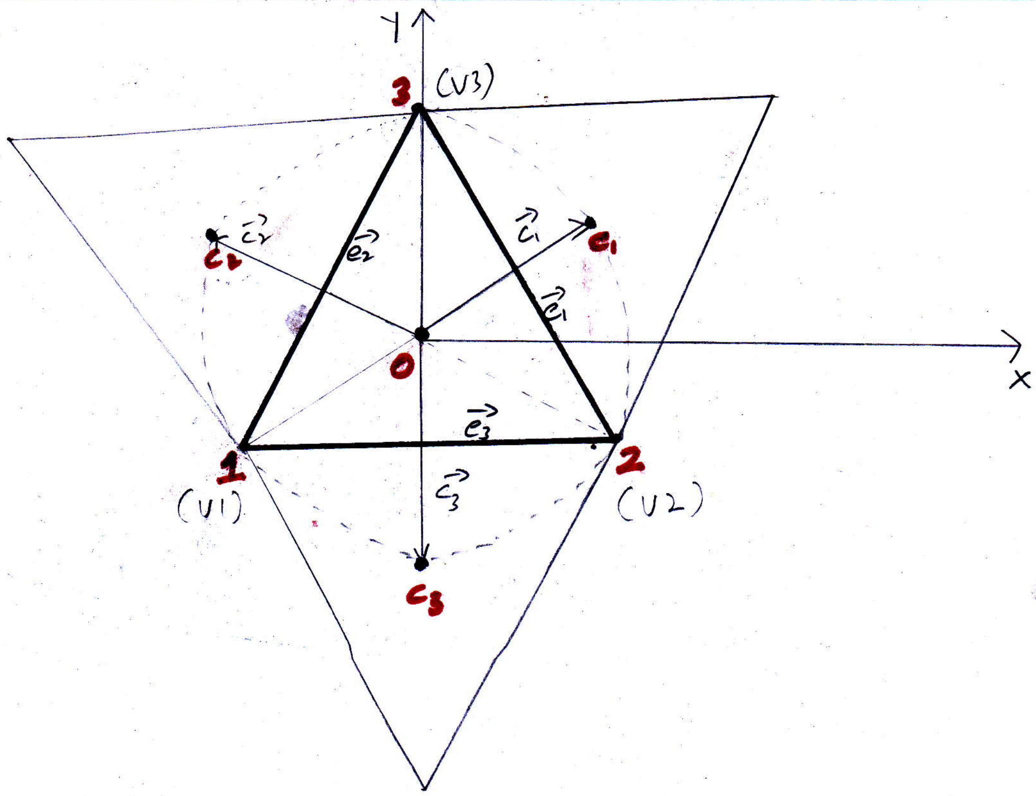

figure 1:

O is the origin that lies at the centeroid of the base triangle. The three neighbors have their centers at c_1, c_2 and c_3 that lie opposite to vertices 1, 2 and 3 respectively of base triangle.

The circumradius of all triangles is R. length of each edge of the triangles is a.

$$a = \sqrt{3} R \quad \text{— (1)}$$

Note that the centers of each neighboring triangles lie on the circumcircle of base triangle. This means,

$$|\vec{c_1}| = |\vec{c_2}| = |\vec{c_3}| = R \quad \text{— (2)}$$

The three vertices of the triangle will henceforth be represented by V1, V2 and V3 respectively instead of just 1, 2 and 3.

With the coordinate system as shown in figure 1, the three vertices have co-ordinates as:

$$V1 \equiv \left(-\frac{\sqrt{3}}{2} R, -\frac{R}{2}\right)$$

$$V2 \equiv \left(\frac{\sqrt{3}}{2} R, -\frac{R}{2}\right) \quad \text{—— (3)}$$

$$V3 \equiv (0, R)$$

The co-ordinates of centers of three neighboring triangles are

$$C_1 \equiv \left(\frac{\sqrt{3}}{2} R, \frac{R}{2}\right)$$

$$C_2 \equiv \left(-\frac{\sqrt{3}}{2} R, \frac{R}{2}\right) \quad \text{—— (4)}$$

$$C_3 \equiv (0, -R)$$

We model the scalar conservative equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = 0 \quad \text{— (5)}$$

Written in finite volume discreate form, the equation becomes

$$\frac{dU}{dt} + Q(U) - D(U) = 0 \quad \text{— (6)}$$

here $Q(U)$ is the FU integral of convective fluxes and $D(U)$ is the FU integral of Artificial dissipation fluxes.

$$Q(U) \equiv \frac{1}{h} \sum_{k=1}^{3} (\vec{V} \cdot \vec{S_k}) U_{ck} \quad \text{— (7)}$$

$h \equiv$ area of base triangle $\equiv \frac{3\sqrt{3}}{4} R^2$

$\vec{V} = A\hat{i} + B\hat{j}$ where A, B are convective velocities along x & y axis.

