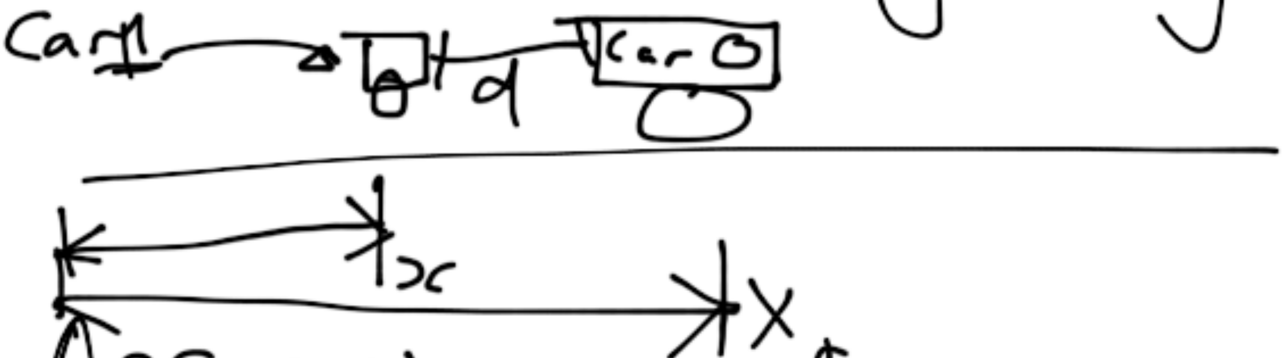
To-do:

☐ Sign up for pre-lab Jan 20.

Ch1:

What is ctrl eng?

Ex: Automated highway.



Assume:

- car only moves in x-dir
- we can directly assign vel.
- we have sensors to measure in front.

car0 moves at the speed lim. $v_0 > 0$

Obj: set the vel. of car1 to maintain a safe dist $d_{safe} > 0$ away from car_0

$$u = V_0 + k\,(d_{safe} - d)$$

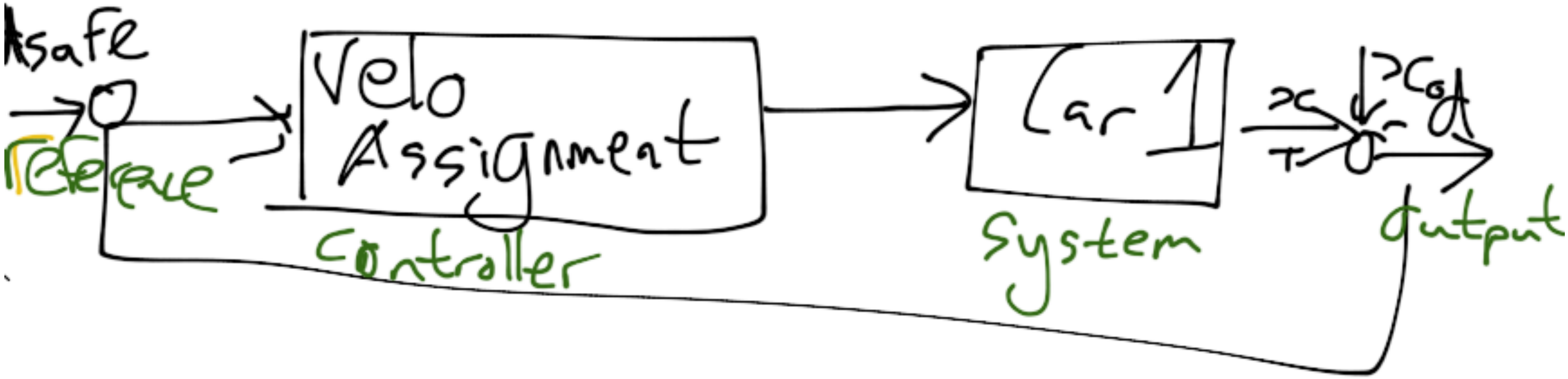
in this example, any $k<0$ works.

Ex: Queing System.

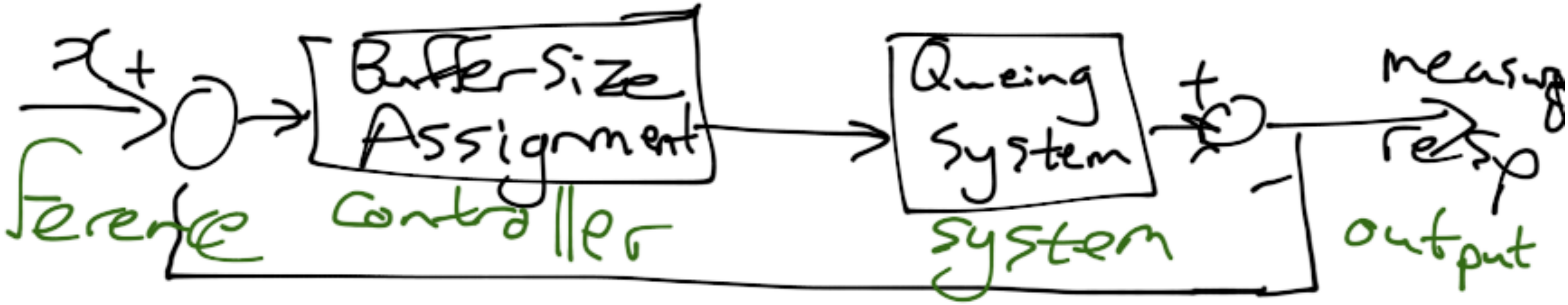
Assumption:

- Requests that don't fit buffer are redirected.
- two types of req's: buying & browsing.

Obj: Assign the buffer size to max the # of completed requests while ensuring "n" buying' req's are serviced in $x > 6$ seconds or less

