數學、統計及保險學系
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AMS1001

Chapter 1 - Preliminaries

Numbers and Symbols

(A) N – Natural numbers

e.g., 1, 2, 3, …

(B) Z – Integers

e.g., …, -3, -2, -1, 0, 1, 2, 3, …

(C) Z^+ – Positive Integers

e.g., 1, 2, 3, …

(D) Z^- – Negative Integers

e.g., …, -3, -2, -1

Numbers and Symbols

(E) Q – Rational numbers p/q, p and q are integers

and q is not 0.

e.g., 4/5, 3/7, -6/5, …

An irrational number is any real number that cannot

be expressed as a ratio of integers.

e.g., π, $\sqrt{2}$

(F) R: Real numbers include all the rational
numbers and irrational numbers

Binomial Expansion

(A) Notation of Binomial Coefficient C_r^n

The number of combinations of selecting *r* objects out of *n* objects is given by

$$C_r^n = \frac{n!}{(n-r)!\,r!}$$

where $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$

Remark: There are other forms of C_r^n, such as $\binom{n}{r}$ and ${}_nC_r$.

eg. 1 $$C_3^8 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = 56$$

(B) Notation of Summation

The notation Σ (a Greek capital letter read as "sigma") denotes the sum of terms of a sequence.

Let x_i denote the ith term of a sequence

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

eg. $x_1 = 7, x_2 = 5, x_3 = 8$

Then $\sum_{i=1}^{3} x_i = \underline{\underline{20}}$

Summation

eg. 2 $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

eg. 3 $\sum_{r=5}^{99} \frac{2r}{r+1} = \frac{2(5)}{5+1} + \frac{2(6)}{6+1} + \cdots + \frac{2(99)}{99+1}$

eg. 4 $\sum_{i=5}^{20} (2i+1)^r = 11^r + 13^r + 15^r + \cdots + 41^r$

(C) Properties of Summation Notation

(i) $\sum_{r=1}^{n}(ax_r \pm by_r) = a\sum_{r=1}^{n} x_r \pm b\sum_{r=1}^{n} y_r$

(ii) $$\sum_{r=1}^{n}(ax_r \pm by_r)^2 = a^2\sum_{r=1}^{n}(x_r)^2 \pm 2ab\sum_{r=1}^{n} x_r y_r + b^2\sum_{r=1}^{n}(y_r)^2$$

$$\sum_{r=1}^{n}(ax_r + by_r) = (ax_1 + by_1) + (ax_2 + by_2) + \cdots + (ax_n + by_n)$$

$$= (ax_1 + ax_2 + \cdots + ax_n) + (by_1 + by_2 + \cdots + by_n)$$

$$= a\sum_{r=1}^{n} x_r + b\sum_{r=1}^{n} y_r$$

Since

$$(ax_r \pm by_r)^2 = a^2 x_r{}^2 \pm 2abx_r y_r + b^2 y_r{}^2$$

Then

$$\sum_{r=1}^{n}(ax_r \pm by_r)^2 = \sum_{r=1}^{n}\left(a^2 x_r{}^2 \pm 2abx_r y_r + b^2 y_r{}^2\right)$$

$$= a^2 \sum_{r=1}^{n} x_r{}^2 \pm 2ab \sum_{r=1}^{n} x_r y_r + b^2 \sum_{r=1}^{n} y_r{}^2$$

Binomial Theorem

For Positive Integral Index (positive integer *n*)

$$(x+y)^n = \underbrace{\overbrace{C_0^n x^n y^0}^{r=0}}_{1^{\text{st}}\text{ term}} + \underbrace{\overbrace{C_1^n x^{n-1} y^1}^{r=1}}_{2^{\text{nd}}\text{ term}} + C_2^n x^{n-2} y^2 + \cdots + \underbrace{\overbrace{C_{n-1}^n x^1 y^{n-1}}^{r=n-1}}_{n^{\text{th}}\text{ term}} + C_n^n x^0 y^n$$

$$= \sum_{r=0}^{n} C_r^n x^{n-r} y^r$$

The $(r+1)^{\text{th}}$ term = $C_r^n x^{n-r} y^r$ **is called the general term.**

Remarks:

(1) There are $(n + 1)$ terms for the expansion of $(x + y)^n$

(2) The Binomial expansion of $(x + y)^n$ can be expressed in ascending or descending powers of x.

eg. 5 Expand $(2 + 3x)^3$ in *ascending* powers of x up to x^3

Note that

$$(x+y)^n = C_0^n x^n y^0 + C_1^n x^{n-1} y^1 + C_2^n x^{n-2} y^2 + \cdots + C_{n-1}^n x^1 y^{n-1} + C_n^n x^0 y^n$$

$$(2+3x)^3 = C_0^3 (2)^3 (3x)^0 + C_1^3 (2)^2 (3x) + C_2^3 (2)^1 (3x)^2 + C_3^3 (2)^0 (3x)^3$$

$$= 8 + 36x + 54x^2 + 27x^3$$

eg. 6 Expand $(2-3x)^5(1+2x^2)^4$ in *ascending* powers of *x* up to degree 3

$$(2-3x)^5 = [2+(-3x)]^5$$

$$= C_0^5 2^5(-3x)^0 + C_1^5 2^4(-3x)^1 + C_2^5 2^3(-3x)^2$$

$$+ C_3^5 2^2(-3x)^3 + C_4^5 2^1(-3x)^4 + C_5^5 2^0(-3x)^5$$

$$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$$

$$(1+2x^2)^4 = 1 + 4(2x^2) + 6(2x^2)^2 + 4(2x^2)^3 + (2x^2)^4$$

$$= 1 + 8x^2 + 24x^4 + 32x^6 + 16x^8$$

Hence, $(2-3x)^5(1+2x^2)^4$

$$= (32 - 240x + 720x^2 - 1080x^3 + \dots)(1 + 8x^2 + \dots)$$

$$= 32(1+8x^2+\dots) - 240x(1+8x^2+\dots)$$
$$+ 720x^2(1+\dots) - 1080x^3(1+\dots)$$

$$= 32 + 256x^2 - 240x - 1920x^3 + 720x^2 - 1080x^3 + \dots$$

$$= 32 - 240x + 976x^2 - 3000x^3 + \dots$$

Special example: (OPTIONAL)

Find the coefficient of x^2 in the expansion of

$(1 - x + 2x^2)^5$

$$(1-x+2x^2)^5 = \left(1+[-x(1-2x)]\right)^5$$

$$= C_0^5 1^5 \left[-x(1-2x)\right]^0 + C_1^5 1^4 \left[-x(1-2x)\right]^1$$

$$+ C_2^5 1^3 \left[-x(1-2x)\right]^2 + \cdots + C_5^5 1^0 \left[-x(1-2x)\right]^5$$

$$= 1+5[-x(1-2x)]+10[-x(1-2x)]^2+\cdots$$

$$= 1-5x+10x^2+10x^2+\cdots$$

$$= 1-5x+20x^2+\cdots$$

Therefore, the coefficient of x^2 is 20

(OPTIONAL)

Question: Would you expand in the following way ?

$$\left(1 - x + 2x^2\right)^5 = \left[\left(1 - x\right) + 2x^2\right]^5$$

$$= \left(1 - x\right)^5 + 5(1 - x)^4 (2x^2) + \frac{5(4)}{2!}(1 - x)^3 (2x^2)^2 + \cdots$$

$$= \cdots$$

$$= \cdots$$

(C) Finding Specific Terms

Example : For $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Suppose we want to find the coefficient of x^2y term.

Can we find it without expanding the whole expression?

Yes!

Using the general term, i.e., the $(r + 1)^{\text{th}}$ term

eg. 7 Write down & simplify the general term for the expression $\left(x-\frac{1}{x}\right)^{10}$. Hence, find the middle term of it.

Recall that the general term for $(x+y)^n$ is $C_r^n x^{n-r} y^r$

$\therefore$ **The $(r+1)^{th}$ term for** $\left(x-\frac{1}{x}\right)^{10}$ is $C_r^{10} x^{10-r}\left(-\frac{1}{x}\right)^r = C_r^{10}(-1)^r x^{10-2r}$

There are totally 11 terms and the middle term is the 6^{th} term. Thus, we put $r = 5$

i.e. $C_5^{10} x^5\left(-\frac{1}{x}\right)^5 = \underline{\underline{-252}}$

eg. 8 Find the term independent of x (i.e. the constant term) and the coefficient of x^3 in the expansion of $\left(2x-\frac{3}{x^2}\right)^9$

Sol: The general term = $C_r^9(2x)^{9-r}\left(-\frac{3}{x^2}\right)^r = C_r^9(2x)^{9-r}(-3)^r(x^{-2})^r$

$$= C_r^9 2^{9-r}(-3)^r x^{9-3r}$$

For the constant term, we set $9 - 3r = 0$ to get $r = 3$

Thus, the constant term = $C_3^9 2^6(-3)^3 x^0 = -145152$

The coefficient of x^3 in the expansion of $\left(2x - \frac{3}{x^2}\right)^9$

Sol: The general term = $C_r^9 2^{9-r}(-3)^r x^{9-3r}$

For the term in x^3, we set $9 - 3r = 3$ to get $r = 2$

Thus, the coefficient of x^3 is

$$C_2^9 2^7 (-3)^2 = 41472$$

The Number e

Let us introduce and understand Euler's number e in the following way!

Imagine that a bank offers an annual interest rate of 100% and that you deposit $ 1 for one year.

