

AMATH 301 Final A

Skyler Riki Hallinan

TOTAL POINTS

125 / 150

QUESTION 1

1 Question 1 14 / 15

- 0 pts Correct

✓ - 1 pts One line with a small bug

- 2 pts 2 bugs

- 5 pts Iteration line mistakes

- 6 pts Forward Euler correct, but $f(x)$ incorrect.

- 7 pts All math correct, but not in runnable Matlab syntax.

- 10 pts Initializing variable errors, and/or missing or incorrect Forward Euler algorithm.

- 12 pts No runnable code, but some theory that is correct.

- 15 pts Not an answer.

💬 I think you meant "dxdt(x(k))"

QUESTION 2

2 Question 2 7 / 10

- 0 pts Correct

- 1 pts Missed 'econ'

- 3 pts Line for svd command

✓ - 3 pts Dimensions for all three U, S, and V

- 3 pts Reconstruction line

- 10 pts No answer

QUESTION 3

Question 3 15 pts

3.1 Part a 7 / 7

✓ - 0 pts Correct

- 3 pts Only calculated one inner product by hand

- 4 pts Computed outer products instead of inner

- 4 pts Multiplied incorrectly (dimensions make sense)

- 5 pts Multiplied incorrectly (dimensions don't make sense)

- 5 pts Did not calculate by hand

- 0.5 pts Minor arithmetic error

- 1 pts Arithmetic error (several errors or no work shown)

- 1 pts Used .* with row/column

- 1 pts Found outer product in code

- 1.5 pts Did not use transpose in code

- 0.5 pts No * symbol in code

- 2 pts No code

- 0.5 pts Minor syntax error

- 6 pts Said the formula does not make sense

- 7 pts Did not answer question

3.2 Part b 8 / 8

✓ - 0 pts Correct

- 3 pts Only calculated one outer product by hand

- 4 pts Computed inner products instead of outer

- 4 pts Did not multiply correctly (dimensions make sense)

- 5 pts Did not multiply correctly (dimensions don't match)

- 5 pts Did not calculate by hand

- 0.5 pts Minor arithmetic error

- 0.5 pts Missing * symbol in code

- 1 pts Used .* with row/column

- 1 pts $(u^*v)^*$ instead of u^*v^*

- 0.5 pts Minor syntax error

- 1.5 pts Did not use transpose in code

- 2 pts No code

- 1 pts Wrong rank / No rank

- 7 pts Said the formula does not make sense

- 8 pts Did not answer question

QUESTION 4

Question 4 18 pts

4.1 Part a 5.5 / 6

✓ - 0 pts Correct

- 1.5 pts Minor problem with nature of correction.
- 1.5 pts Minor problem with location of correction.
- 3 pts Major error with location of correction.
- 3 pts Major problem with nature of correction.
- 6 pts Blank/No credit

- 0.5 Point adjustment

 I think you know where it needs to go, based on your description. But you still need to specify the exact location (right after line 14).

4.2 Part b 6 / 6

✓ - 0 pts Correct

- 1.5 pts Minor problem with nature of correction.
- 1.5 pts Minor problem with location of correction.
- 3 pts Major problem with nature of correction.
- 3 pts Major problem with location of correction.
- 6 pts Blank/No credit.

4.3 Part c 3 / 6

- 0 pts Correct

- 1.5 pts Minor problem with nature of correction

- 1.5 pts Minor problem with location of correction

✓ - 3 pts Major problem with nature of correction.

- 3 pts Major problem with location of correction

- 6 pts Blank/no credit

QUESTION 5

Question 5 16 pts

5.1 Part a 5.5 / 8

- 0 pts Correct

- 0.5 pts Minor syntax error.

- 0.5 pts Other minor error

✓ - 1 pts Syntax error

- 1.5 pts Data not loaded in

✓ - 1.5 pts Incorrect calculation.

- 3 pts Incorrect Calculation

- 5 pts Incorrect Calculation

- 8 pts Blank/No Credit

 You want load, not import. Also, you need to divide by the length of the vector before taking the square root.

