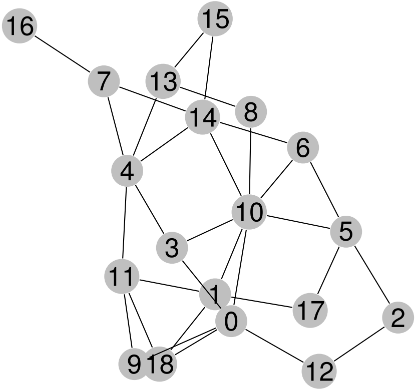
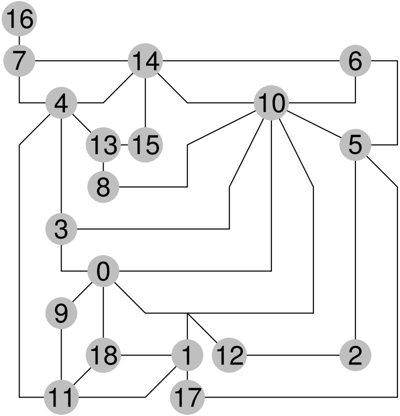
Welcher dieser Graphen ist besser lesbar?



# Optimal Crossing Minimization

## Proseminar Graph Drawing

Sebastian Weiß

Technische Universität München

26. Juni 2015

$\mathcal{NP}$

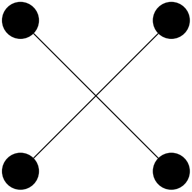
$$\mathcal{NP}$$

Heuristiken

$$\mathcal{NP}$$

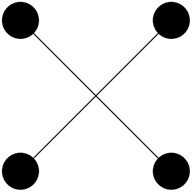
Heuristiken Optimale Verfahren

Nicht planar



# Idee

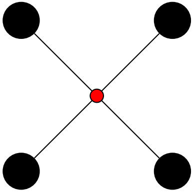
Nicht planar



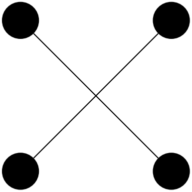
Dummy Node

$\longrightarrow$

Planar



Nicht planar



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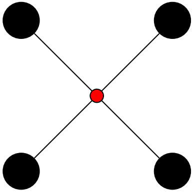
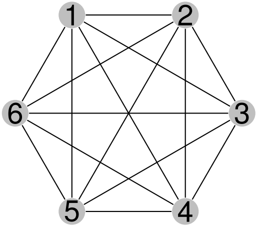
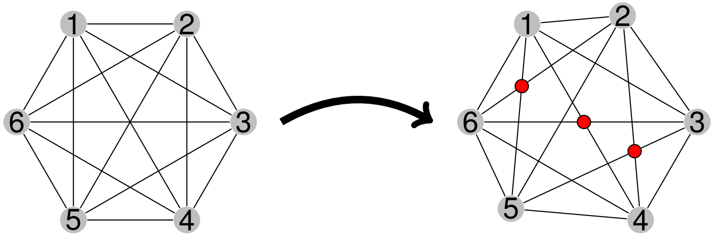


Ziel: Möglichst wenige Dummy-Nodes

# Beispiel: K6



# Beispiel: K6



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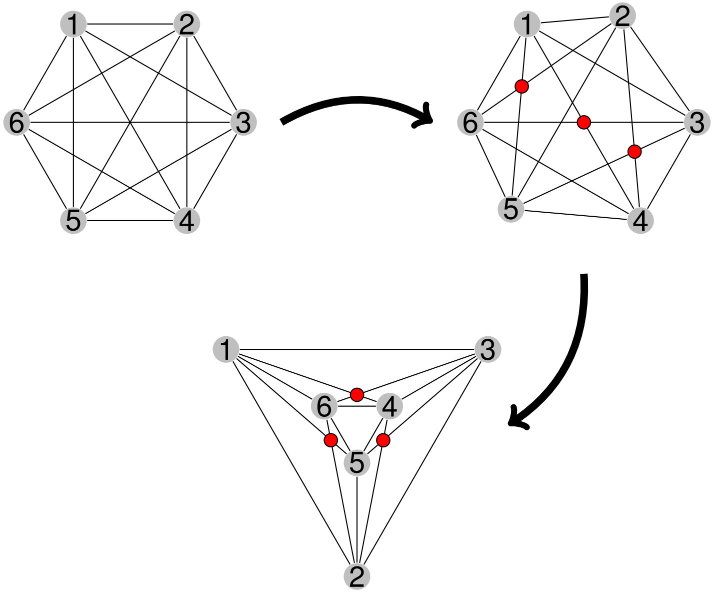


