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Abstract

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Introduction

Chapter 1

Lambda Calculus

The Lambda Calculus is an example of a formal system which consists a language of lambda terms and some auxiliary notions such as *free variables* and *subterms*, and the transformation theory. The core of the theory is the notion of substitution - the driving force behind function application.

1.1 λ-terms

The expressions in lambda calculus can be formalized into following:

Definition 1.1 (λ-TERMS) λ-terms can be defined by the following rules:

- a variable, x, itself is a lambda term
- if M is a lambda term, and x is a variable, then $(\lambda x.M)$ is a lambda term
- if M and N are both lambda terms, then *(MN)* is a lambda term

A lambda term is valid if and only if it can be obtained by repeated application of these rules. However, some parentheses can be omitted in certain forms. For example the leftmost, outermost brackets are usually omitted.

A lambda abstraction $\lambda x.M$ takes a single input x and returns a term M. Thus, it defines an **anonymous** function. For example $\lambda x.x^2$ is a lambda abstraction for the function $f(x) = x^2$ using the term x^2 for M, then we can say that $f = \lambda x.x^2$. The function is anonymous since we write $(x^2)2$ rather than $f3$.

An application (MN) applies the input term N to the function M. For example $(\lambda x.x^2)2$ is an application that applies 2 to the function $f(x) = x^2$, it return 2^2 which equals 4.

As mentioned above, leftmost, outermost brackets are usually omitted. Therefore, $M_1M_2(M_3M_4)$ stands for $((M_1 \cdot M_2)(M_3 \cdot M_4))$. Similarly, λ can be omitted in repeated abstractions, for example $\lambda x_1x_2.M$ stands for $\lambda x_1.(\lambda x_2.M)$.

Definition 1.2 (free and bound variables) The set of free and bound variables are defined inductively by the following function:

$$\begin{array}{lcllcl} FV(x) & = & x & BV(x) & = & \emptyset \\ FV(\lambda x.M) & = & FV(M_1) \cup x & BV(\lambda x.M) & = & FV(M) \backslash y \\ \lambda y.MFV(MN) & = & FV(M) \cup FV(N) & BV(MN) & = & BV(M) \cup BV(N) \end{array}$$

For example, the lambda term $\lambda x.x$ has no free variables but a bound variable x. The function $\lambda x.x + y$ only has a single free variable y and a bound variable x. Notice that, the sets of bound and free variables are not necessarily disjoint; x occurs both bound and free in:

$$x(\lambda xy.x)$$

$$
\begin{array}{lrll}
(1) & x[x := N] & = N & \\
(2) & y[x := N] & = y, & if\ y \neq x \\
(3) & (\lambda y.M)[x := N] & = \lambda y.M, & if\ y = x \\
(4) & (\lambda y.M)[x := N] & = \lambda y.(M[x := N]), & if\ y \notin FV(N)\ \&\ y \neq x \\
(5) & (\lambda y.M)[x := N] & = \lambda z.(M[y := z])[x := N], & if\ y \in FV(N)\ \&\ y \neq x, z\ new \\
(6) & (M_1 M_2)[x := N] & = (M_1[x := N])(M_2[x := N]) &
\end{array}
$$

Definition 1.3 *(α-equivalent) M is α-equivalent to N, written $M \equiv_\alpha N$, if N results from M by a series of changes of bound variable.*

The notion of alpha-equivalent(or congruence) is a basic form of equivalence defined in lambda calculus. It captures the property that the particular choice of a bound variable in a lambda abstraction does not matter. For instance, $\lambda x.x$ and $\lambda y.y$ are α-congruent lambda terms which represent the same function. Notice that, the term-variable x and y are not α-equivalent terms since they are not bound in a lambda abstraction.

With the property of alpha equivalence, we can define the *α-conversion*:

$$\lambda x.M =_\alpha \lambda z.(M[z := x])$$

In some sense, the α-conversion is defined by the spirit of the (5) substitution rule. If we have a function $\lambda x.M$, then we can apply a substitution to this function: $(\lambda x.M)[z := x]$. According to the fifth substitution rule, it is unfolded to $(\lambda z.M[z := x])$. Using the α-conversion, we can always rename the bound variables of a term.

Definition 1.4 *(Variable Convention) If M_1,...,M_n occur in a certain context then in these terms all bound variables are chosen to be different from free variables.*

The alpha-conversion is built based on the variable convention. Alpha convertion is used to allow bound variable names to be changed. For example, alpha-conversion of $\lambda x.x$ might get $\lambda y.y$. Terms that differ only by alpha-conversion are called alpha-equivalence(or alpha-congruence) as mentioned in the definition 1.3. By the assumption that free and bound variables are always different(variable convention), and that alpha-conversion will take place whenever a variable is both free and bound. The definition of term substitution becomes:

$$
\begin{array}{rll}
x[x := N] & = N & \\
y[x := N] & = y, & if\ y \neq x \\
(\lambda y.M)[x := N] & = \lambda y.(M[x := N]) & \\
(M_1 M_2)[x := N] & = (M_1[x := N])(M_2[x := N]) &
\end{array}
$$

By this definition of term substitution, the *variable capture* is avoided. Since that y is bound in $\lambda y.M$ of the context:

$$\lambda y.M[x := N]$$

y can only be bound in N according to variable convention rule, otherwise, if y is also free in N, it would be renamed by another variable.

Here is an example of its use:

$$
\begin{array}{ll}
 & (\lambda xyz.xzy)(\lambda xz.x) \\
= & \lambda yz.(\lambda xz.x)zy \\
= & \lambda yz.(\lambda xw.x)zy \ \ by\ the\ variable\ convention \\
= & \lambda yz.(\lambda w.z)y \\
= & \lambda yz.z
\end{array}
$$

1.2 β-reduction

There are various notions of reduction for λ-terms, but the principle one is β-reduction. β-reduction is the one-step reduction relation, written as $' \rightarrow'_\beta$.