5.2 Part b 8 / 8

✓ - 0 pts Correct

- 0.5 pts Minor syntax error
- 1.5 pts Syntax error
- 1.5 pts Incorrect calculation.
- 3 pts Incorrect calculation
- 6 pts Incorrect calculation
- 8 pts Blank/no credit

QUESTION 6

Question 6 16 pts

6.1 Part a 0 / 8

- 0 pts Correct

- 5 pts Substituted $v = x'$, but did not successfully isolate v'

- 3 pts Found correct v' equation, then made an incorrect system

- 2 pts Did not include equation $x' = v$ (but did use this substitution elsewhere)

- 1 pts Correct equations, but still written with x'' or x' instead of v' or v

- 2 pts Used an additional variable/equation for x''

- 2 pts Set $x' = \text{constant}$ (but used correct substitution elsewhere)

- ✓ - 8 pts Did not substitute $v = x'$ (completely wrong approach)

- 2 pts Did not rewrite initial conditions correctly

- 1 pts Correct initial conditions, but not written in terms of v

- 8 pts Did not answer the question

- 0.5 pts Algebra error

6.2 Part b 6 / 8

- 0 pts Correct

- 0.5 pts Used stiff solver

- 1 pts Used an invalid solver (or no solver)

- ✓ - 1 pts Did not use a function of the form $@(t,v)$

- 2 pts Wrong/missing function body

- 1 pts Missing $x' = v$ in function (otherwise correct)

- 2 pts Did not define function in code (but did write equations)

- 1 pts Used a vector instead of a function

- 1 pts Wrong/missing time span
- 1 pts Wrong/missing initial condition
- 1 pts Wrong output syntax
- ✓ - 1 pts Did not find x5 correctly
 - 0.5 pts Minor syntax error
 - 1 pts Major syntax error
 - 8 pts Did not answer question
- 💬 This is the wrong system, but it matches your answer from part (a).

QUESTION 7

7 Question 7 15 / 15

- ✓ - 0 pts Correct
- 3 pts Incorrect A / not defined
- 0 pts Incorrect dt / not defined
- 2 pts Incorrect b / not defined
- 2 pts Didn't use permutation (LUP)
- 2 pts Incorrect lu() input (A+l)
- 2 pts Didn't use P*b
- 2 pts Incorrect y definition (L\((P*b))
- 2 pts Incorrect x definition (U\y)
- 1 pts Matlab syntax error
- 15 pts not attempted or no points

QUESTION 8

8 Question 8 14 / 18

- 0 pts correct
- 3 pts says not approximate
- 4 pts incorrect leading error term
- ✓ - 2 pts Incorrect order accuracy
 - 2 pts incorrect LET coeff
 - 18 pts not attempted or equivalent
 - 1 pts minor arithmetic error
- ✓ - 2 pts Error term calculations
 - 6 pts no work

QUESTION 9

9 Question 9 11 / 11

- ✓ - 0 pts Correct
- 2 pts Used wrong tolerance
- 2 pts Did not specify answer (y)

- 4 pts Treated output cn as a function
- 4 pts Not enough output variables
- 4 pts Found wrong output
- 3 pts Did not define u or m
- 1 pts Syntax error(s)
- 3 pts Wrong u and/or m
- 4 pts Used undefined variable tol
- 11 pts Did not answer question

QUESTION 10

Question 10 16 pts

10.1 Part a 7 / 7

- ✓ - 0 pts Correct
- 1 pts One error
- 3 pts indexing errors and/or missing boundary points
- 5 pts Some code written, but incorrect algorithm and/or output
- 7 pts Not an answer

10.2 Part b 4 / 4

- ✓ - 0 pts Correct
- 1 pts Wrong output variable.
- 2 pts X and Y ordering incorrect.
- 2 pts Extra operations
- 3 pts Wrong arguments to trapz()
- 4 pts No answer

10.3 Part c 4 / 5

- 0 pts Correct
- ✓ - 1 pts One error
 - 2 pts LHR instead of RHR
 - 3 pts Code written uses wrong algorithm and/or will not run.
 - 5 pts Not an answer
- 💬 Index should have started at 2.

AMATH 301 Final Exam

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Question	Points	Score
1	15	
2	10	
3	15	
4	18	
5	16	
6	16	
7	15	
8	18	
9	11	
10	16	
Total:	150	

Please write your name on every page you want us to grade.

Many of these questions ask you to write short segments of code. You should try your best to write functioning Matlab code. That means proper use of arithmetic symbols, parentheses, brackets, builtin function names, semicolons, etc.

