Duality Theory in LP

Linear Programming

A model consisting of linear relationships representing a firm's objective and resource constraints

LP is a mathematical modeling technique used to determine a level of operational activity in order to achieve an objective, subject to restrictions called constraints

LP Model Formulation

- Decision variables
  - mathematical symbols representing levels of activity of an operation
- Objective function
  - a linear relationship reflecting the objective of an operation
  - most frequent objective of business firms is to *maximize profit*
  - most frequent objective of individual operational units (such as a production or packaging department) is to *minimize cost*
- Constraint
  - a linear relationship representing a restriction on decision making

LP Model Formulation (cont.)

Max/min $\quad z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\ \quad\quad\quad\quad : \\ a_{m1}x1 + a_{m2}x_2 + ... + a_{mn}x_n \ (\leq, =, \geq) \ b_m \end{cases}$$

x_j = decision variables
b_i = constraint levels
c_j = objective function coefficients
a_{ij} = constraint coefficients

Geometry of the Prototype Example

$$
\begin{array}{llll}
\text{Max} & 3\,P1 + 5\,P2 & & \\
\text{s.t.} & P1 \quad + & \le 4 & \text{(Plant 1)} \\
 & \qquad\quad 2\,P2 & \le 12 & \text{(Plant 2)} \\
 & 3\,P1 + 2\,P2 & \le 18 & \text{(Plant 3)} \\
 & P1,\ P2 & \ge 0 & \text{(nonnegativity)}
\end{array}
$$

P2

Every point in this nonnegative quadrant is associated with a specific production alternative.

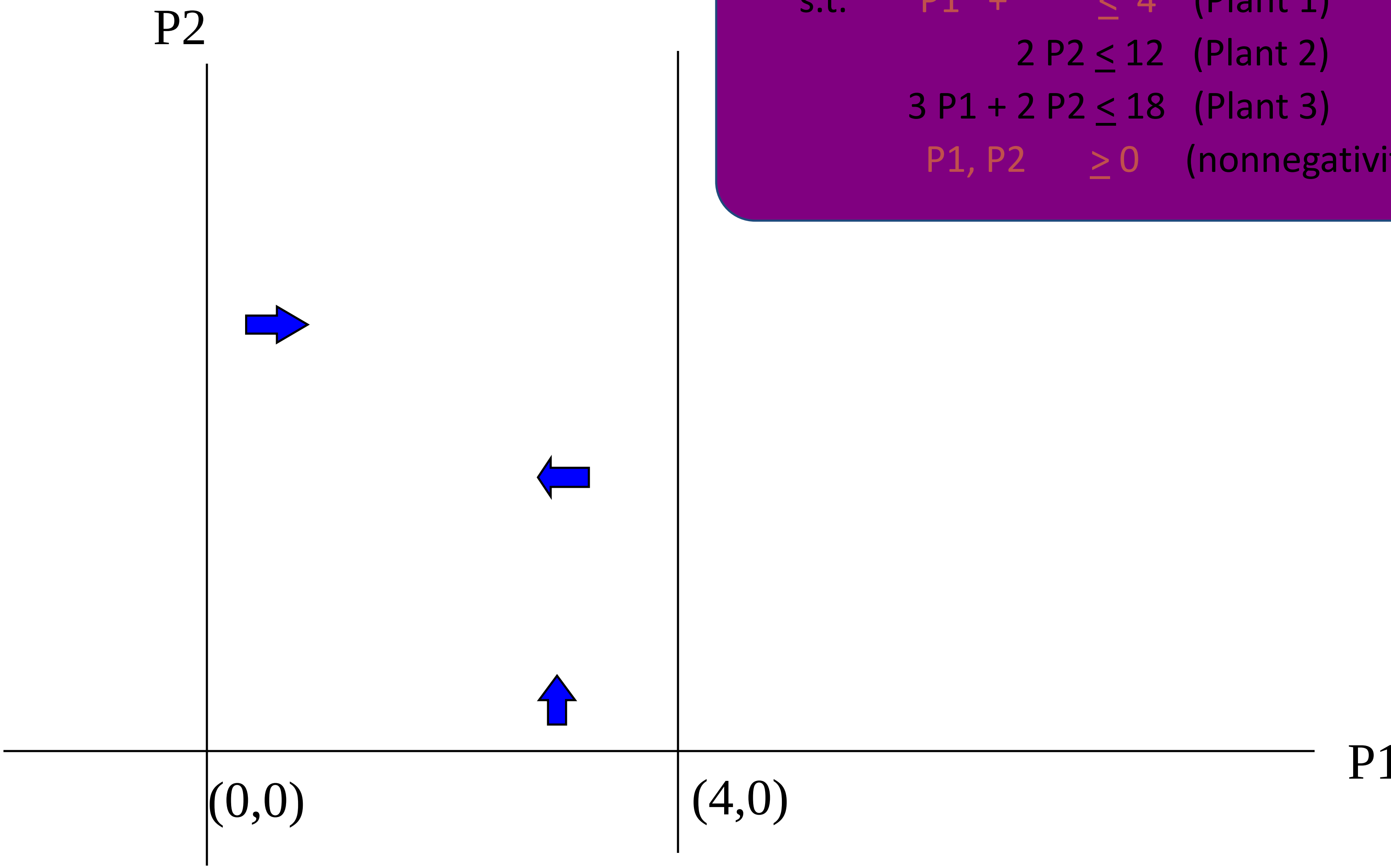
(point = decision = solution)

0

P1

Geometry of the Prototype Example

$$
\begin{array}{llll}
\text{Max} & 3\,P1 + 5\,P2 & & \\
\text{s.t.} & P1 \;+ & \leq 4 & \text{(Plant 1)} \\
 & 2\,P2 & \leq 12 & \text{(Plant 2)} \\
 & 3\,P1 + 2\,P2 & \leq 18 & \text{(Plant 3)} \\
 & P1,\ P2 & \geq 0 & \text{(nonnegativity)}
\end{array}
$$



