Linear Algebra - Pre-Course Exercises

The following exercises aim to refresh your applied linear algebra knowledge. Please write down detailed solutions, not just the end result…

It is highly preferable to write down your answers using an electronic format (LaTeX, MS Word etc.) but if that's a big burden you can also scan a hand-written solution, as long as it is tidy and clear.

Please upload your solution to Hive as a PDF.

Enjoy!

1. Find all solutions to the following systems of linear equations:

   a.
$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 4 \\ x_1 - x_2 &= -2 \\ -x_1 + x_2 + 3x_3 &= 4 \end{aligned}$$

   b.
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 5 \\ x_1 + x_2 + 3x_3 &= 3 \\ 2x_1 + x_2 + 5x_3 &= 2 \end{aligned}$$

2. Find a basis for the solutions of the following system of linear equations:

$$\begin{aligned} 2x_1 - x_2 + x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 + 3x_4 &= 0 \\ 5x_1 - 5x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

3. Find a basis for the following subspace of $\mathbb{R}^4$:

$$Span\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \\ 3 \end{bmatrix}\right)$$

4. Find the spanning set for the nullspace of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

5. For each of the following matrices, determine whether it is invertible, and if it is, compute its inverse matrix.
   a. $A_1 = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$
   b. $A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$
   c. $A_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

6. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & x \\ 2 & 1 & x^2 \end{bmatrix}$$

   a. Compute the determinant of A (your answer will be in terms of x)
   b. For what values of x is the matrix A invertible?

7. Consider the following basis for $\mathbb{R}^2$:

$$B = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

   Find the coordinates of the vector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ in terms of B.

8. Find the eigenvalues and the corresponding eigenvectors of the following matrix:

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Let A be the following matrix:

$$A = \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix}$$

   Find a general formula for the entries of A^n
   Hint: Diagonalize A.

10. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors:

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

   Hint: Use the Gram-Schmidt algorithm.