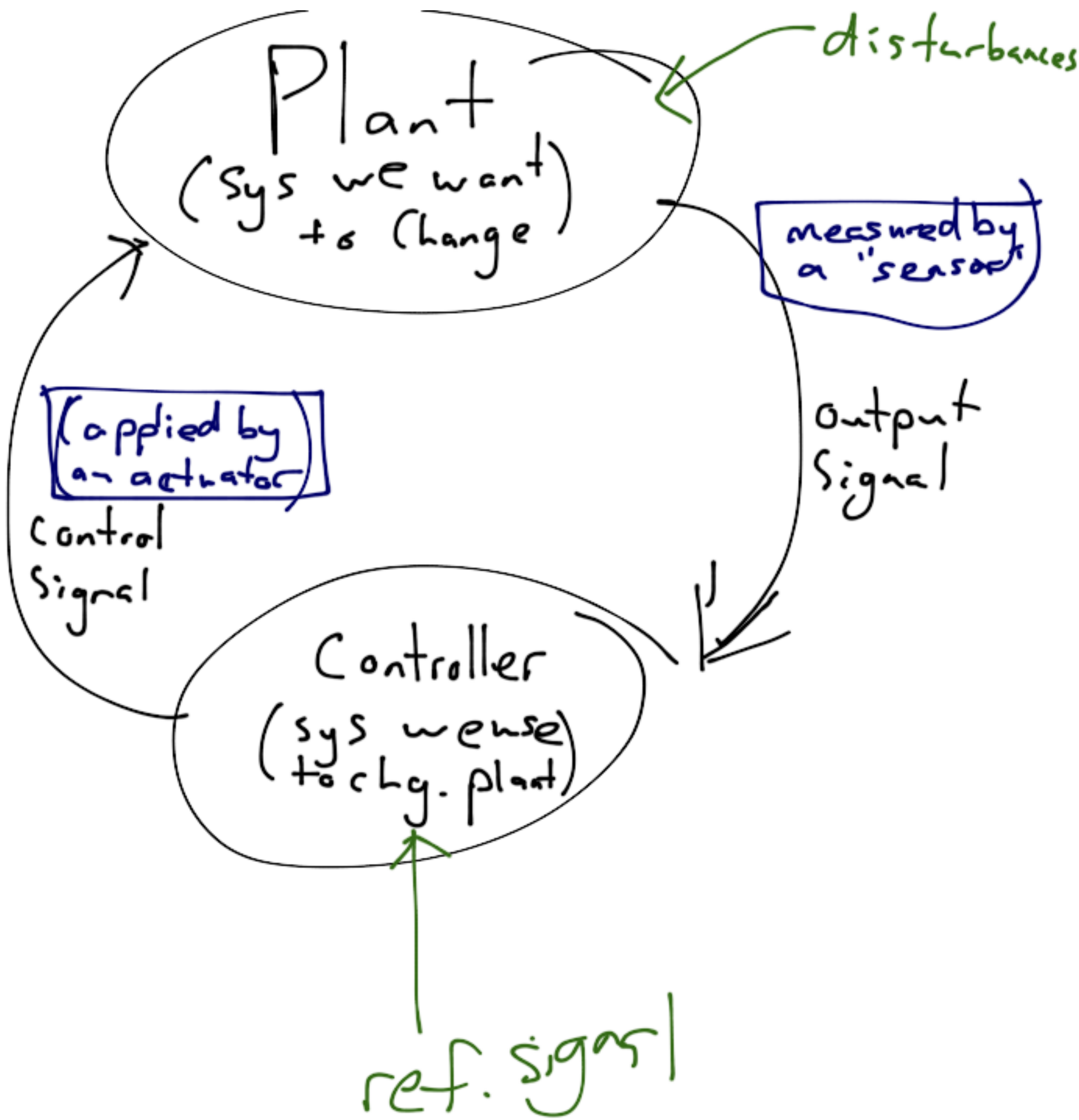


safe
reference
Velo Assignment
controller
Car 1
system
output



+
reference
Buffer Size Assignment
controller
Queing System
system
+
-
measured resp
output

system = plant | controller


Plant
(Sys we want to Change)
disturbances
measured by a "sensor"
output Signal
(applied by an actuator)
control Signal
Controller
(sys we use to chg. plant)
ref. signal

Ctrl Eng Design Cycle:

1. get specs:
   - closed-loop stability
   - good steady-state tracking
   - disturbance rejection
   - transient performance.
2. Model the plant
   - "model" = mathematical Model
   - typically ≥ 1 differential eq'ns

$$\frac{dx(t)}{dt} = u(t)$$

   - experiments are oft used to det the numerical values of plant params → "system identification"

3. Obtain transfer f'n of plant

Classical control (this course, PID controllers) requires we have a transfer function for the plant.

e.g. car:

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \mathcal{L}\left(\frac{u(t)}{dt}\right) \Rightarrow sX(s) = U(s)$$

$$\Rightarrow \frac{X(s)}{U(s)} = \frac{1}{s} = P(s)$$

P(S) is the transfer function of the plant.

N.B.: For discrete time, use Z-transforms instead (not on course, but...)

not on course

$\longrightarrow$

4. Design the controller:

- Controller is also a transfer f'n:
- It corresp. to the D.E that relates the control signal to the output signal.

Can check PID controller:

$$C(s) = \underset{(P)}{-2} \quad \underset{(I)}{-2/s} \quad \underset{D}{-\frac{s}{s+0.1}}$$

5. Simulation

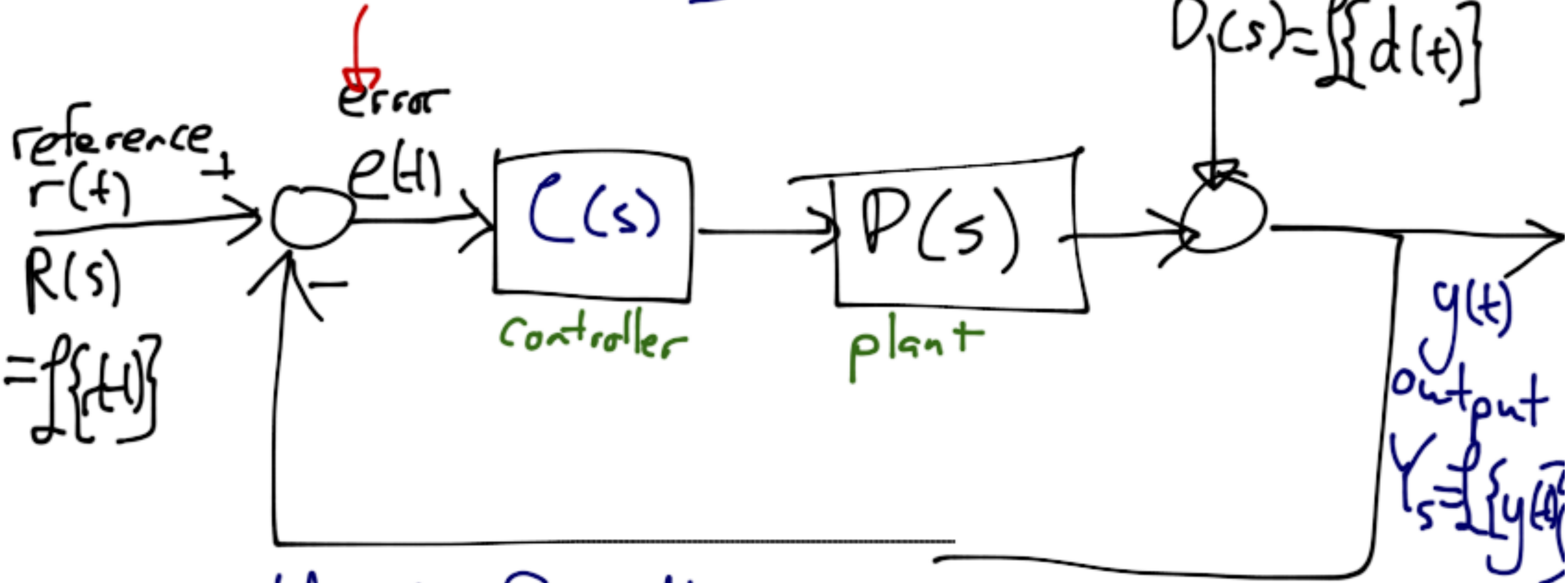
6. Implement Controller

- Can build a sys w/transfer f'n matches the one from step 4.
- Realistically the controller is implemented on a computer as a difference eq'n.

