The 1-Resolution Property of Algebraic Stacks

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The Resolution Property

Definition

An algebraic stack is said to have the *resolution property* if every quasi-coherent sheaf of finite-type is the quotient of a vector bundle.

The Resolution Property

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Theorem ([Totaro, 2004, Gross, 2017])

Let $\mathcal{X}$ be a quasi-compact quasi-separated algebraic stack. Then the following are equivalent:

- *$\mathcal{X}$ has affine stabilizer groups at closed points and satisfies the resolution property.*
- *$\mathcal{X} = [U/GL_n]$ where U is a quasi-affine scheme with an action of GL_n. In particular, $\mathcal{X}$ has affine diagonal.*

The Resolution Property

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So, this property becomes interesting only in the non-projective situation where, however, very little is known about it.

The 1-Resolution Property

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We will say that such a V is *special*.

The 1-Resolution Property

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Let $X \to Y$ be a quasi-affine morphism of algebraic stacks. If Y has the 1-resolution property then so does X.

These results also hold for the resolution property.

Hall and Rydh considered this property for algebraic stacks and posed the following question ([Hall and Rydh, 2017, 7.4]):

Question

Does every algebraic stack with the 1-resolution property admit a finite flat covering by a quasi-affine scheme? Moreover, is every algebraic space with the 1-resolution property quasi-affine?

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In this talk we will try and answer this question under moderate hypotheses. We have the following theorem (joint with Amit Hogadi and Siddharth Mathur):

The 1-Resolution Property

Theorem ([DHM20])

Let $\mathcal{X}$ be a Noetherian, quasi-excellent and normal algebraic stack whose stabilizers at closed points are affine. Then we have:

- *$\mathcal{X}$ has the 1-resolution property if and only if there exists a finite flat covering $U \to \mathcal{X}$ with U a quasi-affine scheme.*

A result of David Rydh gives us the following corollary.

Corollary

An algebraic space X (Noetherian, quasi-excellent, and normal) has the 1-resolution property if and only if it is quasi-affine.

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- BG_m and BG_a do not have the 1-resolution property.
- A DVR with a doubled point does not have the 1-resolution property.
- Any projective variety over a field with the 1-resolution property is a set of points.

The 1-Resolution Property

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We can glue the maps f_1 and f_2 along $\operatorname{Spec} K$, and get map f from $\operatorname{Spec} R$ with a doubled point to X.
Using the pullback property, we see that if a scheme X has the 1-resolution property, it must be separated.

The 1-Resolution Property

More generally, we have the following:

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Lemma ([DHM20])

Let $\mathcal{X}$ be a qcqs algebraic stack whose stabilizer groups at closed points are affine. If $\mathcal{X}$ has the 1-resolution property then $\mathcal{X}$ is separated, and hence has finite diagonal.

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Theorem ([DHM, 2020])

Let $\mathcal{X}$ be a Noetherian, quasi-excellent and normal algebraic stack whose stabilizers at closed points are affine. Then we have $\mathcal{X}$ has the 1-resolution property if and only if there exists a finite flat covering $U \to \mathcal{X}$ with U a quasi-affine scheme.

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- Prove that if $\mathcal{X}$ is a scheme, it is quasi-affine.

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- Prove that if $\mathcal{X}$ is a scheme, it is quasi-affine.
- Starting from a proper flat covering, use the slicing strategy of Kresch-Vistoli as in [Kresch and Vistoli, 2004] to produce a finite flat scheme covering, and conclude from the scheme case.

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The following result extends their argument to the arithmetic setting of stacks with finite diagonal over $\operatorname{Spec} Z$.

Theorem ([DHM20])

Let $\mathcal{X}$ be a quasi-compact algebraic stack with finite diagonal and denote by X the corresponding coarse moduli space. Suppose that $\mathcal{X}$ has the resolution property and that X admits an ample line bundle. Then $\mathcal{X}$ admits a finite flat cover $Z \to \mathcal{X}$ where Z is a scheme with an ample line bundle.

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Observe that this technique needs the existence of an ample line bundle on the coarse space.
Our hypotheses (Noetherian, normal and quasi-excellent) on $\mathcal{X}$ ensure this.

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This, however, only works in characteristic zero, since it uses the fact that GL_n is linearly reductive in characteristic zero.

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- $\mathcal{X}$ can be written as a limit $\lim_\lambda \mathcal{X}_\lambda$ where each $\mathcal{X}_\lambda$ is an algebraic stack of finite type over $\operatorname{Spec}\mathbb{Q}$ and all the maps $\mathcal{X} \to \mathcal{X}_\lambda$ are affine and schematically dominant.

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- The special vector bundle on $\mathcal{X}$ descend to a special vector bundle at some finite stage $\mathcal{X}_\alpha$ for which the result has been proved.
- Base change the finite flat cover to $\mathcal{X}$, which is quasi-affine because the map $\mathcal{X} \to \mathcal{X}_\alpha$ is affine.

This gives us the following strengthening of our theorem.

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Theorem ([DHM20])

Let $\mathcal{X}/\operatorname{Spec}\mathbb{Q}$ be a qcqs integral normal algebraic stack whose stabilizers at closed points are affine. Then we have:

- *$\mathcal{X}$ has the 1-resolution property if and only if there exists a finite flat covering $U \to \mathcal{X}$ with U a quasi-affine scheme.*

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- *$\mathcal{X}$ has the 1-resolution property if and only if there exists a finite flat covering $U \to \mathcal{X}$ with U a quasi-affine scheme.*

In particular, if such an $\mathcal{X}$ is an algebraic space then it must already be quasi-affine.

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- As it stands, answering Question 6 in complete generality is beyond the scope of our methods and would require new and innovative techniques.
- Another direction is to try and remove the normality hypothesis. It is not clear how to do this even in the scheme case. To that, we ask the following questions. A positive answer to these question will help generalise the above theorems to the non-normal setting.

Question

Let X, Y, Z be quasi-affine schemes which are finite-type over an excellent Noetherian scheme. Suppose we have a closed immersion $Y \to X$ and a finite morphism $Y \to Z$, then does the resulting pushout $X \cup_Y Z$ admit an ample family of line bundles?

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Let X, Y, Z be quasi-affine schemes which are finite-type over an excellent Noetherian scheme. Suppose we have a closed immersion $Y \to X$ and a finite morphism $Y \to Z$, then does the resulting pushout $X \cup_Y Z$ admit an ample family of line bundles?

Question

Let $\mathcal{X}$ be an algebraic stack which is finite-type over an excellent base and suppose $\mathcal{X}$ has the 1-resolution property. Does the coarse moduli space of $\mathcal{X}$ have the 1-resolution property?

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