SEMINAR ON MODULI THEORY

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These are notes for the first few lectures. The aim is to gather sufficient algebraic geometry background for discussing moduli theory. A lot of LaTeX code in this document has been shamelessly copied from the stacks project repository on GitHub.

1. Schemes

For the sake of completeness we begin by reviewing the definition of a locally ringed space.

Definition 1.1. Locally ringed spaces.

(1) A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.
(2) Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the *local ring of X at x*. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of X at x* is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
(3) A *morphism of locally ringed spaces* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We know that affine schemes are locally ringed spaces: we take $\operatorname{Spec} R$ with the zariski topology and for any principal open set $D(f)$ we assign the ring R_f. So, any ring R produces the sheaf $\widetilde{R}$ on $\operatorname{Spec} R$. This is called the tilde construction. (sanity check: if you can do this, then you should be able to construct a sheaf on $\operatorname{Spec} R$ for any R-module M).

Definition 1.2. A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Definition 1.3. Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of modules on X is a sheaf $\mathcal{F}$ on X such that for every open set U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module. We say that a sheaf of modules $\mathcal{F}$ is *quasi-coherent* if for every affine open $U \simeq \operatorname{Spec}(R)$, the sheaf $\mathcal{F}|_U$ on U is of the form $\widetilde{M}$ for some R-module M.

Make special note of the next lemma. This basically lets us reduce problems about schemes to statement about affine schemes (ergo, ring theory), whenever the problem at hand is of a *local* nature. Ravi Vakil calls this *affine communication lemma.*

Lemma 1.4. Let X be a scheme. Let P be a local property of rings. The following are equivalent:

(1) The scheme X is locally P.
(2) For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.

(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P.

Moreover, if X is locally P then every open subscheme is locally P.

This is how commutative algebra meets geometry. Often, the properties that we want to consider are "globalised" versions of statements about rings. Some examples, normality, reduced, etc.[1]

1.1. **Two ways of Gluing $\mathbb{A}^1 \setminus \{0\}$.** Take two copies of $\mathbb{A}^1 := \operatorname{Spec} k[x]$[2]. Let $U := \operatorname{Spec} k[x, 1/x]$, be the complement of the origin in $\mathbb{A}^1$.

$$
\begin{tikzcd}
\mathbb{A}^1 & \mathbb{A}^1 \\
U \arrow[u, hook] \arrow[r, "\sim"] & U \arrow[u, hook]
\end{tikzcd}
$$

Giving this information is that same giving a scheme which is looks like $\mathbb{A}^1$ around every point (why?). We consider two possible choices for the identification on U:

$$
\begin{aligned}
x &\mapsto x \\
x &\mapsto 1/x
\end{aligned}
$$

Example 1.5. The first choice gives us a scheme which is like $\mathbb{A}^1$ everywhere except at the origin where it is now two points instead of one. Notice that the ring of global section of this scheme is $k[x]$ (a global section is same as giving polynomials $f, g \in k[x]$, one for each copy of $\mathbb{A}^1$ which are equal on U; conclude form this).

Example 1.6. The second choice gives us the pojective line $\mathbb{P}^1$. This is looks like $\mathbb{A}^1$ with "a point added at infinity". We will now compute its global sections.

Let $f, g \in \mathbb{A}^1$ be two polynomials such that $f(x) = g(1/x)$ in $k[x, 1/x]$. Then straightforward algebra shows that this can happen only when f, g are constant, i.e, $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$.

1.2. **A DVR with double origin.** Similar to $\mathbb{A}^1$ with double origin, we can glue two copies of a DVR. Let R be a discrete valuation ring. Then $\operatorname{Spec} R$ has exactly two point: the generic point (zero ideal) and the closed point (maximal ideal). The generic point is open in $\operatorname{Spec} R$ and is given by $\operatorname{Spec} K$, where K is quotient field of R. As for $\mathbb{A}^1$, the ring of global sections of a DVR with double origin is R.

Furthermore, any coherent sheaf (a quasi-coherent sheaf which is a finitely generated module on each copy of R) is determined by a pair (n, T) where n is a positive integer and $T \in Gl_n(K)$. Since R is a principal ideal domain, any finitely generated R-module M is a direct sum of its free and torsion parts. Thus, if M, N are two finitely generated R-modules,there exists an isomorphism (given by a K-linear map) $M \otimes K \simeq N \otimes K$ if and only if the rank of their free parts is the same.

All the above are examples of normal, reduced and Noetherian scheme. An example of a non-Noetherian scheme is $\operatorname{Spec} R[x_1, x_2, \ldots]$.

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[1] You can also "globalise" morphisms of rings, but now you have two schemes to work locally on. We'll do this soon.

