Seminar on Moduli Theory

Lecture 18

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Last Time

1. Flattening Stratification

Flattening stratification

Theorem

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$ with S a Noetherian scheme. Then the set I of Hilbert polynomials of $\mathcal{F}$ on the fibers of $\mathbb{P}^n_S \to S$ is a finite set. Moreover, for each $f \in I$, there exists a locally closed subscheme $S_f \subset S$ such that the following conditions hold:

1. $|S| = \coprod_f |S_f|$, *set-theoretically;*
2. *Fix an $f \in \mathbb{Q}[\lambda]$. For any morphism $\phi : T \to S$ the pullback $\phi^*\mathcal{F}$ is flat on $\mathbb{P}^n_T$ with Hilbert polynomial f if and only if $\phi : T \to S$ factors through S_f.*

Semi-continuity theorem

Theorem

Let $\pi : X \to S$ be a projective morphism with S Noetherian. Let $\mathcal{F}$ be a coherent sheaf on X which is flat over $\mathcal{O}_S$. Then the following hold:

1. *The function $s \mapsto h^i(X_s, \mathcal{F}_s)$ is upper semi-continuous on S.*
2. *The function $s \mapsto \sum_i (-1)^i h^i(X_s, \mathcal{F}_s)$ is locally constant.*
3. *If for some integer i, there is some $d \geq 0$ such that $h^i(X_s, \mathcal{F}_s) = d$ for all $s \in S$, then $R^i\pi_*\mathcal{F}$ is locally free of rank d and $(R^{i-1}\pi_*\mathcal{F})_s \to H^{i-1}(X, \mathcal{F}_s)$ is an isomorphism for all $s \in S$.*

Semi-continuity theorem

Theorem

Let $\pi : X \to S$ be a projective morphism with S Noetherian. Let $\mathcal{F}$ be a coherent sheaf on X which is flat over $\mathcal{O}_S$. Then the following hold:

4. *If $(R^i\pi_*\mathcal{F})_s \to H^i(X, \mathcal{F}_s)$ is surjective for some integer i and some $s \in S$, then there is a open neighbourhood $U \subset S$ of s such that for any quasi-coherent sheaf $\mathcal{G}$, the natural homomorphism*

$$(R^i\pi_*\mathcal{F}_{X_U}) \otimes_{\mathcal{O}_U} \mathcal{G} \to R^i\pi_{U*}(\mathcal{F}_{X_U} \otimes_{\mathcal{O}_{X_U}} \pi_U^*\mathcal{G})$$

is an isomorphism. In particular, $(R^i\pi_*\mathcal{F})_{s'} \to H^i(X_{s'}, \mathcal{F}_{s'})$ is an isomorphism for all $s' \in U$.

Semi-continuity theorem

Theorem

Let $\pi : X \to S$ be a projective morphism with S Noetherian. Let $\mathcal{F}$ be a coherent sheaf on X which is flat over $\mathcal{O}_S$. Then the following hold:

5. *If for some integer i and some point $s \in S$, the map $(R^i\pi_*\mathcal{F})_s \to H^i(X_s, \mathcal{F}_s)$ is surjective, the following two conditions are equivalent:*
   - (a) $(R^{i-1}\pi_*\mathcal{F})_s \to H^{i-1}(X_s, \mathcal{F}_s)$ *is surjective.*
   - (b) *The sheaf $R^i\pi_*\mathcal{F}$ is locally free in a neighbourhood of $s \in S$.*

Castelnuovo Mumford regularity

Definition

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Let m be an integer. $\mathcal{F}$ is said to be *m-regular* if we have

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0 \text{ for each } i \geq 1.$$

Theorem (Mumford)

Given any non-negative integers p and n, there exists a polynomial $F_{p,n}$ in $n+1$-variables with the following property:
If $\mathcal{F} \subset \bigoplus^p \mathcal{O}_{\mathbb{P}^n_k}$ is any coherent subsheaf with Hilbert polynomial

$$\chi(\mathcal{F}, r) = \sum_{i=0}^{n} a_i \binom{r}{i},$$

then $\mathcal{F}$ is $F_{p,n}(a_0, \ldots, a_n)$-regular.

Theorem (Altman-Kleiman)

Let S be a Noetherian scheme and V a vector bundle on S. Let $\pi : X \to S$ be a closed subscheme of $\mathbb{P}(V)$ and $L := \mathcal{O}_{\mathbb{P}(V)}|_X$. Let W be a vector bundle on S and ν an integer. Consider a coherent quotient $\pi^*(W)(\nu) \to E$. Then for any $\Phi \in \mathbb{Q}[\lambda]$, the quot functor $\mathfrak{Quot}^{\Phi}_{E/X/S}$ is representable by a scheme.

Moreover, this scheme can be embedded in $\mathbb{P}(F)$ for some vector bundle F over S.

Reduction to the case of $\mathfrak{Quot}^{\Phi}_{\pi_* W/\mathbb{P}(V)/S}$

Use of m-regularity

Use of semi-continuity

Functor to Grassmannian

Use of flattening stratification

Properness