Efficient Estimation of Distribution-free dynamics in the Bradley-Terry Model

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Abstract

We propose a time-varying convex generalization of the original Bradley-Terry model. Our model directly captures the temporal dependence structure of the pairwise comparison data without explicit temporal distribution assumptions. This enables the modeling of discrete-time dynamic global rankings of N distinct objects. Different choices of the convex penalization term provide a control on the degree of smoothing in the time-varying global rankings of the N distinct objects. Furthermore this directly enables analysis on sparse time-varying pairwise comparison data. We also prove that a relatively weak condition is necessary and sufficient to guarantee the existence and uniqueness of the solution of our model. We implement various convex optimization algorithms to efficiently estimate the model parameters under the ℓ_2^2, ℓ_2, and ℓ_1 convex penalization norms. We conclude by thoroughly testing the practical effectiveness of our model under both simulated and real world settings, including ranking 5 seasons of National Football League (NFL) team data. Our generalized time-varying Bradley-Terry model thus provides a useful benchmarking tool for other feature-rich dynamic ranking models since it only relies on the time-varying score pairwise ranking information.

1 Introduction and Prior Work

1.1 Pairwise Comparison Data and the Bradley-Terry Model

Pairwise comparison data is very common in daily life especially in cases where the goal is to rank several objects. Rather than directly ranking all objects simultaneously it is usually much easier and more efficient to first obtain results of pairwise comparisons. The pairwise comparisons can then be used to derive a *global* ranking across all individuals in a principled manner. One such statistical model for deriving global rankings using pairwise comparisons was presented by statisticians R. A. Bradley and M. E. Terry in their classic 1952 paper [3], and thereafter commonly referred to as the Bradley-Terry model in statistical literature. A similar model was also studied by Zermelo dating back to 1929 [21]. The Bradley-Terry model is one of the most popular models to analyze paired comparison data due to its interpretable setup and computational efficiency in estimation. The Bradley-Terry model along with its various generalizations has been studied and applied in various ranking applications across many broad domains. This includes the ranking of sports teams ([12],[5],[7]), scientific journals ([16],[17]), and the quality of several brands ([1],[15]).

In order to describe the original Bradley-Terry model [3], suppose that we have N distinct objects, each with a **[SS: positive?]** score or index $(s_i)_{i\in[N]}$ showing their power of competing with each other at a single point in time (static). This model assumes that the comparisons between different pairs are independent and the results of comparisons between a given pair, object i and object j, are independent and identically distributed as Bernoulli random variables, with success probability

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defined as

$$\mathbb{P}(i \text{ beats } j) = \frac{s_i}{s_i + s_j} \tag{1}$$

A common way to parameterize $(s_i)_{i\in[N]}$ is to assume that $s_i = \exp(\beta_i)$, in this case, equation (1) is usually expressed as $\mathsf{logit}(\mathbb{P}(i \text{ beats } j)) = \beta_i - \beta_j$, where $\mathsf{logit}(x) := \log \frac{x}{1-x}$. An important assumption in the original Bradley-Terry model is that comparisons of different pairs are independent. However, in practice such strong independence assumptions are unlikely to hold, see [2],[6],[4],[17] for some discussions. A typical way to deal with such data dependence in the pairwise comparison scores $(s_i)_{i\in[N]}$ is to use quasi-likelihood approaches [19],[17]. Fortunately under the setting of the original Bradley-Terry model [3] the log-quasi-likelihood and usual log-likelihood are the same, in terms of optimizing for the target parameters $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_N)$ as discussed further in [19].

1.2 The Time-varying (dynamic) Bradley-Terry Model

In many applications it is very common to observe paired comparison data spanning over multiple (discrete) time periods. A natural question of interest is then to understand how the *global* rankings *change* over time. For example in sports analytics the performance of teams often changes from season to season and thus explicitly incorporating the time-varying dependence into the model is crucial. In particular the paper [7] considers a state-space generalization of the Bradley-Terry model to modelling the sports tournaments data. In a similar manner bayesian frameworks for the dynamic Bradley-Terry model are studied further in [9]. Such dynamic analysis of paired comparison data is becoming increasingly important because of the rapid growth of openly available time-dependent paired comparison data.