For annual compounding:

$$A = 1 \times (1 + 100\%)^{1} = \$\,2$$

For quarterly compounding:

$$A = 1 \times (1 + \tfrac{1}{4})^{4} = \$\,2.4414$$

In general, if the compounding is done *n* times a year, the amount *A* is

$$A = \left(1 + \frac{1}{n}\right)^n \text{ dollars}$$

Compounding	n	$(1+\frac{1}{n})^n$ (to 6 d.p.)
Annually	1	2
Semiannually	2	2.25
Quarterly	4	2.441406
Daily	365	2.714567
Hourly	8760	2.718125
	10,000	2.718146
	100,000	2.718146
	1,000,000	2.718280
	10,000,000	2.718282

As n becomes larger and larger $(n \to \infty)$, the value $(1+\frac{1}{n})^n$

approaches to a limiting value which is equal to 2.718281828….

This irrational number is denoted by e.
(just like the number 3.14159… is denoted by π)

An irrational number e is defined as:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{r!} + \ldots$$

$$= 2.718281828 \ldots$$

Laws of Exponents

(1) $b^r \times b^s = b^{r+s}$

(2) $b^r \div b^s = b^{r-s}$

(3) $b^{-r} = \dfrac{1}{b^r}$

(4) $(b^r)^s = b^{rs}$

(5) $(ab)^r = a^r b^r$

(6) $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$

Definition of Exponential Function

A function in the form

$$f(x) = b^x$$

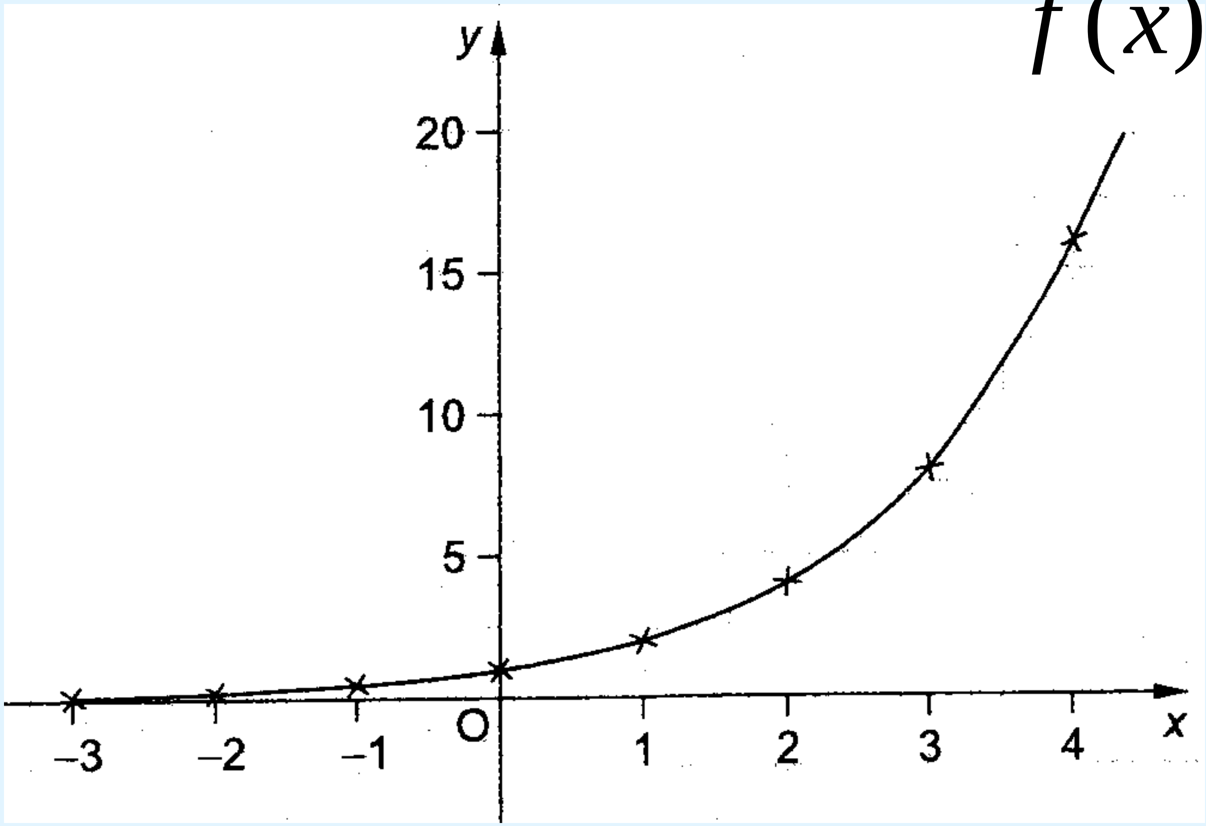
is called an exponential function, with *b* as the *base* and a positive constant other than 1.

Since the variable *x* is the *exponent* (power/index), so the function is called an exponential function.

