A nonnegative $m \times n$ matrix A is called *completely decomposable* if there exist partitions α_1, α_2 of $\{1, \cdots, m\}$ and β_1, β_2 of $\{1, \cdots, n\}$ into nonvacuous sets such that $A[\alpha_1, \beta_2]$ and $A[\alpha_2, \beta_1]$ are zero matrices. Here we use the notation that $A[\alpha, \beta]$ is the submatrix of A whose rows are indexed by α and whose columns are indexed by β, the rows and columns in $A[\alpha, \beta]$ appearing in the same order as in A. If $m = n$, the matrix A is called *completely reducible* if there exists a partition α_1, α_2 of $\{1, \cdots, m\}$ into nonvacuous sets such that $A[\alpha_1, \alpha_2]$ and $A[\alpha_2, \alpha_1]$ are zero matrices.

Completely decomposable

$A = \underbrace{\begin{bmatrix} \; & \; \end{bmatrix}}_{n} \Big\}\, m$ $[m] = I_1 \cup I_2$ $[n] = J_1 \cup J_2$

$A_{I_1 J_2}$ and $A_{I_2 J_1}$ are 0 matrices.

ie $\exists$ permutation matrices $\underset{S_m}{P}, \underset{S_n}{Q}$ such that

$$PAQ = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right]$$

Relation to direct sum?

In terms of maps: clear

Special case : $m = n$

Completely reducible : we want it to be completely decomposable with $I_1 = J_1$

Generalizing theorems of Sinkhorn and Knopp [10] and Brualdi, Parter, and Schneider [1], Menon [7] proved the following theorem: Let A be an $m \times n$ nonnegative matrix and let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be positive vectors with $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. Let $\mathfrak{A}(R, S)$ denote the class of all $m \times n$ nonnegative matrices with row sum vector R and column sum vector S. Then there exist diagonal matrices D_1 and D_2 with positive main diagonals such that D_1AD_2 is in $\mathfrak{A}(R, S)$ if and only if there is a matrix in $\mathfrak{A}(R, S)$ which has the same zero pattern as A. (We say that a matrix

Consider the space

$\mathcal{U}(R, C) = \{ A \in \mathbb{R}^{m \times n} \mid A \cdot \mathbb{1} = R,\ A^T \cdot \mathbb{1} = C \}$

closed set in $\mathbb{R}^{m \times n}$

Aim: To find D_1, D_2 s.t $D_1 A D_2 \in \mathcal{U}(R, S)$: (Not !)

* Such matrices D_1 and D_2 exist iff $\exists$ a matrix in $\mathcal{U}(R,S)$ with the same sparsity pattern as A.

ie zero patter

[NOTE: A matrix B has the same sparsity pattern as A if $b_{ij} = 0 \Leftrightarrow a_{ij} = 0$]

EQUIVALENCE ! Not just same sub-pattern

UNIQUENESS: If A is not completely decomposable, then we have ! upto scalar factors

If, in addition, A is not completely decomposable, the diagonal matrices D_1, D_2 are unique up to positive scalar factor: if U_1AU_2 is in $\mathfrak{A}(R, S)$ then there exists $\delta > 0$ such that $U_1 = \delta D_1$, $U_2 = \delta^{-1} D_2$. Brualdi [2] proved that given A there exists a matrix in $\mathfrak{A}(R, S)$ with the same zero pattern as A if and only if the following condition is satisfied.

$\Rightarrow$ All positive matrices have unique matrix scaling factors.

Equivalent conditions to (*) (Brualdi)

(1) *For all partitions α_1, α_2 of $\{1, \cdots, m\}$ and β_1, β_2 of $\{1, \cdots, n\}$ into nonempty sets such that $A[\alpha_1, \beta_2]$ is a zero matrix, $\sum_{j \in \beta_1} s_j \geq \sum_{i \in \alpha_1} r_i$ with equality holding if and only if $A[\alpha_2, \beta_1]$ is also a zero matrix.*

equivalent condition to saying that $\exists$ an equivalent matrix in $\mathcal{U}_{R,C}$ with same sparsity pattern $\rightarrow$ $\forall$ partitions α_1, α_2 of $[m]$ and β_1, β_2 of $[n]$

$$\begin{array}{cc} \begin{array}{c}\alpha_1\\ \alpha_2\end{array} & \left[\begin{array}{c|c} \ & 0 \\ \hline \ & \ \end{array}\right] \\ & \begin{array}{cc}\beta_1 & \beta_2\end{array}\end{array} = PAQ \rightsquigarrow \sum_{j\in\beta_1} s_j \geq \sum_{i\in\alpha_1} r_i$$

with "=" holding iff $A[\alpha_2, \beta_1]$ is also a 0-matrix.

Combining results. such diagonal matrices exist iff * holds!