Error Propagation

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Outline

Error Types: input, method, roundoff, trunction, modeling, machine

Numeric impression in formulas

μ and σ for linear case

μ and σ in non-linear case

(Physical) Errors in digital cameras

- Parallax
- What is a “point”, how is it mapped?

Input, method, roundoff, truncation, modeling, machine + human errors

Input error:

- Given numbers are no machine numbers (e.g. sqrt(2))

While running (method error)

- accumulate round off errors per calculation

Truncation error

- Systematic errors when stopping an approximation too early

Modeling error

- Too strong idealizations

Machine + Human

- Hardware errors, programming errors

Some rules for measurements

Rule for Stating Uncertainties:

(Measured value of x) = $x_{best} \pm \delta x$

Experimental uncertainties should almost always be **rounded to one significant figure**

x_{best} = best estimate for x

δx = uncertainty or error in the measurement

Rule for Stating Answers:

Fractional uncertainty:

The **last significant figure** in any stated answer should usually be **of the same order of magnitude** (in the same decimal position) as the **uncertainty.**

$= \delta x / |x_{best}|$

Approximate correspondence
between significant figures and fractional uncertainties.

Number of significant figures	Corresponding fractional uncertainty is between	or roughly
1	10% and 100%	50%
2	1% and 10%	5%
3	0.1% and 1%	0.5%

Uncertainty in Experiements

Counting Experiment:

The uncertainty in any counted number of random events, as an estimate of the true average number, is the square root of the counted number.

(average numer of events in time T)=$\nu \pm \sqrt{\nu}$

Example: 14 births in 2 weeks =>

(average births in a two-week period) = 14 ± 4

Uncertainties: „+“,“-“,“*“,“/“

Sums and diffs:

If q = x + · · · + z - (u + · · · + w), then

$$\delta q \begin{cases} a) = \sqrt{(\delta x)^2 + (\delta y)^2 \cdots (\delta w)^2} \\ b) \leq \delta x + \delta y \cdots + \delta w \end{cases}$$

Products and Quotients:

$$if\ q = \frac{x \times y \times \cdots \times z}{u \times v \times \cdots \times w}, \delta x, \delta y, \cdots, \delta w\ uncertainties$$

$$\frac{\delta q}{|q|} \begin{cases} a) = \sqrt{(\frac{\delta x}{x})^2 + (\frac{\delta y}{y})^2 \cdots + (\frac{\delta w}{w})^2} \\ b) \leq \frac{\delta x}{|x|} + \frac{\delta y}{|y|} \cdots + \frac{\delta w}{|w|} \end{cases}$$

Case a):

Independend and random

Case b):

always

Uncertainties, special cases

If $q = Bx$, where B is known exactly, then

$q = |B|\ \delta x$.

If q is a function of one variable, q(x), then

$\delta q = |dq/dx|\ \delta x$

If q is a power, $q = x^n$, then

$\delta q/|q| = |n|\ \delta x/|x|$.

Differential Error Analysis

$input\ data: x \in IR^m,\ output\ y \in IR^n,\ algorithm\ y = \varphi(x).$

$Let\ \Delta_x\ be\ the\ vector\ of\ absolute\ data\ error\ in\ x\ and$

$$JAC(\varphi) := \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_n}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_m} \end{pmatrix} \in IR^{n\ x\ m}$$

$Then\ for\ the\ absolut\ output\ error\ it\ holds\ (to\ first\ order):$

$$\Delta_y \doteq JAC(\varphi)\Delta_x$$

$if\ we\ calculate\ in\ absence\ of\ round\ off\ errros.$

Example curvature Radius R

$$R(v_r, v_l) = \frac{d}{2}\frac{v_r + v_l}{v_r - v_l}$$

$$\frac{\partial R}{\partial v_r} = \frac{d}{2}\left[\frac{1(v_r - v_l) - (v_r + v_l)1}{(v_r - v_l)^2}\right] = -d\,\frac{v_l}{(v_r - v_l)^2}$$

$$\frac{\partial R}{\partial v_l} = \frac{d}{2}\left[\frac{1(v_r - v_l) + (v_r + v_l)1}{(v_r - v_l)^2}\right] = -d\,\frac{v_r}{(v_r - v_l)^2}$$

$$\Delta R = -d\left[\frac{v_l}{(v_r - v_l)^2}\Delta v_r + \frac{v_r}{(v_r - v_l)^2}\Delta v_l\right]$$

