmap of the average final OBPI portfolio value as a function of N and K, confirming that lower values of both N and K yield higher returns. This heatmap is based on 10 simulations per parameter pair, which explains the observed irregularities; increasing the number of simulations would produce a smoother representation.

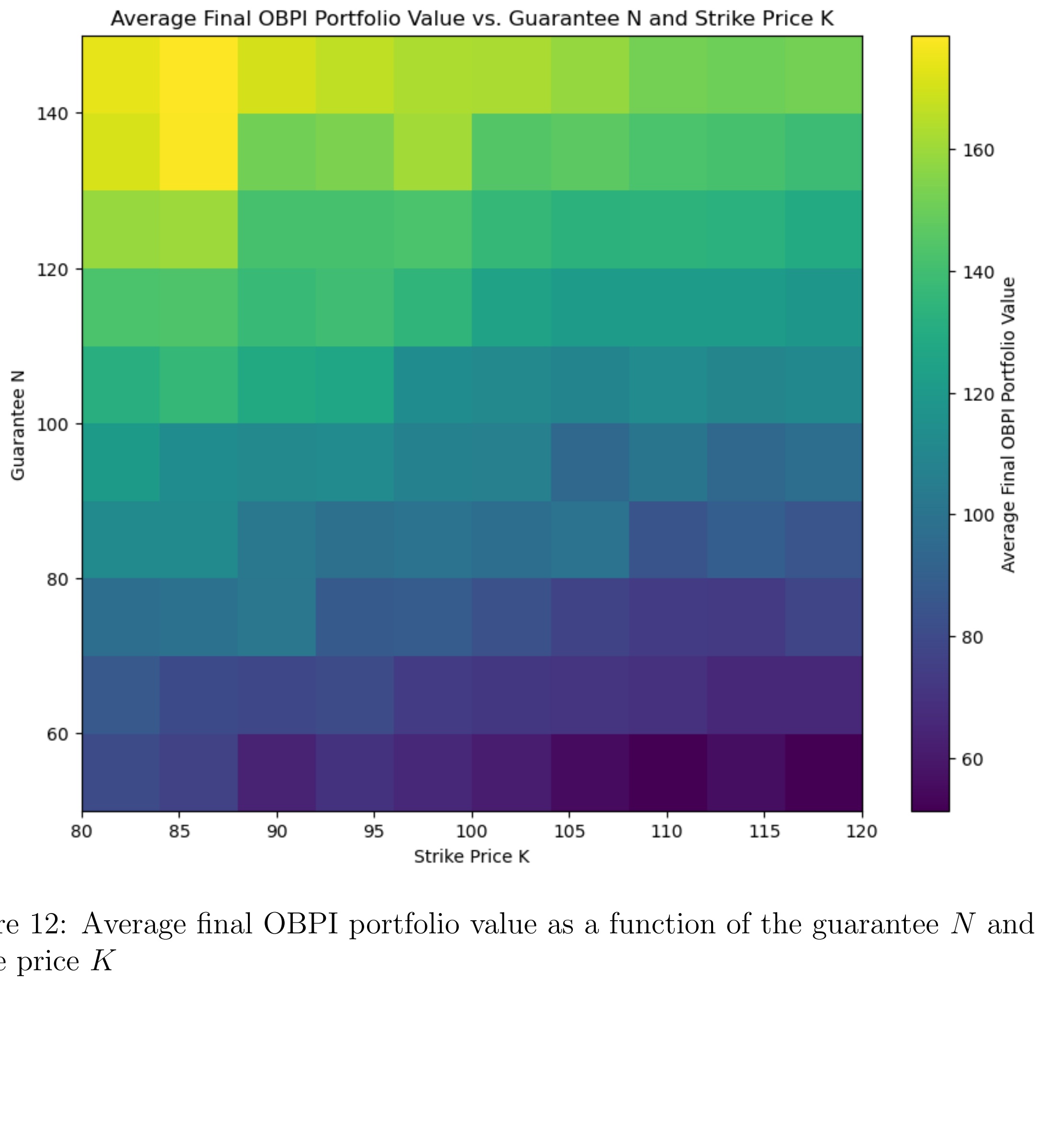

Figure 12: Average final OBPI portfolio value as a function of the guarantee N and the strike price K

5.4 Limitations and Potential Enhancements

While the OBPI strategy robustly ensures a minimum payoff at maturity by allocating funds between a risk-free asset and a call option, it has several limitations. The strategy employs a static allocation, allocating a fixed amount to the risk-free asset and the remaining funds to the call option—without adapting to changing market conditions. As a result, if market conditions improve, the strategy might underperform due to insufficient participation in the upside.