(ECE 481)

Basic Unity Feedback

$$E(s) = \mathcal{L}\{e(t)\}$$



Unity feedback control system.

Ch 2: Mathematical Models of Systems

For Controller design we want a "good" model ⇒ simple & effective

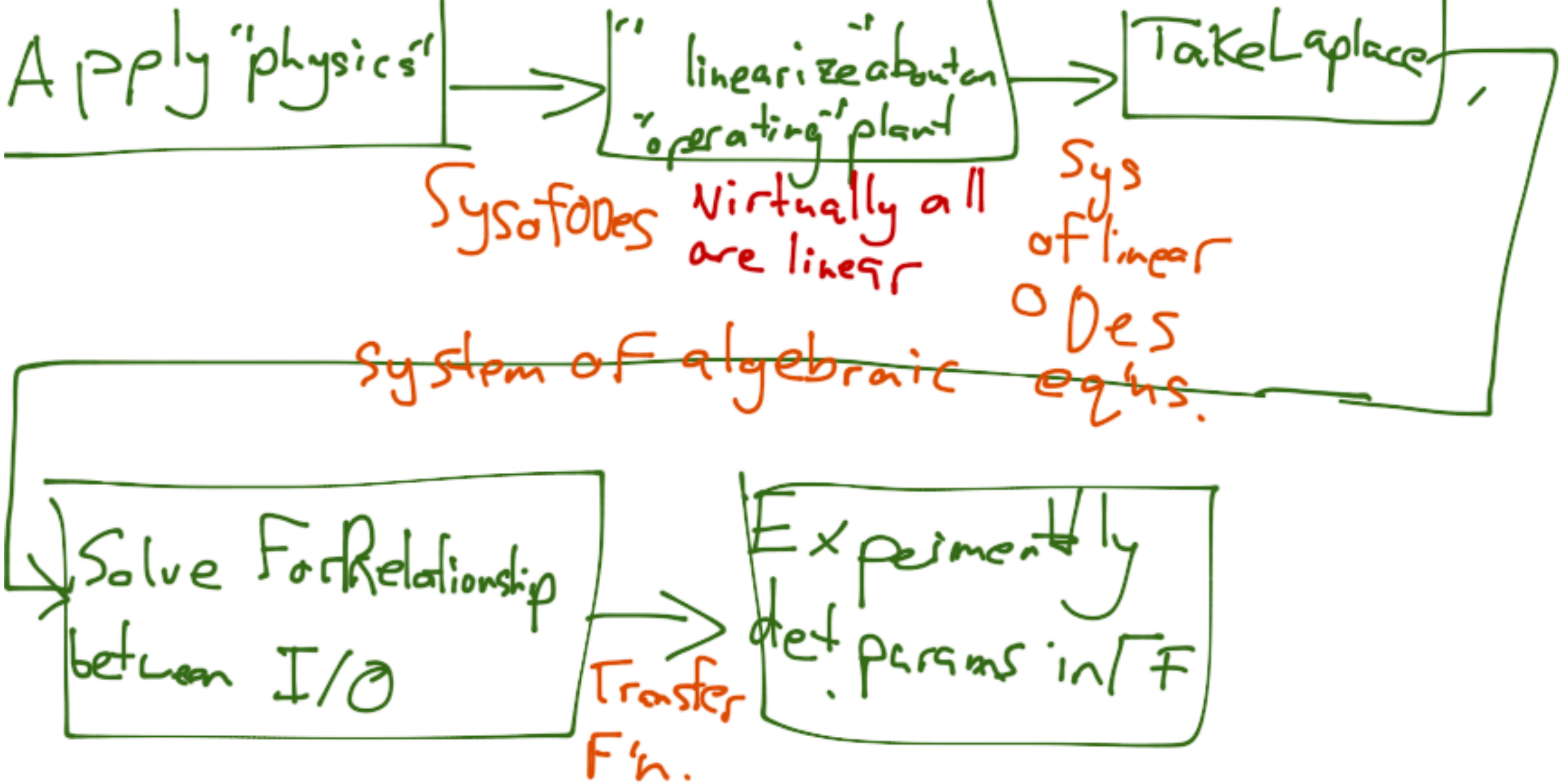
2.2: Comments on Modelling

→ A model is a set of equations used to represent a physical system.
⇒ Never perfect

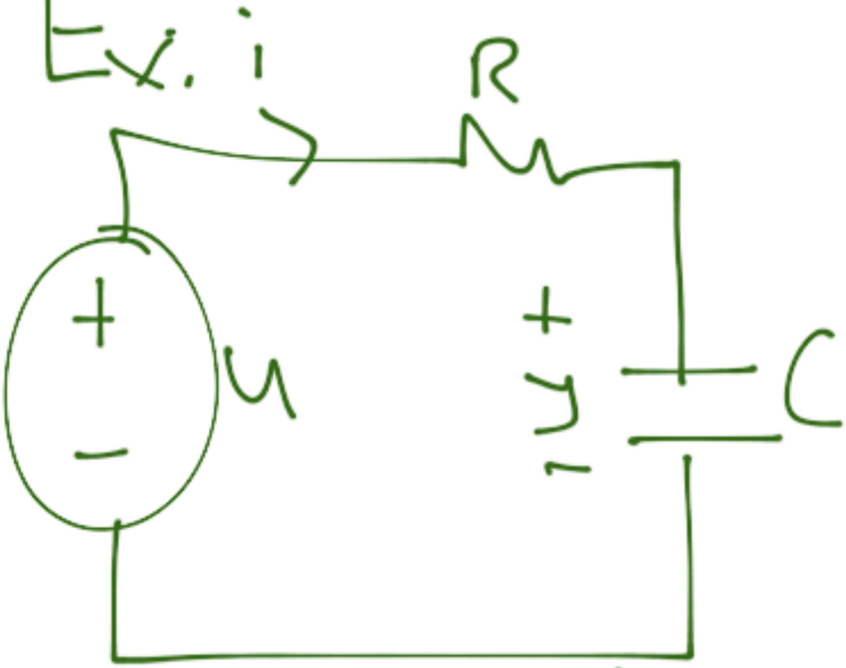
→ models for design are usually simpler rather than accurate. But most are a tradeoff.

