MTH391 Report
Simultaneous Triangularization

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Abstract

We dealt with literature on Simultaneous Triangularization which talked about the criteria a set of operators would satisfy so as to be triangularizable. We gradually made the collection of operators more specific- beginning with a set of operators that had properties inherited by quotients, we then moved on to an Algebra of operators, claiming that a sufficient condition for an Algebra to be triangularizable was that it was proper; We began with first proving the Burnside's theorem and showed how algebra could be triangularizable when it satisfied certain properties, like being the operators being nilpotent (the converse didn't hold though), then a more general algebra was taken- where the commutator $(AB-BA)$ was nilpotent, which was further generalized. Then we read about triangularizability of a pair of operators in the McCoy's Theorem which was generalized using Laffey's theorem
We further saw the definition of a Radical of an Algebra and how it happened to be the intersection of all Maximal Right Ideals of the Algebra (Or similarly, Left Sided Ideals) and hence two-sided ideals We used the concept of a radical to show that a unital subalgebra of the linear operators on a vector space was triangularizable iff $\mathcal{A}/Rad\mathcal{A}$ was commutative

1 Some basic Definitions

1.1 Invariant Subspace

A subspace $\mathcal{M}$ is invariant for a collection $\mathcal{C}$ of linear transformations if $Ax \in \mathcal{M}$ whenever $A \in \mathcal{C}$

1.2 Non Trivial Subspace

A subspace different from 0 and the whole space

1.3 Reducible Subspace

If the subspace has a nontrivial invariant subspace. Irreducible subspace is defined similarly

1.4 Triangularizable Collection

If the vector space has a basis with respect to which the all the transformations have an upper triangular matrix

Triangularizability happens to be equivalent to the existence of a chain of invariant subspaces

$$\{0\} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots \mathcal{M}_n = \mathcal{V}$$

In this report, we'd be using $\triangle^l$ as a short form for triangularizable

1.5 Quotient Space

If $\mathcal{V}$ is a vector space and $\mathcal{N}$ is a subspace of $\mathcal{V}$, then the quotient space $\mathcal{V}/\mathcal{N}$ is the collection of cosets $[x] = x + \mathcal{N} = x + z : z \in \mathcal{N}$ for $x \in V$, with $[x] + [y]$ defined as $[x + y]$ and $\lambda[x]$ defined as $[\lambda x]$ for scalars λ. If A is a linear transformation on V and N is invariant under A,then the quotient transformation $\tilde{A}$ on $\mathcal{V}/\mathcal{N}$ is defined by $\tilde{A}$[x] = [Ax] for each x E V (the invariance of N under A ensures that $\tilde{A}$ is well-defined on the cosets). If $\mathcal{C}$ is a collection of linear transformations on $\mathcal{V}$, and if $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces for $\mathcal{C}$ with $\mathcal{N} \subset \mathcal{M}$, then the collection of quotients of $\mathcal{C}$ with respect to $\{\mathcal{M}, \mathcal{N}\}$ is the set of all quotient transformations A on $\mathcal{M}/\mathcal{N}$. A property is inherited by quotients if every collection of quotients of a collection satisfying the property also satisfies the property. (Note that this implies, in particular, that the property is inherited by restrictions since a restriction to $\mathcal{M}$ is a quotient with respect to $\{\mathcal{M}, 0\}$)

1.6 Non Commutative Polynomial

A non commutative polynomial p in $\{A_1, \dots A_k\}$ is any linear combination of words in the transformations

For example, for $\{A_1, A_2\}$, $A_1A_2 + A_2A_1A_2$ is a non commutative polynomial

1.7 Quasi nilpotent matrix

A quasinilpotent matrix is one such that $\sigma(A) \subseteq \{0\}$

The term was taken to ease defining in the case where $\mathcal{A}$ is a banach algebra. There $\sigma(A) = \{0\}$ iff $\lim_{n\to\infty} \|A^n\|^{1/n} = 0$ from the spectral radius theorem, which says the following

Spectral Radius Theorem

Theorem: For every $A \in \mathcal{B}(\mathcal{V})$, the spectral radius, denoted by $\rho(A)$ and defined as $sup\{|\lambda| : \lambda \in \sigma(A)\}$ is given by

$$\rho(A) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$$

1.8 Algebra

Any collection of linear transformations is called an Algebra if it's closed under addition, multiplication and multiplication by scalars
A **unital** Algebra has the identity operator as well
We'll use $\mathcal{B}(\mathcal{V})$ to denote algebras from $\mathcal{V}$ to $\mathcal{V}$

1.9 Radicals of an Algebra

The radical of an Algebra $\mathcal{A}$ is defined as

$$\{A \in \mathcal{A} : \sigma(AB) \subseteq \{0\}\ \forall B \in \mathcal{A}\}$$

From the lemma 2.6, we get the result

$$Rad\mathcal{A} = \{A \in \mathcal{A} : \sigma(BA) \subseteq \{0\}\ \forall B \in \mathcal{A}\}$$

2 Some of the Theorems/Lemmas Used

2.1 The Triangularization Lemma

This Lemma would be used several times in the report
Lemma: Let $\mathcal{P}$ be a set of properties, each of which is inherited by quotients. Then every collection of transformations on a space of dimension greater than 1 that satisfies $\mathcal{P}$ is reducible, then every collection of transformations satisfying $\mathcal{P}$ is $\triangle^l$.
Another way of putting this would be Reducible$\Rightarrow$ $\triangle^{able}$

Proof. Zorn's Lemma □

Claim: There exists a $\lambda \in \mathbb{C}$ s.t $(T_0A_0 - \lambda)$ is not invertible

