

# Seminar on Moduli Theory

## Lecture 1

Neeraj Deshmukh

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For the sake of completeness we begin by reviewing the definition of a locally ringed space.

## Definition

Locally ringed spaces.

- 1 A *locally ringed space*  $(X, \mathcal{O}_X)$  is a pair consisting of a topological space  $X$  and a sheaf of rings  $\mathcal{O}_X$  all of whose stalks are local rings.
- 2 Given a locally ringed space  $(X, \mathcal{O}_X)$  we say that  $\mathcal{O}_{X,x}$  is the *local ring of  $X$  at  $x$* . We denote  $\mathfrak{m}_{X,x}$  or simply  $\mathfrak{m}_x$  the maximal ideal of  $\mathcal{O}_{X,x}$ . Moreover, the *residue field of  $X$  at  $x$*  is the residue field  $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ .
- 3 A *morphism of locally ringed spaces*  $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a morphism of ringed spaces such that for all  $x \in X$  the induced ring map  $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$  is a local ring map.

# Tilde construction

We know that affine schemes are locally ringed spaces.

## Definition

A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted  $Sch$ .

## Definition

Let  $(X, \mathcal{O}_X)$  be a scheme. A sheaf of modules on  $X$  is a sheaf  $\mathcal{F}$  on  $X$  such that for every open set  $U$ ,  $\mathcal{F}(U)$  is an  $\mathcal{O}_X(U)$ -module. We say that a sheaf of modules  $\mathcal{F}$  is *quasi-coherent* if for every affine open  $U \simeq \operatorname{Spec}(R)$ , the sheaf  $\mathcal{F}|_U$  on  $U$  is of the form  $\widetilde{M}$  for some  $R$ -module  $M$ .

# Affine Communication Lemma

## Lemma

*Let  $X$  be a scheme. Let  $P$  be a local property of rings. The following are equivalent:*

- ① *The scheme  $X$  is locally  $P$ .*
- ② *For every affine open  $U \subset X$  the property  $P(\mathcal{O}_X(U))$  holds.*
- ③ *There exists an affine open covering  $X = \bigcup U_i$  such that each  $\mathcal{O}_X(U_i)$  satisfies  $P$ .*
- ④ *There exists an open covering  $X = \bigcup X_j$  such that each open subscheme  $X_j$  is locally  $P$ .*

*Moreover, if  $X$  is locally  $P$  then every open subscheme is locally  $P$ .*

# Examples

Two ways of gluing  $\mathbb{A}^1 \setminus \{0\}$ :  $x \mapsto x$  or  $x \mapsto 1/x$ .

- 1 Double origin: What are global sections? what are quasi-coherent sheaves?
- 2  $\mathbb{P}^1$ : What are global sections?

# More examples

A normal scheme, a reduced scheme and a Noetherian scheme.



## More Examples

Something non-noetherian:  $\text{Spec } R[x_1, x_2, \dots]$ .

# Line Bundles on $\mathbb{P}^1$

Locally on an affine open, this should be a free module of rank one. Let's construct one such line bundle (non-trivial, of course).

## A slightly more involved scheme: $\mathbb{P}^n$ .

Let  $D(x_i) := k[x_{0/i}, x_{1/i}, \dots, x_{\hat{i}/i}, \dots, x_{n/i}]$ . We have a maps  $\phi_{ij} : D(x_i)_{x_j} \rightarrow D(x_j)_{x_i}$ , given by  $x_{k/i} \mapsto x_{k/j}$ .

# More examples

- 1  $V_+(x^2 + y^2 + z^2)$  over  $\mathbb{R}$  and  $\mathbb{C}$ .
- 2 Blow-up of  $\mathbb{A}^2$  at the origin.
- 3 An example of a scheme without a closed point.

## Definition

Let  $\mathcal{P}$  be a property of morphisms of schemes. Let  $f : X \rightarrow Y$  be a morphism which satisfies  $\mathcal{P}$ . Then,

- 1 We say that  $\mathcal{P}$  is *affine-local on the target* if given any affine open cover  $\{V_i\}$  of  $Y$ ,  $f : X \rightarrow Y$  has  $\mathcal{P}$  if and only if the restriction  $f : f^{-1}(V_i) \rightarrow V_i$  has  $\mathcal{P}$  for each  $i$ .
- 2 We say that  $\mathcal{P}$  is *affine-local on the source* if given any affine open cover  $\{U_i\}$  of  $X$ ,  $X \rightarrow Y$  has  $\mathcal{P}$  if and only if the composite  $U_i \rightarrow Y$  has  $\mathcal{P}$  for each  $i$ .

Using *affine communication lemma* one can then show that it suffices to check the above statements on single affine open cover.

Something flat, something finitely presented/finite type, something finite.

Not all properties are like this. For example, separatedness, properness, quasi-compactness, etc.

Something ramified, something smooth, something singular.