Additionally, the cost of the call option reduces the capital available for capturing gains, particularly when the option premium is high. The effectiveness of OBPI is also sensitive to the choice of the guarantee N and the strike K: a high N leads to a larger allocation to the risk-free asset (thus limiting upside potential), and a high K decreases the likelihood that the option will be in-the-money.

Potential enhancements include incorporating dynamic adjustments—such as an adaptive mechanism to modify the strike or adjust the allocation between the risk-free asset and the option in response to market volatility. Such improvements could allow OBPI to capture market gains more effectively while still preserving capital.

In summary, although OBPI provides solid capital protection, its static nature and sensitivity to parameter choices can limit performance. Enhancing the strategy with dynamic adjustments could result in a more flexible and responsive approach to managing risk and capturing market upside. This motivates the need for more adaptive portfolio insurance methods, such as CPPI with a dynamic floor. Unlike OBPI, where the allocation is determined once at inception, the CPPI dynamic floor strategy adjusts the protective floor over time in response to market conditions. This adaptability allows for a more responsive risk management approach, balancing downside protection with increased participation in market gains. In the following section, we explore the CPPI strategy with a dynamic floor, which introduces time-varying adjustments to further optimize portfolio performance.

6 CPPI with a Dynamic Floor

The CPPI strategy with a dynamic floor enhances the standard CPPI approach by allowing the protective floor to adjust over time in response to changing market conditions. Instead of using a static floor

$$P_t = Ne^{-r(T-t)},$$

we introduce an adjustment term Δ_t so that the dynamic floor becomes

$$F_t = Ne^{-r(T-t)} + \Delta_t.$$

The adjustment term Δ_t is designed to capture changes in the risky asset's price. For example, one simple formulation is:

$$\Delta_t = \delta \left(1 - \frac{S_t}{S_0}\right),$$

where δ is a fixed sensitivity parameter and S_0 is the initial price of the risky asset. This formulation causes the floor to increase when S_t declines—thereby reducing the cushion $C_t = V_t - F_t$ and limiting exposure to the risky asset—and to decrease when S_t rises, which enlarges the cushion and allows for greater risky exposure.

More adaptive approaches may also incorporate a measure of market volatility. For instance, one might use:

$$\Delta_t = \delta \left(1 - \frac{S_t}{S_0}\right) + \kappa\,\sigma_t,$$

where σ_t represents the rolling volatility and κ adjusts its impact. Such a mechanism would enable the floor to respond not only to price changes but also to changes in market uncertainty.

The available cushion is then defined as:

$$C_t = V_t - F_t,$$

and the exposure to the risky asset is given by:

$$E_t = m\,C_t,$$

with the remaining funds, $V_t - E_t$, invested in the risk-free asset. The portfolio is rebalanced continuously so that if V_t falls below F_t, the entire portfolio is shifted into the risk-free asset to preserve capital.

While this dynamic floor mechanism provides enhanced flexibility by adapting the protective floor to current market conditions, it introduces additional complexity. In practice, techniques such as smoothing Δ_t (for example, using an exponential moving average of past prices) may be necessary to prevent excessive fluctuations in the floor.

6.1 Simulation under GBM

Using parameters $S_0 = 1000$, $\mu = 0.08$, $\sigma = 0.15$, $T = 1$ year, and $dt = \frac{1}{252}$, $V_0 = 1000$, $r = 0.02$, $m = 3$, $N = 900$, $\Delta = 50$, the CPPI dynamic strategy is applied on simulated GBM paths. Figure 13 displays the evolution of the portfolio value V_t, the dynamic floor F_t, and the cushion C_t over time under GBM.

