

Seminar on Moduli Theory

Lecture 1

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For the sake of completeness we begin by reviewing the definition of a locally ringed space.

Definition

Locally ringed spaces.

- 1 A *locally ringed space* (X, \mathcal{O}_X) is a pair consisting of a topological space X and a sheaf of rings \mathcal{O}_X all of whose stalks are local rings.
- 2 Given a locally ringed space (X, \mathcal{O}_X) we say that $\mathcal{O}_{X,x}$ is the *local ring of X at x* . We denote $\mathfrak{m}_{X,x}$ or simply \mathfrak{m}_x the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of X at x* is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
- 3 A *morphism of locally ringed spaces* $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is a local ring map.

Tilde construction

We know that affine schemes are locally ringed spaces.

Definition

A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted Sch .

Definition

Let (X, \mathcal{O}_X) be a scheme. A sheaf of modules on X is a sheaf \mathcal{F} on X such that for every open set U , $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module. We say that a sheaf of modules \mathcal{F} is *quasi-coherent* if for every affine open $U \simeq \operatorname{Spec}(R)$, the sheaf $\mathcal{F}|_U$ on U is of the form \tilde{M} for some R -module M .

Affine Communication Lemma

Lemma

Let X be a scheme. Let P be a local property of rings. The following are equivalent:

- ① *The scheme X is locally P .*
- ② *For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.*
- ③ *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P .*
- ④ *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P .*

Moreover, if X is locally P then every open subscheme is locally P .

Examples

Two ways of gluing $\mathbb{A}^1 \setminus \{0\}$: $x \mapsto x$ or $x \mapsto 1/x$.

- 1 Double origin: What are global sections? what are quasi-coherent sheaves?
- 2 \mathbb{P}^1 : What are global sections?

More examples

A normal scheme, a reduced scheme and a Noetherian scheme.

More Examples

Something non-noetherian: $\operatorname{Spec} R[x_1, x_2, \dots]$.

A slightly more involved scheme: \mathbb{P}^n .

Let $D(x_i) := k[x_{0/i}, x_{1/i}, \dots, x_{\hat{i}/i}, \dots, x_{n/i}]$. We have a maps $\phi_{ij} : D(x_i)_{x_j} \rightarrow D(x_j)_{x_i}$, given by $x_{k/i} \mapsto x_{k/j}$.

Line Bundles on \mathbb{P}^1

Locally on an affine open, this should be a free module of rank one. Let's construct one such line bundle (non-trivial, of course).

More examples

- 1 $V_+(x^2 + y^2 + z^2)$ over \mathbb{R} and \mathbb{C} .
- 2 Blow-up of \mathbb{A}^2 at the origin.
- 3 An example of a scheme without a closed point.

Definition

Let \mathcal{P} be a property of morphisms of schemes. Let $f : X \rightarrow Y$ be a morphism which satisfies \mathcal{P} . Then,

- 1 We say that \mathcal{P} is *affine-local on the target* if given any affine open cover $\{V_i\}$ of Y , $f : X \rightarrow Y$ has \mathcal{P} if and only if the restriction $f : f^{-1}(V_i) \rightarrow V_i$ has \mathcal{P} for each i .
- 2 We say that \mathcal{P} is *affine-local on the source* if given any affine open cover $\{U_i\}$ of X , $X \rightarrow Y$ has \mathcal{P} if and only if the composite $U_i \rightarrow Y$ has \mathcal{P} for each i .

Using *affine communication lemma* one can then show that it suffices to check the above statements on single affine open cover.

Something flat, something finitely presented/finite type, something finite.

Not all properties are like this. For example, separatedness, properness, quasi-compactness, etc.

Something ramified, something smooth, something singular.