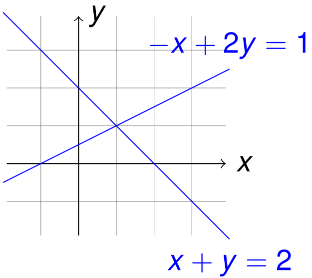
Eine Variable für jede mögliche Kreuzung

$$x_{\{e,f\}} \in \{0,1\} \,;\, e,f \in E$$

Minimiere $\sum x_{\{e,f\}}$

Lineare Gleichungssysteme

$$\begin{aligned} x + y &= 2 \\ -x + 2y &= 1 \end{aligned}$$



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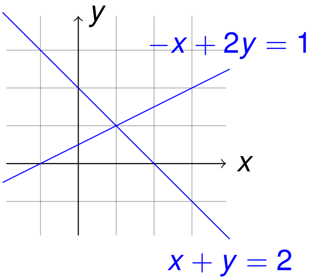
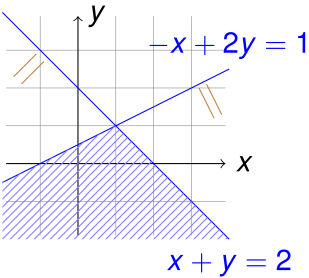
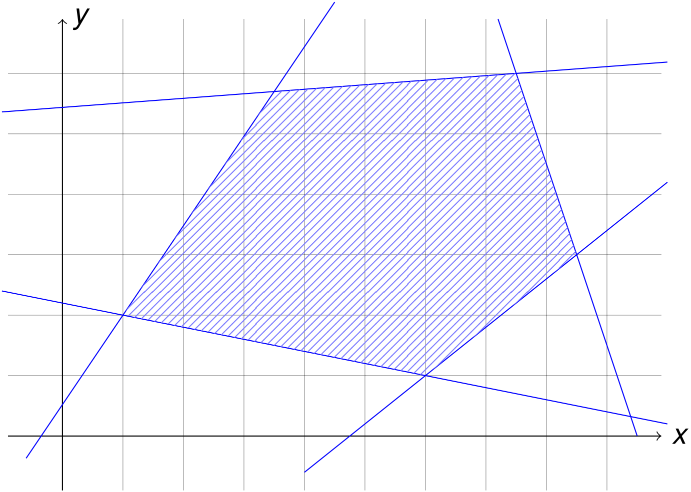



Lineare Ungleichungssysteme

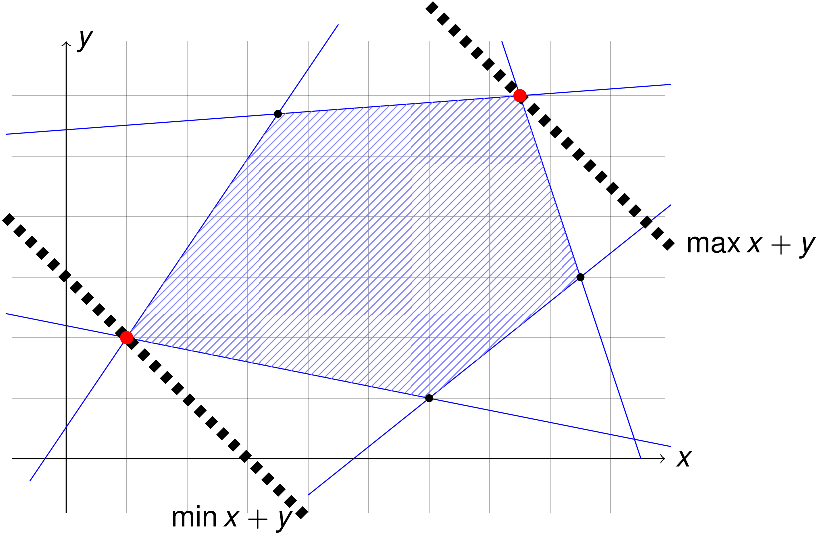
$$\begin{aligned} x + y &\leq 2 \\ -x + 2y &\leq 1 \end{aligned}$$