Our main focus in this paper is to tackle the problem of efficiently estimating the parameters in the time-varying Bradley-Terry model under a frequentist framework. Our frequentist approach is to assume a distribution-free approach in the changes in parameters over time in order to perform a useful dynamic estimation with minimal assumptions and relying only on the time-varying pairwise comparison information. The remainder of this paper discusses our model in detail and is organized as follows. We firstly formulate our overall time-varying Bradley-Terry estimation requirement as a convex optimization problem of our proposed model solution under various penalty norms. We then describe the relatively weak necessary and sufficient conditions to guarantee uniqueness of our model. We proceed to apply various well known convex optimization algorithms to derive efficient estimation of time-varying parameters with provable convergence guarantees, exploiting specific structure of the convex penalty terms as applicable. We then provide two strategies to determining our penalization parameter, λ, using a data-driven and heuristic approaches. Finally we conclude with applying our model on synthetic datasets and two real-world examples of our technique using NFL and NASCAR datasets.

2 Our proposed Time-varying Bradley-Terry Model

In our time-varying frequentist setup of the Bradley-Terry model, we generalize the approach taken in the original paper [3] estimating parameters of interest via maximizing the likelihood (or mimimizing negative likelihood) over the discrete observed time points $\{1, 2, \ldots, T\} =: [T]$. The parameters of interest, $\boldsymbol{\beta}^{(t)} \in \mathbb{R}^N$, are now given for each time point $t \in [T]$. We assume that the pairwise comparison between a given pair, object i and j, at a given time point t is determined by the corresponding parameter $\boldsymbol{\beta}^{(t)}$ so that

$$\mathsf{logit}(\mathbb{P}(i \text{ beats } j \text{ at time } t)) = \boldsymbol{\beta}_i^{(t)} - \boldsymbol{\beta}_j^{(t)}. \tag{2}$$

Given a $\boldsymbol{\beta}^{(t)}$, by equation (2) we can derive the log-likelihood $\ell_t(\boldsymbol{\beta}^{(t)})$. As the size of data in each t is much smaller than the global data, we might want to *smooth* the parameter $\boldsymbol{\beta}^{(t)}$ across different time points and to leverage the dynamic structure in the global data. Our proposal is to include the penalization term $\lambda \sum_{t=1}^{T-1} h(\boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}^{(t+1)})$ for some convex penalty function h that can effectively smooth the estimation of parameter by penalizing large differences in subsequent time points. In sum, the time varying setup of the Bradley-Terry model can be framed as the following convex optimization problem:

$$\min_{\{\boldsymbol{\beta}^{(t)}\}_{t\in[T]}} -\sum_{t=1}^{T} \ell_t(\boldsymbol{\beta}^{(t)}) + \lambda \sum_{t=1}^{T-1} h(\boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}^{(t+1)}), \text{ s.t. } \sum_{i=1}^{N} \beta_i^{(1)} = 0 \tag{3}$$

where $\lambda \geq 0$, and $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ is the convex penalty function. We firstly note that if if $T = 1$ and $\lambda = 0$ then equation (3) reduces to the same objective as the original (static) Bradley-Terry model [3]. As such our model represents a generalization of the original Bradley-Terry model to the time-varying setting. Furthermore we include an additional constraint, namely $\sum_{i=1}^{N} \beta_i^{(1)} = 0$ which sets the sum of the estimated parameters to 0 at the initial time point $t = 1$. This artificial constraint is used to guarantee the existence and uniqueness of the global solution set in the convex formulation discussed further in section 3. Since we are only concerned with obtaining *relative* global object rankings this constraint still maintains our ultimate goal and further ensures that equation (3) is invariant to constant shifts in the parameter. In our project we consider 3 cases for the convex penalty function h, namely $h = \|\cdot\|_1$, $h = \|\cdot\|_2$, and $h = \|\cdot\|_2^2$. **[SS: What about mentioning changepoint detection?]**

3 Existence and uniqueness of solution

The existence or the uniqueness of solutions for the model (3) is not guaranteed in general **[SS: without the additional constraint?]**. This is an innate property of the original Bradley-Terry model. As pointed out by Ford Jr. [8] the Bradley-Terry model requires a sufficient amount of pairwise comparisons so that there is enough information of relative performance between any pair of two entries for estimation purposes. For example, if there is an entry which has never been compared to the others, there is no information which the model can exploit to assign a score for the entry, so its score could be arbitrary. In addition if there are several entries which have never outperformed the others then Bradley-Terry model would assign negative infinity for the performance of these entries, and we would not be able to compare among them for global ranking purposes.

