ON THE MOTIVIC HOMOTOPY TYPE OF ALGEBRAIC STACKS

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Abstract. We construct smooth presentations of algebraic stacks that are local epimorphisms in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category. As a consequence we show that the motive of a smooth stack (in Voevodsky's triangulated category of motives) has many of the same properties as the motive of a smooth scheme.

1. Introduction

An important technique in motivic homotopy theory of algebraic stacks is reduction to the scheme case by means of homotopical descent. This is possible, for instance, when the stacks in question are Nisnevich locally quotient stacks. The results in [AHR] (and further generalisation in [AHLHR]) show that stacks with linearly reductive stabilisers are Nisnevich locally quotient stacks.

In this note we establish a certain homotopy descent result for any quasi-separated algebraic stack. This will allows us to conclude that the various formalisms of motives and motivic homotopy theory[1] produce the correct results for algebraic stacks. To wit, we will show that the motive of a smooth algebraic stack has similar properties as the motive of a smooth scheme. This generalises the results in [CDH] that were established for Nisnevich locally quotient stacks. Another improvement, albeit minor, that we can make is to show that for algebraic stacks with speparated diagonal the existing notions of stable homotopy category coincide: in [Cho], it is shown that the stable motivic homotopy category of [Cho] and the lisse-extended category of [KR] are equivalent when the stack admits a smooth presentation with a Nisnevich local section; we will show that all quasi-separated stacks admit such a presentation. Following [Pir], we will call such coverings *smooth-Nisnevich coverings.*

Definition 1.1. A *smooth-Nisnevich morphism* of algebraic stacks is a morphism $f : \mathcal{Y} \to \mathcal{X}$ such that f is smooth, surjective and any morphism $\operatorname{Spec} k \to \mathcal{X}$ from the spectrum of a field k lifts to $\mathcal{Y}$. We say that f is *smooth-Nisnevich covering* when $\mathcal{Y}$ is an algebraic space.

When f is étale such covering are called Nisnevich coverings in literature.

Smooth-Nisnevich coverings were first discussed in [Pir], where it is shown that such coverings exist for finite type stacks with affine stabilisers that are defined over an infinite field. The aim of this note is extend this result to all finite type stacks with affine stabilisers. The following is our main geometric result.

Theorem 1.2. *Let $\mathcal{X}$ be a quasi-separated algebraic stack with separated diagonal over a scheme S.*
(1) *There exists a smooth-Nisnevich covering $X \to \mathcal{X}$ with X a scheme.*
(2) *Assume S is Noetherian and that $\mathcal{X}$ is of finite type over S with affine stabilisers, then X can also be chosen to be of finite type over S.*

Note that the argument for Part (1) of the above theorem is essentially contained in [LMB, §6.7]. Part (2) is new in the mixed charactersitic setting and is built from a modification of Pirisi's argument [Pir, Proposition 3.6] which works over an infinite field.

The existence of such covers has implications for motivic homotopy theory of algebraic stacks. More specifically, it shows that the Nisnevich homotopy type of an algebraic stack can be described by a

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[1] In the case of stacks one has to distinguish between *genuine* vs *Kan-extended* motivic spectra. For instance K-theory is *genuine* but Chow groups are *Kan-extended*. In this note we will be concerned with the latter kind of objects.

simplicial scheme (or simplicial algebraic space). This makes many homotopical descent arguments accessible for algebraic stacks.

Theorem 1.3. *Let $p : X \to \mathcal{X}$ be a smooth-Nisnevich covering over a field k. Let $X_\bullet$ denote the Čech nerve of p. Then the morphism $p_\bullet : X_\bullet \to \mathcal{X}$ induces an equivalence in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category, $\mathcal{H}(k)$.*

A consequence of the above result is that the motive of a smooth algebraic stack continues to enjoy the same properties as the motive of a smooth scheme (see Section 3). So far this was only known for stacks which satisfied local structure theorems in the sense of [AHR, AHLHR] or in a different direction for motives of smooth stacks in the étale topology. We describe various results of this kind in the text.

Remark 1.4. Another application of Theorem 1.2 is the following: In [KM], the authors use Theorem 1.2(2) to apply Elkik's approximation technique to the Picard stack and study the following question of Grothendieck: when is the map

$$H^2(X, \mathbb{G}_m) \to \varprojlim H^2(X_n, \mathbb{G}_m)$$

injective for a proper morphism $X \to \operatorname{Spec} A$, where $(A, \mathfrak{m})$ be a complete Noetherian ring and $X_n := X \times_A \operatorname{Spec} A/\mathfrak{m}^{n+1}$ is the n-th infinitesimal thickening.