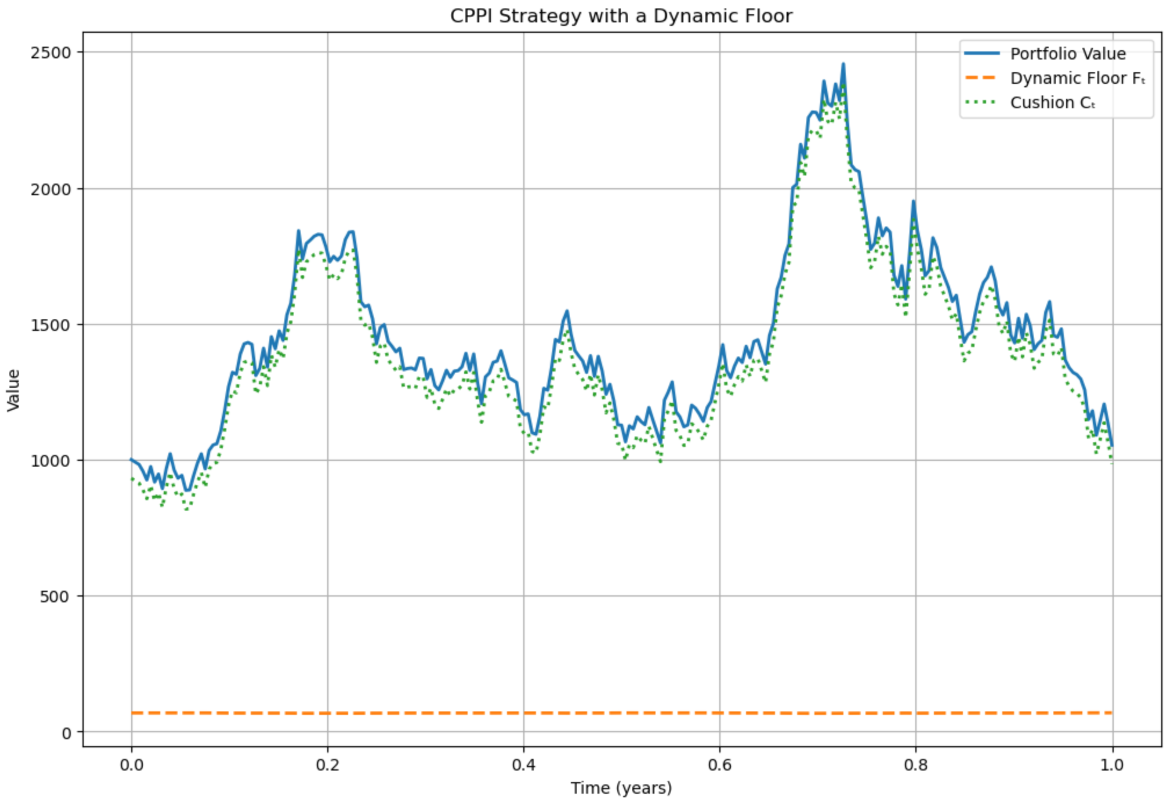

Figure 13: CPPI with a Dynamic Floor under GBM

6.2 Simulation under Jump Diffusion

Using the same parameters as in Figure 13, along with parameters $\lambda = 10$, $\text{Jump}_\mu = -0.1$, and $\text{Jump}_{\sigma^2} = 0.3$, the simulation incorporates both continuous evolution and sudden price jumps. The dynamic CPPI strategy is then applied to these paths. Figure 14 illustrates the portfolio evolution under jump diffusion.

