Probability theory

1) binomial distribution

$p(y) = \binom{n}{y} p^y q^{n-y}$

The first k-1 tosses has 1 head & the last toss is heads

$p = .5$
$q = 1 - p = .5$

$\binom{k-1}{1} \cdot (.5)^1 (.5)^{k-2}$

$(.5)\ (k-1) \cdot (.5)^{k-1}$

Chances of the last toss being heads

$$\boxed{= (k-1) \cdot (.5)^k}$$

2) The probability that at least 2 people chose the same number is 1 minus (-) the probability no one ~~[illegible]~~ chooses the same number

$\cancel{p(y \geq 2) = 1 - p(\ldots)\, p(y=0)}$

$$p(y \geq 2) = 1 - p(y < 2)$$

$p(y < 2) = \cancel{\ldots}$

$$\frac{\left(\frac{k!}{(k-n+1)!}\right)}{k^n}$$

$\cancel{p(y<2)}$

$$p(y \geq 2) = 1 - \frac{\left(\frac{k!}{(k-n+1)!}\right)}{k^n}$$

3)

$p(t=0)$ - tested negative

$p(t=1)$ = test positive

$p(i=0)$ = not infected

$p(i=1)$ = is infected

$$p(i=1/t=1) = \frac{p(i=1, t=1)}{p(t=1)}$$

$$p(t=1/i=1) = .99$$

$$p(t=1) = (1/10{,}000)(.99) + \left(\frac{9{,}999}{10{,}000}\right)(.01)$$

$$= .010098$$

$$p(i=1) = .0001$$

$$p(i=1/t=1) = \frac{p(t=1/i=1) \cdot p(i=1)}{p(t=1)}$$

$$= \frac{.99 \cdot .0001}{.010098} = .0098$$

$$\boxed{p(i=1/t=1) = .0098}$$

4.

A) $E(x+y) = E(X) + E(Y)$

$$E(X) = \sum_{x} x\,p(x)$$

$$E(x+y) = \sum_{x,y} (x+y)\,p(x,y)$$

$$= \sum_{x,y} x\,p(x,y) + \sum_{x,y} y\,p(x,y)$$

$$= \sum_{x} x \underbrace{\sum_{y} p(x,y)} + \sum_{y} y \underbrace{\sum_{x} p(x,y)}$$

$$\underbrace{\sum_{x} x\,p(x)} + \underbrace{\sum_{y} y\,p(y)}$$

$$E(x) + E(y)$$

$$E(x+y) = E(X) + E(Y)$$

$$E(\alpha Z) = \alpha E(Z)$$

B)

$$E(Y) = \int Y f(y)\,dy$$

$$E(\alpha Z) = \int \alpha Z f(\alpha Z)\,dz$$

$$= \alpha \int Z \underbrace{f(\alpha Z)\,dz}_{f(Z)}$$

$$= \alpha \underbrace{\int Z f(Z)\,dz}_{E(Z)}$$

$$= \alpha E(Z)$$

$$E(\alpha Z) = \alpha E(Z)$$

c)

~~$E(xy) = \iint xy$~~

$$E(xy) = \sum_x \sum_y xy \, \cancel{f(x,y)} \, p(x,y)$$

$$E(xy) = \sum_x p(x) \sum_y p(y)$$

$$E(xy) = E(x) \cdot E(y)$$

D)

$$\underline{Var(aX+\beta) = a^2 var(X)}$$

$$\cancel{Var(aX+\beta)} = Var(aX) + \cancel{Var(\beta)}$$

$$Var(aX+\beta) = E((ax+b)^2) - (E(ax+b))^2$$

$$= E(a^2x^2 + 2abx + b^2) -$$

$$(a E(x) + b)^2$$

$$= a^2 E(x^2) + 2ab E(x) + b^2 - a^2 (E(x))^2$$

$$- 2ab E(X) - b^2$$

$$= a^2 \underbrace{\left(E(x^2) - (E(x))^2\right)}_{a^2 Var(X)}$$

e)

$$\mathrm{Var}(X+Y) =$$

$$E((X+y)^2) - (E(X+y))^2$$

$$= E(X^2+2xy+y^2) - (E(X))^2 - 2E(X)E(Y) - (E(Y))^2$$

$$= \underbrace{E(X^2) - (E(X))^2} + \underbrace{E(Y) - (E(Y))^2}$$

$$\mathrm{Var}(X) + \mathrm{Var}(Y)$$

$$\mathrm{Var}(X+y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

5) $\cancel{F(y) = \ldots}$

$$F(y) = \prod_{i=1}^{n} P(X_i \leq y)$$

$$= y^n$$

$$E(Y) = \int_0^1 y\, f(y)\, dy$$

$$= \int_0^1 y^{n+1}\, dy$$

$$= \left. \frac{y^{n+2}}{n+2} \right]_0^1$$

$$= \frac{1}{n+2}$$

6a) $\cancel{E(X_n) = p}$

$E(X) = p \cdot n$

$E(X_n) = p \cdot n$

6b) $var(X) = npq$

$var(X_n) = n \cdot p \cdot (1-p)$