Graphs of Exponential Functions and their Properties **(OPTIONAL)**

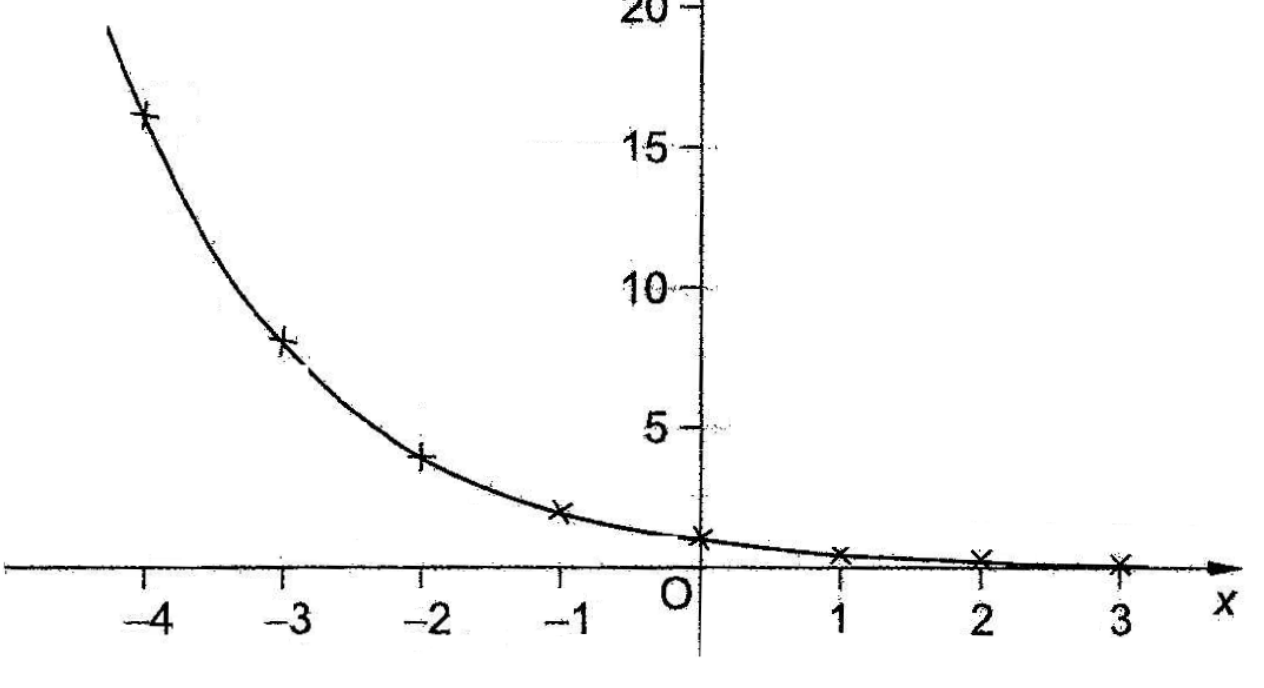
eg. 9

$$f(x) = 2^x$$

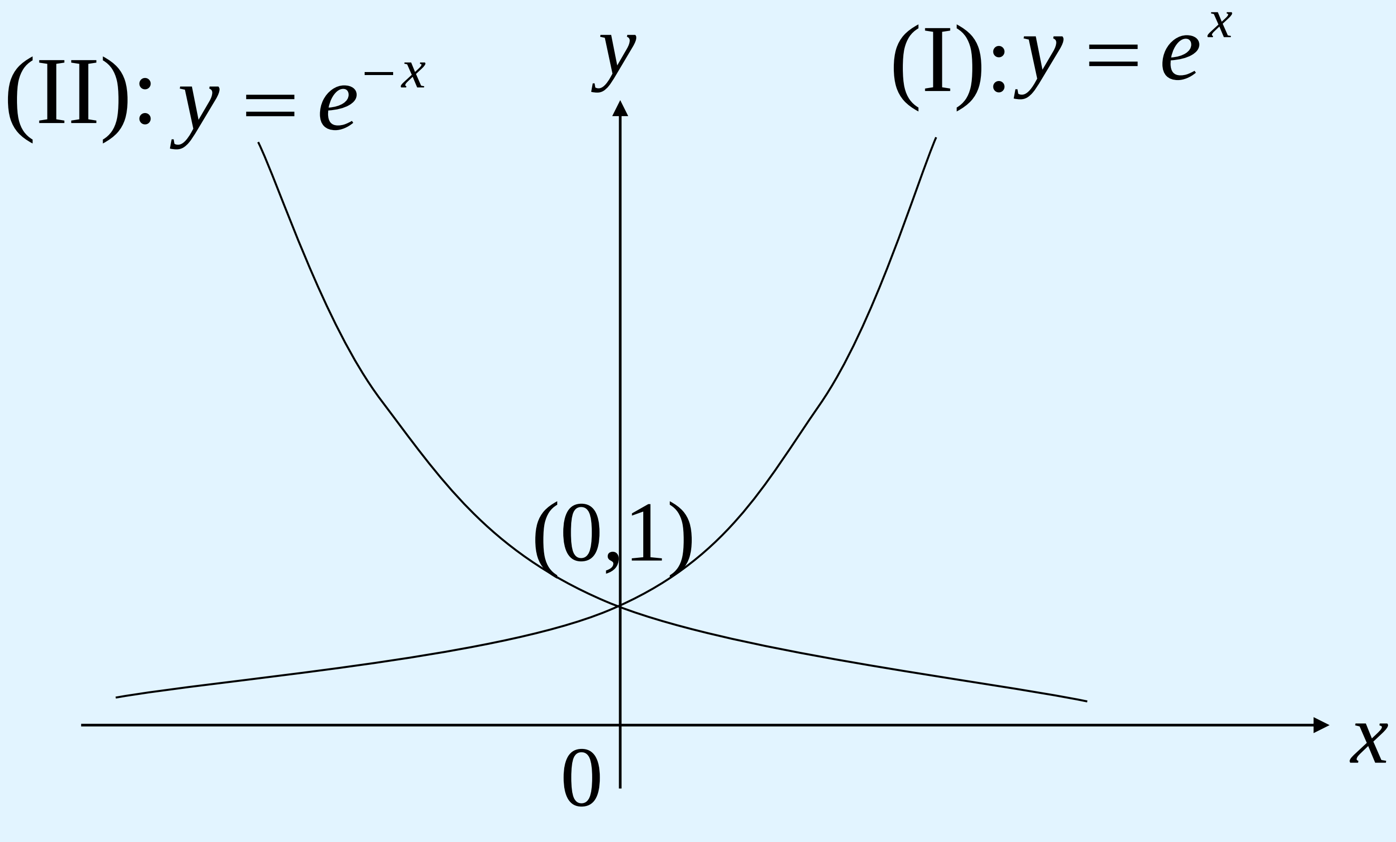


eg. 9 (OPTIONAL)

$$f(x) = (\frac{1}{2})^{x} = 2^{-x}$$



If we change the base 2 to e, then we can get similar results and the function is called the natural exponential function. **(OPTIONAL)**



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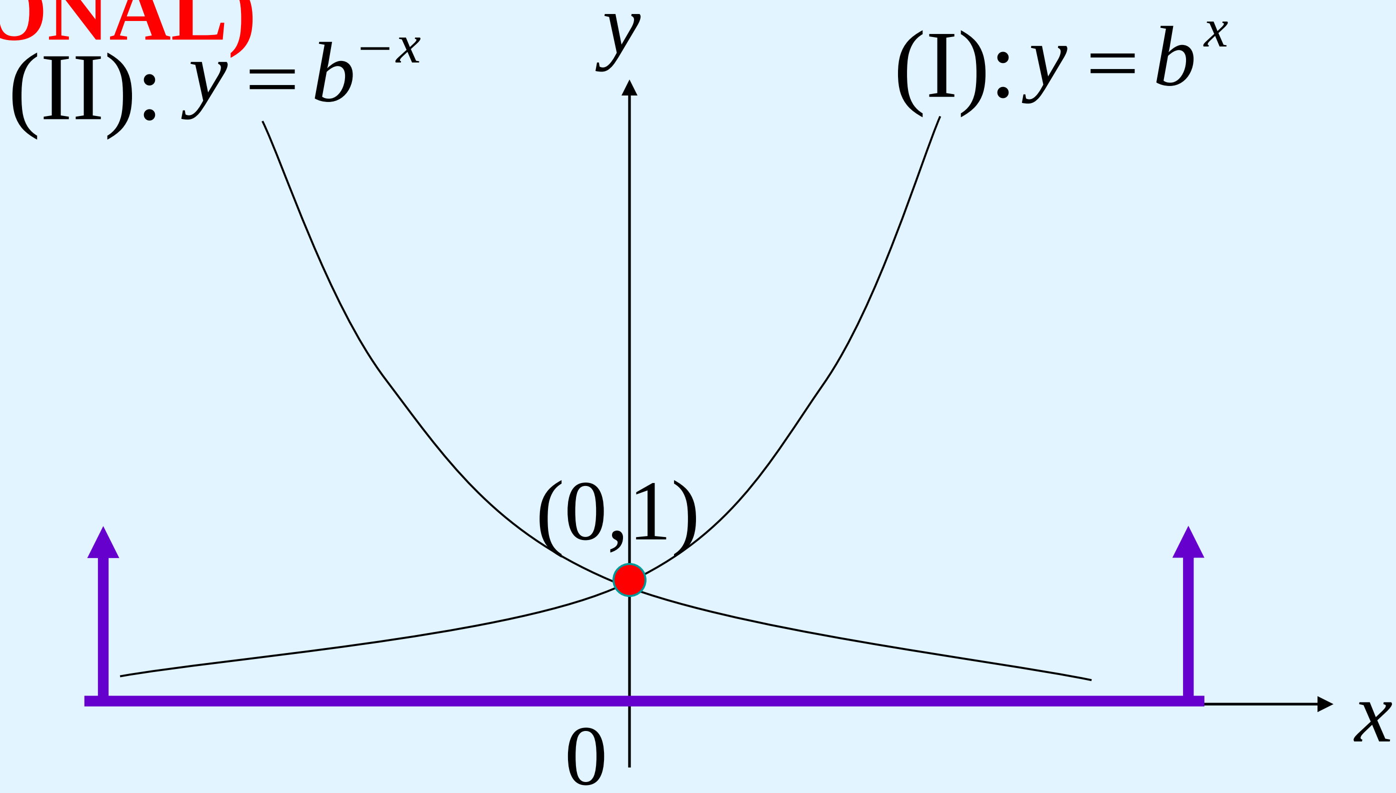
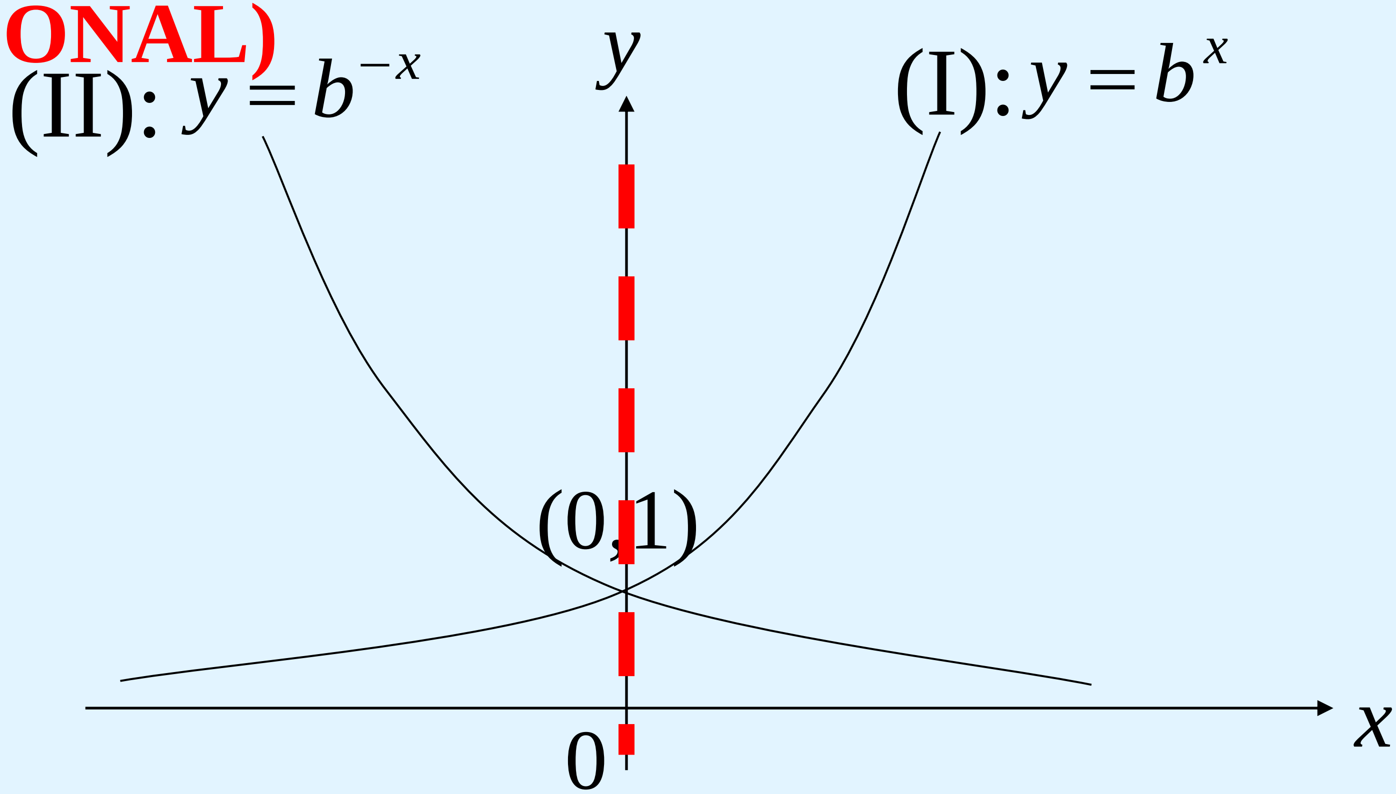