Geometry of the Prototype Example

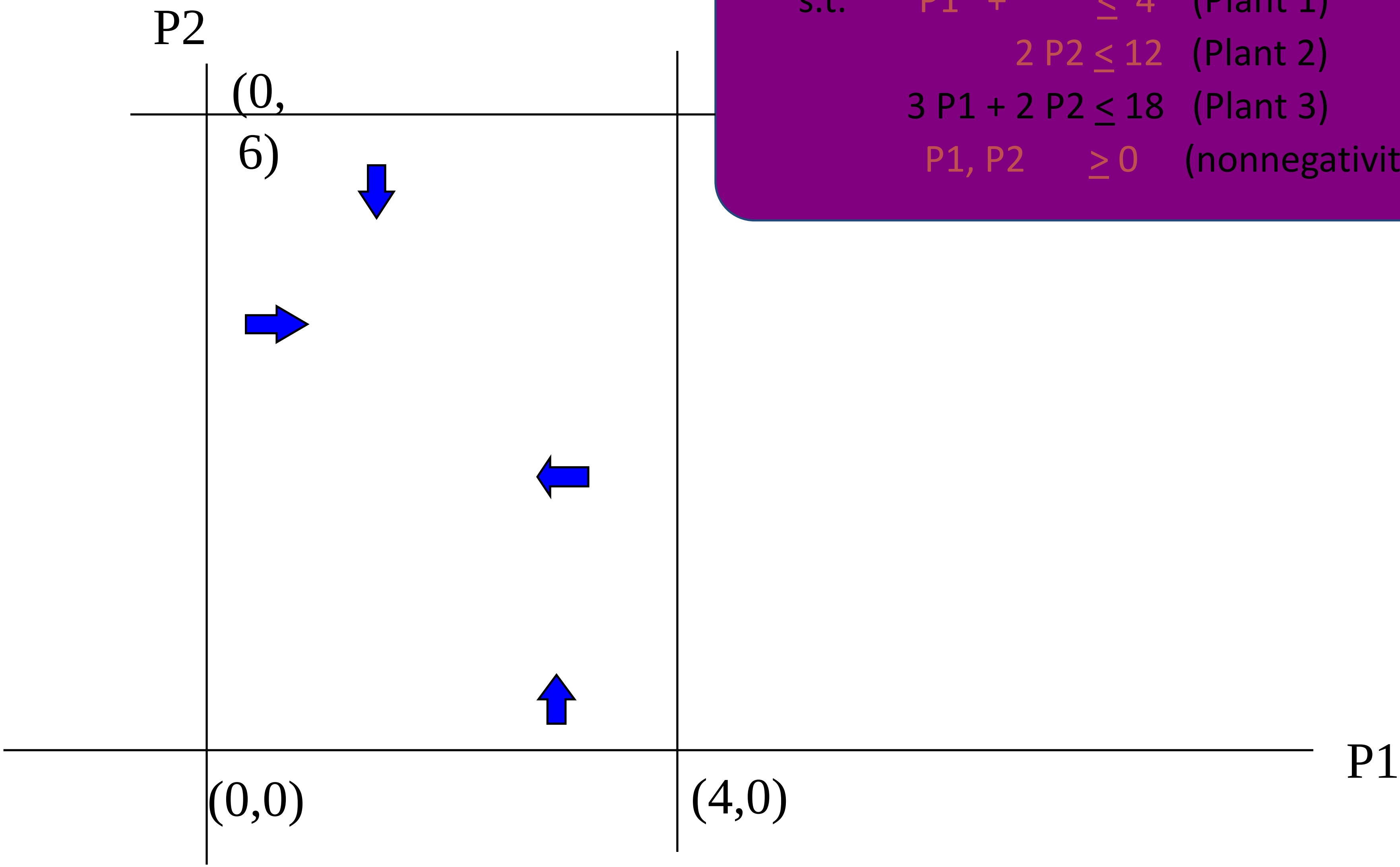
Max 3 P1 + 5 P2

s.t. P1 + $\leq$ 4 (Plant 1)

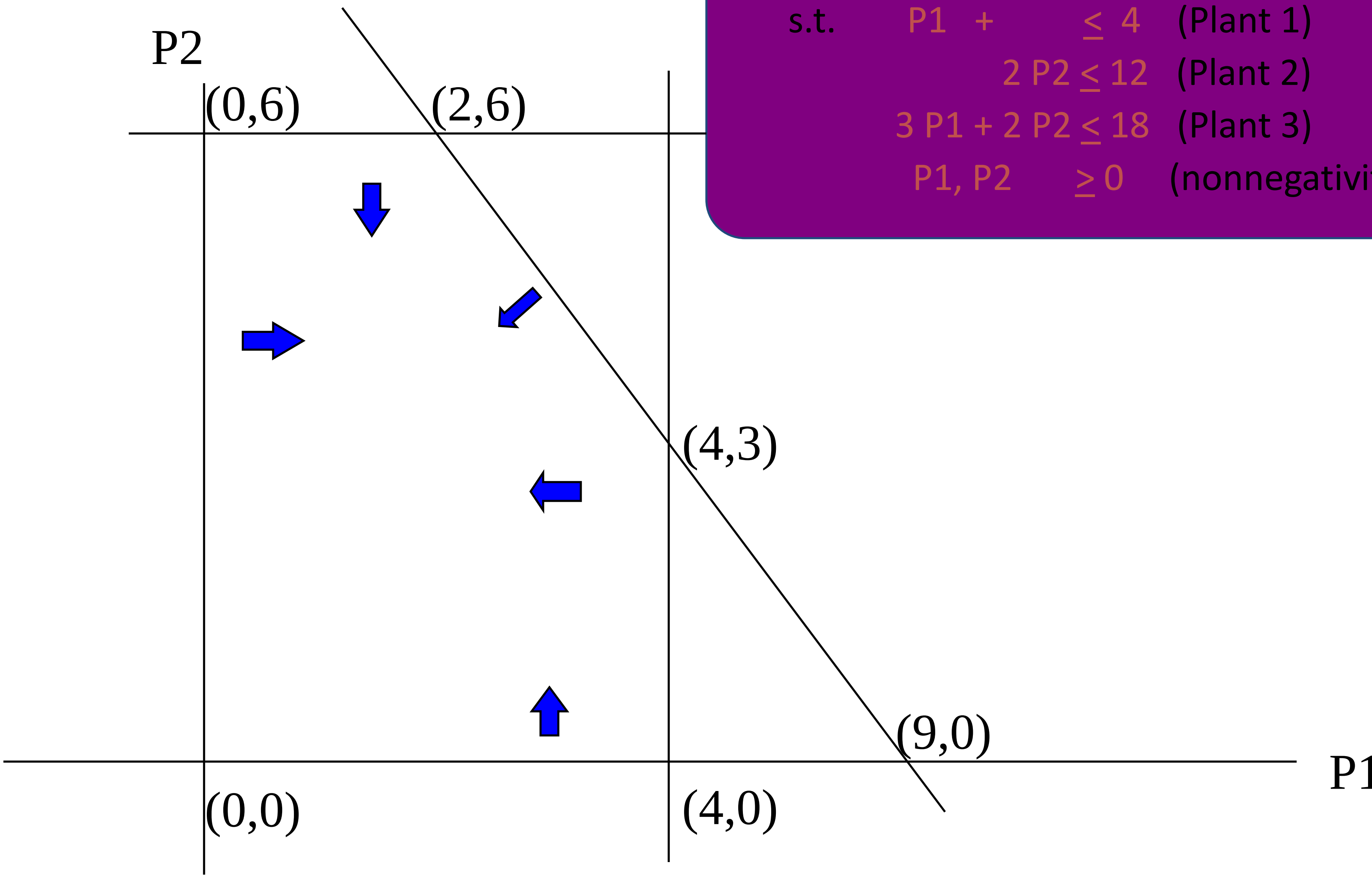
2 P2 $\leq$ 12 (Plant 2)

3 P1 + 2 P2 $\leq$ 18 (Plant 3)

P1, P2 $\geq$ 0 (nonnegativity)



Geometry of the Prototype Example



Max	$3\,P1 + 5\,P2$	
s.t.	$P1 \ + \qquad \leq 4$	(Plant 1)
	$2\,P2 \leq 12$	(Plant 2)
	$3\,P1 + 2\,P2 \leq 18$	(Plant 3)
	$P1, P2 \geq 0$	(nonnegativity)

