Seminar on Moduli Theory

Lecture 1

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For the sake of completeness we begin by reviewing the definition of a locally ringed space.

Definition

Locally ringed spaces.

1. A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.
2. Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the *local ring of* X *at* x. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of* X *at* x is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
3. A *morphism of locally ringed spaces* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

Tilde construction

We know that affine schemes are locally ringed spaces.

Definition

A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Definition

Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of modules on X is a sheaf $\mathcal{F}$ on X such that for every open set U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module. We say that a sheaf of modules $\mathcal{F}$ is *quasi-coherent* if for every affine open $U \simeq \mathrm{Spec}\,(R)$, the sheaf $\mathcal{F}|_U$ on U is of the form $\widetilde{M}$ for some R-module M.

Affine Communication Lemma

Lemma

Let X be a scheme. Let P be a local property of rings. The following are equivalent:

1. *The scheme X is locally P.*
2. *For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P.*

Moreover, if X is locally P then every open subscheme is locally P.

Examples

Two ways of gluing $\mathbb{A}^1 \setminus \{0\}$: $x \mapsto x$ or $x \mapsto 1/x$.

1. Double origin: What are global sections? what are quasi-coherent sheaves?
2. $\mathbb{P}^1$: What are global sections?

More examples

A normal scheme, a reduced scheme and a Noetherian scheme.

More Examples

Something non-noetherian: $\operatorname{Spec} R[x_1, x_2, \ldots]$.

A slightly more involved scheme: $\mathbb{P}^n$.

Let $D(x_i) := k[x_{0/i}, x_{1/i}, \ldots, \hat{x_{i/i}}, \ldots, x_{n/i}]$. We have a maps $\phi_{ij} : D(x_i)_{x_j} \to D(x_j)_{x_i}$, given by $x_{k/i} \mapsto x_{k/j}$.

Line Bundles on $\mathbb{P}^1$

Locally on an affine open, this should be a free module of rank one. Let's contruct one such line bundle (non-trivial, of course).

More examples

1. $V_+(x^2 + y^2 + z^2)$ over $\mathbb{R}$ and $\mathbb{C}$.
2. Blow-up of $\mathbb{A}^2$ at the origin.
3. An example of a scheme without a closed point.

Definition

Let $\mathcal{P}$ be a property of morphisms of schemes. Let $f : X \to Y$ be a morphism which satisfies $\mathcal{P}$. Then,

1. We say that $\mathcal{P}$ is *affine-local on the target* if given any affine open cover $\{V_i\}$ of Y, $f : X \to Y$ has $\mathcal{P}$ if and only if the restriction $f : f^{-1}(V_i) \to V_i$ has $\mathcal{P}$ for each i.
2. We say that $\mathcal{P}$ is *affine-local on the source* if given any affine open cover $\{U_i\}$ of X, $X \to Y$ has $\mathcal{P}$ if and only if the composite $U_i \to Y$ has $\mathcal{P}$ for each i.

Using *affine communication lemma* one can then show that it suffices to check the above statements on single affine open cover.

Something flat, something finitely presented/finite type, something finite.

Not all properties are like this. For example, separatedness, properness, quasi-compactness, etc.

Something ramified, something smooth, something singular.