In 1957 Ford Jr. [8] discovered the necessary and sufficient conditions which guarantee the uniqueness of the solution in the original Bradley-Terry model. The two equivalent conditions are:

Condition (1)**.** In every possible partition of the objects into two nonempty subsets, some object in the second set has been preferred at least once to some object in the first set

Condition (2)**.** For each ordered pair (i, j), there exists a sequence of indices $i_0 = i, i_1, \ldots, i_n = j$ such that

$$x_{i_{k-1},i_k} > 0 \tag{4}$$

for $k = 1, \ldots, n$.

We prove that these conditions can also be used to guarantee the uniqueness of the solution in our time-varying Bradley-Terry model.

Theorem. *Given data $\{x_{i,j}^{(t)}\}_{i,j,t}$ satisfies the previous condition, let $x_{i,j} = \sum_{t=1}^{T} x_{i,j}^{(t)}$. If $\{x_{i,j}\}$ satisfies the condition above, then*

1. *If h is continuous and $h(x) \to \infty$ as $x \to \infty$, the solution $\boldsymbol{\beta}^\star$ for* (2) *is attainable in $\mathbb{R}^{N\times T}$; and*

2. *With squared-ℓ_2 penalty, $h = \|\cdot\|_2^2$, the uniqueness of solution classes for* (3).

Hence, in our proposed time-varying Bradley-Terry we do not require the strong conditions 3 to hold at each time point, but simply require the aggregated conditions in 3 to hold. This is a quite weak condition for that it is satisfied even when each object does does not have pairwise comparisons with all other objects at every time point. For example, even if one player did not play any game in a season, as far as we have a game record on another season (with at least one win and one lose respectively), we can assign a rank to this player in the missed season. This minimal data requirement for our proposed model is in our view a key advantage to other frequentist approaches which could be alternatively considered. One such approach would be to fit a separate Bradley-Terry model at each discrete time point and then 'smooth' rankings using, say, kernel smoothing techniques. However such an approach would require the stronger conditions in [8] to hold at *every time point* rather than the much weaker aggregated conditions required for our model to hold. **[SS: HB - are you fine with this wording?]**

4 Optimization Methods

We note that the objective function in equation (3) can be expressed as follows:

$$f(\boldsymbol{\beta}) = \underbrace{g(\boldsymbol{\beta})}_{\text{negative log-likelihood}} + \underbrace{H(\boldsymbol{\beta})}_{\text{convex penalty}} \tag{5}$$

where

$$g(\boldsymbol{\beta}) = -\sum_{i=1}^{T} l_t(\boldsymbol{\beta}^{(t)}), \text{ and } H(\boldsymbol{\beta}) = \lambda \sum_{t=1}^{T-1} h(\boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}^{(t+1)}). \tag{6}$$

Since g is convex differentiable and H is convex for the convex penalty h the proximal gradient descent method (PGD) is a natural optimization strategy for this problem. In particular using the ℓ_2^2-norm for the smoothing penalty, we can derive the closed form of the proximal operator and run PGD with an inexpensive computational cost. However in the case of other smoothing penalties such as using the ℓ_1-norm or non-squared ℓ_2-norm, the proximal operator for h does not have a closed form thereby reducing the computational feasibility of the PGD method in such settings. We consider both fixed descent step sizes s_i and also determined by the backtracking line search with the initial step size s_{init}. Furthermore we also perform PGD with and without Nesterov acceleration [13] and assess impact on the convergence rate thereof. Below we detail the closed form for the proximal operator with the squared ℓ_2-penalty and discuss the computational feasibility under more general ℓ_1-norm and ℓ_2-norm smoothing penalties.

4.1 Efficient PGD for ℓ_2^2-penalty

The proximal operator for the step size $s > 0$ and the smoothing penalty H is defined as follows:

$$\text{prox}_{s,H}(\boldsymbol{\beta}') = \arg\max_{\boldsymbol{\beta} \in \mathbb{R}^{N \times T}} \left[\frac{1}{2s} \|\boldsymbol{\beta} - \boldsymbol{\beta}'\|_2^2 + \lambda \sum_{t=1}^{T-1} \|\boldsymbol{\beta}^{(t+1)} - \boldsymbol{\beta}^{(t)}\|_2^2 \right] \tag{7}$$

Hence, we can decompose the optimization in the proximal operator into the marginal optimizations for $i = 1 \dots N$ so that if $\boldsymbol{\beta}^+ = \text{prox}_{s,H}(\boldsymbol{\beta}')$, then

$$\boldsymbol{\beta}_i^+ = \arg\max_{\boldsymbol{\beta}_i \in \mathbb{R}^T} \left[\frac{1}{2s} \|\boldsymbol{\beta}_i^{(t)} - \boldsymbol{\beta}_i^{'(t)}\|_2^2 + \lambda \sum_{t=1}^{T-1} (\boldsymbol{\beta}_i^{(t+1)} - \boldsymbol{\beta}_i^{(t)})^2 \right]. \tag{8}$$

