① Let $\tilde{A}$ be a random matrix and $A = \mathbb{E}[\tilde{A}]$

PROBLEM SETUP: Doubly stochastic normalisation

Scaling factors: $\tilde{A}$ | $(\tilde{x}, \tilde{y})$ A | (x, y)

(Normalised)

Question 1: Given $\dfrac{|\tilde{x} - x|}{x}, \dfrac{|\tilde{y} - y|}{y} \leq c \cdot \mathbb{1}$ (Assume $c < 1$)

(Defined elementwise

can we bound row and column sums of approximate scaling

Ans: For any i, j $\sum_i x_i A_{ij} y_j$ $\sum_j x_i A_{ij} y_j = 1$

$$(1-c) \leq \frac{\tilde{x}_i}{x_i}, \frac{\tilde{y}_j}{y_j} \leq (1+c)$$

$$\left| \sum_i x_i \tilde{A}_{ij} y_j - 1 \right| = \left| \sum_i x_i \tilde{A}_{ij} y_j - \sum_i \tilde{x}_i \tilde{A}_{ij} \tilde{y}_j \right|$$

$$= \left| \sum_i \tilde{A}_{ij} (x_i y_j - \tilde{x}_i \tilde{y}_j) \right|$$

$$= \left| \sum_i x_i (\tilde{A}_{ij} - A_{ij}) y_j \right|$$

consider each entry of A to be sampled from a distn. with mean 1.

$$x_i = y_j = \frac{1}{\sqrt{n}} \quad \forall i, j$$

Yet to use that $\|\tilde{x}\|_1 = \|\tilde{y}\|_1$

(?) $\left| \frac{1}{n} \sum_i (\tilde{A}_{ij}) - 1 \right|$

Also: $\dfrac{1-c}{\sqrt{n}} \leq \tilde{x}_i, \tilde{y}_j \leq \dfrac{1+c}{\sqrt{n}}$

$$\sum_i \tilde{x}_i \tilde{A}_{ij} \tilde{y}_j = 1 \qquad \sum_j \tilde{x}_i \tilde{A}_{ij} \tilde{y}_j = 1$$

Does this imply bounded entries.

Boundedness of entries $\Rightarrow$ Boundedness of scaling factors
+ approximate scaling $\Leftarrow_2$

$$\sum_i \tilde{x}_i \tilde{A}_{ij} = \frac{1}{\tilde{y}_j}$$

$$\Rightarrow \left(\sum_i \tilde{A}_{ij}\right) \cdot \left(\min_i \tilde{x}_i\right) \leq \frac{1}{\tilde{y}_j} \quad \text{and} \quad \left(\sum_j \tilde{A}_{ij}\right)\left(\min_j \tilde{y}_j\right) \leq \frac{1}{\tilde{x}_i}$$

Similarly, $\left(\sum_j A_{ij}\right) \max_j \tilde{x}_i \geq \frac{1}{\tilde{y}_j}$

$$\sum_i \tilde{A}_{ij} \leq \left(\frac{\sqrt{n}}{1-c}\right)^2 \quad \forall i \qquad \sum_j \tilde{A}_{ij} \geq \left(\frac{\sqrt{n}}{1+c}\right)^2$$

$$\Rightarrow \min_{i,j} \tilde{A}_{ij} \leq \frac{1}{(1-c)^2} \quad \text{and} \quad \max_{i,j} \tilde{A}_{ij} \geq \frac{1}{(1+c)^2}$$

$c \to 0 \qquad \tilde{A}_{ij} \to 1$

Very weak bounds on entries of $\tilde{A}$ $\rightarrow$ How effective are these ?

Boundedness of scaling factors and approximate scaling

$$\frac{n}{(1+c)^2} \leq \sum_i \tilde{A}_{ij} \leq \frac{n}{(1-c)^2}$$

What does this mean. How useful is this expression ?

$$\frac{1}{(1+c)^2} - 1 \leq \frac{1}{n}\left(\sum_i \tilde{A}_{ij}\right) - 1 \leq \frac{1}{(1-c)^2} - 1$$

$\boxed{\text{Similarly for column sum}}$

Level of approximate scaling

? ($\Rightarrow$) Require Boundedness ? (

ANALYSING WORKFLOW OF LANDA'S PAPER:

Assumption : Boundedness $\rightsquigarrow$ We know that the scaling factors $\boxed{\frac{\tilde{x}_i}{x_i}, \frac{\tilde{y}_j}{y_j}}$: are bounded

Now, we find the probability with which (x, y) δ-approximately scale $\tilde{A}$ $\longrightarrow$ Here, we use the boundedness of entries to bound values of (x, y) $\longrightarrow$ We can instead just use boundedness assumption in expectation

$\longrightarrow$ Alternatively, if we assume some model, we know what the exact values are.

Next step in our proof: The deterministic step.

Taking an instance $\tilde{A}$ which is approximately scaled to a δ-level by (x, y) $\rightsquigarrow$ we construct exact scaling factors $(\tilde{x}, \tilde{y})$

And here we use the boundedness away from 0

ie That $\min \tilde{A}_{ij} \geq a > 0$ a.s

and also the boundedness of scaling Factors.