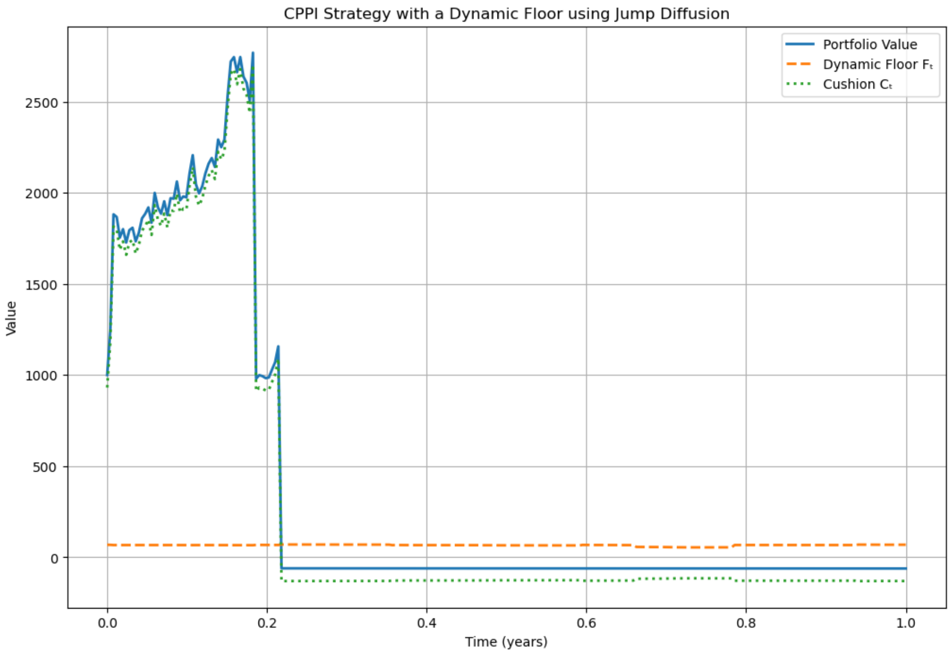

Figure 14: CPPI with a Dynamic Floor under Jump Diffusion

6.3 Parameter Sensitivity Analysis: Multiplier m and Guarantee N

In the dynamic CPPI strategy, the protective floor is given by

$$F_t = Ne^{-r(T-t)} + \Delta_t,$$

where the adjustment term Δ_t allows the floor to adapt to changes in market conditions. This adaptation moderates the direct influence of the guarantee N on the available cushion $C_t = V_t - F_t$. Consequently, while a lower N still results in a larger cushion and thus higher exposure to the risky asset, the dynamic adjustment diminishes the overall sensitivity to N compared to a static floor approach.

Conversely, the multiplier m directly scales the allocation to the risky asset. A higher m amplifies the investment in the risky asset, leading to a significantly higher average final portfolio value albeit with increased risk. In our simulations, we observe that increasing m consistently boosts portfolio performance, while variations in N yield only a modest effect, as the dynamic floor mechanism effectively dampens its impact.

Figure 15 illustrates a heatmap of the average final portfolio value as a function of m and N, based on 100 simulations per parameter pair. The results clearly show that larger m values result in higher portfolio values, whereas changes in N have a relatively minor impact with very high N values forcing an almost complete allocation to the risk-free asset. The limited number of simulations explains the slight irregularities in the heatmap; a higher number of simulations would likely produce a smoother representation.

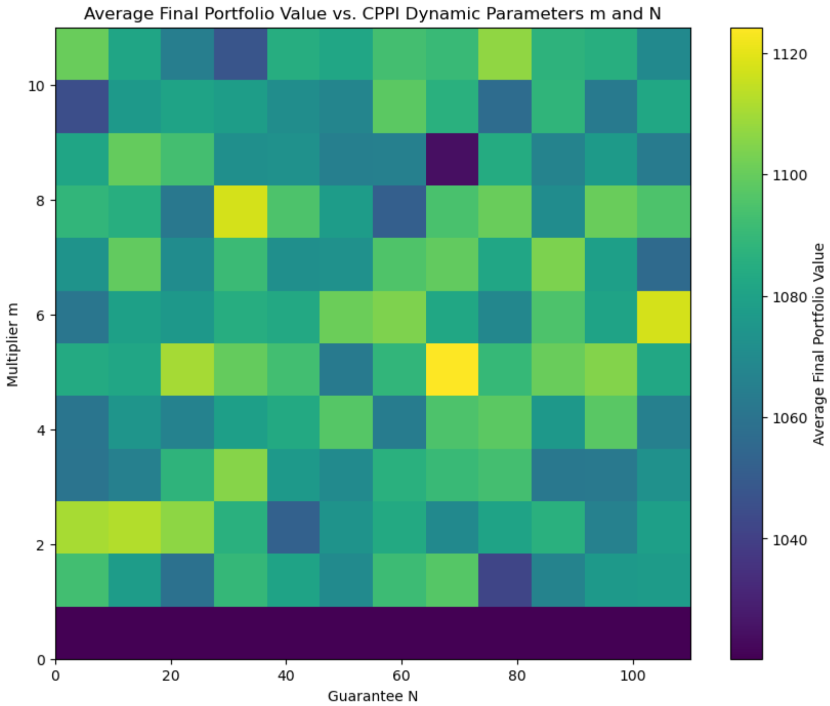

Figure 15: Average final portfolio value versus CPPI dynamic parameters m and N

6.4 Limitations and Potential Enhancements

Although the dynamic CPPI strategy improves upon the basic CPPI model by allowing the protective floor to adjust over time, it still has certain limitations. The dynamic adjustment term Δ_t is typically modeled in a simple manner (e.g., as a linear function of the relative change in the risky asset's price). This fixed functional form may not fully capture the complexities of market dynamics, and if the papameter controlling Δ_t is not optimally calibrated, the floor may overreact or underreact to market movements.