Hence, $\nabla_{\boldsymbol{\beta}_i} \mathcal{F}_i = 0$ if and only if

$$\begin{bmatrix} 1+2s\lambda & -2s\lambda & & \\ -2s\lambda & 1+4s\lambda & \ddots & \\ & \ddots & \ddots & -2s\lambda \\ & & -2s\lambda & 1+4s\lambda \end{bmatrix} \boldsymbol{\beta}_i^+ = \boldsymbol{\beta}_i'. \tag{9}$$

Hence by solving the linear system (9) for each $i = 1, \dots, N$, we get the output of the proximal operator for the squared l_2-norm penalty. By exploiting the sparse *tri-diagonal* structure of the linear system we can solve it in $O(n)$ computations which is much more efficient that just using matrix inversion.

4.2 ADMM for ℓ_1-penalty and ℓ_2-penalty

On the other hand, getting the proximal operator for general smoothing penalty functions is nontrivial. In case of l_1-norm penalty, for instance, the problem in (8) is the 1-dimensional fused-lasso problem, which does not have a closed-form solution and requires nontrivial amount of computation to approximate the optimum. In these two cases, we can use ADMM to get the solution as detailed in the following section.

Alternating Direction Method of Multipliers (ADMM) Let $\boldsymbol{\beta} \in \mathbb{R}^{TN} = (\boldsymbol{\beta}^{(1)^\top}, \cdots, \boldsymbol{\beta}^{(T)^\top})^\top$ **[SS: WL - could we just write this using vec?]** be the matrix of scores of all individuals at all time points. We can define $\boldsymbol{\theta} \in \mathbb{R}^{(T-1)N}$ whose entries are given by $\theta(t \cdot N + i) = \mu_i^{t+2} - \mu_i^{t+1}$ for $1 \leq i \leq N,\ 0 \leq t \leq T-2$. By introducing a matrix $A \in \mathbb{R}^{[(T-1)N] \times TN}$ we can then express $\boldsymbol{\theta}$ by $\boldsymbol{\theta} = A\boldsymbol{\mu}$ and rewrite the optimization problem (3) as

$$\begin{aligned} &\text{minimize} \quad -\sum_{t=1}^{T} L_t(\boldsymbol{\beta}^{(t)}) + \lambda \tilde{h}(\boldsymbol{\theta}), \\ &\text{subject to } A\boldsymbol{\beta} = \boldsymbol{\theta}, \end{aligned} \tag{10}$$

where $\tilde{h} = \sum_{t=1}^{T-1} h(\boldsymbol{\theta}^{I_t})$ with the index set $I_t = \{i : (t-1) \cdot N + 1 \leq i \leq t \cdot N\}$.

The ADMM scheme can be written as

$$\begin{aligned} \boldsymbol{\beta}^{k+1} &= \arg\min_{\boldsymbol{\beta}} -L_T(\boldsymbol{\beta}) + (A\boldsymbol{\beta})^\top \boldsymbol{\mu}^k + \frac{\eta}{2}\|A\boldsymbol{\beta} - \boldsymbol{\theta}^k\|^2, \\ \boldsymbol{\theta}^{k+1} &= \arg\min_{\boldsymbol{\theta}} \lambda \tilde{h}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \boldsymbol{\mu}^k + \frac{\eta}{2}\|A\boldsymbol{\beta}^{k+1} - \boldsymbol{\theta}\|^2, \\ \boldsymbol{\mu}^{k+1} &= \boldsymbol{\mu}^k + \eta(A\boldsymbol{\beta}^{k+1} - \boldsymbol{\theta}^{k+1}). \end{aligned} \tag{11}$$

The update of $\boldsymbol{\theta}$ is simply the proximal operator of $\tilde{h}$. The update of $\boldsymbol{\beta}$ is a convex optimization problem of a well-behaved function with no constraints and can be solved by some basic methods, like Newton's method.

5 Choosing λ (smoothing penalty parameter) in practical settings

In our convex formulation of the time-varying Bradley Terry model (2) we note that the penalty coefficient λ is effectively a global smoothing parameter for the fitted β_t values between subsequent time periods under the various penalty norms. Increasing λ thus increases the penalty on the difference between subsequent β_t values over a single time period and thus leads to β_t values (and hence the global rankings) becoming smoothed together.

