Lecture 7 - Linear Programming

Lecturer: Debmalya Panigrahi   *Scribe: Hieu Bui*

1   Overview

In this lecture, we cover basics of linear programming, linear-programming duality, Farkas's lemma, and complementary slackness. In addition, we demonstrate how to express the maximum-flow problem as a linear program.

2   Linear Programming

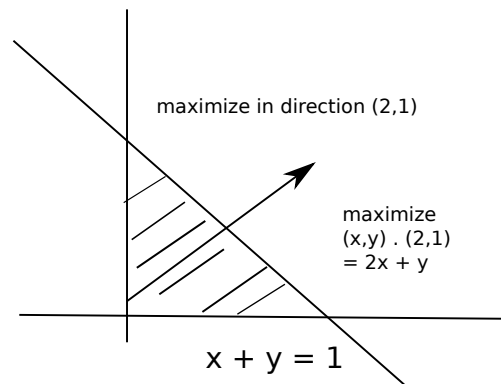Let us consider the following linear program with two variables:

$$\text{maximize} \quad (2x+y) \tag{1}$$
$$\text{subject to}$$
$$x+y \quad \leq \quad 1 \tag{2}$$
$$x \quad \geq \quad 0 \tag{3}$$
$$y \quad \geq \quad 0 \tag{4}$$

Any setting of the variable x and y that satisfies all the constraints (2)-(4) a feasible solution to the linear program. The following graph shows the constraints in the Cartesian coordinate system. The set of feasible

solutions (dashed in the figure) forms a convex region in the two-dimensional space. This convex region is called the feasible region. Each constraint (2)-(4) creates a hyperplane which splits space at two half spaces. The intersection of half-spaces form a polyhedron which is equivalent to the convex region.

Next, we express the maximum-flow problem as a linear program. Given a directed graph $G = (V, E)$ in which each edge $(x, y) \in E$ has a nonnegative capacity $c(x, y) \geq 0$, and two distinguished vertices: a source

s and a sink t. A flow is a nonnegative real-valued function that satisfies the capacity constraint and flow conservation. A maximum flow is a flow that satisfies these constraints and maximizes the flow value. A flow satisfies linear constraints and the value of a row is a linear function. We can express the maximum-flow problem as a linear program:

$$\text{maximize} \quad \sum_x f(s,x) - \sum_y f(y,s)$$

$$\text{subject to}$$

$$
\begin{array}{rcll}
f(x,y) & \leq & u(x,y) & \forall (x,y) \in E \\
\sum_y f(x,y) & = & \sum_w f(w,x) & \forall x \neq \{s,t\} \\
f(x,y) & \geq & 0 & \forall (x,y) \in E
\end{array}
$$

2.1 Canonical and standard forms of LP

To describe properties of and algorithms for linear programs, it is convenient to express them in canonical forms. A linear program in standard form is the maximization of a linear function subject to linear inequalities. A linear program in canonical (slack) form is the maximization of a linear function subject to linear equalities.

Canonical and standard forms of the minimization linear program:

$$
\begin{array}{cc}
\min c^T x & \min c^T x \\
Ax = b & Ax \geq b \\
x \geq 0 & x \geq 0
\end{array}
$$

Canonical and standard forms of the maximization linear program:

$$
\begin{array}{cc}
\max c^T x & \max c^T x \\
Ax = b & Ax \leq b \\
x \geq 0 & x \geq 0
\end{array}
$$

In canonical form, all the constraints are equalities, whereas in standard form, all the constraints are inequalities. The following rules can be used to convert any arbitrary LP to canonical form: (1) If it's a maximization problem, negate c and minimize, (2) if a constraint is less than equal ($\leq$), negate it and convert it to greater than equal ($\geq$), and (3) if a constraint is =, add two constraints with less than equal ($\leq$) and greater than equal ($\geq$).

3 Linear-Programming Duality

Given a linear program in which the objective is to minimize, we shall describe how to formulate a dual linear program in which the objective is to maximize and whose optimal value is identical to that of the original linear program. When referring to dual programs, we call the original linear program the primal.

Given a primal linear program in standard form as the following:

$$\min c^T x \qquad (5)$$
$$Ax \geq b \qquad ((6)$$
$$x \geq 0. \qquad (7)$$

we define the dual linear program as

$$\max b^T y \qquad (8)$$
$$A^T y \leq c \qquad ((9)$$
$$y \geq 0. \qquad (10)$$

Lemma 1. *Weak linear-programming duality: Let x^* be any feasible solution to the primal linear program in (5)-(7) and let y^* be any feasible solution to the dual linear program in (8)-(10). Then, we have $c^T x^* \geq b^T y^*$.*

Proof.

$$
\begin{aligned}
Ax^* &\geq b \\
\Rightarrow y^{*^T} Ax^* &\geq y^{*^T} b & (y^* \geq 0) \\
\Rightarrow x^{*^T} A^T y^* &\geq b^T y^* & (y^{*^T} b = b^T y^*) \\
\Rightarrow x^{*^T} c &\geq b^T y^* & (A^T y^* \leq c) \\
\Rightarrow c^T x^* &\geq b^T y^*
\end{aligned}
$$

$\square$

We have shown weak duality (i.e., the dual objective (maximum objective function) is a lower bound on the primal objective (minimum objective function)). Similarly, the dual objective (minimum objective function) is an upper bound on the primal objective (maximum objective function). In fact duals are reversible and dual of dual is primal.

