To-do:

- [ ] Sign up for pre-lab Jan 20.

Ch1:

What is ctrl eng?

Ex: Automated highway

Assume:

- car only moves in x-dir
- we can directly assign velo.
- we have censors to measure in front.

car0 moves at the speed lim. $v_0 > 0$

Obj: set the velo. of car1 to maintain a safe dist $d_{safe} > 0$ away from car_0

$$u = V_0 + k(d_{safe} - d)$$

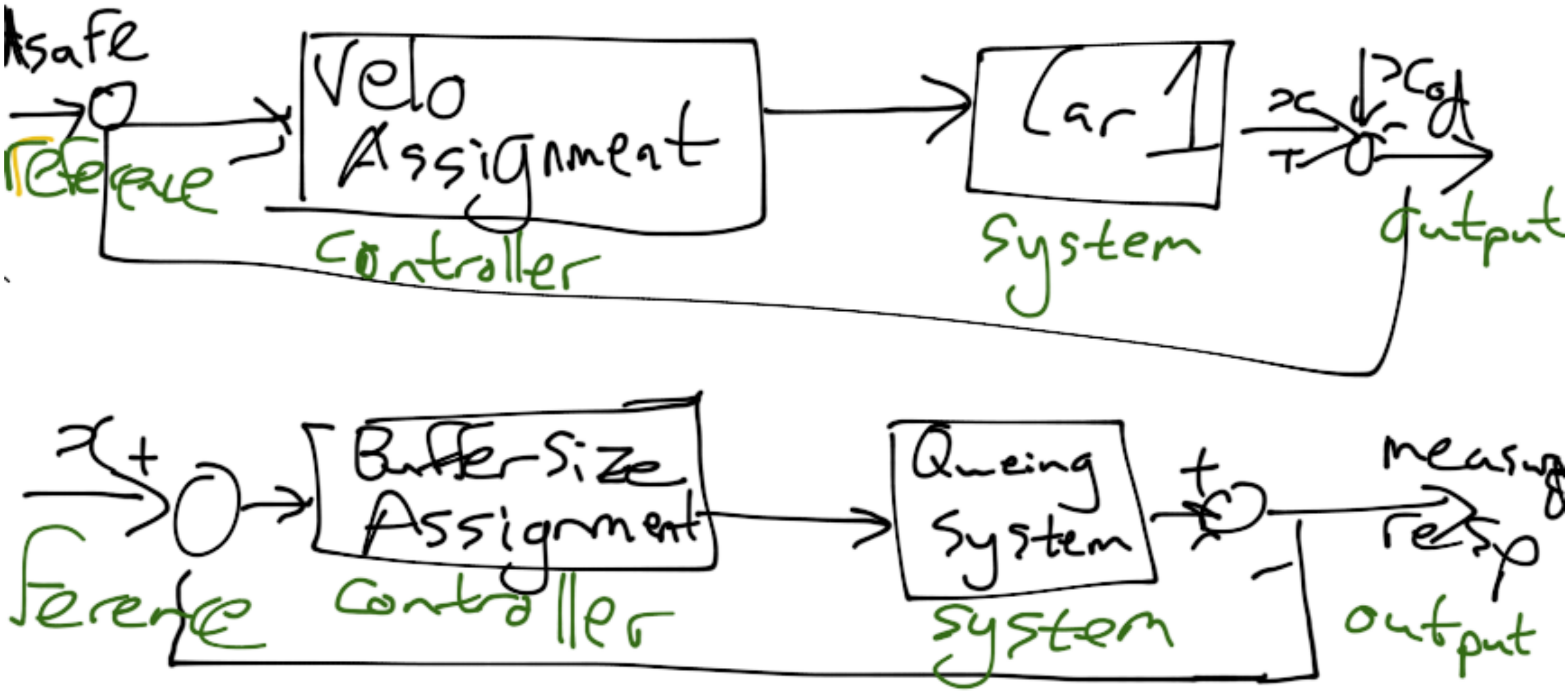
in this example, any $k<0$ works.

Ex: Queing System.

Assumption:

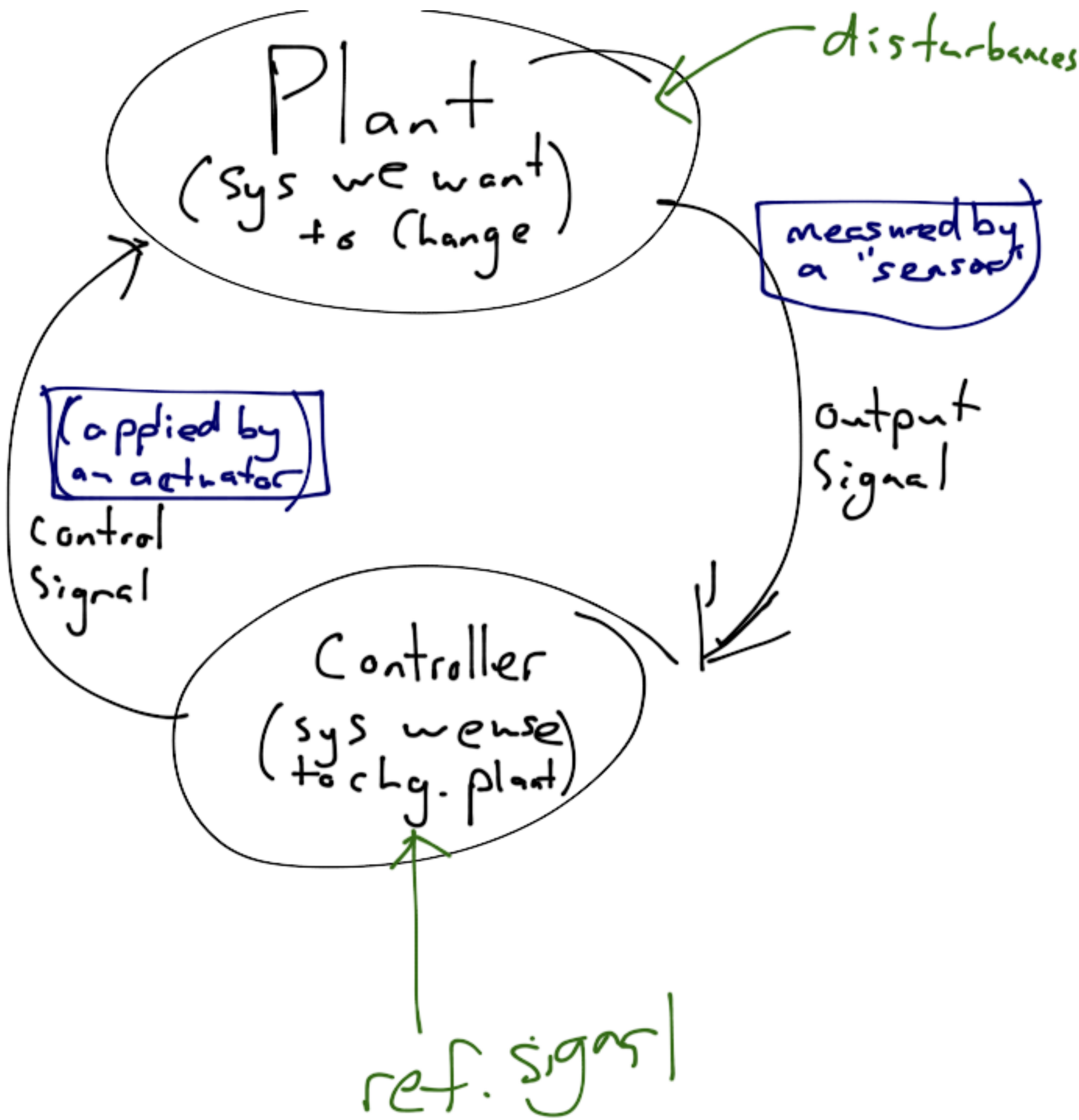
- requests that don't fit buffer are redirected.
- two types of req's: buying & browsing.

Obj: Assign the buffer size to max the # of completed requests while ensuring "buying" req's are serviced in $x > 6$ seconds or less



safe
reference
Velo Assignment
controller
Car 1
system
output
Buffer Size Assignment
controller
Queing System
system
+
-
measured resp
output

system = plant | controller


Plant
(Sys we want to Change)
disturbances
measured by a "sensor"
output Signal
(applied by an actuator)
control Signal
Controller
(sys we use to chg. plant)
ref. signal

Ctrl Eng Design Cycle:

1. get specs:
   - closed-loop stability
   - good steady-state tracking
   - disturbance rejection
   - transient performance.
2. Model the plant
   - "model" = mathematical Model
   - typically ≥ 1 differential eq'ns

$$\frac{dx(t)}{dt} = u(t)$$

   - experiments are oft used to det the numerical values of plant params → "system identification"

3. <u>Obtain transfer f'n of plant</u>

Classical control (this course, PID controllers) requires we have a transfer function for the plant.

e.g. car:

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \mathcal{L}\left(\frac{u(t)}{dt}\right) \Rightarrow sX(s) = U(s)$$

$$\Rightarrow \frac{X(s)}{U(s)} = \frac{1}{s} = P(s)$$

P(s) is the transfer function of the plant.

N.B.: For discrete time, use Z-transforms instead (not on course, but...)

not on course

4. Design the controller:

- Controller is also a transfer f'n:
- It corresp. to the D.E that relates the control signal to the output signal.

Can check PID controller:

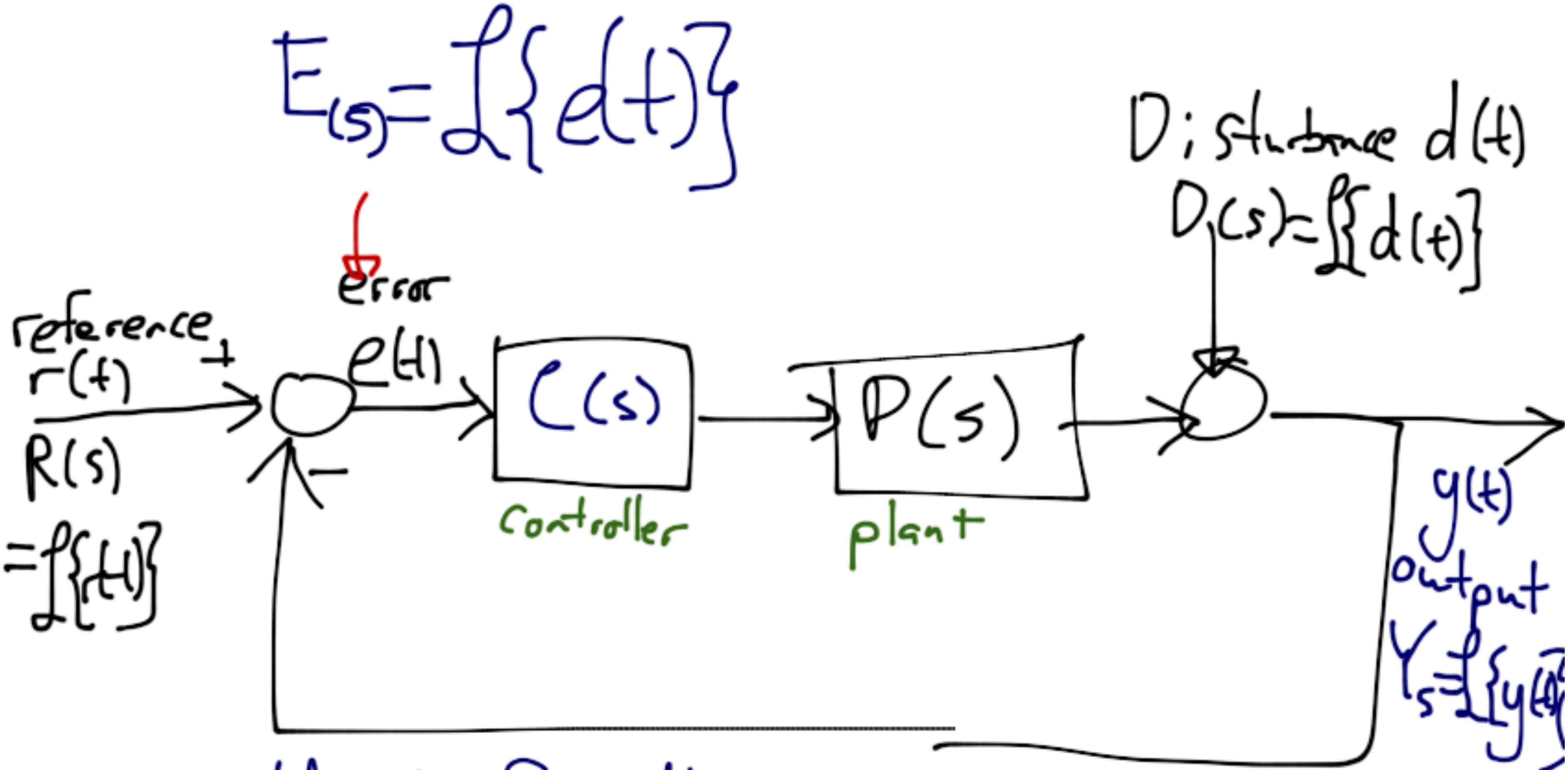
$$C(s) = \underset{(P)}{-2} \quad \underset{(I)}{-2/s} \quad \underset{D}{-\frac{s}{s+0.1}}$$

5. Simulation

6. Implement Controller

- Can build a sys w/transfer f'n matches the one from step 4.
- Realistically the controller is implemented on a computer as a difference eq'n. (ECE481)

Basic Unity Feedback



Unity feedback control system.

Ch 2: Mathematical Models of Systems

For Controller design we want a "good" model ⇒ simple & effective

2.2: Comments on Modelling

→ A model is a set of equations used to represent a physical system.
⇒ Never perfect

→ models for design are usually simpler rather than accurate. But most are a tradeoff.

→ There are two modelling approaches:
- → statistical (experimental)
- → Analytical (science + theory)

→ Use a modular approach - break pbm into sub-sections

→

Apply "physics" → "linearize about an "operating" plant" → Take Laplace

Sys of ODEs

Virtually all are linear

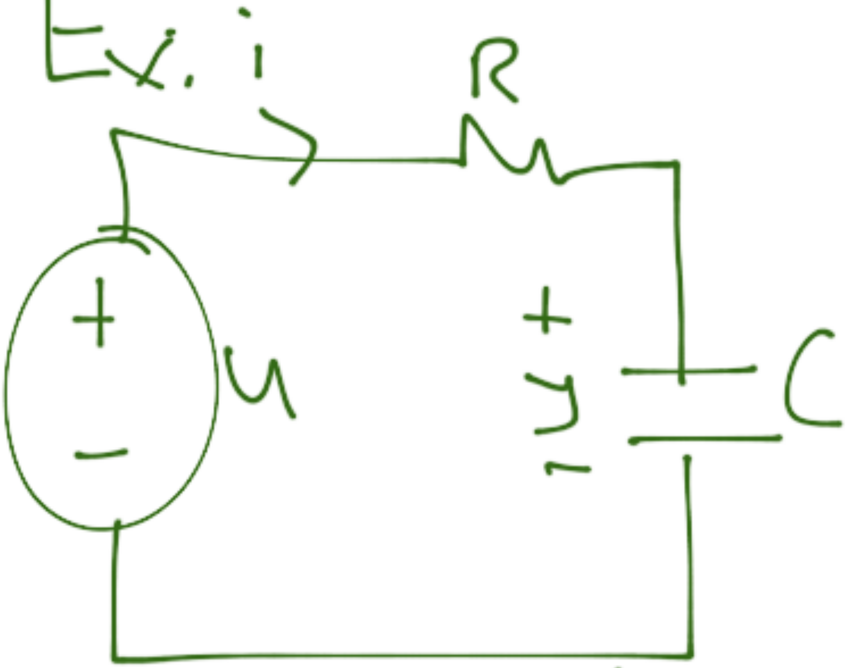
Sys of linear ODEs

system of algebraic eq'ns.

Solve For Relationship between I/O → Experimentally det. params in TF

Transfer F'n.

Ex.



RC Circuit.

Ways to model Systems:

(a) Linear ODE:

$$-u + V_R + V_c = 0$$

$$V_R = Ri, \quad V_c = y, \quad i = C\frac{dV_c}{dt}$$

$$\boxed{\Rightarrow -u + RC\frac{dy}{dt} + y = 0}$$

$$\dot{y} = \frac{dy}{dt} \text{ (derp)}$$

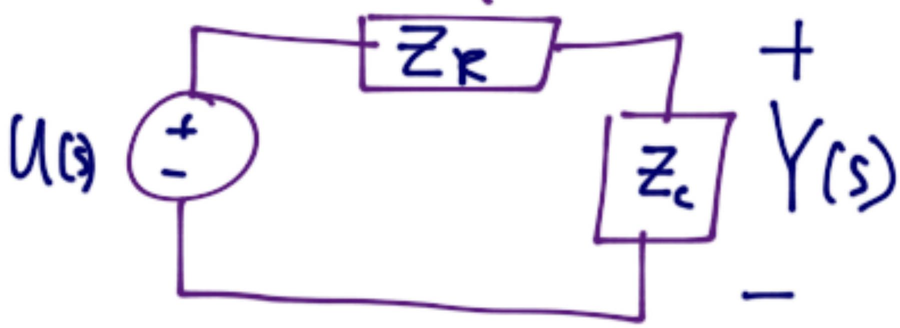
b) Transfer Fn:

i) L.T of ODE (assume 0 initial cond't)

$$-U(s)+sRCY(s)+Y(s)=0$$

$$\Rightarrow \boxed{\frac{Y(s)}{U(s)}=\frac{1}{RCs+1}}$$

ii) use impedence and the voltage divider rule.



$$\left.\begin{array}{l} Z_R=R \\ Z_C=\frac{1}{Cs} \end{array}\right\} \text{impedance.}$$

$$Y(s)=\frac{Z_c}{Z_c+Z_R}U(s) \quad \text{(voltage divider Rule)}$$

$$\Rightarrow Y(s)=\frac{1}{RCs+1}U(s)$$

Circuits Review:

$$V = \frac{I}{R}$$

KCL: $\sum V_{loop} = 0$

KVL: $\sum I_{node} = 0$

	Voltage	Current	Laplace
C	$v(t)$	$C\frac{dV}{dt}$	$V_{(s)} = \frac{I(s)}{sC}$
L	$L\frac{dI}{dt}$	$I(t)$	$V_{(s)} = sLI(s)$
R	IR	$\frac{V}{R}$	$V_{(s)} = RI(s)$

self-added

c) Convolution:

Let $G(s) = \frac{1}{RCs+1}$,

then $g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{RC} e^{-\frac{t}{RC}}$

the whole sys output can be expressed as:

$$y(t) = g(t) * u(t) = \underbrace{\int_0^t g(t-\tau)u(\tau)d\tau}_{\text{convolution}}$$

(d) Bode Plot:

→ next chapter

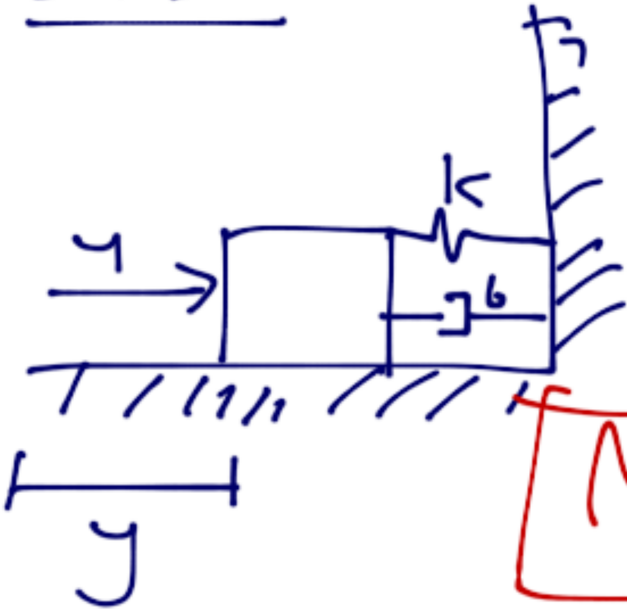
(e) state model:

→ 1+ 1st order ODEs.

$$\boxed{\begin{aligned} \dot{x} &= \frac{-1}{RC}x + \frac{1}{RC}u \\ y &= x \end{aligned}}$$

(x = voltage on capacitor)

Ex1:



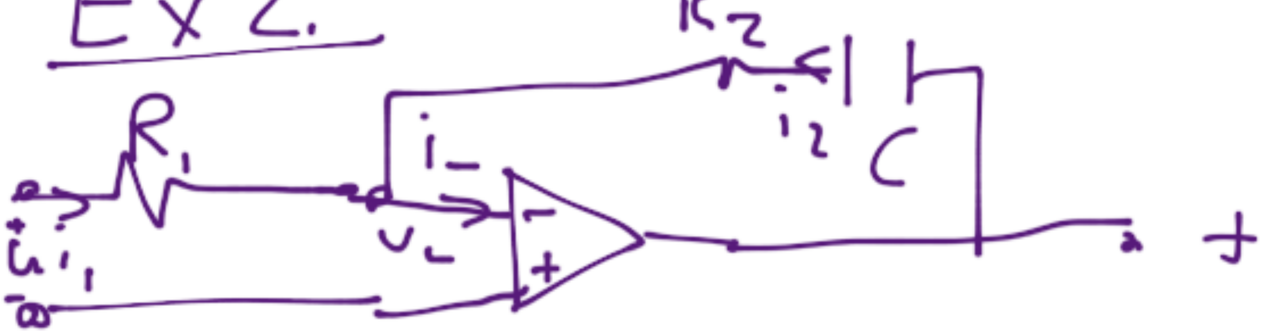
Newtons

$M\ddot{y} = \sum \text{Forces}$

$= u - F_{spring} - F_{damper}$

$$\boxed{M\ddot{y} = u - ky - b\dot{y}}$$

Ex2:



Ideal op-amp

$i_- = 0, \quad v_- = v_+$

KCL: $i_1 + i_2 = 0$,

using impedences:

$$0 = \frac{U(s) - V_-(s)}{Z_{R_1}} + \frac{Y - V_-(s)}{Z_{R_2} + Z_C}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = -\frac{R_2}{R_1} - \frac{1}{R_1 C s}$$

Summary:

The main point of Modeling is to obtain an approx model of the system through analysis & experiments.

A system can be expressed in multiple ways:

ODE

$\updownarrow$

Transfer Function

$\updownarrow$

Convolution

$\updownarrow$

State Model

State x at time $t=t_0$, $x(t_0)$ encapsulates all the system dynamics up to time t_0.

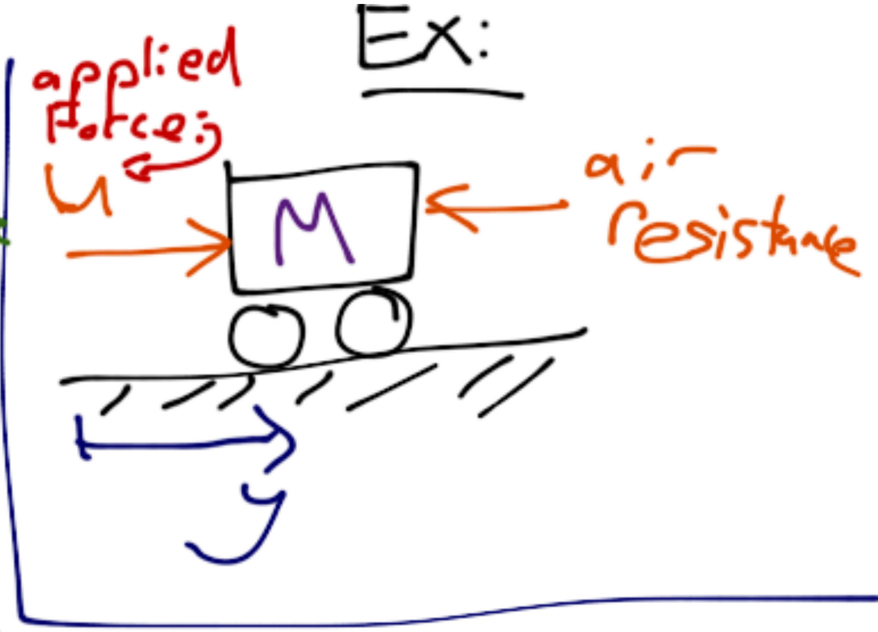
For any times t_0, t_1, $t_0 < t_1$, knowing $x(t_0)$ and knowing the applied control $\{u(t) : t_0 \leq t \leq t_1\}$ we can compute $x(t_1)$ and hence $y(t_1)$

State Models:

Typically air resistance creates an air force that depends on $\dot{y}$. Let's say this force is a possibly nonlinear function $D(\dot{y})$

eg: $D(\dot{y}) = \dot{y}^2$

Ex:



Newton's 2nd Law: $M\ddot{y} = u - D(\dot{y})$

Can we find a Transfer Function?

No. Not if $D(\dot{y})$ is non-linear.

Let's Def the State Variables:

$x_1 = y,\ x_2 = \dot{y},\ \underbrace{x}_{\text{state of system}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$

System Of equations:

$$\left.\dot{X} = \begin{bmatrix} X_2 \\ \frac{1}{M}u - \frac{1}{M}D(x_2) \end{bmatrix}\right\} \text{state equations}$$

$$\left.y = x_1\right\} \text{output eq'n}$$

Sometimes "u" too.

of the form $\dot{X} = f(x, u),\ y = h(x)$

Idea: in the case that $D(x_2)$ is not linear, we define a system of equations that we can solve.

System of eq'ns: $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = f(x, u)$

$y = h(x, u)$

const (not a function)

useful for the case that

$\dot{y}^a$ (or others) have $a > 1$ $\mathcal{L}$ doesn't work for these functions and we need to do it this way.

In the case that $D(x_2)$ is linear,
e.g.: $D(x_2) = d x_2$ (d is a const)

then,

$$\dot{x} = f(x,u) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \frac{-d}{M} \end{bmatrix}}_{\frac{\text{constant}}{\text{matrix}}\ \text{"}A\text{"}} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{\frac{\text{constant}}{\text{matrix}}\ \text{"}B\text{"}} u$$

so: $f(x,u) = Ax + Bu = \dot{x}$

In general, we have:

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

$C = [0 \ \ 1]$;

Linear Time invariant state mode

$$\text{if } \begin{bmatrix} a & b \\ c & d \end{bmatrix} x = \begin{bmatrix} a x_1 + b x_2 \\ c x_1 + d x_2 \end{bmatrix}$$

The idea is that many states have the model

$$\dot{x} = f(x,u), \quad f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

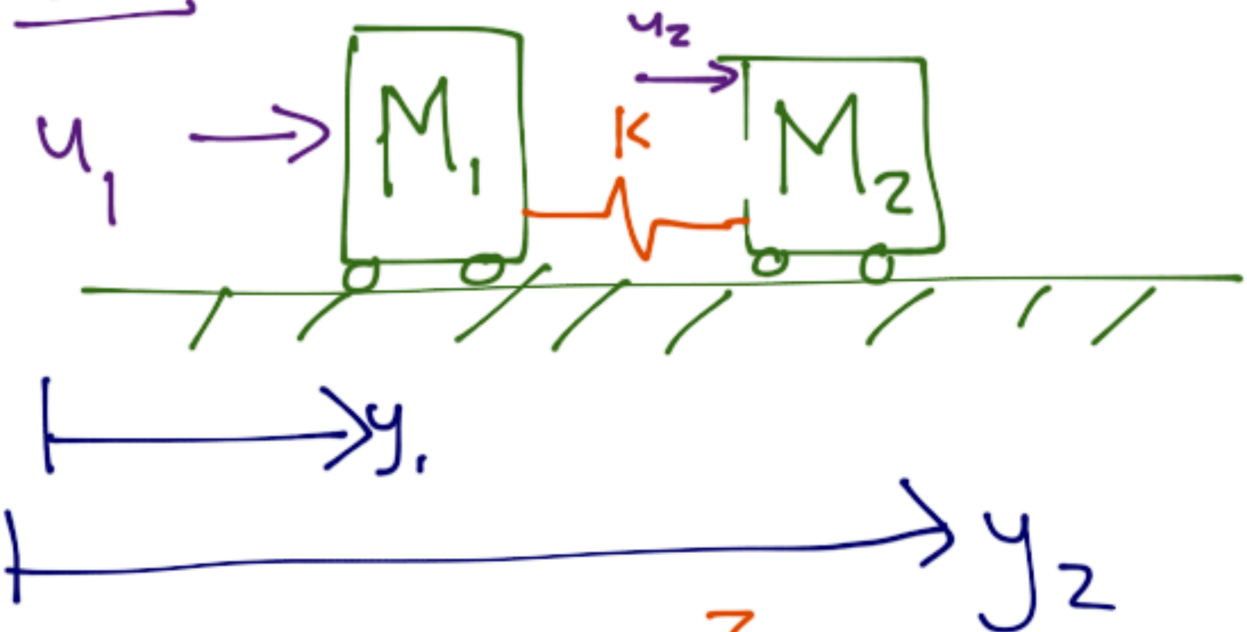
$$y = h(x,u), \quad h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$$

output (y) dimension p

input (u) dimension m

state (x) dimension n

Ex:



$u = (u_1, u_2) \in \mathbb{R}^2 \Rightarrow m = 2$

$y = [y_1, y_2] \in \mathbb{R}^2 \Rightarrow p = 2$

$x = [y_1, \dot{y}_1, y_2, \dot{y}_2] \in \mathbb{R}^4 \Rightarrow n = 4$

the linear time-invariant special-case:

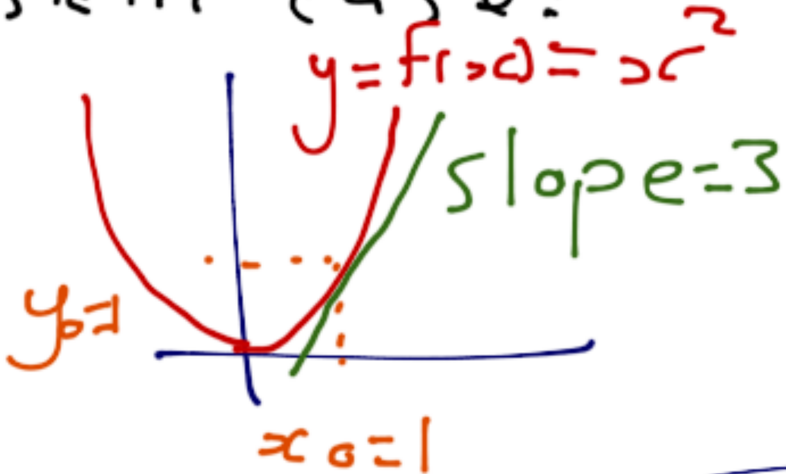
$\dot{x} = Ax + Bu$ $\quad A \in \mathbb{R}^{n \times n}$ $\quad B \in \mathbb{R}^{n \times m}$

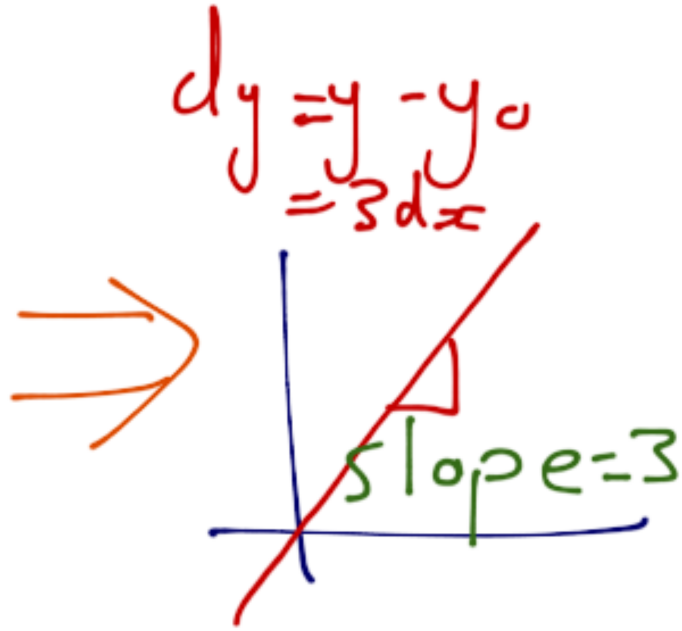
$y = Cx + Du$ $\quad C \in \mathbb{R}^{p \times n}$ $\quad D \in \mathbb{R}^{p \times m}$

in this course we <u>only</u> deal w/ time-invariant systems.

