§ UNDERSTANDING PERTURBATIVE BOUNDS – Á LA LI ET AL

Lemma 6: Given $v_1, v_2 \in B_{P_2}$ s.t $\langle v_1, v_2 \rangle = 0$, and $\bar{v}_1, \bar{v}_2 \in B_{P_2}$ s.t

$\|\bar{v}_1 - v_1\|_2 \leq \eta_0$ and $\|\bar{v}_2 - v_2\|_2 \leq \eta_0$

Then, we have $\rho_{Fin}\left([A^{v_1, v_2}], [A^{\bar{v}_1, \bar{v}_2}] \right) \leq 2\kappa(A)\, \eta_0$

Lemma 9: (d-mode rank-r Tensor case) → sufficient : But is it achievable ? via G-S

$\theta_2^r, \theta_3^r, \cdots \theta_d^r \;\; \forall\, r \in [R] \rightsquigarrow$ Partial orthogonality *

$\theta_2^r \in \mathbb{R}^{n_2}$, $\theta_3^r \in \mathbb{R}^{n_3}$, $\theta_d^r \in \mathbb{R}^{n_d}$

Why? Absorbed into span

* What do we actually need here ? For any $r, s \in [R]$ $\exists$ index i, $i \neq 1$

such that $\langle \theta_i^r, \theta_i^q \rangle = 0 \rightsquigarrow$ Atleast Orthogonal in 1-co-ordinate

Further, suppose that $\exists\, i \in [d] \setminus [1]$ and $r \in [R]$, we can find

vectors $\bar{\theta}_i^r \in \mathbb{R}^{n_i}$ s.t $\|\bar{\theta}_i^r - \theta_i^r\| \leq \eta_0$

For full orthogonalisation

Define $\rho_{Fin}\left(\boxed{\quad}, \boxed{\quad} \right) \leq 2 R_k(A) \left((1+\eta_0)^{D-1} - 1 \right)$

Consider $d = 3$ First. $\quad [A^{vw}_{rq}]_{r,q \in [R]} \in \mathbb{R}^{n \times n_1 R^2}$

$\quad [A^{\bar{v}, \bar{w}}_{r,q}]_{r,q \in [R]}$

Li's perturbative bound:

$$\rho_{Fin}\left([A^{vw}_{r,q}], [A^{\bar{v}\bar{w}}_{rq}] \right) \leq \frac{\left\| [A^{vw}_{rq}] - [A^{\bar{v}\bar{w}}_{rq}] \right\|_2}{\sigma_{min}[A^{vw}_{rq}]}$$

Upper-Bounding Numerator:

$$\left\| [A^{vw}_{rq}] - [A^{\bar{v}\bar{w}}_{rq}] \right\|_2$$

$$A^{vw}_{rq} = \sum_{jk} A^{jk} v_j^r w_k^q$$

$$A^{\bar{v}\bar{w}}_{rq} = \sum_{jk} A^{jk} \bar{v}_j^r \bar{w}_k^q$$

$$\left[A^{vw}_{r\alpha}\right] - \left[A^{\bar{v}\bar{w}}_{r\alpha}\right] = \left[\sum_{j,k} A^{jk} \left(v_j^{\ r} w_k^{\ \alpha} - \bar{v}_j^{\ r} \bar{w}_k^{\ \alpha}\right)\right]_{r\alpha}$$

Does r, α Difference matter?

No!

IV $\quad \leq 2R^2 \sigma_{max}(A)\left((1+\eta_0)^2 - 1\right)$

Extending to Full rank would give us $\quad \leq 2R^{D-1}\sigma_{max}(A)\left((1+\eta_0)^{D-1}-1\right)$

Lower-Bounding Denominator

(Explain this part rigorously

$$\sigma_{min}\left(\left[A^{vw}_{r\alpha}\right]\right) = \min_{u \in \mathbb{R}^{R^2 n_1 \times 1}} \left\|\left[A^{vw}_{r\alpha}\right]u\right\|_2 \overset{?}{=} \left\| A \cdot \sum_{r=1}^{R}\sum_{\alpha=1}^{R} \underset{n_3}{w^\alpha} \circ \underset{n_2}{v^r} \circ \underset{n_1}{u^{r\alpha}} \right\|$$

$\underset{\mathrm{I}}{\downarrow}$ $\quad u \in \mathbb{R}^{R^2 n_1 \times 1}$ $\quad \in \mathbb{R}^{n \times n_1 n_2 n_3}$

$A^{vw}_{r\alpha} \in \mathbb{R}^{n \times n_1 R^2}$

$$\underset{\mathrm{II}}{\geq} \sigma_{min}(A) \left\| \sum_{r=1}^{R}\sum_{\alpha=1}^{R} \underset{n_3}{w^\alpha} \circ \underset{n_2}{v^r} \circ \underset{n_1}{u^{r\alpha}} \right\| \underset{\mathrm{III}}{=} \sigma_{min}(A)$$

Reference: Courant-Fisher Theorem!

Suppose $A \in \mathbb{C}^{m \times n}$ has singular values $\sigma_1, \cdots \sigma_n$

$$\sigma_k = \min_{\dim(S)=n-k+1} \max_{\substack{x \in S \\ x \neq 0 \\ \|x\|=1}} \|Ax\|_2 = \max_{\dim(S)=k} \min_{\substack{x \in S \\ x \neq 0 \\ \|x\|=1}} \|Ax\|_2$$

$\Rightarrow \sigma_1 = \max_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \|Ax\|_2$ ~ Weirstrass' Theorem

SOCP?

