Linear and Combinatorial Optimization

Lecture 2: The Simplex method

- Basic solution.
- The Simplex method (standardform, b>0).
  1. Repetition of basic solution.
  2. One step in the Simplex algorithm.
  3. Unbounded problems.
  4. Tableau form.
  5. Optimality Criterion.
- Degeneracy and cycling.
- Artificial variables.

Properties of LP

Theorem 2.1. *A linear programming problem is a convex optimization problem.*

Proof. Exercise □

Theorem 2.2. *Let S be the set of feasible solutions to a general LP problem.*

1. *If S is non-empty and bounded, then an optimal solution to the problem exists and occurs at an extreme point.*
2. *If S is non-empty and* not *bounded and if an optimal solution the the problem exists, then an optimal solution occurs at an extreme point.*
3. *If an optimal solution to the problem does* not *exist, then either S is empty or S is unbounded.*

Observe that the feasible set S is usually infinite, but the number of extreme points is finite.

A simple algorithm for solving LP

Exhaustive search:

- Identify all extreme points
- Calculate the objective value for all extreme points
- Pick the largest value

Drawback: Although the number of extreme points is finite, it might be very large.

Extreme points in LP

Study a LP-problem in canonical form.

$$\max z = c^T x \tag{1}$$

$$Ax = b \tag{2}$$

$$x \geq 0 \tag{3}$$

A $m \times s$ matrix with $m \leq s$ and rank m.

Theorem 2.3. *Suppose that $A = [A_1 A_2]$ and there exists x_2 such that $A_2 x_2 = b, x_2 > 0$. Then, the columns of A_2 are linearly dependent if and only if $x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ is not an extreme point in S.*

Proof: ($\Rightarrow$) If the columns of A_2 are linearly dependent, then there exists a vector $c \neq 0$ such that $A_2 c = 0$. Then

$$A_2 \underbrace{(x_2 + \epsilon c)}_{v_2} = b$$

$$A_2 \underbrace{(x_2 - \epsilon c)}_{w_2} = b$$

Since all elements in x_2 are positive, one can choose ϵ small enough so $v_2 > 0$ and $w_2 > 0$. Then $x = \begin{pmatrix} 0 & x_2 \end{pmatrix}^T$ lies on the line segment between $v = \begin{pmatrix} 0 & v_2 \end{pmatrix}^T$ and $w = \begin{pmatrix} 0 & w_2 \end{pmatrix}^T$ which are both feasible points in S.

($\Leftarrow$) If x is not an extreme point, then there exist v and w with $Av = b$ and $Aw = b$ and a scalar $0 < \lambda < 1$ such that $x = \lambda v + (1 - \lambda) w$. Then

$$0 = \underbrace{\lambda}_{>0} \underbrace{v_1}_{\geq 0} + \underbrace{(1-\lambda)}_{>0} \underbrace{w_1}_{\geq 0}, \quad \Rightarrow \quad v_1 = 0, \, w_1 = 0$$

$$x_2 = \lambda v_2 + (1 - \lambda) w_2.$$

Since $b = Av = A_2 v_2 = A_2 x_2$ so $A_2 \underbrace{(x_2 - v_2)}_{\neq 0} = 0$,

hence the columns in A_2 are linearly dependent. ■

Important consequences

Corollary 2.1. $x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ *is an extreme point in S if and only if the columns of A_2 are linearly independent.*

Corollary 2.2. *Suppose x is an extreme point with r non-zero elements. Then $r \leq m$ and the corresponding columns may be extended to a basis by adding columns in A.*

Corollary 2.3. *At most m elements in x are non-zero for extreme points.*

Basic and non-basic variables

For an LP-problem in canonical form, choose m basis variables such that corresponding columns in A form a basis. By reordering

$$\begin{pmatrix} B & N \end{pmatrix} \begin{pmatrix} x_B \\ 0 \end{pmatrix} = Bx_B = b$$

Solve for x_b. Then $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is called a **basic solution**. In addition, if $x \geq 0$ then x is called a **basic feasible solution**.

Definition 2.1. *The variables in x_B are called* basic variables *and the variables in x_N are called* non-basic variables*.*

Number of basic feasible solutions

Since the number of basic feasible solutions can be at most $\binom{s}{m}$:

Corollary 2.4. *An LP-problem in canonical form has a finite number of basic feasible solutions.*

Every extreme point in a standard LP-problem corresponds to an extreme point in the corresponding canonical LP-problem and vice versa.

Corollary 2.5. *An LP-problem in standard form has a finite number of basic feasible solutions.*

Algorithm idea: Exhaustive search.

Constructing a basic solution

Consider the canonical LP:

$$\begin{cases} \max z = c^T x \\ Ax = b \\ x \geq 0 \end{cases} \tag{4}$$

Choose m linearly independent columns of A with the corresponding components of x and reorder so that:

$$\begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ 0 \end{bmatrix} = Bx_B = b \quad \Rightarrow \quad x_B = B^{-1}b$$

Now $x = (x_B^T \quad 0)^T$ is a *basic solution* to (4).

If $x_B \geq 0$ then x is a *basic feasible solution* since it fulfils the constraints in (4).

A basic feasible solution to (4) is an extremepoint to $S = \{x | Ax = b, x \geq 0\}$.