Definition 1.5 *(β-reduction) For λ-terms M and N, we say that M β-reduces in one step to N, written as $M \rightarrow_\beta N$:*

$$(\lambda x.M)N \rightarrow_\beta M[x := N]$$

$$\frac{M \rightarrow_\beta N}{MZ \rightarrow_\beta NZ}$$

$$\frac{M \rightarrow_\beta N}{ZM \rightarrow_\beta ZM}$$

$$\frac{M \rightarrow_\beta N}{\lambda x.M \rightarrow_\beta \lambda x.N}$$

The reduction relation, written $' \twoheadrightarrow'_\beta$, is the reflexive, transitive closure of the one-step reduction relation. The one-step reduction relation allows a single step of reduction, while the reduction relation allows many steps. The reflexive transitive closure is defined as follows:

$$\frac{M \rightarrow_\beta N}{M \twoheadrightarrow_\beta N}$$

$$M \twoheadrightarrow_\beta M$$

$$\frac{M \twoheadrightarrow_\beta N \quad N \twoheadrightarrow_\beta Z}{M \twoheadrightarrow_\beta Z}$$

For the notion $'M \rightarrow_\beta N'$, it is read as 'M β-reduces to N'. The first rule defines that the reduction relation is reflexive to one-step reduction relation, while the third rule indicates the transitivity of reduction relation.
Finally, the β-equality, written as '$=_\beta$'. For λ-terms A and B, we say that $A =_\beta B$ if either $A \equiv B$ or there exists a sequence of reduction starting with A, ending with B. It is the equivalence relation generated by $\twoheadrightarrow_\beta$:

$$\frac{M \twoheadrightarrow_\beta N}{M =_\beta N}$$

$$\frac{M =_\beta N}{N =_\beta M}$$

$$\frac{M =_\beta N \quad N =_\beta Z}{M =_\beta Z}$$

For the notion $'M =_\beta N'$, we say 'M β-equivalent to N'.
Following is a simple example which applies β-reduction:

$$\begin{aligned} & (\lambda x.x^2)3 \\ = \; & (x^2)[x := 3] \\ = \; & 3^2 \\ = \; & 9 \end{aligned}$$

Definition 1.6 *(β-redex) A β-redex of a λ-term M is a subterm of M of the form $(\lambda x.P)Q$. A term M is called in β-normal form if it has no β-redex.*

A β-redex is essentially a candidate for an application of β-reduction. A term M has a β-normal form if there exists a term N such that N is in β-normal form and $M \twoheadrightarrow_\beta N$.

1.3 Head Normal Form

A λ-term in head normal form is generally in the form:

$$\lambda x_1 \ldots x_n.xM_1 \ldots M_m \qquad n, m \geqslant 0$$

In this case x is called the head variable. If $M \equiv \lambda x_1 \ldots x_n.(\lambda x.M_0)M_1 \ldots M_m$ where $n \geqslant 0$, $m \geqslant 1$ then the subterm $(\lambda x.M_0)M_1$ is called the head redex of M. Following are some examples of λ-terms in head normal form:

$$(1)\ xM \quad (2)\ \lambda x.x \quad (3)\ \lambda xy.x((\lambda z.z)y)$$

Normal order: Notice that, an expression in head normal form may contain redexes in argument positions whereas a normal form may not.

1.4 Reduction strategies

Recall that a term is said to be in β-normal form if it has no β-redexes, that is, subterms of the shape (λx.M)N. A term has a β-normal form if it can be reduced to a term in β in β-normal form. It is clear that if a term has a β-normal form, then we can get to the β-normal form by exhaustively reducing all β-redexes of the term, then reducing all β-redexes of all resulting terms, and so forth until we get the β-normal form. A **β-reduction strategy** is a function that selects, whenever a term has multiple β-redexes, which one should be reduced first. If a term is in β-normal form, then no β-reduction is to be done. Notice that, a β-reduction strategy may or may not have the property that adhering to the strategy will ensure that we(eventually) reach a β-normal form, if one exists.

In general, there are several different β-reductions possible for a given term. In this project, there are 5 different reduction strategies studied and implemented: normal order, call-by-name, call-by-value, head reduction, and applicative order. Figure X generally summarizes and compares the property of different strategies:

Reduction Strategy	Reduction Order	Reach β-normal form	Form Reached
Normal Order	leftmost outermost	Yes	normal form
Call-by-name	leftmost outermost	No	weak head normal form
Call-by-value	leftmost innermost	No	weak normal form
Head Reduction	inside lambda abstractions	No	head normal form
Applicative Order	leftmost innermost	Yes	normal form

Table 1.1: Reduction Strategies

Normal order: We model lambda terms as Haskell constructed data, representing variable names by character:

```
type Name = Char

data Term = Var Name | Abs Name Term | App Term Term
            deriving (Show, Eq)
```

Operational Semantics

A sequence of computational steps of a valid program is the operational semantics for the programming language. The final step of a terminating sequence returns the resulting value of the

program. Operational semantics can be classified into two types: **structural operational semantics(SOS or small-step semantics)** and **natural semantics(NS or big-step semantics)**. The structural operational semantics describes each single step of a computation in a program, while the natural semantics describes the overall resulting value generated by the program. Following, we use both SOS and NS to define the behavior of each reduction strategy. The small-step semantics gives reduction rules of the strategy, when the big-step semantics describes how the final term is reached.

Recursion

Although a reduction strategy may or may not have the property that ensures it reaches a β-normal form, it would recursively contract the selected redex until no redex(decided by specific reduction strategy) can be contracted. All the following Haskell functions defined conrresponding to the reduction strategy model the small-step semantics. Therefore, in order to recursively reduce a λ-term, an auxiliary function `recur_eval :: Term -> IO()` is defined:

```
recur_eval :: Term -> IO()
recur_eval t | evalcbn t == t = putStrLn("")
             | otherwise = do
                         putStrLn("==> "++ tostring (evalcbn t))
                         recur_eval (evalcbn t)
```

The Haskell function above defines the recursive reduction by Call-by-Name strategy. It uses the small-step function defined in Section 1.4.2. The recursion will terminate if the input λ-term cannot be reduced anymore, otherwise, it would print-out the term after a single step reduction and continue reducing the resulting term.

1.4.1 Normal Order Reduction

The normal order reduction $e \xrightarrow{no} e'$ continually applies te rule for β-reduction on the redex in leftmost outermost position until no more β-reduction can be performed. At that point, the resulting term is in normal form. When reducing an application $(e_1 e_2)$, the function term e_1 must be reduced using call-by-name. Since the strategy is outermost, when e_1 is reduced to an abstraction $(\lambda x.e)$, then the redex $((\lambda x.e)e_2)$ must be reduced before redexes in e.

$$\llbracket x \rrbracket^{ls}_{no} = x$$

$$\frac{\llbracket e \rrbracket^{ls}_{no} = e'}{\llbracket (\lambda x.e) \rrbracket^{ls}_{no} = (\lambda x.e')}$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbn} = (\lambda x.e) \qquad \llbracket e[e_2/x] \rrbracket^{ls}_{no} = e'}{\llbracket (e_1 e_2) \rrbracket^{ls}_{no} = e'}$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbn} = e_1' \neq (\lambda x.e) \qquad \llbracket e_1' \rrbracket^{ls}_{no} = e_1'' \qquad \llbracket e_2 \rrbracket^{ls}_{no} = e_2'}{\llbracket (e_1 e_2) \rrbracket^{ls}_{no} = (e_1'' e_2')}$$

Normal Order: Big-step operational semantics

$$\frac{}{(\lambda x.M)N \to_\beta M[N/x]} \quad \frac{M \to_\beta N}{MP \to_\beta NP} \quad \frac{M \to_\beta N}{PM \to_\beta PN}(P \textit{ contains no redex}) \quad \frac{M \to_\beta N}{(\lambda x.M) \to_\beta (\lambda x.N)}$$

Normal Order: Small-step operational semantics

Normal order reduction is *normalizing*, since it reduces the λ-term until there is no redex, redex in abstractions is also contracted. So, the normal order reduction will terminate with the normal form as result.