$\vec{S_k}$ = area vector associated with k^{th} edge of base triangle.

Now, $|\vec{S}_k| = a$ because each triangle is equilateral triangle

$\therefore \vec{S}_1 = a \dfrac{\vec{C}_1}{|\vec{C}_1|}$; This is because ~~edge~~ line joining centers of two neighboring triangles is perpendicular to the common edge for equilateral triangles.

$$\therefore \vec{S}_k = a \frac{\vec{C}_k}{|\vec{C}_k|}$$

but $|\vec{C}_k| = R$ and $a = \sqrt{3}\,R$, thus

$$\vec{S}_k = \sqrt{3}\,\vec{C}_k \qquad — (8)$$

U_{ek} is the value of scalar at the mid point of the k^{th} edge.

$$\therefore U_{e1} = \frac{U_0 + U_1}{2}$$

or general form : $U_{ek} = \frac{1}{2}(U_0 + U_k) \qquad — (9)$

Artificial dissipation term is composed of 2^{nd} and 4^{th} order term; for the time being we will concentrate only on 2^{nd} order term.

$$\therefore D_2(U) \equiv \frac{\varepsilon_2}{h} \sum_{k=1}^{3} (|\vec{V}\cdot\vec{S}_k|)(U_k - U_0) \qquad — (10)$$

Fourier decomposition of the scalar quantity :

$$U_0 \equiv \hat{U}\, e^{i(\vec{k}\cdot\vec{C}_0 - \omega t)} \qquad — (11)$$

here, $\vec{k} \equiv k_x \hat{i} + k_y \hat{j}$ is the wave number vector

$\vec{C}_0$ = position vector of base triangle's center

ω = pulsation

$\hat{U}$ = Fourier coefficient

Similarly :

$$U_1 = \hat{U}\, e^{i(\vec{k}\cdot(\vec{c}_0 + \vec{c}_1) - \omega t)}$$

$$= U_0\, e^{i\vec{k}\cdot\vec{c}_1}$$

$\therefore$ Generally $\quad U_k = U_0\, e^{i\vec{k}\cdot\vec{c}_k} \qquad — (12)$

Note that in our placement of co-ordinate axis, $\vec{c}_0$ is origin.

Using (9) and (12) we can write edge mid point quantities as

$$U_{ek} = \frac{1}{2}\left(U_0 + U_0\, e^{i\vec{k}\cdot\vec{c}_k}\right)$$

$$= \frac{U_0}{2}\left(1 + e^{i\vec{k}\cdot\vec{c}_k}\right) \qquad — (13)$$

Similarly,

$$U_k - U_0 \equiv U_0\, e^{i\vec{k}\cdot\vec{c}_k} - U_0 = U_0\left(e^{i\vec{k}\cdot\vec{c}_k} - 1\right) \qquad — (14)$$

Expanding $Q(U)$; From (7)

$$Q(U) = \frac{1}{h}\left[(\vec{V}\cdot\vec{S}_1)\, U_{e1} + (\vec{V}\cdot\vec{S}_2)\, U_{e2} + (\vec{V}\cdot\vec{S}_3)\, U_{e3}\right]$$

$$= \frac{U_0}{2h}\left[(\vec{V}\cdot\vec{S}_1)\left(1 + e^{i\vec{k}\cdot\vec{c}_1}\right) + (\vec{V}\cdot\vec{S}_2)\left(1 + e^{i\vec{k}\cdot\vec{c}_2}\right) + (\vec{V}\cdot\vec{S}_3)\left(1 + e^{i\vec{k}\cdot\vec{c}_3}\right)\right]$$

$$= \frac{U_0}{2h}\left[\vec{V}\cdot(\vec{S}_1 + \vec{S}_2 + \vec{S}_3) + \sum_{k=1}^{3}(\vec{V}\cdot\vec{S}_k)\, e^{i\vec{k}\cdot\vec{c}_k}\right]$$

$$= \frac{U_0}{2h}\left[\sum_{k=1}^{3}(\vec{V}\cdot\vec{S}_k)\, e^{i\vec{k}\cdot\vec{c}_k}\right] \qquad — (15)$$

Note that $\vec{k}$ should not be confused with the summation index k in the above expression.

