The 1-Resolution Property of Algebraic Stacks

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1 Introduction

In this talk, we will discuss the 1-resolution property for algebraic stacks. While this notion is fairly limited in its scope of enquiry, it does have one interesting feature. We will show that it is equivalent to the existence of a finite flat cover by quasi-affine scheme.
We will begin with some preliminary discussion of the (1-)resolution property.

1.1 The Resolution Property

Definition 1.1. An algebraic stack is said to have the *resolution property* if every quasi-coherent sheaf of finite-type is the quotient of a vector bundle.

The Totaro–Gross Theorem relates the resolution property to quotients by group actions.

Theorem 1.2 ([Tot04, Gro17])**.** *Let $\mathcal{X}$ be a quasi-compact quasi-separated algebraic stack. Then the following are equivalent:*

- *$\mathcal{X}$ has affine stabilizer groups at closed points and satisfies the resolution property.*
- *$\mathcal{X} = [U/GL_n]$ where U is a quasi-affine scheme with an action of GL_n. In particular, $\mathcal{X}$ has affine diagonal.*

This is a remarkable result, because it is gives new information even for schemes!

Note that the resolution property always holds for (quasi-)projective schemes. So, this property becomes interesting only in the non-projective situation where, however, very little is known about it.

In general, questions about the resolution propery are hard. In fact, there is no known example of a separated scheme or a Deligne-Mumford stack which does not have the resolution property (see, however, [EHKV01] for an example of a non-quotient stack).

1.2 The 1-Resolution Property

We have the following definition.

Definition 1.3. An algebraic stack $\mathcal{X}$ is said to have the *1-resolution property* if it admits a vector bundle V such that every quasi-coherent sheaf of finite-type on $\mathcal{X}$ is a quotient of $V^{\oplus n}$ for some natural number n.

We will say that such a V is *special*.

We note the following properties of the (1-)resolution property along morphisms of stacks.

The 1-resolution property descends along finite finitely presented covers.

Lemma 1.4 ([Gro17]). *Let $X \to Y$ be a faithfully flat finite and finitely presented morphism of algebraic stacks. If X has the 1-resolution property then so does Y.*

And can be pulled back along quasi-affine morphisms.

Lemma 1.5 ([Gro17]). *Let $X \to Y$ be a quasi-affine morphism of algebraic stacks. If Y has the 1-resolution property then so does X.*

These results also hold for the resolution property.

Make special note of 1.5, because we will frequently use it in the sequel to "dévissage" our problems to simpler situations.

Hall and Rydh considered this property for algebraic stacks and posed the following question ([HR17, 7.4]):

Question 1.6. *Does every algebraic stack with the 1-resolution property admit a finite flat covering by a quasi-affine scheme? Moreover, is every algebraic space with the 1-resolution property quasi-affine?*

In this talk we will try and answer this question under moderate hypotheses. We have the following theorem (joint with Amit Hogadi and Siddharth Mathur):

Theorem 1.7 ([DHM20]). *Let $\mathcal{X}$ be a Noetherian, quasi-excellent and normal algebraic stack whose stabilizers at closed points are affine. Then we have:*

- *$\mathcal{X}$ has the 1-resolution property if and only if there exists a finite flat covering $U \to \mathcal{X}$ with U a quasi-affine scheme.*

By [Ryd11, Lem C.1], any algebraic space which admits a finite flat covering by a quasi-affine scheme is automatically quasi-affine. Hence, we have the following corollary.

Corollary 1.8. *An algebraic space X (Noetherian, quasi-excellent, and normal) has the* 1*-resolution property if and only if it is quasi-affine.*

Before we discuss the proof of the theorem, it is perhaps instructive to have a few (non)-examples at hand. Interesting examples of schemes with the 1-resolution property are difficult to find (this is not unreasonable, since they are all ultimately expected to be quasi-affine!).

However, we do have some enlightening non-examples.

- For finite group G, BG has the 1-resolution property. In fact, BG has the 1-resolution property if and only if $dim(G) = 0$. This implies that an algebraic stack with 1-resolution property has quasi-finite diagonal.
- BG_m and BG_a do not have the 1-resolution property.
- A DVR with a doubled point does not have the 1-resolution property.
- Any projective variety over a field with the 1-resolution property is a set of points.
- There is an example of a non-quasi-affine variety whose normalisation is quasi-affine (see [Sta18, Tag 0271]). This variety does not have the 1-resolution property.

We can take the example of the DVR with a doubled point further. Let X be a scheme with two maps from a DVR $f_1, f_2 : \operatorname{Spec} R \to X$ which agree on the quotient field $K \supset R$. We can glue the maps f_1 and f_2 along $\operatorname{Spec} K$, and get map f from $\operatorname{Spec} R$ with a doubled point to X. Using the pullback property, we see that if a scheme X has the 1-resolution property, it must be separated.

Since, the isotropy group of any point on a stack $\mathcal{X}$ must be zero dimensional in the presence of the 1-resolution property (because $dim(G) = 0$ if BG has the 1-resolution property), we see that such a stack has quasi-finite diagonal. [Ryd15, Theorem B] tells us that it admits a finite (*but not flat*) scheme covering. Then, the following lemma tells us that $\mathcal{X}$ is also separated.

Lemma 1.9. *Let $\mathcal{X}$ be an algebraic stack and $a : X \to \mathcal{X}$ be an integral surjective map from a scheme X. Then if X is separated then $\mathcal{X}$ is as well.*

Thus, we have proved the following:

Lemma 1.10 ([DHM20])**.** *Let $\mathcal{X}$ be a qcqs algebraic stack whose stabilizer groups at closed points are affine. If $\mathcal{X}$ has the 1-resolution property then $\mathcal{X}$ is separated, and hence has finite diagonal.*

This implies that $\mathcal{X}$ admits a coarse moduli space [KM97]. This will play an imporant role in our proof of 1.7, as we will use the coarse space to produce a finite flat cover.