0 0.01 0.02 0.03

Name: Skye Hollenbeck
NetID: 1732227

$$\Delta t = 0.01 \quad \text{time} = 0 : \Delta t : 1$$

1. (15 points) Write Matlab code to solve the initial value problem $\dot{x} = x^2 + 1$ from $t = 0$ to $t = 1$ with a time step of $\Delta t = 0.01$ and the initial condition $x(0) = -1$. Use the forward Euler method. (You should write your own solver, not use a builtin function.) Your answer should be a row vector named x .

$$dx/dt = @(x)(x^2 + 1)$$

$$x_0 = -1$$

$$\Delta t = 0.01$$

$$x = zeros(1, 101)$$

$$x(1) = x_0$$

$$N = 100$$

$$\text{for } k=1:N$$

$$x(k+1) = x(k) + \Delta t * dx/dt(k)$$

end

2. (10 points) Suppose that A is a 5×7 matrix. Write a single line of Matlab code using the `svd` command to find the reduced singular value decomposition of A . (You can assume that A is already defined.) What are the dimensions of the matrices returned by `svd`? Write another line of Matlab code to reconstruct A (exactly) using the resulting matrices.

$$[U, S, V] = \text{svd}(A, 'econ')$$

Dimensions: S is 5×5 , U and V are 5×7

$$A = U * S * V'$$

$\begin{matrix} \text{v}_1 & \text{v}_2 \\ \text{v}_3 & \end{matrix}$ $\begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix}$ $\begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix}$
 3. (15 points) Let $\begin{matrix} 3 \times 1 \text{ vector} \\ \text{vector} \end{matrix}$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Decide if each of the following expressions makes sense. If it does, calculate the result by hand and write a *single line* of Matlab code to compute the result. (You can assume that \mathbf{u} , \mathbf{v} , \mathbf{x} and \mathbf{y} have already been defined in Matlab.) If the result is a matrix, say what the rank of the matrix is.

$$(a) \mathbf{u}^T \mathbf{v} + \mathbf{x}^T \mathbf{y}$$

$$(1 \times 3) \times (3 \times 1) + (1 \times 3) \times (3 \times 1) \quad \text{Adding inner products}$$

$$1 \times 1 + 1 \times 1 \rightarrow \text{works}$$

to give value but

$$\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -1(1) + 1(0) - 1(2) \end{bmatrix} + \begin{bmatrix} 2(2) + 1(1) + 3(3) \end{bmatrix} \rightarrow \begin{bmatrix} -3 \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \end{bmatrix}$$

Matlab: $\mathbf{u}.^T * \mathbf{v} + \mathbf{x}.^T * \mathbf{y}$

$$(b) \mathbf{u}\mathbf{v}^T + \mathbf{x}\mathbf{y}^T$$

$$(3 \times 1)(1 \times 3) + (3 \times 1)(1 \times 3)$$

$$(3 \times 3) + (3 \times 3) \rightarrow \text{Rank 2 Matrix}$$

$$\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 2 \\ 2 & -1 & 1 \\ 6 & -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 0 \\ 3 & -1 & 3 \\ 5 & -3 & 1 \end{bmatrix}$$

Matlab: $\mathbf{u} * \mathbf{v}.^T + \mathbf{x} * \mathbf{y}.^T$

4. (18 points) The following code is meant to implement the section search to find the maximum of the function $f(x)$ given below. It is easy to check that the maximum is at $x = 1$. This code is written in two different files, called `section.m` and `fun.m`. These files are saved in the same directory. The code has several errors, which we will fix in turn.

$$f(x) = \begin{cases} -x^2 & \text{if } x < -2 \\ 2x & \text{if } -2 \leq x \leq 1 \\ 3 - x^3 & \text{if } x > 1. \end{cases}$$

FileName: `section.m`

```

1   clear all; close all; clc;
2
3   a = -4; b = 4;
4   c = 0.6;
5   tolerance = 1e-8;
6
7   for k = 1:100
8       x = c*a + (1 - c)*b;
9       y = (1 - c)*a + c*b;
10
11      if fun(x) < fun(y)
12          b = y;
13      else
14          a = x;
15      if (b - a) < tolerance
16          break
17      end
18  end

```

$$x = 0.6 > a^4 [1]$$

Filename: `fun.m`

```

1   function y = fun(x)
3   if x < -2
4       y = -x^2;
5   elseif (x >= -2) && (x <= 1)
6       y = 2x;
7   else
8       y = 3 - x^3;
9   end
10 end

```

The numbers on the left are just line numbers, not code. For each of the following parts (starting on the following page), clearly indicate any lines that you would need to change or add and what changes you would make. Be sure to indicate which file(s) need changes. Only fix the problem from the given part, not all problems that you see.

- (a) The code from the previous page does not work. When you run `section.m`, Matlab produces the following error message:

Error: File: section.m Line: 7 Column: 1

At least one END is missing: the statement may begin here.

What is causing this error, and what would you have to change in order to fix it?

There is a missing end inside the for loop that starts at line 7 and ends at 18. There needs to be an end inserted at the end of the if else statement inside the for loop.