Geometry of the Prototype Example

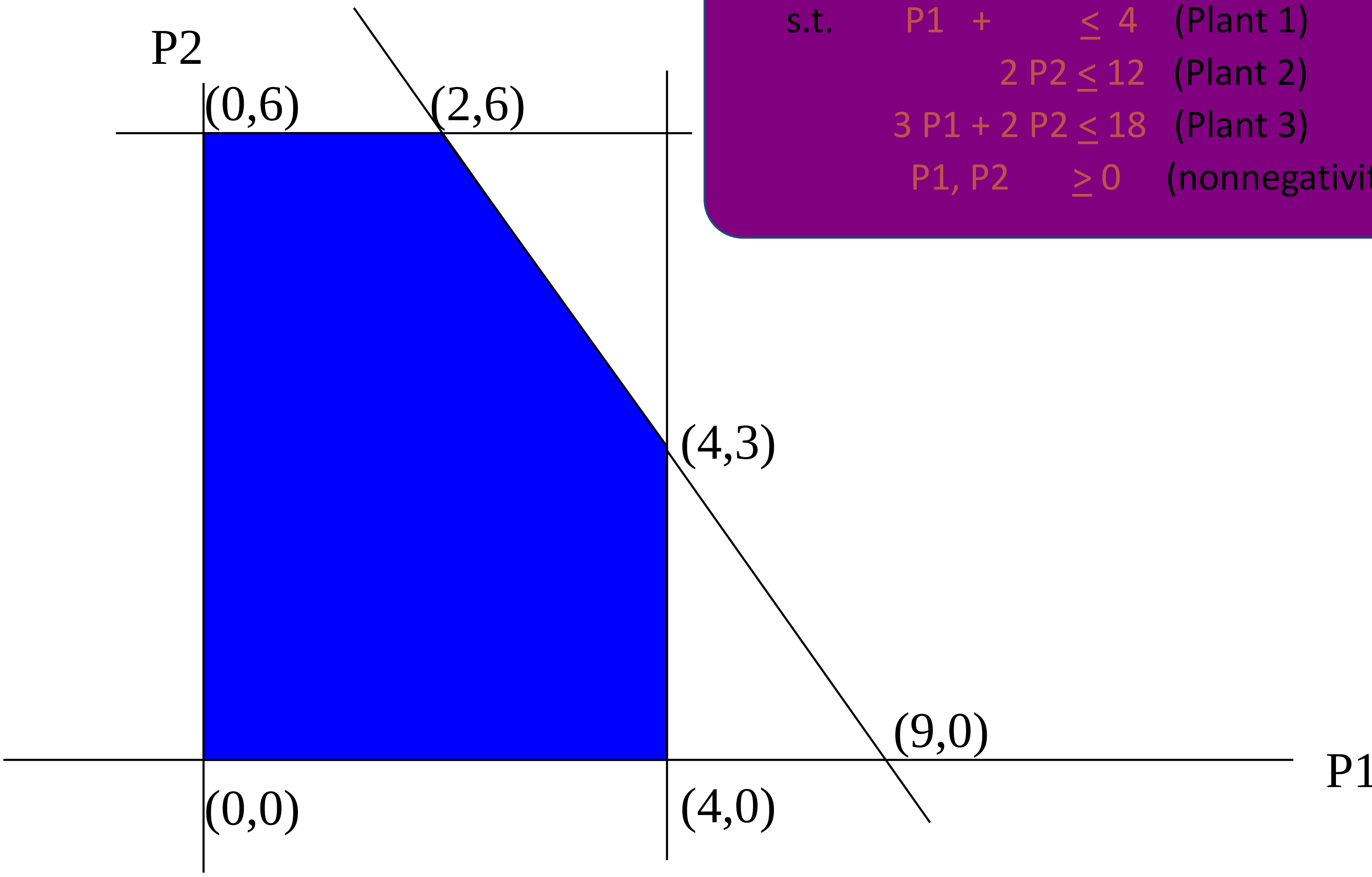
Max 3 P1 + 5 P2

s.t. P1 + ≤ 4 (Plant 1)

2 P2 ≤ 12 (Plant 2)

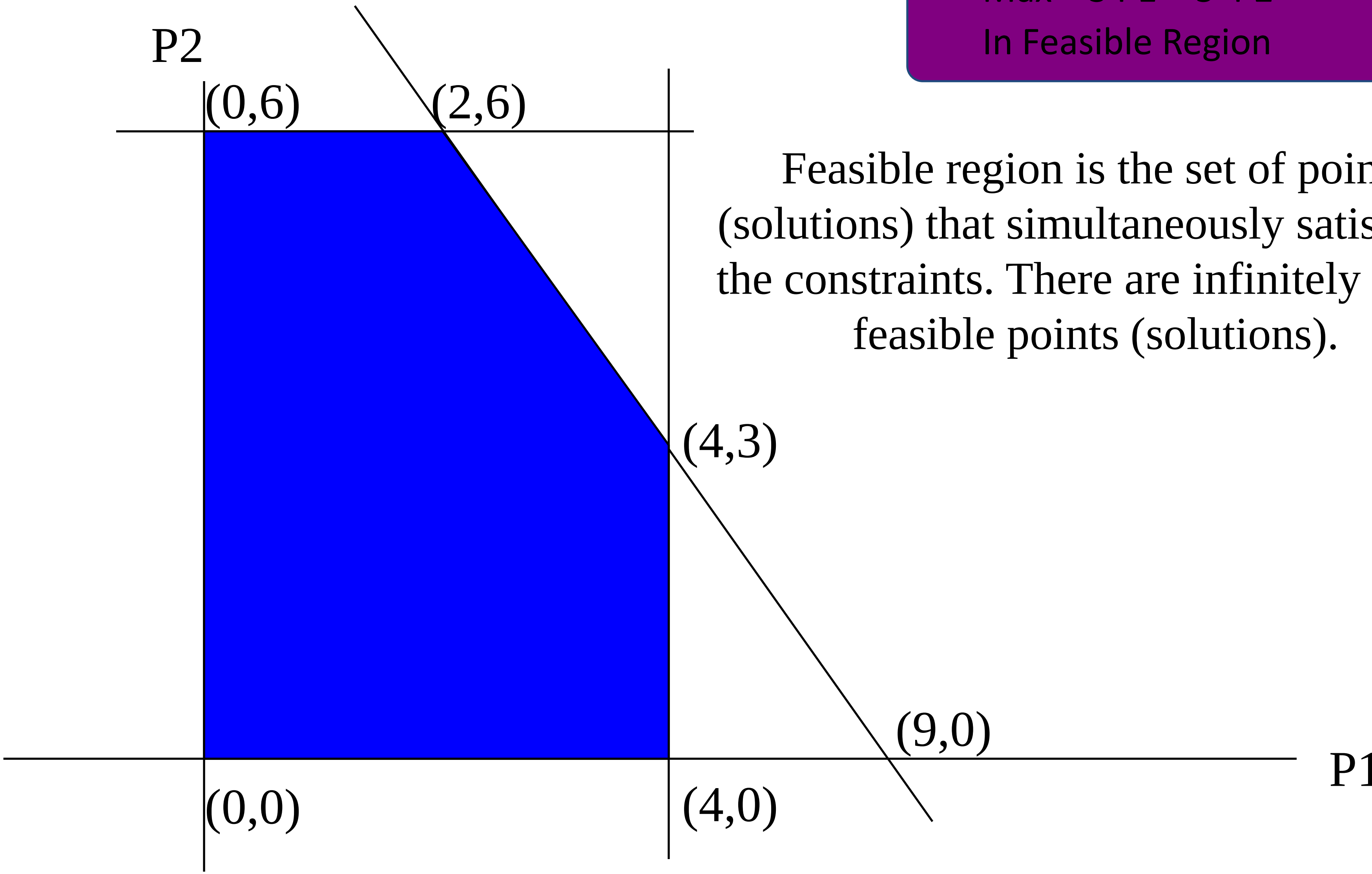
3 P1 + 2 P2 ≤ 18 (Plant 3)