The corresponding Haskell function `evalnormal :: Term -> Term` below implements the normal order reduction. Notice that, it uses the function evalcbn defined in 1.4.2:

```
evalnormal :: Term -> Term
evalnormal (Var x) = Var x
evalnormal (Abs x t) = (Abs x (evalnormal t))
evalnormal (App (Abs x t) v) = subs t (Var x, v)
evalnormal (App x t) | x == evalcbn x = (App x (evalnormal t))
                     | otherwise = (App (evalcbn x) t)
```

The first function clause above handles variables x. It is not a β-reduction step, since a variable is not a redex. It returns itself which indicates no reduction can be done. It also conforms the pattern matching mechanism in Haskell. Similarly, the second function clause handles abstractions $(\lambda x.e)$ and implements the fourth SOS rule. The third function handles applications $(e_1 e_2)$ when e_1 is an abstraction, then the substitution will take place. It applies an argument to the function and returns the resulting value. Finally, the fourth function handles applications $(e_1 e_2)$ when e_1 is not an abstraction. Since the normal reduction is leftmost outermost, it reduces the function e_1 first. If e_1 cannot be reduced(returns itself), then it goes to the argument e_2.

Folllowing is the running example of normal order reduction in Haskell. We use "/" to represent the λ symbol:

```
    (/a.a)(/b.b)((/x.x)(/y.(/z.z)w))
==> (/b.b)((/x.x)(/y.(/z.z)w))
==> (/x.x)(/y.(/z.z)w)
==> /y.(/z.z)w
==> /y.w
Reduced to:/y.w
```

As we can see, it uses leftmost outermost strategy. The leftmost outermost redex $(\lambda a.a)(\lambda b.b)$ is first reduced. Then the whole reduced term is a redex, the argument $((\lambda x.x)(\lambda y.(\lambda z.z)w))$ substitutes the bound variables in the abstraction $(\lambda b.b)$. After that, the same substitution take place. Finally, it reaches into the redex $(\lambda z.z)w$ in the abstraction $\lambda y.(\lambda z.z)w$ and generates the resulting term $\lambda y.w$

1.4.2 Call-by-Name Reduction

The Call-by-Name reduction $e \xrightarrow{cbn} e'$ reduces the leftmost outermost redex not insde a lambda abstraction first. That is, the arguments to a function are not evaluated before the function is called. The redex in e of the asbtraction $(\lambda x.e)$ will never be reduced.

$$\llbracket x \rrbracket^{ls}_{cbn} = x$$

$$\llbracket (\lambda x.e) \rrbracket^{ls}_{cbn} = (\lambda x.e)$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbn} = (\lambda x.e) \qquad \llbracket e[e_2/x] \rrbracket^{ls}_{cbn} = e'}{\llbracket (e_1 e_2) \rrbracket^{ls}_{cbn} = e'}$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbn} = e_1' \neq (\lambda x.e)}{\llbracket (e_1 e_2) \rrbracket^{ls}_{cbn} = (e_1' e_2)}$$

Call-by-Name: Big-step operational semantics

$$\frac{}{(\lambda x.M)N \rightarrow_\beta M[N/x]} \qquad \frac{M \rightarrow_\beta N}{MP \rightarrow_\beta NP}$$

Call-by-Name: Small-step operational semantics

The Call-by-Name reduction generates term in weak head normal form. A lambda expression is in weak head normal form if it is a head normal form or any lambda abstraction.

The corresponding Haskell function `evalcbn :: Term -> Term` below implements the Call-by-Name reduction:

```
evalcbn :: Term -> Term
evalcbn (Var x) = Var x
evalcbn (Abs x t) = (Abs x t)
evalcbn (App (Abs x t) y) = subs t (Var x, y)
evalcbn (App x y) | x == evalcbn x = (App x y)
                  | otherwise = (App (evalcbn x) y)
```

The first two function clauses handle the variables and abstractions, it returns itself since they cannot be reduced under Call-by-Name reduction. The third function implements the first SOS rule which performs substitution. The last function implements the second SOS rule: if the function e_1 of the application (e_1e_2) cannot be reduced, then the argument e_2 will never be reduced.

The same example in 1.4.1 run in Haskell by the Call-by-Name strategy is as follows:

```
    (/a.a)(/b.b)((/x.x)(/y.(/z.z)w))
==> (/b.b)((/x.x)(/y.(/z.z)w))
==> (/x.x)(/y.(/z.z)w)
==> /y.(/z.z)w
Reduced to:/y.(/z.z)w
```

It is easy to see that the first three reduction steps are the same as normal order reduction. In this example, it stops at $\lambda y.(\lambda z.z)w$. Since the only difference between normal order and call-by-name is the redex inside abstractions will never be reduced in call-by-name.

1.4.3 Call-by-Value Reduction

Call-by-Value reduction is the most common reduction strategy. In Call-by-Value, the argument expression is evaluated, and the resulting value is bound to the corresponding variable in the function. It first reduces the leftmost innermost redex not inside an abstraction. It never reduces a redex when the argument is not a *value*. It differs from Call-by-Name only by reducing the argument e_2 of the application (e_1e_2) before the substitution take place.

$$\llbracket x \rrbracket^{ls}_{cbv} = x$$

$$\llbracket (\lambda x.e) \rrbracket^{ls}_{cbn} = (\lambda x.e)$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbv} = (\lambda x.e) \qquad \llbracket e_2 \rrbracket^{ls}_{cbv} = e_2' \qquad \llbracket e[e_2'/x] \rrbracket^{ls}_{cbv} = e'}{\llbracket (e_1e_2) \rrbracket^{ls}_{cbv} = e'}$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbv} = e_1' \neq (\lambda x.e) \qquad \llbracket e_2 \rrbracket^{ls}_{cbv} = e_2'}{\llbracket (e_1e_2) \rrbracket^{ls}_{cbv} = (e_1'e_2')}$$

Call-by-Value: Big-step operational semantics

$$\frac{}{(\lambda x.M)N \rightarrow_\beta M[N/x]} \quad \frac{M \rightarrow_\beta N}{MP \rightarrow_\beta NP}(M \textit{ not an abstraction}) \quad \frac{M \rightarrow_\beta N}{VM \rightarrow_\beta VN}(M \textit{ not an abstraction})$$

Call-by-Value: Small-step operational semantics

Call-by-Value generates terms in weak head normal form only. The implementation of the rules by an Haskell function `evalcbv :: Term -> Term` is as follows:

```
evalcbv :: Term -> Term
evalcbv (Var x) = Var x
evalcbv (Abs x t) = (Abs x t)
evalcbv (App (Abs x t) y) = subs t (Var x, evalcbv y)
evalcbv (App x y) | x == evalcbn x = (App x (evalcbv y))
                  | otherwise = (App (evalcbn x) y)
```

The functions are similar to Call-by-Name, the only difference is the argument is reduced before the substitution take place as indicated in the function clause 3.