- (b) After correcting the above error, your code still does not work. When you run `section.m`, Matlab produces the following error message:

Error: File: fun.m Line: 6 Column: 10

Unexpected MATLAB expression.

Error in section (line 11)

if $\text{fun}(x) < \text{fun}(y)$

What is causing this error, and what would you have to change in order to fix it?

*There is an error in the function file fun.m on line 6. There is an error because they have $y = 2x$ and they are missing the asterisk to indicate multiplication, should be $y = 2^*x$ instead.*

- (c) After correcting both of the above errors, your code appears to run successfully and stops after 41 steps. However, the final value of x is extremely close to -4 , which is not the correct maximum value. What is causing this problem, and what would you have to change in order to fix it?

The initial bounds a and b have negative function evaluations when they are called. This means that using a and b would not allow for a proper convergence towards a minimum. Change initial a and b values. Or, instead

root mean
square

5. (16 points) For the rest of this problem, assume that you have a file called `data.mat` in the same directory as the rest of your code. This file contains two vectors `x` and `y` of the same length. We will try find parameters `a`, `b` and `c` such that the curve $y = \cos(a * x) + \sin(b * x) + c$ best fits the data.

- (a) First, complete the following function that will compute the root-mean-square error between the curve $y = \cos(a * x) + \sin(b * x) + c$ and the data given in `data.mat`. Note that `a`, `b` and `c` are arguments in the function but `x` and `y` are not.

`function E = rms_error(a, b, c)`

`import (data, mat)`

$$y_{\text{calc}} = \cos(a * x) + \sin(b * x) + c$$

$$E = \sqrt{\sum((y_{\text{calc}} - y)^2)}$$

`end`

local
minimizew

- (b) Now write code to calculate the coefficients `a`, `b` and `c` that minimize the root-mean-square error from part (a). Use an initial guess of $a = -1$, $b = 1$ and $c = 0$. You should name your results `a`, `b` and `c`. You can assume that the function from part (a) is saved as `rms_error.m` in the same directory as this code and the file `data.mat`.

$$\text{guess} = [-1, 1, 0];$$

$$\text{wrap} = @(v)(\text{rms_error}(v(1), v(2), v(3)))$$

$$[x, fval] = \text{fminsearch}(\text{wrap}, \text{guess})$$

$$a = x(1);$$

$$b = x(2);$$

$$c = x(3);$$

9, 3, 4, 6

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6. (16 points) Consider the second order initial value problem

$$\ddot{x} - 5(1-x^2)\dot{x} + x = 0,$$

$$\text{where } x(0) = 1 \text{ and } \dot{x}(0) = 0.$$

$$\ddot{x} = 5x(1-x^2)\dot{x} + x$$

$$\dot{x} = 0$$

- (a) Rewrite this equation (by hand) as a system of first order equations. Be sure to rewrite the initial conditions as well. Show your work!

$$\text{At } t=0, \quad \dot{x} - 5(1-1)\dot{x} + 1 = 0 \\ \dot{x} = +1$$

$$-5(1-x^2)\dot{x} + x = -1$$

$$\begin{aligned} \ddot{x} &= 5(1-x^2)\dot{x} \\ x &= \end{aligned}$$

$$\ddot{x} = -5(0)\dot{x} + 1$$

$$\dot{x} = \frac{-1-x}{5(1-x^2)}$$

- (b) This equation is known to *not* be stiff. Write Matlab code using an appropriate builtin solver to solve this system of differential equations from $t = 0$ to $t = 5$. Find the value of x at time $t = 5$ and name your answer $x5$.

$$\text{guess} = [1 \ 0]$$

$$v = @ (x) \left(\left[5 * (1 - x(1)^2) \times (1) \right] \left[(-1 - x(1)) / (5 * (1 - x(1)^2)) \right] \right)$$

$$T = 5;$$

$$[t_out, x_out] = \text{ode45}(v, [0 T], \text{guess})$$

$$x5 = x_out(\text{end})$$

$$Ux = y$$

$$LU = PA$$

$$LUx = Pb$$

$$Ly = Pb$$

Name: Shayler Holloman
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7. (15 points) Consider the system of equations $(A + I)x = b$, where

$$A = \frac{1}{(\Delta t)^2} \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 3/(\Delta t)^2 \\ 0 \\ 0 \\ 0 \\ -1/(\Delta t)^2 \end{pmatrix},$$

$\Delta t = 0.2$ and I is the 5×5 identity matrix.