1. The y -intercepts of both curves are equal to 1.

2. The whole curves lie above the x -axis.

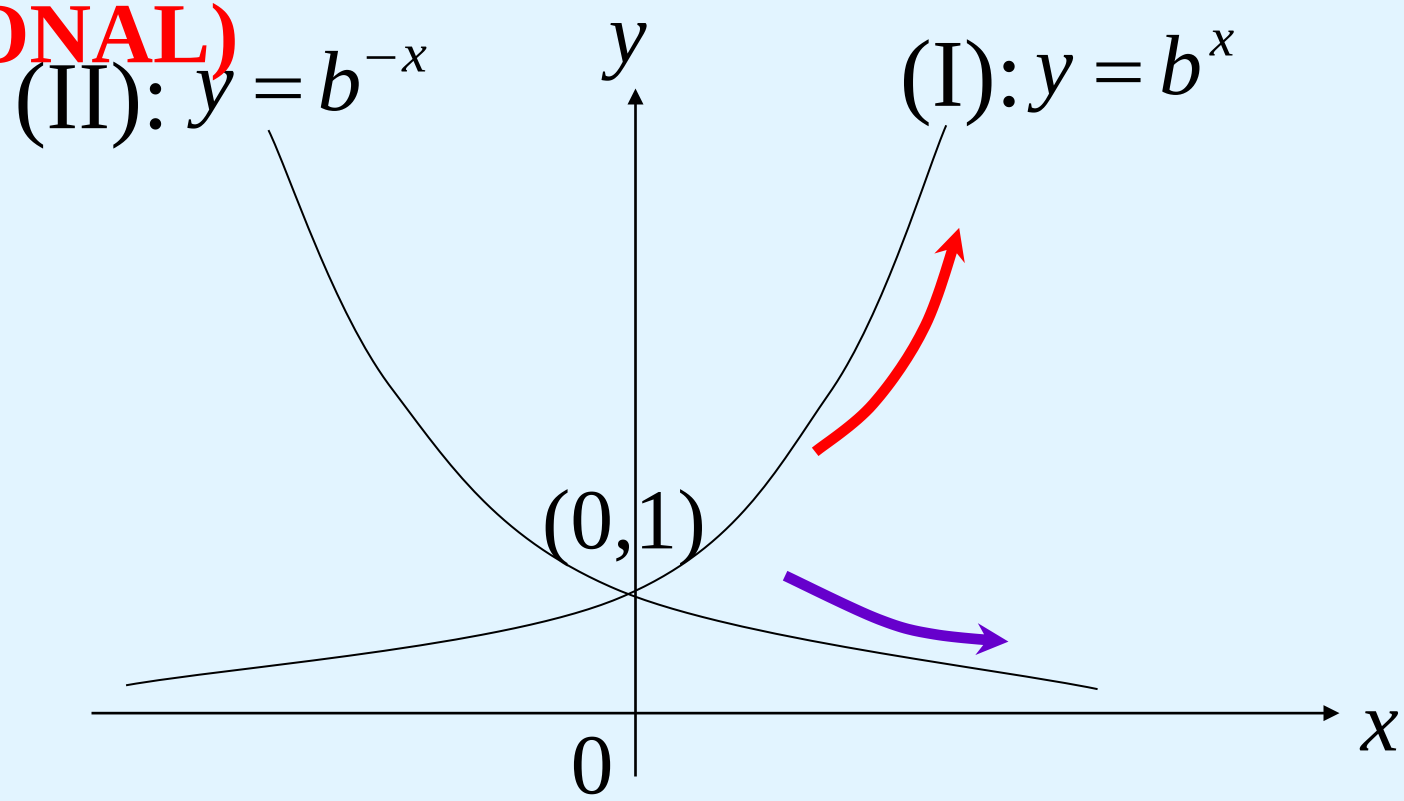
   That means $b^{x} > 0$ for all x.

(OPTIONAL)



3. The graph of $y=b^{-x}$ is the reflection of the graph $y=b^{x}$ about the y-axis.

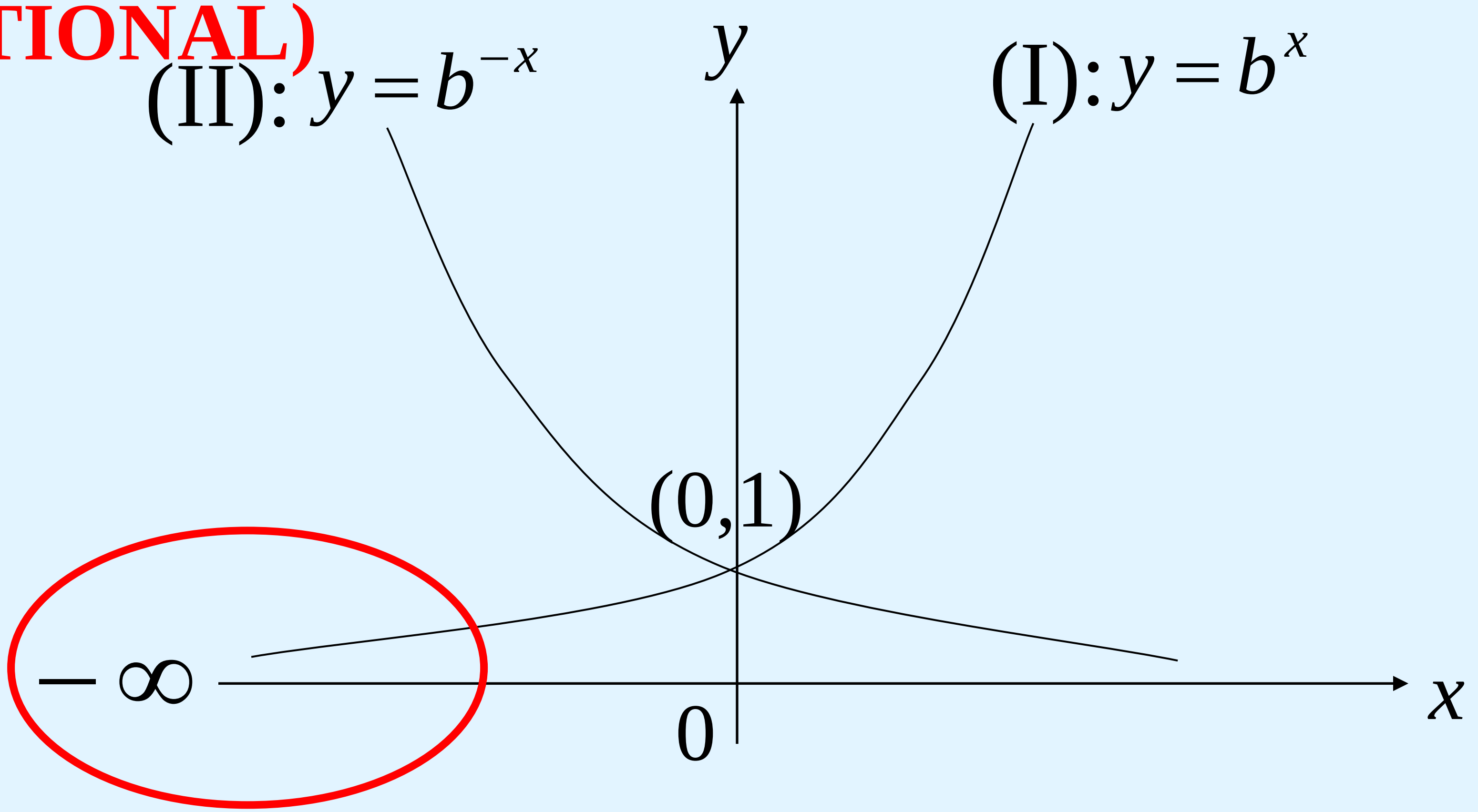
(OPTIONAL)



4. The graph (I) is increasing, while the graph (II) is decreasing.

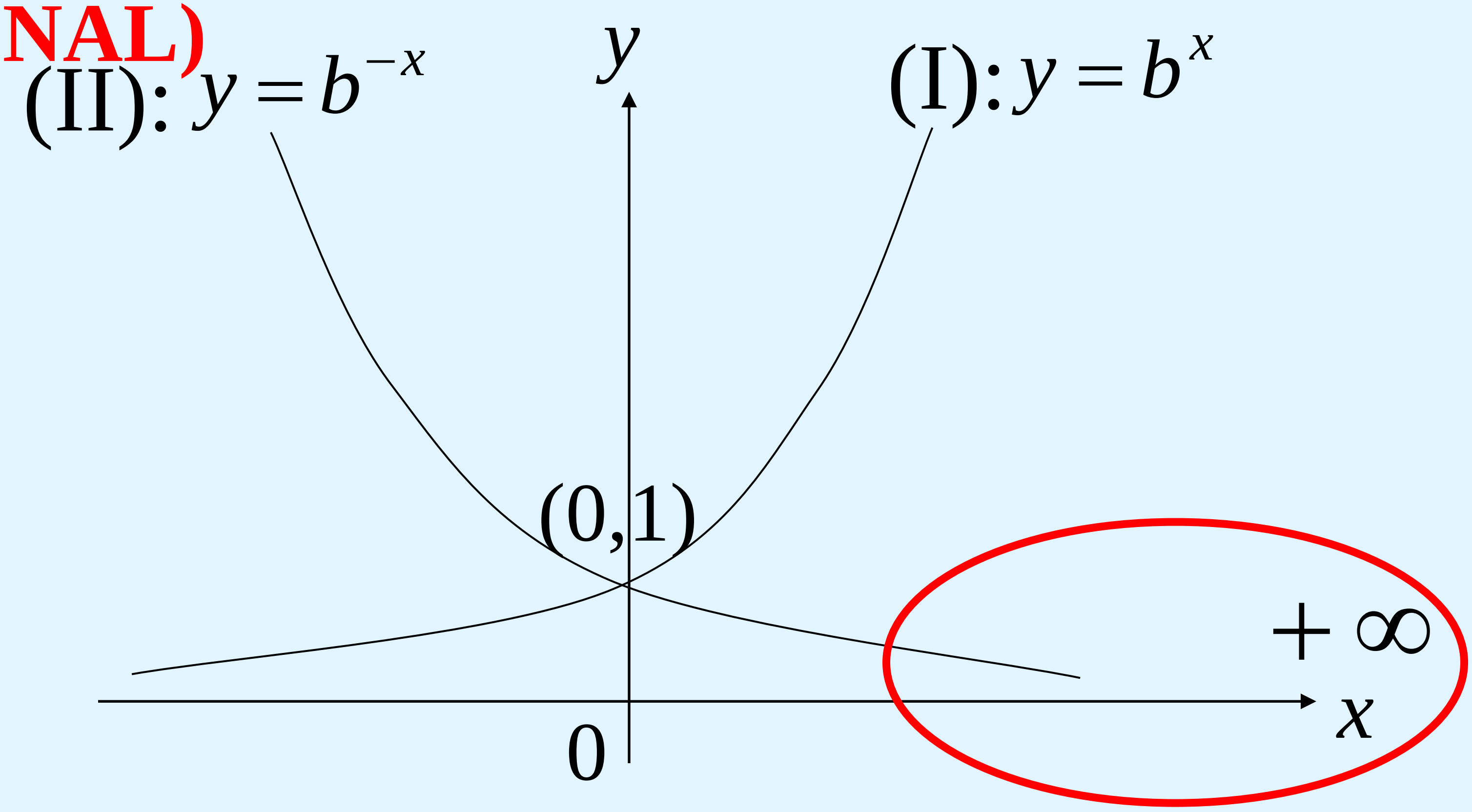
Both graphs are concave upward (will discuss later).