Furthermore, even with a dynamic floor, the strategy still relies on continuous rebalancing, which may not be practical in real-world settings due to transaction costs and discrete trading intervals. In periods of extreme volatility, the dynamic floor may not adjust rapidly enough to fully protect the portfolio, potentially exposing it to gap risk.

Potential enhancements include:

- Incorporating a more adaptive form for Δ_t, such as one that also considers market volatility or momentum.
- Implementing smoothing techniques to avoid over-adjustment of the floor.
- Adjusting the rebalancing frequency to better reflect realistic trading conditions while mitigating transaction costs.

In summary, while the dynamic CPPI strategy offers a more flexible and responsive approach compared to its static counterpart, further improvements such as an adaptive adjustment mechanism for the floor could enhance its effectiveness in managing risk and capturing market upside.

7 Application on Real-World Data

In this section, we apply the portfolio insurance strategies developed in this report to real-world stock data. Historical data for SPY, AAPL, MSFT, GOOGL, and AMZN are obtained from Yahoo Finance over a 5-year period. The following subsections detail the data presentation and the application of various strategies.

7.1 Data Presentation

We use historical daily data for SPY, AAPL, MSFT, GOOGL, and AMZN spanning 5 years. The data set includes the adjusted closing prices, which are used to capture the overall performance of each stock. Figure 16 shows the evolution of these stocks over the selected period.

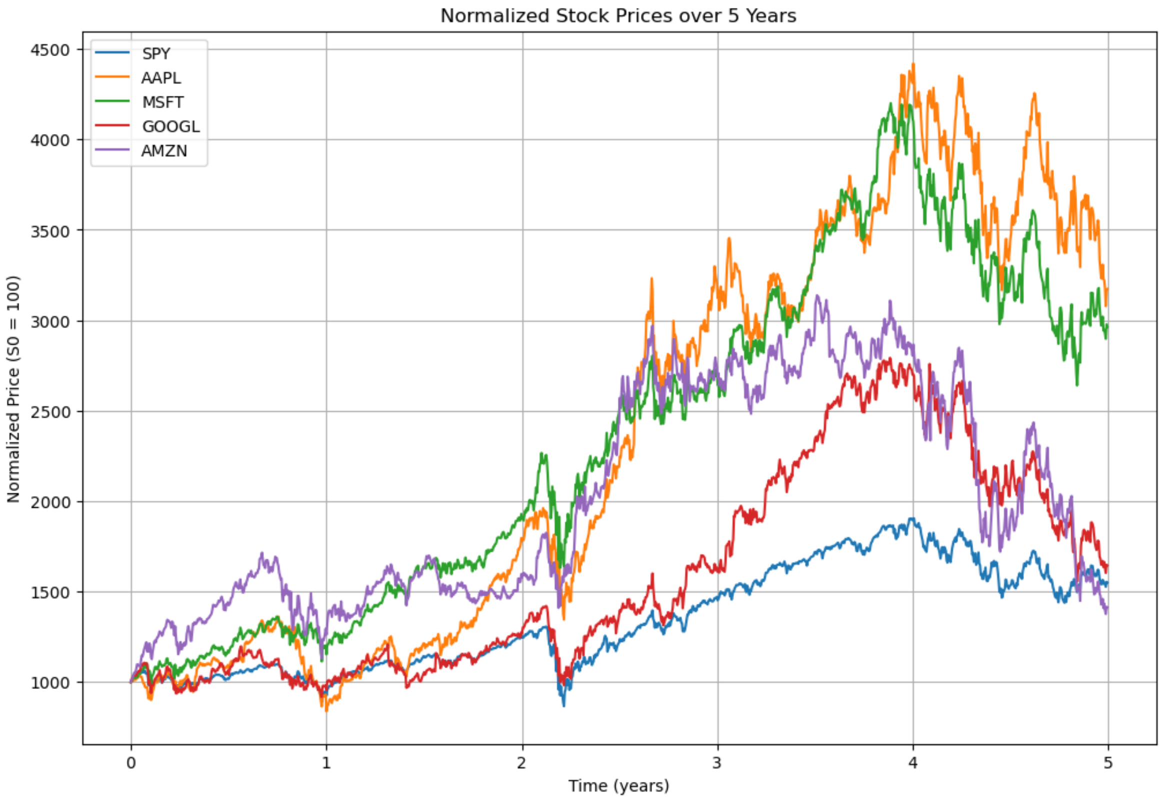

Figure 16: Historical evolution of SPY, AAPL, MSFT, GOOGL, and AMZN over 5 years.

