Shalom Azar

1.
$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ 1 & -1 & 0 & -2 \\ -1 & 1 & 3 & 4 \end{array}\right]$$

$r_3 + r_2 \rightarrow r_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ 1 & -1 & 0 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

$-\frac{2}{3}(r_3) + r_1 \rightarrow r_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 8/3 \\ 1 & -1 & 0 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

$-r_2 + r_1 \rightarrow r_1$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 14/3 \\ 1 & -1 & 0 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

~~$-r_1 + r_2 \rightarrow r_2$~~ $\rightarrow$ next page

$-r_1 + r_2 \to r_2$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 14/3 \\ 1 & 0 & 0 & -20/3 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

$-r_1 \to r_1$

$1/3\, r_3 \to r_3$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -14/3 \\ 1 & 0 & 0 & -20/3 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$$

$r_1 \leftrightarrow r_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \cancel{-14/3}\ -20/3 \\ 0 & 1 & 0 & -14/3 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$$

A) $\boxed{-20/3,\ -14/3,\ 2/3}$

B)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 2 \end{array}\right]$$

$-r_1 + r_2 \to r_2$
$-2r_1 + r_3 \to r_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & -3 & -3 & -8 \end{array}\right]$$

$-3r_2 + r_3 \to r_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

B) No Solution

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 & 0 \\ 5 & -5 & 2 & 3 & 0 \end{array}\right]$$

$-\frac{1}{2}r_1 + r_2 \to r_2$

~~-2 25~~ $-\frac{5}{2}r_1 + r_3 \to r_3$

$$\left.\begin{array}{cccc|c} 2 & -1 & 1 & 2 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} & 2 & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & -2 & 0 \end{array}\right.$$

$-r_2 + r_3 \to r_3$

$$\left.\begin{array}{cccc|c} 2 & -1 & 1 & 2 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} & 2 & 0 \\ 0 & 0 & -1 & -4 & 0 \end{array}\right.$$

2.

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

3.
$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 2 & 5 & 1 & -1 & 0 \\ 1 & 4 & -1 & -2 & 0 \\ -1 & -5 & 2 & 3 & 0 \end{array}\right]$$

$-2r_1 + r_2 \to r_2$

$-r_1 + r_3 \to r_3$

$r_1 + r_4 \to r_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 0 & 5 & -5 & -5 & 0 \\ 0 & 4 & -4 & -4 & 0 \\ 0 & -5 & 5 & -1 & 0 \end{array}\right]$$

$-4/5\, r_2 + r_3 \to r_3$

$r_2 + r_4 \to r_4$

Basis = {1, 2, 4}

r_1, r_2, r_4

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 0 & 5 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \end{array}\right]$$

$x + 2y = 1$

$2x + 4y = 2$

$x + 2y = 1$

~~$x = 1 - 2y$~~

$1 - 2y, y$

$x + 2y + z = 0$

$x = -2y - z$

~~$-2y - z$~~ $y = \frac{-x - z}{2}$

$z = -x - 2y$

~~$S/2$~~

4. Spanning set

$(0,0,0) \in N(A)$

5. a. $\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

a. $\frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = -\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$

A) $= \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

B) $A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

$$\det(A_2) = \begin{array}{ccccc} 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \\ 3 & 1 & 3 & 3 & 1 \end{array}$$

$$(1)(1)(3) + (2)(2)(3) + (1)(2)(1)$$
$$- (1)(1)(3) - (1)(2)(1) - (2)(2)(3)$$

$\det(A_2) = 0$

A_2 is not invertible

5c) $A_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

$$\begin{matrix} -1 & 1 & 0 & -1 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ -1 & -2 & 1 & -1 & -2 \end{matrix}$$

$(-1)(2)(1) + (1)(1)(-1) + (0)(1)(2)$
$- (0)(2)(-1) - (-1)(1)(-2) - (1)(1)(1)$

$-2 - 1 - 2 - 1$

$\det(A_2) = -6$

Matrix of minors Matrix with negatives

$$\begin{bmatrix} 4 & 5 & 0 \\ 1 & -1 & 3 \\ 1 & -1 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -5 & 0 \\ -1 & -1 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

Transpose matrix $\Rightarrow \begin{bmatrix} 4 & -1 & 1 \\ -5 & -1 & 1 \\ 0 & -3 & -3 \end{bmatrix}$

$$A_3^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & -1 & 1 \\ 5 & -1 & 1 \\ 0 & -3 & -3 \end{bmatrix}$$

$$A_3^{-1} = \begin{bmatrix} -4/6 & 1/6 & -1/6 \\ \frac{-5}{6} & 1/6 & -1/6 \\ 0 & 3/6 & 3/6 \end{bmatrix}$$

6.
$$\begin{matrix} 1 & 2 & 2 & 1 & 2 \\ 1 & 1 & x & 1 & 1 \\ 2 & 1 & x^2 & 2 & 1 \end{matrix}$$

$x^2 + 4x + 2 - 4 - x - 2x^2$

$\det(A) = -x^2 + 3x - 2$

$0 = -1(x^2 - 3x + 2)$

$0 = -1(x+2)(x+1)$

$\boxed{x \neq -1, -2}$

7. $\left[\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & 1 & 2 \end{array}\right]$

$-r_2 + r_1 \to r_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & 1 & 2 \end{array}\right]$$

$-2r_1 + r_2 \to r_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \end{array}\right]$$

$$[X]_B = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

8.