(OPTIONAL)



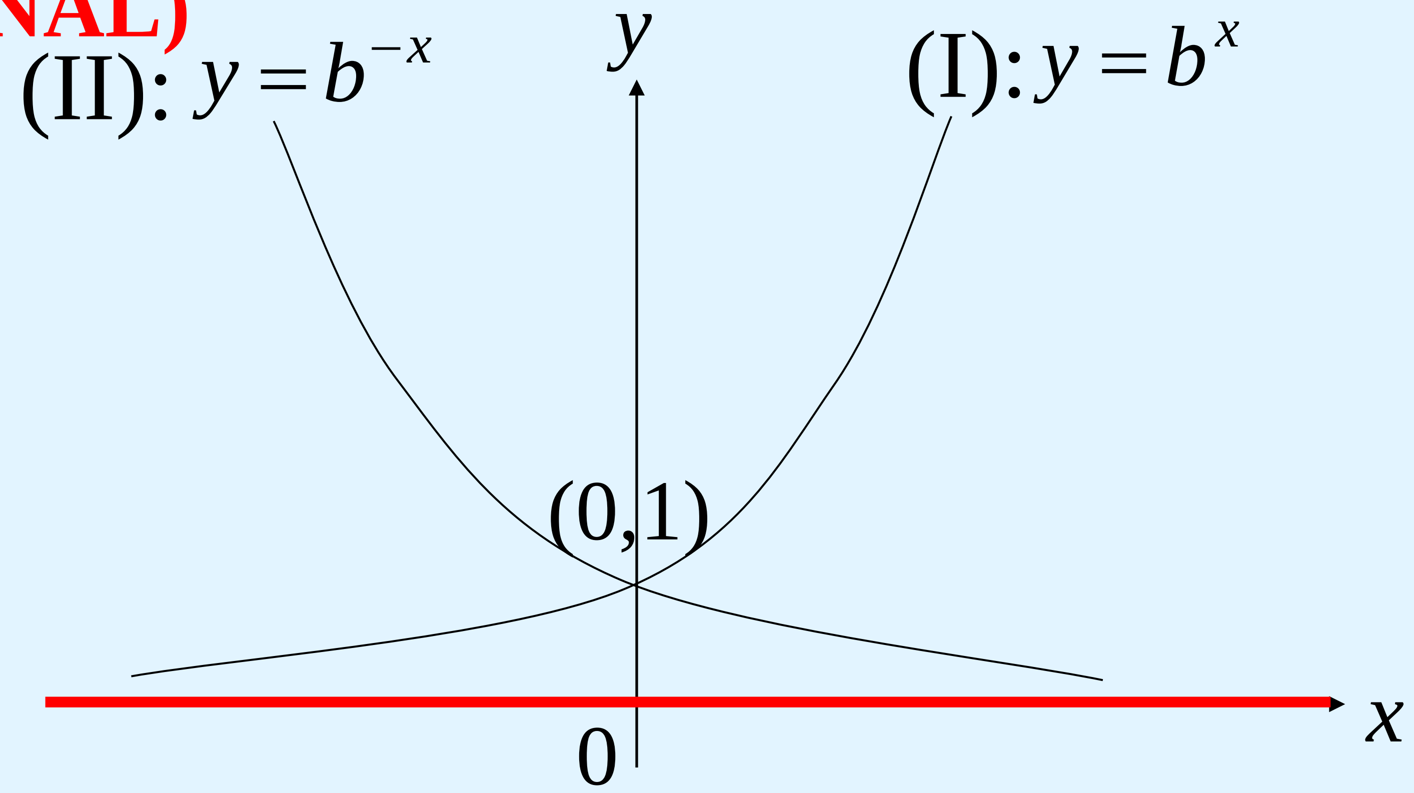
5. For graph (I), as x tends to $-\infty$, the graph is closer and closer to the x -axis.

(OPTIONAL)



6. For graph (II), as x tends to $+\infty$, the graph is closer and closer to the x -axis.

(OPTIONAL)



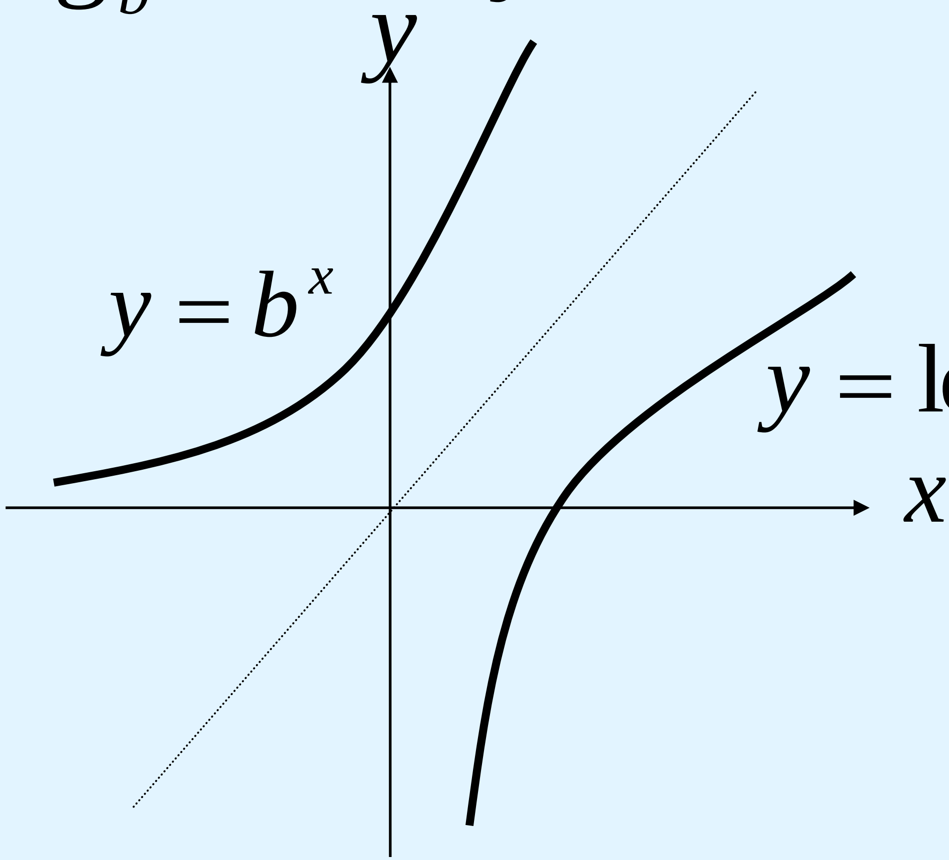
7. We call $y = 0$ (x -axis) is the *horizontal asymptote* for both graphs.