→ There are two modelling approaches:
- → statistical (experimental)
- → Analytical (science+theory)

→ Use a modular approach - break pbm into sub-sections


Apply "physics"
Sys of ODEs
"linearize about an "operating" plant
Virtually all are linear
Take Laplace
Sys of linear ODEs
system of algebraic eq'ns.
Solve For Relationship between I/O
Transfer F'n.
Experimentally det. params in TF

Ex.



RC Circuit.

Ways to model Systems:

(a) Linear ODE:

$$-u + V_R + V_c = 0$$

$$V_R = Ri, \quad V_c = y, \quad i = C\frac{dV_c}{dt}$$

$$\boxed{\Rightarrow -u + RC\frac{dy}{dt} + y = 0}$$

$$\dot{y} = \frac{dy}{dt} \text{ (derp)}$$

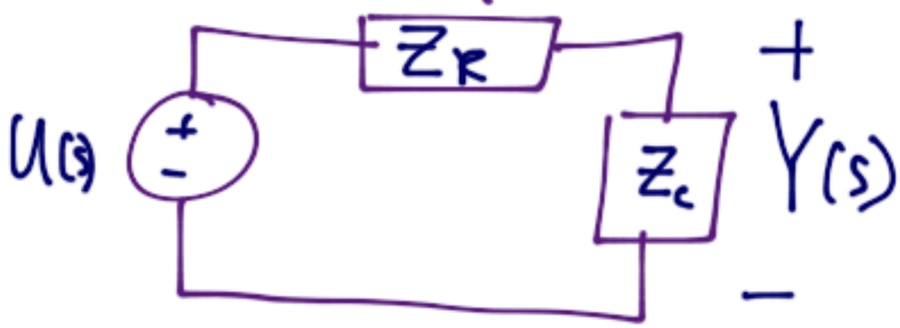
b) Transfer Fn:

i) L.T of ODE (assume 0 initial cond't)

$$-U(s) + sRCY(s) + Y(s) = 0$$

$$\boxed{\Rightarrow \frac{Y(s)}{u(s)} = \frac{1}{RCs+1}}$$

ii) use impedence and the voltage divider rule.



$$\left.\begin{aligned} Z_R &= R \\ Z_C &= \frac{1}{Cs} \end{aligned}\right\} \text{impedance.}$$

$$Y(s) = \frac{Z_c}{Z_c + Z_R} U(s) \quad \text{(voltage divider Rule)}$$

$$\Rightarrow Y(s) = \frac{1}{RCs+1} U(s)$$

Circuits Review:

$V = \frac{I}{R}$

KCL: $\sum V_{loop} = 0$

KVL: $\sum I_{node} = 0$

	Voltage	Current	Laplace
C	$v(t)$	$C\frac{dV}{dt}$	$V_{(s)} = \frac{I(s)}{sC}$
L	$L\frac{dI}{dt}$	$I(t)$	$V_{(s)} = sLI(s)$
R	IR	$\frac{V}{R}$	$V_{(s)} = RI(s)$

self-added

c) Convolution:

Let $G(s) = \frac{1}{RCs+1}$,

then $g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{RC} e^{-\frac{t}{RC}}$

the whole sys output can be expressed as:

$$y(t) = g(t) * u(t) = \underbrace{\int_0^t g(t-\tau)\,u(\tau)\,d\tau}_{\text{convolution}}$$

(d) Bode Plot:

→ next chapter

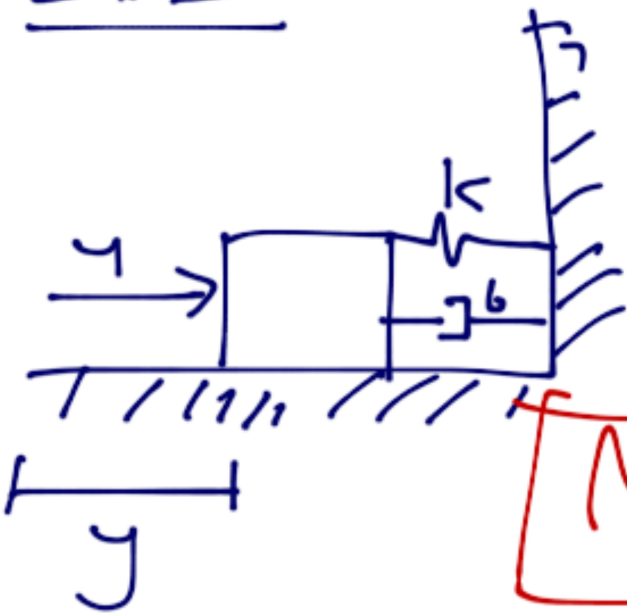
(e) state model:

→ 1+ 1st order ODEs.

$$\boxed{\begin{aligned} \dot{x} &= \frac{-1}{RC}x + \frac{1}{RC}u \\ y &= x \end{aligned}}$$

(x = voltage on capacitor)

Ex 1:



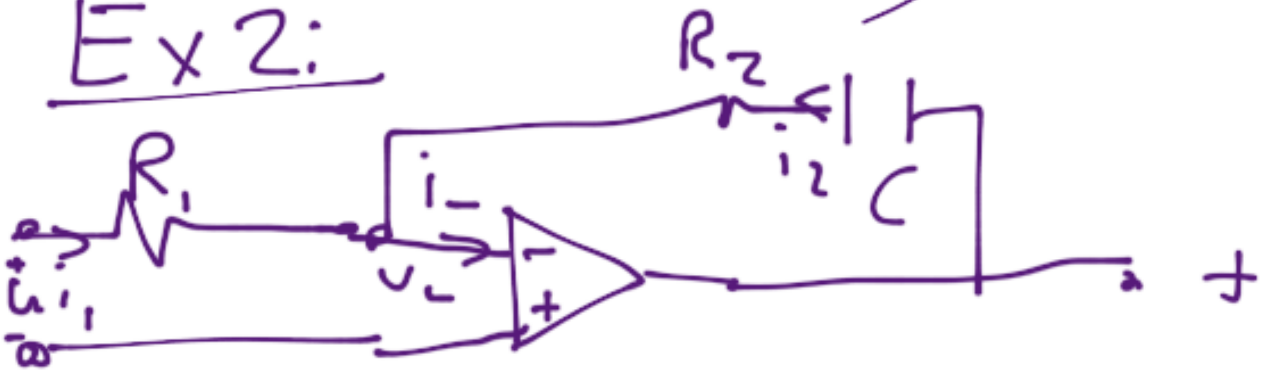
Newtons

$M\ddot{y} = \sum \text{Forces}$

$= u - F_{spring} - F_{damper}$

$$\boxed{M\ddot{y} = u - ky - b\dot{y}}$$

Ex 2:



Ideal op-amp

$i_- = 0, \quad v_- = v_+$

KCL: $i_1 + i_2 = 0$,

using impedences:

$0 = \frac{U(s) - V_-(s)}{Z_{R_1}} + \frac{Y - V_-(s)}{Z_{R_2} + Z_C}$

$$\Rightarrow \frac{Y(s)}{U(s)} = -\frac{R_2}{R_1} - \frac{1}{R_1 C s}$$

Summary:

The main point of Modeling is to obtain an approx model of the system through analysis & experiments.

