Seminar on Moduli Theory

Lecture 16

Neeraj Deshmukh

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Last Week

1. Hironaka's examples
2. Hilbert scheme of a proper scheme need not be representable

Castelnuovo-Mumford regularity

Definition

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Let m be an integer. $\mathcal{F}$ is said to be *m-regular* if we have

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0 \text{ for each } i \geq 1.$$

Lemma (Castelnuovo)

Let $\mathcal{F}$ be a m-regular on $\mathbb{P}^n_k$. Then the following statements hold:

1. *The canonical map* $H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n_k, \mathcal{F}(r)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(r+1))$ *is surjective whenever* $r \geq m$.
2. $H^i(\mathbb{P}^n_k, \mathcal{F}(r)) = 0$ *whenever* $i \geq 1$ *and* $r \geq m - i$. *That is, if* $\mathcal{F}$ *is m-regular then it also* m'*-regular for all* $m' \geq m$.
3. *The sheaf* $\mathcal{F}(r)$ *is generated by global sections, and all its higher cohomologies vanish, whenever* $r \geq m$.

(2) $H^i(\mathbb{P}^n_k, \mathcal{F}(r)) = 0$ whenever $i \geq 1$ and $r \geq m - i$. That is, if $\mathcal{F}$ is m-regular then it also m'-regular for all $m' \geq m$.

(1) The canonical map
$H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n_k, \mathcal{F}(r)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(r+1))$ is surjective whenever $r \geq m$.

The sheaf $\mathcal{F}(r)$ is generated by global sections, and all its higher cohomologies vanish, whenever $r \geq m$.

Theorem (Mumford)

Given any non-negative integers p and n, there exists a polynomial $F_{p,n}$ in $n+1$-variables with the following property:
If $\mathcal{F} \subset \bigoplus^p \mathcal{O}_{\mathbb{P}^n_k}$ is any coherent subsheaf with Hilbert polynomial

$$\chi(\mathcal{F}, r) = \sum_{i=0}^{n} a_i \binom{r}{i},$$

then $\mathcal{F}$ is $F_{p,n}(a_0, \ldots, a_n)$-regular.

Idea of Proof