The sample example run in Haskell by the Call-by-Value strategy is as follows:

```
    (/a.a)(/b.b)((/x.x)(/y.(/z.z)w))
==> (/b.b)((/x.x)(/y.(/z.z)w))
==> (/b.b)(/y.(/z.z)w)
==> /y.(/z.z)w
Reduced to:/y.(/z.z)w
```

As we can see, the reduction procedure is different from previous. It differs from previous two strategies at the second reduction step. Since the call-by-value reduction always reduces the argument before the substitution take place, it reduces the redex in the argument $((\lambda x.x)(\lambda y.(\lambda z.z)w))$ before it replaces the bound variables in the function $(\lambda b.b)$.

1.4.4 Head Reduction

The head reduction performs reductions inside lambda abstractions, but only in head position. Notice that, the *head reduction* strategy introduced in this project is the same as defined by Barendregt [1], however it differs from Sessoft's [4] *head spine reduction*. In the leftmost head reduction, only head redexes are reduced. A redex $((\lambda x.e_0)e_1)$ is a *head redex* if it is proceded to the left only by lambda abstractions of non-redexes, as in $\lambda x_1 ... \lambda x_n.(\lambda x.e_0)e_1 ... e_m$ when $n \geqslant 0$ *and* $m \geqslant 1$.

$$\llbracket x \rrbracket^{ls}_{hr} = x$$

$$\frac{\llbracket e \rrbracket^{ls}_{hr} = e'}{\llbracket (\lambda x.e) \rrbracket^{ls}_{hr} = (\lambda x.e')}$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbn} = (\lambda x.e) \qquad \llbracket e[e_2/x] \rrbracket^{ls}_{hr} = e'}{\llbracket (e_1 e_2) \rrbracket^{ls}_{hr} = e'}$$

$$\frac{\llbracket e_1 \rrbracket^{ls}_{cbn} = e_1' \neq (\lambda x.e)}{\llbracket (e_1 e_2) \rrbracket^{ls}_{hr} = (e_1' e_2)}$$

Head Reduction: Big-step operational semantics

$$\frac{}{(\lambda x.M)N \rightarrow_\beta M[N/x]} \qquad \frac{M \rightarrow_\beta N}{MP \rightarrow_\beta NP} \qquad \frac{M \rightarrow_\beta N}{(\lambda x.M) \rightarrow_\beta (\lambda x.N)}$$

Head Reduction: Small-step operational semantics

The head reduction strategy generates terms in head normal form. Recall that, a term is in head normal form if it has the form: $\lambda x_1 \ldots x_n.xM_1 \ldots M_m \quad n, m \geqslant 0$

```
headreduction :: Term -> Term
headreduction (Var x) = Var x
headreduction (Abs x t) = (Abs x (headreduction t))
headreduction (App (Abs x t) y) = subs t (Var x, y)
headreduction (App x y) | x == evalcbn x = (App x y)
                        | otherwise = (App (evalcbn x) y)
```

The first 3 function clauses are similar as before, it transfers the semantic rules straght forward into Haskell functions. The last function clause uses Call-by-Name function defined in 1.4.2, because it has to avoid premature reduction of inner redexes.

The sample example run by Head Reduction is as follows:

```
    (/a.a)(/b.b)((/x.x)(/y.(/z.z)w))
==> (/b.b)((/x.x)(/y.(/z.z)w))
==> (/x.x)(/y.(/z.z)w)
==> /y.(/z.z)w
==> /y.w
Reduced to:/y.w
```

The reduction procedure is the same as normal order reduction. Notice that, in the third reduction step, the redex $(\lambda z.z)w$ is a head redex in the abstraction $\lambda y.(\lambda z.z)w$.

1.4.5 Applicative Order Reduction

Applicative order reduction $e \xrightarrow{ao} e'$ reduces the leftmost innermost redex first. A function's arguments are always reduced before the function itself. Applicative order always attempts to apply functions to normal forms. It differs from Call-by-Value only by reducing also under abstractions:

$$\llbracket x \rrbracket_{ao}^{ls} = x$$

$$\frac{\llbracket e \rrbracket_{ao}^{ls} = e'}{\llbracket (\lambda x.e) \rrbracket_{ao}^{ls} = (\lambda x.e')}$$

$$\frac{\llbracket e_1 \rrbracket_{ao}^{ls} = (\lambda x.e) \qquad \llbracket e_2 \rrbracket_{ao}^{ls} = e_2' \qquad \llbracket e[e_2'/x] \rrbracket_{ao}^{ls} = e'}{\llbracket (e_1 e_2) \rrbracket_{ao}^{ls} = e'}$$

$$\frac{\llbracket e_1 \rrbracket_{ao}^{ls} = e_1' \neq (\lambda x.e) \qquad \llbracket e_2 \rrbracket_{ao}^{ls} = e_2'}{\llbracket (e_1 e_2) \rrbracket_{ao}^{ls} = (e_1' e_2')}$$

Applicative Order: Big-step operational semantics

$$\frac{}{(\lambda x.M)N \rightarrow_\beta M[N/x]} (M,\ N \textit{ in normal form}) \quad \frac{M \rightarrow_\beta N}{MP \rightarrow_\beta NP} \quad \frac{M \rightarrow_\beta N}{PM \rightarrow_\beta PN} (P \textit{ contains no redex})$$

$$\frac{M \rightarrow_\beta N}{(\lambda x.M) \rightarrow_\beta (\lambda x.N)}$$

Applicative Order: Small-step operational semantics

Applicative orde reduction always generates terms in normal form. Since it also reduce the redexes in abstractions. The applicative order reduction is not normalizing, functions applied to non-normalizing arguments are non-normalizing.

```
apporder :: Term -> Term
apporder (Var x) = Var x
apporder (Abs x t) = (Abs x (apporder t))
apporder (App (Abs x t) y) = subs t (Var x, y)
apporder (App x y) | x == apporder x = (App x (apporder y))
                   | otherwise = (App (apporder x) y)
```

The function clauses are simliar to the Call-by-Value introduced in 1.4.3. The only difference is the redexex inside abstractions are also reduced.

