Seminar on Moduli Theory

Lecture 9

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Last Week

1. Integer-valued polynomials and polynomial-like functions
2. Hilbert function and examples

We want to prove the following theorem about Hilbert polynomials:

Theorem

Let $X \to Y$ be a projective morphism with Y locally Noetherian. If $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then the Hilbert polynomial $\chi(\mathcal{F}_{X_y}, d)$ is locally constant for $y \in Y$.

Theorem (Serre Vanishing)

Let A be a Noetherian ring. Let $\mathcal{F}$ be a coherent sheaf on a projective A-scheme X. Then, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$.

Relative Serre Vanishing

Lemma

Suppose we have a commutative diagram of schemes,

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* (g')^* \mathcal{F}$$

Lemma (Flat base change)

Consider a cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow_{f'} & & \downarrow_{f} \\ S' & \xrightarrow{g} & S \end{array}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{F}' = (g')^*\mathcal{F}$. Assume that g is flat and that f is quasi-compact and quasi-separated. For any $i \geq 0$

1. *the base change map, $g^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* \mathcal{F}'$ is an isomorphism*
2. *if $S = \mathrm{Spec}\,(A)$ and $S' = \mathrm{Spec}\,(B)$, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(X', \mathcal{F}')$.*

Case 1: X is separated

Case 2: X is quasi-separated

Theorem

Let $X \to Y$ be a projective morphism with Y locally Noetherian. If $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then the Hilbert polynomial $\chi(\mathcal{F}_{X_y}, d)$ is locally constant for $y \in Y$.

Lemma

Let $S = \operatorname{Spec} A$ be a Noetherian local ring. Let $\mathcal{F}$ be a coherent sheaf on $X = \mathbb{P}^n_S$. Consider the following statements:

1. *$\mathcal{F}$ is flat over S;*
2. *$H^0(X, \mathcal{F}(m))$ is a free A-module of finite rank, for all $m \gg 0$;*
3. *for any $t \in S$, the Hilbert polynomial $\chi(\mathcal{F}_t, m)$ of $\mathcal{F}_t$ on X_t is independent of t.*

Then we have the implications, (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). Moreover, if S is domain then they are all equivalent.

$(1) \Leftrightarrow (2)$

$(2) \Rightarrow (3)$