7.2 Application of Naïve CPPI on Real Data

The naïve CPPI strategy is applied to the real-world data. In this approach, the portfolio is constructed by allocating a portion of the capital to a zero-coupon bond to secure a target N at maturity, while the remainder is invested in the risky asset. The portfolio evolution over the 5-year period is simulated and the average performance is displayed in Figure 17.

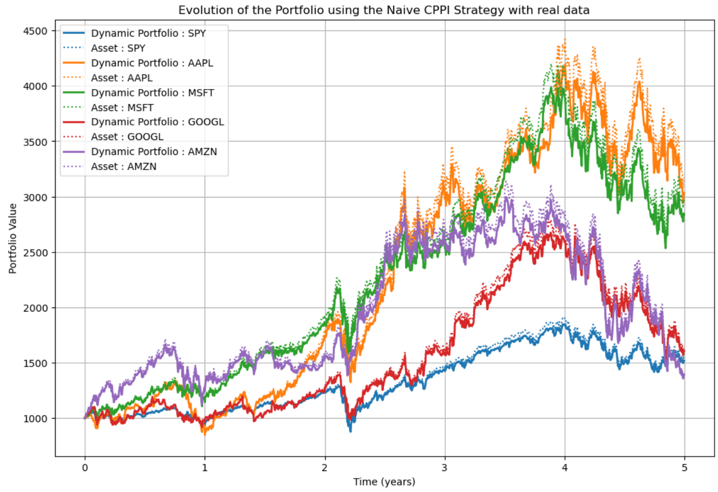

Figure 17: Evolution of the portfolio using the naïve CPPI strategy on real stock data.

The results showing that the naive CPPI strategy underperforms the stock price over most of the simulation are, in fact, expected. By design, a CPPI strategy allocates only the "cushion" ($V_t - \text{floor}(t)$) to the risky asset, while the remainder is invested in a risk-free component to guarantee a minimum value at maturity. Consequently, when the stock is rising, the portfolio never fully participates in the market's upside, since part of the capital is always protected in the risk-free asset. Moreover, with a multiplier of $m = 1$, there is no leverage to amplify the gains. In other words, the strategy's focus on capital protection naturally comes at the cost of reduced upside potential, causing the CPPI portfolio to lag behind a direct investment in the stock in bullish conditions.

7.3 Application of CPPI on Real Data

Next, the standard CPPI strategy is implemented on the same real-world data. The strategy dynamically adjusts the risky asset exposure based on the cushion $C_t = V_t - P_t$ (with $P_t = Ne^{-r(T-t)}$). Figure 18 illustrates the performance of the CPPI strategy applied over the 5-year period.

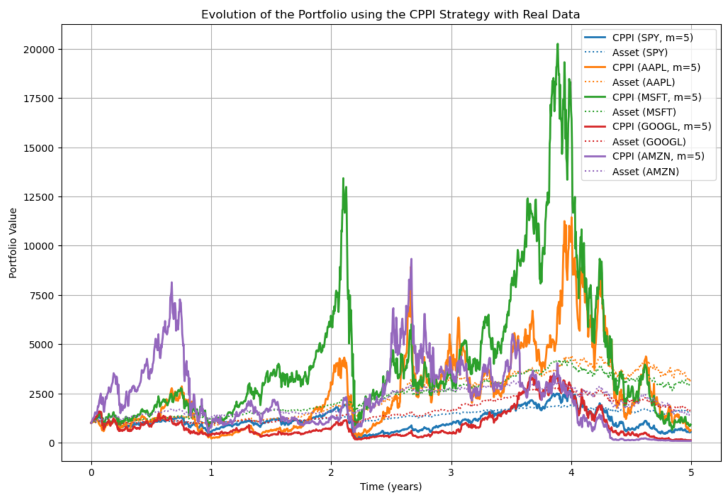

Figure 18: Evolution of the portfolio using the standard CPPI strategy on real stock data.

These results are fully consistent with the theoretical behavior of a CPPI strategy. When the portfolio value falls below the floor, the allocation shifts entirely into the risk-free asset, thereby protecting the guaranteed level. Conversely, when the portfolio value rises sufficiently above the floor, the excess (or cushion) is multiplied by m and allocated to the risky asset, which can lead to rapid growth. This dynamic rebalancing process explains the observed spikes and drops in the portfolio's value, as it continuously balances capital protection with the opportunity to capture upside gains.