Summary (Add J Wei's notes Here)

- States:
- linearization
  - static case:





  - vector case — $dy = \left.\frac{df}{dx}\right|_{x=x_0} dx$
  - Apply this to: $\dot{x} = f(x,u)$
    $y = h(x,u)$

First assume (x_0, u_0) is an eq'bm

$$f(x_0, u_0) = 0$$

$$f(x,u) = f(x_0,u_0) + \underbrace{\left.\frac{df}{dx}\right|_{\substack{x=x_0 \\ u=u_0}}}_{A}(x-x_0) + \underbrace{\left.\frac{df}{du}\right|_{\substack{x=x_0 \\ u=u_0}}}_{B}(u-u_0)$$

+ higher order terms

Now consider a solution "near" the constant solution

$$x(t) = x_0 + \delta x(t)$$
$$u(t) = u_0 + \delta u(t)$$

$$\dot{\delta x} = \frac{d(\delta x(t))}{dt} = \frac{dx(t)}{dt} - \frac{d(x_0)}{dt}$$

$$= f(x,u) - 0$$

$$\approx f(x_0, u_0) + A\delta x + B\delta u$$

$$= A\delta x + B\delta u$$

$$\Rightarrow \dot{\delta x} = A\delta x + B\delta u$$

We can linearize the output eq'n as in the previous example.

$$\delta y = \underbrace{\left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_0\\u=u_0}}}_{C} \delta x + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{\substack{x=x_0\\u=u_0}}}_{D} \delta u$$

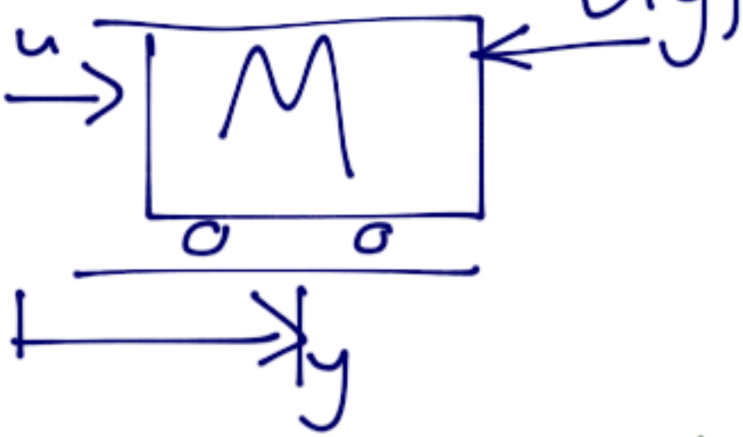
Summary: Linearization

1) Select, if one exists an equilibrium (x_0, u_0)
2) Compute A, B, C, D (Jacobians)
3) The linearized system is:

$$\delta \dot{x} = A\delta x + B\delta u$$
$$\delta y = C\delta x + D\delta u$$

4) Under mild conditions the linearization is a good approx for "small" δx and δu.

Ex:



Cart example @ position $y_0 = 10$

$D(\dot{y}) = k\dot{y}^2,\ K > 0$

i) The model is $\dot{x} = f(x,u) = \begin{bmatrix} x_2 \\ \dfrac{-kx_2^2}{M} + \dfrac{u}{M} \end{bmatrix}$

$y = h(x,u) = x_1$

We need an operating point (x_0, u_0) corresponding to $y = x_1 = 10$.

Solve: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = f(x_0, u_0) = \begin{bmatrix} x_{20} \\ \dfrac{-kx_{20}^2}{M} + \dfrac{u_0}{M} \end{bmatrix}$

$\Rightarrow x_{20} = 0,\ u_0 = 0$

$\Rightarrow (x_0, u_0) = \left(\begin{bmatrix} 10 \\ 0 \end{bmatrix}, 0 \right)$

2.)

$$A = \frac{\delta f}{\delta x}\bigg|_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-2Kx_2}{M} \end{bmatrix}_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 0 & 1 \\ 0 & d \end{bmatrix}$$

$$B = \frac{\delta f}{\delta u}\bigg|_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} \frac{\delta f_1}{\delta u} \\ \frac{\delta f_2}{\delta u} \end{bmatrix}_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

$$C = \frac{\delta h}{\delta x}\bigg|_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} \frac{\delta h}{\delta x_1} & \frac{\delta h}{\delta x_2} \end{bmatrix}_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = \frac{\delta h}{\delta u}\bigg|_{\substack{x=x_0\\u=u_0}} = 0$$

3) Linear Model

$$\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \delta u$$

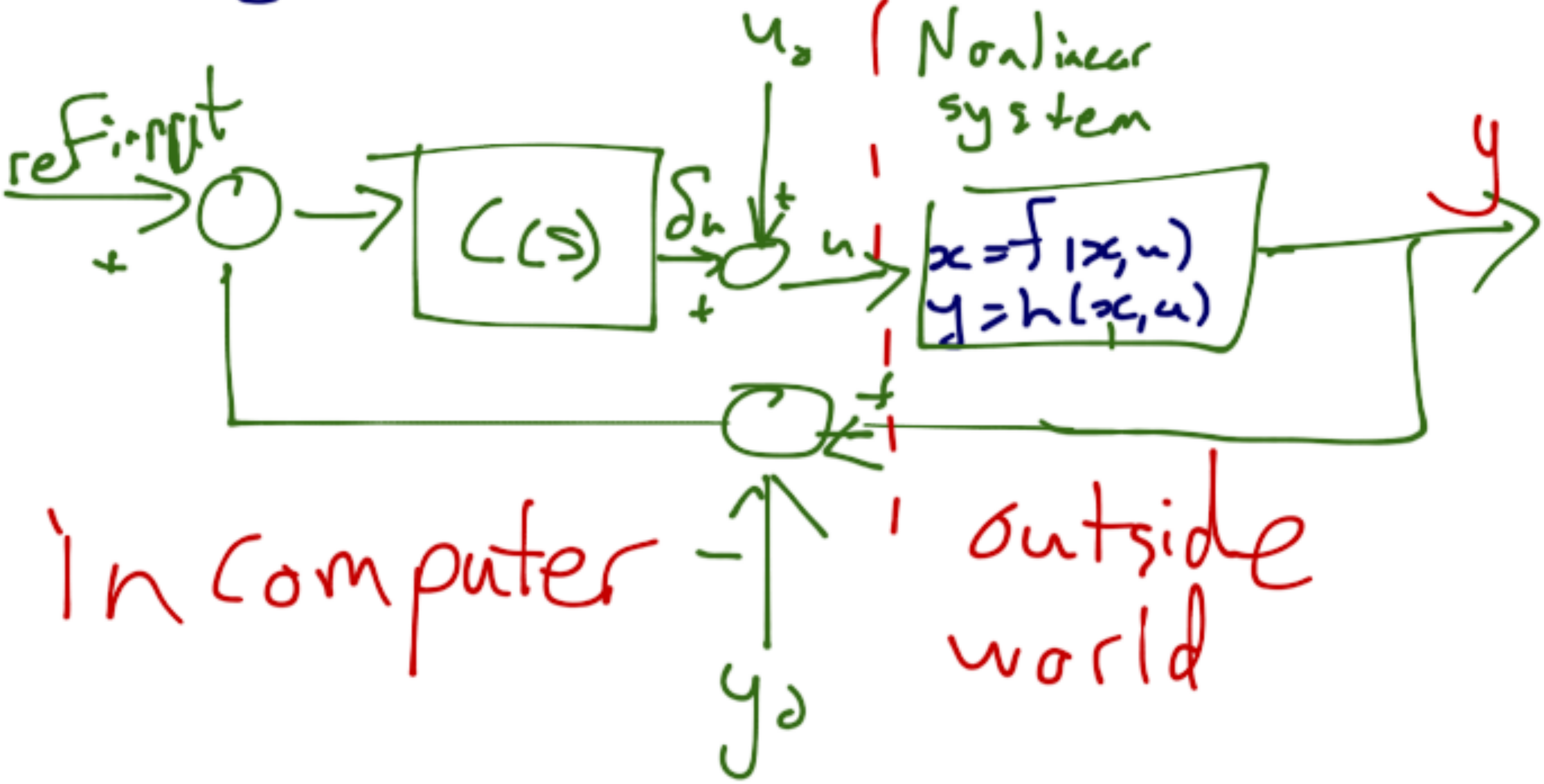
$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

$y = 10$

$u = 0$

$\dot{y} = 0$

Implementing a controller designed using a linearized model:

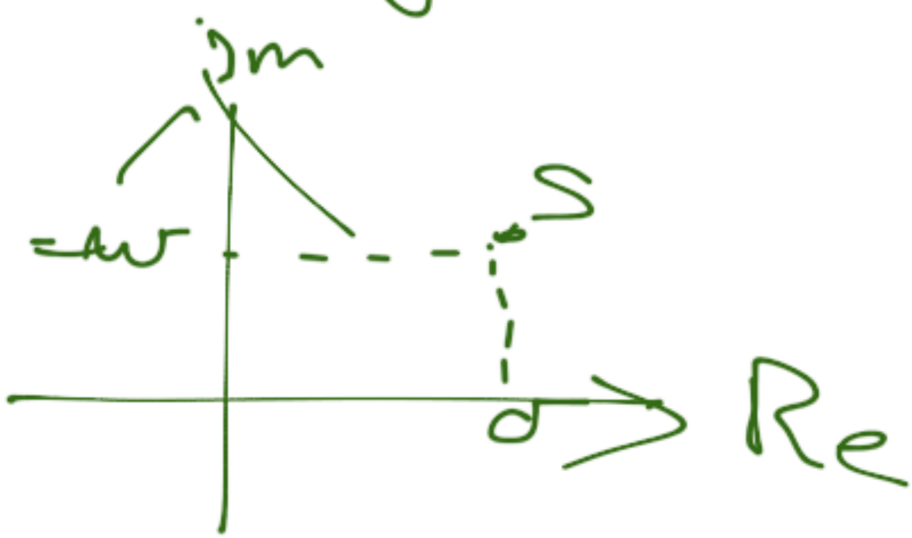


2.8 Laplace transform

Let $f(t)$ be a signal (function) def'd for $t \geq 0$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

here, $s = \sigma + j\omega$



Intuition:

$F(s)$ is a decomposition of $f(t)$ into a weighted sum of complex exponentials

$$e^{-st} = e^{-\sigma t}(\cos\omega t - j\sin\omega t)$$

Summary

- linearize state models

$$\dot{x} = f(x,u) \qquad \delta\dot{x} = A\,\delta x + B\,\delta u$$
$$y = h(x,u) \longrightarrow \delta y = C\,\delta x + D\,\delta u$$

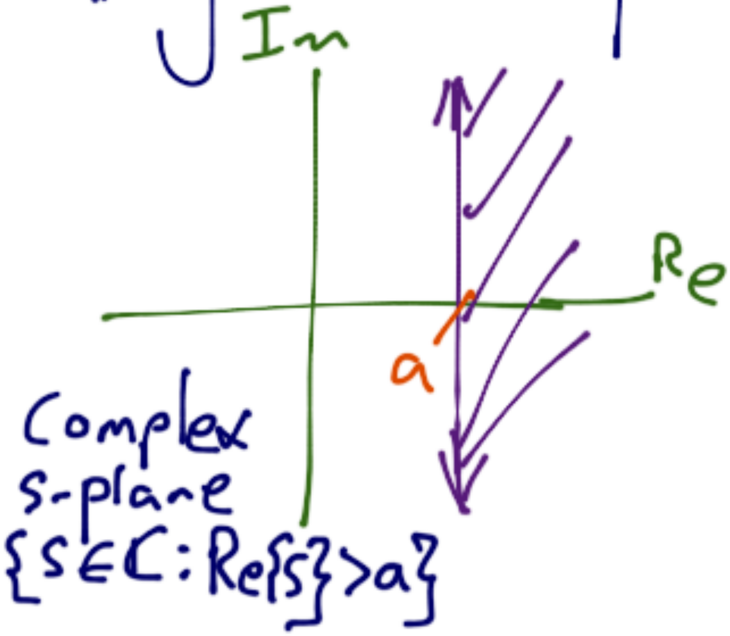
- Approximation is valid near the operating points (x_0, u_0)

$$f(x_0, u_0) = 0$$

– Laplace Transform: $F(s) = \mathcal{L}\{f(t)\} = \int_0 f(t)e^{-st}dt$

there's a region of convergence for some s. Where the region converges.

The $\overbrace{\text{Region of convergence}}^{R.O.C}$ is in the right-half plane:



$f(t) = e^{-at} \Rightarrow \dfrac{1}{s+a}$

$\Rightarrow ROC = Re\{s\} > a$

$f(t) = te^{-at} \Rightarrow \dfrac{1}{(s+a)^2}$

$\Rightarrow ROC = Re\{s\} > a$

$f(t) = e^{t^2} \Rightarrow$ ROC is empty & there is no $\mathcal{L}\{f\}$

LT $\mathcal{L}\{\}$

a is const, f,g are f'ns.

i) $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$

ii) $\mathcal{L}\{af\} = a\mathcal{L}\{f\}$

linear

iii) $\mathcal{L}\{\frac{df}{dt}\} = s\mathcal{L}\{f\} - f'(0)$ derivative

iv) $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$ convolution

v) $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}\mathcal{L}\{f\}$ integral

vi) if $\lim_{t\to\infty} f$ exists & is finite, then

$$\lim_{t\to\infty} f = \lim_{s\to 0} sF(s)$$

Final Value theorem (FVT)

ex: $\longrightarrow$

$f(t)$	$\lim_{t\to\infty} f(t)$	$\mathcal{L}\{f\}$	$\lim_{s\to 0} sF(s)$	FVT Applies?
e^{-t}	0	$\frac{1}{s+1}$	0	✓
$1(t)$ (heavy side)	1	$\frac{1}{s}$	1	✓
te^{-t}	0	$\frac{1}{(s+1)^2}$	0	✓
e^{t}	∞	$\frac{1}{s-1}$	0	✗
$\cos\omega t$ (const)	DNE	$\frac{s}{s^2+\omega^2}$	0	✗

The main idea is that if $sF(s)$ is defined (no div by 0) for any s with $\mathrm{Re}\{s\}>0$, the FVT applies.

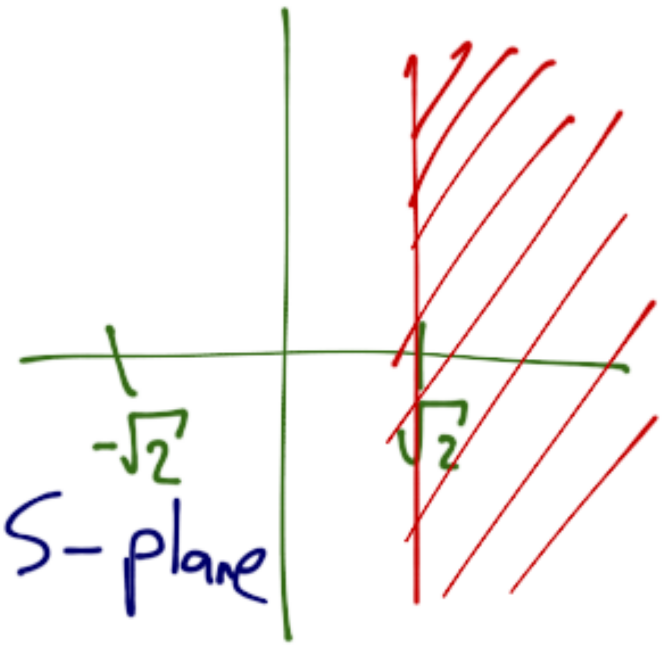
If $s=-j\omega$, the $\mathcal{L}\{f\}$ dne and we cannot apply the final value theorem.

We can use PFE (partial fraction expansion when)

$$F(s) = \frac{n(s)}{d(s)} \Big\} \text{polynoms}$$

$$\rightarrow \deg(d) > \deg(n)$$

Eg: $F(s) = \dfrac{3s+17}{s^2-2}$ find $f(t)$



$$F(s) = \frac{C_1}{s+\sqrt{2}} + \frac{C_2}{s-\sqrt{2}}$$

$$\Rightarrow C_1 = \frac{23}{4}, \quad C_2 = \frac{-11}{4}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{C_1}{s+\sqrt{2}}\right\} + \mathcal{L}^{-1}\left\{\frac{C_2}{s-\sqrt{2}}\right\}$$

$$= \frac{23}{4} e^{-\sqrt{2}t} - \frac{11}{4} e^{\sqrt{2}t}, \quad t \geq 0$$

Ex: $F(s) = \frac{s+1}{s(s+2)^2} \Rightarrow f(t) = ?$ $\boxed{2.3}$

$$F(s) = \frac{c_1}{s} + \frac{c_2}{s+2} + \frac{c_3}{(s+2)^2} = \frac{s+1}{s(s+1)^2}$$

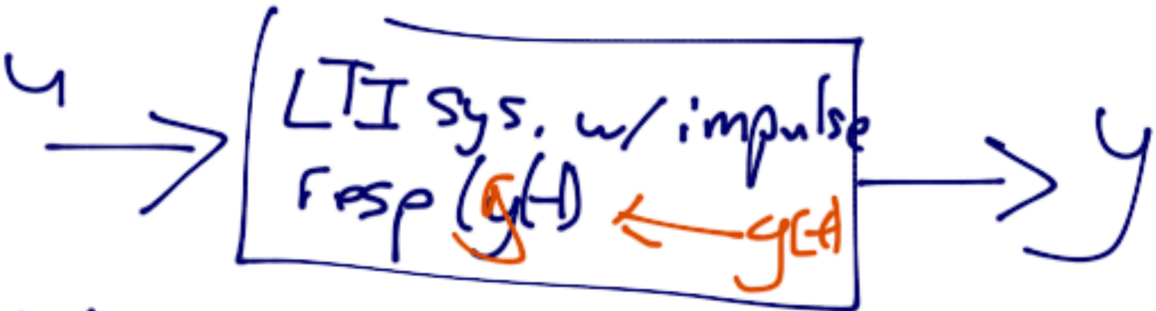
$$\left.\begin{aligned} c_1 &= 1/4 \\ c_2 &= -\frac{1}{4} \\ c_3 &= 1/2 \end{aligned}\right\} f(t) = \frac{1}{4} + -\frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

The idea is that LT are useful for solving initial value functions, like

$$\dot{y} - 2y = t, \quad y(0) = 1$$

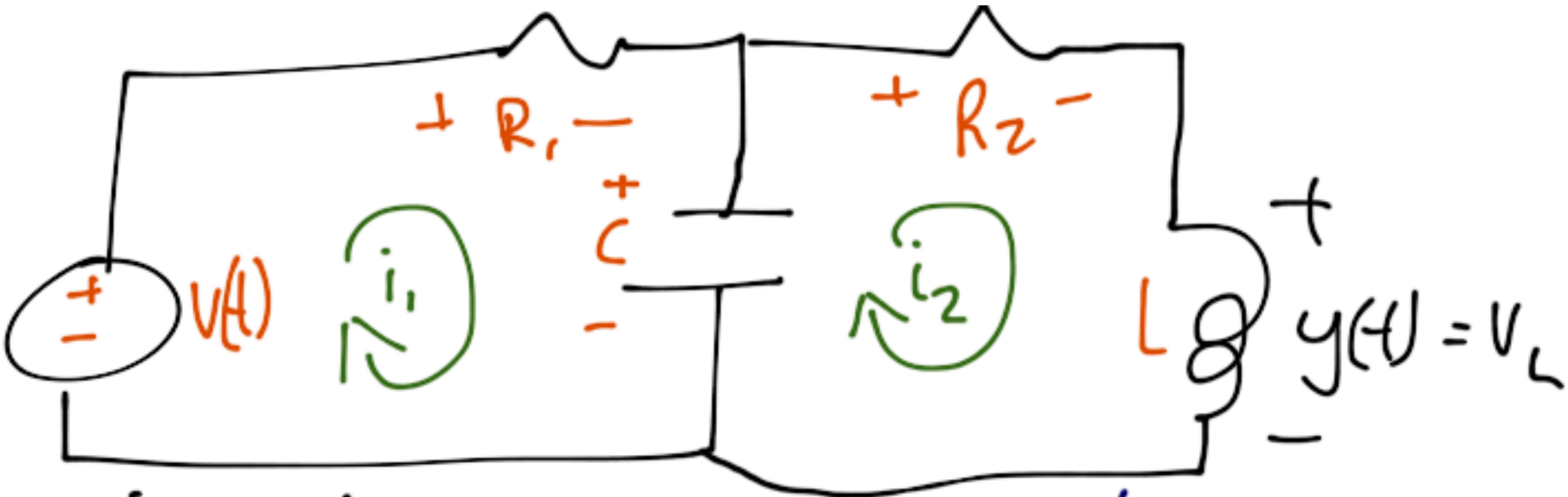
2.4 Transfer Functions:

Consider a linear time-invariant system (LTI) with response $g(t)$



You should have seen

$$y(t) = g(t) * u(t)$$



Loop 1

$$V_c = \frac{1}{C}\int_0^t i_1(\tau) - i_2(\tau)\,d\tau$$

$$-u + R_1 i_1 + V_C = 0 \quad (1)$$

Loop 2

$$-V_C + R_2 i_2 + V_L = 0 \quad (2)$$

$$V_L = L\frac{di_2}{dt} = L\dot{x}_2 \quad (3)$$

b) State Model

X_1 = Voltage accross C
X_2 = current through L
y = voltage accross L
u = applied voltage.

$\dot{x} = f(x,u)$

$y = h(x,u)$

Sub 3: $x_1 = \frac{1}{C}\int_0^t (i_1 - x_2)\,d\tau$

$\dot{x}_1 = \frac{1}{C}(i_1 - x_2)$ (4)

rearr. 1: $i_1 = \frac{1}{R_1}u - \frac{1}{R_1}x_1$ (5)

Sub (5)→(4): $\boxed{\dot{x} = \frac{-x_1}{R_1 C} - \frac{x_2}{C} + \frac{u}{R_1 C}}$

then, $V_L = V_C - R_2 i_2 \Rightarrow \boxed{y = x_1 - R_2 x_2}$

$$\boxed{y = x_1 - R_2 x_2}$$

$$\boxed{\dot{x} = \frac{-x_1}{R_1 C} - \frac{x_2}{C} + \frac{u}{R_1 C}}$$

$A, B, C, D = ?$

$$\dot{x} = f(x,u) = Ax + Bu$$

$$y = h(x,u) = Cx + Du$$

$$A = \frac{\delta f(x,u)}{dx} = \begin{bmatrix} \frac{\delta f_1}{dx_1}, & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1}, & \frac{\delta f_2}{\delta x_2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_1 C}, & \frac{-1}{C} \\ \frac{1}{L}, & \frac{-R_2}{L} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\delta f_1}{\delta u_1} \\ \frac{\delta f_1}{\delta u_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 C} \\ 0 \end{bmatrix}$$

$$C = [0, -R_2], \quad D = [0]$$

7b $^{(A_2)}$ Linearize:

$y_0 = f(x_0)$

$y = f(x)\ x^T A x + b^T x$ at $x = x_0$

$$y = f(x) \simeq f(x_0) + \frac{\delta f}{\delta x} \underbrace{(x - x_0)}_{\delta x}$$

$$\underbrace{y - f(x_0)}_{\delta y} = \frac{\delta f}{\delta x} \delta x$$

For $x, z \in \mathbb{R}^n$

$$\frac{\delta x^T z}{\delta x} = \frac{\delta z^T x}{\delta x}$$

$$[x_1 \ \ x_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = x_1 z_1 + x_2 z_2$$

fact 1

Summary

- Radius of convergence for $\mathcal{L}$
  → at $S \geq$ imaginary of $\mathcal{L}$
- properties of L.T (FVT)
- inverse Laplace Transform (pfe)
- transfer functions.