P1, P2 ≥ 0 (nonnegativity)



Max 3 P1 + 5 P2

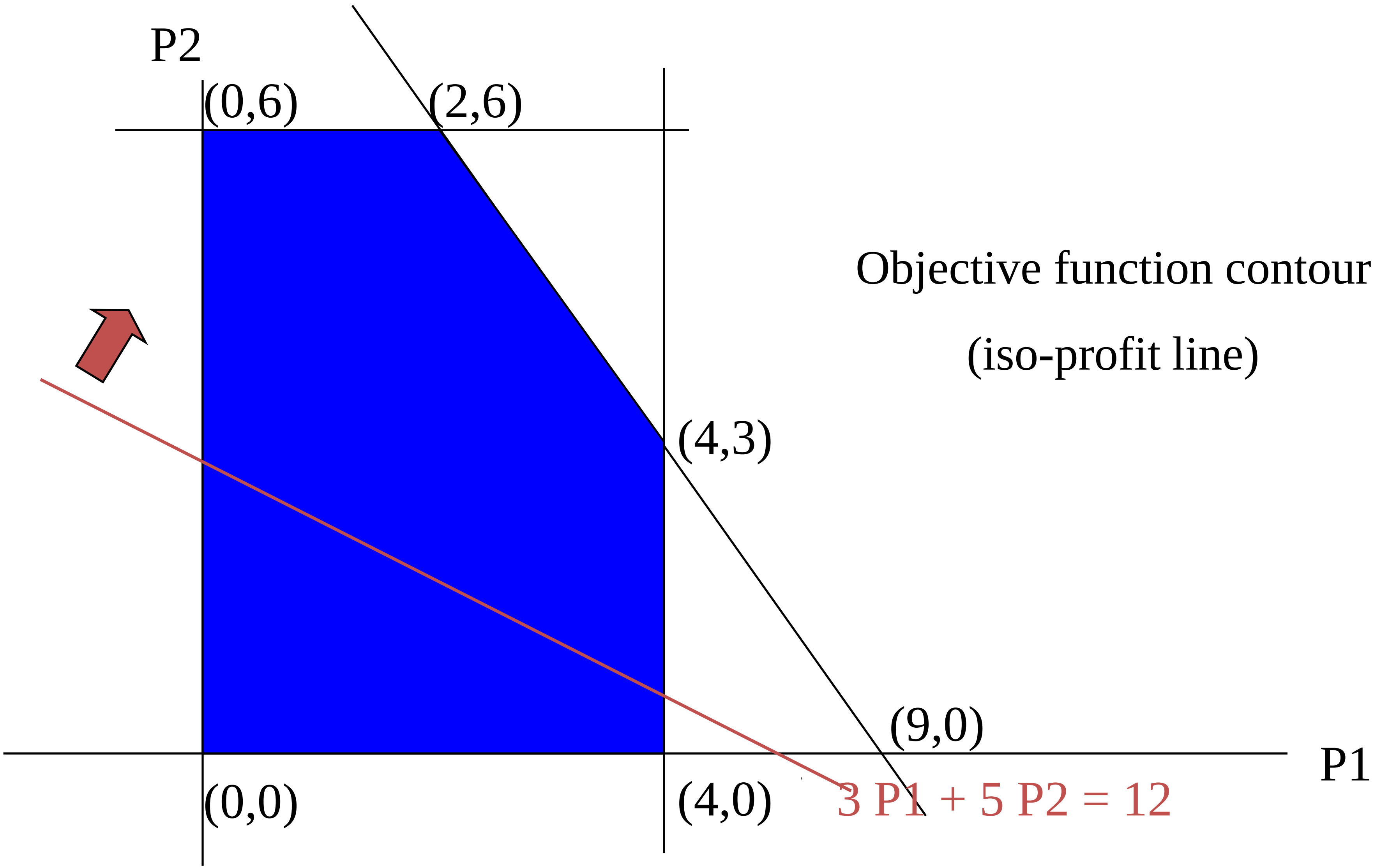
In Feasible Region



Feasible region is the set of points (solutions) that simultaneously satisfy all the constraints. There are infinitely many feasible points (solutions).