Conventions. We work with algebraic stacks in the sense of [LMB], i.e, all stacks are assumed to be quasi-separated with separated diagonal.

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2. Smooth-Nisnevich coverings

In order to prove Theorem 1.2, we will use the following results from [LMB, §6]

2.1. **Moduli of finite étale covers.** Fix a base scheme S. For a separated and representable S-morphism $\pi : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, we recall the construction and basic properties of the stacks $SEC_d(\mathcal{X}/\mathcal{Y})$ and $ET_d(\mathcal{X}/\mathcal{Y})$.

Consider the d-fold fibre product $(\mathcal{X}/\mathcal{Y})^d := \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \ldots \times_{\mathcal{Y}} \mathcal{X}$. For any S-scheme U, an object $u : U \to (\mathcal{X}/\mathcal{Y})^d$ is equivalent to the data of a morphism $U \to \mathcal{Y}$ together with d sections $x_1, \ldots, x_d$ of the morphism $\mathcal{X} \times_{\mathcal{Y}} U \to U$. Since π is separated, the locus where any two such section agree is closed. Thus, the complement of this locus is an open substack which is denoted by $SEC_d(\mathcal{X}/\mathcal{Y})$.

The symmetric group S_d acts naturally on $(\mathcal{X}/\mathcal{Y})^d$ and the restriction of this action to $SEC_d(\mathcal{X}/\mathcal{Y})$ is free. We denote the quotient stack by

$$ET_d(\mathcal{X}/\mathcal{Y}) := [SEC_d(\mathcal{X}/\mathcal{Y})/S_d].$$

Almost by construction SEC_d and ET_d satisfy the following properties:

Proposition 2.1 ([LMB, Propositions 6.6.2, 6.6.3])**.** *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a representable and separated morphism of algebraic stacks over S. Then*

(1) *The constructions $SEC_d(\mathcal{X}/\mathcal{Y})$ and $ET_d(\mathcal{X}/\mathcal{Y})$ behave well with respect to base change. That is, given any morphism $U \to \mathcal{Y}$, we have cononical isomorphisms $SEC_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{Y}} U \simeq SEC_d(\mathcal{X} \times_{\mathcal{Y}} U/U)$ and $ET_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{Y}} U \simeq ET_d(\mathcal{X} \times_{\mathcal{Y}} U/U)$.*
(2) *The natural morphisms $SEC_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ and $ET_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ are representable and separated. Moreover, they are smooth (respectively, étale) if π is.*

(3) *If $\mathcal{X}$ is an algebraic space, then $SEC_d(\mathcal{X}/\mathcal{Y})$ is an algebraic space and, therefore, $ET_d(\mathcal{X}/\mathcal{Y})$ is a Deligne-Mumford stack.*
(4) *For any S-scheme U, a U-point of $ET_d(\mathcal{X}/\mathcal{Y})$ is equivalent to a morphism $U \to \mathcal{Y}$ together with a closed subscheme $Z \subset U \times_{\mathcal{Y}} \mathcal{X}$ such that $Z \to U$ is finite étale of degree d.*
(5) *If π is finite étale of degree d, then $ET_d(\mathcal{X}/\mathcal{Y}) \simeq \mathcal{Y}$.*

2.2. **Proof of Theorem 1.2.** For part (2) of the theorem, we will need the following result about stacks admitting stratification by global quotient stacks as found in [Kre].

Definition 2.2. Let $\mathcal{X}$ be an algebraic stack over S. We say that $\mathcal{X}$ admits a stratification by global quotient stacks if there exists a finite chain $\mathcal{X} \supset \mathcal{X}_1 \supset \ldots \mathcal{X}_n$ of closed substacks such that for each i, $\mathcal{X}_i \setminus \mathcal{X}_{i-1}$ is a global quotient stack over S.

