Noetherian approximation and limits of schemes

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In this talk we will discuss absolute Noetherian approximation and limit results for schemes. These are the standard techniques used to eliminate Noetherian hypothesis from theorem statements. We will discuss the precise statements of these results, and also look at how they are applied in practice. Standard references are [DG67, Part 4 §8], [TT90, Appendix C] and [Sta18]. We will primarily follow the exposition in the Stacks project [Sta18, Tag 01YT].

A lot of the LaTeX code in this document has been shamelessly copied from the stacks project repository on GitHub[1].

1 Colimits and limits

Let us begin by recalling what a limit is. The dual notion is that of a colimit.

Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the *index category* or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

Definition 1.2. A *limit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\lim_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $p_i : \lim_{\mathcal{I}} M \to M_i$ such that

1. for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
2. for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ (indexed by $i \in \mathcal{I}$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to \lim_{\mathcal{I}} M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibred products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

Definition 1.4. A *colimit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\operatorname{colim}_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to \operatorname{colim}_{\mathcal{I}} M$ such that

[1] Thank you Aise Johan de Jong *et al.* for TeX-ing all that math!

1. for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
2. for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ (indexed by $i \in \mathcal{I}$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : \operatorname{colim}_{\mathcal{I}} M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

While the above definitions are valid in arbitrary generality, we often restrict ourselves to the situation where the indexing category has nice properties, like being a set. This makes the (co)limits easy to construct.

Definition 1.6. Let $(I, \leq)$ be a directed[2] set. Let $\mathcal{C}$ be a category.

1. A *system over I in $\mathcal{C}$*, sometimes called a *inductive system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \leq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \mathrm{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \leq i' \leq i''$.
2. An *inverse system over I in $\mathcal{C}$*, sometimes called a *projective system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i' \leq i$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \mathrm{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i'' \leq i' \leq i$. (Note reversal of inequalities.)

Our primary interest is limits in the cateogory of schemes. Hence, the following *colimit* is important for us:

Example 1.7 (colimit of rings). Let $(R_i, \mu_{i,j})$ be a directed system of rings indexed by I. We define the limit as

$$\operatorname{colim} R_i = \{(\sqcup_i R_i)/\sim \; \mid x_i \sim \mu_{i,j}(x_i) \text{ for any } i \in I \text{ and } x_i \in M_i\}$$

Since I is directed, we define addition and multiplication using the fact that any elements $x_i \in R_i$ and $x_j \in R_j$ eventually land in a common ring R_k. The equivalence relation used to define $\operatorname{colim} R_i$ ensures that these operations are well-defined.

2 Limit constructions on Schemes

We will now describe limit constructions in the category of schemes.

2.1 Standard limit constructions

In what follows we will often deal with the following setup.

Setup: Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes indexed over I. Assume that the morphisms $f_{ii'} : S_i \to S_i$ are affine. We will be interested in the limit $S = \lim_i S_i$.

[2] This means that given $i, j \in I$ there exists a k such that $i, j \leq k$

Remark 2.1. *If the S_i's are affine, then we automatically have a corresponding direct system of ring $(R_i, f_{ii'})$ such that* $\operatorname{Spec} R_i = S_i$*. Then*

$$\begin{aligned}\lim_i S_i &= \lim \operatorname{Spec} R_i \\ &= \operatorname{Spec}(\operatorname{colim} R_i)\end{aligned}$$

Thus, $\lim_i S_i$ *exists.*

More generally, we have the following:

Lemma 2.2. *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,*

1. *each of the morphisms $f_i : S \to S_i$ is affine,*
2. *for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have*

$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$

in the category of schemes.

The proof builds on the remark above. Fix an element $0 \in I$. Then, for all $i \geq 0$, we can write $S_i = \operatorname{Spec}_{S_0} \mathcal{A}_i$, where $\mathcal{A}_i = (f_{i0})_* \mathcal{O}_{S_i}$. And we construct S as $\operatorname{Spec}_{S_0} \mathcal{A}$, where $\mathcal{A} = \operatorname{colim}_{i \geq 0} \mathcal{A}_i$. The $\mathcal{A}_i$'s are quasi coherent sheaves on S_0. Hence, their colimit $\mathcal{A}$ exists and is quasi-coherent (since this is true for rings!).

