Seminar on Moduli Theory

Lecture 2

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Last Week

1. Affine Communication Lemma.
2. Two ways of gluing $\mathbb{A}^1$ outside the origin.
3. DVR with a double origin, and sheaves on it.
4. Line bundles on $\mathbb{P}^1$.

A slightly more involved scheme: $\mathbb{P}^n$

Let $D(x_i) := \operatorname{Spec} k[x_{0/i}, x_{1/i}, \ldots, \ldots, x_{n/i}]/(x_{i/i} - 1)$. If we invert one of the variables, say $x_{j/i}$, we can write an isomorphism $D(x_i)_{x_{j/i}} \cong D(x_j)_{x_{i/j}}$ given by the maps

$$\phi_{ij} : x_{k/i} \mapsto x_{k/j}/x_{i/j} \ \ \& \ \ \phi_{ji} : x_{k/j} \mapsto x_{k/i}/x_{j/i}$$

A classical interlude

Let k be a field. Consider $k^{n+1} \setminus (0, 0, \ldots, 0)$. We define $\mathbb{P}^n$ to be:

$$\mathbb{P}^n := \{(x_i) \mid (x_i) \simeq (y_i) \text{ if there is a } \lambda \in k^\times \text{ such that } x_i = \lambda y_i\}$$

Motivating $\mathbb{P}roj$

Consider $k[x_0, x_1, \ldots, x_n]$, now thought of as a graded ring with the grading given by degrees of monomials.
$\mathbb{P}roj(k[x_0, x_1, \ldots, x_n])$ is the set of those homogeneous prime ideals which do not contain the ideal $(x_0, x_1, \ldots, x_n)$. The resulting scheme is $\mathbb{P}^n$.

Here's an alternative description of $\mathbb{P}^1$ using degree 2 hypersufaces

Relation to our original construction of $\mathbb{P}^1$

Caution: While using affine opens corresponding to degree 2 element, it may seem like it should be sufficient to just invert x^2, and y^2 to get an open cover of $\mathbb{P}^1$. This is false!

More Examples

$V_+(x^2 + y^2 + z^2)$ over $\mathbb{R}$ and $\mathbb{C}$.

More Examples

Blow-up of $\mathbb{A}^2$ at the origin.

More Examples

An example of a scheme without a closed point.

Morphisms

Definition

Let $\mathcal{P}$ be a property of morphisms of schemes. Let $f : X \to Y$ be a morphism which satisfies $\mathcal{P}$. Then,

1. We say that $\mathcal{P}$ is *affine-local on the target* if given any affine open cover $\{V_i\}$ of Y, $f : X \to Y$ has $\mathcal{P}$ if and only if the restriction $f : f^{-1}(V_i) \to V_i$ has $\mathcal{P}$ for each i.
2. We say that $\mathcal{P}$ is *affine-local on the source* if given any affine open cover $\{U_i\}$ of X, $X \to Y$ has $\mathcal{P}$ if and only if the composite $U_i \to Y$ has $\mathcal{P}$ for each i.

Using *affine communication lemma* one can then show that it suffices to check the above statements on single affine open cover.

Something flat, something finitely presented/finite type, something finite.

Not all properties are like this. For example, separatedness, properness, quasi-compactness, etc.

Something ramified, something smooth, something singular.