y
x

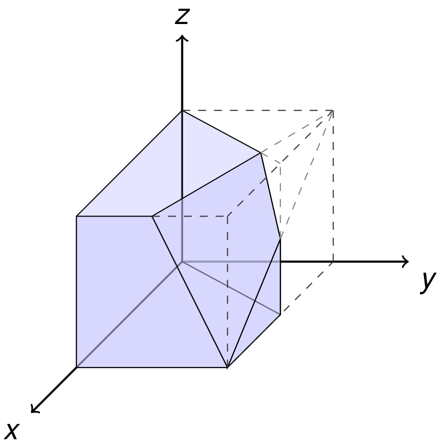
# Gültige Region + Zielfunktion





$$\max x + y + z$$

$$\begin{aligned} x, y, z &\geq 0 \\ x, y, z &\leq 1 \\ x + 2y + z &\leq 3 \\ -2x + y &\leq 0 \end{aligned}$$



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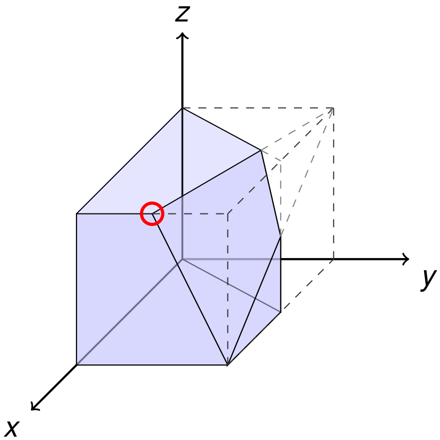


$\Rightarrow$ MIN/MAX sind immer Vertices

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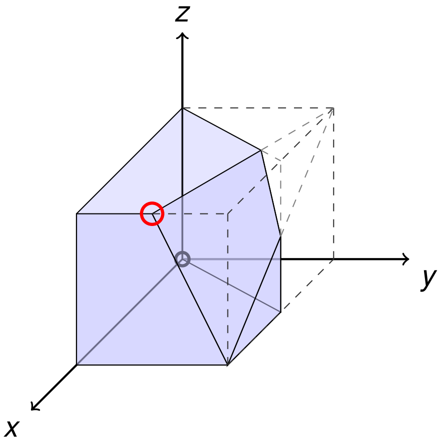


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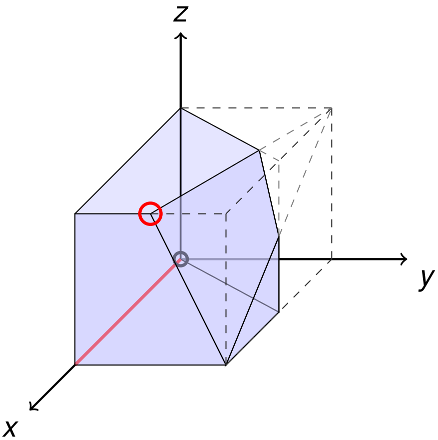


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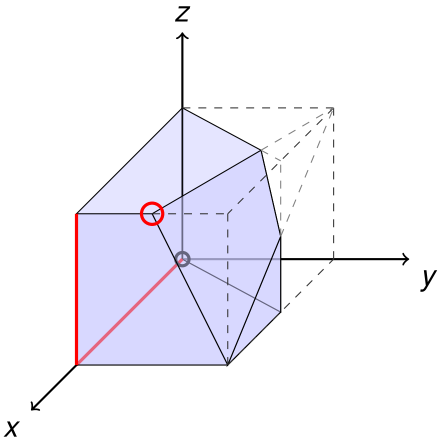