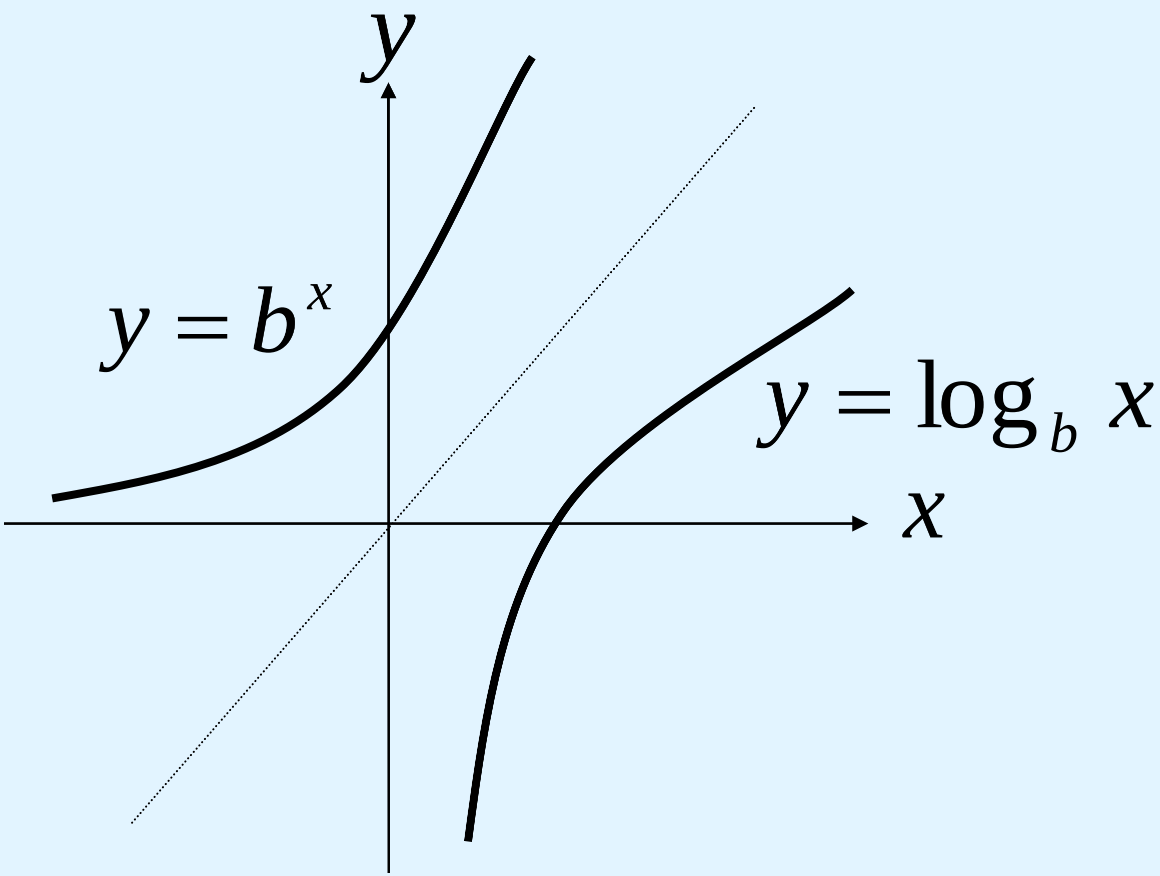
The Logarithmic Functions

(A) Definition of Logarithmic Function

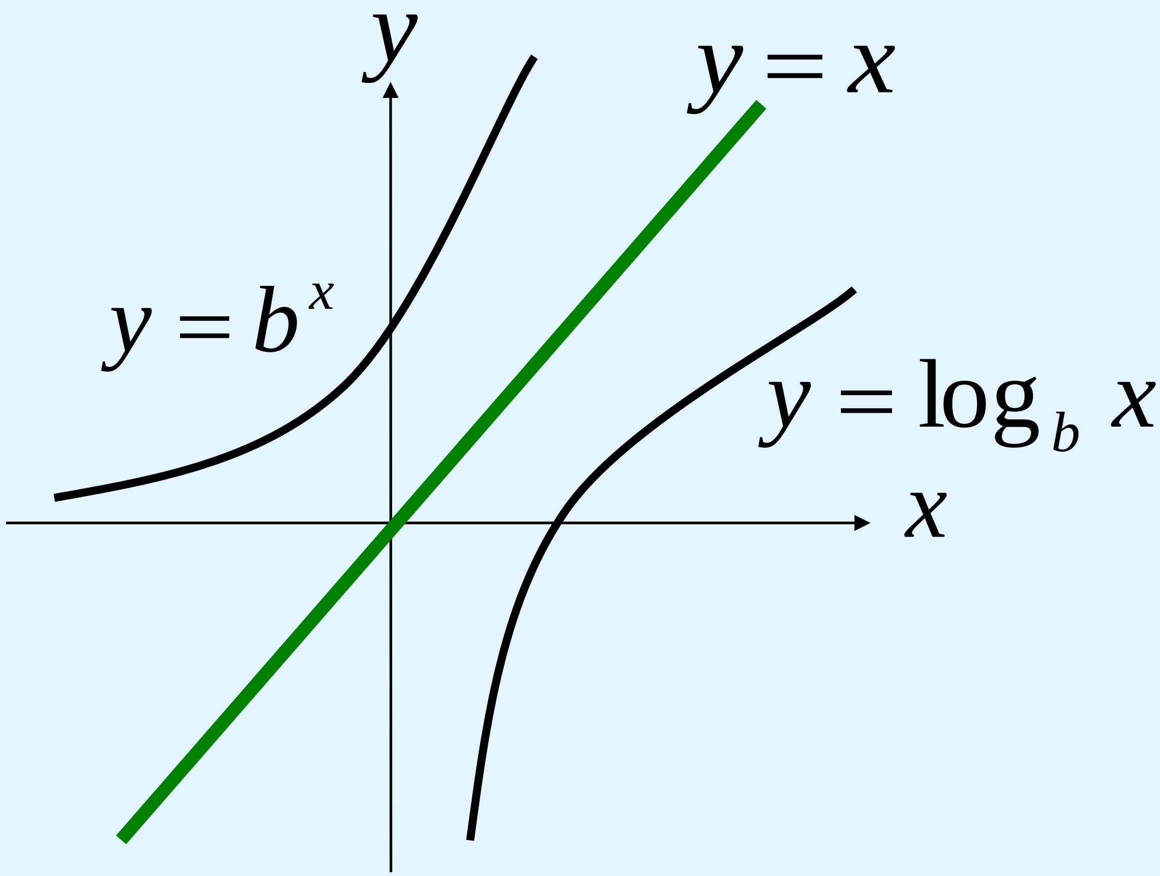
Consider the functions

$y = \log_b x$ and $y = b^x$.





The two functions are inverse functions of each other.



They are the mirror image to each other about the line $y = x$.

In general,

$y = \log_b x$ is equivalent to $x = b^y$.

where b is the base of the logarithmic function, $b > 0$ and $b \neq 1$

In particular,

If the base $b = e$, then $\log_e y$ is called the natural logarithmic function and is denoted by $\ln y$.

$$\boxed{\log_e y = \ln y}$$

eg. 10

$$y = \log_3 81$$

It means the exponent y with base 3 will give 81.

$$y = \log_3 81$$

$$3^y = 81$$

$$y = 4$$

eg. 11 Evaluate $\log_{\sqrt{3}} 9$

Let $y = \log_{\sqrt{3}} 9$

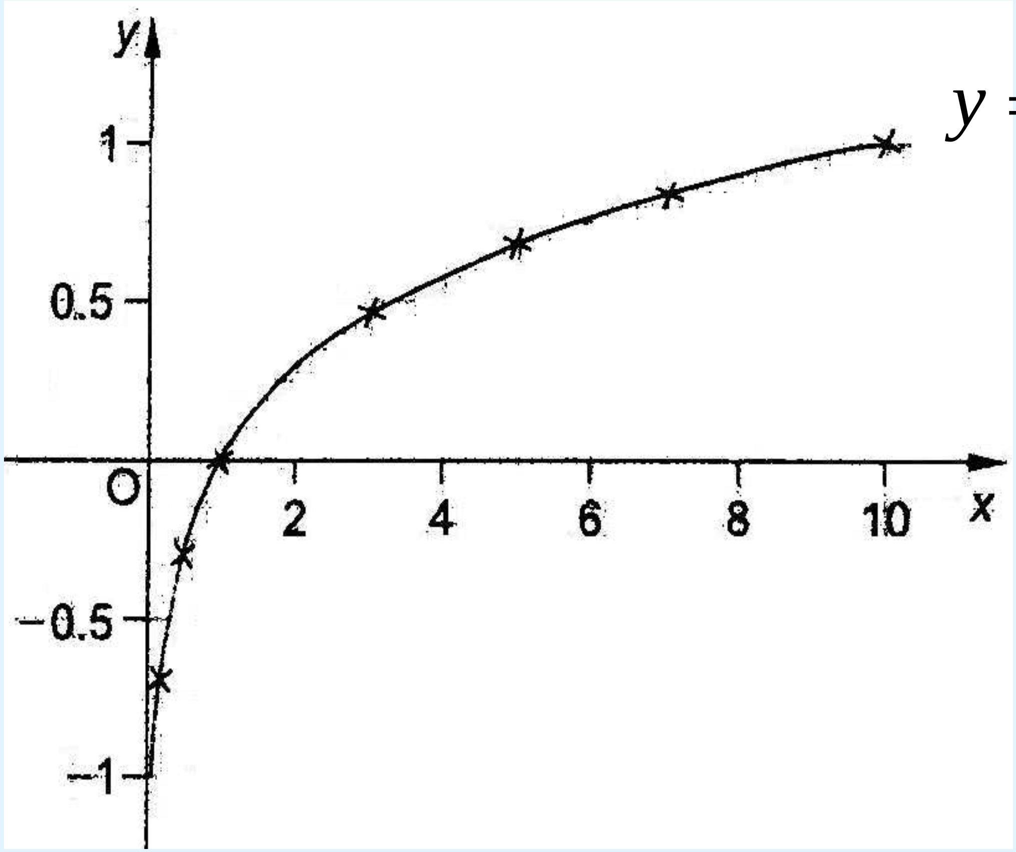
$$(\sqrt{3})^y = 9$$

$$3^{\frac{y}{2}} = 3^2$$

$$y = 4$$

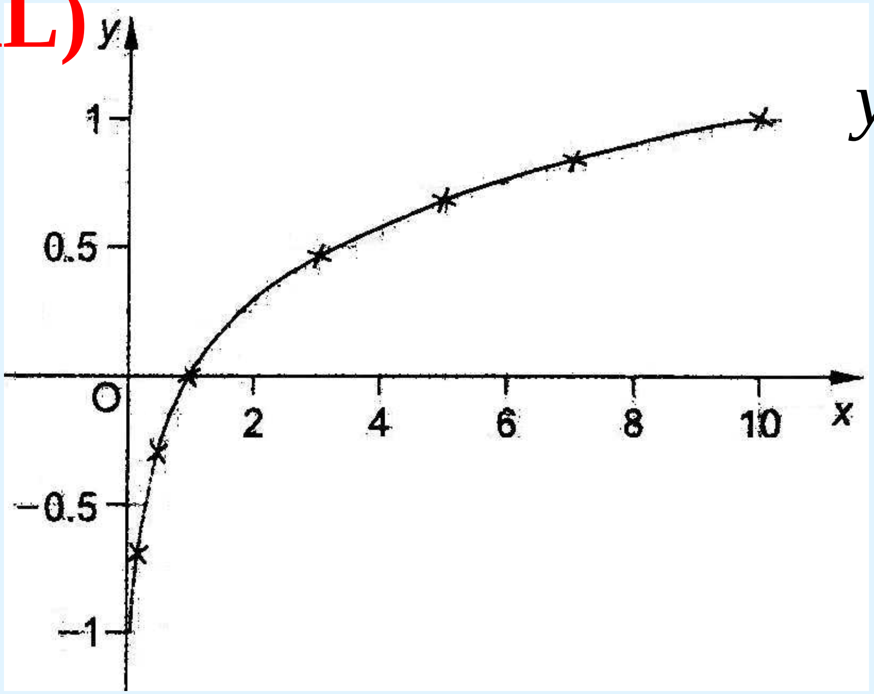
Graphs of Logarithmic Functions and Their Properties **(OPTIONAL)**

eg. 12 Sketch $y = \log_{10} x$.