When $v_l \approx v_r$ then R is very imprecise

Input errors in v_l and v_r are grossly amplified

Theorem

E[] for linear maps

Linear case, y is a linear map of x :

$$\vec{y} = F(\vec{x}) = A\vec{x} + \vec{b} \Rightarrow$$

$$E[\vec{y}] = E[A\vec{x} + \vec{b}]$$

$$= \int\int\cdots\int (A\vec{x} + \vec{b})\, p(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n \quad \textit{take component } j:$$

$$E[y_j] = \int\int\cdots\int \left(\sum_i a_{ij} x_i \right) p(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n +$$

$$\int\int\cdots\int b_j\, p(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

$$= \sum_i a_{ij} \int\int\cdots\int x_i\, p(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n + b_j$$

$$= \sum_i a_{ij} E[x_i] + b_j \Rightarrow E[\vec{y}] = AE\left[\vec{x}\right] + \vec{b}$$

Covariances for linear case

$$\operatorname{cov}(y)$$

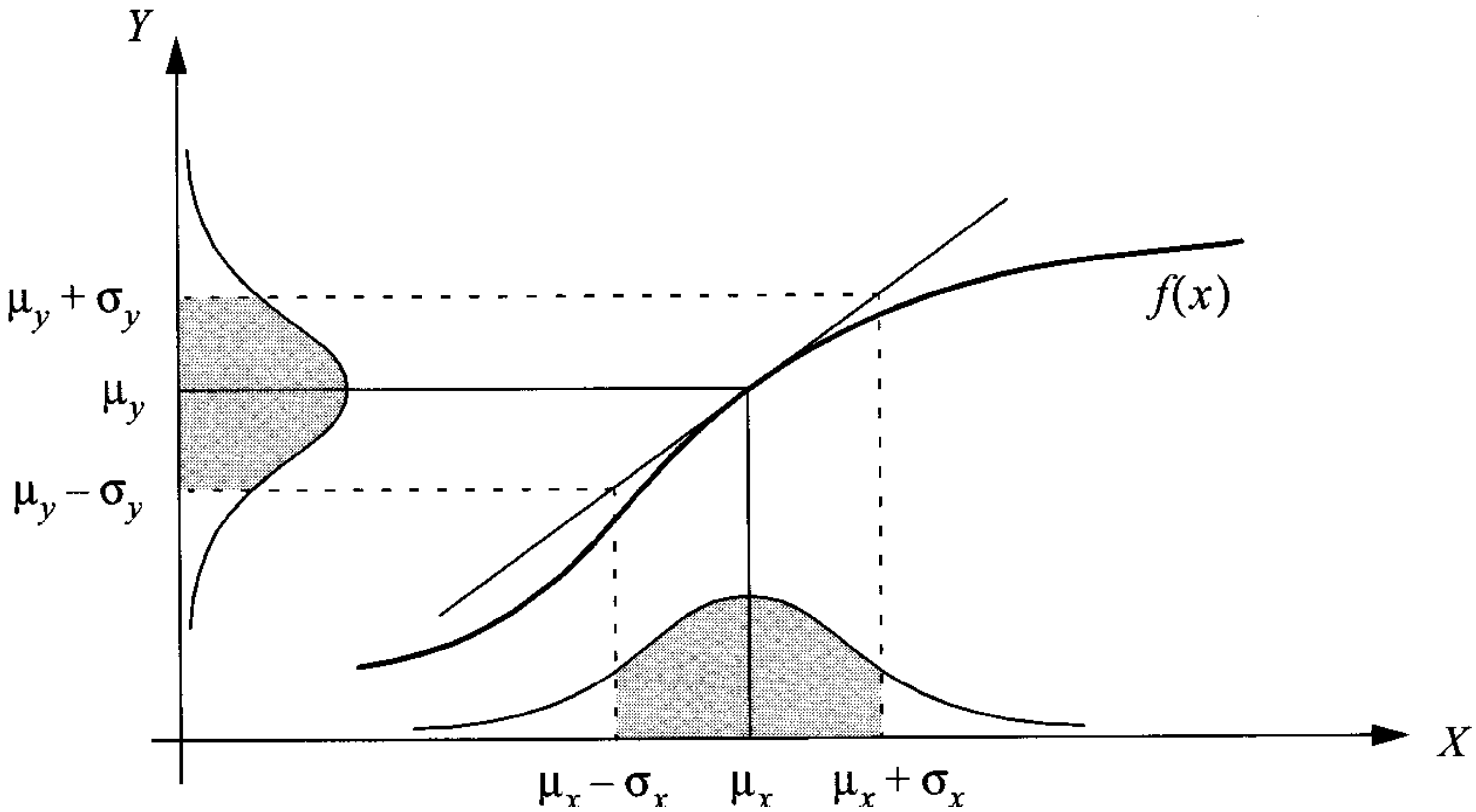
$$\overset{def}{=} E[((A\vec{x}+\vec{b}) - (AE[\vec{x}]+\vec{b}))((A\vec{x}+\vec{b}) - (AE[\vec{x}]+\vec{b}))^T]$$