A system can be expressed in multiple ways:

ODE

$\updownarrow$

Transfer Function

$\updownarrow$

Convolution

$\updownarrow$

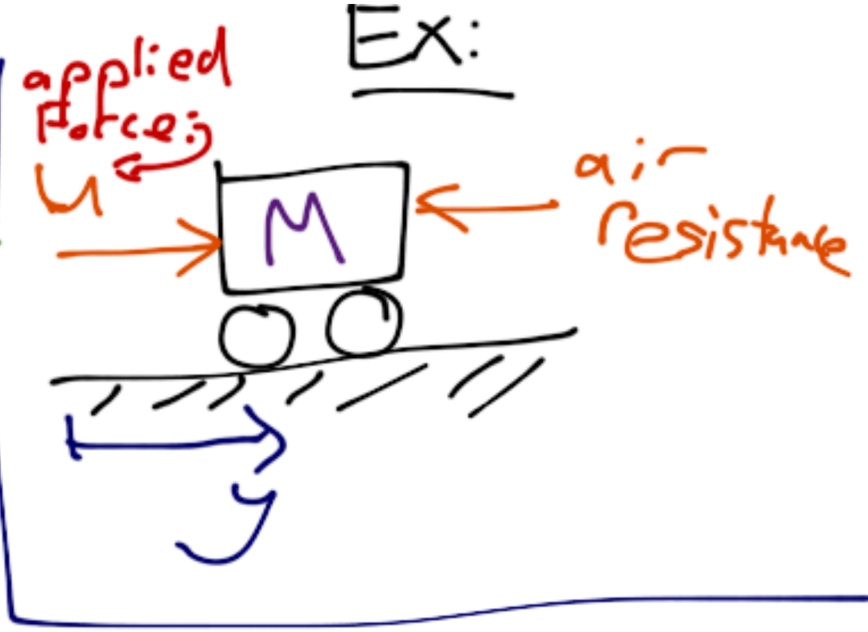
State Model

State x at time $t = t_0$, $x(t_0)$ encapsulates all the system dynamics up to time t_0.

For any times t_0, t_1, $t_0 < t_1$, knowing $x(t_0)$ and knowing the applied control $\{u(t) : t_0 \le t \le t_1\}$ we can compute $x(t_1)$ and hence $y(t_1)$

State Models:

Ex:



Typically air resistance creates an air force that depends on $\dot{y}$. Let's say this force is a possibly non-linear function $D(\dot{y})$

eg: $D(\dot{y}) = \dot{y}^2$

Newton's 2nd Law: $M\ddot{y} = u - D(\dot{y})$

Can we find a Transfer Function?

No. Not if $D(\dot{y})$ is non-linear.

Let's Def the State Variables:

$x_1 = y$, $x_2 = \dot{y}$, $\underbrace{x}_{\text{state of System}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$

System Of equations:

$$\left.\dot{X} = \begin{bmatrix} X_2 \\ \frac{1}{M}u - \frac{1}{M}D(x_2) \end{bmatrix}\right\} \text{state equations}$$

$$\left. y = x_1 \right\} \text{output eq'n}$$

of the form $\dot{X} = f(x, u)$, $y = h(\overbrace{x}^{\text{Sometimes "u" too.}})$

Idea: in the case that $D(x_2)$ is not linear, we define a system of equations that we can ~~so~~ solve.

System of eq'ns: $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = f(x, u)$

$y = h(x, u)$

const (not a function)

useful for the case that

$\dot{y}^a$ (or others) have $a > 1$ $\mathcal{L}$ doesn't work for these functions and we need to do it this way.

In the case that $D(x_2)$ is linear,

e.g: $D(x_2) = d x_2$ (d is a const)

then,

$$\dot{x} = f(x,u) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \frac{-d}{M} \end{bmatrix}}_{\frac{\text{constant}}{\text{matrix}}\ \text{"}A\text{"}} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{\frac{\text{constant}}{\text{matrix}}\ \text{"}B\text{"}} u$$

so: $f(x,u) = Ax + Bu = \dot{x}$

In general, we have:

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

$C = [0 \ \ 1]$;

Linear Time invariant state mode

if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} x = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$

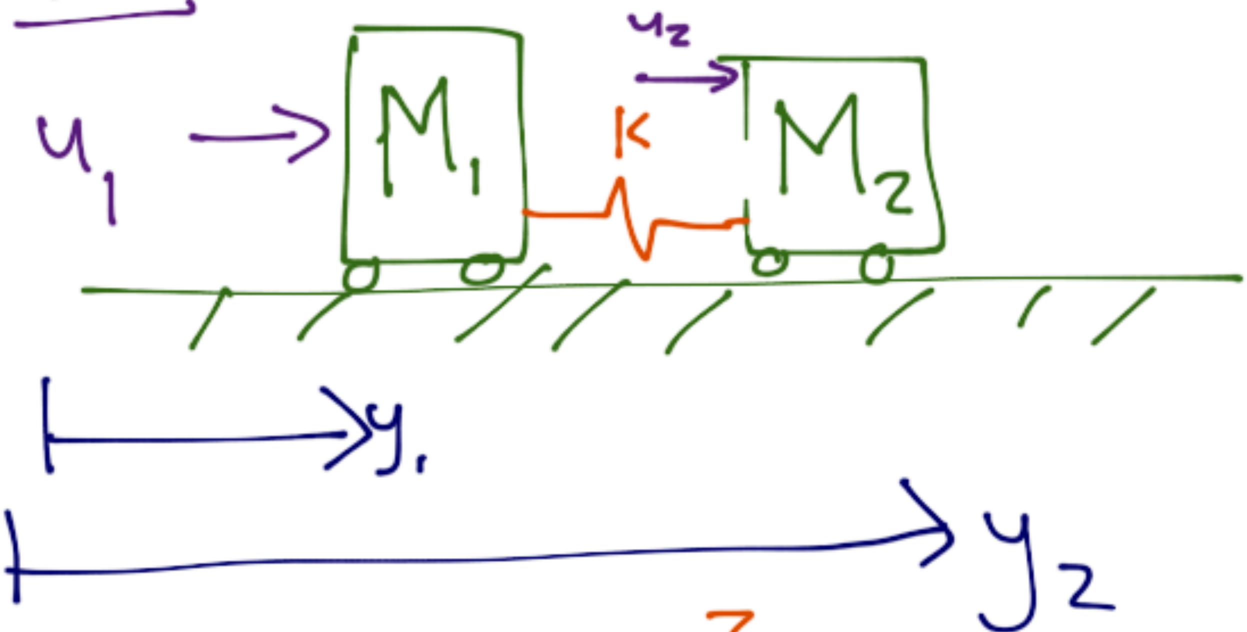
The idea is that many states have the model

$$\dot{x} = f(x, u), \quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

$$y = h(x, u), \quad h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$$

output (y) dimension p
input (u) dimension m
state (x) dimension n

Ex:



$u=(u_1,u_2)\in\mathbb{R}^2 \Rightarrow m=2$

$y=[y_1,y_2]\in\mathbb{R}^2 \Rightarrow p=2$

$x=[y_1,\dot{y}_1,y_2,\dot{y}_2]\in\mathbb{R}^4 \Rightarrow n=4$

the linear time-invariant special-case:

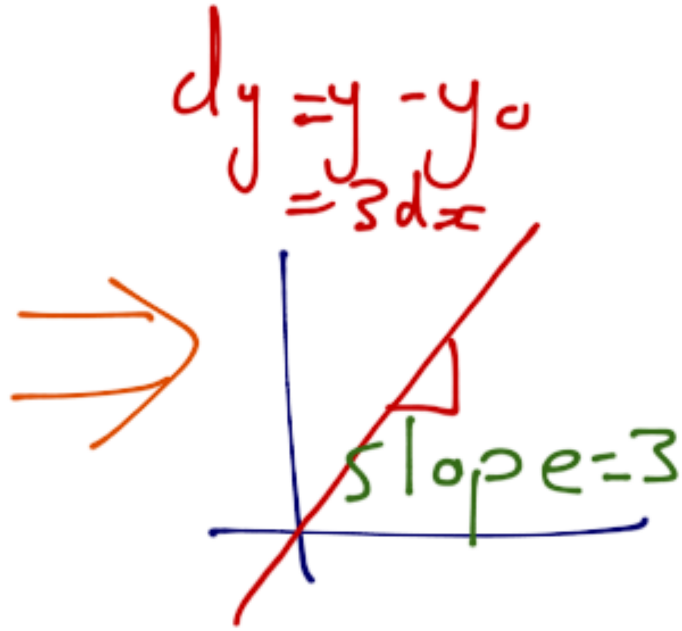
$\dot{x}=Ax+Bu$ $\quad A\in\mathbb{R}^{n\times n}$ $\quad B\in\mathbb{R}^{n\times m}$

$y=Cx+Du$ $\quad C\in\mathbb{R}^{p\times n}$ $\quad D\in\mathbb{R}^{p\times m}$

in this course we only deal w/ time-invariant systems.

Summary (Add JWei's notes Here)

- States:
- linearization
  - static case:



- vector case — $dy = \left.\frac{df}{dx}\right|_{x=x_0} dx$

- Apply this to: $\dot{x} = f(x,u)$

  $y = h(x,u)$

First assume (x_0, u_0) is an eq'bm

$$f(x_0, u_0) = 0$$

$$f(x,u) = f(x_0,u_0) + \underbrace{\left.\frac{df}{dx}\right|_{\substack{x=x_0\\u=u_0}}}_{A}(x-x_0) + \underbrace{\left.\frac{df}{du}\right|_{\substack{x=x_0\\u=u_0}}}_{B}(u-u_0)$$

+ higher order terms

Now consider a solution "near" the constant solution

$$x(t) = x_0 + \delta x(t)$$
$$u(t) = u_0 + \delta u(t)$$

$$\dot{\delta x} = \frac{d(\delta x(t))}{dt} = \frac{dx(t)}{dt} - \frac{d(x_0)}{dt}$$

$$= f(x,u) - 0$$

$$\approx f(x_0, u_0) + A\delta x + B\delta u$$

$$= A\delta x + B\delta u$$

$$\Rightarrow \dot{\delta x} = A\delta x + B\delta u$$

We can linearize the output eq'n as in the previous example.

$$\delta y = \underbrace{\frac{\partial h}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}}}_{C} \delta x + \underbrace{\frac{\partial h}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}}}_{D} \delta u$$

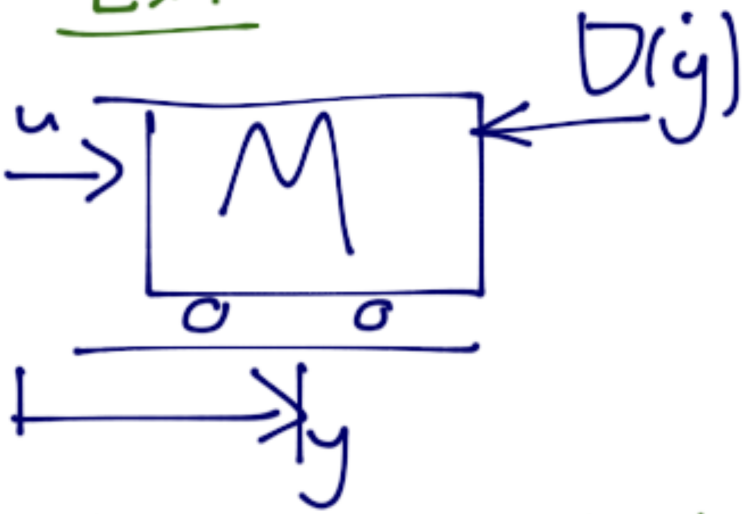
Summary: Linearization

1) Select, if one exists an equilibrium (x_0, u_0)
2) Compute A, B, C, D (Jacobians)
3) The linearized system is:

$$\boxed{\begin{aligned} \delta \dot{x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u \end{aligned}}$$

4) Under mild conditions the linearization is a good approx for 'small' δx and δu.

Ex:



Cart example @ position $y_0 = 10$

$D(\dot{y}) = k\dot{y}^2, \; k > 10$

i) The model is $\dot{x} = f(x,u) = \begin{bmatrix} x_2 \\ \dfrac{-kx_2^2}{M} + \dfrac{u}{M} \end{bmatrix}$

$y = h(x,u) = x_1$

We need an operating point (x_0, u_0) corresponding to $y = x_1 = 10$.