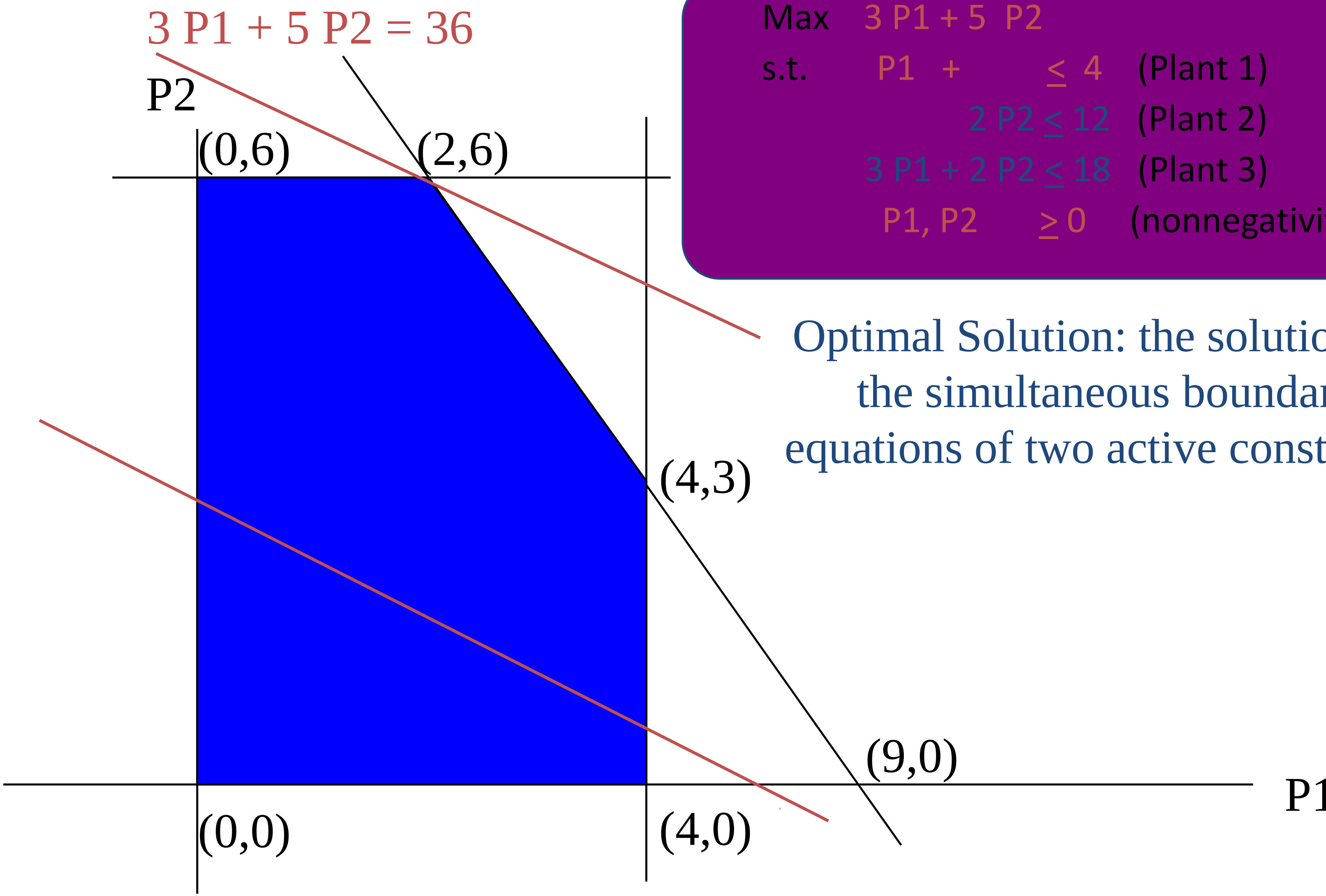
Geometry of the Prototype Example

Max 3 P1 + 5 P2



Geometry of the Prototype Example

Max	$3 P1 + 5 P2$	
s.t.	$P1 \; + \qquad \leq 4$	(Plant 1)
	$2 P2 \leq 12$	(Plant 2)
	$3 P1 + 2 P2 \leq 18$	(Plant 3)
	$P1, P2 \quad \geq 0$	(nonnegativity)



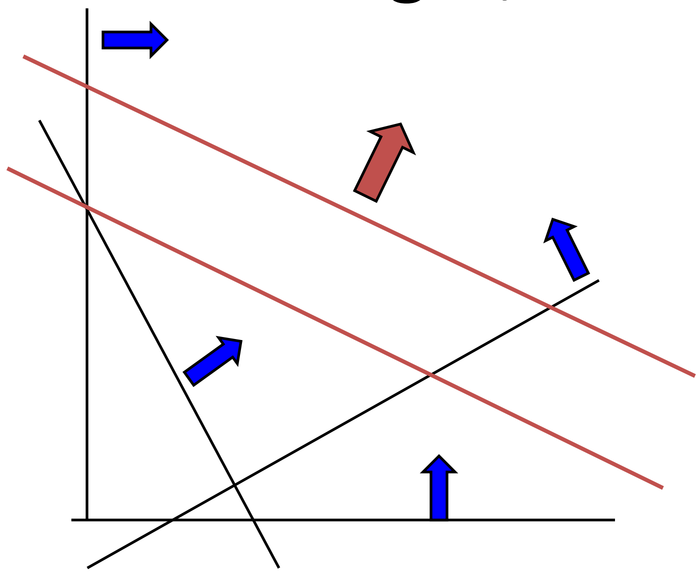
Optimal Solution: the solution for the simultaneous boundary equations of two active constraints

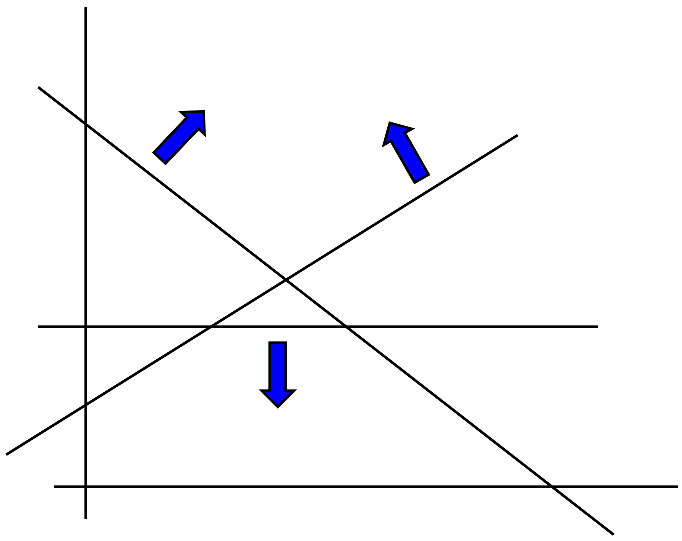
LP Terminology

- **solution (decision, point):** any specification of values for all decision variables, regardless of whether it is a desirable or even allowable choice
- **feasible solution**: a solution for which all the constraints are satisfied.
- **feasible region (constraint set, feasible set):** the collection of all feasible solution
- **optimal solution (optimum):** a feasible solution that has the most favorable value of the objective function
- **optimal (objective) value**: the value of the objective function evaluated at an optimal solution

Unbounded or Infeasible Case

- On the left, the objective function is unbounded
- On the right, the feasible set is empty





LP Model: Example

	RESOURCE REQUIREMENTS		
PRODUCT	*Labor (hr/unit)*	*Clay (lb/unit)*	*Revenue ($/unit)*
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day

Decision variables

x_1 = number of bowls to produce

x_2 = number of mugs to produce

LP Formulation: Example

Maximize $Z = \$40\, x_1 + 50\, x_2$

Subject to

$$
\begin{aligned}
x_1 + 2x_2 &\le 40 \text{ hr} \quad \text{(labor constraint)} \\
4x_1 + 3x_2 &\le 120 \text{ lb} \quad \text{(clay constraint)} \\
x_1, x_2 &\ge 0
\end{aligned}
$$

