Seminar on Moduli Theory

Lecture 6

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October 2, 2020

Last Week

1. Presheaves, and Yoneda Lemma.
2. Sites and sheaves.
3. Topologies on $(Ring)^{opp}$ and Sch.

Lemma

We have the follow inclusion of topologies:

$$\textit{Zariski} \subset \textit{Étale} \subset \textit{fppf} \subset \textit{fpqc}.$$

Defining Schemes without locally ringed spaces.

Representable morphisms of functors. (Morphisms of representable functors are always representable!)

A useful lemma about the diagonal

Lemma

Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:

1. *the diagonal $\Delta : F \to F \times F$ is representable,*
2. *for every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable,*
3. *for every pair U, V in $\mathcal{C}$ and any $\xi \in F(U)$, $\xi' \in F(V)$ the fibre product $h_U \times_{\xi, F, \xi'} h_V$ is representable.*

Schemes (with affine diagonal)

Schemes

Characterising fpqc sheaf property

Lemma

Let $F : Sch \to Sets$ be a presheaf. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies

1. *the sheaf property for every Zariski covering, and*
2. *the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.*

Characterising fpqc sheaf property

Theorem (Grothendieck)

Every representable functor satisfies the sheaf property in the fpqc topology.

Amitsur's Lemma

Let $f : A \to B$ be a faithfully flat ring map. Then, the following sequence of A-modules is exact:

$$0 \to A \xrightarrow{f} B \xrightarrow{e_1 - e_2} B \otimes_A B$$

What happens at $B \otimes_A B$?

subcanonical sites and a non-canonical site