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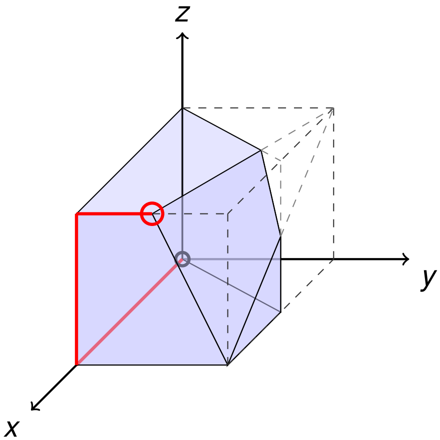


$\Rightarrow$ MIN/MAX sind immer Vertices
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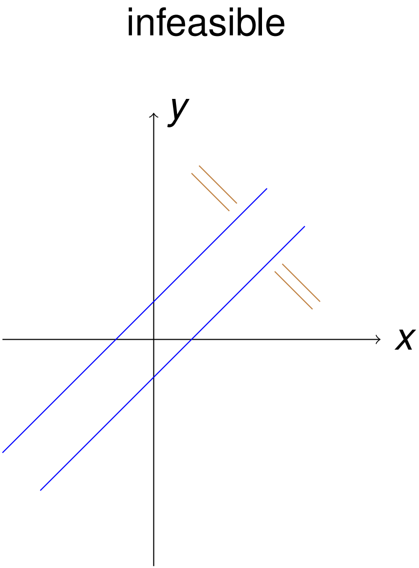
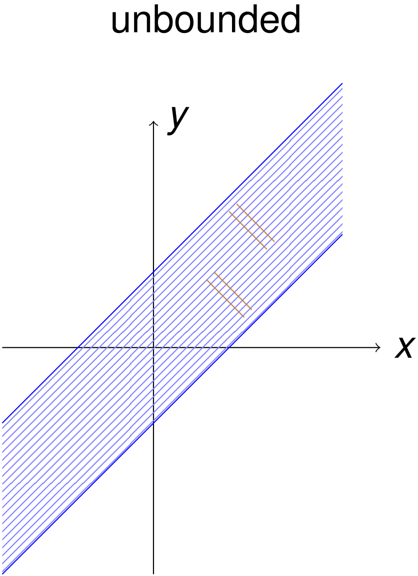
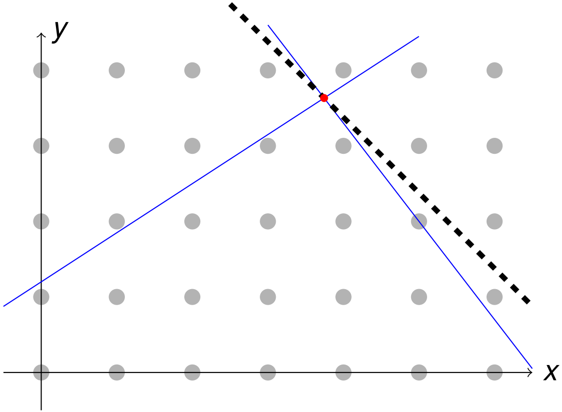
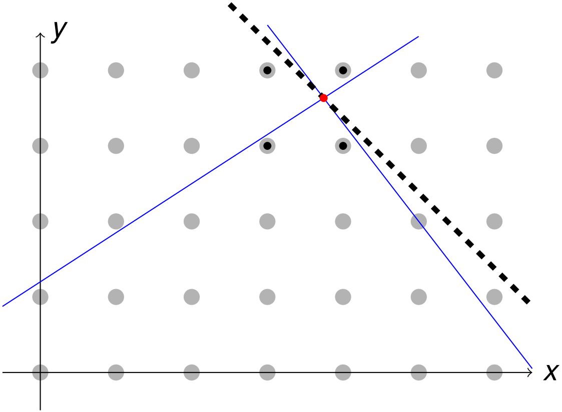
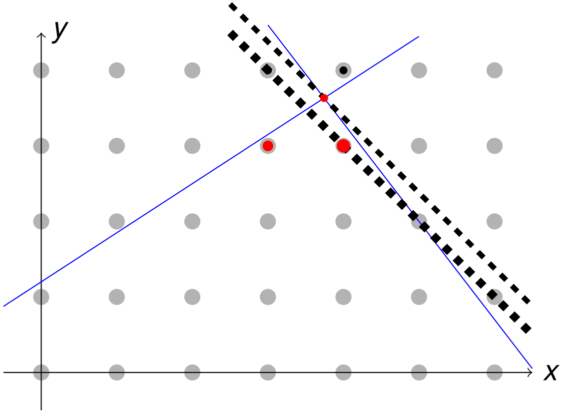
# Spezialfälle