Theorem 2.3. [Kre, Theorem 3.5.9] *Let $\mathcal{X}$ be an algebraic stack of finite type over a Noetherian scheme S. Assume that $\mathcal{X}$ has affine stabilizers at geometric points. Then $\mathcal{X}$ admits a stratification global quotient stacks over S.*

We also need the following result whose proof can also be found in [LMB, Théorème 6.1]

Proposition 2.4. *Let $\mathcal{Y}$ be a Deligne-Mumford stack. Assume that $\mathcal{Y}$ is of the form $[Y/G]$ with Y an algebraic space and G a finite group. Then there exists a smooth covering $X \to \mathcal{Y}$ such that given any semi-local ring $\mathcal{O}$ any $\mathcal{O}$-valued point of $\mathcal{Y}$ lifts to X.*

We will now prove Theorem 1.2.

Proof of 1.2. For both part (a) and part (b), we will show that we can find such a covering with X an algebraic space. As any algebraic space admits a Nisnevich covering by a scheme (see [Knu, Theorem 6.4]),we conclude by replacing X with a Nisnevich covering by a scheme.
Proof of part (1). The argument for (1) is essentially in [LMB, §6.7] and we reproduce it here for completeness. It suffices to consider the case when $\mathcal{X}$ is quasi-compact. Let $Y \to \mathcal{X}$ be a smooth presentation with Y affine. Let K be a field and choose a K-valued point $\operatorname{Spec} K \to \mathcal{X}$. Then there exists a finite separable extension $L \supset K$ and a closed immersion $f : \operatorname{Spec} L \to Y$. Let $d := [L : K]$. Denote by $ET_d(Y/\mathcal{X})$ the moduli of degree d étale covers. Then the closed immersion $f : \operatorname{Spec} L \to X$ defines a K-point of $ET_d(XY/\mathcal{X})$ giving us a commutative diagram,

$$\begin{tikzcd} & ET_d(Y/\mathcal{X}) \arrow[d] \\ \operatorname{Spec} K \arrow[ur] \arrow[r] & \mathcal{X} \end{tikzcd}$$

As $ET_d(Y/\mathcal{X}) := [SEC_d(Y/\mathcal{X})/S_d]$ and Y is affine, $SEC_d(Y/\mathcal{X})$ is an algebraic space. Applying the previous proposition, we see that $\operatorname{Spec} K$ lifts to $X_d := SEC_d(Y/\mathcal{X}) \times^{S_d} GL_n$. Then the countable collection $\{X_n\}_n$ gives a smooth-Nisnevich covering $\cup_n X_n \to \mathcal{X}$. As every algebraic space admits a Nisnevich covering by a scheme we can also assume that the family $\{X_n\}$ is a countable collection of schemes.

Proof of part (2). Assume $\mathcal{X}$ is finite type with affine stabilisers and that S is Noetherian. Let $\mathcal{A} := \{X_n\}_n$ be the countable family constructed in part (1), and $p : \cup_n X_n \to \mathcal{X}$ be the smooth-Nisnevich covering. By the Theorem 2.3, $\mathcal{X}$ admits a stratification by global quotient stacks. Thus, it suffices to show that for each stratum $\mathcal{Y}_i := \mathcal{X}_i \setminus \mathcal{X}_{i-1}$, there is a finite collection $X_{i_1}, X_{i_2}, \ldots, X_{i_k} \in \mathcal{A}$ such that $\cup_k (X_{i_k} \times_{\mathcal{X}} \mathcal{Y}_i) \to \mathcal{Y}_i$ is a smooth-Nisnevich covering. The union of all such X_{i_k} is a finite type scheme that is a smooth-Nisnevich covering of $\mathcal{X}$. Thus, it suffices to consider the case of global quotient stacks.

Let $[Y/GL_n]$ be a global quotient stack. Let $\cup_n X_n \to [Y/GL_n]$ be the countable family of part (1). We have the following cartesian diagram where each arrow is a smooth-Nisnevich covering:

$$\begin{tikzcd} \cup_n(X_n \times_{[Y/GL_n]} Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ \cup_n X_n \arrow[r] & {[Y/GL_n]} \end{tikzcd}$$

Denote $\cup_n(X_n \times_{[Y/GL_n]} Y) := \cup_n Z_n$. Since $\cup_n Z_n \to Y$ is smooth-Nisnevich, the generic point of Y admits a lift to some Z_k. Thus, there is an open subset $U \subset Y$ that lifts to Z_k. We may now consider the complement of U, and apply the same procedure. This gives a descending sequence $Y \subset Y_1 \subset \dots$ whose generic points lift to $\cup_n Z_n$. Since Y is Noetherian, there exists an N such that $\cup_{n \le N} Z_n$ stabilises.