It suffices to compute the limit over $i \geq 0$, because I is a directed set. If j is some element of I not related to 0, we can find a k such that $k \geq j$ and $k \geq 0$.

This construction has many nice properties. For example, S inherits the inverse limit topology. Moreover, the construction of S commutes with formation of fibre products over S_0.

Lemma 2.3. *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = \lim_i S_i$. Let $0 \in I$. Suppose that T is a scheme over S_0. Then*

$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

You can say more things. For example, assume that the inverse system is taken over quasi-compact and quasi-separated schemes. Properties of the limit are often exhibited at some finite stage in the inverse system. The following "meta-theorem" is often true.

Meta-theorem 1: Let **P** be a property of schemes, and $(S_i, f_{ii'})$ be an inverse system of *qcqs* schemes with affine bonding maps. If S has **P**, then there exists an $i_0 \in I$ such that S_i has **P** for all $i \geq i_0$.

Such a statement holds for many properties of schemes: quasi-affine, affine, separated, etc.

We also have situations where it is sufficient to find *some* i for which S_i has $\mathbf{P}$. This is also useful in practice.

Meta-theorem 2: Let $\mathbf{P}$ be a property of schemes, and $(S_i, f_{ii'})$ be an inverse system of *qcqs* schemes with affine bonding maps. If S has $\mathbf{P}$, then there exists an $i \in I$ such that S_i has $\mathbf{P}$.

For example, this is true for an ample line bundle on S: Let $\mathcal{L}_0$ be a line bundle on S_0. If the pullback $\mathcal{L}$ to S is ample, then for some $i \in I$ the pullback $\mathcal{L}_i$ to S_i is ample.

The above discussion also holds for properties of morphisms. That is, consider the following situation:

Relative Setup: Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms (Lemma 2.2). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S.

Observe that f_i's define an inverse system with limit f. Thus, we can approximate properties of f by the f_i's. The Meta-theorems 1 & 2 also "hold" for properties of morphisms of schemes.

2.2 Absolute Noetherian approximation

Following is the statement of absolute Noetherian approximation[3] (see [Sta18, Tag 07RN] for a more general version).

Lemma 2.4. *Let S be a quasi-compact and quasi-separated scheme. There exist a directed set I and an inverse system of schemes $(S_i, f_{ii'})$ over I such that*

1. *the transition morphisms $f_{ii'}$ are affine*
2. *each S_i is of finite type over $\mathbb{Z}$, and*
3. *$S = \lim_i S_i$.*

As you can see, this lets us write any quasi-compact and quasi-separated scheme as an inverse limit of Noetherian schemes. Coupled with "Meta-theorems 1 & 2", this allows us to reduce statements on S to statements on the Noetherian schemes S_i. Moreover, since the projection maps $S \to S_i$ are affine, we can pullback various properties of S_i to S. We will see more about this in the applications.

Sketch of proof. The proof is by induction on an affine cover of S. Note that if S is affine, then this is just a statement about commutative rings. Any commutative ring can be written as an inverse limits over its finite type subrings. Let $S = U \cup V$, where $U = U_1 \cup \ldots \cup U_{n-1}$ is covered by $n-1$ affine opens, and V

[3] The adjective "absolute" refers to the fact that we are working over $\mathbb{Z}$

is affine. Assume by induction that the statement holds for U, i.e, $U = \lim_i U_i$. Then, $W := U \cap V$ is a quasi-compact open in U, and we can assume that $W = \lim_i W_i$ over an inverse system of quasi-compact opens of U_i's. Also, $W \subset V$ is quasi-affine. The idea now is to leverage the W sitting inside V to construct an inverse system $\{V_i\}$ for V of finite type $\mathbb{Z}$-algebras, which patches with the inverse system $\{U_i\}$ along the W_i's. □

You might wonder if something similar is true for the relative case. In the relative case (over a base B), one can approximate quasi-compact quasi-separated objects over B by *finitely presented* object over B[4]:

Lemma 2.5. *Let $f : S \to B$ be a morphism of schemes. Assume that*

1. *S is quasi-compact and quasi-separated, and*
2. *B is quasi-separated.*

Then $S = \lim S_i$ is a limit of a directed system of schemes S_i of finite presentation over B with affine transition morphisms over B.