$y = \log_{10} x$

(OPTIONAL)



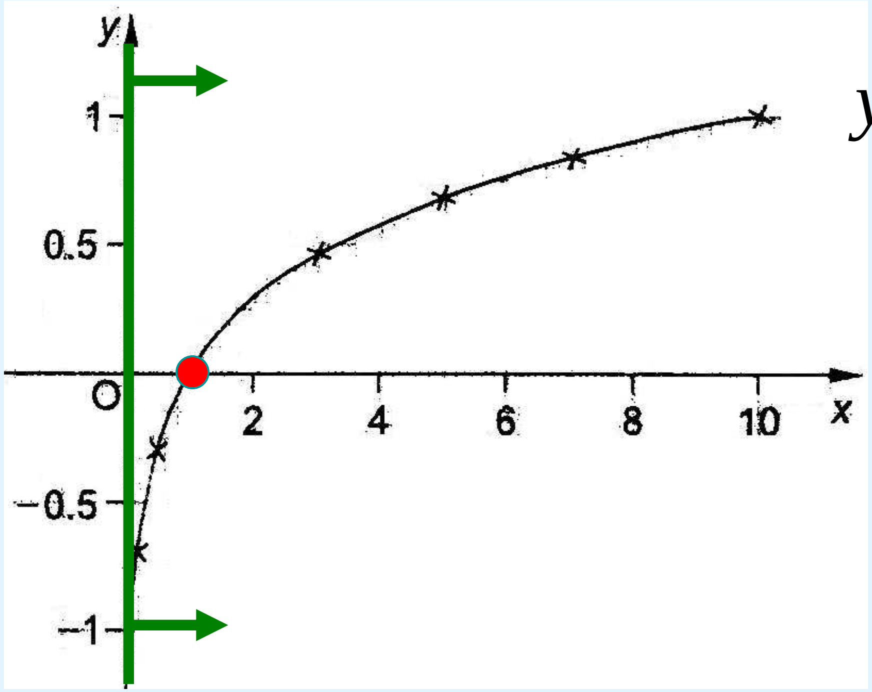
$y = \log_{10} x$

For any base $b > 1$, the shape of the graph of $y = \log_b x$ is similar.

For example: $y = \log_2 x$, $y = \ln x$

(OPTIONAL)

Features of the Graph $(b > 1)$

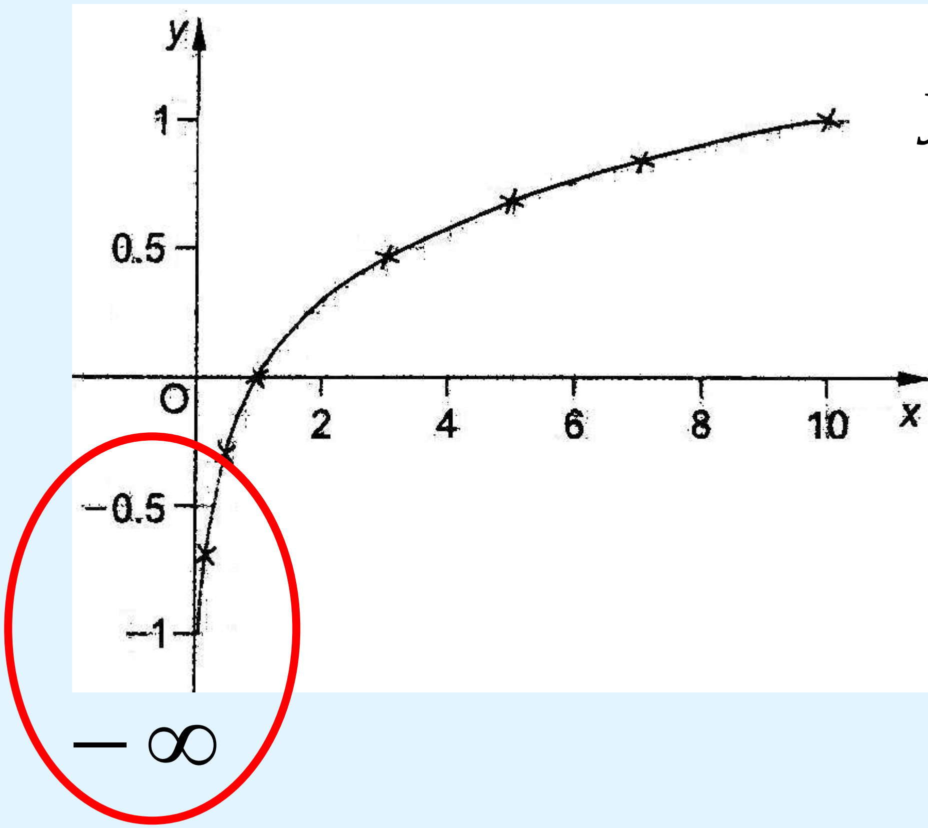


$y = \log_{10} x$

1. It cuts the x-intercept (1,0).
2. Whole graph lies on the right of the y-axis.

(OPTIONAL)

Features of the Graph $(b > 1)$



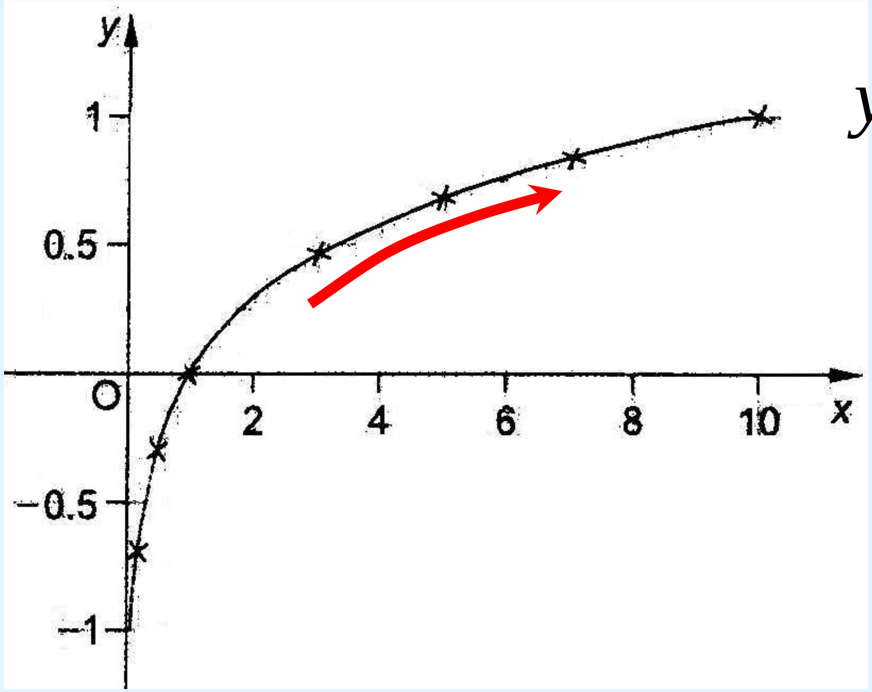
$y = \log_{10} x$

3. As $x \to 0$, $y \to -\infty$.

y - axis is the vertical asymptote

(OPTIONAL)

Features of the Graph $(b > 1)$



$y = \log_{10} x$

4. The curve is increasing.
5. The curve is concave downward (will discuss later).

Remarks:

(1) For the function $y = \log_b x$,

$x > 0$ and $-\infty < y < \infty$.