We will show that the map $\cup_{n \le N} Z_n \to Y$ is a smooth-Nisnevich covering. Let $x : \operatorname{Spec} k \to Y$ be a morphism. By construction, there exists an i such that the morphism x factors as $\operatorname{Spec} k \to Y_i \setminus Y_{i-1} \subset Y$. Then the lifting $Y_i \setminus Y_{i-1} \subset Z_i \subset \cup_{n \le N} Z_n$, gives a lift of $x : \operatorname{Spec} k \to Y$. Further, as $Y \to [Y/GL_n]$ is a smooth-Nisnevich covering, it is easy to see that $\cup_{n \le N} X_n \to [Y/GL_n]$ is also a smooth-Nisnevich covering. □

Remark 2.5. We note that the proof of part (2) above is essentially the same argument as in [Pir, Proposition 3.6]. A careful choice of presentations for the strata allows us to to relax the characteristic zero assumption in [Pir, Proposition 3.6] and also deal with the mixed characteristic situation.

Remark 2.6. The argument for part (1) is extracted from the proof of [LMB, Theorem 6.7] which states for any field k and a k-point of $\mathcal{X}$ there exists smooth affine neighbourhood that lifts the given k-point. While it is tempting to take union over all such neighbourhood to get a smooth-Nisnevich covering, it is unclear to me how to do this if the stack is not defined over a field. This is due the fact that there is no notion of a residue field for algebraic stacks, hence there is no concept of "minimal" fields through which any field valued point factors.

In fact smooth-Nisnevich coverings also lift Hensel local points by the following corollary.

Corollary 2.7. *Let $X \to \mathcal{X}$ be a smooth-Nisnevich covering. Let $\operatorname{Spec} \mathcal{O}$ be the spectrum of a Henselian local ring $\mathcal{O}$. Then any morphism $\operatorname{Spec} \mathcal{O} \to \mathcal{X}$ lifts to a morphism $\operatorname{Spec} \mathcal{O} \to X$.*

Proof. Let k be the residue field of $\mathcal{O}$ and $X_{\mathcal{O}} := X \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O}$ denote the base change of X to $\operatorname{Spec} \mathcal{O}$. Then the composite $\operatorname{Spec} k \to \operatorname{Spec} \mathcal{O} \to \mathcal{X}$ lifts to a morphism $\operatorname{Spec} k \to X$. This also gives us a morphism $\operatorname{Spec} k \to X_{\mathcal{O}}$. Now, as $X_{\mathcal{O}} \to \operatorname{Spec} \mathcal{O}$ is smooth, we have a surjection of sets $X_{\mathcal{O}}(\mathcal{O}) \twoheadrightarrow X_{\mathcal{O}}(k)$. Thus, we have a section $\operatorname{Spec} \mathcal{O} \to X_{\mathcal{O}}$. Composing with the projection map $X_{\mathcal{O}} \to X$ gives us the desire lift. □

2.3. **The non-separated case.** In general, the proof of theorem 1.2(1) breaks down in the non-separated situation. However, we will describe a particular situation where the statement hold. This is essentially based on [Čes, Appendix B]. We make the following definition

Definition 2.8. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks defined over a scheme S. We say that $f : \mathcal{X} \to \mathcal{Y}$ is S-fibrewise separated, if for every $s \in S$, the fibre $\mathcal{X}_s \to \mathcal{Y}_s$ is separated.

Proposition 2.9. *Let $\mathcal{X}$ be an algebraic stack over a scheme S. Suppose that the diagonal $\Delta_{\mathcal{X}/S}$ is S-fibrewise separated. Then there exists a smooth-Nisnevich morphism $f : X \to \mathcal{X}$ with X a scheme.*

Proof. The proof essentially follows from [Čes, Appendix B]. One just needs to do careful book-keeping to ensure that the end result is a set.

