MTH392 Report

Shambhavi Singh
13644

Abstract

This report deals with literature that on the Prime Number Theorem and Dirichlet Theorem on Prime Numbers in an Arithmetic Progression It also talks about the prime number theorem for Arithmetic Progressions It briefly covers the Prime Number Theorem, which talks about the distribution of primes among natural numbers. It shows that as the numbers grow bigger, the occurrence of prime numbers decreases. We'd restrict ourselves to the outline of the proof

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1 Introduction

Analytic Number Theory is a branch of number theory which incorporates Real and Complex analysis to study properties of integers and prime numbers. The Dirichlet L-Series, which is used to prove the Dirichlet Theorem on Primes in an Arithmetic Progression and the Riemann-Zeta function, which is used to prove the Prime Number Theorem fall in this field.
It also deals with the Prime number theorem for prime numbers in an AP which talks about the uniform distribution of primes among the reduced residue classes mod k. This arises from the fact that the the relation between the sum of log(p)/p of primes in an Arithmetic Progression is independent of the residue class being talked about

2 Some basic Definitions

In this section we shall define a few of the basic structures used in the report

2.1 Arithmetical Functions

Definition 2.1. *Any function that goes from the set of natural numbers to $\mathbb{C}$, that is*

$$f : \mathbb{N} \to \mathbb{C}$$

some arithmetical functions are given below

$$\begin{aligned} u(n) =& 1 \\ N(n) =& n \\ I(n) =& \left[\frac{1}{n}\right] = \begin{cases} 1 & if n = 1 \\ 0 & otherwise \end{cases} \end{aligned}$$

2.2 Möbius Function

Definition 2.2. *Suppose* $x = {p_1}^{a_1} {p_2}^{a_2} \dots {p_n}^{a_n}$

$$\mu(x) = \begin{cases} (-1)^{a_1} & if\ a_i = 1\ \forall\ i = 1, \dots, n \\ 0 & otherwise \end{cases}$$

2.3 Mangoldt Function

Definition 2.3. *Suppose* $x = {p_1}^{a_1} {p_2}^{a_2} \dots {p_n}^{a_n}$

$$\Lambda(x) = \begin{cases} logp & if\ x = {p_1}^{a_1} \\ 0 & otherwise \end{cases}$$

We also define the Chebyshev function as the following

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

2.4 Euler Totient Function $\varphi(n)$

Definition 2.4. *The Euler totient function basically counts the number of primes that are smaller than a number x*

$$\varphi(x) = \sum_{p \leq x} 1$$

2.5 Dirichlet Multiplication

Definition 2.5. *Dirichlet multiplication is denoted by * and defined as $h = f * g$ for 2 arithmetical functions f and g such that*

$$h = \sum_{d|n} f(\frac{n}{d})g(d)$$

2.6 Dirichlet Characters

Definition 2.6. $\chi_1, \chi_2, \ldots, \chi_{\phi(k)}$ *are the dirichlet characters mod k where χ_1 is the principal character. Dirichlet characters are defined for a group G of reduced residue classes modulo k. Corresponding to each character f of G we define $\chi = chi_f$ as follows:*

$$\chi(n) = \begin{cases} f(\widehat{n}) & \text{if } (n,k) = 1 \\ 0 & \text{if } (n,k) > 1 \end{cases}$$

$$\chi_1(n) = \begin{cases} 1 & \text{if } (n,k) = 1 \\ 0 & \text{if } (n,k) > 1 \end{cases}$$

Dirichlet Characters are completely multiplicative and periodic with period k

2.7 L functions $L(s, \chi), \zeta(s)$ and $\Gamma(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ s \in \mathbb{C}$$

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi^s(n)}{n^s},\ s \in \mathbb{C}$$

