MTH399 Presentation Report

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Application of diagonalization: Recurrence Relations

0.1 2×2 case

Special CASE: Fibonacci Series

We know that the fibonacci series is given by:

$$R_{n+1} = R_n + R_{n-1}$$

To compute the n+1th term in terms of R_0 and R_1
We can express it through a matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} R_{n+1} \\ R_n \end{bmatrix}$$

This is of the form $A^n R = X$
The eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_1 = \frac{1-\sqrt{5}}{2}$
So $A^n = U \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^n U^{-1}$
So we can say that the nth term will be some linear combination of the power of the 2 eigenvalues

$$\begin{aligned}
R_n &= a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n \\
R_0 &= a + b = 0 \\
&\Rightarrow a = -b \\
R_1 &= a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1 \\
&\Rightarrow a = \frac{1}{\sqrt{5}}, b = -\frac{1}{\sqrt{5}} \\
\therefore R_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\
&= \frac{1}{\sqrt{5}} \sum_{i=1,odd}^{n} \left[\binom{n}{i} \frac{\sqrt{5}^i}{2^n}\right]
\end{aligned}$$

CASE I: $\lambda_1 \neq \lambda_2$

$$R_{n+1} = cR_n + dR_{n-1}$$

We can express it through a matrix

$$\begin{bmatrix} c & d \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} R_{n+1} \\ R_n \end{bmatrix}$$

This is of the form $A^n R = X$
The eigenvalues are $\lambda_1 = \frac{c+\sqrt{c^2+4d}}{2}, \lambda_1 = \frac{c-\sqrt{c^2+4d}}{2}$ Here the matrix can easily diagonalized ($A = UDU^{-1}$) where U is the matrix containing its eigenvectors and D is a diagonal matrix comprising the eigenvalues

$$A = UDU^{-1} = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^{-1}$$

Hence $A^n = UD^nU^{-1} = U \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} U^{-1}$
So we can say that the nth term will be some linear combination of the power of the 2 eigenvalues

$$R_n = a\lambda_1^n + b\lambda_2^n$$
$$R_0 = a + b$$
$$R_1 = a\lambda_1 + b\lambda_2$$

Therefore

$$a = \frac{R_1 - \lambda_2 R_0}{\lambda_1 - \lambda_2}, b = \frac{R_1 - \lambda_1 R_0}{\lambda_2 - \lambda_1}$$
$$\therefore R_n = \frac{1}{\lambda_1 - \lambda_2} \left[(R_1 - \lambda_2 R_0)\lambda_1^n - (R_1 - \lambda_1 R_0)\lambda_2^n \right]$$
$$R_n = (\lambda_2^n - \lambda_1^n) \left[R_0 - \frac{R_1}{\lambda_1 - \lambda_2} \right]$$

CASE II: $\lambda_1 = \lambda_2$

For the eigenvalues to be equal, we can see from the previous case that $d = -\frac{c^2}{4}$
Hence we have

$$R_{n+1} = cR_n - \frac{c^2}{4} R_{n-1}$$

, giving the matrix representation

$$\begin{bmatrix} c & \frac{-c^2}{4} \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} R_{n+1} \\ R_n \end{bmatrix}$$

On computation we found

$$A^n = \begin{bmatrix} (n+1)\frac{c^n}{2^n} & -n\frac{c^{n+1}}{2^{n+1}} \\ (n)\frac{c^{n-1}}{2^{n-1}} & -(n-1)\frac{c^n}{2^n} \end{bmatrix}$$

Since we know $\lambda = c/2$, the matrix can be re-written as

$$A^n = \lambda^{n-1} \begin{bmatrix} (n+1)\lambda & -n\lambda^2 \\ (n) & -(n-1)\lambda \end{bmatrix}$$

Hence, $R_n = \lambda^{n-1}[nR_1 - (n-1)\lambda R_0] = n\lambda^{n-1}R_1 - (n-1)\lambda^n R_0$
Another way of finding the n^{th} term we use the jordan canonical form, thus

$$A = UDU^{-1} = U \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} U^{-1}$$

$$\Rightarrow A^n = UD^nU^{-1} = U \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} U^{-1}$$

So we can find the nth term as a general combination of the $\lambda^n, n\lambda^{n-1}$

$$\begin{aligned}
R_n &= a\lambda^n + bn\lambda^{n-1} \\
R_0 &= a \\
R_1 &= a\lambda + b \\
\Rightarrow b &= R_1 - R_0\lambda \\
\Rightarrow R_n &= \lambda^n R_0 + n(R_1 - R_0\lambda)\lambda^{n-1} \\
&= n\lambda^{n-1}R_1 - (n-1)\lambda^n R_0
\end{aligned}$$