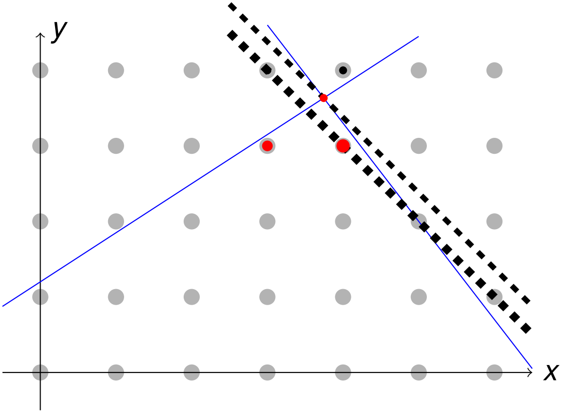



y
x


y
x


y
x



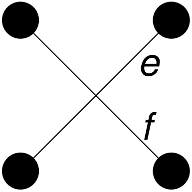

$x_i \in \mathbb{R} \Rightarrow \mathcal{P}$ $\quad$ $x_i \in \mathbb{Z} \Rightarrow \mathcal{NP}$

# Minimierung mit LP

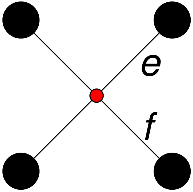


## Eine Variable für jede mögliche Kreuzung

$$x_{\{e,f\}} \in \{0,1\}\,;\, e,f \in E$$

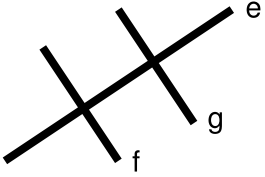



$$\xrightarrow{x_{\{e,f\}}=1}$$

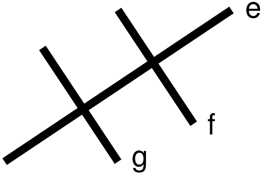



Minimiere $\sum x_{\{e,f\}}$

$$x_{\{e,f\}} = 1 \text{ und } x_{\{e,g\}} = 1$$



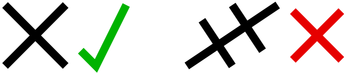
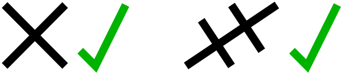

ODER




$\mathcal{NP}$-hart

| **SOCM**<br>*subdivision-based optimal crossing minimization* | **OOCM**<br>*ordering-based optimal crossing minimization* |
| --- | --- |
| | |

# Zwei Ansätze

| **SOCM**<br>*subdivision-based optimal crossing minimization* | **OOCM**<br>*ordering-based optimal crossing minimization* |
|---|---|
| nur einfache Kreuzungen | mehrfache Kantenkreuzungen |
|  |  |

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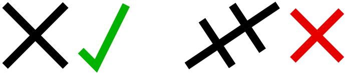
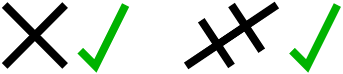
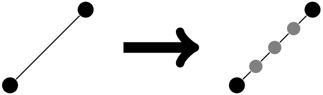
|  |  |
| Aufspalten von Kanten<br> | Zusätzliche Variablen für die Reihenfolge |

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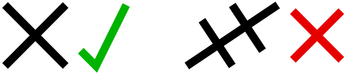
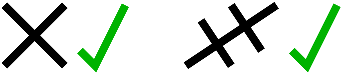
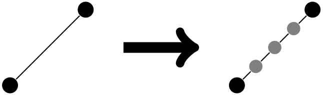
|  |  |
| Aufspalten von Kanten<br> | Zusätzliche Variablen für die Reihenfolge |
| $\Rightarrow \Theta(\lvert E \rvert^4)$ Variablen | $\Rightarrow \Theta(\lvert E \rvert^3)$ Variablen |