χ is a character of a finite abelian group, or as in our case, the dirichlet character

$$L'(1, \chi) = \sum_{n=1}^{\infty} -\frac{\chi(n) log\ n}{n}$$

$$\Gamma = \int_0^{\infty} x^{s-1} e^s$$

3 Fundamental Theorem of Arithmetic

3.1 The fundamental theorem of Arithmetic

Theorem 3.1. *Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.*

Proof. Using induction on n
*Induction Step:*The theorem is true for n=2
Induction Hypothesis: suppose it's true for all integers less than n and greater than 1
CASE 1: n is prime, We are done
CASE 2: n is composite
Suppose n has 2 factorizations

$$n = p_1 p_2 \dots p_s = q_1 q_2 \dots q_t$$

We wish to show that $s = t$ and that each p equals some q. As p_1 divides n, it must divide at least on of the q_i. WLOG, we can assume $p \mid q$. Since both p_1 and q_1 are primes $p_1 = q_1$. Now proceed similarly for

$$n/p_1 = p_2 \dots p_s = q_2 \dots p_t$$

If $s > 1$ and $t > 1$, $1 < n/p_1$. The induction hypothesis tells us that the 2 factorizations of n/p_1 must be identical upto the order of the factors, thus $s = t$ and the factors are identical □

We can also re-write the factorization by combining the recurring primes. that is

$$n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$$

3.2 Divergence of the series $\sum_{n=1}^{\infty} 1/p_n$

Theorem 3.2. *The infinite series $\sum_{n=1}^{\infty} 1/p_n$ diverges*

Proof. Suppose the series converges. Thus the tail of the series must be arbitrarily small, hence

$$\sum_{m=k+1}^{\infty} \frac{1}{p_m} < \frac{1}{2}$$

Let $Q = p_1 \dots p_k$ and consider $1 + nQ$ for $n = 1, 2, \dots$ None of these is div. by any of the primes $p_1 \dots p_k$. Hence all $1 + nQ$ must have $p_{k+1}, p_{k+2}, \dots$. Thus for each r

$$\sum_{n=1}^{r} \frac{1}{1 + nQ} \leq \sum_{t=1}^{\infty} \left(\sum_{m=k+1}^{\infty} \frac{1}{p_m} \right)^t < \sum_{t=1}^{\infty} \left(\frac{1}{2} \right)^t$$

This is true as the in the middle includes among its terms, all the terms on the left and the 2nd inequality is by our assumption. Thus the series $\sum_{n=1}^{r} \frac{1}{1+nQ}$ has bdd partial sums and hence converges, but

$$\sum_{n=1}^{\infty} \frac{1}{2nQ} \leq \sum_{n=1}^{\infty} \frac{1}{1+nQ}$$

and the LHS series diverges. Hence we have a contradiction. □

4 Abel's Identity

Theorem 4.1. *For any arithmetical function a(n), we have*

$$A(x) = \sum_{n \leq x} a(n)$$

where $A(x) = 0$ if $x > 1$. Assuming f has a continuous derivative in the interval $[y, x]$ where $0 < y < x$, then we have

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt$$

Proof. Let $k = [x], m = [y]$ so that $A(x) = A(k), A(y) = A(m)$, then

$$\begin{aligned}
\sum_{y<n\leq x} a(n)f(n) &= \sum_{n=m+1}^{k} a(n)f(n) = \sum_{n=m+1}^{k} \{A(n) - A(n-1)\}f(n) \\
&= \sum_{n=m+1}^{k} A(n)f(n) - \sum_{n=m+1}^{k-1} A(n)f(n+1) \\
&= \sum_{n=m+1}^{k-1} A(n)\{f(n) - f(n+1)\} + A(k)f(k) - A(m)f(m+1) \\
&= -\sum_{n=m+1}^{k-1} A(n) \int_n^{n+1} f'(t)dt + A(k)f(k) - A(m)f(m+1) \\
&= -\sum_{n=m+1}^{k-1} \int_n^{n+1} A(t)f'(t)dt + A(k)f(k) - A(m)f(m+1) \\
&= -\int_{m+1}^{k} A(t)f'(t) + A(x)f(x) - \int_k^x A(t)f'(t)dt - A(y)f(y) - \int_y^{m+1} A(t)f'(t)dt \\
&= A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt
\end{aligned}$$

□

5 Equivalent Statements of the Prime Number Theorem

The prime number talks about the behaviour of $\pi(x)$ as x goes to infinity. Gauss and Legendre separately conjectured by the inspection of table of primes that $\pi(x)$ is asymptotic to $x/log(x)$, that is,

$$\lim_{x\to\infty} \frac{\pi(x)logx}{x} = 1$$

We find out that an equivalent statement to the prime number theorem is the following

$$\sum_{n\leq x} \Lambda(n) \sim x \text{ as } x \to \infty$$

From the definition of the Chebyshev's function, we can rewrite the above formula as

