Seminar on Moduli Theory

What to expect (and not to)

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What is it?

Moduli theory tries to construct nice parameter spaces for geometric objects or *families*.
For example, a classical question of this kind is:

Question

What is the parameter space of all lines passing through the origin in the complex plane $\mathbb{A}^2_{\mathbb{C}}$*?*

We can ask this question for any collection of geometric objects - like flags of a vector space, *n* points in space, curves of a given genus, vector bundles on said curves, etc.

Food for Thought

A more interesting version of the above problem is the following:

Question

What is the parameter space of all lines in $\mathbb{A}^2_{\mathbb{C}}$?

Enter Grothendieck..

This is the point category theory meets algebraic geometry. Grothendieck realised that the correct setting to think about moduli problems is the language of functors.

Definition

A moduli problem is a functor $F : Sch^{op} \rightarrow Sets$ on the category of schemes to sets.

If there exists a scheme X and a natural isomorphism $F \simeq Hom(\_, X)$ then we say that the moduli problem is representable.

What we hope to achieve

The aim of this seminar is to develop sufficient algebraic geometry in order to prove the following theorem:

Theorem

Let X be projective over a noetherian base S. Consider the functor which for any scheme T is given by

$$Hilb_{X/S}(T) = \left\{ closed\, Y \subset T \times_S X \,\middle|\, Y \text{ is flat and proper over } T \right\}.$$

Then, $Hilb_{X/S}$ is representable by a scheme which is projective over S.

This is known as the Hilbert scheme of X over S. This is an important object in modern moduli theory, since representability of many moduli functors can reduced to showing that they define certain locally closed subsets of the Hilbert scheme.

What We Will Definitely Do

1. Theory of Hilbert(-Samuel) polynomials (I'm specifically including this because I don't understand this very well).
2. Basics of Grothendick topologies and descent. (Some of you may already be familiar with this, but a recap might be a good idea).
3. Show that the Hilbert scheme of points is representable. There is a nice section in the Stacks project on this.
4. Prove the above theorem.

This is a rough list and things may get added/modified as we get into the details.

What We May Eventually Do

These things are contingent on factors of time and interest.

- Worry about removing the projectivity and Noetherian hypothesis in above theorem.

The most general statement about the Hilbert functor can be stated for finitely presented morphisms of algebraic spaces.