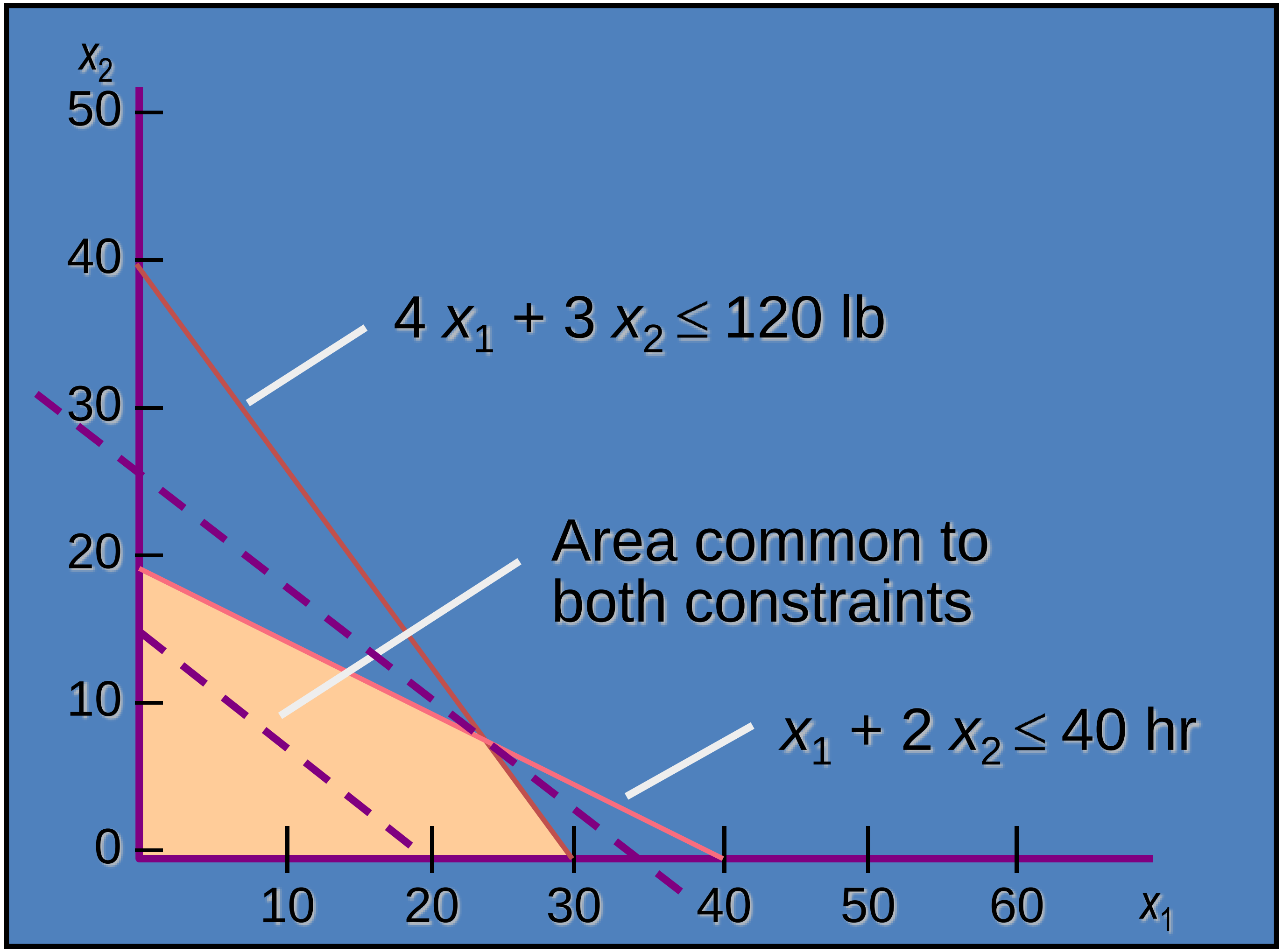
Solution is $x_1 = 24$ bowls $x_2 = 8$ mugs

Revenue = \$1,360

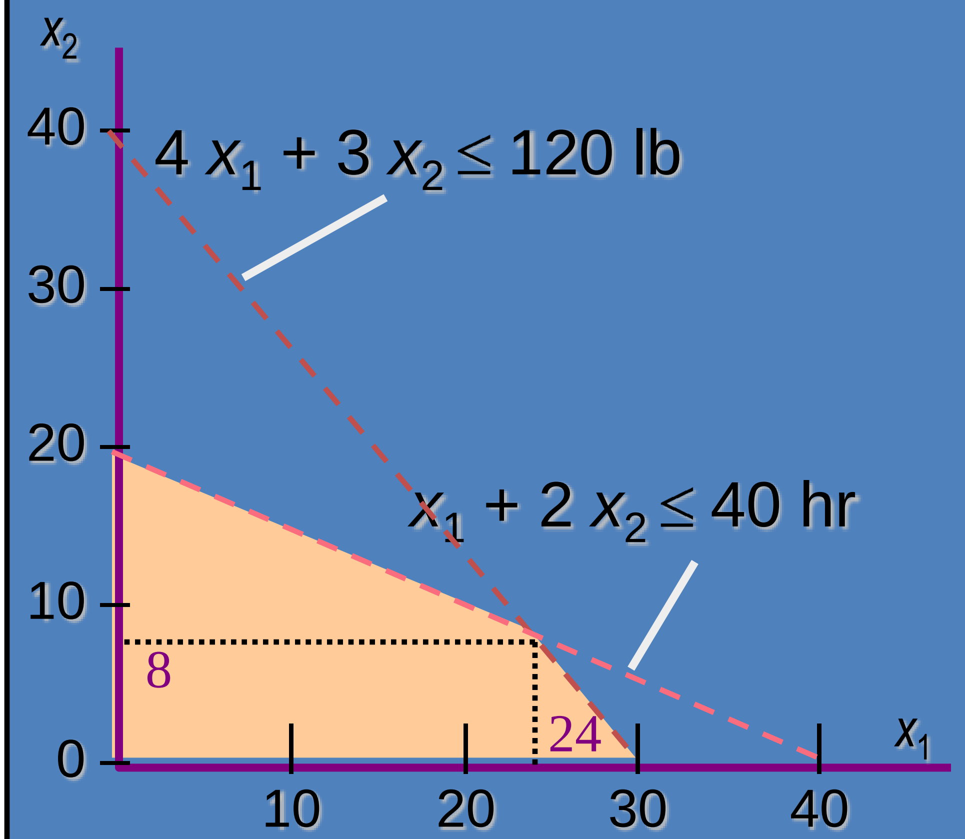
Graphical Solution Method

1. Plot model constraint on a set of coordinates in a plane
2. Identify the feasible solution space on the graph where all constraints are satisfied simultaneously
3. Plot objective function to find the point on boundary of this space that maximizes (or minimizes) value of objective function