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$$

to show this to be equivalet, we proceed the following way

Definition 5.1. *We define Chebyshev's ϑ- function as*

$$\vartheta(x) = \sum_{p\leq x} log\ p$$

Lemma 5.1. *For $x \geq 2$ we have*

$$\vartheta(x) = \pi(x)logx - \int_2^x \frac{\pi(t)}{t}$$
$$\pi(x) = \frac{\vartheta(x)}{log\ x} + \int_2^x \frac{\vartheta(t)}{tlog^2\ t}$$

Theorem 5.2. *The following relations are logically equivalent*

$$\lim_{x\to\infty} \frac{\pi(x)logx}{x} = 1$$
$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$$
$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$$

6 Dirichlet Theorem on Primes in an AP and its consequences

6.1 Dirichlet Theorem on Primes in an Arithmetic Progression

We first proved this for specific cases, viz. primes of the form 4n+1 and 4n-1, being infinite in number

The crux of the proof lies in showing the L function for a dirichlet character has non-zero value for any character
One proves the same for Real valued characters and Complex valued characters separately
For Real valued characters, we show the following relations

Lemma 6.1. *For any real-valued nonprincipal character χ mod k, let*

$$A(n) = \sum_{d|n} \chi(d),\ B(n) = \sum_{n\leq x} \frac{A(n)}{\sqrt{n}}$$

Then we have

1. $B(x) \to \infty$ *as* $x \to \infty$
2. $B(x) = 2\sqrt{x}L(1,\chi) + O(1)$

For Complex valued characters we first consider the fact that they're always going to occur in pairs hence if we consider $N(k)$ to be the number of character χ mod k such that $L(1,\chi) = 0$ then proving the following lemma works

Lemma 6.2.

$$\sum_{\substack{p\leq x\\ p\equiv 1(modk)}} \frac{logp}{p} = \frac{1-N(k)}{\varphi(k)} logx + O(1)$$

Here N(k) being an even value cannot be anything but 0 as that would make the RHS negative as $x \to \infty$ whereas the LHS is positive which would be a contradiction
To prove this lemma, we first have to prove another bunch of lemmas

Lemma 6.3. *For $x > 1$ we have*

$$\sum_{\substack{p\leq x\\ p\equiv h(modk)}} \frac{logp}{p} = \frac{1}{\varphi(k)} logx + \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \overline{\chi}_r(h) \sum_{p\leq x} \frac{\chi_r(p)logp}{p} + O(1)$$

This also proves the Dirichlet Theorem if we're able to show that $\sum_{p\leq x} \frac{\chi_r(p)logp}{p} = O(1)$ which is a result of the following lemmas

Lemma 6.4.

$$\sum_{p\leq x} \frac{\chi_r(p)logp}{p} = -L'(1,\chi) \sum_{n\leq x} \frac{\mu(n)\chi(n)}{n} + O(1)$$

Lemma 6.5.

$$L(1,\chi) \sum_{n\leq x} \frac{\mu(n)\chi(n)}{n} = O(1)$$

Which would imply that $\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}=O(1)$ if $L(1,\chi)\neq 0$ for all $\chi\neq\chi_1$ The proof for non vanishing of complex valued characters needs one more lemma:-

Lemma 6.6. *If $\chi\neq\chi_1$ and $L(1,\chi)=0$ we have*

$$L'(1,\chi)\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}=logx+O(1)$$

We proved this series of lemmas to finally come to conclude Dirichlet's theorem, an equivalent statement of which is the following

Theorem 6.7. *If $k>0$ and $(h,k)=1$ we have for all $x>1$,*

$$\sum_{\substack{p\leq x\\ p\equiv h(modk)}}\frac{logp}{p}=\frac{1}{\varphi(k)}logx+O(1)$$

where the sum is extended over those primes $p\leq x$ which are congruent to h mod k.