$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A - I(\lambda)) = 0$

$\begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda, & 4, & 2 \\ 1, & -1-\lambda, & 1 \\ 0, & 0, & 1-\lambda \end{bmatrix} \begin{matrix} 2-\lambda & 4 \\ 1 & -1-\lambda \\ 0 & 0 \end{matrix}$

$\det(A - \lambda I) = (2-\lambda)(-1-\lambda)(1-\lambda)$
$+ (4)(1)(0)$
$+ (2)(1)(0)$
$- (4)(1)(1-\lambda)$
$- (2-\lambda)(1)(0)$
$- (2)(1-\lambda)(0)$

$(2-\lambda)(-1-\lambda)(1-\lambda)$
$-(4)(1)(1-\lambda)$

$\boxed{\begin{matrix} (1-\lambda)(-2+\lambda-2\lambda+\lambda^2) \\ -2+\lambda-2\lambda+\lambda^2 \\ +2\lambda-\lambda^2+2\lambda^2-\lambda^3 \\ -4+4\lambda \end{matrix}}$

~~$-\lambda^3+2\lambda^2+5\lambda-6=0$~~

~~$\lambda = -1, 1$~~

Go to back

$$(1-\lambda)(-2+\lambda-2\lambda+\lambda^2)$$

$$-4+4\lambda$$

$$(1-\lambda)(\lambda^2-\lambda-2)-4(1-\lambda)$$

$$(1-\lambda)(\cancel{\lambda^2-2\lambda}\ \lambda^2-\lambda-6)$$

$$\cancel{\lambda^2-2\lambda}\ (1-\lambda)(\lambda-3)(\lambda+2)$$

$$= \cancel{-1(\lambda)}\ -1(\lambda-1)(\lambda-3)(\lambda+2)$$

Eigenvalues $= \lambda = 1, 3, -2$

Eigenvectors

$$\underbrace{(A-(\lambda)I)}_{B}\,\bar{x} = \bar{0}$$

$$B = \begin{bmatrix} 2-\lambda & 8 & 2 \\ 1 & -1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$-r_1 + r_2 \to r_2$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -6 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2/3\, r_2 + r_1 \to r_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4/3 & 0 \\ 0 & -6 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 4/3\, z = 0$

$-6y - z = 0$

if $x = 1$, $z = -3/4$

if $z = -3/4$,

$-6y + 3/4 = 0$

$y - 3/24 = 0$

$y = 3/24$ or $1/8$

eigenvector for $\lambda = 1$

$\cancel{1, 4/18, -4/3}$ $\left(1, 1/8, -3/4\right)$

$\lambda = 3$

$$A - \lambda I = \begin{bmatrix} -1 & 4 & 2 \\ 1 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 4 & 2 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$r_1 + r_2 \to r_2$

$$\left[\begin{array}{ccc|c} -1 & 4 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$r_1 + r_3 \to r$

$r_2 + 3/2\, r_3 \to r_2$

$$\left[\begin{array}{ccc|c} -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$(4, 1, 0)$

$-x + 4y = 0$

$-2z = 0$

$z = 0$

$x = 4y$

y

One eigenvector for $\lambda = 3$ is $(4,1,0)$

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for $\lambda = -2$

$$A - \lambda I = \begin{bmatrix} 4 & 4 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$r_1 + \frac{2}{3} r_3 \to r_1$

$r_2 + \frac{1}{3} r_3 \to r_2$

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$-\frac{1}{4} r_1 + r_2 \to r_2$

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$4x + 4y = 0$

$x = -y$

$z = 0$

One eigenvector for $\lambda = \cancel{2}\ -2$ is $(1,-1,0)$

9. $\underbrace{A - \lambda I}_{B} = \begin{bmatrix} 2-\lambda & 4 \\ -2 & -4-\lambda \end{bmatrix}$

$\det(B) = (2-\lambda)(-4-\lambda) - (+4)(-2)$

$\rightarrow -8 - 2\lambda + 4\lambda + \lambda^2 + 8$

$\lambda^2 + 2\lambda = 0$

$\lambda(\lambda + 2) = 0$

$\cancel{\lambda = 0} \quad \lambda = \{0, -2\}$

$\boxed{A^D = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}}$

when $\lambda = 0$

$A - \lambda I = \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$r_1 + r_2 \rightarrow r_2$ $\qquad 2x + 4y = 0$