Solve: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = f(x_0, u_0) = \begin{bmatrix} x_{20} \\ \dfrac{-kx_{20}^2}{M} + \dfrac{u_0}{M} \end{bmatrix}$

$\Rightarrow x_{20} = 0, u_0 = 0$

$\Rightarrow (x_0, u_0) = \left(\begin{bmatrix} 10 \\ 0 \end{bmatrix}, 0 \right)$

2.)

$$A = \frac{\delta f}{\delta x}\bigg|_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-2Kx_2}{M} \end{bmatrix}_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 0 & 1 \\ 0 & d \end{bmatrix}$$

$$B = \frac{\delta f}{\delta u}\bigg|_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} \frac{\delta f_1}{\delta u} \\ \frac{\delta f_2}{\delta u} \end{bmatrix}_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

$$C = \frac{\delta h}{\delta x}\bigg|_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} \frac{\delta h}{\delta x_1} & \frac{\delta h}{\delta x_2} \end{bmatrix}_{\substack{x=x_0\\u=u_0}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = \frac{\delta h}{\delta u}\bigg|_{\substack{x=x_0\\u=u_0}} = 0$$

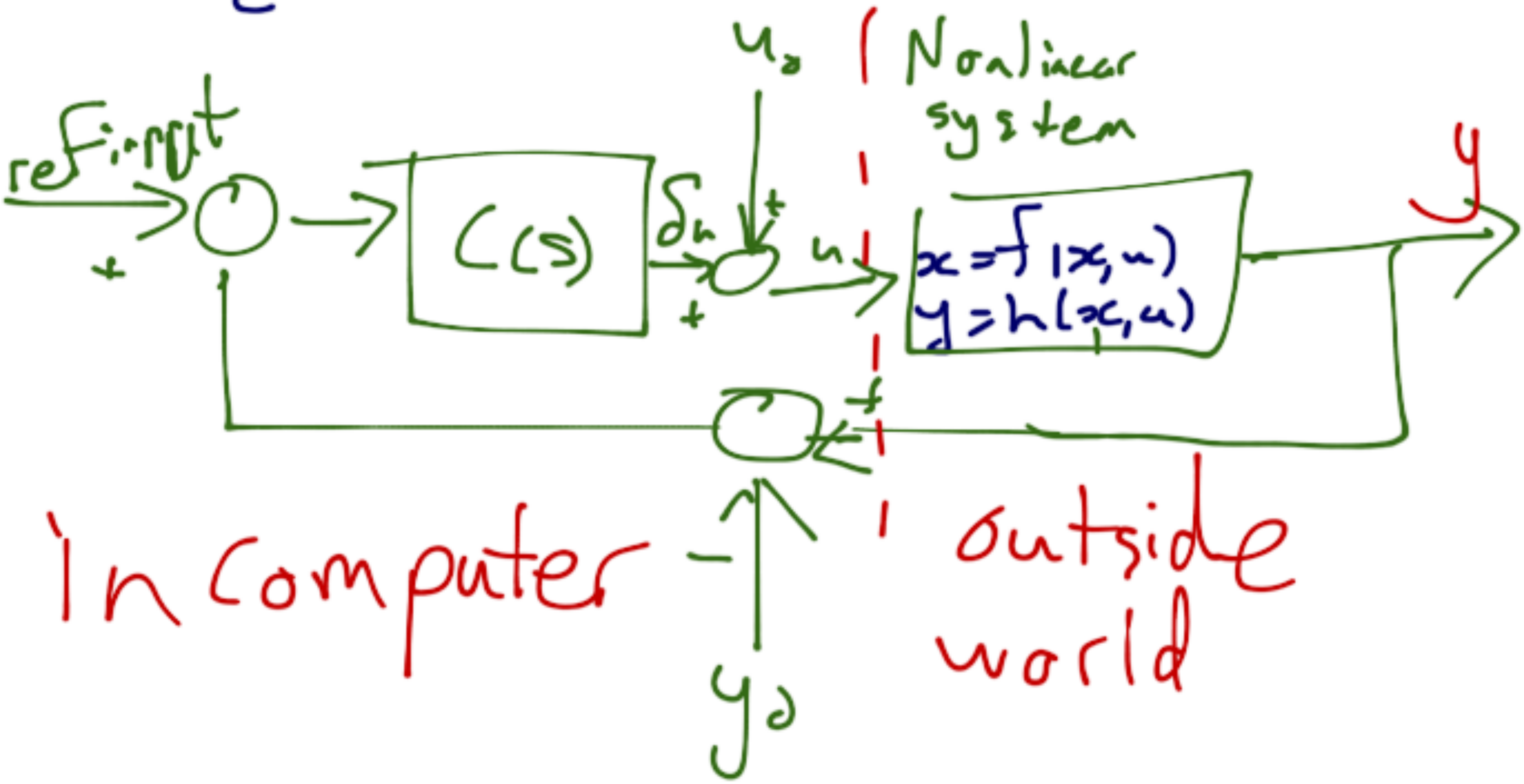
3) Linear Model

$$\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

$y = 10$
$u = 0$
$\dot{y} = 0$

Implementing a controller designed using a linearized model:

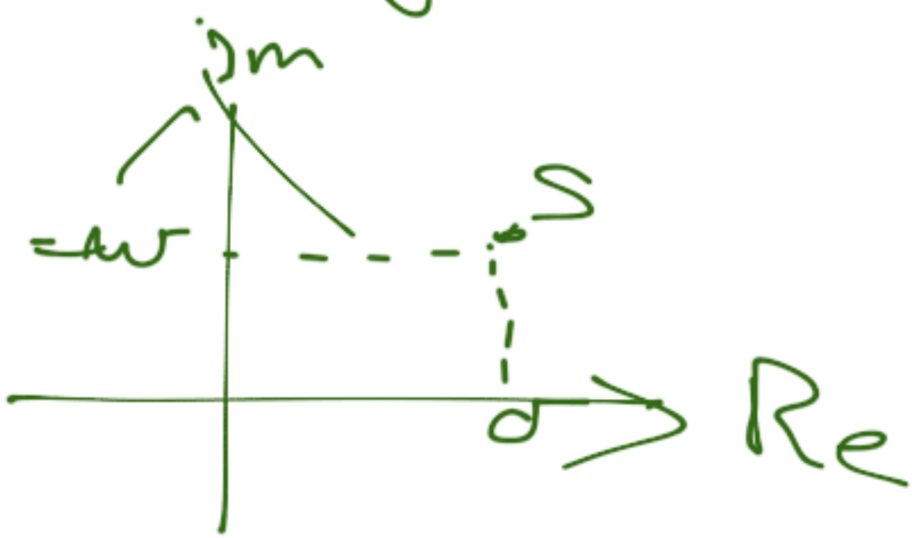


2.8 Laplace transform

Let $f(t)$ be a signal (function) def'd for $t \geq 0$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

here, $s = \sigma + j\omega$



Intuition:

$F(s)$ is a decomposition of $f(t)$ into a weighted sum of complex exponentials

$$e^{-st} = e^{-\sigma t}(\cos\omega t - j\sin\omega t)$$

Summary

- linearize state models

$$\dot{x} = f(x,u) \qquad \delta\dot{x} = A\,\delta x + B\,\delta u$$
$$y = h(x,u) \longrightarrow \delta y = C\,\delta x + D\,\delta u$$

