Seminar on Moduli Theory

Lecture 19

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Last Time

1. Proof of representability for $X \hookrightarrow \mathbb{P}(V)$

Theorem (Altman-Kleiman)

Let S be a Noetherian scheme and V a vector bundle on S. Let $\pi : X \to S$ be a closed subscheme of $\mathbb{P}(V)$ and $L := \mathcal{O}_{\mathbb{P}(V)}|_X$. Let W be a vector bundle on S and ν an integer. Consider a coherent quotient $\pi^*(W)(\nu) \to E$. Then for any $\Phi \in \mathbb{Q}[\lambda]$, the quot functor $\mathfrak{Quot}^{\Phi}_{E/X/S}$ is representable by a scheme.

Moreover, this scheme can be embedded in $\mathbb{P}(F)$ for some vector bundle F over S.

Steps:

1. Reduction to the case of $\mathfrak{Quot}^{\Phi}_{\pi * W/\mathbb{P}(V)/S}$
2. Use of m-regularity
3. Use of semi-continuity
4. Functor to Grassmannian
5. Use of flattening stratification

Question: What is the Flattening stratification for the coherent sheaf $\mathcal{O}_{X_{red}}$ on X?

Properness of $\mathfrak{Quot}^{\Phi}_{\pi * W/\mathbb{P}(V)/S}$

Theorem (Grothendieck)

Let $\pi : X \to S$ be a projective morphism with S Noetherian. Then for any coherent sheaf E on X and any polynomial $\phi \in \mathbb{Q}[t]$, the functor $\mathfrak{Quot}^{\phi(t)}_{E/X/S}$ is representable by a projective S-scheme.

The quasi-projective case

Fact: If $Z \to S$ is a projective morphism with S Noetherian, and if $X \subset Z$ is an open subset, then $\mathfrak{Quot}_{E_X/X/S}$ is an open subfunctor of $\mathfrak{Quot}_{E/Z/S}$

The quasi-projective case