The sample example run by Applicative Order reduction in Haskell is as follows:

```
    (/a.a)(/b.b)((/x.x)(/y.(/z.z)w))
==> (/b.b)((/x.x)(/y.(/z.z)w))
==> (/x.x)(/y.(/z.z)w)
==> /y.(/z.z)w
==> /y.w
Reduced to:/y.w
```

It is the same as normal order reduction and the resulting term $\lambda y.w$ is in normal form. Although applicative order reduction always reduces the leftmost innermost redex, there is no redex on leftmost has an inner redex. If we make the leftmost abstraction $(\lambda a.a)$ become $(\lambda a.(\lambda c.c)w)$, then the first reduction step would reduce the inner redex $(\lambda c.c)w$ first:

```
    (/a.(/c.c)w)(/b.b)((/x.x)(/y.(/z.z)w))
==> (/a.w)(/b.b)((/x.x)(/y.(/z.z)w))
==> (/b.b)((/x.x)(/y.(/z.z)w))
==> (/x.x)(/y.(/z.z)w)
==> /y.(/z.z)w
==> /y.w
Reduced to:/y.w
```

1.5 Comparisons between Different Reduction Strategies

Chapter 2

The Curry type assignment system

A type system consists a set of rules that assign a property called a *type* to the various constructs — such as variable, expression, functions or modules. There are several reasons to add types to a program, one of the most important reason to add type properties to a program is to reduce bugs. It is possible to build an abstract interpretation of programs by treating terms as objects. The abstractions can be regarded as the interface of objects, by checking whether the interfaces have been connected properly we can decide whether the program is defined in a consistent way. Moreover, type systems provide a form of documentation and improve the readability of a program. Since a program can be abstracted, there are less information in the structure of a program. Furthermore, it also enables independent compilation before the implementation of a program runs. The types of functions and arguments can be checked during the generation of codes. Type-checking during implementation enables dynamic debugging which largely reduces the debugging workload at run-time. If a program is error-free, it is safe to run:"Typed programs cannot go wrong". It also allows multiple dispatch which is the feature of some objected-oriented programming languages in which a function or method can be dynamically dispatched based on the run time type of more than one of its arguments.

An exmaple of a type system is the Java language. A Java program consists of classes which contains a set of function definitions. A function is invoked by an instance of a class that the function belongs to. The interface of a function states the type of the return value, the name of the function and a list of values that are passed to the function. The code of an invoking function states the object instance on which this particular method is to be invoked, and the name of this function along with the names of variables that hold values to pass to it. The Java compiler checks the type of each argument that passed into the function, against the type declared for each varaible in the interface of the invoked function. If the types do not match, the compiler throws a compile-time error.

The lambda calculus as treated in Chapter 1 is referred to as a *type-free* theory. Because every expression may be applied to every other expression. For example, the $\lambda x.x$ may be applied to any argument x and generates the same x.

The typed version of lambda calculus(or called Combinatory Logic, a variant of the lambda calculus) is introduced essentially in Curry[2] and in Church(1940). Types are objects of a syntactic nature and may be assigned to lambda terms. If M is a lambda term and a type A is assigned to M, then we say *'M has type A'*; the notation used is '$M : A'$. For example, in a typed system one has $\lambda x.x : (A \to A)$, which means the function $\lambda x.x$ should take an argument in type A and return a value of type A. In general, $A \to B$ is the type of functions from A to B.

2.1 The System $\lambda \to$-Curry

In Curry and Feys(1958) and Curry et al. the type assignment theory was modified in a natural way to the lambda calculus assigning elements of a given set $\mathbb{T}$ of types to type free lambda terms. For this reason these calculi are sometimes called systems of type assignment. If the type $\sigma \in \mathbb{T}$ is assigned to the term $M \in \Lambda$ which writes as $\vdash M : \sigma$, sometime with a under subcript such

as $\vdash_\lambda$ to denote the particular system. Usually a set of assumptions Γ is needed to derive a type assignment write as $\Gamma \vdash M : \sigma$(read as 'Γ yields M in σ').

Definition 2.1 (i) The set of *types*, notation $\mathbb{T}$,

Notation. (i) If $\sigma_1, ..., \sigma_n \in \mathbb{T}$ then

$$\sigma_1 \to \sigma_2 \to ... \to \sigma_n$$

stands for

$$(\sigma_1 \to (\sigma_2 \to ... \to (\sigma_{n-1} \to \sigma_n)))$$

we use association to the right.

(ii) $\alpha, \beta, \gamma, ...$ denote arbitrary type variables.

Definition 2.2 (i) A *statement* is of the form $M : \sigma$, where $M \in \Lambda$ and $\sigma \in \mathbb{T}$(pronounced as 'M *in* σ'). M is called the *subject* and A the *predicate* of $M : A$.

(ii) A *context* Γ is a set of statements with only distinct (term) variables as subjects; In [5], the notion $\Gamma, M : \sigma$ is used for the context $\Gamma \cup \{M : \sigma\}$ where either $M : \sigma \in \Gamma$ or M does not occur in Γ, and we use $M : \sigma$ as shorthand for $\emptyset, M : \sigma$.

Definition 2.3 (CF. [[2], [3]]) Curry type assignment and *derivations* can be defined using following derivation rules.

$$(1) : \frac{}{\Gamma, M : \sigma \vdash_c M : \sigma}$$

$$(2) : \frac{\Gamma, x : \sigma \vdash_c M : \beta}{\Gamma \vdash_c (\lambda x.M) : \sigma \to \beta}$$

$$(3) : \frac{\Gamma \vdash_c M_1 : \sigma \to \beta \qquad \Gamma \vdash_c M_2 : \sigma}{\Gamma \vdash_c (M_1 M_2) : \beta}$$

***Example* 2.1.1** (i)Let $\sigma \in \mathbb{T}$. Then $\Gamma \vdash \lambda f x.f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma$, is shown by the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, f : \sigma \to \sigma \vdash f : \sigma \to \sigma}\,(1) \quad \dfrac{\dfrac{}{\Gamma, f : \sigma \to \sigma \vdash f : \sigma \to \sigma} \quad \dfrac{}{\Gamma, x : \sigma \vdash x : \sigma}}{\Gamma \vdash fx : \sigma}}{\Gamma \vdash f(fx) : \sigma}}{\Gamma \vdash \lambda x.f(fx) : \sigma \to \sigma}\,(1)}{\Gamma \vdash \lambda f x.f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma}\,(2)$$

(ii)Notice that, we cannot type *'self-application'* xx. In order to type the application xx, the derivation should be as following:

$$\frac{\dfrac{}{\Gamma \vdash x : A \to B} \quad \dfrac{}{\Gamma \vdash x : A}}{\Gamma \vdash xx : B}$$

As we can see, the term variable x is assigned to two types: $A \to B$ and A, and either of them can be substituted by the other one. So, $A \to B \neq A$, and there are two statements of subject x in context Γ with distinct predicates. Those two statements would be conflict according to the definition of context. Therefore, the 'self-application' is untypable.

