Seminar on Moduli Theory

Lecture 17

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Last Time

1. Castelnuovo-Mumford regularity

Castelnuovo Mumford regularity

Definition

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Let m be an integer. $\mathcal{F}$ is said to be *m-regular* if we have

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0 \text{ for each } i \geq 1.$$

Theorem (Mumford)

Given any non-negative integers p and n, there exists a polynomial $F_{p,n}$ in $n+1$-variables with the following property:
If $\mathcal{F} \subset \bigoplus^p \mathcal{O}_{\mathbb{P}^n_k}$ is any coherent subsheaf with Hilbert polynomial

$$\chi(\mathcal{F}, r) = \sum_{i=0}^{n} a_i \binom{r}{i},$$

then $\mathcal{F}$ is $F_{p,n}(a_0, \ldots, a_n)$-regular.

Theorem (Grothendieck)

Let $\pi : X \to S$ be a projective morphism with S Noetherian. Then for any coherent sheaf E on X and any polynomial $\phi \in \mathbb{Q}[t]$, the functor $\mathfrak{Quot}^{\phi(t)}_{E/X/S}$ is representable by a projective S-scheme.

Three key ingredients:

(1)

(2)

(3)

Flattening stratification

Theorem

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$ with S a Noetherian scheme. Then the set I of Hilbert polynomials of $\mathcal{F}$ on the fibers of $\mathbb{P}^n_S \to S$ is a finite set. Moreover, for each $f \in I$, there exists a locally closed subscheme $S_f \subset S$ such that the following conditions hold:

1. $|S| = \coprod_f |S_f|$, *set-theoretically;*
2. *Fix an $f \in \mathbb{Q}[\lambda]$. For any morphism $\phi : T \to S$ the pullback $\phi^*\mathcal{F}$ is flat on $\mathbb{P}^n_T$ with Hilbert polynomial f if and only if $\phi : T \to S$ factors through S_f.*

Special case when $n = 0$

Behaviour of the fibres $\mathcal{F}_s$ via generic flatness

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Applying special case to get stratification for pushforward sheaves

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The correct subscheme structure on $W_{e_0,\ldots,e_n}$