Seminar on Moduli Theory

Lecture 4

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Last Week

1. Forms of $\mathbb{P}^1$.
2. Blow-ups.
3. Scheme without a closed point.

Morphisms

Definition

Let $\mathcal{P}$ be a property of morphisms of schemes. Let $f : X \to Y$ be a morphism which satisfies $\mathcal{P}$. Then,

1. We say that $\mathcal{P}$ is *affine-local on the target* if given any affine open cover $\{V_i\}$ of Y, $f : X \to Y$ has $\mathcal{P}$ if and only if the restriction $f : f^{-1}(V_i) \to V_i$ has $\mathcal{P}$ for each i.
2. We say that $\mathcal{P}$ is *affine-local on the source* if given any affine open cover $\{U_i\}$ of X, $X \to Y$ has $\mathcal{P}$ if and only if the composite $U_i \to Y$ has $\mathcal{P}$ for each i.

Using *affine communication lemma* one can then show that it suffices to check the above statements on single affine open cover.

Something flat, something finitely presented/finite type, something finite.

Not all properties are like this. For example, separatedness, properness, quasi-compactness, etc.

Something ramified, something smooth, something singular.

Open $\Rightarrow$ étale $\Rightarrow$ fppf $\Rightarrow$ fpqc

What are morphisms to $\mathbb{P}^n$?

Amitsur's Lemma

Let $f : A \to B$ be a faithfully flat ring map. Then, the following sequence of A-modules is exact:

$$0 \to A \xrightarrow{f} B \xrightarrow{e_1 - e_2} B \otimes_A B$$