Write Matlab code to solve this system using LU decomposition. You should explicitly keep track of the permutation matrix and you should not assume that any Matlab variables have already been defined. Your final answer should be named x .

$$A = \begin{pmatrix} 1/0.04 & & & & \\ & -2, 1, 0, 0, 0 & & & \\ & 1, -2, 1, 0, 0 & & & \\ & 0, 1, -2, 1, 0 & & & \\ & 0, 0, 1, -2, 1 & & & \\ & 0, 0, 0, 1, -2 & & & \end{pmatrix}$$

$$A = A + \text{eye}(5);$$

$$[L, U, P] = lu(A);$$

$$b = [3/0.04; 0; 0; 0; -1/0.04]$$

$$y = L \setminus P^* b;$$

$$x = U \setminus y;$$

8. (18 points) Your friend just told you that the following difference scheme is an approximation for $f'(x)$:

$$f'(x) \approx \frac{-f(x - 2\Delta x) + 3f(x) + 4f(x + \Delta x)}{6\Delta x}$$

Confirm that this scheme really does approximate $f'(x)$ as Δx goes to zero. Find the leading error term and the order of accuracy of this method. (Write the order using big-oh notation.) Show your work!

Taylor series: $f(x) + \frac{\Delta x}{2!} f'(x) - \frac{\Delta x^2}{3!} f''(x)$

$$f'(x) \approx \frac{1}{6\Delta x} \left(\left(f(x) - \frac{2\Delta x}{2!} f'(x) + \frac{\Delta x^2}{3!} f''(x) \right) - 3f(x) + 4 \left(f(x) + \frac{\Delta x}{2!} f'(x) - \frac{\Delta x^2}{3!} f''(x) \right) \right)$$

Error
 Leading Term: $f'(x) - \frac{5\Delta x^3}{6} f'''(x)$

Order of Accuracy: $O(\Delta x^3)$

9. (11 points) The Jacobi elliptic functions $sn(u, m)$, $cn(u, m)$ and $dn(u, m)$ are defined in terms of the integral

$$u = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}.$$

The Jacobi elliptic functions are

$$sn(u, m) = \sin(\phi(u, m)), \quad cn(u, m) = \cos(\phi(u, m)) \text{ and} \\ dn(u, m) = \sqrt{1 - m \sin^2(\phi(u, m))}.$$

Matlab has a builtin command called `ellipj` that efficiently evaluates these functions. You probably haven't seen this function before, but with access to the help file you should be able to use it in Matlab code. Below is a portion of the help file for `ellipj`.

`ellipj`

Jacobi elliptic functions

Syntax

`[SN, CN, DN] = ellipj(U, M)`
`[SN, CN, DN] = ellipj(U, M, tol)`

Description

`[SN, CN, DN] = ellipj(U, M)` returns the Jacobi elliptic functions sn , cn and dn evaluated at the corresponding elements of arguments U and M .

Inputs U and M must be the same size, or either U or M must be scalar.

`[SN, CN, DN] = ellipj(U, M, tol)` computes the Jacobi elliptic functions to accuracy tol . The default value of tol is `eps`. Increase tol for a less accurate but more quickly computed answer.

Use the function `ellipj` to find $cn(0.75, 0.3)$ using the default tolerance. Name your final answer y .

$$[SN, CN, DN] = \text{ellipj}(0.75, 0.3)$$

$$y = CN$$

10. (16 points) Suppose that x and y are two 100×1 vectors representing points on the function $y(x)$, where the entries of x are strictly increasing and evenly spaced dx apart. You can assume that dx is small. Throughout the problem, you can assume that x , y and dx are already defined, but you should define any other variables you need.

- (a) Write Matlab code to approximate the derivative y' at each point of x . Your answer should be in a vector named $yprime$. Use a central difference scheme whenever possible. You can use any order of accuracy.

$$yprime = zeros(100, 1)$$

$$yprime(1) = (y(1) - y(2)) / dx$$

$$yprime(100) = (y(100) - y(99)) / dx$$

$$\text{for } k = 2 : 98$$

$$yprime(k) = (y(k+1) - y(k-1)) / (2 * dx)$$

end

- (b) Write a *single line* of Matlab code to approximate the integral of $y(x)$ (over the entire x interval) using the trapz command. Save your answer as TRAP.

$$TRAP = \text{trapz}(x, y)$$

% Save ("TRAP.dat", "TRAP", "ascii");

- (c) Write Matlab code to approximate the integral of $y(x)$ (over the entire x interval) using the right hand rule. Save your answer as LHR.

$$LHR = 0$$

$$\text{for } k = 1 : 100$$

$$LHR = LHR + y(k) * dx$$

end

% LHR