(2) $\ln x$ has all the properties of $\log_b x$

Laws of Logarithms

If x, y, and b are positive real numbers, $b \neq 1$, then

1. $\log_b 1 = 0$
2. $\log_b (xy) = \log_b x + \log_b y$
3. $\log_b \left(\frac{x}{y} \right) = \log_b x - \log_b y$
4. $\log_b \left(x^n \right) = n \log_b x$

5. $$\log_b b^x = x$$

6. $$b^{\log_b x} = x$$

eg. 13

Without using calculator, find

$$(\log_4 \frac{1}{64}) \times (\log_2 \sqrt{8})$$

$$= \left(\log_4 \frac{1}{4^3}\right) \times \left[\log_2 (2^3)^{\frac{1}{2}}\right]$$

$$= \left(\log_4 4^{-3}\right) \times \left[\log_2 (2^{\frac{3}{2}})\right]$$

$$= \left(\log_4 4^{-3}\right) \times \left[\log_2 (2^{\frac{3}{2}})\right]$$

$$= (-3)(\log_4 4) \times \frac{3}{2} (\log_2 2)$$

$$= (-3)(1) \times \frac{3}{2} (1)$$

$$= -\frac{9}{2}$$

eg. 14

$$\log_6(x+1)+\log_6(x-1)-\log_6(x^2-1)$$

$$=\log_6\frac{(x+1)(x-1)}{x^2-1}$$

$$=\log_6\frac{x^2-1}{x^2-1}$$

$$=\log_6 1$$

$$=0$$

Equations Involving Logarithmic and Exponential Functions

eg. 15 Solve the following equations for x :

$$e^{2x} + 3e^{-2x} - 4 = 0$$

Sol:

$$e^{2x} + 3\left(e^{2x}\right)^{-1} - 4 = 0$$

Let $y = e^{2x}$

$$y + 3y^{-1} - 4 = 0$$

Quadratic equation:

$$y + 3y^{-1} - 4 = 0$$

$$y^2 - 4y + 3 = 0$$

$$(y-1)(y-3) = 0$$

$$y = 1 \quad \text{or} \quad y = 3$$

$$e^{2x} = 1 \quad \text{or} \quad e^{2x} = 3$$

$$e^{2x} = 1 \quad \text{or} \quad e^{2x} = 3$$

Taking "ln" on both sides:

$$\ln e^{2x} = \ln 1 \quad \text{or} \quad \ln e^{2x} = \ln 3$$

$$2x = \ln 1 \quad \text{or} \quad 2x = \ln 3$$

$$x = 0 \quad \text{or} \quad x = \frac{\ln 3}{2}$$

eg. 16 Solve the following equations for x :

(a) $$2\ln(x-1)=\ln(x+1)$$

$$\ln(x-1)^2=\ln(x+1)$$

Sol:

Equality Property:

$$(x-1)^2=x+1$$

$$(x-1)^2 = x+1$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$x = 0$ or $x = 3$

[rejected; since

ln(-1) is undefined]

(b) [log + ln]

$$\log_2[\ln(x+1)] = 2$$

Sol:

$$\ln(x+1) = 2^2$$

$$\ln(x+1) = 4$$

$$e^{\ln(x+1)} = e^4$$

$$e^{\ln(x+1)} = e^4$$

Recall: $e^{\ln x} = x$

$$x+1 = e^4$$

$$x = e^4 - 1$$

(c) [ln + e]

$$\ln(e^{-0.02x}+8)-\ln(e^{-0.01x}-2)=\ln(2e^{-0.01x}+1)$$

Sol:

$$\ln(e^{-0.02x}+8)=\ln[(2e^{-0.01x}+1)(e^{-0.01x}-2)]$$

Equality Property:

$$e^{-0.02x}+8=(2e^{-0.01x}+1)(e^{-0.01x}-2)$$

$$e^{-0.02x}+8=(2e^{-0.01x}+1)(e^{-0.01x}-2)$$

$$e^{-0.02x}+8=2e^{-0.02x}-3e^{-0.01x}-2$$

$$e^{-0.02x}-3e^{-0.01x}-10=0$$

$$(e^{-0.01x})^2-3e^{-0.01x}-10=0$$

$$(e^{-0.01x})^2 - 3e^{-0.01x} - 10 = 0$$

Let $y = e^{-0.01x}$

$$y^2 - 3y - 10 = 0$$

$$(y-5)(y+2) = 0$$

$$y = 5 \quad \text{or} \quad y = -2$$

$$y = 5 \quad \text{or} \quad y = -2$$

$$e^{-0.01x} = 5 \quad \text{or} \quad e^{-0.01x} = -2$$

$$\ln e^{-0.01x} = \ln 5$$

[rejected; since $b^x > 0$ for all x]

$$-0.01x = \ln 5$$

$$x = -100 \ln 5$$

(d) [log + b^x]

$$\log_{16}(2^{x+1}-3)+\log_{16}(2^{-x}+1)=\frac{1}{2}$$

Sol:

$$\log_{16}[(2^{x+1}-3)(2^{-x}+1)]=\frac{1}{2}$$

$$(2^{x+1}-3)(2^{-x}+1)=16^{\frac{1}{2}}$$

$$(2^{x+1}-3)(2^{-x}+1)=4$$

$$(2^{x+1}-3)(2^{-x}+1)=4$$

$$[2(2^x)-3][\left(2^x\right)^{-1}+1]=4$$

Let $y=2^x$

$$(2y-3)(\frac{1}{y}+1)=4$$

$$2y-\frac{3}{y}-5=0$$

$$2y - \frac{3}{y} - 5 = 0$$

$$2y^2 - 5y - 3 = 0$$

$$(y - 3)(2y + 1) = 0$$

$$y = 3 \quad \text{or} \quad y = -\frac{1}{2}$$

$$y = 3 \quad \text{or} \quad y = -\frac{1}{2}$$

$$2^x = 3 \quad \text{or} \quad 2^x = -\frac{1}{2}$$

$$x \ln 2 = \ln 3$$

$$x = \frac{\ln 3}{\ln 2}$$

[rejected;
since $b^x > 0$
for all x]

Applications of Exponential and Logarithmic Functions

(A) Exponential Growth and Decay

The Growth Model: $Q(t) = Q_0 e^{kt}$

where Q_0 is the initial value ($t = 0$) of Q and k is a positive constant.

The Decay Model: $Q(t) = Q_0 e^{-kt}$

where Q_0 is the initial value ($t = 0$) of Q and k is a positive constant.

eg. 17 The population density x miles from the center of a city is given by a function of the form $Q(x) = Ae^{-kx}$. Find this function if it is known that the population density at the center of the city is 15,000 people per square miles and the density 10 miles from the center is 9,000 people per square miles.

Sol.

For simplicity, we shall express the density in units of 1,000 people per square miles. Thus, we have: Q(0) = 15 and Q(10) = 9.

$$15 = Ae^{-k(0)} \quad \text{and} \quad 9 = Ae^{-k(10)}$$

$$A = 15 \quad \text{and} \quad 9 = Ae^{-10k}$$

$$9 = 15e^{-10k} \Rightarrow k = -\frac{1}{10}\ln\frac{3}{5}$$

Hence, the function is $Q(x) = 15e^{(\frac{1}{10}\ln\frac{3}{5})x}$

eg. 18 If $1,000 is invested at 8% annual interest, compounded quarterly, how long will it take for the investment to double? Would the doubling time change if the principal were something other than $1,000?

Sol.

With a principal of \$1,000, the balance after t years is $B(t)=1000\left(1+\frac{0.08}{4}\right)^{4t}$. For the investment to double, we have

$$2000=1000\left(1+\frac{0.08}{4}\right)^{4t}$$

$$\Rightarrow 2=1.02^{4t}$$

$$\Rightarrow t\approx 8.75$$

Hence, it takes 8.75 years to double the investment.

Sol.

If a principal is P_0 dollars instead of \$1,000, then

$$2P_0 = P_0\left(1+\frac{0.08}{4}\right)^{4t}$$

$$\Rightarrow 2 = 1.02^{4t}$$

$$\Rightarrow t \approx 8.75$$

Hence, it still takes 8.75 years to double the investment.

Mathematical Induction

Let us consider the following example:

$$1 = 1^2$$

$$1 + 3 = 2^2$$

$$1 + 3 + 5 = 3^2$$

$$1 + 3 + 5 + 7 = 4^2$$

One may deduce that for any given positive integer n:

$$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$$

(Arithmetic sum)

Such a mathematical process of deducing the general result is called induction. However, the validity of the above identity is questionable since it has not been tested for all positive integers. To resolve such a difficulty, a mathematical procedure known as Mathematical Induction (M.I.) is introduced.

eg. 19 Using mathematical induction to prove that

$$1+3+5+7+\ldots+(2n-1)=n^2$$

where *n* is a positive integer.

Sol.

Let P(*n*) be $1+3+5+7+\ldots+(2n-1)=n^2$

When n = 1, L.H.S. = 1, R.H.S. = 1^2 = 1

$\therefore$ P(1) is true

Assume P(k) is true for some positive integer k

i.e.,

$$1+3+5+7+\ldots+(2k-1)=k^2$$

When $n = k + 1$

L.H.S. $=1+3+5+7+\ldots+(2k-1)+(2(k+1)-1)$

$$=k^2+(2(k+1)-1)$$

$$=k^2+2k+1$$

$$=(k+1)^2$$

= R.H.S.

$\therefore$ By the principle of M.I., P(n) is true for all positive integers n.

eg. 20 Using Mathematical induction to prove that

$$1\times 2+2\times 3+2^2\times 4+\ldots+2^{n-1}(n+1)=2^n(n)$$

where n is a positive integer.

Sol.

Let P(n) be

$$1\times 2+2\times 3+2^2\times 4+\ldots+2^{n-1}(n+1)=2^n(n)$$

When n = 1, L.H.S. = 1(2) = 2, R.H.S. = 2^1 (1) = 2

$\therefore$ P(1) is true

Assume P(k) is true for some positive integer k

i.e., $$1\times 2+2\times 3+2^2\times 4+\ldots+2^{k-1}(k+1)=2^k(k)$$

When $n = k + 1$

L.H.S. $$=1\times 2+2\times 3+\ldots+2^{k-1}(k+1)+2^{(k+1)-1}((k+1)+1)$$

$$=2^k(k)+2^k(k+2)$$

$$=2^k(2k+2)$$

$$=2^{k+1}(k+1)$$

= R.H.S.

$\therefore$ By the principle of M.I., P(n) is true for all positive integers n.