2 Proof of the Theorem

We will now sketch a proof of theorem 1.7. We will try and explain the "big" ideas, rather than get bogged down in the details.

The 'if' direction is just descent for the 1-resolution property along finite faithfully flat maps (see [Gro17, Prop. 2.13]) combined with the fact that on a quasi-affine scheme U every coherent sheaf is globally generated (i.e, $\mathcal{O}_U$ is special). So, we need only be concerned with proving the 'only if' direction. The idea quite straightforward. We will produce a a finite flat scheme cover for $\mathcal{X}$.

As the 1-resolution property pullbacks along quasi-affine maps, this scheme also has the 1-resolution property. And we will show that this scheme is quasi-affine.

Usually, obtaining a finite or a flat cover is not difficult. But finding one that *both* finite and flat can be challenging. One usually needs to employ projective techniques for this: start with a cover that proper flat and then use generic hypersurfaces to cut down the dimension and maintain flatness. Such method are not directly accessible in the stacky setting, as yet. However, it was observed by Kresch and Vistoli [KV04] that if a stack admits a coarse moduli space with an ample line bundle, then this slicing strategy can be employed on coarse moduli and then pulled back to the stack.

Strategy of Proof

- Prove that if $\mathcal{X}$ is a scheme, it is quasi-affine.
- Starting from a proper flat covering, use the slicing strategy of Kresch-Vistoli as in [KV04] to produce a finite flat scheme covering, and conclude from the scheme case.

Essentially the only way check that a scheme is quasi-affine is to show that the structure sheaf is ample. In the presence of the 1-resolution property, it boils down to the following:

Lemma 2.1. *Let X be a quasi-compact scheme with the* 1*-resolution property. Let $\mathcal{E}$ be a special vector bundle. Then X is quasi-affine if and only if $\mathcal{E}$ is globally generated.*

The proof in the scheme case is an argument by Noetherian induction which requires openness of the regular locus. This is ensured by the quasi-excellence hypothesis.

The original argument of Kresch and Vistoli works for any Deligne-Mumford stack over a field. The following result extends their argument to the arithmetic setting of stacks with finite diagonal over Spec Z.

Theorem 2.2 ([DHM20])**.** *Let $\mathcal{X}$ be a quasi-compact algebraic stack with finite diagonal and denote by X the corresponding coarse moduli space. Suppose that $\mathcal{X}$ has the resolution property and that X admits an ample line bundle. Then $\mathcal{X}$ admits a finite flat cover $Z \to \mathcal{X}$ where Z is a scheme with an ample line bundle.*

Observe that this technique needs the existence of an ample line bundle on the coarse space. Our hypotheses (Noetherian, normal and quasi-excellent) on $\mathcal{X}$ ensure this.

3 Approximation

We can use limit arguments as in [Ryd15] to eliminate the Noetherian, and quasi-excellence hypotheses on $\mathcal{X}$. This, however, only works in characteristic zero, since it uses the fact that GL_n is linearly reductive in characteristic zero.

Approximation

The argument roughly goes as follows:

- $\mathcal{X}$ can be written as a limit $\lim_\lambda \mathcal{X}_\lambda$ where each $\mathcal{X}_\lambda$ is an algebraic stack of finite type over $\operatorname{Spec}\mathbb{Q}$ and all the maps $\mathcal{X} \to \mathcal{X}_\lambda$ are affine and schematically dominant.
- The special vector bundle on $\mathcal{X}$ descend to a special vector bundle at some finite stage $\mathcal{X}_\alpha$ for which the result has been proved.
- Base change the finite flat cover to $\mathcal{X}$, which is quasi-affine because the map $\mathcal{X} \to \mathcal{X}_\alpha$ is affine.

This gives us the following strengthening of our theorem.

Theorem 3.1 ([DHM20])**.** *Let $\mathcal{X}/\operatorname{Spec}\mathbb{Q}$ be a qcqs integral normal algebraic stack whose stabilizers at closed points are affine. Then we have:*

- *$\mathcal{X}$ has the 1-resolution property if and only if there exists a finite flat covering $U \to \mathcal{X}$ with U a quasi-affine scheme.*

In particular, if such an $\mathcal{X}$ is an algebraic space then it must already be quasi-affine.

4 Beyond the normal

As it stands, answering Question 1.6 in complete generality is beyond the scope of our methods and would require new and innovative techniques. Even version of Theorem 3.1 in characteristic p is not accessible at the moment. This requires some techniques to replace certain GL_n-bundles with bundles which have a linearly reductive structure group in suitable manner.

Another direction is to try and remove the normality hypothesis. It is not clear how to do this even in the scheme case. To that, we ask the following questions. A positive answer to these question will help generalise the above theorems to the non-normal setting.

Question 4.1. *Let X, Y, Z be quasi-affine schemes which are finite-type over an excellent Noetherian scheme. Suppose we have a closed immersion $Y \to X$ and a finite morphism $Y \to Z$, then does the resulting pushout $X \cup_Y Z$ admit an ample family of line bundles?*

Question 4.2. *Let $\mathcal{X}$ be an algebraic stack which is finite-type over an excellent base and suppose $\mathcal{X}$ has the 1-resolution property. Does the coarse moduli space of $\mathcal{X}$ have the 1-resolution property?*

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