Idea of proof. Note that the image $f(S) \subset B$ is quasi-compact, so we can replace B with $f(S)$. The details are a bit involved, but the idea is to apply Lemma 2.4 on both S and B, and construct an inverse system $\{X_a \times_{f_{a,b}, S_b} S\}$ where the morphisms $f_{a,b} : X_a \to S_b$ commute with the projection maps. This works because all the limits involved are filtered. □

3 Applications

We will now look at a few applications of the above constructions.

General principle of approximation techniques

Approximation techniques usually work as follows:

1. Approximate the given object by Noetherian objects or, if we are working over an arbitrary base B, by finitely presented objects over B.
2. Use the Meta-theorems to descend properties of the limit to some finite stage in the inverse system. This reduces the problem to the situation where everything is finitely presented.
3. Pullback along the projection maps to conclude the original case. Since the projection maps are affine, most properties can be pulled back along it.

[4] Since the base need not be Noetherian (unlike $\mathbb{Z}$), we have to replace finite type with finitely presented.

3.1 Chevalley's affineness theorem

Consider the following result of Chevalley characterising affine schemes:

Theorem 3.1 (Chevalley)**.** *Let $X \to S$ be a finite surjective morphism of schemes with S Noetherian. If X is affine then so is S.*

We will use approximation and remove the Noetherian assumption on S:

Theorem 3.2 (Non-Noetherian Chevalley)**.** *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and finite, and assume that X is affine. Then S is affine.*

Sketch of Proof. Since f is finite surjective and X is affine, we see that S is quasi-compact and separated. Note that since S is not Noetherian, f is not autoamtically of finite presentation despite being finite.

The idea now is to apply approximation simulataneously for f and S, i.e, we will approximate f by a surjective, finite and finitely presented morphism, and at the same time approximate S by a Noetherian scheme. If $S = \lim_{i \in I} S_i$ is an approximation of S, then since the projection maps $S \to S_i$ are affine, it suffices to prove that some S_i is affine,

We can write $X = \lim_a X_a$ with $X_a \to S$ finite and of finite presentation. Moreover, we can arrange that X_a is affine for some $a \in A$. Replacing X by X_a we may assume that $X \to S$ is surjective, finite, of finite presentation and that X is affine.

We may write $S = \lim_{i \in I} S_i$ as a directed limits of schemes of finite type over $\mathbf{Z}$. After shrinking I, we can assume that there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and such that $X = \lim_i X_i$. We can further assume that X_i is affine and that $X_i \to S_i$ is finite for all $i \in I$. And now we are in the Noetherian case. $\square$

In fact, we can do slightly better.

Theorem 3.3. *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and integral, and assume that X is affine. Then S is affine.*

The idea is that we can approximate an integral surjective map by finite surjective maps. That is, we can write $X = \lim_i X_i$ with $X_i \to S$ finite, surjective and X_i affine for all i sufficiently large. Now use Theorem 3.2.

3.2 The 1-Resolution property

The following notion was first defined by Jack Hall and David Rydh in [HR17]:

Definition 3.4. A scheme X is said to have the *1-resolution property* if it admits a vector bundle V such that every quasi-coherent sheaf of finite-type on X is a quotient of $V^{\oplus n}$ for some natural number n. We say that such a V is *special.*

Theorem 3.5 (D-Hogadi)**.** *(See [DHM20, Theorem 1.3]) Let X be a Noetherian, quasi-excellent and normal scheme. Then X has the 1-resolution property if and only if it is quasi-affine scheme.*

This result was motivated by a question asked by Jack Hall and David Rydh in [HR17, 7.4].