We will briefly sketch the steps of the argument (see [Čes, Appendix B] for details):

Step 1: Let $X \to \mathcal{X}$ be a smooth atlas. Consider the n-fold product $(X/\mathcal{X})^n$ with the action of S_n given by permuting the coordinates. The associated quotient stack $[(X/\mathcal{X})^n/S_n] \to \mathcal{X}$ is smooth over $\mathcal{X}$ with separated diagonal. The mophism $[(X/\mathcal{X})^n/S_n] \to \mathcal{X}$ behaves well with respect to base change. Let $Y_n \to [(X/\mathcal{X})^n/S_n]$ be a smooth-nisnevich covering. We will show that $\sqcup Y_n \to \mathcal{X}$ has the required property. Moreover, note that it suffices to prove that any k-point of $\mathcal{X}$ lifts to $[(X/\mathcal{X})^n/S_n]$ for some n.

Step 2: Let $x : \operatorname{Spec} k \to \mathcal{X}$ be a k-point. Let $s : \operatorname{Spec} k \to S$ be its image in S, and consider the fibre of $\mathcal{X}$ over s, $\mathcal{X}_s \to \operatorname{Spec} k$. As $\Delta_{\mathcal{X}/S}$ is fibrewise separated, the morphism $\operatorname{Spec} k \to \mathcal{X}_s$ admits a lift to $Et_n(X_s/\mathcal{X}_s)$ for some n. Further, $Et_n(X_s/\mathcal{X}_s) \to [(X/\mathcal{X})^n/S_n]_s$ is an open immersion, giving us a map, $\operatorname{Spec} k \to [(X/\mathcal{X})^n/S_n]_s \to [(X/\mathcal{X})^n/S_n]$, as required. □

Remark 2.10. The covering constructed above can be very large. We have described this construction more as an idle curiosity than anything else. In practice, the case of qcqs algebraic stacks with separated diagonal suffices for most applications.

3. Applications to Motivic homotopy theory of algebraic stacks

We will now mention some applications of Theorem 2.3 to the motivic homotopy theory of algebraic stacks. The proof of all the statements are the same as in [CDH] using the smooth-Nisnevich covering constructed in Theorem 1.2.

We will now recall the notion of the (unstable) $\mathbb{A}^1$-homotopy category over a field.

Fix a base field k. Let Sm/k denote the category of smooth schemes over k. Let $\Delta^{op}PSh(Sm/k)$ be the category of simplicial presheaves on Sm/k. $\Delta^{op}PSh(Sm/k)$ has a local model structure with respect to the Nisnevich topology (see [Jar]). A morphism $f : X \to Y$ in $\Delta^{op}PSh(Sm/k)$ is a weak equivalence if the induced morphisms on stalks (for the Nisnevich topology) are weak equivalences of simplicial sets. Cofibrations are monomorphisms, and fibrations are characterised by the right lifting property.

We Bousfield localise this model structure with respect to the class of maps $X \times \mathbb{A}^1 \to X$ (see [MV, 3.2]). The resulting model structure is called the Nisnevich motivic model structure. Denote by $\mathcal{H}(k)$ the resulting homotopy category. This is the (unstable) $\mathbb{A}^1$-homotopy category for smooth schemes over k.

Let us also rapidly recall the definition of Voevodsky's triangulated category of motive $\mathbf{DM}^{eff}(k,\mathbb{Z})$. Let Cor_k denote the category of finite correspondences whose objects are smooth separated schemes over k. For any two X, Y, the morphisms of Cor_k are given by $Cor(X,Y)$ which is the free abelian group generated by irreducible closed subschemes $W \subset X \times Y$ that are finite and surjective over X. An additive functor $F : Cor_k^{op} \to \mathbf{Ab}$ is called a presheaf with transfers. Let $PST(k,\mathbb{Z})$ denote the category presheaves with transfers. For any smooth scheme X, let $\mathbb{Z}_{tr}(X)$ be the presheaf with tranfers which on any smooth scheme Y is defined as

$$\mathbb{Z}_{tr}(X)(Y) := Cor(X,Y)$$

Let $K(PST(k,\mathbb{Z}))$ denote the category of complexes of presheaves with transfers. The category $K(PST(k,\mathbb{Z}))$ also has a Nisnevich motivic model structure which is defined analogously as in the case of $\Delta^{op}PSh(Sm/k)$. We denote the associated homotopy category by $\mathbf{DM}^{eff}(k,\mathbb{Z})$. This is Voevodsky's triangulated category of mixed motives in the Nisnevich topology (for details, see [MVW]).