7.4 Application of OBPI on Real Data

We also implement the Option-Based Portfolio Insurance (OBPI) strategy on the real data. In OBPI, the portfolio is constructed by investing in a zero-coupon bond and purchasing a European call option on the risky asset. The resulting portfolio performance over 5 years is shown in Figure 19.

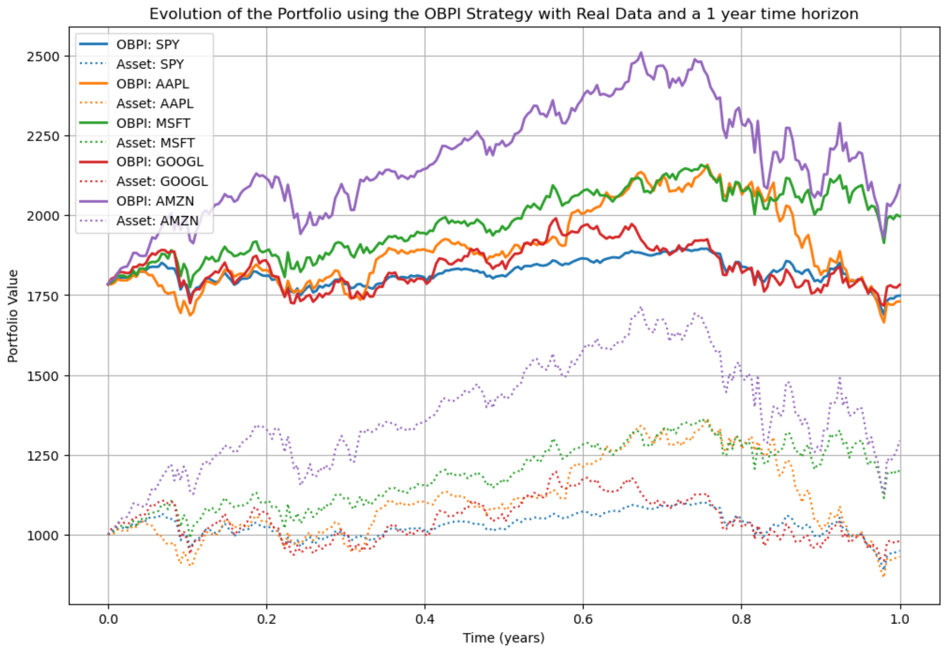

Figure 19: Evolution of the portfolio using the OBPI strategy on real stock data.

These results align with the theoretical expectations of an Option-Based Portfolio Insurance (OBPI) strategy. At inception, a portion of the initial capital is allocated to a zero-coupon bond that guarantees a payoff of N at maturity, while the remainder is used to purchase a European call option on the underlying asset. Over time, the portfolio value reflects both the deterministic growth of the bond component and the evolving price of the call option. When the underlying asset's price is sufficiently above the strike K, the call component becomes valuable, boosting the overall portfolio. Conversely, if the asset remains below or around K, the call's contribution is limited, and the portfolio's performance remains closer to the guaranteed level. This behavior ensures that the portfolio never falls below N at maturity, while still allowing for participation in upside market movements.

7.5 Application of CPPI with Dynamic Floor on Real Data

Finally, the CPPI strategy with a dynamic floor is applied to the real data. In this variant, the floor is adjusted over time according to a dynamic parameter, which allows for a more flexible protection mechanism. As a result, the allocation to the risky asset is more responsive to market conditions. Figure 20 displays the portfolio evolution under this strategy over the 5-year period.