Using approximation we can eliminate the Noetherian hypothesis from the above theorem but only in characteristic 0.

Theorem 3.8 (D-Hogadi-Mathur)**.** *(See [DHM20, Theorem 1.5]) Let $X/\operatorname{Spec}\mathbb{Q}$ be a qcqs integral normal scheme. Then X has the 1-resolution property if and only if it is quasi-affine scheme.*

Idea. The argument goes as follows:

Step 1: X can be written as a limit $\lim_\lambda X_\lambda$ where each X_λ is of finite type over $\operatorname{Spec}\mathbb{Q}$ and all the maps $X \to X_\lambda$ are affine and schematically dominant. We can choose X_λ's to be normal.

Step 2: The special vector bundle on X descends to a special vector bundle at some finite stage X_α (This step needs characteristic zero!). But X_α is finite type ove $\mathbb{Q}$, so is quasi-affine by the Noetherian case.

Step 3: Since the projection map $X \to X_\alpha$ is affine, the result follows.

□

A fact used in Step 2 is that GL_n is linearly reductive in characteristic 0, i.e, every representation can be written as a direct sum of irreducible representations. This is not true characterisitic p, so the argument breaks down.

Question 3.11. *Is it still possible to do approximation in characteristic p?*

3.3 Chow's lemma

Consider the Noetherian version of Chow's Lemma [DG67, II Theorem 5.6.1(a)].

Theorem 3.12 (Chow)**.** *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

We can prove this result with the relaxed hypothesis that S is quasi-compact and quasi-separated. However, the price paid is that we have to assume that X has finitely many irreducible components.

Theorem 3.13 (Non-Noetherian Chow). *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that X has finitely many irreducible components. Then there exists an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

If X does not have finitely many irreducible components, then we can still prove Theorem 3.13, but we have to forego the quasi-projective open dense U.

Sketch of proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of X into irreducible components. Let $\eta_k \in Z_k$ be the generic point.

By approximation, we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Apply Noetherian approxiamtion to S and write $S = \lim_i S_i$ as a directed limit of Noetherian schemes. Further, we can find an index $i \in I$ and a scheme Y_i over S_i such that $Y_i \to S_i$ separated and of finite presentation, so that $Y = S \times_{S_i} Y_i$. For $i' \geq i$, write $Y_{i'} = S_{i'} \times_{S_i} Y_i$. Then $Y = \lim_{i' \geq i} Y_{i'}$, by Lemma 2.3. For every $j \geq i$, we have the following diagram

$$\begin{tikzcd} \eta_k \in Z_k \arrow[r] & X \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d] & Y_j \arrow[d] \\ & & S \arrow[r] & S_j \end{tikzcd}$$

Denote $h : X \to Y_j$ the composition. Let $Y' \subset Y_j$ be the scheme theoretic image of $h : X \to Y_j$. Y' is separated and of finite presentation over S_j which is Noetherian. We will now apply the Noetherian case to $Y' \to S_j$, and then pullback the result to $X \to S$.

The only thing to ensure is that the images of the generic points η_k do not specialise to each other, i.e, the closures $\overline{h(\eta_k)}$ are distinct components in Y'. Again by using approximation we can find a j large enough such that this is true.

Now, apply Theorem 3.12 to the morphism $Y' \to S_j$. This gives a diagram

$$\begin{tikzcd} Y' \arrow[dr] & Y^* \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[dl] \\ & S_j & \end{tikzcd}$$

such that π is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. Base change the above diagram to S along the projection map $S \to S_j$, and take $X' := X \times_{Y'} Y^*$ and $U = h^{-1}(V)$. Thus, we get maps $X \xleftarrow{\pi} X' \to \mathbf{P}^n_S$, such that π is an isomorphism on $U \subset X$. □

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