# OOCM: Variablen

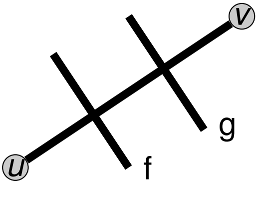
$x_{\{e,f\}} \in \{0,1\} = 1$ , falls Kante $e$ und $f$ sich kreuzen

# OOCM: Variablen

$x_{\{e,f\}} \in \{0, 1\} = 1$ , falls Kante $e$ und $f$ sich kreuzen

$y_{e,f,g} \in \{0, 1\} = 1$ , falls

- $f$ und $g$ Kante $e$ schneiden
- $f$ ist näher an dem ersten Knoten von $e$ ist als $g$




$$e = (u, v)$$
$$x_{\{e,f\}} = 1, x_{\{e,g\}} = 1$$
$$y_{e,f,g} = 1, y_{e,g,f} = 0$$

# OOCM: Zielfunktion

$$\text{minimiere} \sum_{\{e,f\} \in \binom{E}{2}} x_{\{e,f\}}$$

# OOCM: Lineare Ordnung (LO)

- $x_{\{e,f\}} \geq y_{e,f,g}, \quad x_{\{e,g\}} \geq y_{e,f,g} \quad \forall (e,f,g) \in E^3$

  Falls eine Reihenfolge, dann auch eine Doppelkreuzung

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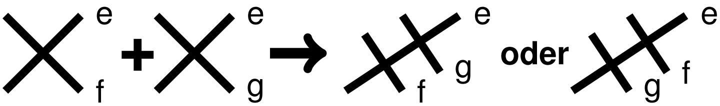
  Falls eine Reihenfolge, dann auch eine Doppelkreuzung

- $1 + y_{e,f,g} + y_{e,g,f} \geq x_{\{e,f\}} + x_{\{e,g\}} \quad \forall (e,f,g) \in E^3$

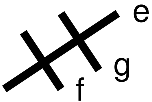


# OOCM: Lineare Ordnung (LO)

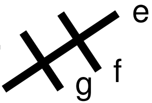


- $y_{e,f,g} + y_{e,g,f} \leq 1 \ \ \forall (e,f,g) \in E^3$

**entweder**




**oder**



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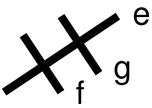
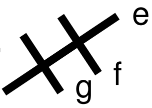
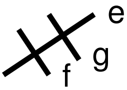
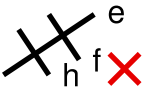
**entweder** 

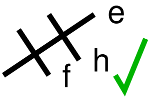
**oder** 


- $y_{e,f,g} + y_{e,g,h} + y_{e,h,f} \leq 2 \ \ \forall (e,f,g,h) \in E^4$



**und** 

→ 



Bis jetzt: Alle $x$ und $y$ können 0 sein.

> *Ein Graph ist genau dann nicht planar, wenn er eine Unterteilung des $K_5$ oder $K_{3,3}$ als Teilgraph enhält*

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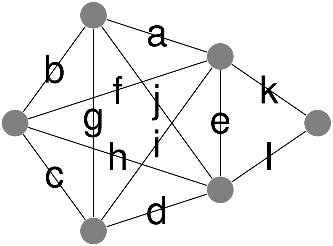