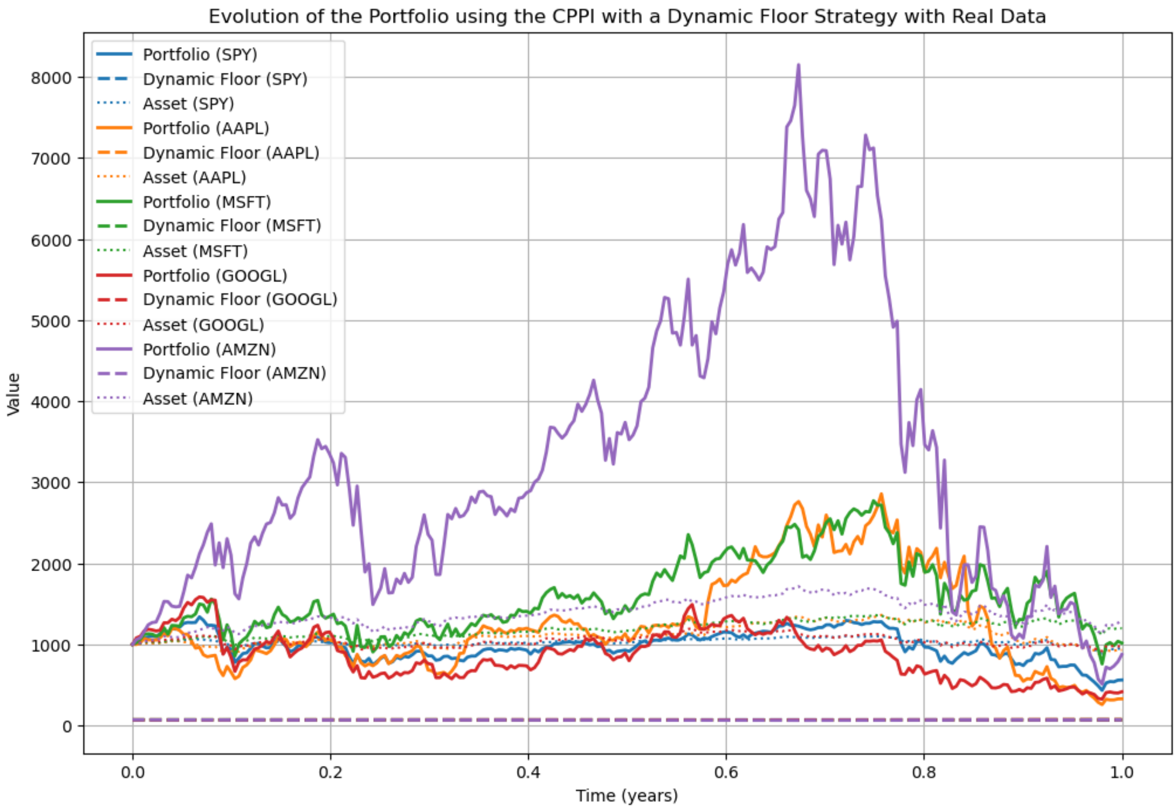

Figure 20: Evolution of the portfolio using the CPPI strategy with a dynamic floor on real stock data.

These outcomes align with the theoretical principles of a CPPI strategy that employs a dynamically adjusted floor. Specifically, the total floor at time t is given by a static component $F_{\text{static}}(t) = N\, e^{-r(T-t)}$ plus an adjustment term $\Delta_t = \text{delta\_param} \times \big(1 - S_t/S_0\big)$. When the portfolio value V_t falls below this floor, the strategy invests entirely in the risk-free asset, thereby protecting the capital. Conversely, when V_t exceeds the floor, the difference $(V_t - F_t)$ is multiplied by m and allocated to the risky asset. This mechanism explains the observed spikes (when the cushion is large and the portfolio heavily invests in the risky asset) and the subsequent drawdowns or plateaus (when a drop in the risky asset price forces the portfolio back toward the risk-free component). Overall, the graph reflects a balance between capturing upside potential and preserving the dynamic floor, in line with the theoretical behavior of a CPPI strategy with an adaptive protection level.

8 Conclusion

This study develops a numerical framework for analyzing portfolio insurance strategies, focusing on CPPI and OBPI. We model asset price dynamics using both Geometric Brownian Motion and Jump Diffusion to capture continuous market movements as well as abrupt shocks.

Our examination of the naive CPPI strategy reveals that dynamically adjusting the exposure to the risky asset via the cushion $C_t = V_t - Ne^{-r(T-t)}$ can be effective. Lower guarantees N and higher multipliers m increase the risky allocation and the expected portfolio returns, whereas very high N essentially shifts the portfolio toward a fully risk-free investment.

In parallel, the OBPI strategy, which allocates capital between a risk-free bond and a European call option, demonstrates that lowering the guarantee and the strike price K enhances the upside potential of the portfolio.

We also introduce a CPPI approach with a dynamic floor, incorporating an adjustment term Δ_t to adapt to market conditions. This dynamic feature reduces reliance on the specific choice of N and makes the multiplier m the primary driver of portfolio performance.

Overall, the results validated through both simulated and real-world data underscore the importance of careful parameter calibration. They further suggest that incorporating adaptive elements into portfolio insurance strategies can offer greater flexibility and resilience in the face of varying market environments.

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