Naturally the question remains on how to tune λ meaningfully in practical applications. We note that in practice the end user of the time-varying Bradley Terry model typically seeks the global time-varying team rankings rather than the individual fitted β_t coefficients used to derive pairwise comparisons. As such the usual approach of tuning λ to fit the β_t values and then assessing the associated impact on the time-varying global rankings changes is a rather indirect approach to controlling the degree of smoothing in the sought after time-varying global rankings.

5.1 Data Driven Approach - Sample Splitting and LOOCV

Ranking objects from pairwise comparisons of them, we want time-varying Bradley-Terry model to sort objects by means of win rates: a higher ranked object wins more likely than a lower ranked object. If we could successfully predict a win rate between any pair of objects, then the scores on which the prediction is made will provide a proper rank on our purpose. Hence, we choose λ giving the best prediction on pair-wise win rates. In regime of prediction, leave-one-out cross-validation has been provably successful without aids of human heuristics [11]. Here, we briefly describe how leave-one-out cross-validation(LOOCV) applies to time-varying Bradley-Terry model and then propose several techniques to reduce computational cost for LOOCV.

In general settings where we have i.i.d. samples, LOOCV assesses the performance of a predictive model by holding out one of the i.i.d. samples. In our case, each pairwise comparison is i.i.d. sample if we take the compared objects and the time point on which they are compared as covariates to the result. Let (t_k, i_k, j_k) denote k-th pair-wise comparison where object i_k won object j_k at time point t_k for $k = 1, \ldots, K$. Then, for a smoothing penalty parameter λ, LOOCV applies to our model as follows:

1. For $k = 1, \ldots, K$, repeat:
   (a) Solve (3) with λ on the dataset where the k-th comparison is held-out.

(b) Calculate the negative log-likelihood of the previous solution to (t_k, i_k, j_k).

2. Take the average of the negative log-likelihoods to get nll_λ as a loss in the predictive performance of time-varying Bradley-Terry model with λ on our dataset.

Solving (3) with every candidate λ on every left-one-out dataset costs tremendous computational resource. To reduce the cost, we can approximate the exhaustive LOOCV with a stochastic estimate. The exhaustive LOOCV fits the model to every possible left-one-out data and takes an average of them. However, if we have a large number of pair-wise comparisons, an average of much fewer losses makes tiny error. Hence, we can random sample a smaller number of matches to be left-out and get $\hat{nll}_\lambda$ with computational efficiency. Also, we can get quick convergences in optimizations by nearly optimal initial values. We assume that the parameter fitted on the entire dataset differs little to the fitted on left-one-out datasets. These two techniques make LOOCV a practical option for time-varying Bradley-Terry models with hundreds dimensional $\boldsymbol{\beta}$'s.

5.2 Heuristic Approach

[SS: team - do we need this? What are we fitting using this?] We propose a simple heuristic whereby the user controls the degree of smoothing in global rankings directly by specifying a maximal ranking change parameter $\alpha \in \mathbb{N}$ over all N teams and over all T time periods. By specifying this global (integer) parameter α we can search over a suitable finite grid of $\lambda \in \mathbb{R}^+$ values to meet this user specified global maximum team rank change requirement. In this heuristic we note that the user simply specifies a positive integer α indicating the maximum increase/ decrease in ranks over all time periods and teams. We claim that α is much more intuitive for the end user to set to control the global ranking changes to a desired level directly. Here α is set as a global smoothing parameter since controlling the maximum rank change over all T periods we are effectively controlling smoothing with respect to local consecutive time periods as well.

Since α here is integer-valued it is naturally capped at the total number of teams N, since a team can't globally change rankings by more than the total number of teams across any time period. However in practice it will be much lower than this and easier to prescribe by the end user based on reasonable domain knowledge of expected time-varying ranking movements. We acknowledge that this heuristic trades the subjectivity of choosing λ for α but that it controls for the degree of smoothing directly. The limited choice for the end user makes it effective to apply in practical situations.

6 Experiments

6.1 Synthetic Data Generation

We have conducted a number of simulation experiments in which we know the underlying global ranks of all entries by generating data from a synthetic process. We used the synthetic data to implement our proposed time varying Bradley Terry model with the variety of convex optimization algorithms and to compare the output with the ground truth.