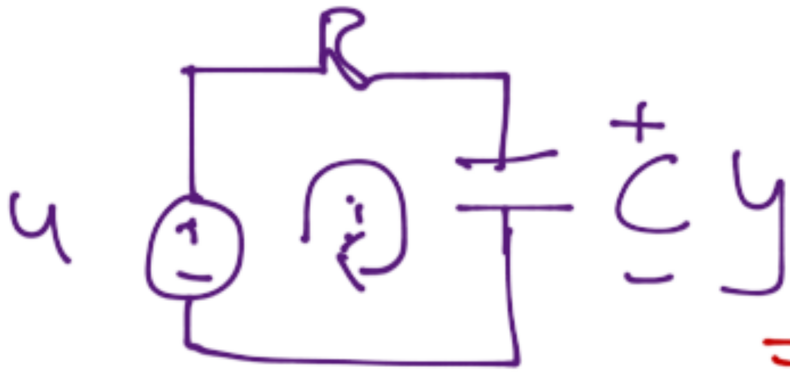
$$y(t) = g(t) * u(t)$$

$$\Big\downarrow \mathcal{L}$$

$$Y(s) = G(s)\,U(s)$$

$$\Big\downarrow \text{Transfer function} \quad \left(\text{ratio of } \frac{\mathcal{L}(y)}{\mathcal{L}(u(t))}\right)$$

$$G(s) = \frac{Y(s)}{U(s)}$$



$$-u + Ri + y = 0$$

$$i = C\frac{dy}{dt}$$

$$\Rightarrow -u + RC\dot{y} + y = 0$$

take $\mathcal{L}$ w/ $y(t_0) = 0$:

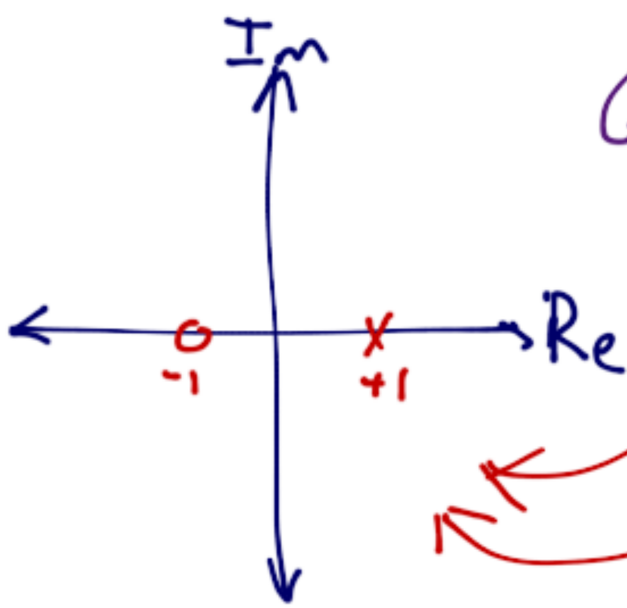
$$-U(s) + sRCY(s) + Y(s) = 0$$

$$\Rightarrow \frac{Y(s)}{U(s)} = G(s) = \frac{1}{RCs+1}$$

$G(s)$	type	Circuit
1	pure gain	$u(t) \to 1 \to y(t)$
$1/s$	integrator	$u(t) \to \boxed{\int} \to y(t)$
$1/s^2$	double integrator	$u(t) \to \boxed{\int} \to \boxed{\int} \to y(t)$
s	differentiator	$u(t) \to \boxed{\frac{d}{dt}} \to y(t)$
e^{-sT}	[$T>0$, constant T second delay]	$u(t) \to \boxed{e^{-sT}} \to y(t) = u(t-T)$

Terminology: $G(s)=\dfrac{n(s)}{d(s)}$

1) rational $\Leftrightarrow$ n & d are polynomials
2) proper $\Leftrightarrow$ $\deg(d) \geq \deg(n)$
3) strictly proper $\Leftrightarrow$ $\deg(d) > \deg(n)$
4) improper $\Leftrightarrow$ $\deg(n) > \deg(g)$
5) Poles are roots of $d(s)$
6) zeroes are roots of $n(s)$



$$G(s)=\frac{s+1}{s-1}$$

o zeroes: $s \in \{-1\}$

X Poles: $s \in \{1\}$

Obtaining the T.F. from linear state model:

2.9.1

$\dot{x} = Ax + Bu \quad y = Cx + Du \quad x \in \mathbb{R}^n$

Take Laplace Transforms w/ zero Initial Coordinates.

$$sX(s) = \begin{bmatrix} sX_1(s) \\ \vdots \\ sX_n(s) \end{bmatrix} = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$\Rightarrow \Big(s\underbrace{\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}}_{I} - A\Big)X(s) = BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1} BU(s)$$

$$G(s) = (sI - A)^{-1} B + D \leftarrow \boxed{\text{The TF from } u \text{ to } Y}$$

In general, the idea is to get $G(s)$, a matrix $p \times m$ $(p = \dim Y,\ m = \dim u)$

Example: $\delta\dot{x} = \underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{A}\delta(x) + \underbrace{\begin{bmatrix}0\\ 1/M\end{bmatrix}}_{B}\delta u$

$$\delta y = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\delta(x)$$

Let $X(s) = \mathcal{L}\{\delta x\}$, $Y(s) = \mathcal{L}\{\delta y\}$, $u(s) = \mathcal{L}\{\delta u\}$

$$sX(s) = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}X(s) + \begin{bmatrix}0\\ 1/m\end{bmatrix}u(s)$$

$$X(s) = \begin{bmatrix}s & -1\\ 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1/m\end{bmatrix}u(s)$$

$$Y(s) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s & -1\\ 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\ \frac{1}{M}\end{bmatrix}u(s) = \frac{1}{M}\cdot\frac{1}{s^2}U(s)$$

Summary: (getting a TF)

1. use Theory/science to get eq'ns to model the evolution of the system.
   (in this course, these are nonlinear)

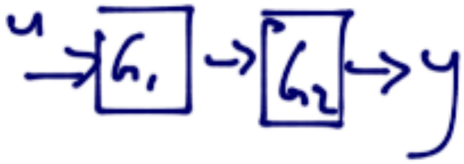
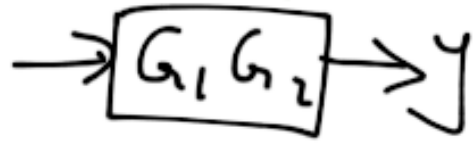
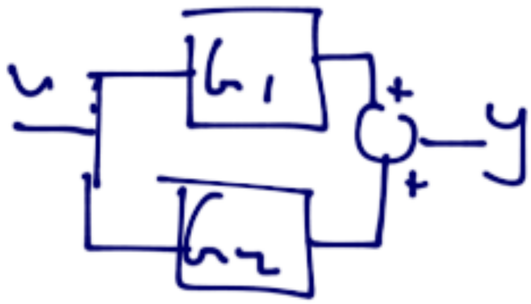
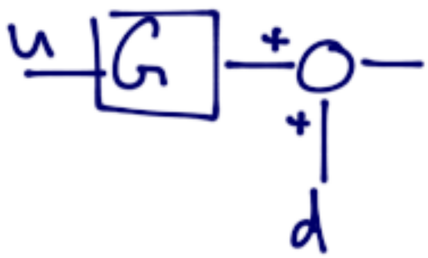
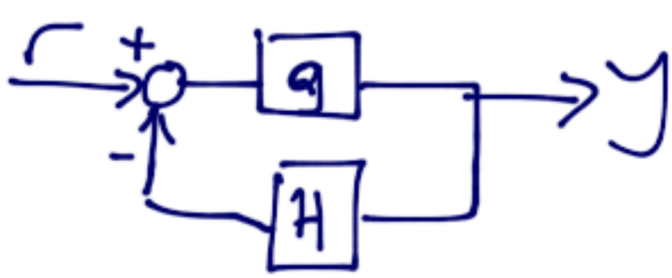
2. Find an equilibrium
   (i.e the ctrl objective)

3. Linearize about eqbm

4. Take LT of linearized eq'ns w/ zero initial conditions.

5. Solve for $Y(s)$ in terms of $U(s)$

2.10 Interconnections

name	from	To
series		
parallel		
moving blocks		
feedback		

$$X(s) = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} U(s)$$

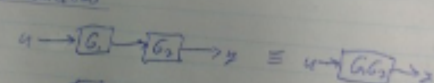
$$Y(s) = [1 \ \ 0] \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} U(s) = \underbrace{\left(\frac{1}{m} \cdot \frac{1}{s^2}\right)}_{G(s)} U(s)$$

Summary of Getting a Transfer Function

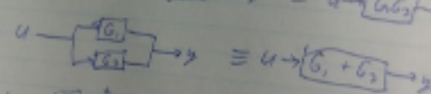
1. Use theory/science to get equations that model the evolution of your system. These are generally nonlinear. In this course they are DEs.
2. Find an equilibrium (usually depends on control objective).
3. Linearize about equilibrium
4. Take LT of linearized equations with zero initial conditions
5. Solve for $Y(s)$ in terms of $U(s)$.

2.10 Interconnections

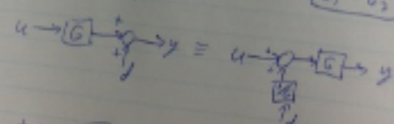
(i) Series



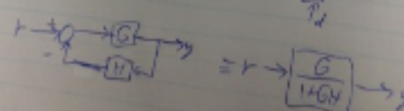
(ii) Parallel



(iii) Moving Block

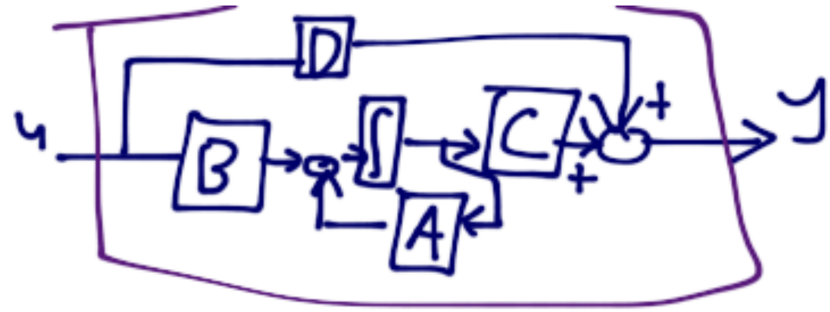


(iv) Feedback



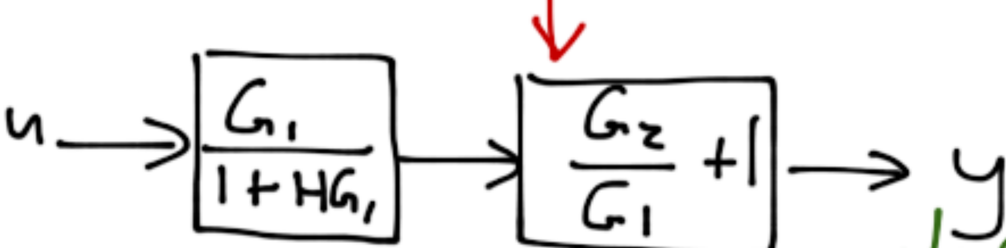
Summary:

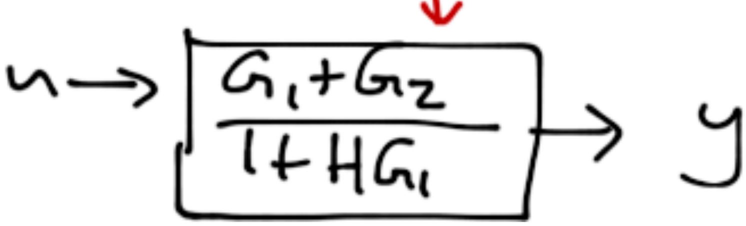
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$



$$\Rightarrow \frac{Y(s)}{u(s)} = G(s) = C(sI-A)^{-1}B + D$$

Ex: we try to move blocks around until we end up with "common" configurations



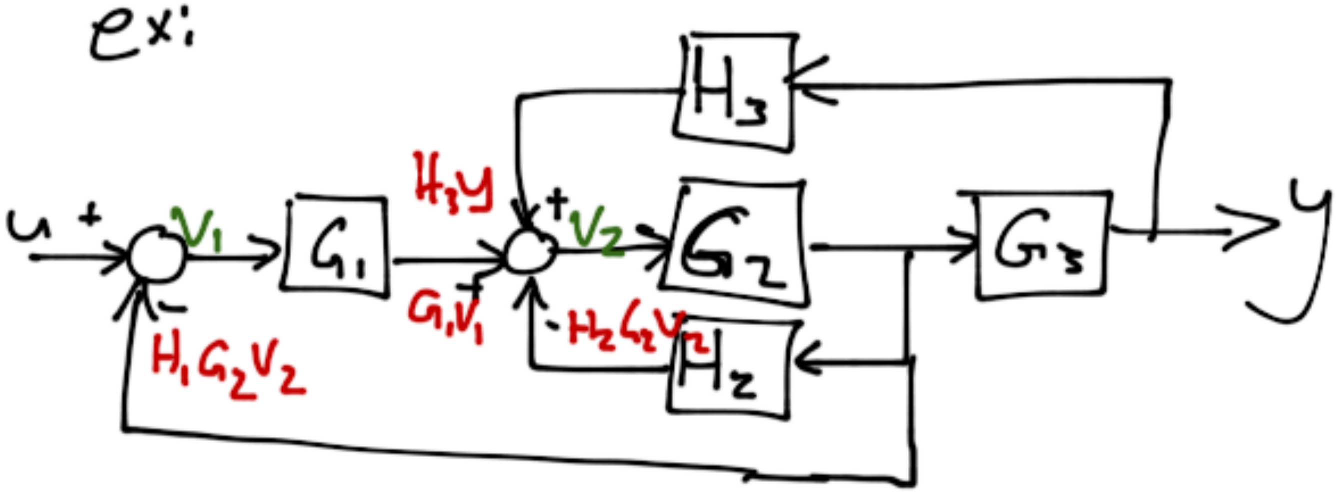


reduce to use known "shapes" to easily solve issues.

Procedure:

1. introduce variables representing the output of summing junctions. ($\to \bigcirc \to V_n$) $\{v_1, \dots, v_n\}$
2. Annotate inputs to the summers.
3. create eq'ns per summer.
4. solve all eq'ns into 1 eq'n in terms of U & Y.

ex:



$$V_1 = U - H_1 G_2 V_2$$

$$V_2 = H_3 Y + G_1 V_1 - H_2 G_2 V_2 \iff \begin{bmatrix} 1 & H_1 G_2 & 0 \\ 0 & 1 + H_2 G_2 & -H_3 \\ 0 & -G_3 G_2 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ y \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}$$

$$y = G_3 G_2 V_2$$

$$\Rightarrow Y(s) = \frac{\det \begin{bmatrix} 1 & H_1 G_2 & u \\ -G_1 & 1 + H_2 G_2 & 0 \\ 0 & -G_3 G_2 & 0 \end{bmatrix}}{\det \begin{bmatrix} 1 & H_1 G_2 & 0 \\ -G_1 & 1 + H_2 G_2 & -H_3 \\ 0 & -G_3 G_2 & 1 \end{bmatrix}} = \frac{G_1 G_2 G_3}{1 + G_2 H_2 - G_2 G_3 H_3 + G_1 G_2 H_1} U(s)$$

Ch 3: Linear-System Theory:

3.1 Bounded input & output stability.

$\dot{x} = Ax + Bu$
$y = Cx + Du$ or $\frac{Y(s)}{U(s)} = G(s)$ $\quad y(t) = g(t) * u(t)$

If $u(t)$ $(t \geq 0)$ is a real-valued signal, it is **bounded** if there exists a constant $b \geq 0$ s.t.

$$\forall t \geq 0 : |u(t)| \leq b$$

bounded:
$u(t) = 1(t)$
$u(t) = \cos(t)$

unbounded:
$u(t) = t$
$u(t) = e^{t}$

Def: BIBO: Bounded input/bounded output.

is bibo $\Leftrightarrow$ stable

Ex:

ex: $\dot{x} = -2x + u$, $y = x$ $\Rightarrow Y(s) = \frac{1}{s+2} U(s)$

$$u \longrightarrow \boxed{G} \longrightarrow y$$

$$g(t) = \mathcal{L}\{G(s)\} = e^{-2t}$$

so $$|y(t)| = \left| \int_0^t g(\tau) u(t-\tau)\, d\tau \right|$$

$$\le \int_0^t | \ldots |$$

$$\le \int_0^t e^{-2t} | \ldots |$$

$$\le \int_0^t e^{-2t} \max_{\lambda \ge 0} |u(\lambda)|$$

$$\le \frac{1}{2} \cdot \max_{\lambda \ge 0} |u(\lambda)|$$

every bounded input for this y creates a bounded output.

THM:

Assume $G(s)$ is rational and is strictly proper. Then the following occurs.

The system is BIBO stable.

$\Updownarrow$

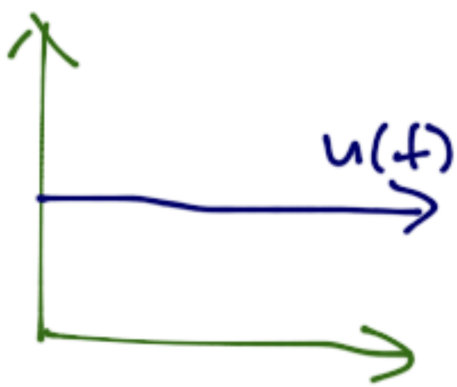
the impulse response $g(t) = \mathcal{L}^{-1}\{G\}$ is absolutely integrable.

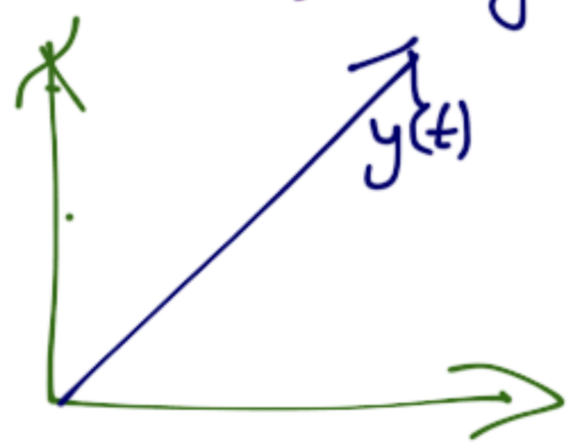
i.e: $\int_0^\infty |g(t)|\,dt < \infty$

i.e. $G(s) = \frac{1}{s}$ since G has a pole at $s=0$, by THM, G is unstable.

$u(t) = 1(t)$
$U(s) = \frac{1}{s}$

$Y(s) = G(s)U(s)$
$Y(s) = \frac{1}{s^2} \Rightarrow y(t) = t$

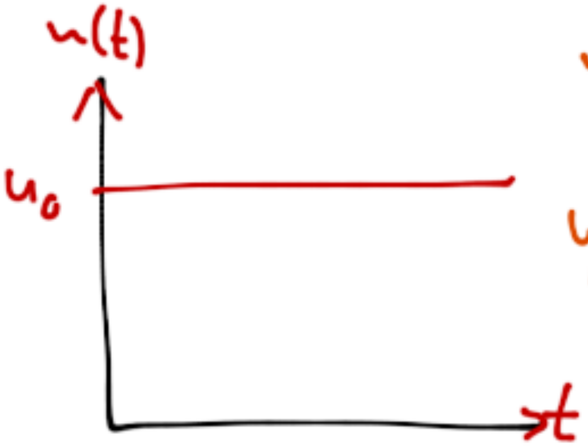




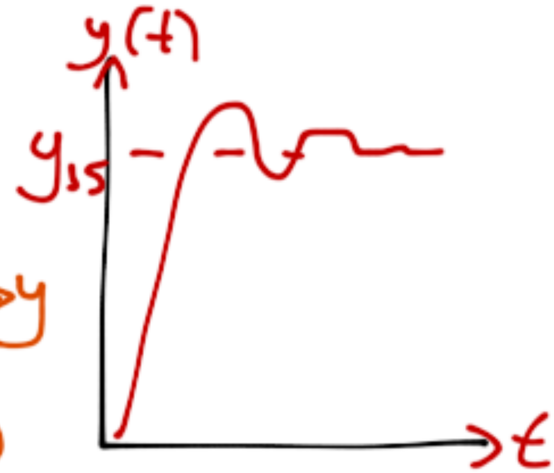
3.2 Steady State gain

A stable Transfer Function with $u(t)=u_0$ for $t \geq 0$ & a const u_0 will have a Steady State Gain (of G):

$$\frac{y_{ss}}{u_0} = \frac{\lim_{t\to\infty} y(t)}{u_0}$$



$Y(s)=G(s)U(s)$



Consider $u_0=1$, $u(t)=\mathbb{1}(t)$,
$\Rightarrow u(s)=\frac{1}{s}$

Since G is stable,

$$\lim_{t\to\infty} y(t) \overset{FVT}{=} \lim_{s\to 0} sY(s) = \lim_{s\to 0} \cancel{s}\, G(s) \frac{1}{\cancel{s}}$$

$$= G(0) \leftarrow \text{the DC gain of } G(s)$$

Thm: Steady State Gain of a stable system equals $G(0)$.

Pf: $u(t) = u_0 \cdot 1(t) \Rightarrow U(s) = \frac{u_0}{s}$.

So $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = u_0 G(0)$

Therefore: $\frac{y_{ss}}{u_0} = G(0)$

$\blacksquare$

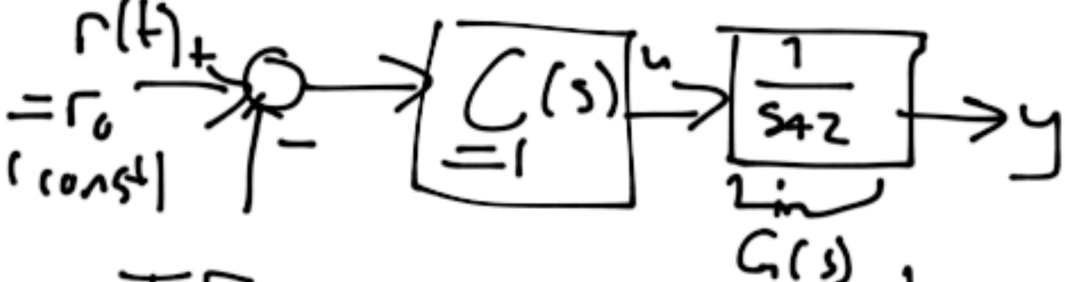
Ex: $\dot{x} = -2x - u \Rightarrow Y(s) = \frac{1}{s+2} u(s)$

$y = x$

Obj: set point ctrl: given a const desired output, find a control flow $u(t)$ so $y(t) \to$ desired const.

attempt 1.

openloop control, $u(t) =$ desired output (const)



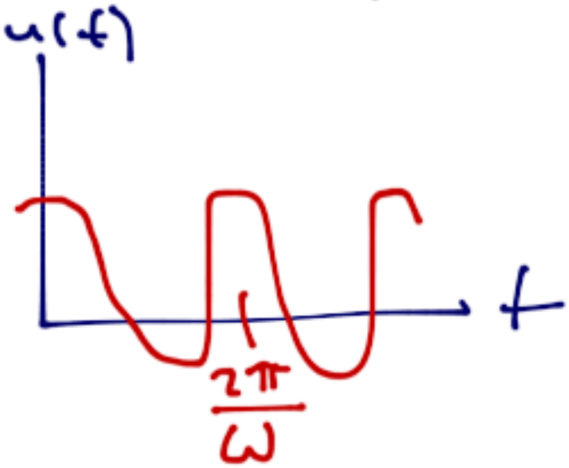
The TF from $r \to y$ is $\frac{1}{s+2}$,

$\lim_{t\to\infty} y(t) = r_0 G(0) \Rightarrow y_{ss} = \frac{1}{2} r_0$ } DC Gain

$\Rightarrow$ DC gain $G(0) = \frac{1}{2} \Rightarrow$ SS output for constant reference has $\frac{1}{2}$ the magnitude. Our Cntroller cannot provide set point control.

3.3 Freq. Response

Consider a stable TF, $G(s)$ and let $u(t) = \cos(\omega t)$, $t \geq 0$, w/ real constant



$y_{ss}(t) = ?$

For LTI systems, the SS. output is also sinosoidal w/ the same freq:

$$y_{ss}(t) = A \cos(\omega t + \phi)$$

A Bode plot: plot of A v.s. ω (magnitude)

& ϕ vs ω (phase)

$y(t) = g(t) * u(t)$ or $Y(s) = G(s)U(s)$

Fact: complex exp. are eigenfns:

$u(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

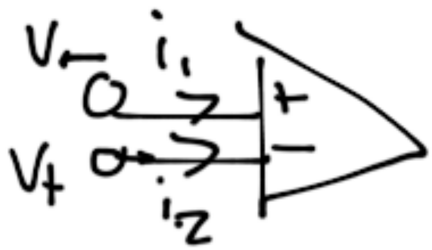
then $y(t) = \int_{-\infty}^{\infty} g(\tau)\, u(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} g(\tau) e^{j\omega(t-\tau)}\, d\tau$

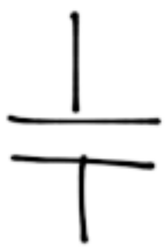
$= \underbrace{\left(\int_{-\infty}^{\infty} g(\tau) e^{-j\omega\tau}\, d\tau\right)}_{\text{looks like } \mathcal{L}\{g(t)\}'} e^{j\omega t}$

w/ $s = j\omega \Rightarrow \mathbb{F}\{g(t)\}$.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow M^{-1} = \frac{1}{\underbrace{da-bc}_{\det(M)}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



$$\Rightarrow i_- = i_+ = 0, \quad V_- = V_+$$

 C =  $Z_C = \frac{1}{sC}$

 L =  $Z_L = sL$

Summary:

BIBO stable $\Leftrightarrow$ $G(s)$'s poles are in $\mathbb{C}^{-} := \{s \in \mathbb{C} : \mathrm{Re}(s) < 0\}$

- Steady-state gain = DC Gain $G(0)$
- freq. response.

$$\underset{=\cos(\omega t)}{u(t)} \longrightarrow \boxed{G(s)} \longrightarrow y_{ss}(t) = A\cos(\omega t + \phi)$$

Stable

Fact 1:

$$u(t) = e^{j\omega t}$$

$$y(t) = \left(\int_{-\infty}^{\infty} g(\tau)\, e^{-j\omega \tau}\, d\tau\right) e^{j\omega t}$$

$$= G(j\omega)\, e^{j\omega t}$$

Aside: for $z \in \mathbb{C}$,

$$z = x + jy$$

$$z = |z|\, e^{j\angle z}$$

$$= |z|\cos(\angle z) + j\sin(\angle z)$$

$$y(t) = |G(j\omega)|\, e^{j\angle G(j\omega)}\, e^{j\omega t}$$

$$= |G(j\omega)|\, e^{j\angle G(j\omega) + j\omega t}$$

$$u(t) = \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\Rightarrow y(t) = \frac{G(j\omega)\, e^{j\omega t}}{2} + \frac{G(-j\omega)\, e^{-j\omega t}}{2}$$

$$\Rightarrow y(t) = \frac{G(j\omega)e^{j\omega t}}{2} + \frac{G(-j\omega)e^{-j\omega t}}{2}$$

$$= \mathrm{Re}\{G(j\omega)e^{j\omega t}\}$$

$$= |G(j\omega)| \cos(\omega t + \angle G(j\omega))$$

<u>Ex:</u> $\dot{x} = -10x + u$ $\Rightarrow Y(s) = \frac{1}{s+10} u(s)$

$y = x$ $\qquad = G(s)u(s)$

If $u(t) = \cos(3t)$, what is the steady state output?

i) sys is BIBO stable $\Rightarrow y(t)$ is bounded,

ii) this is LTI system, (it has a TF),

$\Rightarrow$ looks like $A\cos(3t+\phi)$

$$A = |G(j\omega)|_{\omega=3} = \left|\frac{1}{j\omega+10}\right|_{\omega=3} \approx 0.1$$

$$\phi = \angle G(j\omega)|_{\omega=3} = \angle \frac{1}{j\omega+3}\Big|_{\omega=3} = \angle 1 - \angle j3 - 10$$

$$= 0 - \operatorname{atan}\left(\frac{3}{10}\right) \approx -16.7^\circ = 0.2915 \text{ rad}$$

$$\Rightarrow y_{ss}(t) = 0.1\cos(3t - 0.2915)$$

3.3.1 Bode Plots:

Give us insight about how sys responds to any input

$$20\log|G(j\omega)| \text{ vs } \log(\omega)$$

(magnitude Bode plot)

$$\angle G(j\omega) \text{ vs } \log(\omega)$$

(phase Bode plot)

To sketch Bode plots, we need:

- Pure Gain
- Poles (z=0)
- First order terms
- 2nd order terms
- delays.

Ex:

$$G(s) = \frac{40 s^2 (s-2)\, e^{-s10}}{(s-5)(s^2+4s+100)}$$

$$= \underbrace{\frac{40 \cdot 2}{5 \cdot 100}}_{1} \cdot \frac{\overset{2}{s^2}\,\overset{3}{\left(\frac{s}{2}-1\right)}\,\overset{5}{e^{-s10}}}{\underbrace{\left(\frac{s}{5}-1\right)}_{3}\underbrace{\left(\frac{s^2}{100}+\frac{4}{100}s+1\right)}_{4}}$$

Aside:

$$\text{Log}\left(\frac{AB}{C}\right) = \text{Log}(A) + \text{Log}(B) - \text{Log}(C)$$

$$\angle \frac{AB}{C} = \angle A + \angle B - \angle C$$

Ex:

$$20\,\text{Log}(G(j\omega)) = \overset{1}{20\,\text{Log}\left|\frac{80}{100}\right|} + \overset{2}{2\left(20\,\text{Log}|j\omega|\right)}$$

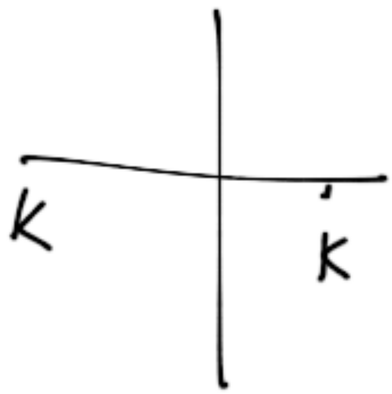
$$+ \overset{3}{20\,\text{Log}\left|\frac{j\omega}{2}-1\right|} + \overset{5}{20\,\text{Log}\left|e^{-j\omega10}\right|}$$

$$- \overset{3}{20\,\text{Log}\left|\frac{j\omega}{5}-1\right|} - \overset{4}{20\,\text{Log}\left|\frac{(j\omega)^2}{100}+4j\omega+1\right|}$$

1) Pure gain $G(s) = K$ ← real constan-

$$20 \log|G(j\omega)| = 20 \log|K|$$

$$\angle G(j\omega) = \angle K$$



Summary:

$$\cos(\omega t) \longrightarrow \underset{\text{stable}}{\boxed{G}} \longrightarrow |G(j\omega)|\cos[\omega t + \angle G(j\omega)]$$

$G(j\omega)$ = freq. resp of G

Bode plots: $\underbrace{20\log|G(j\omega)|}_{dB}$ vs $\log\omega$

$\angle G(j\omega)$ vs $\log\omega$

1. pure gain $G(s) = K$, K real constant.
2. Poles/zeros at the origin. $G(s) = s^n$

$$\Rightarrow 20\log|G(j\omega)| = n\,20\log|j\omega|$$

$$\angle G(j\omega) = n\angle j\omega$$

1. pure gain $G(s) = K$, K real constant.