$$= E[(A\vec{x} - AE[\vec{x}])(A\vec{x} - AE[\vec{x}])^T]$$

$$= E[A(\vec{x} - E[\vec{x}])(\vec{x} - E[\vec{x}])^T A^T]$$

$$= A\operatorname{cov}(x)A^T$$

How Uncertainties get mapped



Use Taylor expansion

μ and cov nonlinear

$$\vec{u} = F(\vec{x})\ (via\ Taylor) \Rightarrow\ E[\vec{u}] = F(E[\vec{x}])$$

$$\mathrm{cov}(\vec{u}) = JAC(F)\big|_{\vec{x}}\ \mathrm{cov}(\vec{x})\ JAC(F)\big|_{\vec{x}}^{T}$$

$$where\ JAC(F)\ is\ the\ Jacobian\ of\ the\ map\ F$$

Example Covariance

assume: Laser scanner measures polar coordinates (d,α), measument of d and α independend normally distributed $d \sim N(\mu_d,\sigma_d^2)$, $\alpha \sim N(\mu_\alpha,\sigma_\alpha^2)$, they have to be mapped to cartesian (x,y) via:

$$F([d,\alpha]^T) \quad = [d \cos(\alpha), d \sin(\alpha)]^T.$$

How does the original covariance matrix change?

[Stefan Gutmann: Dissertation: Robuste Navigation autonomer mobiler Systeme, Okt 2000, Univ. Freiburg]

Solution Covariance

$$\textit{Mapping}:$$

$$F\begin{pmatrix} d \\ \alpha \end{pmatrix} = \begin{pmatrix} d\cos(\alpha) \\ d\sin(\alpha) \end{pmatrix} =: \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\textit{Jacobian}:$$

$$\nabla F = \begin{pmatrix} \cos(\alpha) & -d\sin(\alpha) \\ \sin(\alpha) & d\cos(\alpha) \end{pmatrix}$$

$$\textit{Expected value}:$$

$$\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} = F\begin{pmatrix} \mu_d \\ \mu_\alpha \end{pmatrix} = \begin{pmatrix} \mu_d\cos(\mu_\alpha) \\ \mu_d\sin(\mu_\alpha) \end{pmatrix} = \begin{pmatrix} d\cos(\alpha) \\ d\sin(\alpha) \end{pmatrix}$$