In a similar manner, we can expand $D_2(U)$ using (10)

$$D_2(U_0) = \frac{\varepsilon_2}{h}\left[\sum_{k=1}^{3} |\vec{V}.\vec{S_k}|(U_k - U_0)\right]$$

$$= \frac{\varepsilon_2}{h}\left[\sum_{k=1}^{3} |\vec{V}.\vec{S_k}|\, U_0\left(e^{i\vec{k}.\vec{C_k}} - 1\right)\right]$$

$$= \frac{U_0\varepsilon_2}{h}\left[\sum_{k=1}^{3} |\vec{V}.\vec{S_k}|\, e^{i\vec{k}.\vec{C_k}} - \sum_{k=1}^{3} |\vec{V}.\vec{S_k}|\right] \quad -(16)$$

Using (8), (9), (13) in (15) and applying assumption $A=B$ with

$$\theta_x = k_x R \;;\quad \theta_y = k_y R \quad \text{and} \quad \sigma = \frac{A\Delta t}{R} = \frac{B\Delta t}{R} \quad -(17)$$

we get

$$Q(U) = \frac{2U_0\sigma}{3\Delta t}\left[\left(-\sqrt{3}\sin\left(\frac{\theta_y}{2}\right)\sin\left(\frac{\sqrt{3}\theta_x}{2}\right) + \cos\left(\frac{\theta_y}{2}\right)\cos\left(\frac{\sqrt{3}\theta_x}{2}\right) - \cos\theta_y\right)\right.$$
$$\left. + i\left(\sin\left(\frac{\theta_y}{2}\right)\cos\left(\frac{\sqrt{3}\theta_x}{2}\right) + \sin\theta_y + \sqrt{3}\sin\left(\frac{\sqrt{3}\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right)\right)\right] \quad -(18)$$

And using (8), 9, & 13 in 16 we get

$$D_2(U) = \varepsilon_2\frac{4U_0\sigma}{3\Delta t}\left[\left(-\sin\left(\frac{\theta_y}{2}\right)\sin\left(\frac{\sqrt{3}\theta_x}{2}\right) + \sqrt{3}\cos\left(\frac{\theta_y}{2}\right)\cos\left(\frac{\sqrt{3}\theta_x}{2}\right) + \cos\theta_y - \sqrt{3} - 1\right)\right.$$
$$\left. + i\left(\sqrt{3}\sin\left(\frac{\theta_y}{2}\right)\cos\left(\frac{\sqrt{3}\theta_x}{2}\right) - \sin\theta_y + \sin\left(\frac{\sqrt{3}\theta_x}{2}\right)\cos\left(\frac{\theta_y}{2}\right)\right)\right] \quad -(19)$$

~~From~~ Recalling (6)
~~Replacing (18) & (19) in (6) we get~~

$$\frac{dU}{dt} + Q(U) - D(U) = 0$$

From (18) we write $Q(U) = \frac{2U_0\sigma}{3\Delta t}\tilde{Q}(U)$ where $\tilde{Q}(U)$ is part of eq (18) in square bracket similarly from 19,

$$D_2(U) = \frac{4 U_0 \sigma \varepsilon_2}{3\Delta t} \tilde{D}_2(U)$$ where $\tilde{D}_2(U)$ is again part of eq 19 in square brackets.

Replacing in (6), we get

$$\frac{dU}{dt} + \frac{2 U_0 \sigma}{3\Delta t} \tilde{Q}(U) - \frac{\varepsilon_2 4 U_0 \sigma}{3\Delta t} \tilde{D}_2(U) = 0$$

or $$\frac{dU}{dt} + \frac{\sigma}{\Delta t}\left(\frac{2}{3}\tilde{Q}(U) - \frac{4\varepsilon_2}{3}\tilde{D}_2(U)\right) U_0 = 0 \qquad -(20)$$

Comparing eq (20) with model eq

$$\frac{dU_0}{dt} + \alpha U_0 = 0$$

we get $$\alpha = \frac{\sigma}{3\Delta t}\left(2\tilde{Q}(U) - 4\varepsilon_2 \tilde{D}_2(U)\right) \qquad -(21)$$

For RK3 step defined in Jameson (1981) we have

$$G = \frac{|U_0^{n+1}|}{|U_0^{n}|} = 1 - (\alpha\Delta t) + \frac{(\alpha\Delta t)^2}{2} - \frac{(\alpha\Delta t)^3}{4} \qquad -(22)$$

where α is given by eq (21).

Thus G is a function of $G = G(\sigma, \varepsilon_2, \theta_x, \theta_y)$