Is it achieved? Yes!

compact set, continuous function

Explain this part rigorously

$$\underset{w \in \mathbb{R}^{R^2 n_1 \times 1}}{\min} \|[A^{vw}_{rq}]u\|_2 \overset{?}{=} \underbrace{\Big\|A \cdot \sum_{r=1}^{R}\sum_{q=1}^{R} \underset{n_3}{w^q} \circ \underset{n_2}{v^r} \circ \underset{n_1}{u^{rq}}\Big\|}_{\in \mathbb{R}^{n \times n_1 n_2 n_3}}$$

$\mathbb{R}^{R^2 n_1 \times 1}$

(I)

$$\left[\left(\sum_{jk} A_{jk}\, v_j^{:r} w_k^{:q}\right)_{rq}\right] \begin{bmatrix} u^{rq} \\ \hline \vdots \end{bmatrix} = \sum_{r,q=1}^{R} A \cdot w^q \circ v^r \circ u^{rq}$$

Reversing what was done while constructing the matrix

(I) ✓ Done

(II) $$\Big\| A \cdot \sum_{r,q=1}^{R} w^r \circ v^q \circ u^{rq} \Big\|_2 \underbrace{\geq}_{\text{Why?}} \sigma_{\min}(A) \Big\| \sum_{r,q=1}^{R} w^r \circ v^q \circ u^{rq} \Big\|_2$$

✓ Done

$$\|Ax\|_2 \geq \sigma_{\min}(A)\|x\|_2$$

$$\sigma_{\min}(A) \leq \frac{\|Ax\|_2}{\|x\|_2}$$ ✓ Courant Fisher Theorem

(III) $$\Big\| \sum_{r,q=1}^{R} w^r \circ v^q \circ u^{rq} \Big\|_2$$ ~ Expanding

$$= \sqrt{\left\langle \sum_{r_1,q_1=1}^{R} w^{r_1} \circ v^{q_1} \circ u^{r_1 q_1}, \sum_{r_2,q_2=1}^{R} w^{r_2} \circ v^{q_2} \circ u^{r_2 q_2} \right\rangle}$$

$$= \sqrt{} = 1$$

Goal!

understanding inner product of Kronecker product / cross product.

$$\langle u \circ v, w \circ y \rangle$$

$$= \left\langle \begin{matrix} u_1 v \\ u_2 v \\ \vdots \\ u_n v \end{matrix}, \begin{matrix} w_1 y \\ w_2 y \\ \vdots \\ w_n y \end{matrix} \right\rangle = \begin{bmatrix} u_1 v^T & u_2 v^T \dots & u_n v^T \end{bmatrix} \begin{bmatrix} w_1 y \\ w_2 y \\ \vdots \\ w_n y \end{bmatrix}$$

$$= \sum_{i=1}^{n} u_i w_i \langle v, y \rangle$$

Now using this in our case gives us:

$$\langle u \circ v \circ w, x \circ y \circ z \rangle = \sum_{i=1}^{n} u_i x_i \langle v \circ w, y \circ z \rangle$$

$$= \sum_{i} \sum_{j} u_i x_i v_j y_j \langle w, z \rangle$$

$$\left\langle \sum_{r,q=1}^{R} w^{q} \circ v^{r} \circ u^{rq}, \sum_{\bar{r},\bar{q}=1}^{R} w^{\bar{q}} \circ v^{\bar{r}} \circ u^{\bar{r}\bar{q}} \right\rangle$$

$$= \sum_{r,q,\bar{r},\bar{q}=1}^{R} \langle w^{q} \circ v^{r} \circ u^{rq}, w^{\bar{q}} \circ v^{\bar{r}} \circ u^{\bar{r}\bar{q}} \rangle$$

$$= \sqrt{\sum_{r,q,\bar{r},\bar{q}=1}^{R} \sum_{k=1}^{n_3} w_k^{q} w_k^{\bar{q}} \langle v^{r} \circ u^{rq}, v^{\bar{r}} \circ u^{\bar{r}\bar{q}} \rangle}$$

$$= \sqrt{\left[\sum_{r,q,\bar{r},\bar{q}=1}^{R} \sum_{k=1}^{n_3} \sum_{j=1}^{n_2} w_k^{q} w_k^{\bar{q}} v_j^{r} v_j^{\bar{r}} \langle u^{rq}, u^{\bar{r}\bar{q}} \rangle \right]}$$

$$= \sqrt{\sum_{r=q=1}^{R} \langle u^{rq}, u^{rq} \rangle \underbrace{\left(\sum_{k=1}^{n_3} \sum_{j=1}^{n_2} (w_k^{q})^2 (v_j^{r})^2 \right)}_{=1}}$$

$$+ \sum_{r \neq q} \langle u^{rq}, u^{\bar{r}\bar{q}} \rangle \underbrace{\langle w^{q}, w^{\bar{q}} \rangle}_{\text{Full orthogonalisation not necessary}} \langle v^{r}, v^{\bar{r}} \rangle$$

$$= 1$$

Partial orthogonalisation works in practice

(How do you achieve it in practice though?