2. Poles/zeros at the origin. $G(s) = s^n$

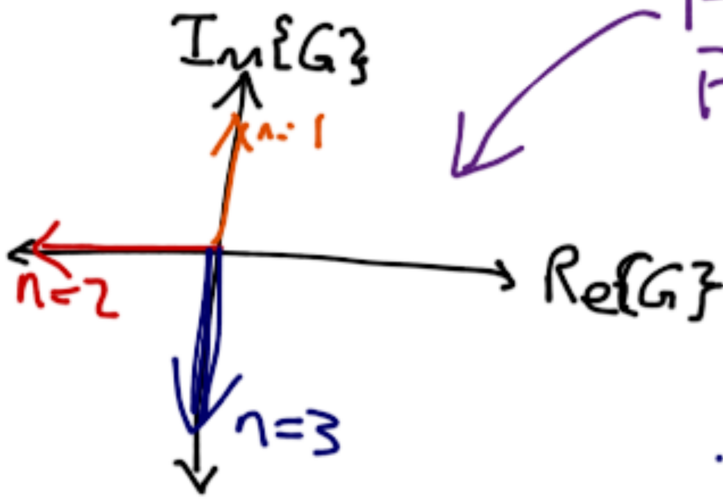
$$\Rightarrow 20\log|G(j\omega)| = n\,20\log|j\omega|$$

$$\angle G(j\omega) = n\angle j\omega$$

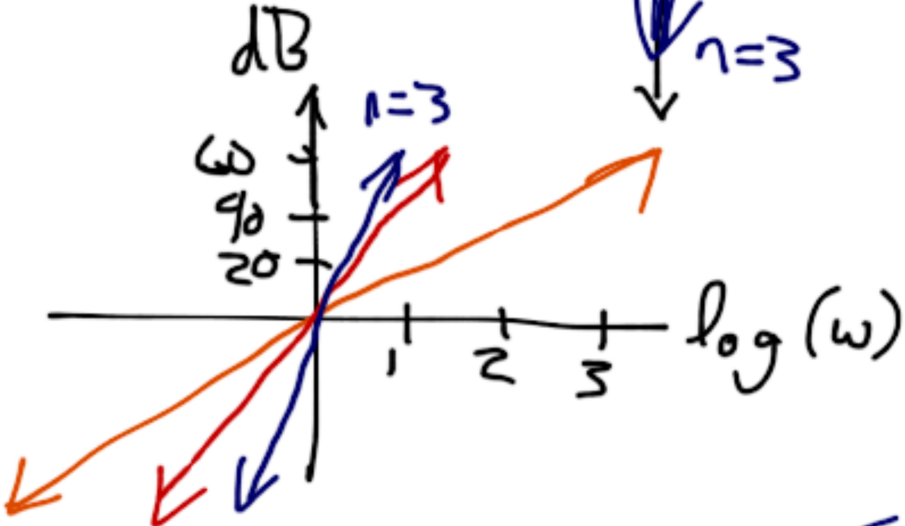
$n=1$:

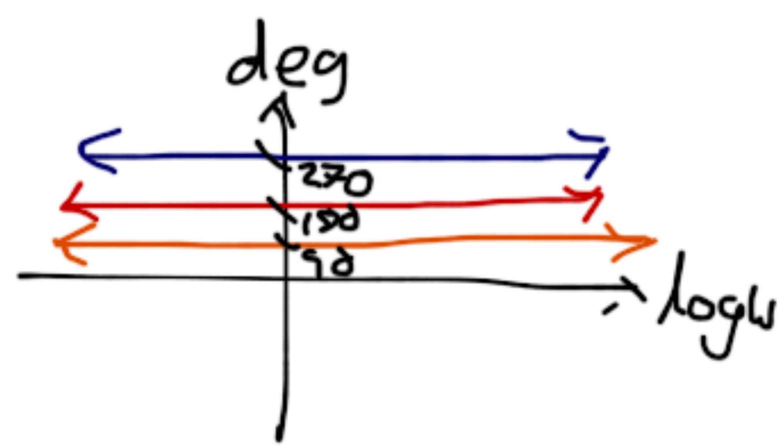
$n=2$

$n=3$



Polar Plot





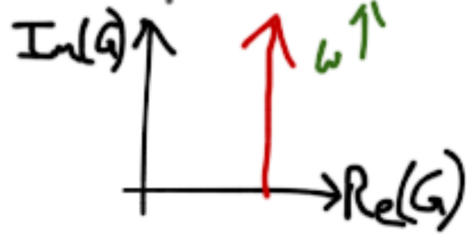
Bode Plot

3. First-order Terms: $G(s) = \left(\frac{s}{\tau}+1\right),\ \tau>0$

$\Rightarrow G(s) = (s+3) = 3\left(\frac{s}{3}+1\right)$

$$20\log|G(j\omega)| = 20\log\left|\frac{j\omega}{\tau}+1\right|$$

$$\angle G(j\omega) = \operatorname{atan}\left(\frac{\omega}{\tau}\right)$$



Approx For Magnitude:

(i) For $\omega \ll \tau$, $\frac{j\omega}{\tau}+1 \simeq 1$

$\rightarrow 20\log|G(j\omega)| = 0$

ii) For $\omega \gg \tau$ $\frac{j\omega}{\tau}+1 \approx \frac{j\omega}{\tau}$

$\rightarrow 20\log|G(j\omega)|$ looks like case #2.

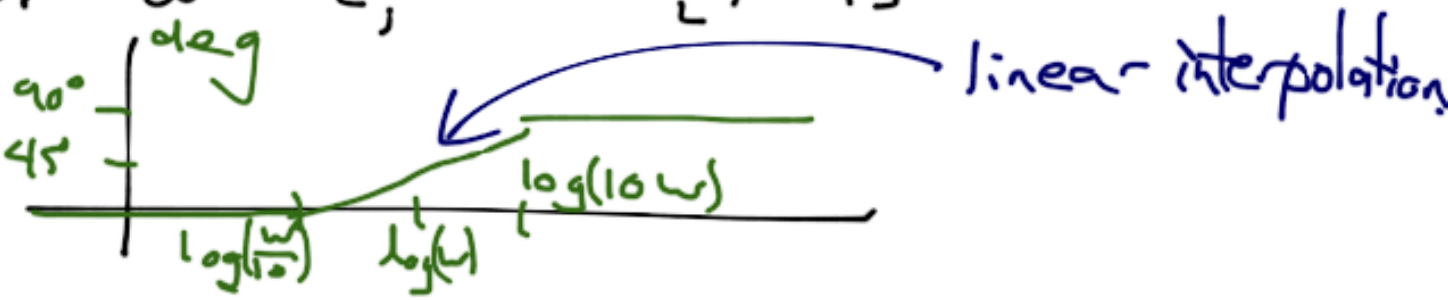
iii) error introduced by approx is largest at $\omega=\tau$

Approx For Phase:

i) for $\omega \ll \tau$, $\operatorname{atan}\left(\frac{\omega}{\tau}\right) \approx 0$

ii) for $\omega \gg \tau$, $\operatorname{atan}\left(\frac{\omega}{\tau}\right) \approx 90^\circ$

iii) for $\omega = \tau$, $\operatorname{atan}\left(\frac{\omega}{\tau}\right) = 45^\circ$



Note: left half-plane

In the LHP, we can write 1^{st} order terms like $\frac{s}{\tau}+1$, $\tau>0$

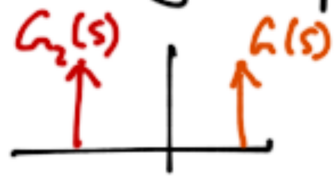
In the RHP, we can write 1^{st} order terms like $\frac{s}{\tau}-1$, $\tau>0$

eg: $G_1(s) = s+3 = 3\left(\frac{s}{3}+1\right)$

$G_2(s) = s-3 = 3\left(\frac{s}{3}-1\right)$

$\rightarrow$ Magnitude plot is the same.

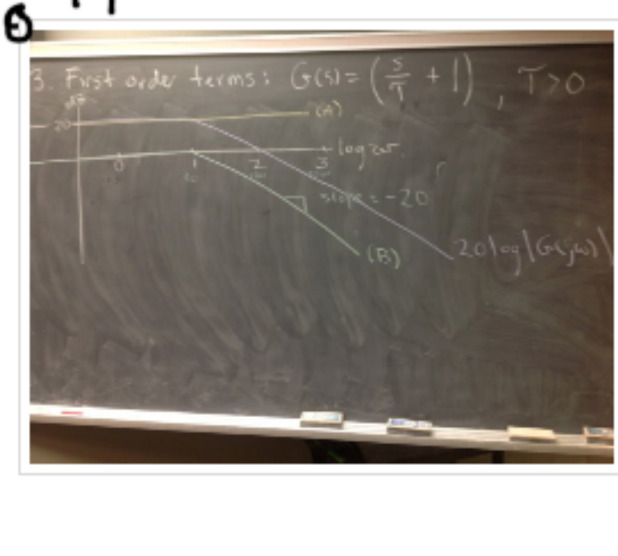
$\rightarrow$ Phase plot changes phase (by 90%) as $\omega \rightarrow \infty$



Ex: $G(s) = \frac{100}{s+10} = \frac{100}{10}\frac{1}{\left(\frac{s}{10}+1\right)}$

$$20\log|G(jw)| = 20\log|10| - 20\log\left|\frac{jw}{10}+1\right|$$

$$\angle G(jw) = \angle 10 - \angle \frac{jw}{10}+1$$



4. 2^{nd} Order Terms:

$$G(s) = \left(\frac{s^2}{\omega_n^2} + \frac{2Zs}{\omega_n} + 1\right), \quad 0 < Z < 1, \ \omega_n > 0$$

i.e: $s^2 + 4s + 100 \Rightarrow \omega_n = 100, \ Z = \frac{1}{5}$

$$\Rightarrow G(j\omega) = \left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2Z\frac{\omega}{\omega_n}$$

1) for $\omega \ll \omega_n$, $G(j\omega) \simeq 1 \Rightarrow \log|G| = \angle G = 0$

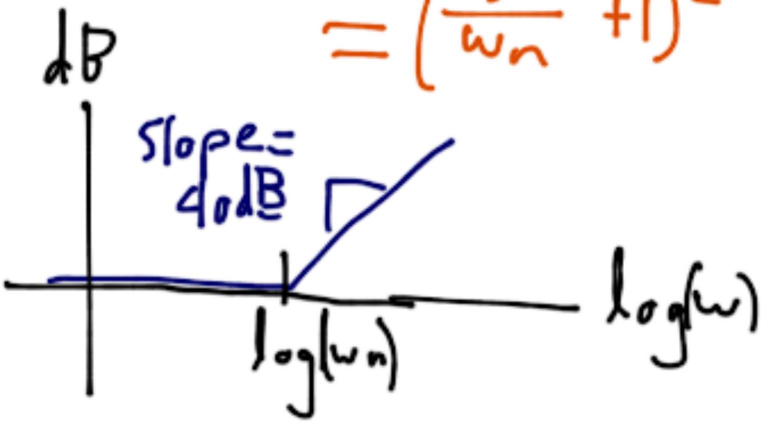
2) for $\omega \gg \omega_n$, $G(j\omega) \simeq \frac{\omega^2}{\omega_n^2} \Rightarrow \log|G| = 2\log\left|\frac{\omega}{\omega_n}\right|$

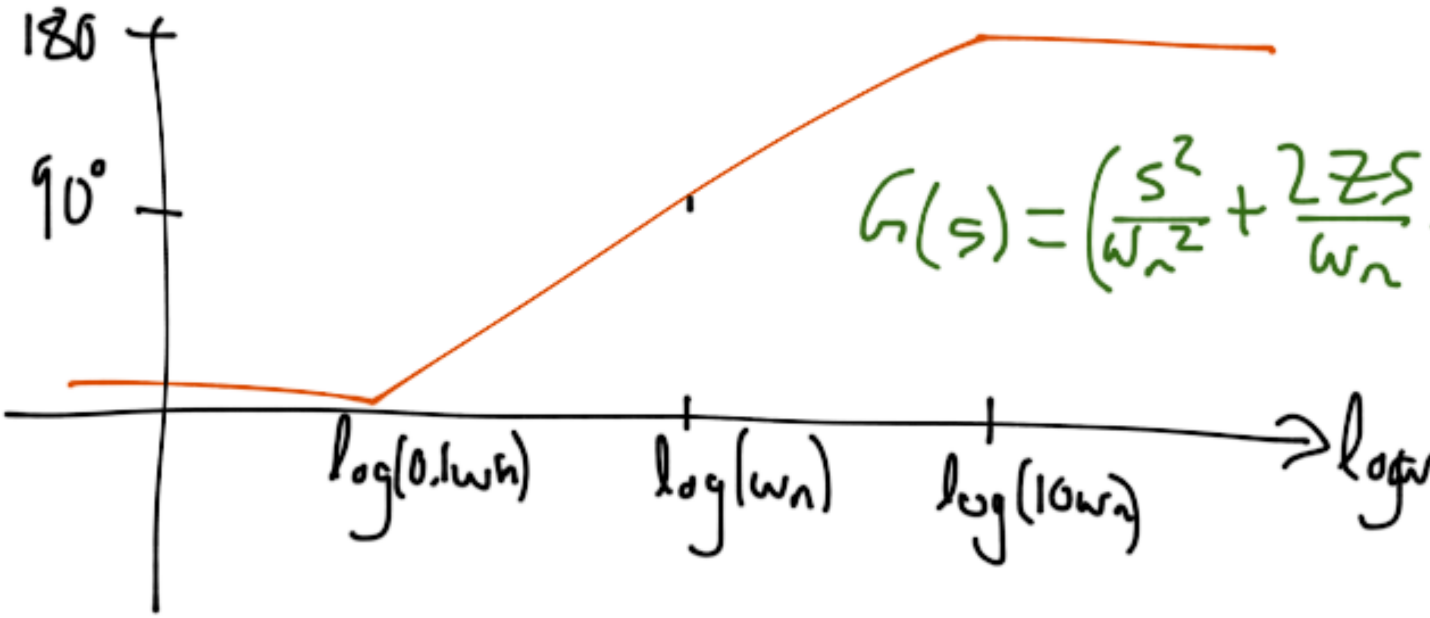
$\angle G \simeq 180$

we can approx G as two 1^{st}-order terms w/ roots at: $s = -\omega_n$, i.e $Z = 1$

$$G = \{ a \# a' : a \in \{0,1\}^*, a \neq a' \}$$

$$G(s) = \frac{s^2}{\omega_n^2} + 2\zeta \frac{1}{\omega_n} s + 1 \approx \frac{s^2}{\omega_n^2} + 2\frac{s}{\omega_n} + 1$$
$$= \left(\frac{s}{\omega_n} + 1\right)^2$$

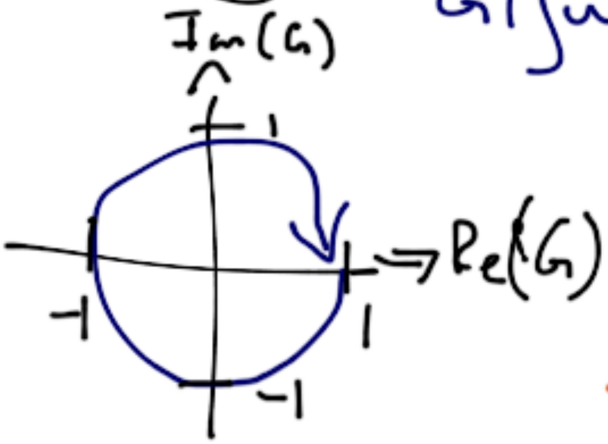


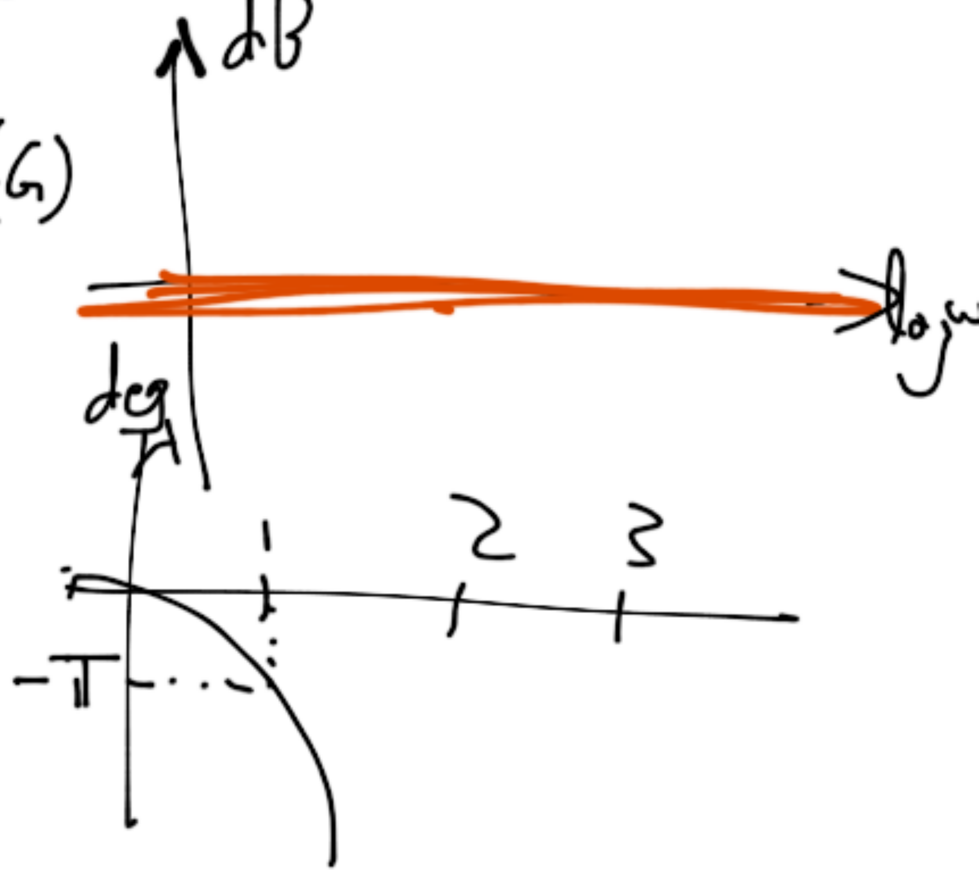


$$G(s) = \left(\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1\right)$$

5. Delay

$$G(j\omega) = e^{-j\omega T} = \cos\omega T - j\sin\omega T$$





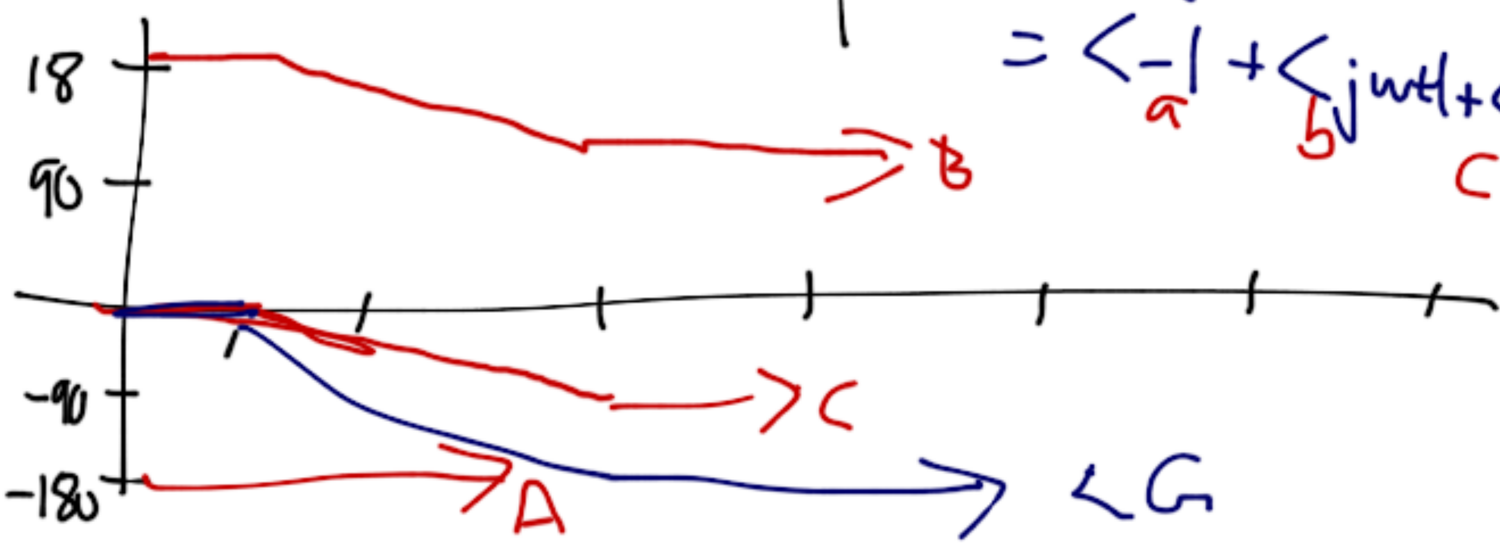
Ex: Allpass TF:

$$\dot{x} = -x + u \qquad \Rightarrow G(s) = \frac{1-s}{s+1} = -1\left(\frac{s-1}{s+1}\right)$$
$$y = 2x - u$$

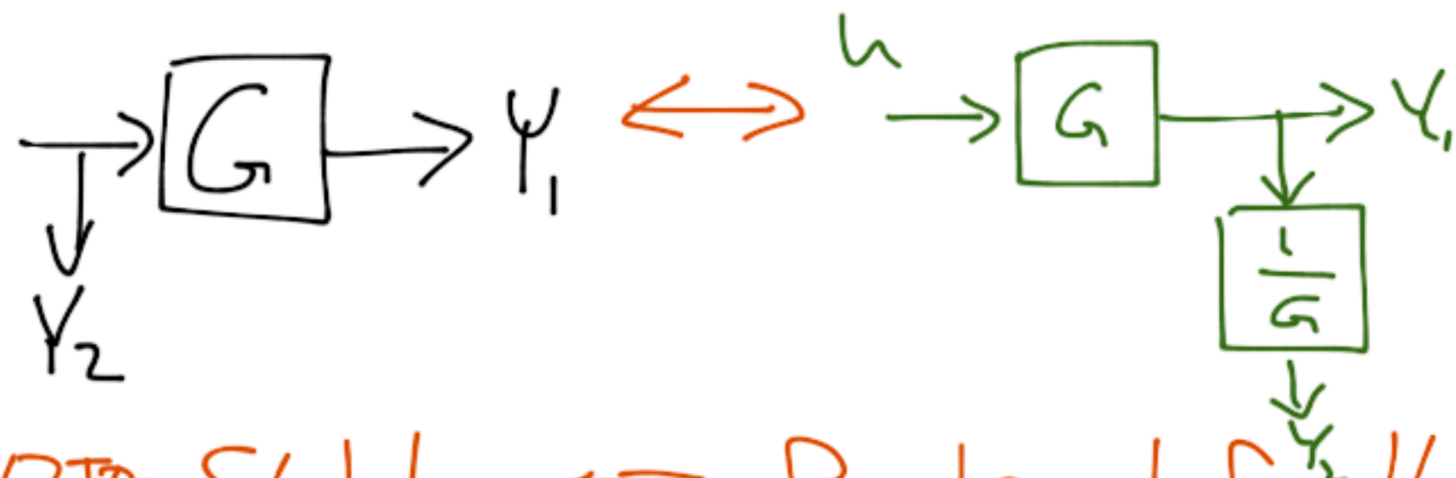
$$20\log|G(j\omega)| = 20\log\overset{a}{|-1|} + \log\overset{b}{|j\omega-1|} - 20\log\overset{c}{|j\omega+1|}$$

$$= 0$$

$$\angle G(j\omega) = \underset{a}{\angle -1} + \underset{b}{\angle j\omega-1} + \underset{c}{\angle j\omega+1}$$



Block Diagram Equivalents:



BIBO Stable $\Longleftrightarrow$ Real part of all poles are negative!

$$Y_{ss} = G(0)\, c_0$$

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} G(s)\, S$$

$$\frac{K}{RCs+1} = \frac{x}{x+1}$$

$$x+1 = x(RCs+1)$$

$$x(RCs+1-1) = 1$$

$$x = \frac{1}{RCs}$$

$$\frac{K}{RCs+1} = \frac{\frac{K}{RCs}}{\frac{K}{RCs}+1}$$

CSPS

Ch 4: 1st & 2nd order systems

1st order:

$$\tau \dot{y} + y = Ku \overset{\mathcal{L}}{\Rightarrow} \frac{Y(s)}{U(s)} = \frac{K}{1+\tau s},$$

constants (real)

or: $\boxed{\dot{x} = \frac{-1}{\tau}x + \frac{K}{\tau}u, \; y = x}$

2nd Order:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K\omega_n^2 u$$

$$\overset{\mathcal{L}}{\Rightarrow} \frac{Y}{U} \quad \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

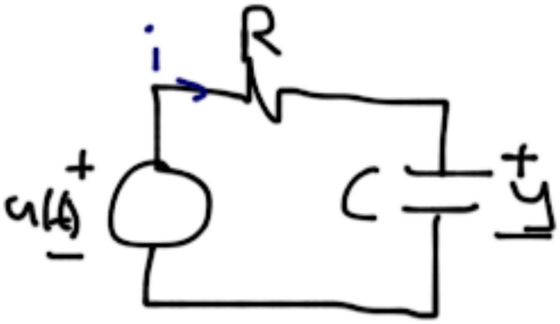
$$\dot{x}_1 = x_2$$

or $\quad \dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2 + K\omega_n^2 u$

$$y = x_1$$

Obj: understand the rel. bw/n pole locations and time response.

Ex: 1^{st} Order, RC Circuit $\boxed{4.1}$



KVL: $-u + Ri + y = 0$

$$\xrightarrow{\mathcal{L}} \frac{Y(s)}{U(s)} = \frac{1}{RCs+1} = \frac{K}{1+\tau s}$$

$$\Rightarrow K = 1, \tau = RC$$

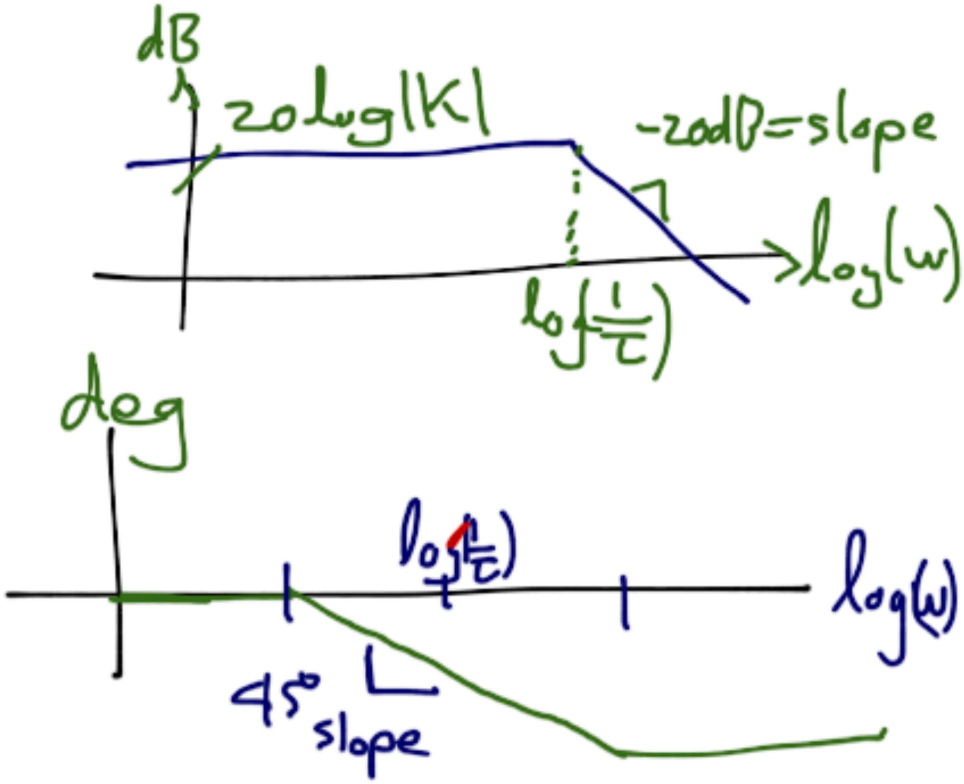
Pole: $s = \frac{-1}{\tau}$

$\Rightarrow$ stable for $\tau > 0$, unstable otherwise

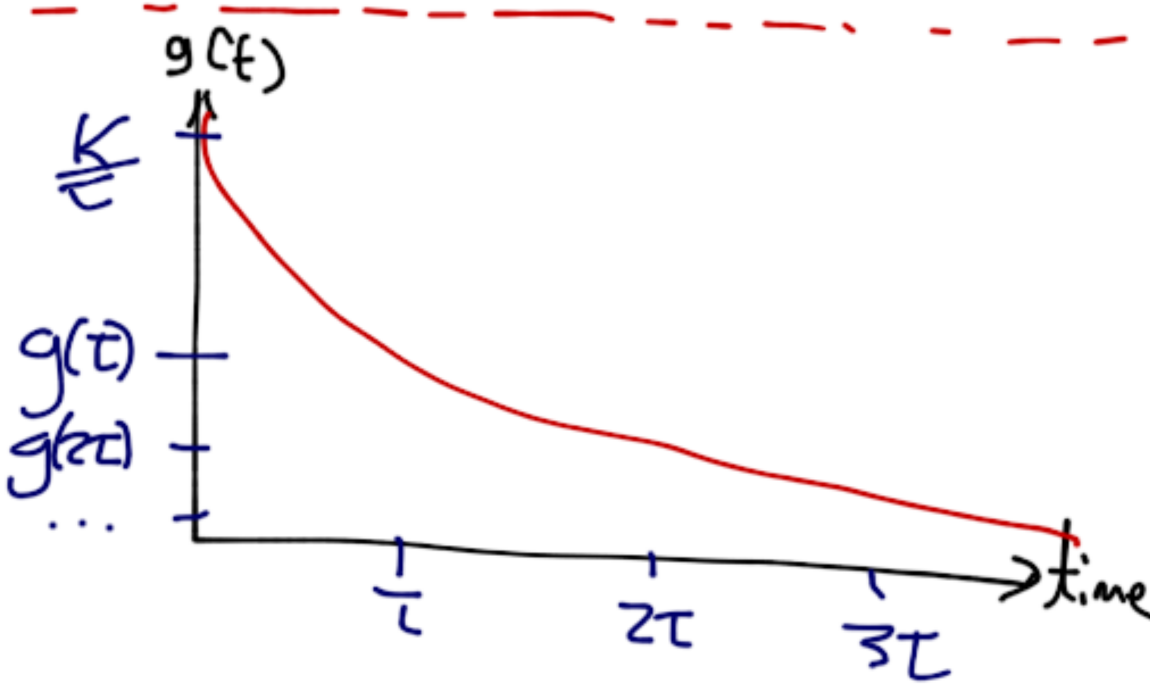
Zeroes: none

DC Gain: $G(0) = K$

For $\tau > 0$:



$G(s) = 1$?