Proof. □

2.2 Spectral Mapping Theorem

Theorem: If $\{A_1, \ldots A_k\}$ is a $\triangle^{able}$ collection of linear transformations and p is any non-commutative polynomial in $\{A_1, \ldots A_k\}$, then

$$\sigma(p(A_1, \ldots A_k)) \subset p(\sigma(A_1), \ldots \sigma(A_k))$$

where $p(\sigma(A_1), \ldots \sigma(A_k))$ denotes the set of all $p(\lambda_1, \ldots, \lambda_k)$ s.t $\lambda_j \in \sigma(A_j) \forall j$

Proof. Since the collection is simultaneously $\triangle^{able}$, we'll have the eigenvalue entries on the main diagonal

Also, any polynomial in an upper triangular collection will give the same polynomial as the entries on the diagonal, that is $p(A_1, \dots A_k) =$

$$p\left(\begin{bmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{n1} \end{bmatrix}, \dots, \begin{bmatrix} a_{1k} & & \\ & \ddots & \\ 0 & & a_{nk} \end{bmatrix}\right) = \begin{bmatrix} p(a_{11}, \dots, a_{1k}) & & \\ & \ddots & \\ 0 & & p(a_{1k}, \dots, a_{nk}) \end{bmatrix}$$

□

2.3 Burnside Theorem

Theorem: The only irreducible algebra of linear transformations of a finite-dimension greater than 1 is the algebra of all linear transformations
In other words, every proper subalgebra of $\mathcal{B}(\mathcal{V})$ is reducible
We won't be proving this in the report, but shall use it

2.4 McCoy's Theorem

Before McCoy's, we'll need the following theorem:-
Theorem: An algebra of linear transformations $\mathcal{A}$ is $\triangle^{able}$ iff each commutator of the form $BC - CB$ with B and C in the algebra is nilpotent
This theorem happens to illustrate the fact that Triangularization is a generalization of commutativity
Now we'll state McCoy's
Theorem: The pair $\triangle^{able}$ iff $p(a, b)(AB - BA)$ is nilpotent for every non-commutative polynomial p

Proof. ($\Rightarrow$) Trivial
($\Leftarrow$) From the $\triangle^n$ lemma, it suffices to prove that nilpotency of the above expression would imply $\{A, B\}$ is reducible (on spaces of dimension greater than 1)
CASE I :$AB = BA$
From $\triangle^n$ lemma, it shows that the collection will be $\triangle^{able}$
CASE II:
Let $(AB - BA)x \neq 0$, choose a linear transformation C (Not necessarily from the Algebra) s.t $C(AB - BA)x = x$. If $\mathcal{A}$ wasn't reducible, $C \in \mathcal{A}$, but then $C(AB - BA)x = x$ is not nilpotent

$$\Rightarrow\Leftarrow$$

Hence $\mathcal{A}$ is reducible □

2.5 Laffey's Theorem

Theorem: If $AB - BA$ has rank 1, then $\{A, B\}$ is $\triangle^{able}$ This was used to sharpen McCoy's Theorem with the following Corollary:-

Corollary: A collection $\mathcal{E}$ of operators is $\triangle^{able}$ iff it has the property that for any integer m and members $R_1, \ldots, R_m, S, T$, the operator $R_1R_2 \ldots R_m(ST - TS)$ is nilpotent

2.6 On Spectrum of Matrices

Lemma: For A and B in a unital Algebra $\mathcal{A}$, $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$
In particular $\sigma(AB) \subseteq \{0\} \Leftrightarrow \sigma(AB) \subseteq \{0\}$

Proof. For this we show that $\lambda - AB$ is invertible implies $\lambda - BA$ is invertible too, given $\lambda \neq 0$
So if $(\lambda - AB)^{-1}$ exists, then

$$\begin{aligned}
(\lambda - BA)&\left(\frac{1}{\lambda}B(\lambda - (AB)^{-1}A + \frac{1}{\lambda}\right) \\
&= B(\lambda - (AB)^{-1}A + \frac{1}{\lambda}BAB(\lambda - AB)^{-1}A + 1 - \frac{1}{\lambda}BA \\
&= B\left((1 - \frac{1}{\lambda}AB)(\lambda - AB)^{-1}A\right) + 1 - \frac{1}{\lambda}BA \\
&= 1
\end{aligned}$$

Similar steps are involved in showing $\left(\frac{1}{\lambda}B(\lambda - (AB)^{-1}A + \frac{1}{\lambda}\right)(\lambda - BA) = 1$ So $\lambda - BA$ is invertible $\square$

3 Attempted Questions

3.1 Example for which $\sigma(AB) \neq \sigma(BA)$

Consider the unilateral shift matrix in the infinite dimension space, where the only non-zero entries are in the super diagonal and are equal to 1, i.e.

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

Then

$$UU^T = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & \vdots & \\ \vdots & \vdots & & \ddots & \end{bmatrix}$$

Hence $\sigma(UU^T) = \{1\}$
However Then

$$U^T U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

Hence $\sigma(UU^T) = \{0, 1\}$
So it can be seen that $\sigma(AB) \neq \sigma(BA)$ in this case

3.2 Algebra of nilpotent matrices

Claim: If a finite operator has the property $\sigma(A) = \{0\}$, then A is nilpotent

Proof. ($\Leftarrow$) Let the characteristic polynomial.
Since we know from the Cayley Hamilton Theorem that a matrix satisfies its characteristic polynomial and for an operator A with only 0 as an eigenvalue, the characteristic equation would be

$$\chi_A = x^n \text{ for some n}$$

Hence putting A in the equation, we get $A^n = 0$, hence A is nilpotent ☐

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- Simultaneous Triangularization by Heydar Radjavi, Peter Rosenthal
- http://math.stackexchange.com