Proposition 3.1. *Let $\mathcal{X}$ be a quasi-separated stack. Let $X \to \mathcal{X}$ be a smooth-Nisnevich covering. Then the associated Čech hypercover $X_\bullet$ is weakly equivalent to $\mathcal{X}$ in the Nisnevich local model structure on the category $\Delta^{op}PSh(Sm/k)$ of simplicial presheaves.*

Proof. By Corollary 2.7, we know that hensel local points lift along smooth-Nisnevich coverings. Adapting the proof of [CDH, Theorem 1.2] gives the result. □

The proof of Corollary 1.3 follows from $\mathbb{A}^1$-localising the above proposition.

Proof of Corollary 1.3. As $\mathbb{A}^1$-localisation preserves simplicial equivalences, the result follows from the previous proposition. □

As a consequence, we have the following result.

Corollary 3.2. *The weak equivalence above induces an equivalence of motives $M(X_\bullet) \simeq M(\mathcal{X})$ in $\mathbf{DM}^{eff}(k,\mathbb{Z})$.*

Proof. Use the functor $M : \mathcal{H}_\bullet(k) \to \mathbf{DM}^{eff}(k,\mathbb{Z})$ (see [CDH] for details). □

We will now state some results about motives and motivic cohomology of smooth stacks.

The proofs are exactly as in [CDH] after replacing GL_n-presentations with smooth-Nisnevich presentations.

Theorem 3.3. *Let $\mathcal{X}$ be a smooth algebraic stack. Then its motive $M(\mathcal{X}) \in \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$ satisfies the following properties:*

(1) *$M(\mathcal{X})$ satisfies Nisnevich descent.*
(2) *(Projective bundle formula) For any vector bundle $\mathcal{E}$ of rank $n+1$ over $\mathcal{X}$, we have a canonical isomorphism*

$$M(\mathbb{P}roj(\mathcal{E})) \simeq \bigoplus_{i=0}^{n} M(\mathcal{X})(i)[2i]$$

(3) *(Blow-up formula) For $\mathcal{Z} \subset \mathcal{X}$ a smooth closed substack of pure codimension c we have*

$$M(Bl_{\mathcal{Z}}(\mathcal{X})) \simeq M(\mathcal{X}) \oplus_{i=0}^{c-1} M(\mathcal{Z})(i)[2i]$$

(4) *(Gysin triangle) For $\mathcal{Z} \subset \mathcal{X}$ a smooth closed substack of codimension c, we have a Gysin triangle:*

$$M(\mathcal{X} \setminus \mathcal{Z}) \to M(\mathcal{X}) \to M(\mathcal{Z})(c)[2c] \to M(\mathcal{X} \setminus \mathcal{Z})[1].$$

Proof. The proofs in [CDH, §4] go through using a smooth-Nisnevich covering $X \to \mathcal{X}$. □

Remark 3.4. Note that the transfer functor $M : \mathcal{H}_{\bullet}(k) \to \mathbf{DM}^{\mathit{eff}}(k, \mathbb{Z})$ allows us to define the motive of *any* smooth stack. What is apriori unclear is whether this motive has any properties (good or bad). This is why in [CDH], the authors work with stacks that are Nisnevich locally quotient stacks in order to use homotopical descent to prove the above formulae for the motive. With Corollary 1.3, those arguments now work for all smooth stacks.

3.1. **Motivic cohomology as hypercohomology.** Recall the definitions of smooth-Nisnevich and smooth-Zariski sites of an algebraic stack

Definition 3.5. Let $\mathcal{X}$ be an algebraic stack. The smooth-Nisnevich (resp. smooth-Zariski) site of $\mathcal{X}$, denoted by $\mathcal{X}_{\text{lis-nis}}$ (resp. $\mathcal{X}_{\text{lis-zar}}$) is the category whose objects consist of pairs (U, p) where p is a smooth morphism $p : U \to \mathcal{X}$ from an algebraic space U and coverings in $\mathcal{X}_{\text{lis-nis}}$ are given by Nisnevich coverings (resp. Zariski coverings) of algebraic spaces.

The following result states that motivic cohomology can be computed as hypercohomology of the motivic complexes $\mathbb{Z}(i)$ on the smooth-Nisnevich site of the stack.