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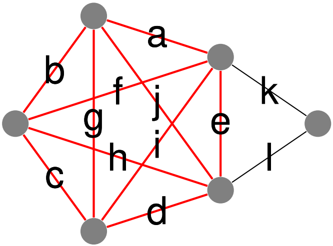
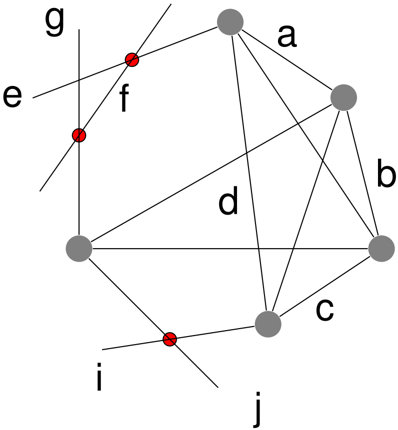



$$x_{\{a,c\}} + x_{\{a,d\}} + \dots$$
$$\dots + x_{\{h,i\}} \geq 1$$

ABER:




$$x_{\{a,c\}} + x_{\{b,d\}} + \ldots$$
$$-x_{\{i,j\}} - y_{(f,g,e)}$$
$$\geq 1 - 2$$

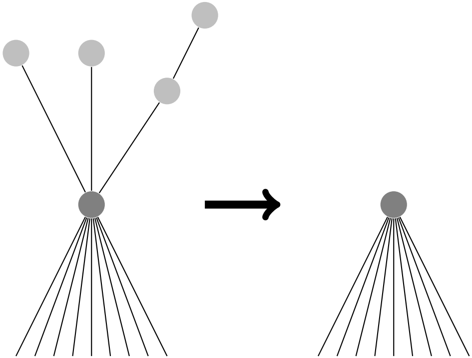
# LP-Modell

- $x_{\{e,f\}}$, $y_{e,f,g}$
- $\min \sum x_{\{e,f\}}$
- LO-Constraints
- Kuratowski-Constraints

$\Rightarrow$ Damit ist das Problem lösbar

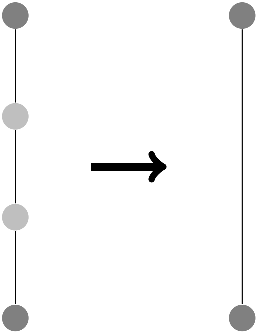
Überspringe Optimierung

Knoten mit deg(v)=1 entfernen

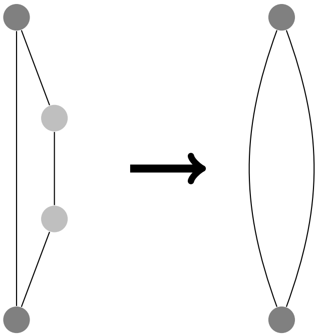


# Optimierung: Nodes entfernen

Knoten mit deg(v)=2 verbinden



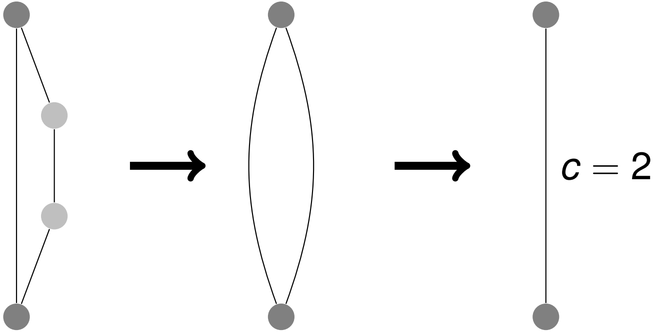
# Spezialfall



# Optimierung: Nodes entfernen

Technische Universität München

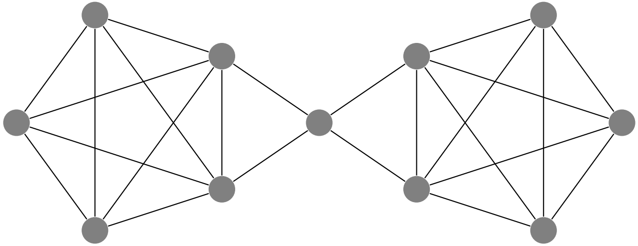
## Spezialfall



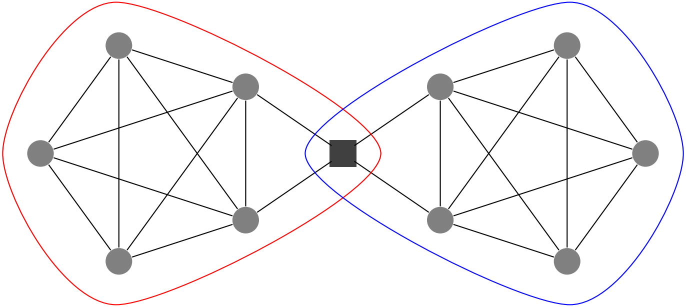

Neue Zielfunktion:

minimiere $$\sum_{\{e,f\}\in\binom{E}{2}} c_e \, c_f \, x_{\{e,f\}}$$

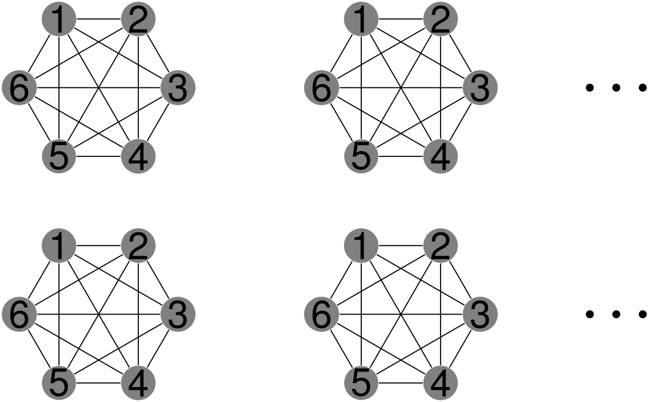
# Biconnected Components getrennt lösen



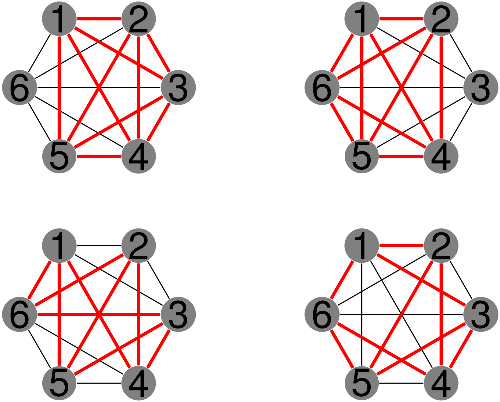
Biconnected Components getrennt lösen



# Mehrere Kuratowski-Teilgraphen auf einmal extrahieren



# Mehrere Kuratowski-Teilgraphen auf einmal extrahieren

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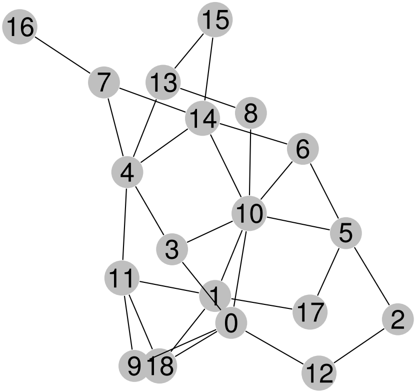
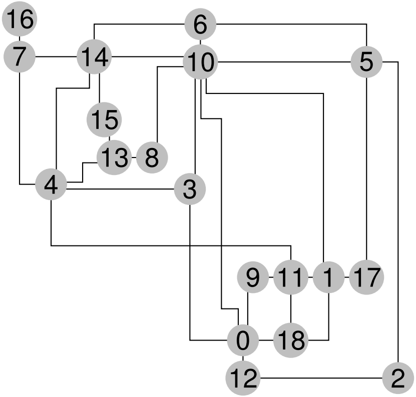
# Fazit



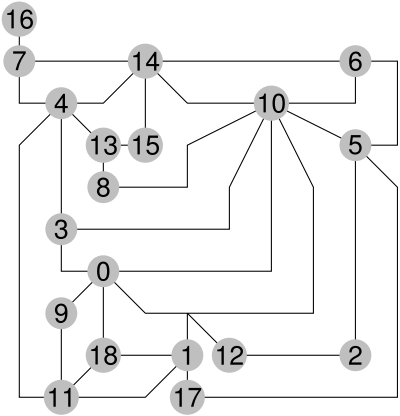
# Fazit

Planarization




< 1 sec

OOCM




32 sec

$\Rightarrow$ Findet optimale Lösung, aber **langsam**

Danke für Eure Aufmerksamkeit