$$\textit{covariance}:$$

$$\operatorname{cov}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \nabla F \operatorname{cov}\left(\begin{pmatrix} d \\ \alpha \end{pmatrix}\right)\nabla F^T =$$

$$\begin{pmatrix} \cos(\alpha) & -d\sin(\alpha) \\ \sin(\alpha) & d\cos(\alpha) \end{pmatrix}\begin{pmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_\alpha^2 \end{pmatrix}\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -d\sin(\alpha) & d\cos(\alpha) \end{pmatrix} =$$

$$\begin{pmatrix} \sigma_d^2\cos^2(\alpha) + d^2\sigma_\alpha^2\sin^2(\alpha) & (\sigma_d^2 - d^2\sigma_\alpha^2)\cos(\alpha)\sin(\alpha) \\ (\sigma_d^2 - d^2\sigma_\alpha^2)\cos(\alpha)\sin(\alpha) & \sigma_d^2\sin^2(\alpha) + d^2\sigma_\alpha^2\cos^2(\alpha) \end{pmatrix}$$

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Observation

Covariance in Error Propagation

Let $F(x,y)$ be given, let N data pairs given: $(x_1,y_1),(x_2,y_2),\ldots,(x_N,y_N)$.

Then we can compute empirical mean $\bar{x}$, S_x , $\bar{y}$ and S_y as usual.

Assume $x_1,\ldots,x_N$ close to $\bar{x}$ (same for y).Then:

$$G_i = G(x_i,y_i) \approx G(\bar{x},\bar{y}) + \left.\frac{\partial G}{\partial x}\right|_{\mu_x}(x_i-\bar{x}) + \left.\frac{\partial G}{\partial y}\right|_{\mu_y}(y_i-\bar{y})$$

$$\bar{G} = \frac{1}{N}\sum G_i = \frac{1}{N}\sum\left(G(\bar{x},\bar{y}) + \left.\frac{\partial G}{\partial x}\right|_{\mu_x}(x_i-\bar{x}) + \left.\frac{\partial G}{\partial y}\right|_{\mu_y}(y_i-\bar{y})\right) =$$

$$= G(\bar{x},\bar{y}) + \frac{1}{N}\left.\frac{\partial G}{\partial x}\right|_{\mu_x}\underbrace{\sum(x_i-\bar{x})}_{=0} + \frac{1}{N}\left.\frac{\partial G}{\partial y}\right|_{\mu_y}\underbrace{\sum(y_i-\bar{y})}_{=0} = G(\bar{x},\bar{y})$$

Standard deviation for G:

$$S_G^2 = \frac{1}{N}\sum (G_i - \bar{G})^2$$

$$\approx \frac{1}{N}\sum \left(\bar{G} + \left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} (x_i - \bar{x}) + \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} (y_i - \bar{y}) - \bar{G}\right)^2 =$$

$$= \left(\left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}}\right)^2 \frac{1}{N}\sum (x_i - \bar{x})^2 + \left(\left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}}\right)^2 \frac{1}{N}\sum (y_i - \bar{y})^2 +$$

$$2\left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} \frac{1}{N}\sum (x_i - \bar{x})(y_i - \bar{y})$$

$$= \left(\left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}}\right)^2 S_x^2 + \left(\left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}}\right)^2 S_y^2 + 2\left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} S_{xy}$$

…cont

S_{xy} is empirical covariance

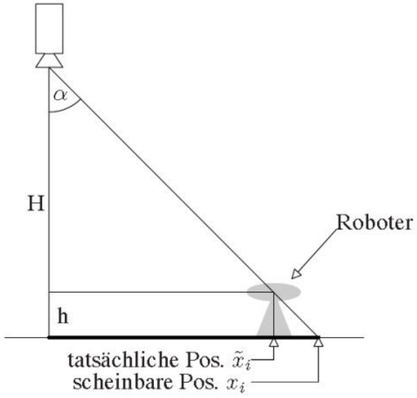
If x ,y independent $S_{xy} \approx 0$

So then it follows:

$$S_G^2 = \sum_i S_{z_i}^2 \left(\frac{\partial G}{\partial z_i} \right)^2$$

Parallax during Observation

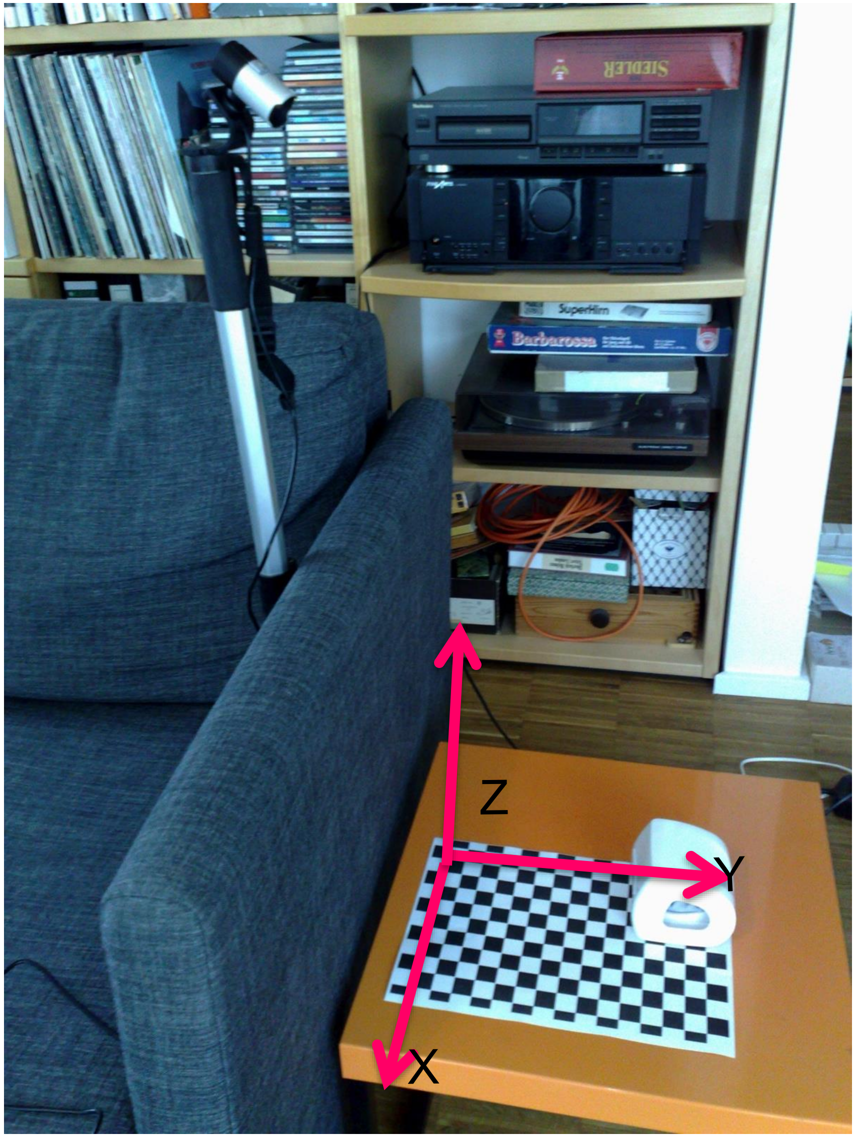
(a) Farbkreise als Trackingmerkmale



(b) Korrektur der Positionsabweichung aufgrund der Perspektive

$$\tilde{x}_i = x_i \left(1 - \frac{h}{H}\right)$$

Camera setup



Make sure that the frame used during calibration (and extrinsic parameter finding) is aligned like shown.

Provide Rc_1 and TC_1

And the current view of robot by the camera

XXc = Rc_1 * XX + Tc_1

„Point“ observations

Like the circles in last image the observered LED is mapped as to many points. Where is the robot?

Q: how about observed points, which are outside depth of field (DOF) and are thus depicted as „circles of confusion“ instead of points?