Corollary 3.6. *Let $\mathcal{X}$ be a smooth stack over a field k. The motivic cohomology of $\mathcal{X}$ agrees with the hypercohomology of the motivic complexes $\mathbb{Z}(j)$ on $\mathcal{X}_{\text{lis-nis}}$. That is,*

$$Ext^i_{D(\mathcal{X})}(\mathbb{Z}, \mathbb{Z}(j)|_{\mathcal{X}}) \simeq Ext^i_{DA(\mathcal{X})}(\mathbb{Z}, \mathbb{Z}(j)|_{\mathcal{X}}) \simeq Hom_{\mathbf{DM}^{\mathit{eff}}(k,\mathbb{Z})}(M(\mathcal{X}), \mathbb{Z}(j)[i]),$$

where $\mathbb{Z}$ denotes the constant sheaf $\mathbb{Z}$ on the $\mathcal{X}_{\text{lis-nis}}$.

Proof. See [CDH, Theorem 5.2]. □

In fact, for smooth separated Deligne-Mumford stacks with schematic coarse space, we can also use the smooth-Zariski site for the above computation. This follows from the following obvious corollary of [KV]:

Corollary 3.7. *Let $\mathcal{X}$ be a smooth separated Deligne-Mumford stack with schematic coarse space. Then $\mathcal{X}$ is Zariski locally a quotient stack.*

Proof. By [KV], it is immediate that $\mathcal{X}$ admits finite flat covers Zarski locally on the coarse space. As resolution property descends along finite flat maps, we are done. □

The above propostion show that for any such stack one can find a GL_n-presentation which is a *Zariski local* equivalence.

We note the following refinement of Corollary 3.6 which is true for smooth separated Deligne-Mumford stacks with schematic coarse space. The proof is that same using a Zariki local presentation (instead of a Nisnevich local presentation).

Corollary 3.8. *Let $\mathcal{X}$ be a smooth separated Deligne Mumford stack over a field k. Assume that the coarse moduli space of $\mathcal{X}$ is a scheme. The motivic cohomology of $\mathcal{X}$ agrees with the hypercohomology of the motivic complexes $\mathbb{Z}(j)$ on $\mathcal{X}_{lis\text{-}zar}$. That is,*

$$Ext^i_{D(\mathcal{X})}(\mathbb{Z}, \mathbb{Z}(j)|_{\mathcal{X}}) \simeq Ext^i_{DA(\mathcal{X})}(\mathbb{Z}, \mathbb{Z}(j)|_{\mathcal{X}}) \simeq Hom_{\mathbf{DM}^{eff}(k,\mathbb{Z})}(M(\mathcal{X}), \mathbb{Z}(j)[i]),$$

where $\mathbb{Z}$ denotes the constant sheaf $\mathbb{Z}$ on the $\mathcal{X}_{lis\text{-}nis}$.

3.2. **Motive cohomology with finite coefficients.** We also have the following comparison theorem relating motivic cohomology with $\mathbb{Z}/n\mathbb{Z}$ to étale cohomology with μ_n-coefficients.

Corollary 3.9. *Let $\mathcal{X}$ be a smooth stack over a field k. Then the homomorphisms*

$$H^{p,q}_M(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \to H^p_{\acute{e}t}(\mathcal{X}, \mu_n^{\otimes q}),$$

are isomorphisms for $p \leq q$ and monomorphisms for $p = q+1$.

Proof. See [CDH, Corollary 5.3]. □

3.3. **Motive with compact support.** In this subsection, we will define the notion of a motive with compact support for smooth algebraic stacks. The definition is very similar to Totaro's construction for quotient stacks using approximation by vector bundles [Tot], except that since we cannot approximate non-quotient stacks by algebraic spaces, we are forced to work with homotopy limits.

Definition 3.10. Let $\mathcal{X}$ be an algebraic stack locally of finite type. Consider a smooth-Nisnevich covering $p : X \to \mathcal{X}$ and let n denote the relative dimension of $p : X \to \mathcal{X}$. Let $p_\bullet : X_\bullet \to \mathcal{X}$ be the Čech nerve of p. Then we define the *compactly supported motive* of $\mathcal{X}$ as

$$M^c(\mathcal{X}) := \mathrm{holim}_i M^c(X_i)(-(i+1)n^2)[-2n^2 i].$$

This definition is a bit weird looking, but it has the following pleasant feature, which also shows that the definition is independent of choices for smooth algebraic stacks.