2.2 Subject Reduction

2.3 Properties of $\lambda \rightarrow$

Definition 2.4 (Type substitution(van Bakel[5])) The substitution $(\varphi \mapsto C)$:, where φ is a type variable and $C \in \mathbb{T}$ is defined as following:

$$\begin{array}{ll} (\varphi \mapsto C)\varphi & = C \\ (\varphi \mapsto C)\varphi' & = \varphi', if\ \varphi' \neq \varphi \\ (\varphi \mapsto C)A \rightarrow B & = ((\varphi \mapsto C)A) \rightarrow ((\varphi \mapsto C)B) \end{array}$$

Definition 2.5 Let Id_s be the substitution that replaces all type variables by themselves:

(i) Robinson's unification algorithm. Unification of Curry types is defined by:

$$\begin{array}{llll} unify & \varphi & \varphi & = (\varphi \mapsto \varphi) \\ unify & \varphi & B & = (\varphi \mapsto B), if\ \varphi\ does\ not\ occur\ in\ B \\ unify & A & \varphi & = unify\ \ \varphi\ \ A \\ unify & (A \rightarrow B) & (C \rightarrow D) & = S_1 \circ S_2 \\ & & & \quad where\ S_1 = unify\ \ A\ \ C \\ & & & \qquad\ \ S_2 = unify\ (S_1 B)\ (S_1 D) \end{array}$$

(ii) UnifyContexts

$$\begin{array}{llll} UnifyContexts & (\Gamma_0, x : A) & (\Gamma_1, x : B) & = S_1 \circ S_2 \\ & & & \quad where\ S_1 = unify\ \ A\ \ B \\ & & & \qquad\ \ S_2 = UnifyContexts\ (S_1\Gamma_0)\ (S_1\Gamma_1) \\ UnifyContexts & (\Gamma_0, x : A) & \Gamma_1 & = UnifyContexts\ \Gamma_0\ \Gamma_1, if\ x\ does\ not\ occur\ in\ \Gamma_1 \\ UnifyContexts & \varnothing & \Gamma_1 & = Id_s \end{array}$$

Definition 2.6 (PPC, Principle Pair Algorithm for Λ)

$$\begin{array}{llll} ppc & x & = & \langle x : \varphi, \varphi \rangle \\ & & & where\ \ \varphi = fresh \\ ppc & (\lambda x.M) & = & \langle \Pi, A \rightarrow P \rangle,\ if\ ppcM = \langle \Pi \cup \{x : A\}, P \rangle \\ & & & \langle \Pi, \varphi \rightarrow P \rangle,\ if\ ppcM = \langle \Pi, P \rangle \quad x \notin \Pi \\ & & & \qquad\qquad where\ \varphi = fresh \\ ppc & (MN) & = & S_1 \circ S_2 \langle \Pi_1 \cup \Pi_2, \varphi \rangle \\ & & & where\ \ \varphi = fresh \\ & & & \qquad \langle \Pi_1, P_1, \varphi \rangle = ppcM \\ & & & \qquad \langle \Pi_2, P_2, \varphi \rangle = ppcN \\ & & & \qquad S_1 = unify\ P_1\ \ (P_2 \rightarrow \varphi) \\ & & & \qquad S_2 = UnifyContexts\ (S_1\Pi_1)\ (S_1\Pi_2) \end{array}$$

Chapter 3

A Purely-Functional Programming Language — Haskell

3.1 Functional Programming

3.2 Haskell

3.3 Model the λ-calculus and the Curry Type Assignment System in Haskell

3.4 Issues and Weakness of Implementation in Haskell

Chapter 4

The λ-calculus and Curry Type Assignment System Implementation in Java

The Java version of implementation is transferred directly from Haskell. However, the semantics and operational order of Haskell is difference from Java, especially the 'where' statement of Haskell. The 'where' declaration is called when the defined variables are used, however, in Java, we need to separate a 'where' block into parts according to where it is called. An example is shown in Section 4.1. This differece is worth taking care of when transfer the Haskell into Java, otherwise, NullPointerException may occur.

In all data types, the method toString() has been implemented – more precisely overridden, as it is a method of the Java primary class `Object`. This function is used to print out the data type as a string. There are more method signatures defined in the interfaces as a reference to instantiable classes.

4.1 Different operational order between Haskell and Java

As mentioned above, the operaional order of Haskell is different from Java, not only the pattern matching, but also the 'where' declarations. Since the code in Haskell cannot be transferred into Java directly, it largely abandoned the simplicity of codes. Following is a piece of code that defines principle pair algorithm:

```
ppc :: Term -> [Char] -> (PPc, [Char])
ppc (Abs x y) r | contains (Var x) pi = ((removeItem (Var x, a) pi, (TP a p)), tl1)
                | otherwise = ((pi, (TP f p)), tl1)
                            where (f, tl) = fresh r                    (1)
                                  ((pi, p), tl1) = ppc y tl            (2)
                                  (_, a) = search (Var x) pi           (3)
```

'Guards' are used as an if-then-else block. There are three declarations in the 'where' block each with a distinct number. For the first condition of ppc function, declaration (1)(2)(3) are called since all the declared variables are used. In the 'otherwise' condition, only declaration (1)(2) are called. In this case, in order to transfer the Haskell function into Java methods, we need to repetitively define the variables which largely reduces the simplicity of codes.

```
public PPC ppc(Term term, String counter){
     ...
     else if(term instanceof Abstraction){
        Variable xv = new Variable(((Abstraction) term).getName());
        PPC receiver = ppc(((Abstraction) term).getTerm(), counter);
```

```
        if(receiver != null){
           if(contains(xv, receiver.getSubject())){
             Type searchType = search(xv, receiver.getSubject()).getPredicate();
             ArrayList<Statement> original = receiver.getSubject().getContext();
             ArrayList<Statement> remove = new ArrayList<>();

             for(Statement s: original){
               if(s.getSubject().equals(xv)) remove.add(s);
             }

             for(Statement rm: remove){
                original.remove(rm);
             }

             TP tp = new TP(searchType, receiver.getPredicate());
             return new PPC(new Context(original), tp, receiver.getCounter());
           }
           else{
              TVar f = new TVar(receiver.getCounter().substring(0, 1));
              return new PPC(receiver.getSubject(), new TP(f, receiver.getPredicate()),
                             receiver.getCounter().substring(1));

           }
         }
         else {
             return null;
         }
      }
      ...
}
```

4.2 λ-calculus Representation

4.2.1 λ-terms

A Lambda term can either be a variable, an abstraction or an application. An interface 'Term' is defined as a reference type, which has three lambda term implementations. This is the generic representation of all lambda terms.

