Curve Average

Project 3 for class CS6491 Computer Graphics

Sebastian Weiss, Can Erdogan

October 29, 2015

I. Objective

Formally, for two curves $A = \{a_0 ... a_{n-1}\}$ and $B = \{b_0 ... b_{n-1}\}$, each with n points and $\forall x \in (A \cup B), x \in \mathcal{R}^3$, the goal is to find a curve C such that for each point $c \in C$, and its closest projections

$$a^c = \underset{a \in A}{\operatorname{argmin}} ||a - c||, \text{ and} \tag{1}$$

$$b^c = \underset{b \in B}{\operatorname{argmin}} ||b - c|| \tag{2}$$

the following holds true:

$$c = \underset{x \in L(a^c, b^c)}{\operatorname{argmin}} ||a^c - x|| \tag{3}$$

where for each point r on line $L(p,q)$, $||r-p|| = ||r-q||$. Moreover, for any three consecutive sampled points c_i, c_j and c_k on curve C, the arc lengths of the curves between their closest projections are the same:

$$D^A(a^{c_i}, a^{c_j}) + D^B(b^{c_i}, b^{c_j}) = D^A(a^{c_j}, a^{c_k}) + D^B(b^{c_j}, b^{c_k}) \tag{4}$$

where $D^Z(x,y)$ is the distance along the curve Z between points $x, y \in Z$.

Semantically, we want to find the curve that is composed of the loci of the smallest spheres that touch the two inputs curves A and B, and sample it such that for each sample on the curve, the distances traveled by the matching samples along their curves is a constant.

II. Input

The input is the six control points for two curves A and B, named A_0 to A_5 and B_0 to B_5, such that the first and the last control points are the same.

III. Overview

The project is composed of the following three main parts: (1) the representation of the curves, (2) the computation of the average curve, and (3) the visualization of the different curve properties. The challenge in the representation is that we want the spline curves to first meet at two end points and then, to have C^1 continuity at the intersection of the splines. For the curve average, we had to ensure the points satisfied the constraints in Equations 1-3. Lastly, the visualization contained multiple challenges including parallel transport, circular arc computation and etc.

IV. Curve Representation

We compose each curve out of piecewise quadratic Hermite and cubic Hermite splines. Each spline is connected at a control point by the same local velocity. This leads to a $\mathcal{C}^1$-smooth curve. We define the tangent/velocity at control point A_i as $T_i = c * (A_{i+1} - A_{i-1})$. The parameter c defines the curvature or the influence of that velocity. In our experiments, we set c to 0.5.

A cubic Hermite spline requires velocities at both end points, a quadratic Hermite spline only at one end point. Therefore, we use the quadratic form for the first and the last part of the curve and the cubic form for all middle parts.

A. *Quadratic Hermite Spline*

Given the points X_0 and X_1 and the tangent T_0, find a quadratic spline $P(t)$ so that the following holds:

$$P(0) = X_0, P(1) = X_1, P'(0) = T_0 \tag{5}$$

Solving this system of equation leads to the following formula:

$$P(t) = (t^2 - 2t + 1)X_0 + (-t^2 + 2t)X_1 + (t^2 - t)T_0 \tag{6}$$

B. *Cubic Hermite Spline*

Given the points X_0 and X_1 and the tangents T_0 and T_1, find a cubic spline $P(t)$ so that the following holds:

$$P(0) = X_0, P(1) = X_1, P'(0) = T_0, P'(1) = T_0 \tag{7}$$

This equation is solved similar to the quadratic case, leading to:

$$P(t) = (2t^3 - 3t^2 + 1)X_0 + (t^3 - 2t^2 + t)T_0 + (-2t^3 + 3t^2)X_1 + (t^3 - t^2)T_1 \tag{8}$$

V. Average Curve Computation

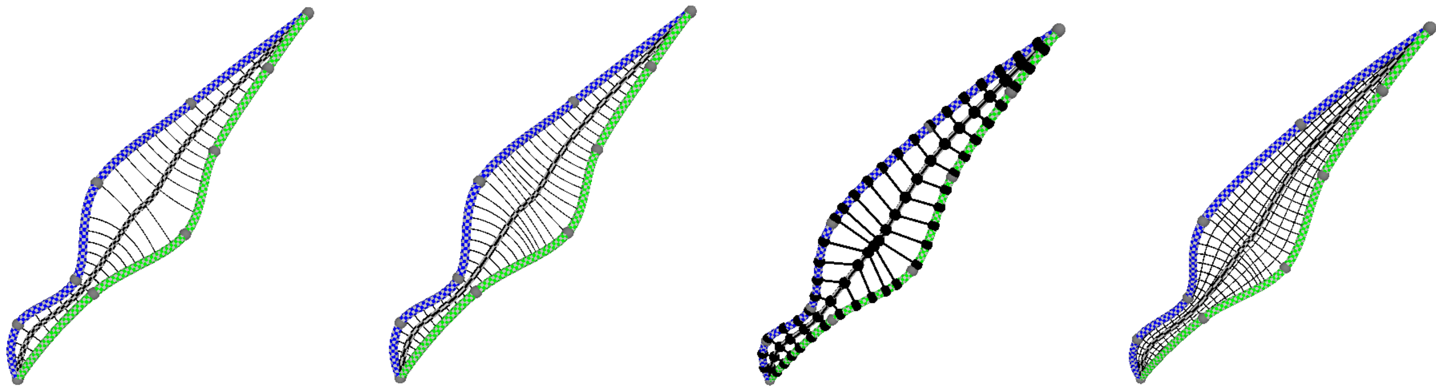
A. *Distance functions*

B. *Average Distance Sampling*

VI. Visualization and Editing

We provide a framework for editing the control curve and visualizing the curve average. More specific, we support the following features:

- Moving the control points of the two input curve
- Displaying the curve average with or without geodesic sampling
- Displaying the closest projections



a) no geodesic sampling b) with geodesic sampling c) the closest projections d) net of circular arcs

Figure 1: Geodesic sampling, closest projections and circular arcs

- Displaying the circular arc and animating the curve average along it
- Showing the inflation tube, the envelope of the smallest spheres

We now describe the single features in more detail

A. *Editing and displaying the control curve*

We display the control points as gray spheres and the two control curves in green and blue. The user can click the control points and change their positions by dragging them over the screen.

B. *Displaying the average curve*

The curve average is displayed by connecting the sample points using piecewise straight tubes. The faces of the tubes are rendered in alternating black and gray so you can see where the trace points are. The user has now the option to toggle geodesic sampling on or off. The effects are shown in Fig.1ab.

As you can see, without geodesic sampling, the distances between the samples of the curve average are almost equispaced, but the closest projections on the control curve, indicated by the circular arcs, vary greatly in the distances. With geodesic sampling turned on, the curve average is sampled in that way that the sum of the distances between two consecutive closest projects on the control curves are constant. However, this leads to large variation of the distances of the curve average samples.

C. *ClosestProjections*

To visualize the closest projections, i.e. the points on the control curve that are closest to the average curve at this point, we draw straight lines between the sample of the curve average and its closest projections. These can be seen in Fig.1c. Note that the lines to both control curves are of the same length and that they stand orthogonal to the control curve at the intersections. These are the properties that define the closest projections and the curve average.

VII. Results

VIII. Future work