$g(t) = \mathcal{L}^{-1}\{G(s)\}$

$= \frac{K}{\tau} e^{-t/\tau}, \; t \geq 0$

Observation

Higher Bandwidth $\left(\frac{1}{\tau}\right)$ $\Longleftrightarrow$ Faster Time response

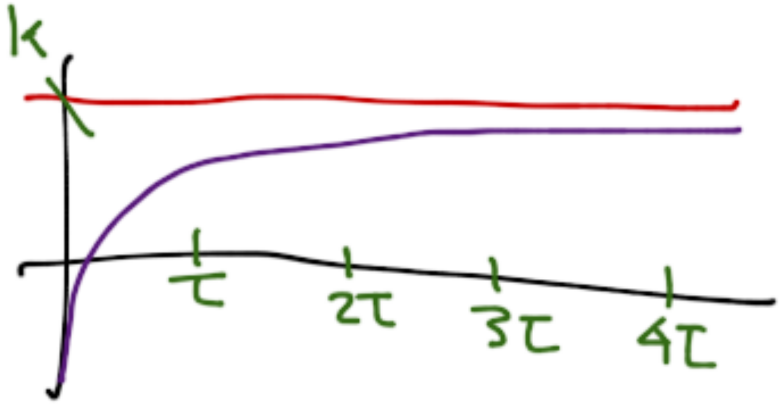
Step Response: $u(t) = \mathbb{1}(t)$

Then,

$$y(t) = \mathcal{L}^{-1}\{G(s)u(s)\} = \mathcal{L}^{-1}\left\{\frac{K}{1+\tau s} \cdot \frac{1}{s}\right\}$$

$$= K\left(1 - e^{-\frac{t}{\tau}}\right), \quad t \geq 0$$

x	$y(x)$
τ	$0.631K$
2τ	$0.85K$
3τ	$0.95K$
4τ	$0.98K$



Observations:

After 4τ seconds, The output settles to 2% of the asymptote (Settling time)

For all $t \geq 0$, $y(t) < y_{ss}$ (no "overshoot")

$y(t)$ is monotonically increasing (no peaking)

4.1 First order $\quad G(s) = \dfrac{K}{1+Ts}$

Summary

- as $T>0$ approaches zero, the time response gets faster.
- as $T>0$ " ", the pole $s = -1/T$ goes further to the left.
- no overshoot or oscillations in the step response
- 2% settling time equals $4T$
- bandwidth is $1/T$.

Ch 4: 1st & 2 order Systems:

$$G(s) = \frac{K}{1+\tau s} \qquad G(s) = \frac{K\omega_n^2}{s^2+2z\omega_n s+\omega_n^2}$$

1^{st} order:

- stable @ $\tau > 0$, pole at $s = \frac{-1}{\tau}$
- bandwidth $\simeq \frac{1}{\tau}$
- no overshoot or oscillations in step resp.

2^{nd} order:

$$m\ddot{y} = u - k_s y - b\dot{y}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\frac{1}{M}}{s^2 + \frac{b}{M}s + \frac{k_s}{M}}$$

$$\omega_n = \sqrt{\frac{K_s}{M}}$$

$$K = \frac{1}{K_s}$$

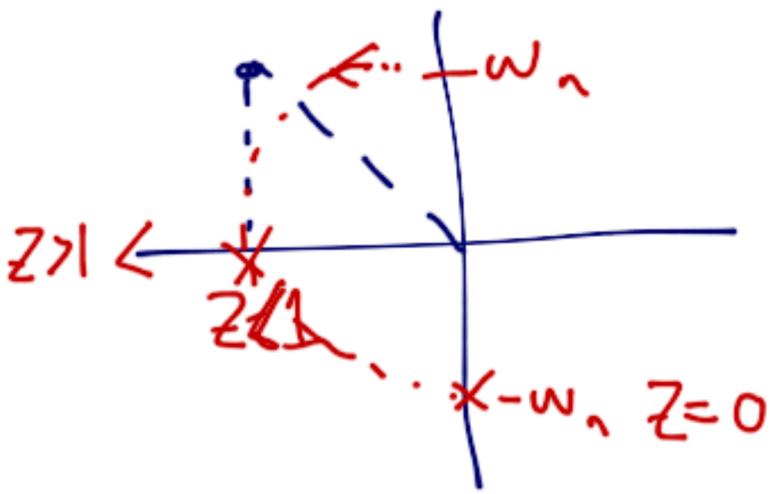
$$z = \frac{b}{2\sqrt{K_s M}}$$

2^{nd} Order Poles

$$s^2 + 2Z\omega_n s + \omega_n^2$$

$$s = -Z\omega_n \pm \sqrt{Z^2 - 1}$$

range:	pole description:
$0 < Z < 1$	Complex Conjugate
$Z = 1$	2 real repeated poles
$1 < Z$	2 real distinct poles



Ex: mass-spring dampener:

$0 < b < 2\sqrt{K_s M}$

Poles at: $s = -z\omega_n \pm j\omega_n\sqrt{1-z^2}$

Magnitude of Poles:

$$\sqrt{(-z\omega_n)^2 + (\omega_n\sqrt{1-z^2})^2}$$
$$= \omega_n$$

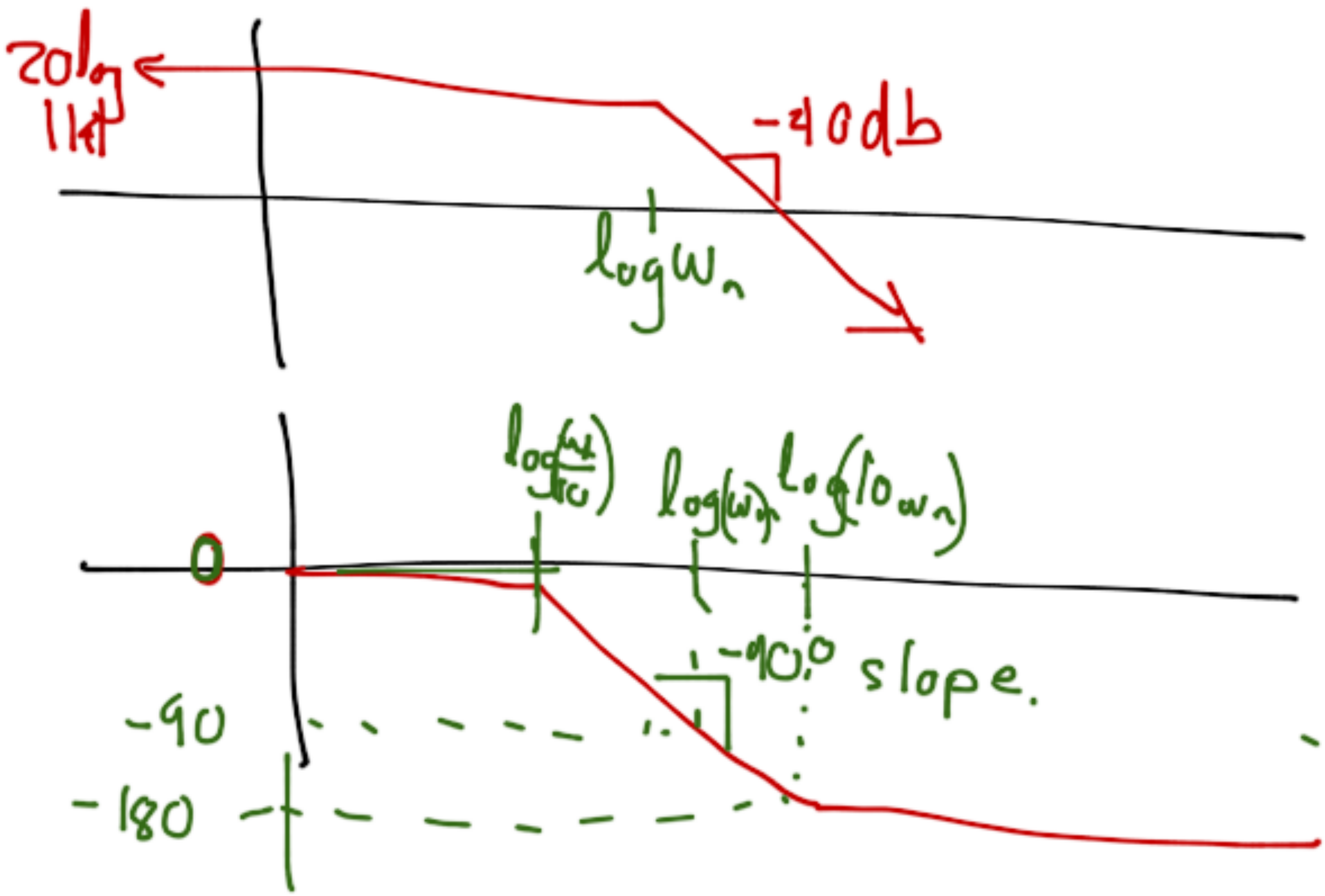
$\cos\theta = \frac{z\omega_n}{\omega_n} = z \Rightarrow \theta = \cos^{-1}(z)$

$\Rightarrow \omega_n$ determines the mag of poles

z determines the θ of poles.

1.e $s = \omega_n e^{j(\pi - \theta)}$
$= \omega_n e^{j(\pi - acos(z))}$

Zeros: none
DC gain: $G(0) = K$
Freq. Response:
Bandwidth $\approx \omega_n$



Impulse Response:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{K\omega_n}{\sqrt{1-z^2}} e^{-z\omega_n t} \sin\left(\omega_n \sqrt{1-z^2}\, t\right)$$

Intuitively: for fixed $z \in (0,1)$,

larger/longer bandwidth (ω_n) $\Leftrightarrow$ faster response

Step Response:

$u(t) = 1(t) \rightarrow \boxed{G(s)} \rightarrow y(t) = ?$

(complex conj. poles, $0 < z < 1$)

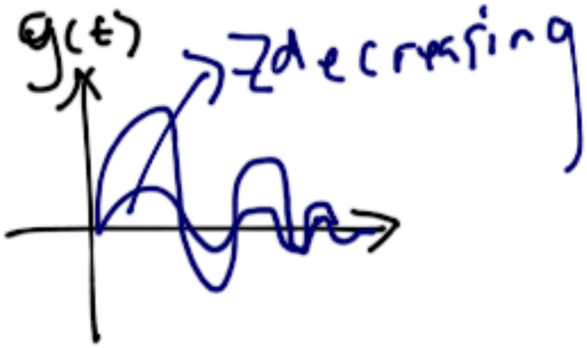
$$y(t) = \mathcal{L}^{-1}\{G(s)\tfrac{1}{s}\}$$

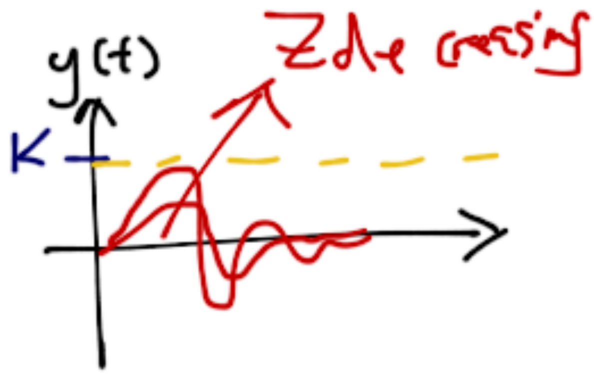
$$= K\left(1 - \frac{1}{\sqrt{1-z^2}} e^{-z\omega_n t} \sin\left(\omega_n \sqrt{1-z^2}\, t + \operatorname{acos}(z)\right)\right)$$

z = damping ratio

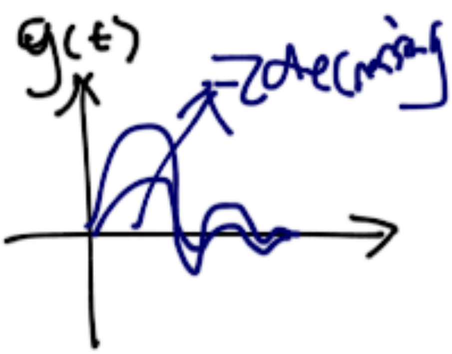
ω_n = undamped natural frequency.

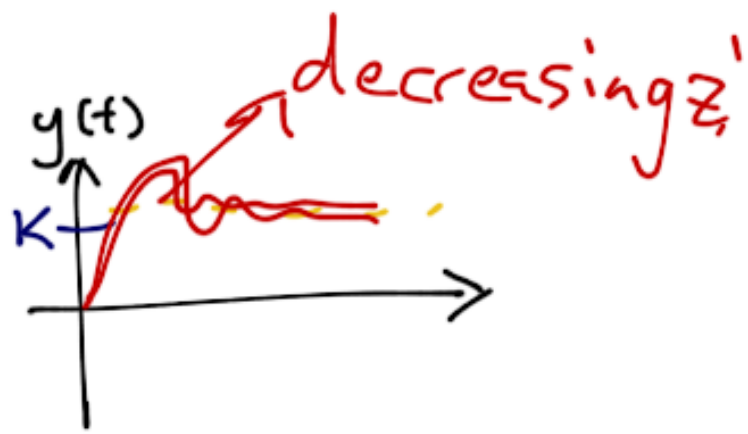
$K = 1, \omega_n = 1$





$K = 1, \omega_n = 5$





Draw a bode Plot: of $G(s)$

Magnitude:

- $20\log(|G(j\omega)|)$ v.s. ω
- rewrite/factor $G(s)$
- seperate it in the log domain
- Draw the Bode Plot for each part
- Final $= \sum_i \text{Bode}_i$

Term	Mag	Phase:
K	$20\log\lvert K\rvert$	$K = \begin{cases} 0 & K>0 \\ \pm 180 & K<0 \end{cases}$
$\frac{1}{s}$	-20dB/dec, $0 @ \omega=1$	-90
s	$+20$dB/dec, $0 @ \omega=1$	$+90$
$\frac{\omega_0}{s+\omega_0}$	0, ω_0, -20db/dec	0, 0.1ω, ω, 10ω, -90
$\frac{s}{\omega_0}+1$	0, 20db/dec, ω_0	0, $\frac{\omega_0}{10}$, ω_0, $10\omega_0$, 90
$\frac{1}{\left(\frac{s}{\omega_0}\right)^2+2\zeta\left(\frac{s}{\omega_0}\right)+1}$	0, ω_0, -40db/dec	0, $\frac{\omega_0}{5^{\zeta}}$, $\omega_0 5^{\zeta}$, -180
$\left(\frac{s}{\omega_0}\right)^2+2\zeta\left(\frac{s}{\omega_0}\right)+1$	ω_0, 40db/dec	$\frac{\omega_0}{5^{\zeta}}$, $\omega_0 \cdot 5^{\zeta}$

Summary

2^{nd} Order Systems

$$G(s) = \frac{K\omega_n^2}{s^2 + 2z\omega_n s + \omega_n^2}$$

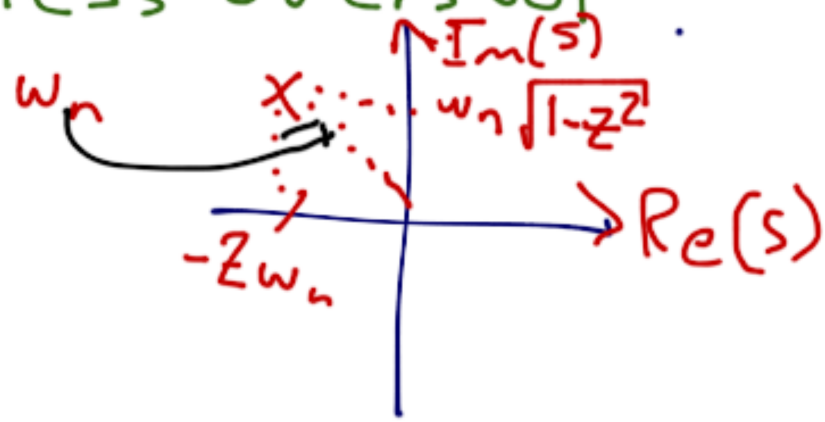
Poles are: $s = -z\omega_n \pm \omega_n\sqrt{z^2-1}$

'underdamped systems ($0 < z < 1$) exhibit oscillitory motion $\rightarrow$ $z \uparrow \Leftrightarrow$ more oscillati

Summary of UnderDamped Systems:

$(0 < Z < 1)$

As $Z \to 1$, the time response oscillates less, and the poles get closer to the real axis w/ less overshoot.



As $Z \to 0$, time resp gets more oscillating, poles approach im axis, more overshoot

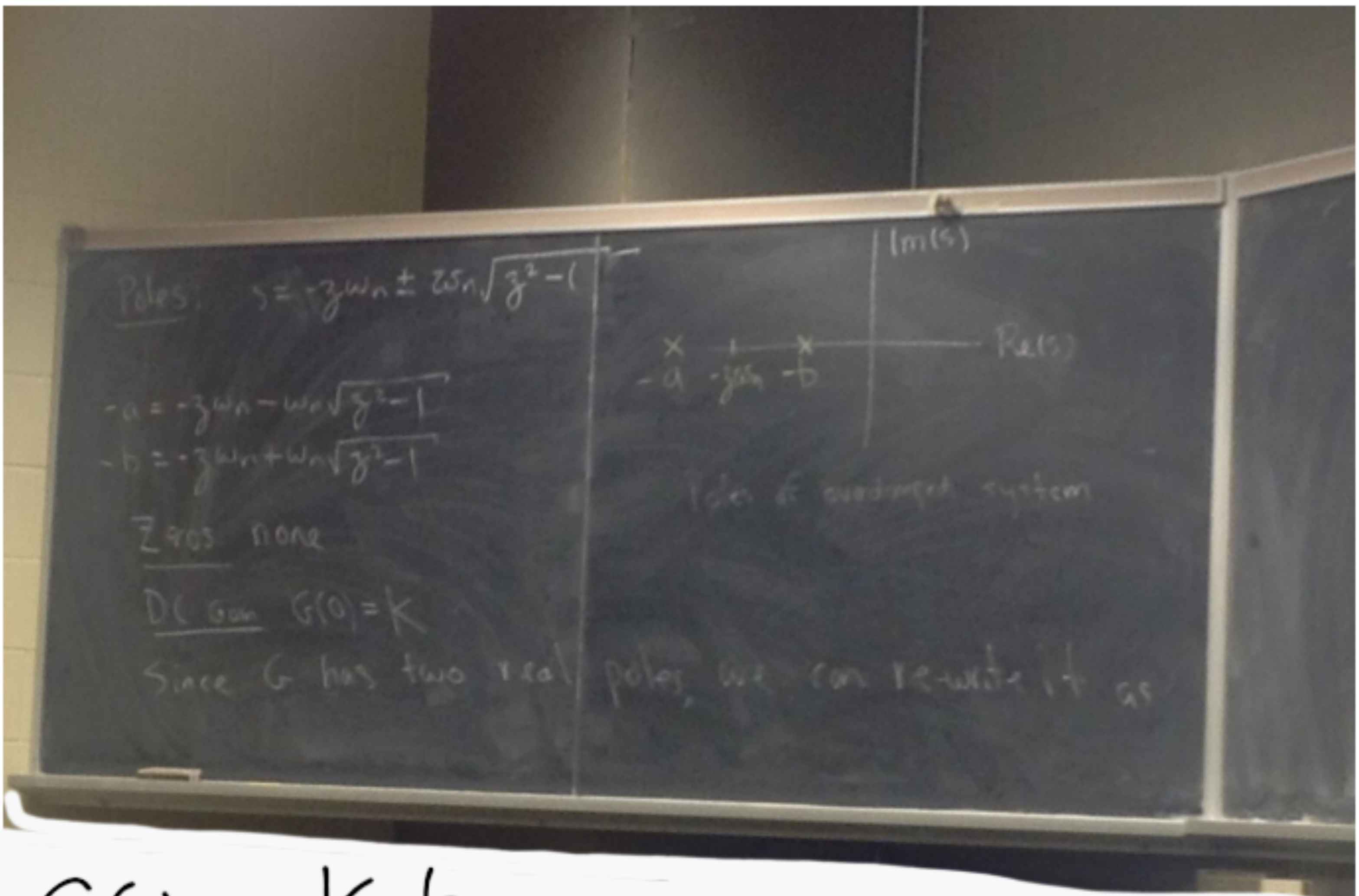
Bandwidth is w_n. As $w_n \uparrow$, response is faster, the poles go left.

Freq $\propto f(\mathrm{Im}(\text{poles}))$
Rate of convergence $\propto g(\mathrm{Im}(\text{poles}))$

You can tell alot about a system's transient resp. by looking at its pole locations.

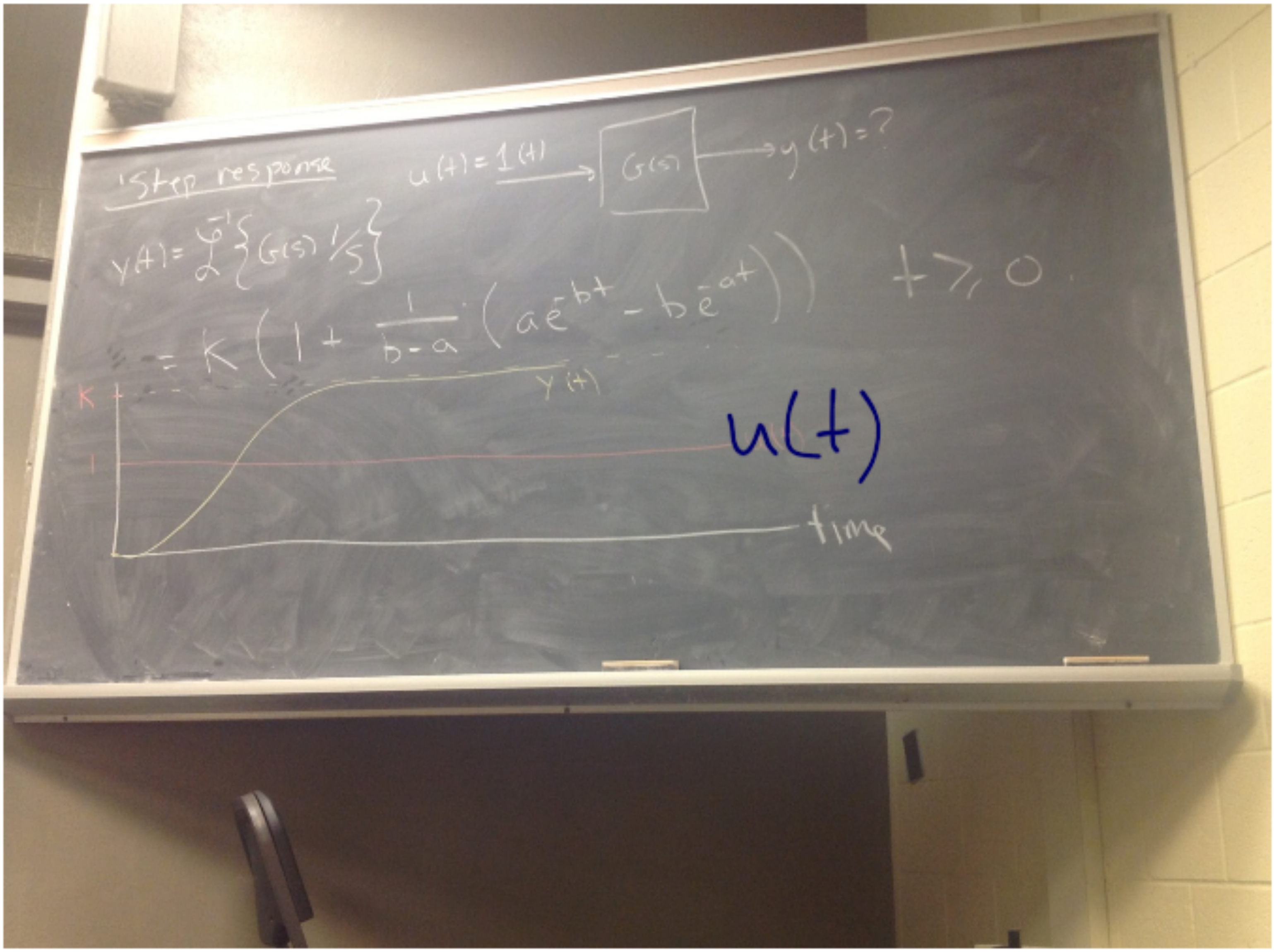
Over-damped Systems: (2 real distinct poles, $\zeta > 1$)

Ex: Mass-spring-damper with $b > 2\sqrt{K_s M}$



$$G(s) = \frac{Kab}{(s+a)(s+b)}$$


Kab
(s+a)(s+b)
20log|K|
-20db/dec
log(a)
log(b)
-40db/dec
Impulse response
g(t)
Bandwidth ≈ b
time
desk.


Step response
u(t) = 1(t)
G(s)
y(t) = ?
y(t) = $\mathcal{L}^{-1}\{G(s)\,1/s\}$
$= K\left(1 + \frac{1}{b-a}\left(ae^{-bt} - be^{-at}\right)\right)$
$t \geq 0$
K
1
y(t)
u(t)
time

In General, over damped systems have no overshoot, no oscillations,

bandwidth = b (closest poles to 0)

the further to the left that the poles are, the faster the response.