Theorem 3.11 (Poincaré Duality)**.** *Let $\mathcal{X}$ be a smooth algebraic stack of dimension d. Then we have an isomorphism,*

$$M^c(\mathcal{X}) = M(\mathcal{X})^\vee(d)[2d].$$

Proof. The proof is a straightforward manipulation of the definition. Tensoring by $\otimes\mathbb{Z}(-d)[-2d]$ we get,

$$\begin{aligned} M^c(\mathcal{X}) \otimes \mathbb{Z}(-d)[-2d] &= (\mathrm{holim}_i M^c(X_i)(-(i+1)n^2)[-2n^2 i]) \otimes \mathbb{Z}(-d)[-2d] \\ &= \mathrm{holim}_i M^c(X_i)(-(i+1)n^2 - d)[-2n^2 i - 2d] \\ &= \mathrm{holim}_i M(X_i)^\vee. \end{aligned}$$

where the last step follows from the fact that each X_i is smooth. Further, since $M(\mathcal{X}) \simeq \mathrm{hocolim}_i(X_i)$, taking dual we get that $M(\mathcal{X})^\vee = \mathrm{holim}_i M(X_i)^\vee$. Thus, we have proved that

$$M^c(\mathcal{X})(-d)[2d] \simeq M(\mathcal{X})^\vee.$$

Tensoring by $\otimes\mathbb{Z}(d)[2d]$, gives the required expression. □

In the following subsection, we note an application to stable motivic homotopy theory of stacks. We will not explain any details to prevent drowning the reader in ∞-categories. Our intention is to simply indicate how the geometric content of Theorem 1.2 can be applied in the world of motivic homotopy theory. The interested reader may consult [Cho] or [KR] for further details.

3.4. **Stable motivic homotopy category of a stack.** Recently, two different definitions of the stable motivic homotopy category for stacks have appeared in literature. The limit-extended category $SH_{\triangleleft}(\mathcal{X})$ in [KR] and the category $SH^{\otimes}_{ext}(\mathcal{X})$ in [Cho] that is extended from schemes to stacks having Nisnevich local sections. In [Cho] it is proved that the two definition are equivalent whenever the stack admits a smooth-Nisnevich covering. As coverings such always exist for quasi-separated stacks by Theorem 1.2, we have a strengthening of [Cho, Corollary 2.5.4].

Definition 3.12. Let $\mathcal{X}$ be an algebraic stack. A smooth presentation $X \to \mathcal{X}$ is said to admit *Nisnevich local sections* if, for any scheme Y and a morphism $Y \to \mathcal{X}$, there exists a Nisnevich covering $Y' \to Y$ with a section $Y' \to Y \times_{\mathcal{X}} X$. Thus, we have the following commutative diagram,

$$\begin{tikzcd} & Y \times_{\mathcal{X}} X \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] \arrow[ur, dotted] & Y \arrow[r] & \mathcal{X} \end{tikzcd}$$

The following lemma shows that the above definition is equivalent to the notion of a smooth-Nisnevich covering.

Lemma 3.13. *Let $\mathcal{X}$ be an algebraic stack. A smooth presentation $f : X \to \mathcal{X}$ is admits Nisnevich local sections if and only if it is a smooth-Nisnevich covering.*

Proof. The only if part is clear. To see the if part, observe that if $X \to \mathcal{X}$ is a smooth-Nisnevich covering, then by Cor 2.7, for any Henselian local ring $\mathcal{O}$ with a map $\operatorname{Spec} \mathcal{O} \to \mathcal{X}$ we have a lift $\operatorname{Spec} \mathcal{O} \to X$. Thus, if $V \to \mathcal{X}$ is a morphism and $\mathcal{O}_{v,V}$ is the Henselian local ring at a point $v \in V$, we have a lift $\operatorname{Spec} \mathcal{O}_{v,V} \to V \times_{\mathcal{X}} X$. By standard limit arguments [EGA], there exists a Nisnevich neighbourhood V' of $v \in V$, and a section $V' \to V \times_{\mathcal{X}} X$. $\square$

Corollary 3.14. *Let $\mathcal{X}$ be a quasi-separated algebraic stack. Then $SH_{\triangleleft}(\mathcal{X}) \simeq SH^{\otimes}_{ext}(\mathcal{X})$, whenever they are defined.*

Proof. The proof in [Cho, Corollary 2.5.4] works verbatim using a smooth-Nisnevich covering $X \to \mathcal{X}$. $\square$

Remark 3.15. Theorem 1.2 also improves [KR, Corollary 12.28] in a similar fashion.

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