$\boxed{v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$ $\qquad x = 2, \; y = -1$

when $\lambda = -2$ $\quad A - \lambda I$

$= \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} \rightarrow \begin{array}{cc|c} 4 & 4 & 0 \\ -2 & -2 & 0 \end{array}$ $\qquad \frac{1}{2} r_1 + r_2 \rightarrow r_2$

$$\left(\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

~~x~~ $x = -y$

A eigen vector is

$V_2 = (1, -1)$

$P = [V_1 \ V_2]$

$$P = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & -1 \\ +1 & 2 \end{bmatrix}$$

$$A^k = P (A^D)^k P^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -1 & -1 \\ +1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & (-2)^k \\ 0 & -(-2)^k \end{bmatrix} \begin{bmatrix} -1 & -1 \\ +1 & 2 \end{bmatrix}$$

$$A^k = \begin{bmatrix} (-2)^k & 2(-2)^k \\ -(-2)^k & -2(-2)^k \end{bmatrix}$$

10. $y_1 = \vec{x}_1$

$y_2 = x_2 - P_{y_1} x_2$

$y_3 = x_3 - P_{y_1} x_3 - P_{y_2} x_3$

$y_4 = x_4 - P_{y_1} x_4 - P_{y_2} x_4 - P_{y_3} x_4$

$P_{y_1} x_2 = \frac{\langle y_1 \rangle \langle y_1 | x_2 \rangle}{\langle y_1 | y_1 \rangle}$ (inner product)

$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$

$y_1 = x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

$y_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ +1 \end{pmatrix} - \frac{(-1)}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \cancel{\begin{pmatrix} 5/4 \\ 5/4 \\ 5/4 \\ -5/4 \end{pmatrix}} \begin{pmatrix} 9/4 \\ -3/4 \\ -3/4 \\ 3/4 \end{pmatrix}$

$y_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \frac{(-1)(9/4) + (-3/4)(2) + (2)(-3/4) + (1)(3/4)}{7} \begin{pmatrix} 9/4 \\ -3/4 \\ \\ 9/4 \end{pmatrix}$

$$\left(-\frac{9}{4}-\frac{6}{4}-\frac{6}{4}+\frac{3}{4}\right)$$

$$y_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} - 7 = \frac{\frac{18}{4}}{7}$$

$$\begin{pmatrix} 9/4 \\ -3/4 \\ -3/4 \\ 3/4 \end{pmatrix}$$

$$y_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \cancel{\frac{18}{28}}\ \frac{9}{14}\begin{pmatrix} \frac{9}{4} \\ -3/4 \\ -3/4 \\ 3/4 \end{pmatrix}$$

$$\begin{pmatrix} -3/2 - \frac{81}{56} \\ 3/2 + 27/56 \\ 3/2 + 27/56 \\ 3/2 - 27/56 \end{pmatrix} \rightarrow \begin{matrix} \frac{84}{56} - \frac{81}{56} \\ \frac{84+27}{56} \\ \frac{84+27}{56} \\ \frac{84-27}{56} \end{matrix} = \begin{pmatrix} \frac{3}{56} \\ \frac{111}{56} \\ \frac{111}{56} \\ \frac{57}{56} \end{pmatrix}$$

$$y_3 = \begin{pmatrix} \frac{3}{56} \\ \frac{111}{56} \\ \frac{111}{56} \\ \frac{57}{56} \end{pmatrix} \qquad y_2 = \begin{pmatrix} \frac{9}{4} \\ -\frac{3}{4} \\ -3/4 \\ 3/4 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \qquad |y_1| = \sqrt{1^2+1^2+1^2+(-1)^2} = \sqrt{4} = \boxed{2}$$

$$|y_2| = \sqrt{\left(\frac{9}{4}\right)^2 + \left(\frac{-3}{4}\right)^2 + \left(\frac{-3}{4}\right)^2 + \left(\frac{3}{4}\right)^2}$$

$$\sqrt{\frac{81}{16} + 3\left(\frac{9}{16}\right)} = \sqrt{\frac{27}{4}} = \frac{3\sqrt{3}}{2}$$

$$|y_3| = \sqrt{\left(\frac{3}{56}\right)^2 + \left(\frac{111}{56}\right)^2 + \left(\frac{111}{56}\right)^2 + \left(\frac{57}{56}\right)^2}$$

$$9 + 2(12321) + 3249$$

$$= \frac{\sqrt{27900}}{\sqrt{56^2}} = \cancel{\frac{\sqrt{27900}}{3136}} \quad \frac{\sqrt{27900}}{56}$$

$$z_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$z_2 = \frac{2}{3\sqrt{3}}\begin{pmatrix} 9/4 \\ -3/4 \\ -3/4 \\ 3/4 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{6} \\ -\sqrt{3}/6 \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$

$$\cancel{\frac{6}{12\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{3}\,6}}$$

$$z_3 = \frac{56}{\sqrt{27900}}\begin{pmatrix} 3/56 \\ 111/56 \\ 111/56 \\ 57/56 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{27900}}{27900} \\ \frac{111\sqrt{27900}}{27900} \\ \frac{111\sqrt{27900}}{27900} \\ \frac{57\sqrt{27900}}{27900} \end{pmatrix}$$