This interface defines the method `equals(Term t)`, which return a boolean states whether it is equal to the input term. This method overrides the `equals(Object o)` method of the Java primary class `Object`. It is essential in the reduction procedure, since we can decide whether a term is reducible by compare the term before reduction and after. If the term is the same as it before reduction, then it cannot be reduced. Therefore, we know when we can stop the reduction procedure.

As mentioned in Section 1.4, there are five reduction strategies enabled in the reducer. So there are five method signatures defined in the interface: evaluateNormal(Bool exsub), evaluateCbn(Bool exsub), evaluateCbv(Bool exsub), headReduction(Bool exsub), applicativeOrder(Bool exsub). Each of these method refers to a specific reduction strategy as the method name.

The method mirror() is used to create a mirror term with exactly the same class variables. This method is used when the substitution is performed. For example, if we have a lambda term $\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$, it is reduced to $\lambda f.f((\lambda x.f(xx))(\lambda x.f(xx)))$. If we do not create a mirror term of $(\lambda x.f(xx))$, both of these two $(\lambda x.f(xx))$ would refer to the same memory space. In other words, the abstraction $(\lambda x.f(xx))$ is used twice to form an application. Therefore, if operations performed on the function, it is also performed on the argument. It is essential to create a mirror

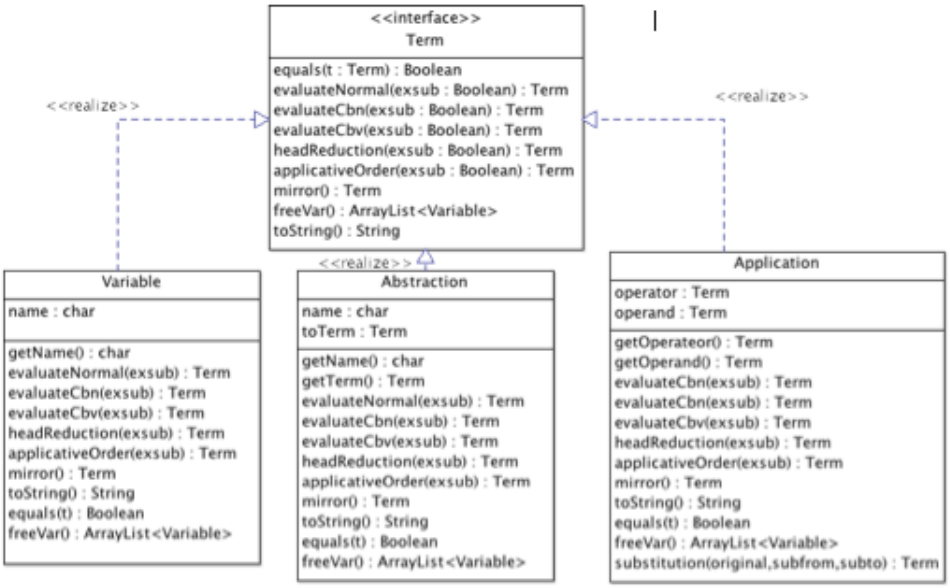


Figure 4.1: Class diagram of Lambda Calculus data structures

term that separate those two terms although they look exactly the same.

The method freeVar() is used to get all the free variables of a λ-term. It is used to perform α-conversion when we substitute bound variables in an abstraction. The free variables are defined in Definition 1.1.

Finally, the toString() method of class Object is overridden to transform a lambda term to a string.

Notice that, in the implementation class `Application`, there is a substitution(Term original, Term subfrom, Term subto):Term method. It is used to substitute bound variables in an abstraction, which is an application function, to the argument. The first parameter of the method is the body of abstraction, the second argument states which bound variable could be substituted and the third argument states what it could be substituted to.

4.2.2 Interpreter

To allow interactions and inputs from users, the program needs to interpret the input term into a data structure in the system. Figure 4.2 illustrates all the methods that used to interpret an input into a lambda term data structure.

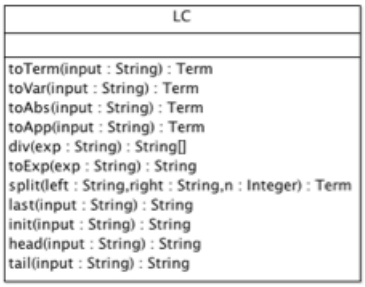


Figure 4.2: Interpretation methods in LC class

The method toTerm(String input) takes a string as input and returns a Term. It calls toVar(String input), toAbs(String input) or toApp(String input) according to type of the outermost term. Then, those three methods iteratively call each other until the whole term is interpreted.

div(String exp) divides an application into the function and argument. Basically, it return an array with 2 string elements. The method toExp(String exp) is used to remove brackets of

applications and returns an expression that is bracker-free. Finally, the most important method split(String left, String right, Int n) implements the bracket removal mechanism. It marks the right most right bracker ')' and find the matching left bracket '(' and extract the term out without brackets.

Since the Java implementation is directly transferred from Haskell, there are also four basic string operations which are Haskell builit-in functions. last(String input) takes a string and returns the last character of the string, it returns itself if the length of input string is 1. init(String input) accepts a string and returns the string without its last character, it returns an empty string if it only contains one character. head(String input) takes a string and returns the first element of the string, it returns itself when the input string length is 1. tail(String input) takes a string and returns the string without its first element, it returns an empty string when the length of input is 1.

Following is the example of how those Haskell built-in function work:

```
init "Hello"   "Hell"          head "Haskell"   "H"
init "A"       ""              head "A"         "A"

last "String"   "g"            tail "Lambda"   "ambda"
last "A"        "A"            tail "A"         ""
```

4.3 Curry Type Assignment System Representation

4.3.1 Types

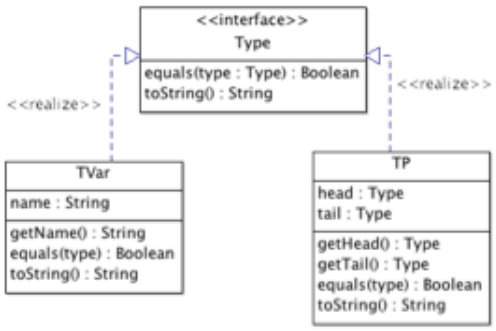
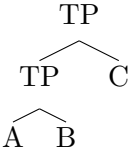


Figure 4.3: Class diagram of type representation

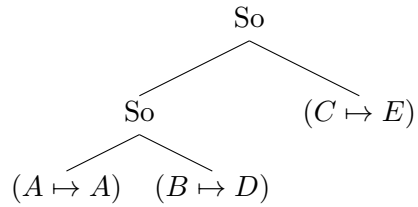
Type assignment system assigns types to λ-terms. In $\mathbb{T}$, a type can either be a single type$(A, B, ...)$ or a complex type$(A \to B)$. Since the complex type is formed by single types, it can be represented using the binary tree structure, where single types are leafs and complex types are branches. For example, the type $(A \to B) \to C$ is represented as TP(TP (TVar A)(TVar B))(TVar C), or in a more intuitionistic way:



Illustrated in Figure 4.3, the interface Type is defined as a reference type. It defines two basic method signatures: equals(Type type) and toString() which override the primary methods in Object. There are two implementation class: TVar and TP which stand for single type and complex type. TVar contains a class variable name which states the type name. It implements method signatures defined in the interface and a get method that returns the type name. The implementation class TP is similar to a Node of a tree structure, which contains two class variables: left branch and right branch. Method signatures and get methods are implemented in the class.