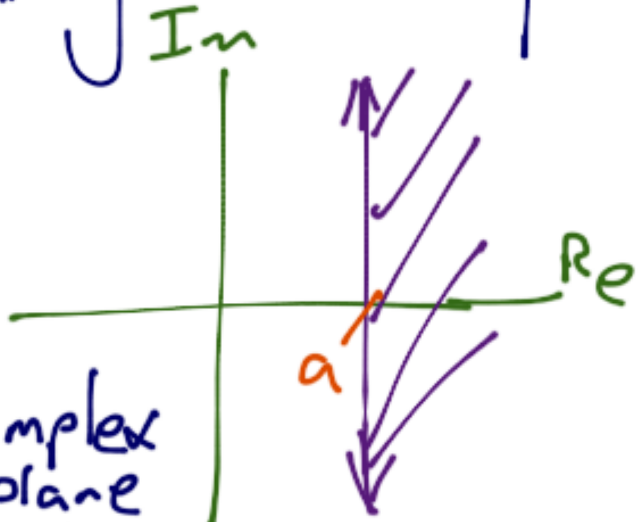
- Approximation is valid near the operating points (x_0, u_0)

$$f(x_0, u_0) = 0$$

- Laplace Transform: $F(s)=\mathcal{L}\{f(t)\}=\int_0 f(t)e^{-st}\,dt$

there's a region of convergence for some s. Where the region converges.

The $\overbrace{\text{Region of convergence}}^{\text{R.O.C}}$ is in the right-half plane:



Complex s-plane
$\{s\in\mathbb{C}: Re\{s\}>a\}$

$f(t)=e^{-at} \Rightarrow \dfrac{1}{s+a}$

$\Rightarrow ROC = Re\{s\}>a$

$f(t)=te^{-at} \Rightarrow \dfrac{1}{(s+a)^2}$

$\Rightarrow ROC = Re\{s\}>a$

$f(t)=e^{t^2} \Rightarrow$ ROC is empty & there is no $\mathcal{L}\{f\}$

LT $\mathcal{L}\{\}$ a is const, f, g are fns.

i) $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$

ii) $\mathcal{L}\{af\} = a\mathcal{L}\{f\}$

linear

iii) $\mathcal{L}\{\frac{df}{dt}\} = s\mathcal{L}\{f\} - f'(0)$ derivative

iv) $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$ convolution

v) $\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{1}{s}\mathcal{L}\{f\}$ integral

vi) if $\lim_{t\to\infty} f$ exists & is finite, then

$$\lim_{t\to\infty} f = \lim_{s\to 0} sF(s)$$

Final Value theorem (FVT)

ex: $\longrightarrow$

$f(t)$	$\lim_{t\to\infty} f(t)$	$\mathcal{L}\{f\}$	$\lim_{s\to 0} sF(s)$	FVT Applies?
e^{-t}	0	$\frac{1}{s+1}$	0	✓
$\mathbb{1}(t)$ (Heavyside)	1	$\frac{1}{s}$	1	✓
te^{-t}	0	$\frac{1}{(s+1)^2}$	0	✓
e^{t}	∞	$\frac{1}{s-1}$	0	✗
$\cos\omega t$ (ω const)	DNE	$\frac{s}{s^2+\omega^2}$	0	✗

The main idea is that if $sF(s)$ is defined (no div by 0) for any s with $Re\{s\}>0$, the FVT applies.

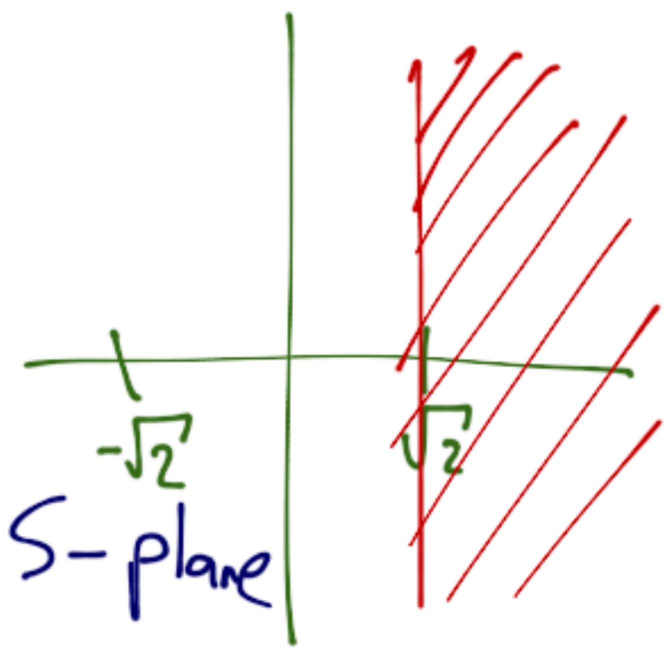
If $s=-j\omega$, the $\mathcal{L}\{f\}$ dne and we cannot apply the final value theorem.

We can use PFE (partial fraction expansion when)

$$F(s) = \frac{n(s)}{d(s)} \Big\} \text{ polynoms}$$

$\Rightarrow \deg(d) > \deg(n)$

<u>Eg:</u> $F(s) = \dfrac{3s+17}{s^2-2}$ find $f(t)$



$$F(s) = \frac{C_1}{s+\sqrt{2}} + \frac{C_2}{s-\sqrt{2}}$$

$$\Rightarrow C_1 = \frac{23}{4}, \quad C_2 = \frac{-11}{4}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{C_1}{s+\sqrt{2}}\right\} + \mathcal{L}^{-1}\left\{\frac{C_2}{s-\sqrt{2}}\right\}$$

$$= \frac{23}{4} e^{-\sqrt{2}t} - \frac{11}{4} e^{\sqrt{2}t}, \quad t \geq 0$$

Ex: $F(s) = \dfrac{s+1}{s(s+2)^2} \Rightarrow f(t) = ?$ $\quad$ [2.3]

$$F(s) = \frac{c_1}{s} + \frac{c_2}{s+2} + \frac{c_3}{(s+2)^2} = \frac{s+1}{s(s+1)^2}$$

$$\left.\begin{aligned} c_1 &= 1/4 \\ c_2 &= -\frac{1}{4} \\ c_3 &= 1/2 \end{aligned}\right\} f(t) = \frac{1}{4} + -\frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

The idea is that LT are useful for solving initial value functions, like

$$\dot{y} - 2y = t, \quad y(0) = 1$$

2.4 Transfer Functions:

Consider a linear tim-invariant system (LTI) with response $g(t)$

$$u \longrightarrow \boxed{\text{LTI sys. w/ impulse resp } g(t)} \longrightarrow y$$

You should have seen

$$y(t) = g(t) * u(t)$$

$X(s) \rightarrow$ the height from the ground. $\rightarrow$ input.