Qn: Does approximate scaling $\Rightarrow$ Existance of exact scaling factors that are close enough when $\min \tilde{A}_{ij} \geq 0$ a.s

Ans. Yes, and we also have a closed form expression for the same

Qn: when do we have approximate scaling?

comes from Hoeffding $\rightarrow$ we can say with what probability we will have approximate scaling.

* ASYMPTOTICS IN L

Qn: what does this paper show?

Qn: what do we want?

why do we care about matrix scaling

- Lemma 5.2 from R,V paper
- Application to spectral clustering
- Community detection → SBM and recovery

Qn: How well do scaling factors of the mean scale $\tilde{A}$

└ How does this depend on properties of $\tilde{A}$

Baby Example: Let $\tilde{A}$ be $n \times n$ matrix; entries iid $\text{Uni}[a,b]$ $\frac{a+b}{2} = c$

- we aren't worried about the existance Qn here

└ Full support :)

$$A = \mathbb{E}[\tilde{A}] = c\begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{pmatrix}$$

Consider doubly stoch. scaling

$$x, y = \left(\left(\frac{1}{\sqrt{cn}}, \ldots\right), \left(\frac{1}{\sqrt{cn}}, \ldots\right)\right)$$

$\bar{A} = \frac{\tilde{A}}{cn}$ ⇝ we want to see how close this is to being doubly stochastically scaled

Qn. How do we measure this?

we also know scaling factors exist for $\tilde{A}$: $\tilde{x}, \tilde{y}$: How far away from (x,y) are they? (again choice to measure this)

$$\mathbb{E}\left(\frac{\tilde{A}}{cn}\right) = \mathbb{1}\mathbb{1}^T$$

$\|\bar{A} - A\|_1$ and $\|\bar{A} - A\|_\infty$: helpful?

↳ max row sum ↳ max col sum

Row sum: $\frac{1}{cn}\sum_i \tilde{a}_{ij} - 1$

Distribution of $\frac{1}{cn}\sum_i \tilde{a}_{ij} - 1$ ~ 0 when n is large $\boxed{\text{LLN}}$

Sum of uniformly distributed random variables

But we also need to worry about the union bound :- Irwin Hall Distn

let $p(n) = \mathbb{P}\left(\left|\frac{1}{cn}\sum_i \tilde{a}_{ij} - 1\right| > \varepsilon\right)$

$\Rightarrow$ $\tilde{A}$ is ε-approximately scaled with probability

$\boxed{1 - 2n\,p(n)}$ $\rightsquigarrow$ Exact scaling factors.

Changing factors: We want $p(n)$ to be small

ideally $\boxed{p(n) - O\left(\frac{1}{n}\right)}$ $\rightsquigarrow$ Probability level "bounded"

§ Going from approximately scaled to exactly scaled

How does this translate to difference in the scaled matrix

$$\underbrace{x\tilde{A}y - \tilde{x}\tilde{A}\tilde{y}}_{\text{Quantifying}} \qquad x\tilde{A}y - x\tilde{A}\tilde{y} + x\tilde{A}\tilde{y} - \tilde{x}\tilde{A}\tilde{y}$$
$$= x\tilde{A}(y-\tilde{y}) + (x-\tilde{x})\tilde{A}\tilde{y}$$

$\tilde{x}\tilde{A}\tilde{y}$: r, c sums are 1

$x\tilde{A}y$: r, c sums $\in [1-\varepsilon, 1+\varepsilon]$

§ Approximate scaling factors

$$\begin{tikzcd}
{[\tilde{A}_{ij}]} \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, phantom, "\text{Doubly stochastic}"] & E[\tilde{A}] = A \arrow[d] \\
(\tilde{x}, \tilde{y}) \arrow[r, leftrightarrow] & (x, y)
\end{tikzcd}$$

$\longrightarrow$ For large n:

↳ we have exact values for A.

But don't have boundedness.

Distributional form : LLN : needs iid (Too weak order of convergence)

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§ Let mean scale approximately

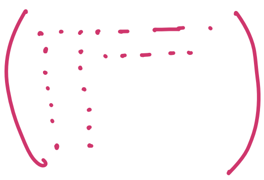
$$\frac{1}{1+\varepsilon} \le \underbrace{\max \tilde{x}_i \min \tilde{x}_i}_{} \le \frac{1}{1-\varepsilon} \quad \rightsquigarrow \text{ Bound on prod}$$

$$\max \tilde{x}_i \ge \frac{1}{\sqrt{1+\varepsilon}}$$

Connection to diagonal entries

$$\min \tilde{x}_i \le \frac{1}{\sqrt{1-\varepsilon}}$$

$$\boxed{\max \tilde{x}_i - \min \tilde{x}_i \ge \frac{1}{\sqrt{1+\varepsilon}} - \frac{1}{\sqrt{1-\varepsilon}}}$$



row-sums , column sums $\longrightarrow 1$

order of convergence

L Asymptotic regime

L Better Bounds

Boundedness of entries $\Longrightarrow$ Boundedness of scaling factors

$$\boxed{\frac{1}{1+\varepsilon} \le \max \tilde{x}_i \min \tilde{x}_i \le \frac{1}{1-\varepsilon}}$$

Let $i = \operatorname{argmax} \tilde{x}_i$
$j = \operatorname{argmin} \tilde{x}_j$

$\downarrow$

Exactly scaled $\|\cdot\|_1$ are equal

? when does ε- approximate scaling $\Rightarrow$ closeness of scaling factors