Given the number of entries N and the number of time points T, the overall synthetic data generation process is as follows:

1. For each $i \in \{1, \ldots, N\}$, simulate $\boldsymbol{\beta}_i^* \in \mathbb{R}^T$ from a distribution of time-series.
2. For each $i \neq j$ and $t = 1, \ldots, T$, generate $n_{ij}^{(t)}$ from some distribution and simulate $x_{ij}^{(t)}$ by

$$x_{ij}^{(t)} \sim \mathrm{Binom}\Big(n_{ij}^{(t)}, \frac{1}{1+\exp\{\beta_j^{(t)} - \beta_i^{(t)}\}}\Big),\ x_{ji}^{(t)} = n_{ij}^{(t)} - x_{ij}^{(t)}$$

For the choice of distribution of $\boldsymbol{\beta}_i^*$'s in step 1, we use Gaussian distribution with a Markov Chain structure, Specifically, for each i we set $\boldsymbol{\beta}_i^{*(0)} \sim \mathcal{N}(0, \sigma_{i,0}^2)$ and generate

$$\boldsymbol{\beta}_i^{*(t+1)} | \boldsymbol{\beta}_i^{*(t)} \sim \mathcal{N}(\gamma \boldsymbol{\beta}_i^{*(t)}, \sigma_{i,t+1}^2), t = 1, \ldots, T-1. \tag{12}$$

where γ is the decay parameter and $\sigma_{i,t}^2$'s are the conditional variances of $\boldsymbol{\beta}_i^{*(t)}$ with respect to the others. An example of simulated $\boldsymbol{\beta}$ is the Figure 1 (left). In this example, there are 20 team entries and 10 time points. The latent parameter $\boldsymbol{\beta}_i^{*(t)}$ is set not to decay (i.e., $\gamma = 1$), and the conditional variance $\sigma_{i,t}^2$ were set to 1 for every $i = 1, \ldots, N$ and $t = 1, \ldots, T$.

6.2 Synthetic Simulation Results

We generated the pairwise comparison data from the latent parameter $\boldsymbol{\beta}^*$ above with $n_{ij}^{(t)} = 10$ for every $i \neq j$ and $t = 1, \ldots, T$ and solved (2) by GD, PGD, and Newton's method. The solution of PGD is shown in Figure 1, (right). We can compare between two figures of $\boldsymbol{\beta}^*$ and $\hat{\boldsymbol{\beta}}$ to see how effectively our formulation of time-varying Bradley-Terry Model fits the synthetic ground truth. Firstly we note that our model is able to recover the comprehensive global ranking of each entry at every time point. In both figures, we can verify that team 20 is uniformly dominating and team 2 is maintaining the lowest position in the majority of the time points. However, we can also find some differences of $\hat{\boldsymbol{\beta}}$ from $\boldsymbol{\beta}^*$ which come from the smoothing property of our model. For instance, the estimated parameter of entry 20 at the first two time points seem overestimated than the ground-truth. These two estimates could be influenced by the following high estimates, and entry 20 is ranked higher than indeed it is for those time points.

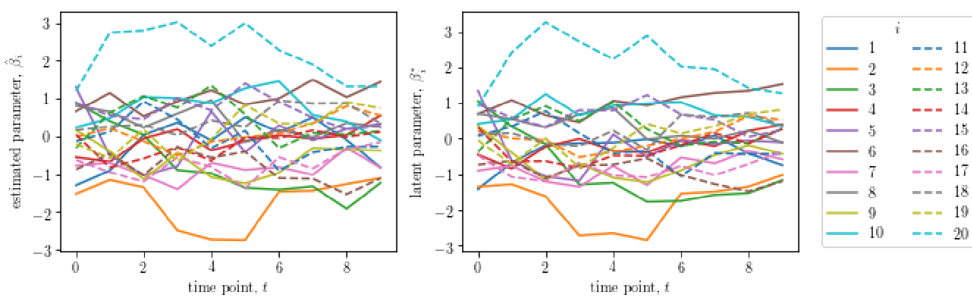

Figure 1: (Left) simulated ground-truth parameter $\boldsymbol{\beta}^*$ and (right) the solution $\hat{\boldsymbol{\beta}}$

We also compare the effects of different types of penalization functions on the solution of (2). With the same ground truth as Figure 1 and $\lambda = 10$, the solutions are shown in Figure 2. As we can see, different choices of the penalty function would lead to different shapes of the solution. Specifically, the squared ℓ_2 norm produces the most smooth paths of β, while the ℓ_1 norm would impose piece-wise constant structure on the paths of β. In applications, the path of β could have various shapes. Therefore, our model is adaptive to different requirements on the shape of β in different applications.