4.5

Critically damped Systems: (2 real, equal poles, $\zeta=1$)

Ex. mass-spring dc per

$$b = 2\sqrt{K_s M}$$

Poles: $s = -\zeta\omega_n$ (repeated)
$= -\omega_n$

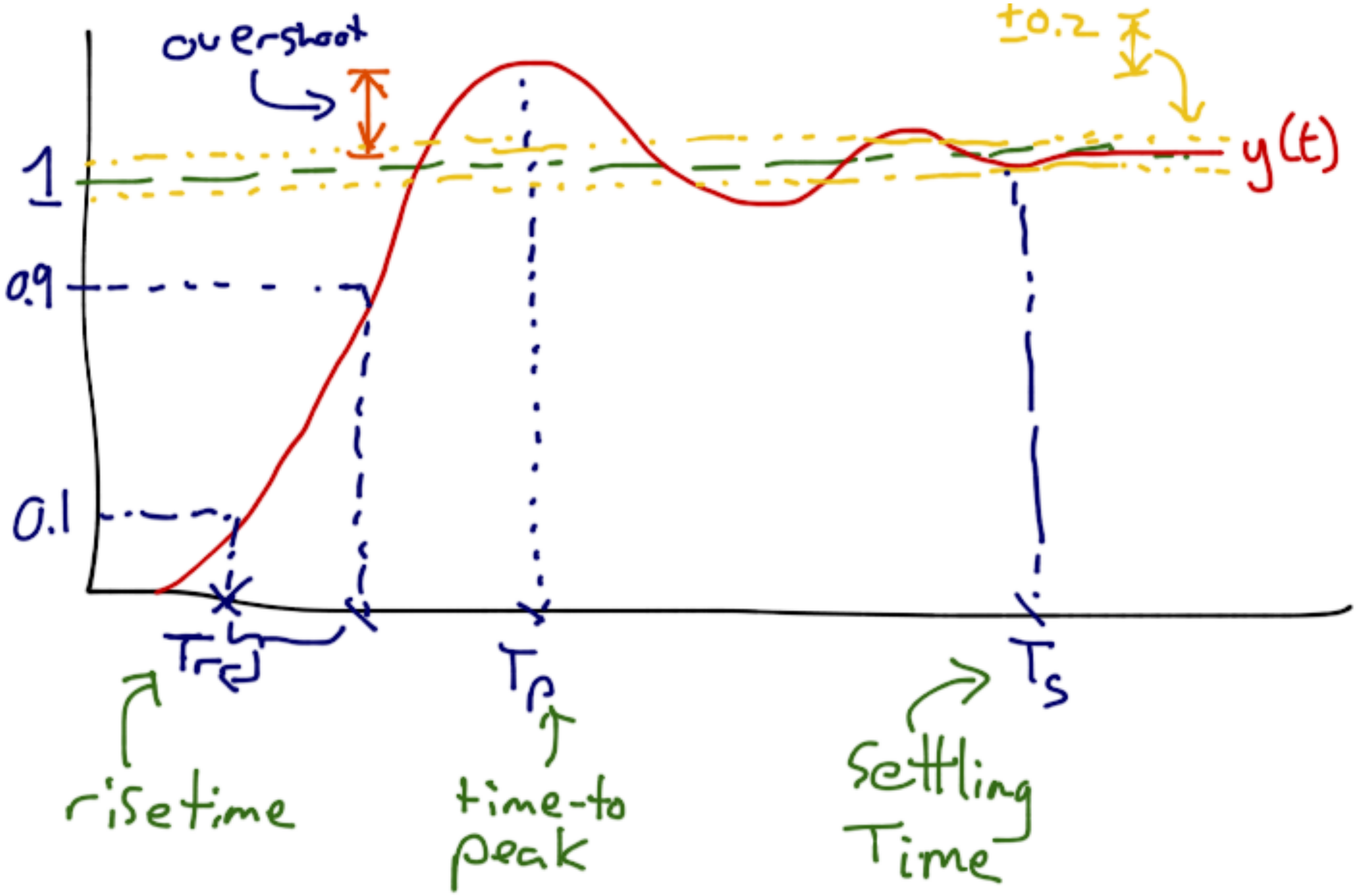
Zeroes: none.

$$G(s) = \frac{Ka^2}{(s+a)^2}$$ [$a = b = \omega_n$ allows this]

By finding the impulse/step response, you'll see that
- no overshoot/oscillation.
- System is slow compared to other cases.

4.6 Characteristics of a step response

- we look common metrics for evaluating the quality of a step response
- the metrics apply to <u>any system</u>
- We will obtain equations for these metrics using prototype 2^{nd} order system
- the value in this:
  1) equations are simple 2) higher order systems can be approx. as 2^{nd} order.



overshoot
±0.2
1
0.9
0.1
y(t)
Tr
Tp
Ts
risetime
time-to peak
Settling Time

Overshoot

- only underdamped systems have overshoot
- to find overshoot: differentiate our expression for the step response $y(t)$ (Section 4.3) then solve for the first $t > 0$ such that $\frac{dy}{dt} = 0$. Plug into $y(t)$.
- In doing so, you get

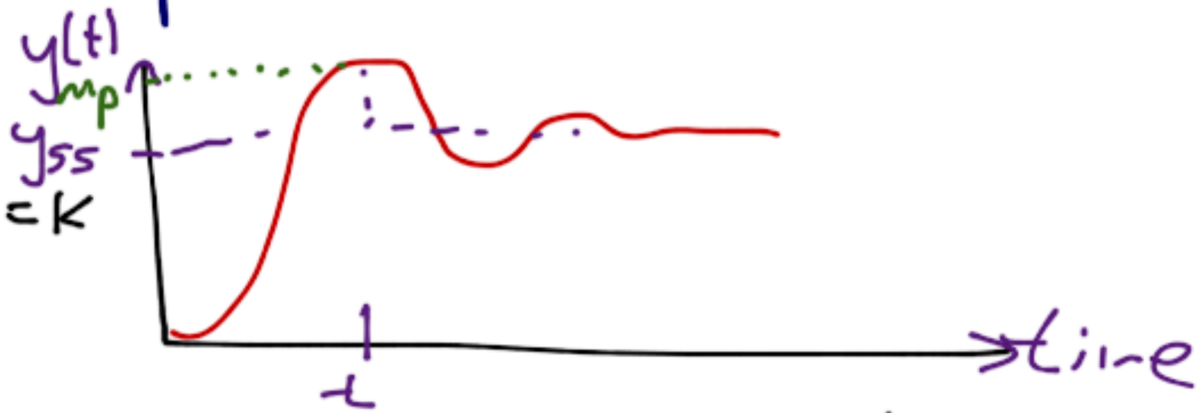
$$\% OS = \frac{M_p - K}{K} = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Summary:

You can tell a lot about a system's transient response by its pole location.

under damped $0 < Z < 1$
over damped $Z > 1$
critically damped $Z = 1$

Step response characteristics.



$$\%OS := \frac{M_p - K}{K} = \exp\left(\frac{-Z\pi}{\sqrt{1-Z^2}}\right)$$

only depends on damping ratio, i.e. the angle of the complex conj. poles.

$Z\uparrow \Leftrightarrow$ less overshoot

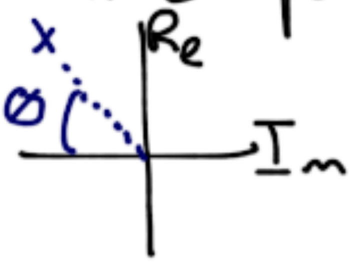
Ex: Mass-spring damper. Find conditions on b, K_s, M so the step response's overshoot is less than or equal to 5%

Spec: $\%OS \leq \%OS_{max} = 0.05$

$$\iff \zeta \geq \frac{-\ln(\%OS_{max})}{\sqrt{\pi^2 + \ln^2(\%OS_{max})}} := \zeta_{min.}$$

In this case, $\zeta = \frac{b}{2\sqrt{K_s M}} \geq \underset{\zeta_{min.}}{0.6901}$

We can visualize this overshoot const. as a constant on pole locations



recall, $\zeta = \cos\theta$

so, $\%OS \leq \%OS_{max}$

$$\iff \theta \leq a\cos(\zeta_{min})$$

$$\iff \theta \leq 46°$$

Settling Time:

Amount of time for response to lie within 2% of its steady state value.

A crude estimate is obtained by looking at the decay rate of $e^{-\zeta\omega_n t}$

$$|e^{-\zeta\omega_n t}| \leq 0.02 \;\Rightarrow\; (\Leftarrow)\; t \geq \frac{4}{\zeta\omega_n}$$

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

recall: poles of 2^{nd} order sys:

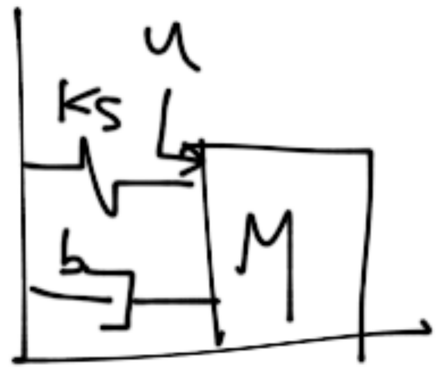
$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Higher Bandwidth (larger ω_n) $\Longleftrightarrow$ Faster response (T_s smaller)

Ex: Mass-spring-damper.

Find cond's b, K_s, M so $T_s \le 3$

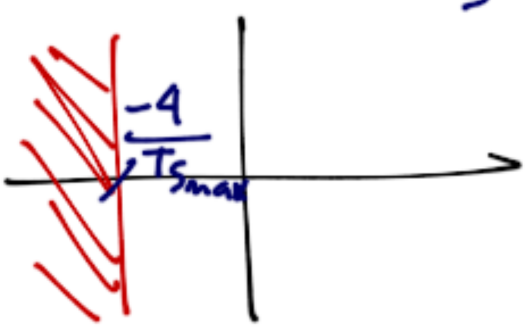
$$T_s \le T_{s_{max}} = 3$$



$$\Rightarrow \frac{4}{z\omega_n} \le 3$$

$$\Rightarrow z\omega_n \ge \frac{4}{3} = \frac{4}{T_{max}}$$

To meet T_s spec, poles need to be to the left of the line



$$Re(s) = \frac{-4}{t_{s_{max}}}$$

In this case,

$$z = \frac{b}{2\sqrt{K_s M}}, \quad \omega_n = \sqrt{\frac{K_s}{M}}$$

$$\Rightarrow \frac{b}{2M} \ge \frac{4}{3}$$

Time To Peak: (T_p)

- Time it takes to reach the peak (max) value
- T_p only depends on imaginary part of poles.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-z^2}}$$

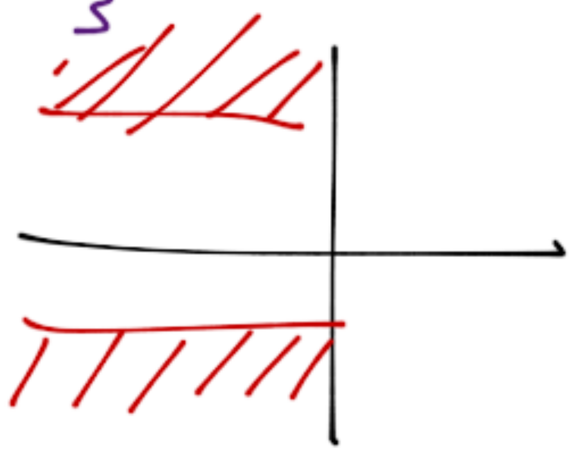
More Bandwidth ($\omega_n \uparrow$) $\iff$ Faster response (smaller T_p)

Ex: Mass-Spring-Damper.

Spec $T_p \leq T_p^{max} = 3$ Seconds

$$\Rightarrow \omega_n \sqrt{1-\zeta^2} \geq \frac{\pi}{3} = \frac{\pi}{T_p^{max}}$$

$$\Rightarrow \sqrt{\frac{k_s}{M} - \frac{b^2}{4M^2}} \geq \frac{\pi}{3}$$

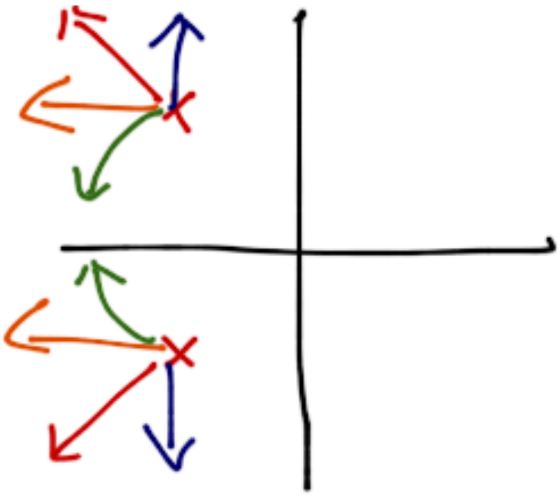


T_r Rise Time:

the time it takes $y(t)$ to go from 10% to 90% of y_{ss} from 0.

$$T_r \approx \frac{2.16\,\zeta + 0.6}{\omega_n}, \quad 0.3 < \zeta < 0.8$$

1
2
3
4



	1	2	3	4
ω_n				
ζ				
%OS				
T_s				
T_p				

$\boxed{\uparrow, \downarrow, \text{ no change.}}$

Fill in

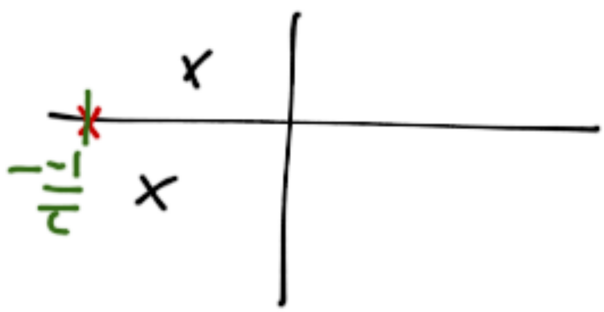
What can we do for non-2^{nd} order systems?

↳ effect of adding poles/zeros.

↳ adding a stable pole

$$G_2(s) = G(s)\,\frac{1}{1+\tau s}$$

$\Longrightarrow$



If the pole at $s = -\frac{1}{\tau}$ is close to the imaginary axis, then the 1^{st} order term dominates the resp.

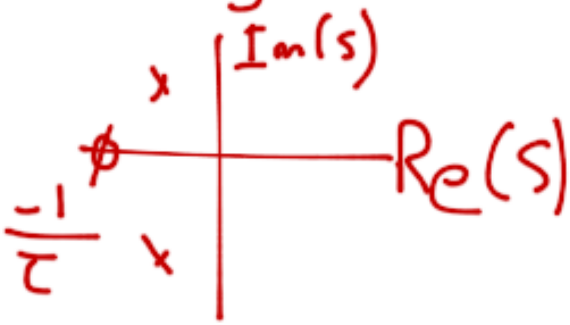
If pole is farther, then $G(s)$ dominates the response.

Adding a minimum phase Zero:

$$G_a(s) = G(s)(1+\tau s), \quad \tau > 0$$

so $\frac{-1}{\tau} < 0$ (in LHP)

$$= \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}(1+\tau s)$$



Step Resp:

$$u(t) = 1(t) \longrightarrow \boxed{G_a(s)} \longrightarrow y(t)$$

$$y(t) = \mathcal{L}^{-1}\left\{G_a \frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{G(s)\frac{1}{s} + \tau s G(s)\frac{1}{s}\right\}$$

$$= \begin{pmatrix}\text{step resp} \\ \text{of } G(s)\end{pmatrix} + \tau\begin{pmatrix}\text{impulse resp} \\ \text{of } G(s)\end{pmatrix}$$

→ As $\tau \to 0$, zero moves to the left, resp approaches step resp of $G(s)$

→ As $\tau \to \infty$, zero moves to the right, resp becomes more prominent.

→ Makes phase plot more positive.

→ Magnitude plot "rolls off" slower, and the bandwidth increases.

→ System is faster, but has more overshoot.

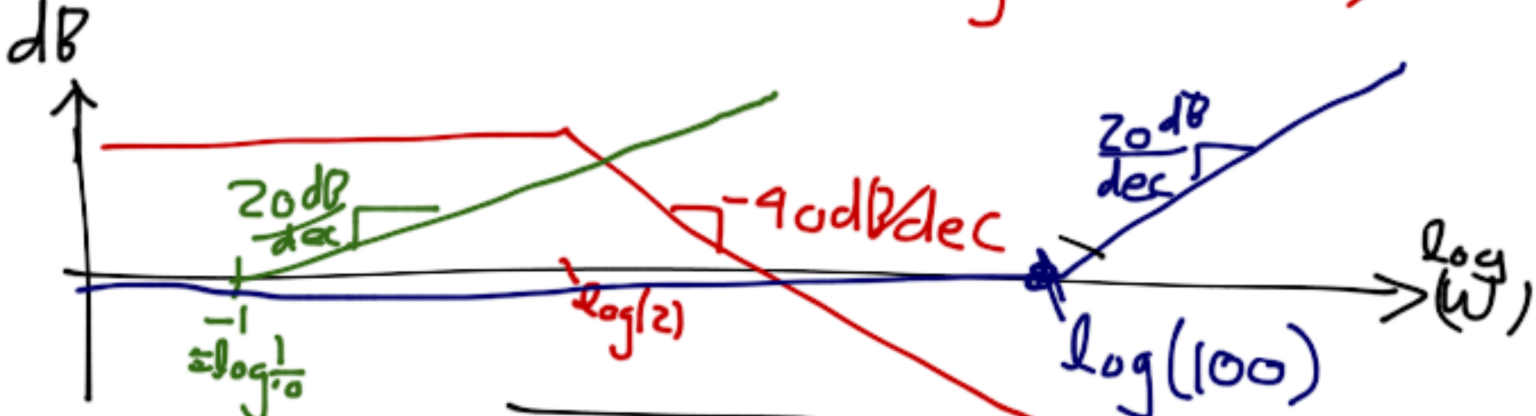
PID controllers have more 0's so knowing these helps when making them.

Ex. $G_a(s) = G(s)(1+\tau s)$, $\tau > 0$

$$G(s) = \frac{8}{s^2+2s+4} \Rightarrow \omega_n = 2, \; \zeta = \frac{1}{2}$$

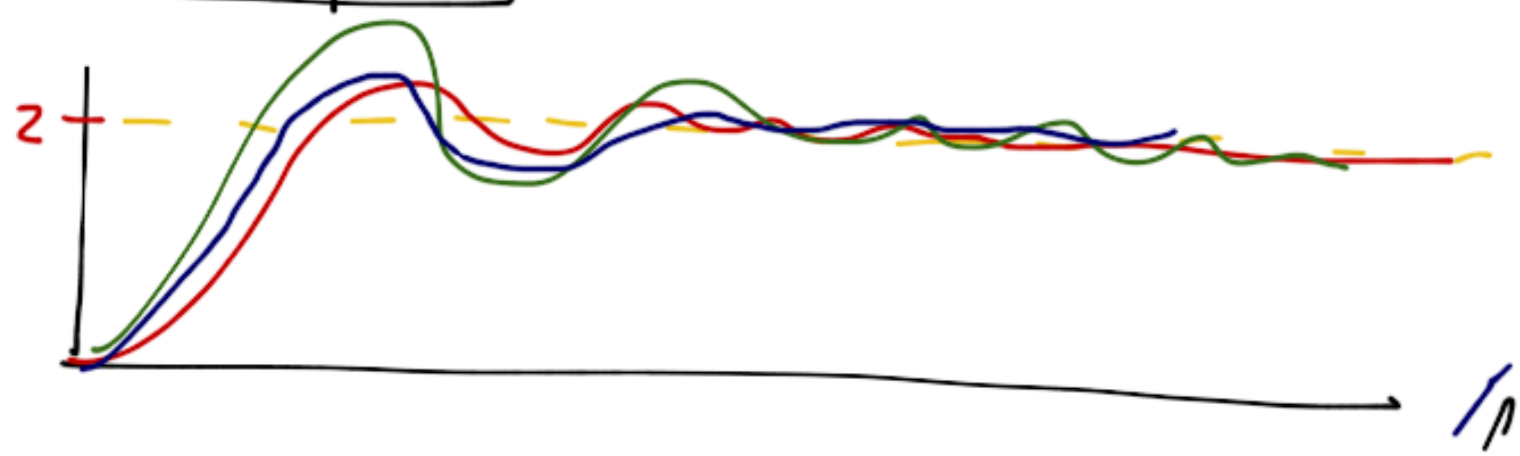
Bode plots for:

- $\tau = 10$
- $\tau = \frac{1}{100}$
- $G(s)$ "basic"/nominal system. $(\tau = 0)$



As $\tau \to 0$, →

Step Response:



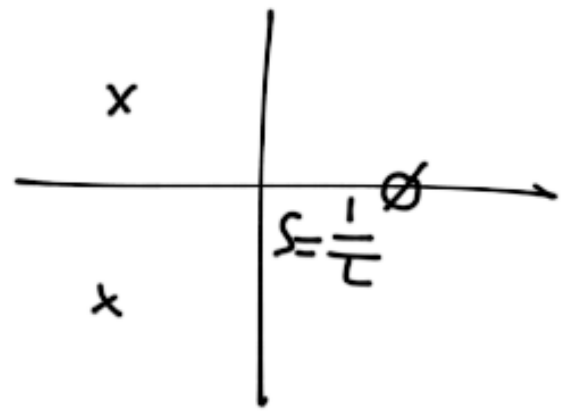
As before, we have:

Closer to im, more prominent.

Further from im, less prominent effect

Adding Non-minimum Phase 0's:

$$G_a(s) = \underbrace{G(s)}_{\substack{\text{2nd order}\\ \text{system}}} \underbrace{(1-\tau s)}_{\substack{\text{non-min}\\ \text{phase zero.}}}, \quad \tau > 0$$



As before:

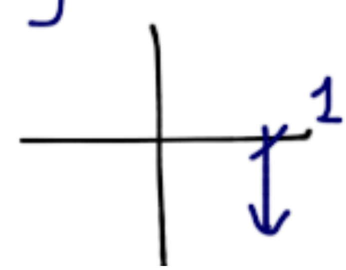
$$y(t) = \begin{pmatrix}\text{step resp}\\ \text{of } G\end{pmatrix} - \tau \begin{pmatrix}\text{impulse}\\ \text{resp of } G\end{pmatrix}$$

As $T \to 0 \Leftrightarrow$ Zero goes to right $\Leftrightarrow$ resp. approaches step resp of G.

As $T \to \infty$ opposite happens, impulse resp. goes the wrong way at first.

- phase $\to$ more neg
- mag has same effect as prev case.
- harder to control

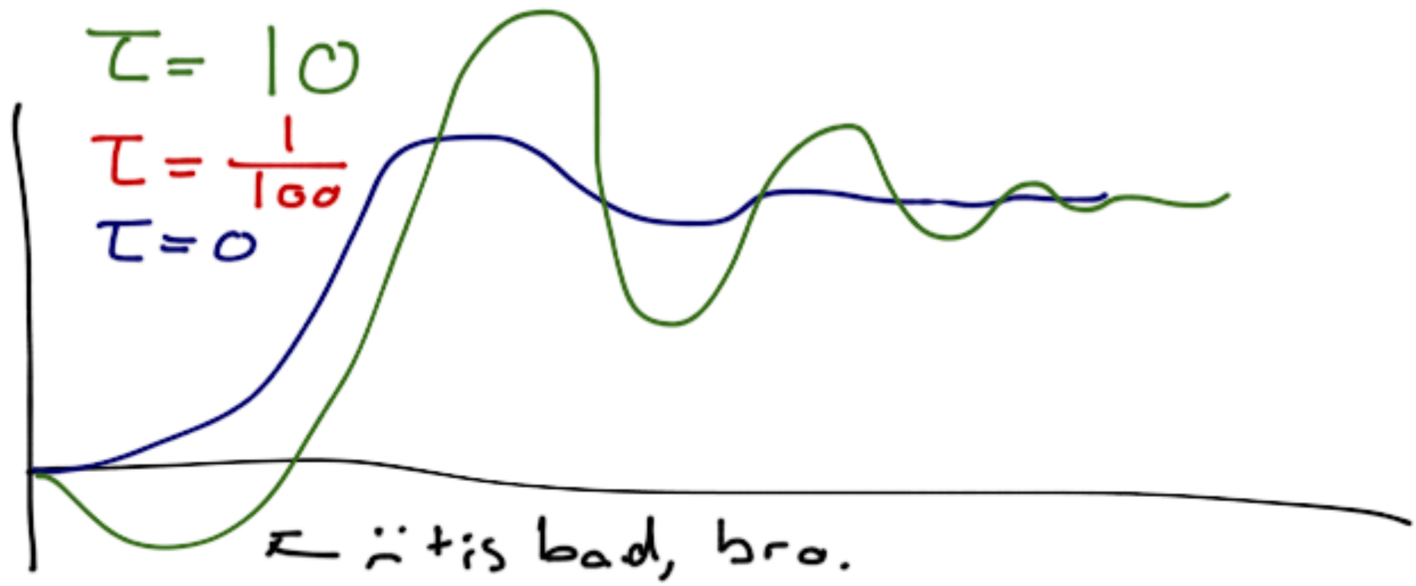
polar plot of $(1 - j\tau w)$ $\to$



Ex. $G_a(s) = G(s)(1 - \tau s)$, $G(s) = \dfrac{8}{s^2+2s+4}$

Bode for

Step response



4.8 Model Reduction:

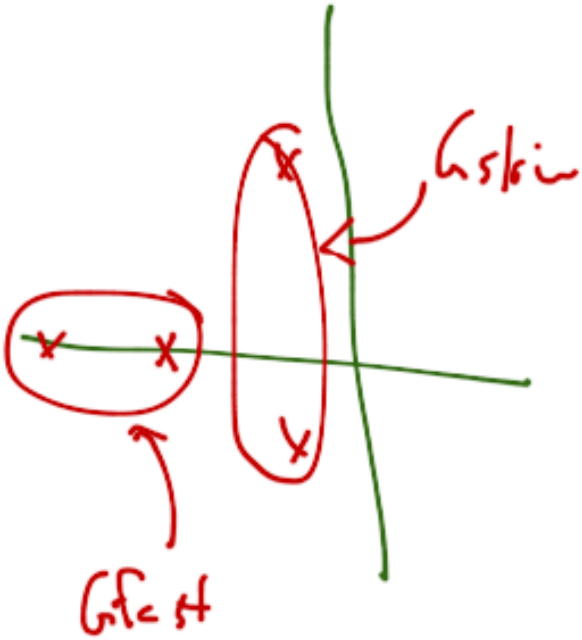
Since, as we've seen the 2^{nd} order model's useful for calculations & evaluating step responses, it is convenient to approximate higher order systems using 2^{nd} order models.

Given a higher order model $G(s)$, the "fast" poles/zeros (poles/zeros far from imag axis) do not affect the step resp. very much.

These extra poles/0's affect the freq resp, but only @ high freq.

Idea: replace poles/zeros that are 5-10x further to the left in $\mathbb{C}$ than the dominant poles/zeros w/ their DC gain

Ex: $G(s) = \dfrac{8(s+20)}{(0.1s+1)(s^2+2s+4)}$



$$= \underbrace{\frac{s+20}{0.1s+1}}_{G_{fast}} \cdot \underbrace{\frac{8}{s^2+2s+4}}_{G_{slow}}$$

The portion of the overall response due to G_{fast} reaches steady-state much faster than that portion from G_{slow}.

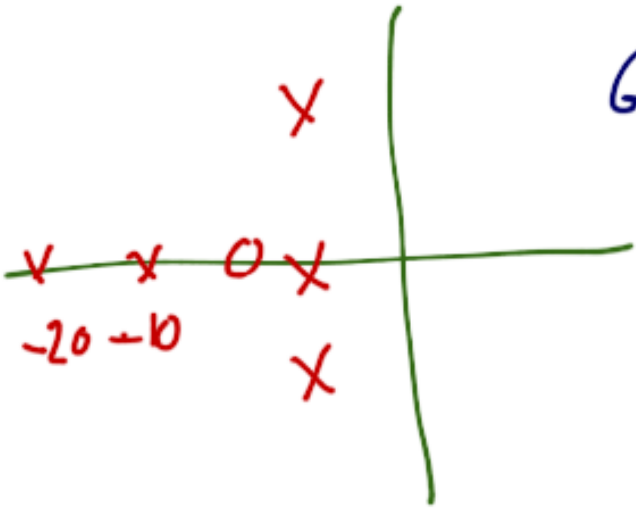
Replace G_{fast} w/ its DC gain.

$$\Rightarrow G_{fast}(0) = 20$$

$$G(s) \simeq \frac{160}{s^2+2s+4}$$

Ex:

$$G(s) = \frac{8(s+20)(s+2)}{(0.1s+1)(s^2+2s+4)(s+1)}$$



$$G(s) \approx \frac{160(s+2)}{(s^2+2s+4)(s+1)}$$

Ch 5: Feedback Control.

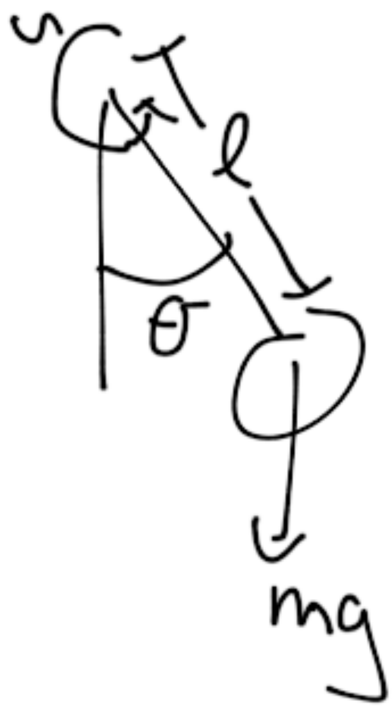
- We want to develop dev. tools for design & analysis of control systems.
  1) Analysis → is controller Good?
  2) Design → find controller to meet specs.
- Most fundamental spec is stability.
  - good perf requires high gain that leads to instability

→ Approaches to design:
  Classical

LATEX.

