Seminar on Moduli Theory

Lecture 5

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Last Week

1. "Affine Communication" for morphisms.
2. Various kinds of morphisms.
3. Morphisms to $\mathbb{P}^n$.

A *presheaf* is a functor. Representable functors and Yoneda.

Three representable and one non-representable functors of *Sch*

Representable morphisms of functors. (Morphisms of representable functors are always representable!)

A useful lemma about the diagonal

Lemma

Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:

1. *the diagonal $\Delta : F \to F \times F$ is representable,*
2. *for every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable,*
3. *for every pair U, V in $\mathcal{C}$ and any $\xi \in F(U)$, $\xi' \in F(V)$ the fibre product $h_U \times_{\xi, F, \xi'} h_V$ is representable.*

We will now discuss some "sheafy jargon".

Definition

Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in \mathrm{Ob}(\mathcal{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i \in I}$ to indicate this.

Definition

A *site* is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i \in I}$, called *coverings of* $\mathcal{C}$, satisfying the following axioms

1. If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
2. If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$.
3. If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

The sheaf condition and the category of sheaves

Various topologies on Affine schemes.

Zariski:

Étale:

fppf :

fpqc:

Various toplogies on schemes

Zariski, étale and fppf are defined almost analogously for schemes.

Kleiman's trick for fpqc morphisms:

Lemma

We have the follow inclusion of topologies:

$$\text{Zariski} \subset \text{Étale} \subset \text{fppf} \subset \text{fpqc}.$$

Why bother with the affine site?

Separated Schemes

Schemes

Characterising fpqc sheaf property

Lemma

Let $F : Sch \to Sets$ be a presheaf. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies

1. *the sheaf property for every Zariski covering, and*
2. *the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.*

Characterising fpqc sheaf property

Theorem (Grothendieck)

Every representable functor satisfies the sheaf property in the fpqc topology.

Amitsur's Lemma

Let $f : A \to B$ be a faithfully flat ring map. Then, the following sequence of A-modules is exact:

$$0 \to A \xrightarrow{f} B \xrightarrow{e_1 - e_2} B \otimes_A B$$

What happens at $B \otimes_A B$?

subcanonical sites and a non-canonical site