The following shows how to obtain the primal-dual pairs of the max-flow problem. Given a primal linear program which can be expressed as

$$
\text{maximize} \sum_{p \in P(s,t)} f_p
$$
$$
\text{subject to} \sum_{p(x,y) \in P} f_p \leq u(x,y) \qquad \forall (x,y) \in E
$$
$$
f_p \geq 0.
$$

We then define the dual linear program as

$$
\text{minimize} \sum_{(x,y) \in E} l(x,y)u(x,y)
$$
$$
\text{subject to} \sum_{(x,y) \in P} l(x,y) \geq 1 \qquad \forall p \in P(s,t)
$$
$$
l(x,y) \geq 0
$$

The length function of minimal volume subjects to $d_l(s,t) \geq 1$. For any (s,t) cut $(S,\bar{S})$, $\{l(x,y) = 1$ if $(x,y) \in (S,\bar{S})$ or $l(x,y) = 0$ otherwise $\}$ is feasible and therefore maxflow $\leq$ mincut by weak duality.

Lemma 2. *Implications of weak duality*
For a primal (P) - dual (D) pair, either of the following must hold:
(1) One of P and D are feasible and bounded
(2) One of P and D is feasible but unbounded; then the other must be infeasible.
(3) Both P and D are infeasible.

Proof. If (2) and (3) are not true, then (1) must be true. □

Theorem 3. *Strong Duality Theorem*
If P and D are feasible, x^* is optimal for P if and only if (1) x^* is feasible for P and (2) there exists y^* feasible for D such that $c^T x^* = b^T y^*$.

Here we are using the standard form primal-dual pair. The primal can be written as

$$
\begin{aligned}
\text{minimize } & c^T x \\
\text{subject to } Ax \;&=\; b \\
x \;&\geq\; 0.
\end{aligned}
$$

The dual can be written as

$$
\begin{aligned}
\text{maximize } & b^T y \\
\text{subject to } A^T y \;&\leq\; c^T \\
y \;&\geq\; 0
\end{aligned}
$$

To prove this theorem, we will need the separating hyperplane theorem.

Theorem 4. *Separating hyperplane theorem*
If x is not inside a close convex set C, then there exists a hyperplane separating x from C.

Lemma 5. *Farka's lemma*
Exactly one of the following is feasible:
(1) $\{x : Ax = b \quad x \geq 0\}$
(2) $\{y : b^T y > 0 \quad A^T y \leq 0\}$

Proof. If (1) is feasible, then feasibility of (2) will violate weak duality of the primal-dual pair.

$$
\begin{aligned}
\text{minimize } 0 && \text{maximize } b^T y \\
\text{subject to } Ax \;=\; b && \text{subject to } A^T y \;\leq\; c^T \\
x \;\geq\; 0 && y \;\geq\; 0
\end{aligned}
$$

Conversely, if (1) is not feasible and $b \notin \{Ax : x \geq 0\}$, then by separating hyperplane theorem, $\exists y$ such that
$y^T b > \max_{x \geq 0}\{y^T Ax\} \geq 0 \quad \Rightarrow \quad b^T y > \max_{x \geq 0}\{x^T A^T y\} \quad (\text{ implies } \quad A^T y \leq 0)$ □

Proof. Finally we prove the strong duality theorem using Farka's lemma. If the two conditions are met, then by weak duality x^* is optimum. Conversely, let x^* be optimum, we want to show that $\{Ax = b \quad cx \leq b^T y \quad A^T y \leq c \quad x \geq 0\}$ is feasible. By Farka's lemma,

$$
\begin{array}{rccc}
\exists y' & A^T y' - c\tilde{c} & \leq & 0 \\
x' \leq 0 & Ax' - b\tilde{c} & = & 0 \\
\tilde{c} \geq 0 & b^T y' - c^T x' & = & 1
\end{array}
$$

$$\tilde{c} = 0 \text{ is violated by weak duality.}$$
$$\text{if } \tilde{c} > 0$$
$$(-y')^T (Ax' - b\tilde{c}) + x'^T (A^T y' - c\tilde{c}) \leq 0$$
$$\tilde{c}(b^T y' - c^T x') \leq 0$$
$$\tilde{c} \leq 0$$

$\square$

Lemma 6. *Complementary Slackness*
At optimality $\quad y^{*^T}(Ax^* - b) = 0 \quad$ *or* $\quad x^{*^T}(A^T y^* - c) = 0.$

Proof.

$$
\begin{aligned}
y^{*^T}(Ax^* - b) &= (y^{*^T}A)x^* - y^{*^T}b \\
&= x^{*^T}(A^T y^*) - b^T y^* \\
&\leq x^{*^T}c^T - b^T y^* \\
&= cx^* - b^T y^* \\
&= 0 \qquad \text{(by strong duality theorem)}
\end{aligned}
$$

$\square$

References
   [1] T. Cormen, C. Leiserson, R. Rivest, C. Stein, Introduction to Algorithms, 3rd Edition, Chapter 29.
   [2] http://courses.csail.mit.edu/6.854/03/scribe/scribe9.ps