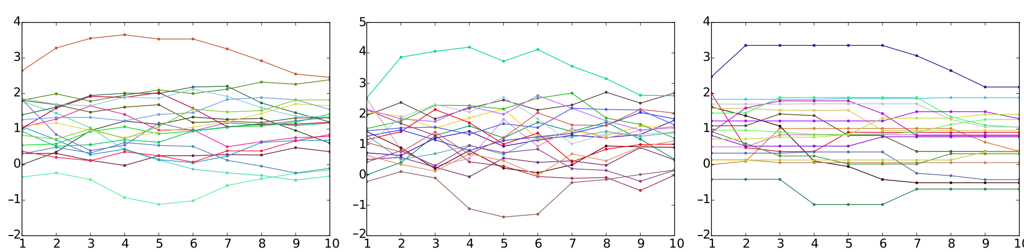

Figure 2: Solutions by different penalties with $\lambda = 10$. From left to right: $h = \|\cdot\|_2^2, \|\cdot\|_2, \|\cdot\|_1$.

The convergence curves of PGD (also with backtracking and Nesterov acceleration), Gradient Descent (GD) and Newton method are shown in Figure 3 with Newton method converging the fastest.

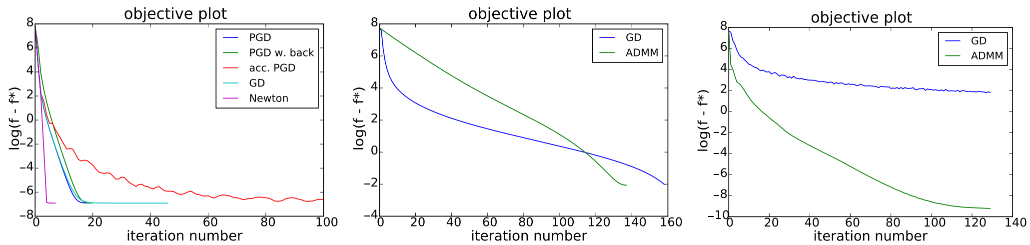

Figure 3: Convergence curves for various algorithms. From left to right: $h = \|\cdot\|_2^2, \|\cdot\|_2, \|\cdot\|_1$.

6.3 Application - NFL Data

We have sourced 5 seasons of National Football League (NFL) data from 2011-2015 (inclusive) from the `nflwar` package [20]. Each season is comprised of $N = 32$ teams playing $T = 16$ games each over the season i.e. $t \in [16]$ in this case. This means that at each point in time t the pairwise matrix of scores across all 32 teams is sparsely populated with only 16 entries. We have fit our time-varying Bradley Terry model over all 16 rounds in the season and set a maximum global rank change α value equal to 3 per our heuristic described in section 5. In order to gauge whether the rankings produced by our model are reasonable compared our season ending rankings (fit over all games played in that season) with the relevant NFL ELO ratings published by the blog `fivethirtyeight.com` [14]. The top 10 season-ending rankings from each method across NFL seasons 2011-2015 are summarized in Table 1:**[SS: Is this now with LOOCV?]**

rank	**2011**		**2012**		**2013**		**2014**		**2015**	
	ELO	**BT**	**BLO**	**BT**	**ELO**	**BT**	**ELO**	**BT**	**ELO**	**BT**
1	GB	GB	NE	DEN	SEA	SF	SEA	SEA	SEA	CAR
2	NE	NO	DEN	NE	SF	SEA	NE	DEN	CAR	ARI
3	NO	NE	GB	SEA	NE	CAR	DEN	GB	ARI	KC
4	PIT	SF	SF	MIN	DEN	ARI	GB	NE	KC	SEA
5	BAL	PIT	ATL	SF	CAR	NE	DAL	DAL	DEN	DEN
6	SF	BAL	SEA	GB	CIN	DEN	PIT	PIT	NE	MIN
7	ATL	DET	NYG	HOU	NO	NO	BAL	ARI	PIT	CIN
8	PHI	ATL	CIN	IND	ARI	IND	IND	IND	CIN	PIT
9	SD	NYG	BAL	WAS	IND	CIN	ARI	DET	GB	GB
10	HOU	SD	HOU	CHI	SD	SD	CIN	BUF	MIN	DET
Av. Diff.	2.8		3.2		2.7		1.9		2.7	