Graphical Solution: Example



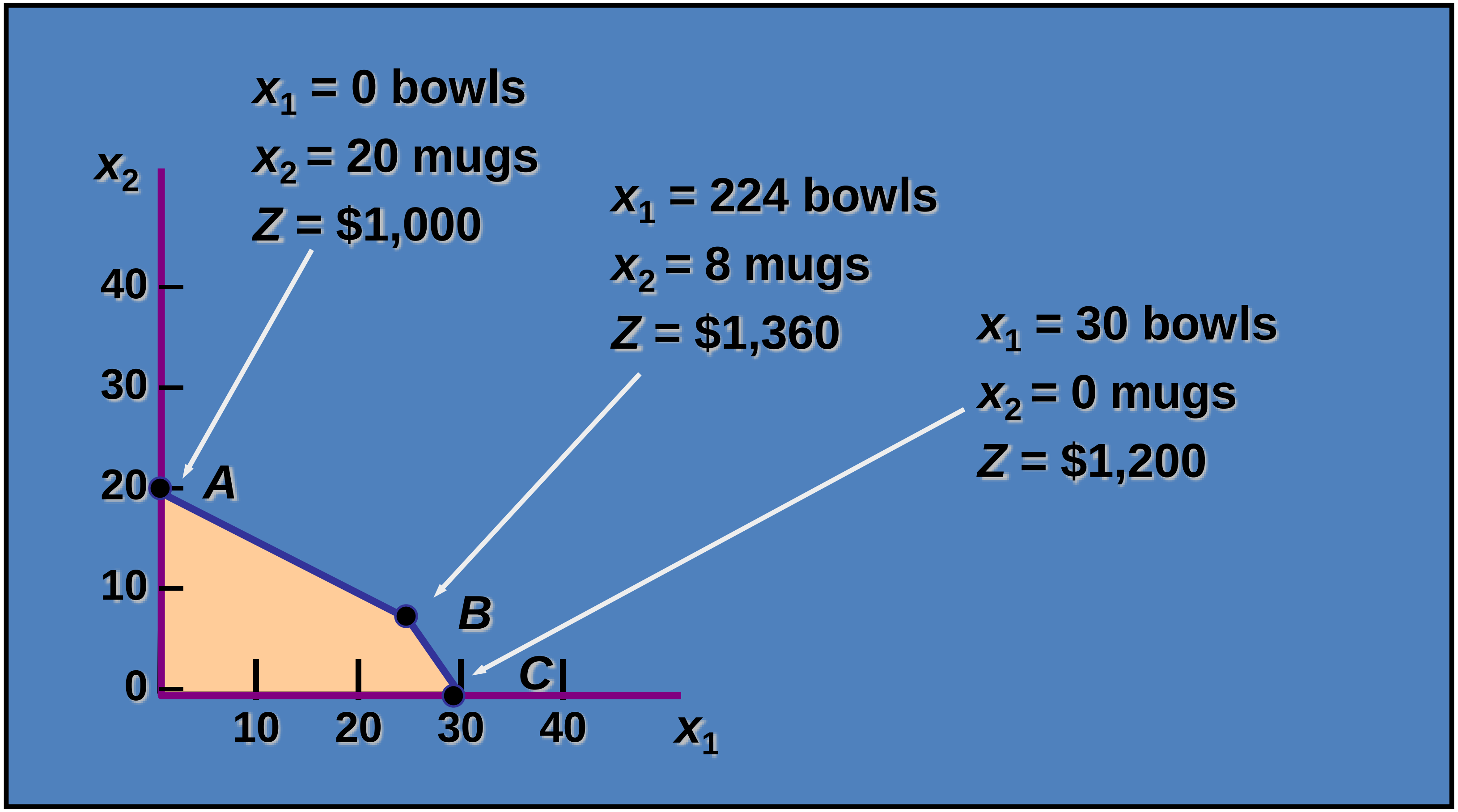
Computing Optimal Values



$$
\begin{array}{rcrcr}
x_1 & + & 2x_2 & = & 40 \\
4x_1 & + & 3x_2 & = & 120 \\
\hline
4x_1 & + & 8x_2 & = & 160 \\
-4x_1 & - & 3x_2 & = & -120 \\
\hline
 & & 5x_2 & = & 40 \\
 & & x_2 & = & 8 \\
 & & & & \\
x_1 & + & 2(8) & = & 40 \\
x_1 & & & = & 24
\end{array}
$$

$$Z = \$50(24) + \$50(8) = \$1{,}360$$

Extreme Corner Points



Theory of Linear Programming

An LP problem falls in one of three cases:

- Problem is *infeasible*: Feasible region is empty.
- Problem is *unbounded*: Feasible region is unbounded towards the optimizing direction.
- Problem is *feasible* and *bounded*: then there exists an optimal point; an optimal point is on the boundary of the feasible region; and there is always at least one optimal corner point (if the feasible region has a corner point).

When the problem is feasible and bounded,

- There may be a unique optimal point or multiple optima (alternative optima).
- If a corner point is not "worse" than all its neighbor corners, then it is optimal.

Duality Theory

The theory of duality is a very elegant and important concept within the field of operations research. This theory was first developed in relation to linear programming, but it has many applications, and perhaps even a more natural and intuitive interpretation, in several related areas such as nonlinear programming, networks and game theory.

Duality Theory

- The notion of duality within linear programming asserts that every linear program has associated with it a related linear program called its **dual**. The original problem in relation to its dual is termed the **primal**.

- it is the **relationship** between the primal and its dual, both on a mathematical and economic level, that is truly the essence of duality theory.

Examples

- There is a small company in Melbourne which has recently become engaged in the production of office furniture. The company manufactures tables, desks and chairs. The production of a table requires 8 kgs of wood and 5 kgs of metal and is sold for $80; a desk uses 6 kgs of wood and 4 kgs of metal and is sold for $60; and a chair requires 4 kgs of both metal and wood and is sold for $50. We would like to determine the revenue maximizing strategy for this company, given that their resources are limited to 100 kgs of wood and 60 kgs of metal.

Problem P1

$$\max_{x} Z = 80x_1 + 60x_2 + 50x_3$$

$$8x_1 + 6x_2 + 4x_3 \le 100$$

$$5x_1 + 4x_2 + 4x_3 \le 60$$

$$x_1, x_2, x_3 \ge 0$$

- Now consider that there is a much bigger company in Melbourne which has been the lone producer of this type of furniture for many years. They don't appreciate the competition from this new company; so they have decided to tender an offer to buy all of their competitor's resources and therefore put them out of business.

- The challenge for this large company then is to develop a linear program which will determine the appropriate amount of money that should be offered for a unit of each type of resource, such that the offer will be acceptable to the smaller company while minimizing the expenditures of the larger company.

Problem D1

$$\min_{y} \; w = 100y_1 + 60y_2$$

$$8y_1 + 5y_2 \geq 80$$

$$6y_1 + 4y_2 \geq 60$$

$$4y_1 + 4y_2 \geq 50$$

$$y_1, y_2 \geq 0$$

Standard form of the Primal Problem

$$\max_{x} Z = \sum_{j=1}^{n} c_j x_j$$

$$\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ \dots \ + a_{1n}x_n \le b_1 \\
a_{21}x_1 + a_{22}x_2 + \ \dots \ + a_{2n}x_n \le b_2 \\
\dots \qquad \dots \qquad \dots \qquad \dots \\
\dots \qquad \dots \qquad \dots \qquad \dots \\
a_{m1}x_1 + a_{m2}x_2 + \ \dots \ + a_{mn}x_n \le b_m \\
x_1, x_2, \dots, x_n \ge 0
\end{array}$$