Diagram 5.1:

Stabilize upright pos'n of pendulum

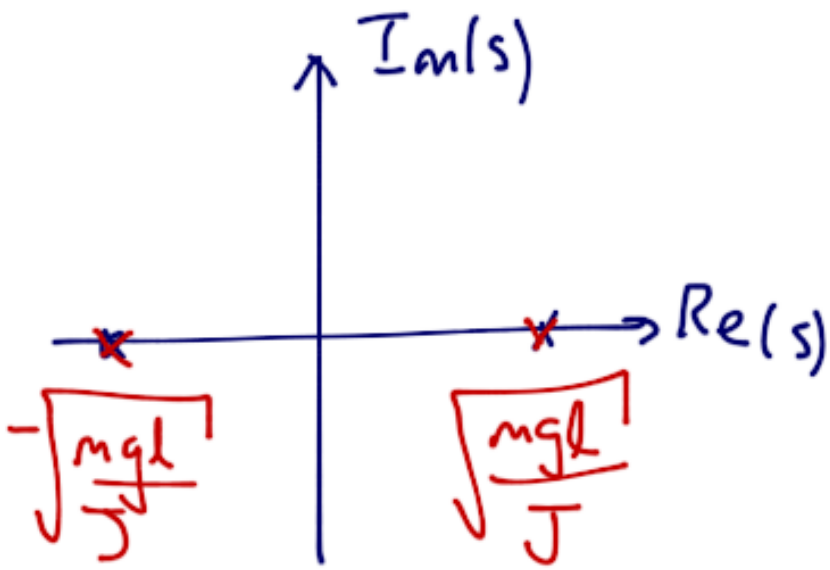


$$\dot{x} = f(x,u)$$

$$y = h(x,u)$$

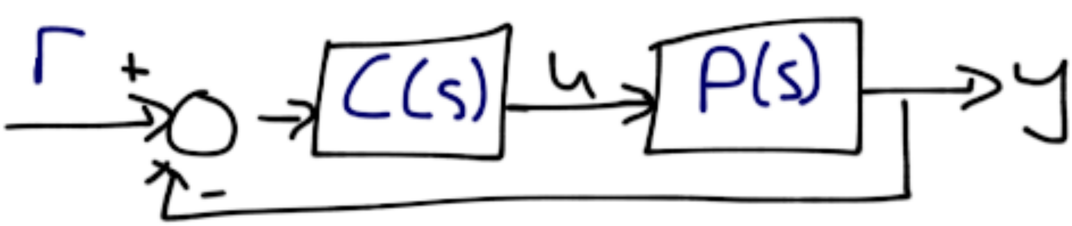
$$x = [\theta, \dot{\theta}]$$

Diagram 5.2a

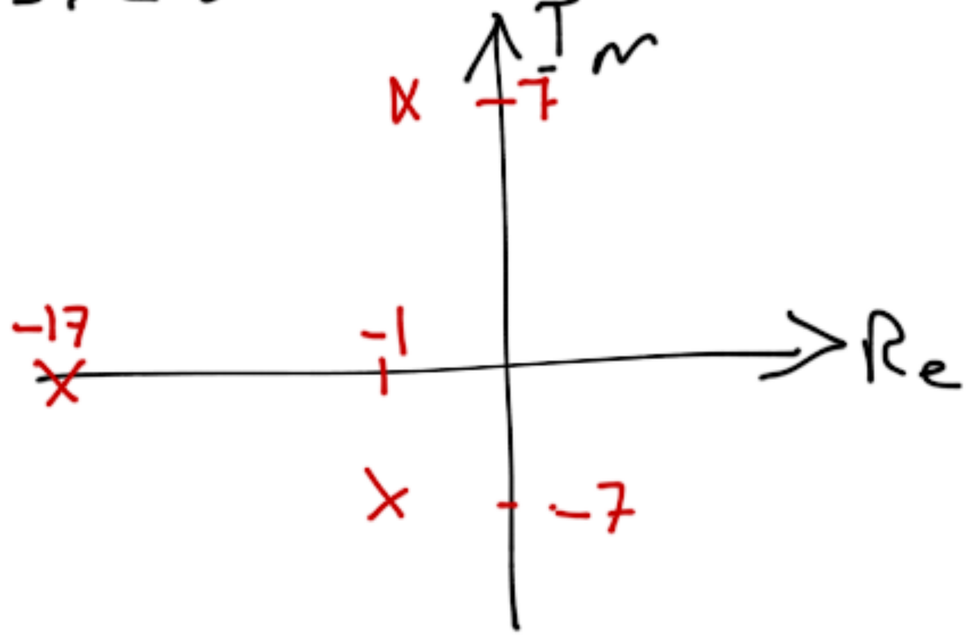


5.2b

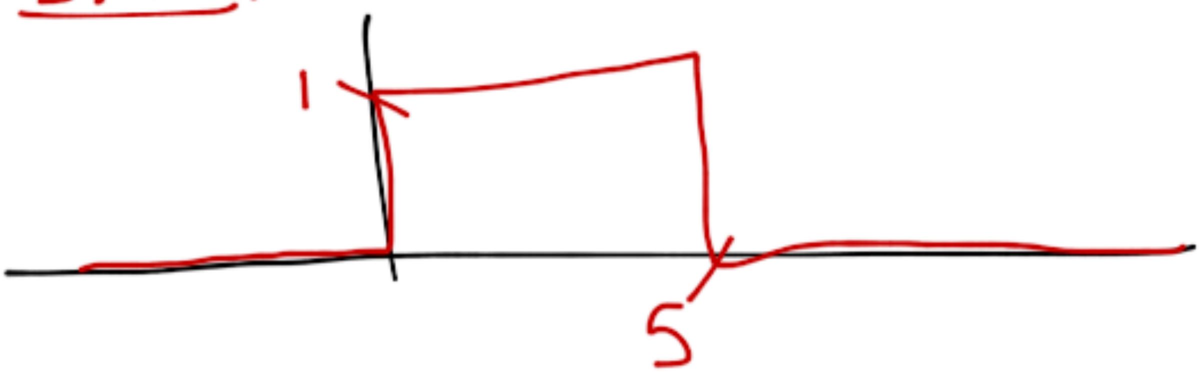
5.2c



5.2D:



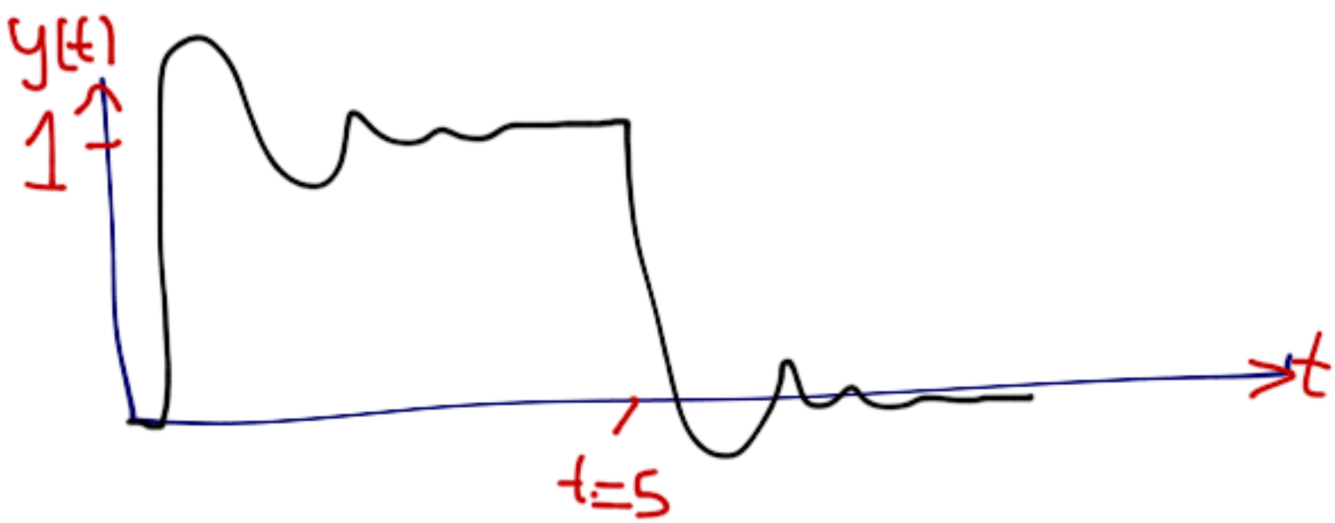
5.2E:



5.2 F:

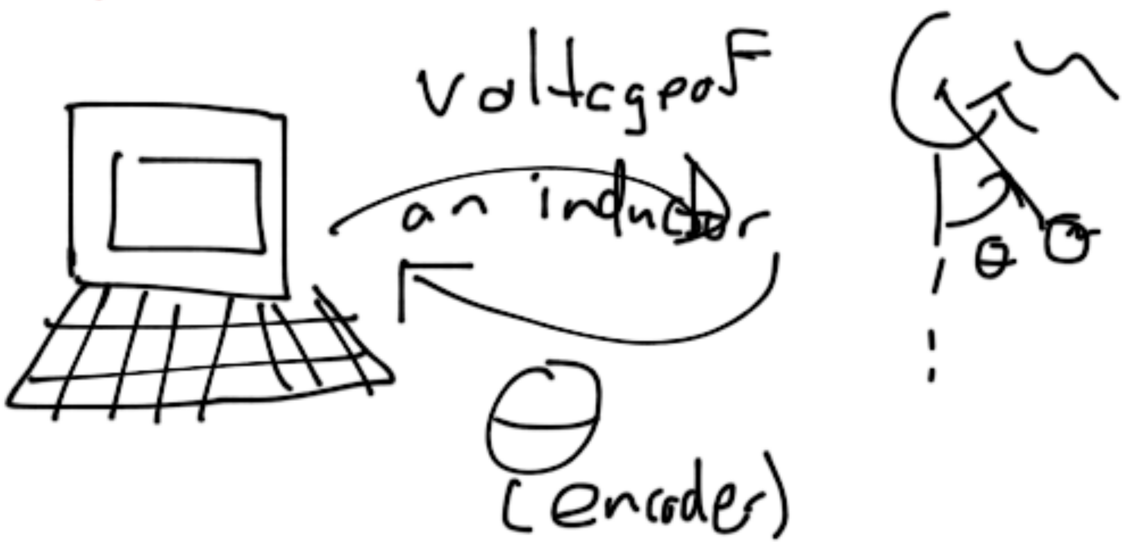
$t \in [0, 5]$ $\quad$ $t \geq 5$

5.2 G:

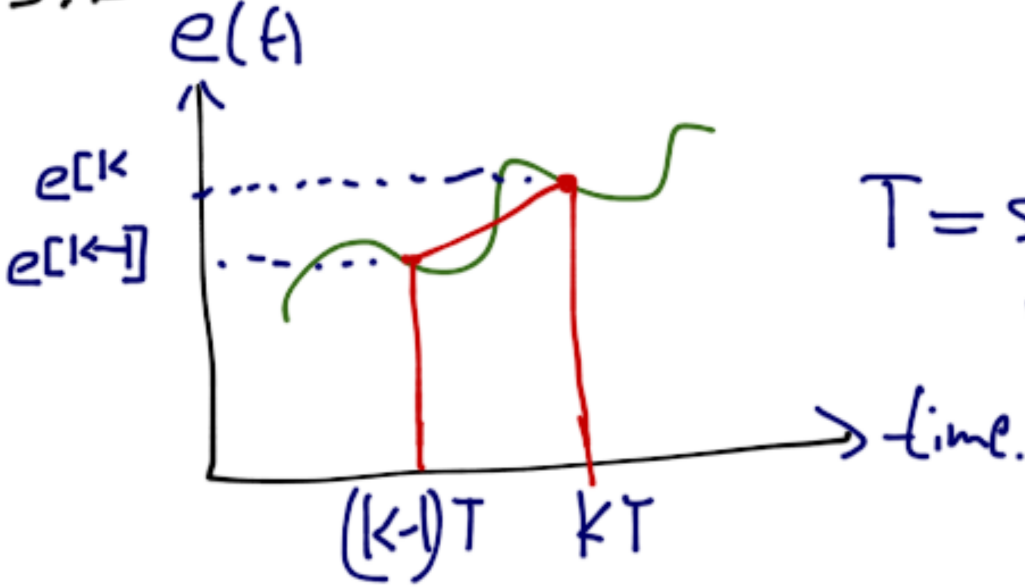


5.1

5.2

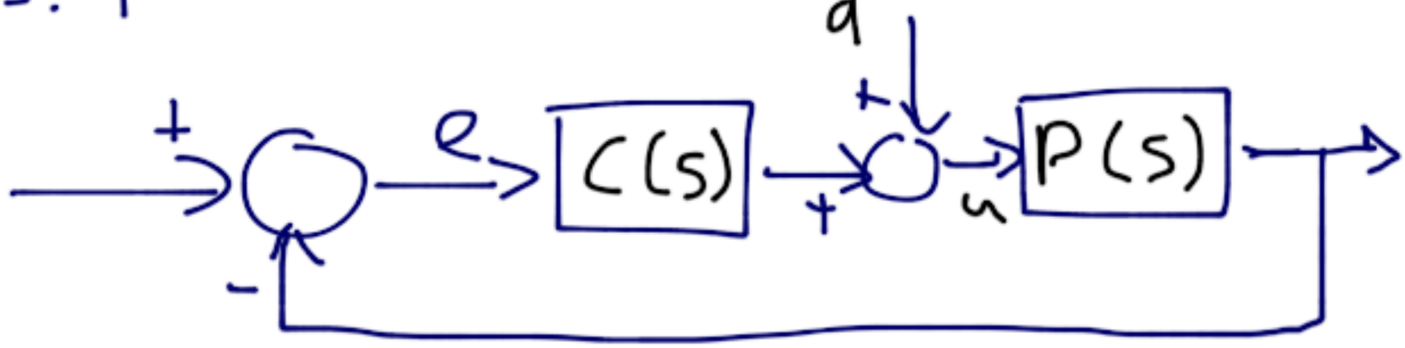


5.3

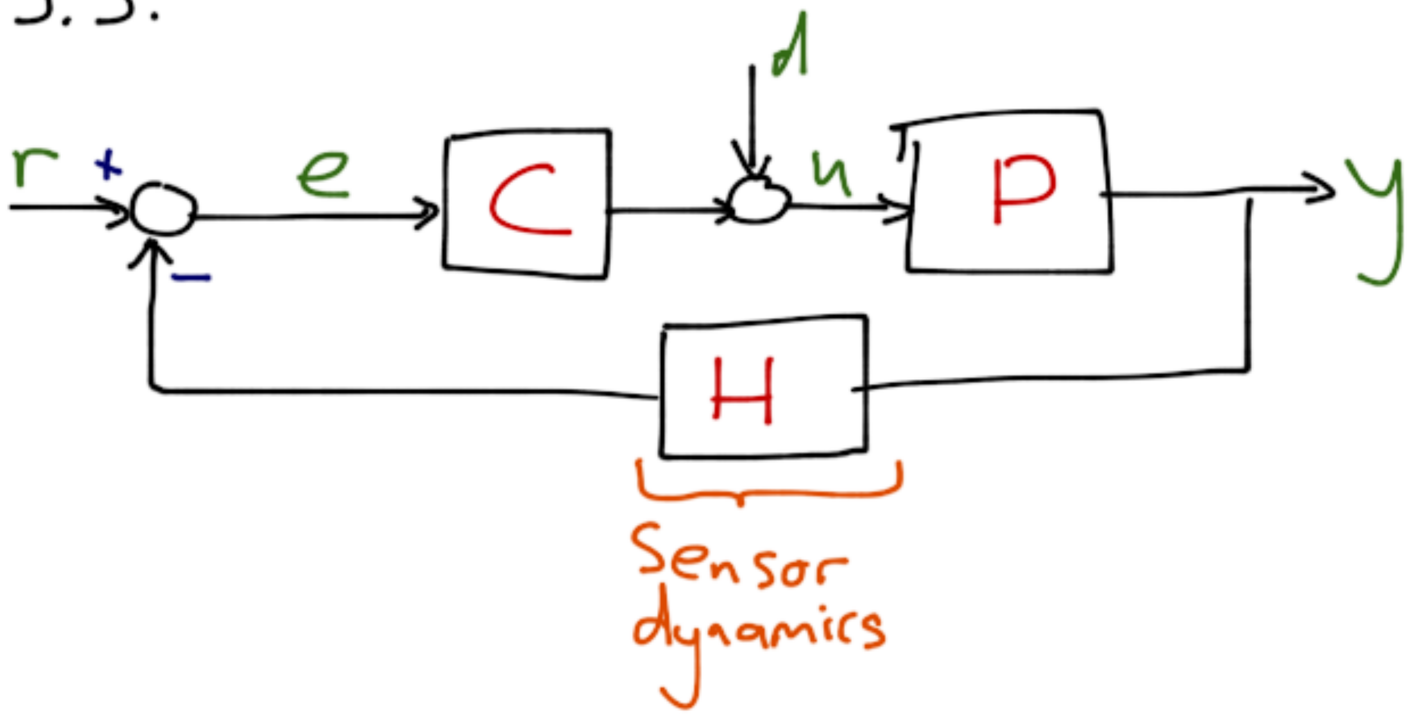


T = sampling period.

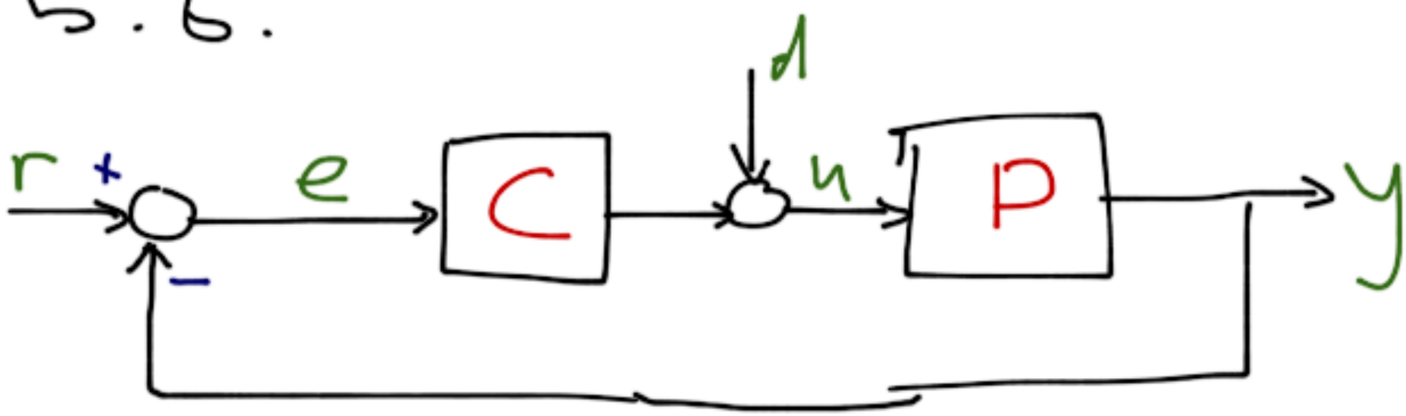
5.4



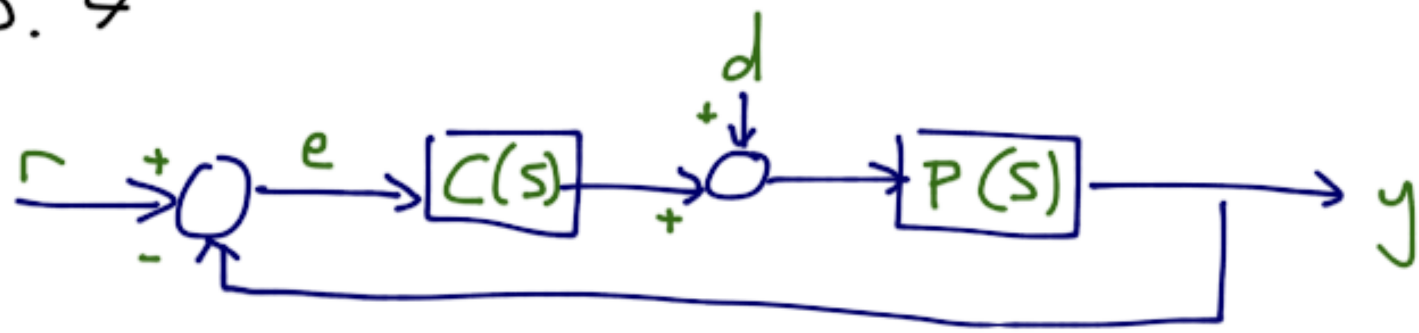
5.5:



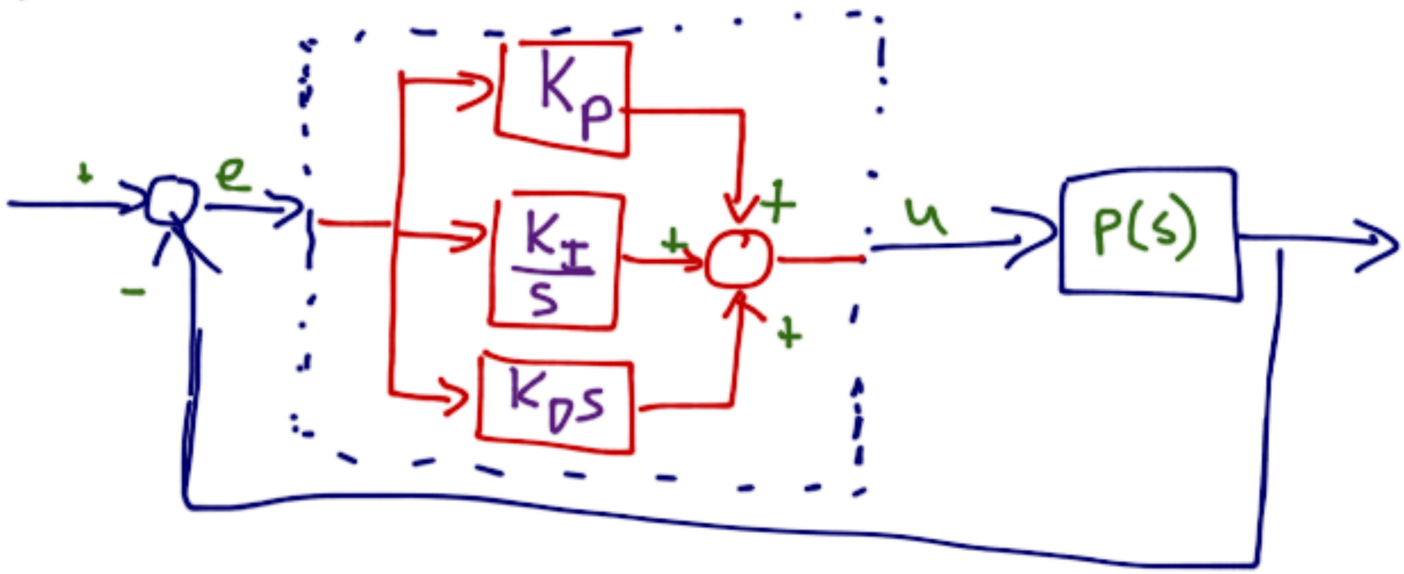
5.6:



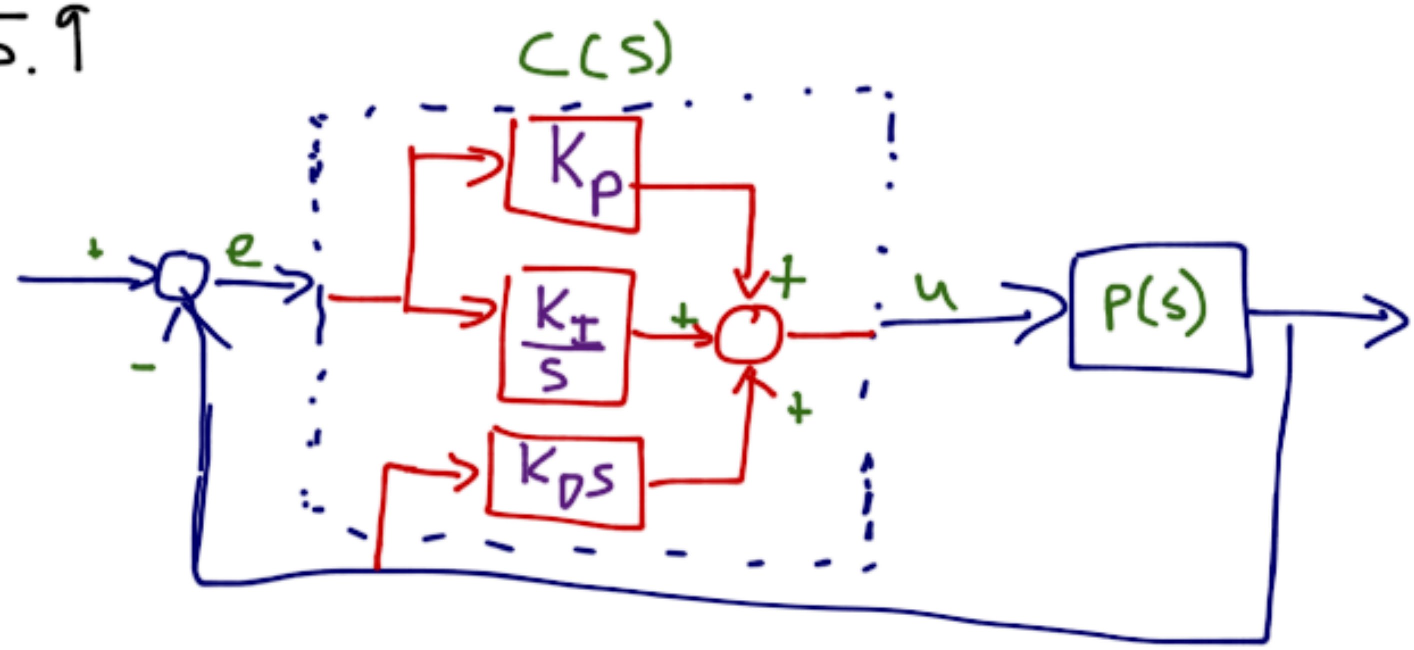
5.7



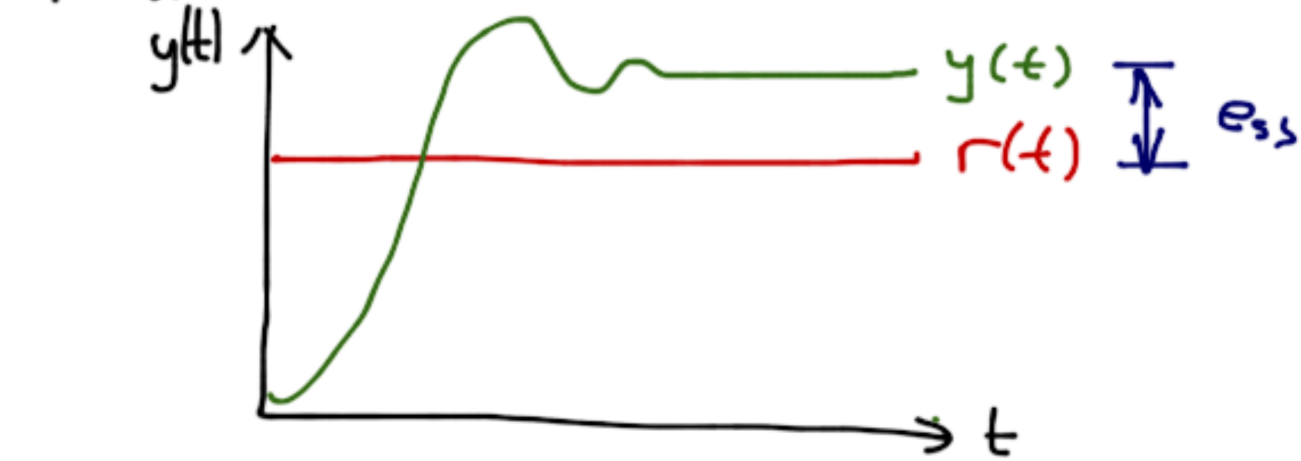
5.8



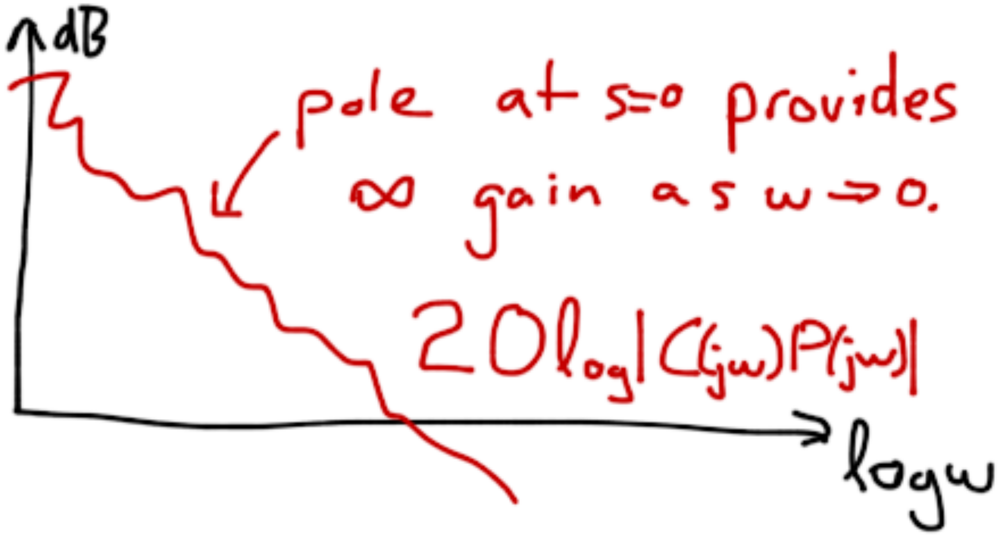
5.9



5.10a

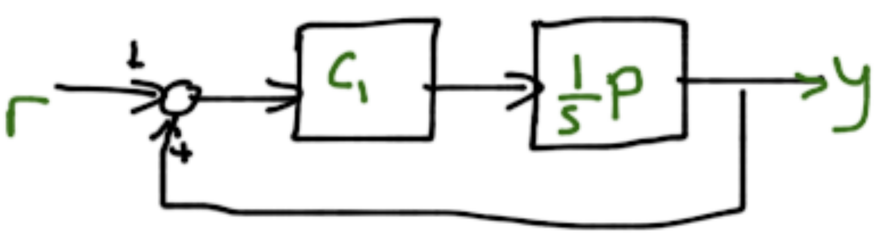


5.11a

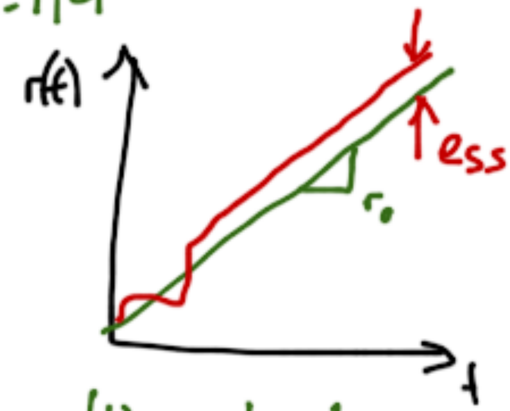


5.11b

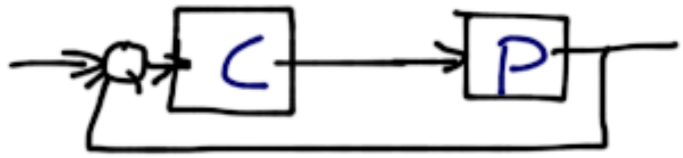
5.11c



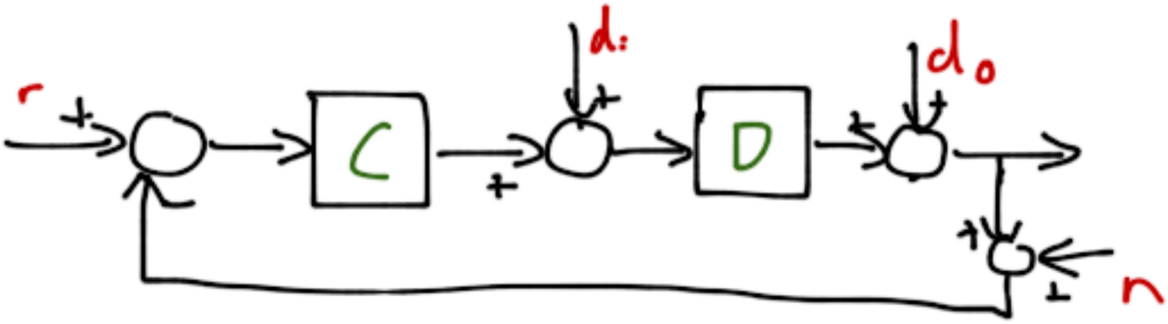
5.11d



$r(t) = r_0 t, \ t \geq 0$

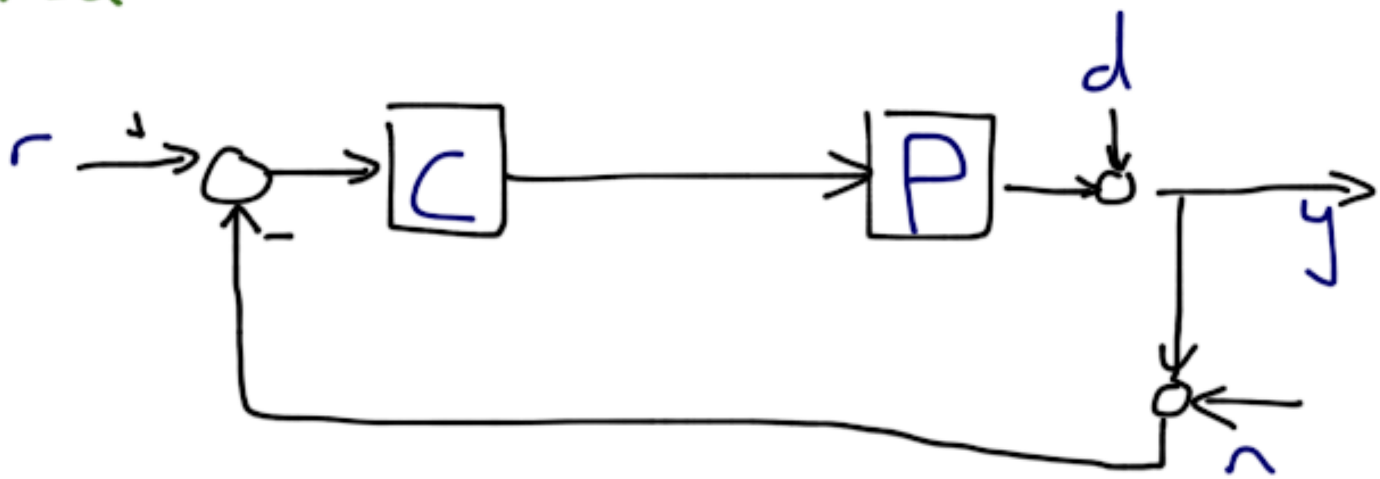


5.12a

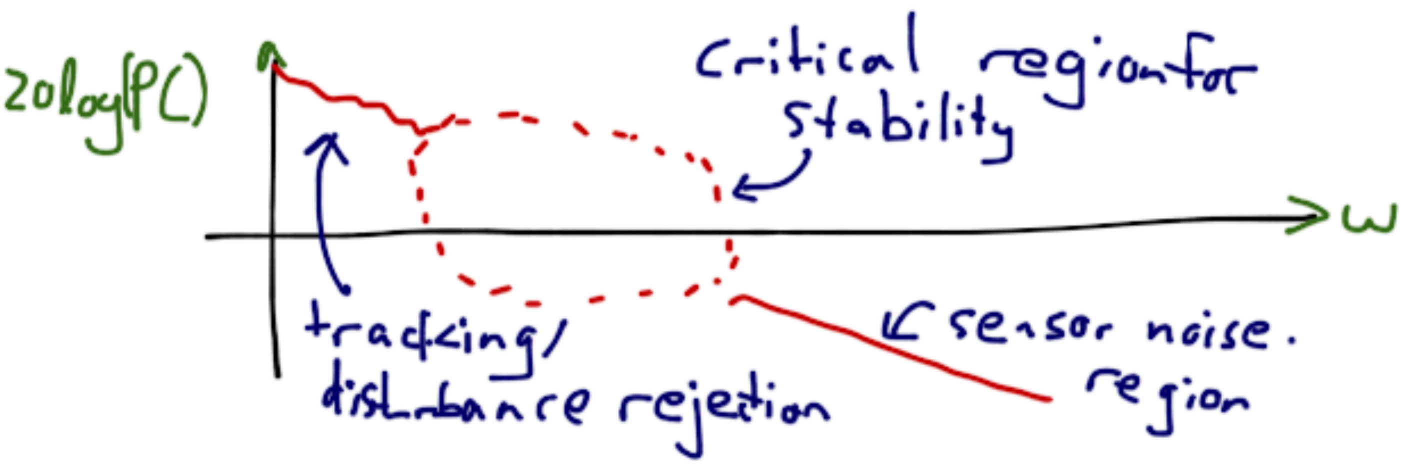


5.12b

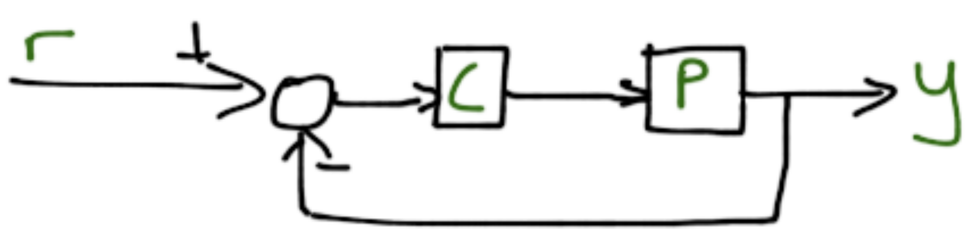
5.13a



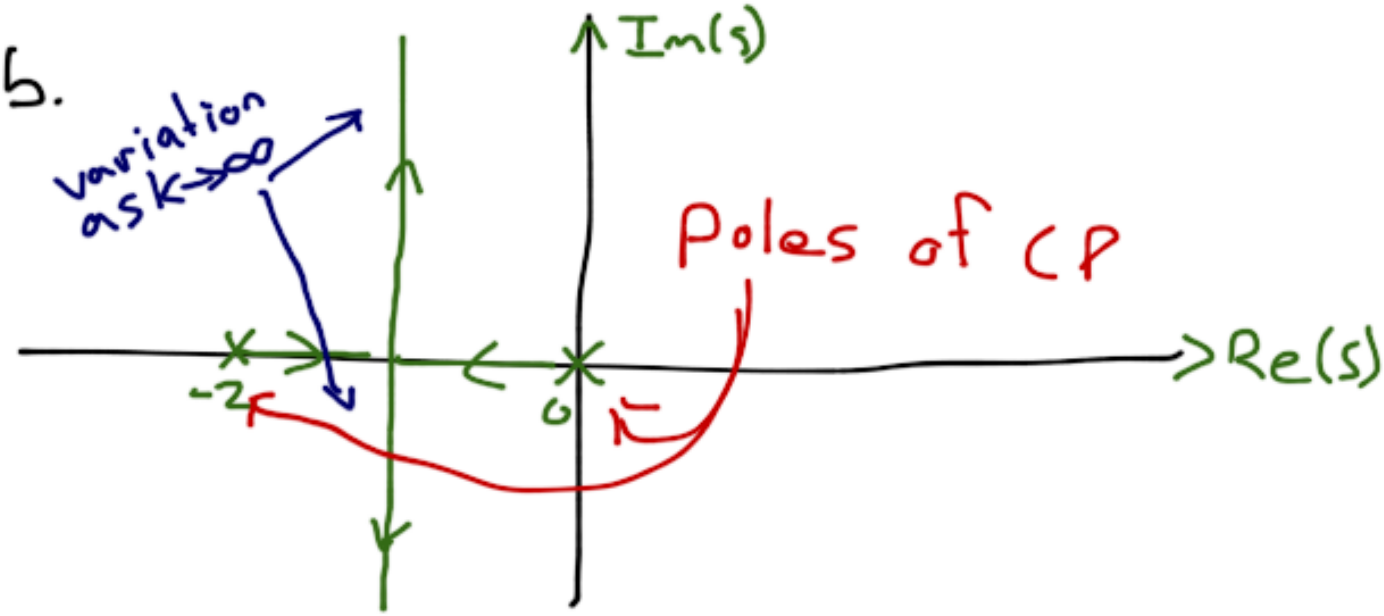
5.13b



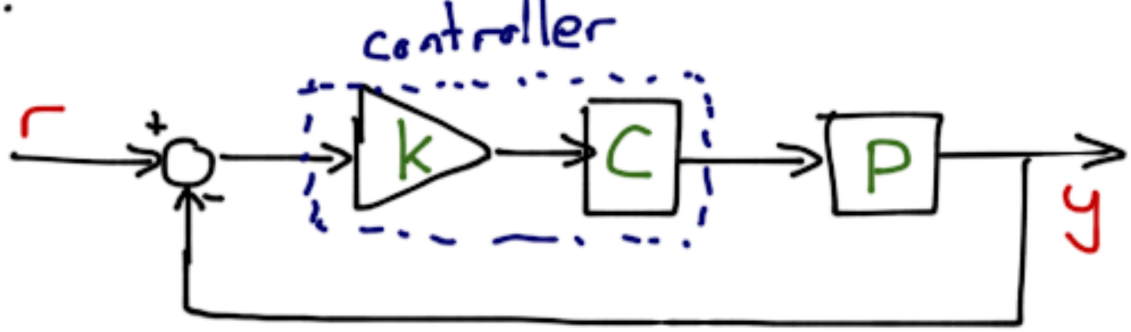
6.1a



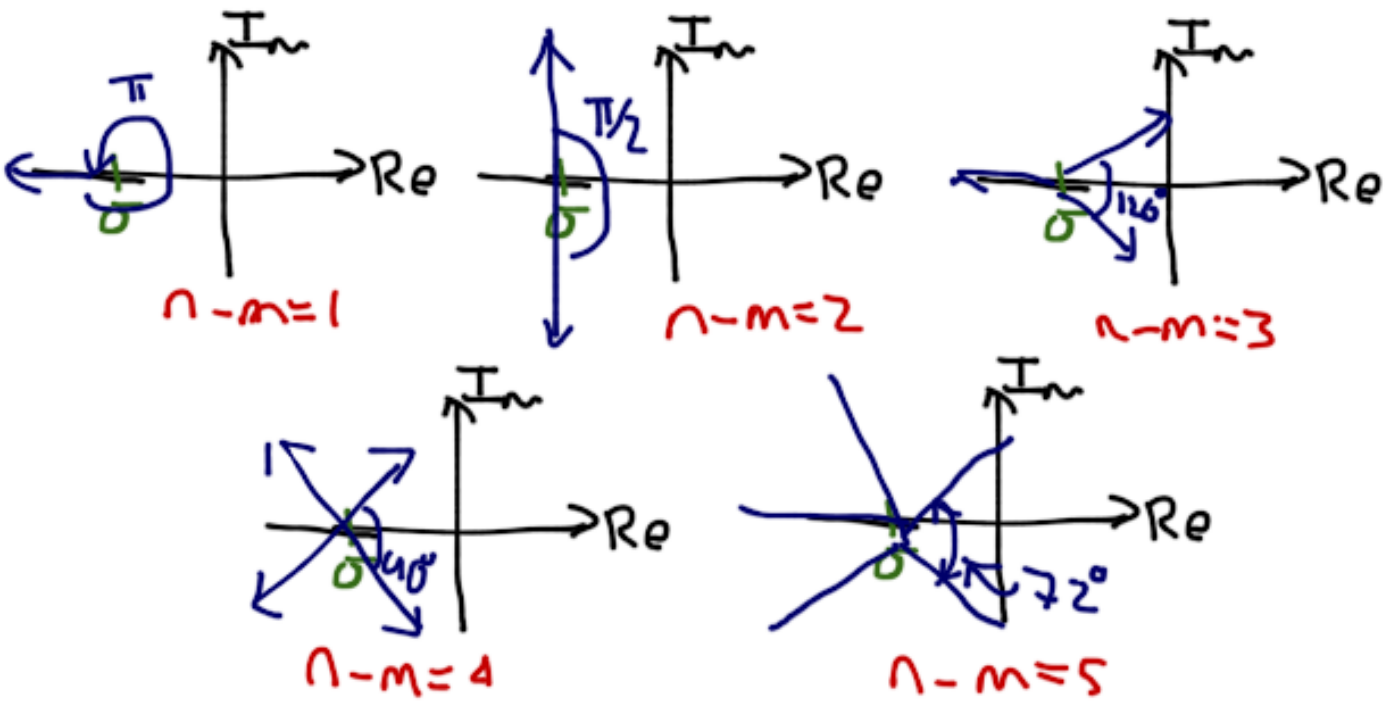
6.1b.


Im(s)
variation as k→∞
Poles of CP
Re(s)
-2
0

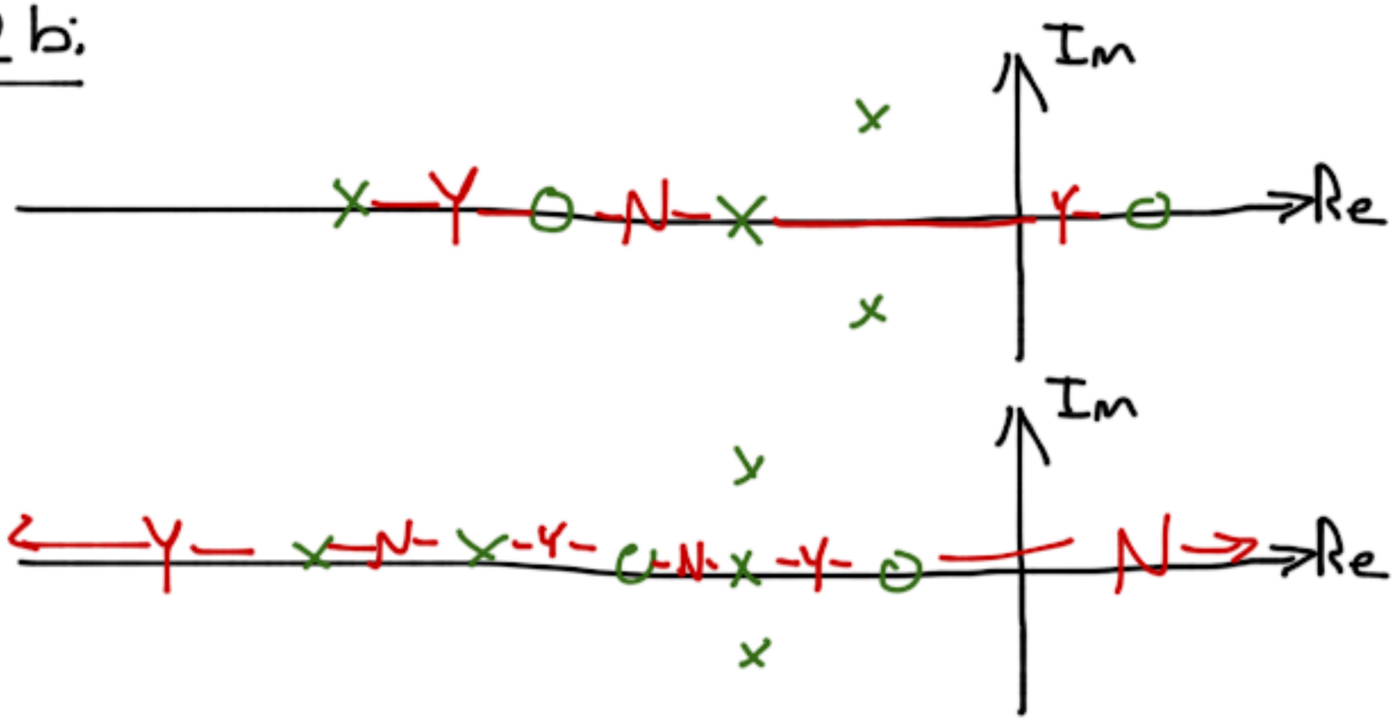
6.1c:


controller
r
+
-
k
C
P
y

6.2a

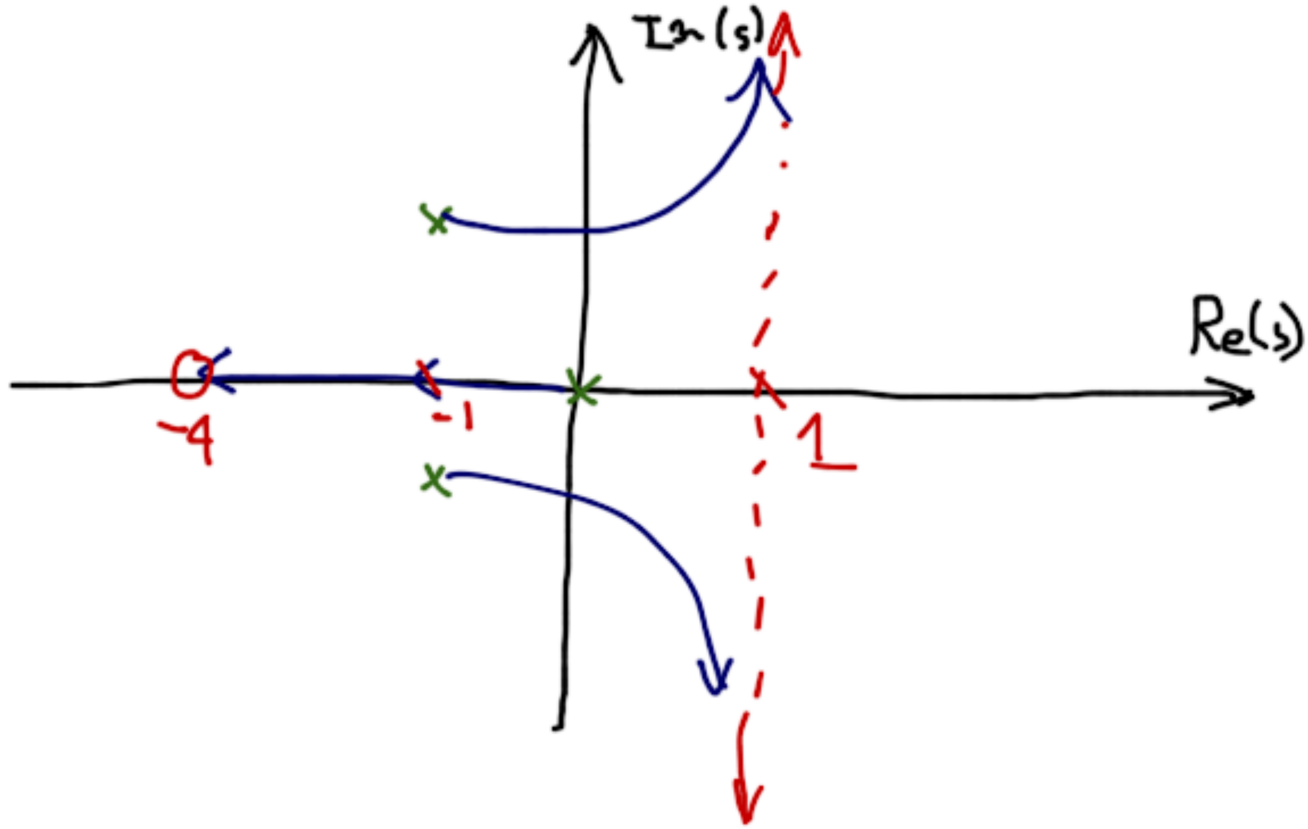

Im
Re
π
σ
n-m=1
π/2
n-m=2
120°
n-m=3
90°
n-m=4
72°
n-m=5

6.2b:

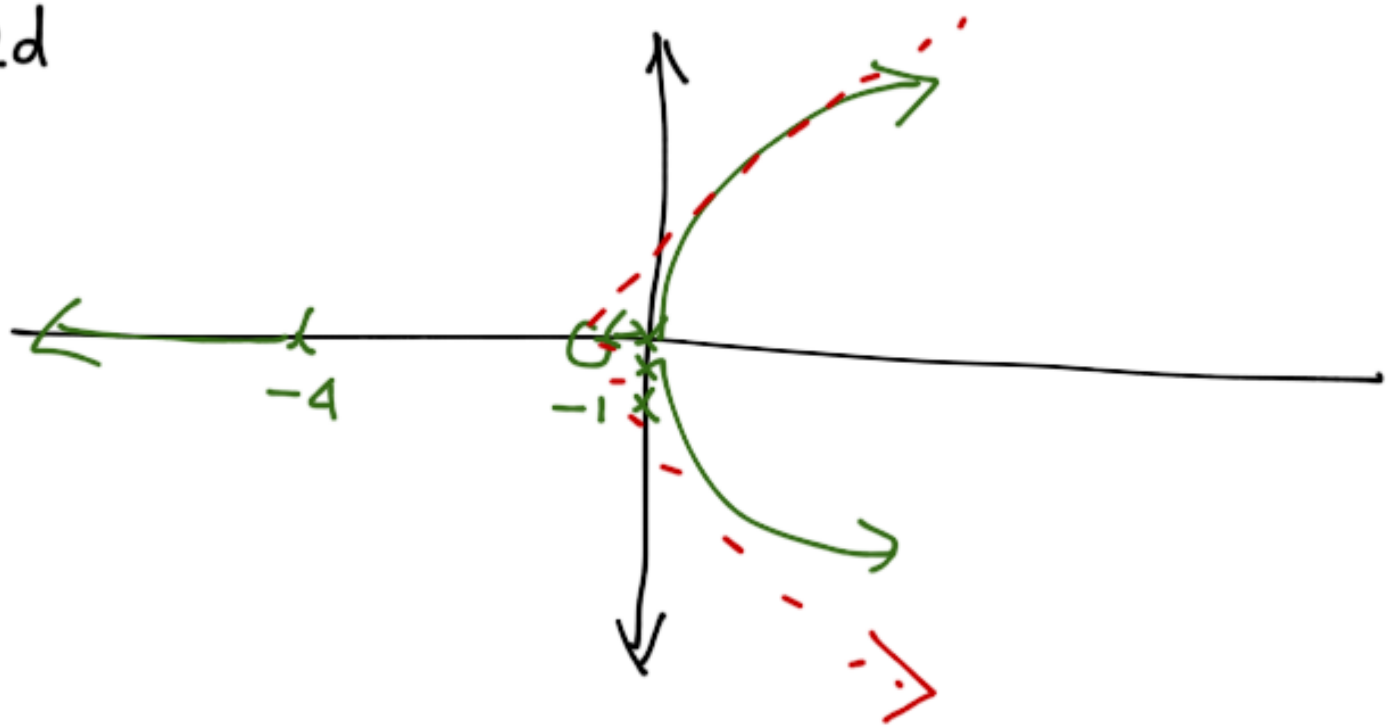


6.2c:

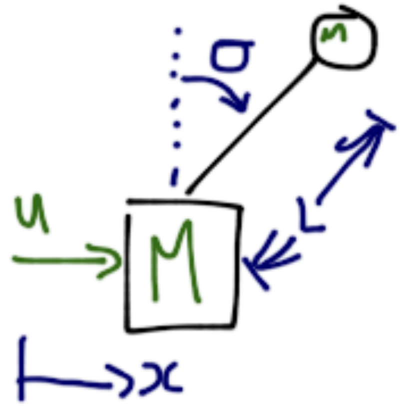
6.2 c



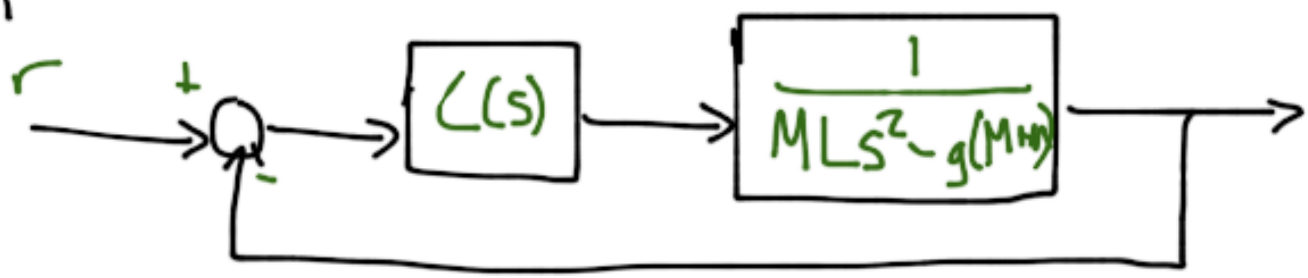
6.2d



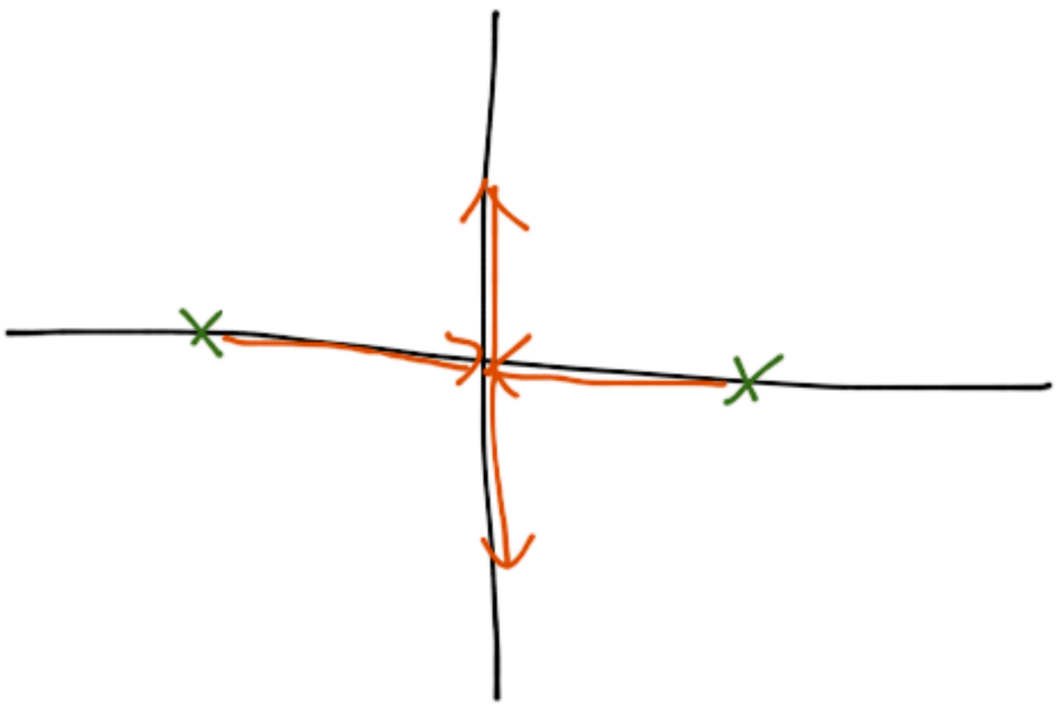
6.2 e



6.2f



6.2g



6.2h