Which clearly indicates the sum on the LHS must go to infinity as RHS goes to infinity as x goes to infinity
One also notices that the term on the RHS is independent of h, hence, one is able to conclude that the primes will be equally distributed in the equivalences classes of mod k
In fact we come up with an an formulation for the prime number theorem for arithmetic progressions, which talks of the distribution of prime numbers in an Arithmetic Progression

6.2 Prime Number Theorem for arithmetic progressions

If $k>0$ and $(a,k)=1$, we define

$$\pi_a=\sum_{\substack{p\leq x\\ p\equiv a(modk)}}1$$

This counts the number of primes in the progression $nk+a,\ n=0,1\ldots$
The prime number theorem for progressions is suggested by the formula of Theorem 5.7
Since the principal term is independent of h, the primes seem to be equally distributed among the $\varphi(k)$ reduced residue classes $modk$

Theorem 6.8.

$$\pi_a\sim\frac{\pi(x)}{\varphi(x)}\ as\ x\to\infty$$

holds for every integer a relatively prime to k, iff

$$\pi_a(x)\sim\pi_b(x)\ as\ x\to\infty$$

whenever $(a,k)=(b,k)=1$

Proof. ($\Rightarrow$) is clear
($\Leftarrow$)

$$\begin{aligned}\pi(x) = &\sum_{p\le x} 1 = A(k) + \sum_{\substack{p\le x\\ p\nmid k}} 1\\ =&A(k) + \sum_{\substack{a=1\\(a,k)=1}}^{k} \sum_{\substack{p\le x\\ p\equiv a(mod k)}} 1 = A(k) + \sum_{\substack{a=1\\(a,k)=1}}^{k} \pi_a(x)\end{aligned}$$

Therefore

$$\frac{\pi(x)-A(k)}{\pi_b(x)} = \sum_{\substack{a=1\\(a,k)=1}}^{k} \frac{\pi_a(x)}{\pi_b(x)}$$

So by assumption, each term in the sum tends to 1 as $x\to\infty$ so the sum tends to $\varphi(k)$, thus

$$\frac{\pi(x)}{\pi_b(x)} - \frac{A(k)}{\pi_b(x)} \to \varphi(k) \text{ as } x\to\infty$$

□

7 Prime Number Theorem

The prime number theorem basically tells us about the distribution of primes and their decreasing instances of occurrence as we move to bigger numbers The crux of the analytic proof of the Prime Number Theorem is in showing that the zeta function doesn't vanish on the line $\sigma = 1$
In this proof, we set out to prove that $\frac{\psi(x)}{x} \to 1$ as $x\to\infty$
We'll just be stating the plan of the proof
Since the function $\psi(x)$ is just a step function we deal with its integral to make it more convenient

$$\psi_1(x) = \int_1^x \psi(t)dt$$

The integral ψ_1 is cts piecewise linear function, we first show that the relation

$$\psi_1(x) \sim \frac{1}{2}x^2 \text{ as } x\to\infty$$

would imply that

$$\psi(x) \sim x \text{ as } x\to\infty$$

For this, we express $\psi_1(x)/x^2$ in terms of the Riemann zeta function by means of the following contour integral

$$\frac{\psi_1(x^2)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s}{\zeta(s)}\right) ds \text{ where } c>1$$

The expression $-\frac{\zeta'(s}{\zeta(s)}$ has a first order pole at $s = 1$ with residue 1. If we subtract this pole, we get the formula

$$\frac{\psi_1(x^2)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s}{\zeta(s)} - \frac{1}{s-1}\right) ds \text{ for } c > 1$$

Let

$$h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta'(s}{\zeta(s)} - \frac{1}{s-1}\right)$$

So

$$\begin{aligned} \psi_1(x^2)x^2 - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 &= \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1} h(s) ds \\ &= \frac{x^{c-1}}{2\pi}\int_{-\infty}^{+\infty} h(c+it)e^{itlogx} dt \end{aligned}$$

To complete the proof, we need to show that RHS goes to 0 as $x \to \infty$
The Riemann-Lebesgue lemma (Fourier Series) states that

$$\lim_{x\to\infty}\int_{-\infty}^{+\infty} f(t)e^{itx} dt = 0$$

if the integral $\int_{-\infty}^{+\infty} |f(t)|dt$ converges which is very much like the integral in the previous equation, we can show that the integral $\int_{-\infty}^{+\infty} |h(c+it)|dt$ converges for $c > 1$ so the integral in tends to 0 as $x \to \infty$. However, the factor $x(c-1)$ causes a problem as for any $c > 1$ it will make the whole series diverge as a whole. So if we're able to prove that $\int_{-\infty}^{+\infty} |h(1+it)|dt$ converges, we'd be able to apply to apply the Riemann-Lebesgue lemma in the following case as well. This would require the study of the nature of the zeta function in the vicinity of $\sigma = 1$

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