Standard form of the Dual Problem

$$\min_{y} \; w = \sum_{i=1}^{m} b_i y_i$$

$$
\begin{array}{lllll}
a_{11}y_1 & + a_{21}y_2 & + \quad \ldots & + a_{m1}y_m & \geq c_1 \\
a_{12}y_1 & + a_{22}y_2 & + \quad \ldots & + a_{m2}y_m & \geq c_2 \\
\ldots & \ldots & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \\
a_{1n}y_1 & + a_{2n}y_2 & + \quad \ldots & + a_{mn}y_m & \geq c_n
\end{array}
$$

$$y_1, y_2, \ldots, y_m \geq 0$$

Definition

Primal Problem

$$z^* := \max_x \; Z = cx$$

$$s.t.$$

$$Ax \le b$$

$$x \ge 0$$

Dual Problem

$$w^* := \min_x \; w = yb$$

$$s.t.$$

$$yA \ge c$$

$$y \ge 0$$

b is not assumed to be non-negative

Primal-Dual relationship

		$x_1 \ge 0$	$x_2 \ge 0$	$x_n \ge 0$		w =
	$y_1 \ge 0$	a_{11}	a_{12}	a_{1n}	$\le$	b_1
Dual	$y_2 \ge 0$	a_{21}	a_{22}	a_{2n}	$\le$	b_2
(min w)	...	...	...		...	...
	$y_m \ge 0$	a_{m1}	a_{m2}	a_{mn}	$\le$	b_n
		$\ge$	$\ge$	$\ge$		
	z =	c_1	c_2	c_n		

Example

$$\max_{x} \; Z = 4x_1 + 10x_2 - 9x_3$$

$$\begin{aligned}
5x_1 - 18x_2 + 5x_3 &\le 15 \\
-8x_1 + 12x_2 - 8x_3 &\le 8 \\
12x_1 - 4x_2 + 8x_3 &\le 10 \\
2x_1 - 5x_3 &\le 5 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

		$x_1 \geq 0$	$x_2 \geq 0$	$x_3 \geq 0$		w=
Dual (min w)	$y_1 \geq 0$	5	-18	5	$\leq$	15
	$y_2 \geq 0$	-8	12	0	$\leq$	8
	$y_3 \geq 0$	12	-4	8	$\leq$	10
	$y_4 \geq 0$	2	0	-5	$\leq$	5
		$\geq$	$\geq$	$\geq$		
	Z=	4	10	-9		

Dual

$$\min_{y}\ w = 15y_1 + 8y_2 + 10y_3 + 5y_4$$

$$\begin{aligned}
5y_1 - 8y_2 + 12y_3 + 2y_4 &\geq 4 \\
-18y_1 + 12y_2 - 4y_3 &\geq 10 \\
5y_1 + 8y_3 - 5y_4 &\geq -9 \\
y_1, y_2, y_3, y_4 &\geq 0
\end{aligned}$$

$$\sum_{i=1}^{k} d_i x_i = e$$

$$\sum_{i=1}^{k} d_i x_i \leq e$$

$$\sum_{i=1}^{k} d_i x_i \geq e$$

$\rightarrow$

$$\sum_{i=1}^{k} d_i x_i \leq e$$

$$-\sum_{i=1}^{k} d_i x_i \leq -e$$

Standard form!

Primal-Dual relationship

Primal Problem	Dual Problem
opt=max	**opt=min**
Constraint i :	**Variable i :**
<= form	**y_i >= 0**
= form	**y_i urs**
Variable j:	**Constraint j:**
x_j >= 0	**>= form**
x_j urs	**= form**

♣ Duality in LP

In LP models, scarce resources are allocated, so they should be, valued

Whenever we solve an LP problem, we solve two problems: the primal resource allocation problem, and the dual resource valuation problem

If the primal problem has n variables and m constraints, the dual problem will have m variables and n constraints

Primal and Dual Algebra

Primal

$$\begin{aligned} Max \quad & \sum_j c_j X_j \\ s.t. \quad & \sum_j a_{ij} X_j \quad \leq \quad b_i \quad i = 1, \ldots, m \\ & X_j \quad \geq \quad 0 \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} Max \quad & C'X \\ s.t. \quad & AX \leq b \\ & X \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{Min} \quad & \sum_i b_i Y_i \\ \text{s.t.} \quad & \sum_i a_{ij} Y_i \quad \geq \quad c_j \quad j = 1, \ldots, n \\ & Y_i \quad \geq \quad 0 \quad i = 1, \ldots, m \end{aligned}$$

$$\begin{aligned} Min \quad & b'Y \\ s.t. \quad & A'Y \geq C \\ & Y \geq 0 \end{aligned}$$

Example

Primal

$$\begin{aligned} &Max\ \ 40x_1 + 30x_2 && (profits) \\ &s.t. \qquad x_1 + \ \ x_2 \le 120 && (land) \\ &\qquad\quad 4x_1 + 2x_2 \le 320 && (labor) \\ &\qquad\quad\ x_1, \quad x_2 \ge \ \ 0 \end{aligned}$$

Dual

$$\begin{aligned} &\qquad\ (land) \quad (labor) \\ &Min\ \ 120y_1 + 320y_2 \\ &s.t. \qquad y_1 + \ \ 4y_2 \ge 40 && (x_1) \\ &\qquad\quad y_1 + \ \ 2y_2 \ge 30 && (x_2) \\ &\qquad\quad y_1, \qquad y_2 \ge \ \ 0 \end{aligned}$$

General Rules for Constructing Dual

1. The number of variables in the dual problem is equal to the number of constraints in the original (primal) problem. The number of constraints in the dual problem is equal to the number of variables in the original problem.

2. Coefficient of the objective function in the dual problem come from the right-hand side of the original problem.

3. If the original problem is a *max* model, the dual is a *min* model; if the original problem is a *min* model, the dual problem is the *max* problem.

4. The coefficient of the first constraint function for the dual problem are the coefficients of the first variable in the constraints for the original problem, and the similarly for other constraints.

5. The right-hand sides of the dual constraints come from the objective function coefficients in the original problem.

Relations between Primal and Dual

1. The dual of the dual problem is again the primal problem.

2. Either of the two problems has an optimal solution if and only if the other does; if one problem is feasible but unbounded, then the other is infeasible; if one is infeasible, then the other is either infeasible or feasible/unbounded.

3. **Weak Duality Theorem**: The objective function value of the primal (dual) to be maximized evaluated at any primal (dual) feasible solution cannot exceed the dual (primal) objective function value evaluated at a dual (primal) feasible solution.

   $c^T x >= b^T y$ (in the standard equality form)

Relations between Primal and Dual (continued)

4. **Strong Duality Theorem**: When there is an optimal solution, the optimal objective value of the primal is the same as the optimal objective value of the dual.

$$c^T x^* = b^T y^*$$

Weak Duality

- DLP provides upper bound (in the case of maximization) to the solution of the PLP.
- Ex) maximum flow vs. minimum cut
- **Weak duality** : any feasible solution to the primal linear program has a value no greater than that of any feasible solution to the dual linear program.
- Lemma : Let x and y be any feasible solution to the PLP and DLP respectively. Then $c^T x \leq y^T b$.

Strong Duality

Strong duality : if PLP is feasible and has a finite optimum then DLP is feasible and has a finite optimum.

Furthermore, if x^* and y^* are optimal solutions for PLP and DLP then $c^T x^* = y^{*T} b$

Four Possible Primal Dual Problems

Dual Primal	Finite optimum	Unbounded	Infeasible
Finite optimum	1	x	x
Unbounded	x	x	2
Infeasible	x	3	4

Jyothimon C
M.Tech Technology Management
University of Kerala

Send your feedbacks and queries to
jyothimonc@yahoo.com