4.3.2 Type substitution

Principal types for λ-terms are defined using the notion of unification of types that defined by Definition 2.5. Similar to types structure in Section 4.3.1, a type substitution can either be a single substitution($\varphi \mapsto B$) or a complex substitution($S_1 \circ S_2$). It could also be represented as a binary tree structure. For example, when we unify $(A \to B) \to C$ and $(A \to D) \to E$, we would get the complex substitution$((A \mapsto A) \circ (B \mapsto D)) \circ (C \mapsto E)$ which also can be represented as a binary tree:



As shown in Figure 4.4, an interface TSub is needed as a reference type. It doesnt have any method signatures, operation on substitutions are defined as auxiliary function in class LC.

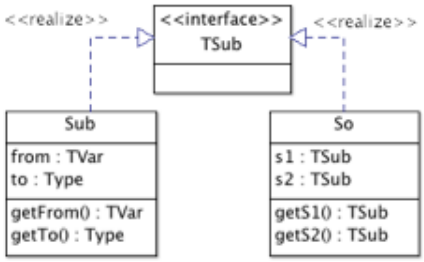


Figure 4.4: Class diagram of type substitutions representation

4.3.3 Other data types

The Curry type assignment system defines other concepts besides types and substitutions. Statement is essential to describe the type assigned to a λ-term; Context is used to collect all statements used for free variables of a term when typing that term; and PPC is a data type that stores context and the type assigned to that term typed.

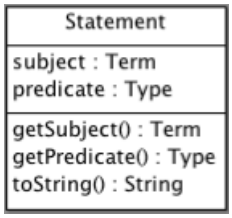


Figure 4.5: Class diagram of Statement

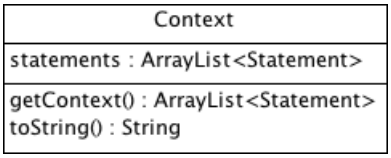


Figure 4.6: Class diagram of Context

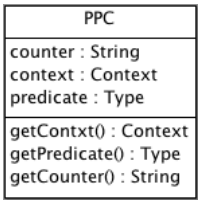


Figure 4.7: Class diagram of PPC

The Statement class has two attributes which are subject and predicate. It contains basic get method that return private attributes, and it also overrides the toString() method to transform a Statement into a string as "subject : predicate"

The Context class only contains an ArrayList of Statement as attribute. A context could have none statement if all the variables are bound. The toString() method returns the context a string in the format: "statement1, statement2, ..."

The PPC class, storing principle pair algorithm results, is a little more complex. As defined in Definition 2.6, the ppc function return a context and the type assigned to the term typed as ⟨context, type⟩. A PPC instance, stores the result of ppc function, therefore it contains two attributes as context and type. In additional, it also has a counter attribute. Since principle pair

algorithm always create fresh types, we should keep track of what type names are available that have not been used. It basically states all the available type names.

4.3.4 Core type assignment mechanism implementation

The Curry type assignment implementation is built based on the lambda calculus implementation. As it extends more operation on λ-terms for type assignment, there are more auxiliary methods defined in class LC.

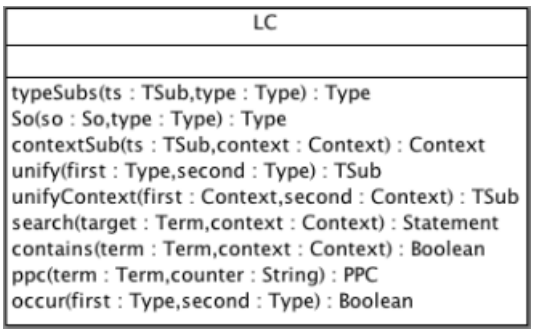


Figure 4.8: Class diagram of Lambda Calculus data structures

The methods listed in Figure 4.8 are the core type assignment mechanism implementations. The top-level method is ppc(Term term, String counter), when a term comes in and needs to be typed, the ppc method is called and it iteratively calls other auxiliary methods. Finally, it will return a PPC instance which contains a context and the typed assigned.

Given a type substitution and a type, the method typeSubs(TSub, Type) substitutes all the type variables according to the subsitution. It implements the type substitution defined in Definition 2.4. Notice that, since a TSub can either be a single substitution Sub or a complex So as defined in Section 4.4, it calls So(So, Type) method that implements the So substitution described in Definition 2.4 (b).

The method contextSub(TSub, Context) implements the context substitution defined in Definition 2.5 (c).

The method unify(Type, Type) implements the type unification algorithm defined in Definition 2.4. And unifyContext(Context, Context) implements the context unification in Definition 2.5 (ii). It should be mentioned that, since Id_s stands for the substitution that replaces all type variables by themselves, it could be simplified as a substitution that only contains the substitution $(A \mapsto A)$. We do not care what the type A is and whether it is a valid type variable, becuase the substitution $(A \mapsto A)$ would not affect any unifications.

The search(Term, Context) methods is an auxiliary method that given a λ-term and a context, it returns the corresponding statement of that term in the context. This method is used in two places. Firstly, it is used when we want to unify the context Γ_0 and Γ_1. We need to find the statements with the same subject in these two context and unify them. Therefore, when we traverse through the context Γ_0, the search method is neccessary to find the corresponding statement in Γ_1. Secondly, it is used in the principal pair algorithm when we type an abstraction. The temporary type assigned to bound variables should be removed from the context. Therefore, given a variable, the search method can find the corresponding statement and further be removed.

The method contains(Term, Context) takes a λ-term and a context as arguments and return a boolean states whether the context contains a statement for the term.

ppc(Term, String) is the core principal pair algorithm implementation method. The definition of principal pair algorithm can be found in Definition 2.6. The second argument in String type is the counter of all available type names.

Finally, the method occur(Type, Type) is used to decide whether a type occurs in another. It is used in the second type unificatin algorithm defined in Definition 2.5.

4.4 Explicit substitution and garbage collection

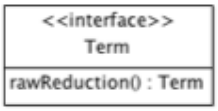


Figure 4.9: Class diagram of Lambda Calculus data structures

4.5 Syntax

Chapter 5

GUI development

5.1 Layout

5.2 Functionalities

Chapter 6

Java vs Haskell, Pros and Cons

6.1 Pros

6.2 Cons

Chapter 7

Conclusion

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