0.2 3×3 case

We'll consider the 3 term case, that is

$$R_{n+2} = aR_{n+1} + bR_n + cR_{n-1}$$

$$\Rightarrow \begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} R_2 \\ R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} R_{n+2} \\ R_{n+1} \\ R_n \end{bmatrix}$$

CASE I: $\lambda_1 \neq \lambda_2 \neq \lambda_3$

Here we can diagonalize the matrix using spectral theorem hence rewriting the nth form matrix as

$$R_{n+2} = aR_{n+1} + bR_n + cR_{n-1}$$

$$U \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} U^{-1} \begin{bmatrix} R_2 \\ R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} R_{n+2} \\ R_{n+1} \\ R_n \end{bmatrix}$$

$$\begin{aligned}
R_n &= \alpha\lambda_1^n + \beta\lambda_2^n + \gamma\lambda_3^n \\
\Rightarrow R_0 &= \alpha + \beta + \gamma \\
R_1 &= \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 \\
R_2 &= \alpha\lambda_1^2 + \beta\lambda_2^2 + \gamma\lambda_3^2
\end{aligned}$$

We have 3 equations for the three unknown coefficients s.t.

$$\begin{bmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_2 \\ R_1 \\ R_0 \end{bmatrix}$$

which gives us a solvable equation set as determinant is non-zero

Tribonacci Case

$$R_{n+2} = R_{n+1} + R_n + R_{n-1}$$

$$R_{n+2} = aR_{n+1} + bR_n + cR_{n-1}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} R_2 \\ R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} R_{n+2} \\ R_{n+1} \\ R_n \end{bmatrix}$$

CASE II: $\lambda_1 = \lambda_2 \neq \lambda_3$

$$\begin{aligned}
&\Rightarrow x^3 - ax^2 - bx - c = (x-\lambda_1)^2(x-\lambda_2) \\
&\Rightarrow a = 2\lambda_1 + \lambda_2 \\
&\quad b = -2\lambda_1\lambda_2 - \lambda_1^2 \\
&\quad c = \lambda_1^2\lambda_2
\end{aligned}$$

This will give us a matrix with 2 jordan blocks

$$U \begin{bmatrix} \lambda_1^n & n\lambda_1^{n-1} & 0 \\ 0 & \lambda_1^n & 0 \\ 0 & 0 & \lambda_2^n \end{bmatrix} U^{-1} \begin{bmatrix} R_2 \\ R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} R_{n+2} \\ R_{n+1} \\ R_n \end{bmatrix}$$

$$\begin{aligned}
&R_n = \alpha\lambda_1^n + \beta n\lambda_1^{n-1} + \gamma\lambda_2^n \\
\Rightarrow &R_0 = \alpha + \gamma \\
&R_1 = \alpha\lambda_1 + \beta + \gamma\lambda_2 \\
&R_2 = \alpha\lambda_1^2 + 2\beta\lambda_1 + \gamma\lambda_2^2
\end{aligned}$$

We have 3 equations for the three unknown coefficients s.t.

$$\begin{bmatrix} \lambda_1^2 & 2\lambda_1 & \lambda_2^2 \\ \lambda_1 & 1 & \lambda_2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} R_2 \\ R_1 \\ R_0 \end{bmatrix}$$

CASE III: $\lambda_1 = \lambda_2 = \lambda_3$

For this case we need to have a perfect cube characteristic equation

$$\begin{aligned}
&\Rightarrow x^3 - ax^2 - bx - c = (x-\lambda)^3 \\
&\Rightarrow a = 3\lambda \\
&\quad b = -3\lambda^2 \\
&\quad c = \lambda^3
\end{aligned}$$

Using the Jordan Canonical form

$$\begin{bmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

So we can say that R_n will have the following form

$$R_n = \alpha\lambda^n + \beta n\lambda^{n-1} + \gamma\binom{n}{2}\lambda^{n-2}$$

$$\begin{aligned}
\Rightarrow\ &R_0 = \alpha & \alpha &= R_0 \\
&R_1 = \alpha\lambda + \beta & \beta &= R_1 - \lambda R_0 \\
&R_2 = \alpha\lambda^2 + 2\beta\lambda + \gamma & \gamma &= \lambda^2 R_0 - 2\lambda R_1 + R_2 \\
&R_n = \frac{(n-1)(n-2)}{2}\lambda^n R_0 - n(n-2)\lambda^{n-1}R_1 + \frac{n(n-1)}{2}\lambda^{n-2}R_2
\end{aligned}$$