[2] For simplicity, assume that k is field, but this is not needed

1.3. **A slightly more involved scheme:** $\mathbb{P}^n$**.** Let $D(x_i) := k[x_{0/i}, x_{1/i}, \ldots, \hat{x_{i/i}}, \ldots, x_{n/i}]$. And, we have a map $D(x_i)_{x_{j/i}} \to D(x_j)_{x_{i/j}}$ given by $x_{k/i} \mapsto x_{k/j}$[3]

If we are over an algebraically closed field, then closed points of $\mathbb{P}^n$ can be written in *homogeneous coordinates* as $[x_0 : x_1 : \ldots : x_n]$, where two such coordinates are the same if they differ by scalar multiplication.

1.4. **Line Bundles on** $\mathbb{P}^1$**:** Locally on an affine open, this should be a free module of rank one. Let's contruct one such line bundle (non-trivial, of course): There are two open sets, $D(x)$ and $D(y)$, on these our line bundle looks like $k[x]$ and $k[y]$, respectively. Now, how do they glue on $k[x, 1/x] \simeq k[y, 1/y]$? Let's use the map which sends $\phi(1) : f(x) \mapsto f(x)y$, since y is $1/x$ in this ring, we see that the global sections are linear polynomials. You construct such a map $\phi(n)$ for every power of y. That will give you degree n monomials. These line bundles are called $\mathcal{O}(n)$'s. Playing around with the algebra of the maps $\phi(n)$ a little will that these line bundles satisfy relations like $\mathcal{O}(n) \otimes \mathcal{O}(m) \simeq \mathcal{O}(m+n)$, and admit duals which are denote by $\mathcal{O}(-n)$.[4] [Discussed till here as of August 28, 2020]

1.5. **Some more examples.**

(1) $V_+(x^2 + y^2 + z^2)$ over $\mathbb{R}$ and $\mathbb{C}$. Over $\mathbb{C}$, one have the following linear change of coordinates, $(x, y, z) \mapsto (x+iy, x-iy, iz)$. Then, $(x+iy)(x-iy)-z^2 = x^2+y^2+(iz)^2$. So, this is the same as $V_+(uv - z^2)$, which is the (2-fold-)Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ given by $[x : y] \mapsto [x^2 : xy : y^2]$. Similarly, the d-fold Veronese embedding is given by $[x : y] \mapsto [x^d : x^{d-1}y : \ldots : xy^{d-1} : y^d]$.
(2) Blow-up of $\mathbb{A}^2$ at the origin. (because everyone should know about blow-ups!)
(3) An example of a scheme without a closed point.

2. Morphisms

As mentioned before, many of the properties of morphisms that we are interested in are "globalised" versions of properties of ring maps. However, we have to first say what it means for morphism of schemes to be a local property. There are three kinds of local properties: local on the source, local on the target, local on the source and target. We will say what this means now:

Definition 2.1. Let $\mathcal{P}$ be a property of morphisms of schemes. Let $f : X \to Y$ be a morphism which satisfies $\mathcal{P}$. Then,

(1) We say that $\mathcal{P}$ is *affine-local on the target* if given any affine open cover $\{V_i\}$ of Y, $f : X \to Y$ has $\mathcal{P}$ if and only if the restriction $f : f^{-1}(V_i) \to V_i$ has $\mathcal{P}$ for each i.
(2) We say that $\mathcal{P}$ is *affine-local on the source* if given any affine open cover $\{U_i\}$ of X, $X \to Y$ has $\mathcal{P}$ if and only if the composite $U_i \to Y$ has $\mathcal{P}$ for each i.

Using *affine communication lemma* one can then show that it suffices to check the above statements on single affine open cover.

An important maxim of Grothendieck was that instead of considering schemes in isolation, we should look at things relative to each other, i.e, everything should be seen as a propery of morphisms. This is mostly true: every property of schemes can be turned into a property of morphisms of schemes.

[3] If you have seen the construction of Grassmannians as smooth manifolds, the same construction also goes through in algebraic geometry.

[4] The line bundle $\mathcal{O}(1)$ is important. To say that a variety is projective, we need to show that something like $\mathcal{O}(1)$ lives on it. Actually, some lesser works, but we will come back to this later.

Examples:

(1) $x \mapsto x^2$ (more, generally x^n). This covers ramified, finitely presented, flat.
(2) A non-quasi-compact, open-immersion. Polynomial ring in infinitely many variables and knock off the origin.
(3) A finite morphism.
(4) A smooth morphism. A non-smooth morphism (nodal curve over $\mathbb{A}^1$).

Open embeddings are locally finite presentation.[5]
Open embedding is étale is fppf is fpqc

[5] This is not true in perfectoid geometry, which is quite sad.