Solving standard LP with $b > 0$

Definition 2.2. *Two basic solutions are* **adjacent** *if all but one basic variable are in common.*

Consider the standard form LP:

$$\begin{cases} \max z = c^T x \\ Ax \leq b \\ x \geq 0 \end{cases} \tag{5}$$

Convert into a canonical LP by introducing slack variables.

An initial basic feasible solution can always be found by choosing the m slack variables as basic variables and setting the other variables to zero, i.e.

$$\hat{x} = (0 \quad b^T)^T, \quad \hat{A} = [A \quad I] \quad \Rightarrow \quad \hat{A}\hat{x} = \hat{b}$$

Choosing a new variable to enter the basis

Divide x into basic variables, x_B, and non-basic variables, $x_N (= 0)$ and divide A similarly into B and N, such that

$$Ax = Bx_B + Nx_N = b \quad \Rightarrow \quad x_B = B^{-1}b - B^{-1}Nx_N$$

Eliminate x_B from the goal function giving

$$\begin{aligned} z = c^T x = c_B^T x_B + c_N^T x_N = \\ = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N = \\ = \hat{z} + \hat{c}_N^T x_N \quad , \end{aligned} \tag{6}$$

where

$$\hat{c}_N = c_N^T - c_B^T B^{-1} N$$

are called *relative costs.*

Observe that if the relative cost $\hat{c}_i$ is positive for some i, the goal function z can be increased by inserting the corresponding non-basic variable in the new basis.

Choosing which variable to leave the basis

Assume that we have selected x_k to enter the basis. The system of equations $Ax = b$ can now be written as

$$x_B + B^{-1}a_k x_k = B^{-1}b \ ,$$

with the solution

$$x_B = B^{-1}b - B^{-1}a_k x_k = \hat{b} - \hat{a}_k x_k$$

where a_k denotes column k in A.

Since we would like to obtain a new *feasible* basis, we need to have

$$x_B = \hat{b} - \hat{a}_k x_k \geq 0 \ .$$

$\hat{a}_{ij} \leq 0$ leads to no decrease
$\hat{a}_{ij} > 0$ leads to decrease
We need to limit the increase in x_k such that none of the x_B will become negative, i.e.

$$x_k \leq \frac{\hat{b}_i}{\hat{a}_{ij}} := \theta \quad (a_{ij} > 0)$$

The Tableau method

We start with the following LP in canonical form

$$\begin{cases} \max z = c^T x \\ Ax = b \\ x \geq 0 \end{cases}$$

Write the goal function as

$$z - c^T x = 0 \ .$$

Assume that we start with the initial basic solution x_D. The tableau is then defined as

Basis	x_B	x_N	z	$\hat{b}$
x_D	B	N	0	b
z	$-c_B^T$	$-c_N^T$	1	0

- Iterate by changing basis.
- Select incoming and outgoing variable.
- Pivot around the corresponding element in the matrix

The Tableau method (ctd.)

We first perform row-operations transforming B to I:

Basis	x_B	x_N	z	$\hat{b}$
x_B	I	$B^{-1}N$	0	$B^{-1}b$
z	$-c_B^T$	$-c_N^T$	1	0

Then we eliminate c_B^T and arrive at the optimum with x_B as basis, giving the tableau:

Basis	x_B	x_N	z	$\hat{b}$
x_B	I	$B^{-1}N$	0	$B^{-1}b$
z	0	$c_B^T B^{-1}N - c_N^T$	1	$c_B^T B^{-1}b$

Observe that the optimum is $x_B = B^{-1}b$, $x_N = 0$ and $z = c_B^T B^{-1}b$.

Observe also that the negative relative costs $c_B^T B^{-1}N - c_N^T$ can be found in the tableau.

The Simplex method for LP-problems in standard form with $b > 0$

- Make tableau for initial basic solution
- Check optimality criterion: If the objective row has zero entries in the columns labeled by basic variables and no negative entries in the columns labeled by nonbasic variables.
- If the solution is not optimal, make tableau for an adjacent and improved solution
  - Selecting entering variable
  - Choosing the departing variable
  - Pivotal column, pivotal row. θ-ratios.
  - How to form the new tableau.

Possible outcomes

- Optimum exists: Objective row has zero entries for basic variables and no negative entries for non-basic variables.
- No finite optimal solution exists: No positive θ-ratios
- No solution exists: Impossible to set up initial tableau since we start with a basic feasible solution.

Degeneracy and cycling

Definition 2.3. *A basic solution in which some basic variables are zero is called* **degenerate***.*

At a degenerate basic solution, it may happen that the objective function does not increase. There is a risk of getting trapped in a cycle.

Bland's rule is a way of choosing pivotal column and pivotal row such that cycling is avoided.

- Selecting the pivotal column: Choose the column with the smallest subscript among those with negative entries.

- Selecting the pivotal row: If two rows have equal θ-ratios, choose the row corresponding to the basic variable with lowest subscript.

Examples

Artificial variables

How to solve an arbitrary LP-problem?

- Two-phase method
  - Introduce artificial variables, eliminate with simplex. This results in a basic feasible solution.
  - Form the Simplex tableau and use the Simplex method.
- Introduce artificial variables and weight them with $M \to \infty$. The Big M method (Optional).

Examples

Repetition - Lecture 2

- Equations and the Simplex tableau.
- The Simplex method.
  - Optimality criterion?
  - How to select a new basic feasible solution?
  - How to detect if the problem is unbounded?
  - How to form the simplex tableau with matrices?
- Degenerate basic solution, cycling, Bland's rule.
- Two-phase method.