Table 1: Bradley-Terry vs. ELO NFL top 10 rankings. Blue: perfect match, yellow: top 10 match

Based on table 1 we observe that we roughly capture between 6 to 9 of the top 10 ELO teams over all 5 seasons. However we can see that there are often misalignment with specific ranking values across both ranking methods. For example in the 2014 season we can see that our rankings are reasonably well aligned and notably a match with Seattle being the number one ranked team by both methods. The 2012 season had slightly more misalignment comparatively across both methods. This is captured in the average ranking difference between ELO and our time varying Bradley-Terry model being 3.2 which is slightly higher than the 2014 season value of 1.9. We observe that the average differences across all seasons between ELO and the Bradley Terry model are uniformly positive indicating that ELO ranks the same teams higher than the Bradley-Terry model on average across all seasons.

We note that it is difficult to interpret the differences in great detail given that the underlying ranking methodologies are fundamentally different. In particular the NFL ELO ranking methodology uses both the pairwise scores between teams (similar to our time-varying Bradley Terry model) but also uses the location information of each game in the modeling process. In this sense we view the comparable top 10 ranking results as an encouraging indication of our model viability in this real world application. We then view our time-varying Bradley Terry model as a useful benchmarking tool for other feature-rich time-varying ranking models since it simply relies on the minimalist time-varying score information for modeling.

6.4 Application - Nascar Data

We use the NASCAR 2002 data provided and used in [10] paper for the fitting majorization-minimization algorithms for generalized Bradley-Terry models. The version of the data we use comprises the results of 36 automobile races for the 2002 United States NASCAR season in which 83 drivers participated in atleast one race over the course of the season. Each race typically consisted of 43 (or sometimes 42) participating drivers. Our cleaning process considered each race as a sepearate time point i.e. $t \in [36]$. At each time point t we calculated the pairwise ranking difference between teams as the input to our model based on their overall ranking in each race. We note that due to different starting positions in such a race that we do not get a idealized pairwise comparison but what we believe to be a reasonable approximation thereof. Importantly we observe that the data at each time period is very sparse given that about half of the drivers participate in a single race on average and provides a useful test case to understand how well the penalization term handles smoothing over such sparse cases.

7 Conclusion

We propose a time-varying convex generalization of the original Bradley-Terry model (2). Our model directly captures the temporal dependence structure of the pairwise comparison data to model time varying global rankings. In particular the convex penalization term enables analysis on sparse time-varying pairwise comparison data. Furthermore depending on the convex penalization norm chosen our model provides a control on the degree of smoothing in the time-varying global rankings. From a theoretical perspective we proved that a relatively weak condition is necessary and sufficient to guarantee the existence and uniqueness of the solution of our time-varying Bradley Terry model. From an algorithmic perspective we implemented various convex optimization algorithms to solve the model efficiently under the squared-ℓ_2, ℓ_2 and ℓ_1 penalization norms. We finally tested the practical effectiveness of our model by separately ranking 5 seasons of National Football League (NFL) team data from 2011-2015. Our NFL ranking results compare favourably to the well-accepted and feature rich NFL ELO model rankings [14]. We thus view our distribution-free time-varying Bradley Terry model as a useful benchmarking tool for other feature-rich time-varying ranking models since it simply relies on the minimalist time-varying score information for modeling.

In this paper, we described a heuristic approach to tuning the model penalty term λ. This heuristic is driven by user preferences in setting the degree of smoothing in the final global rankings and thus model results are affected by the subjective bias of the user. In future work will develop an empirical method of tuning for the λ parameter via a sample splitting approach. This will be particularly useful in practical settings where we don't have ground truth β_t parameter labels.

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8 Appendix

8.1 Proof of Theorem

Throughout this section, let f denote the target loss function:

$$f(\boldsymbol{\beta}) = -\sum_{t=1}^{T} \ell_t(\boldsymbol{\beta}^{(t)}) + \lambda \sum_{t=1}^{T-1} h(\boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}^{(t+1)}) \tag{13}$$

8.1.1 Uniqueness of the solution with squared-ℓ_2 penalty

We can decompose the loss function with squared-ℓ_2 penalty into two parts:

$$f = \sum_{t=1}^{T} L_t + \sum_{i=1}^{N} R_i \tag{14}$$

where $L_t = -\ell_t(\boldsymbol{\beta}^{(t)})$ and $R_i = \lambda \sum_{t=2}^{T} (\boldsymbol{\beta}_i^{(t)} - \boldsymbol{\beta}_i^{(t-1)})$.

Elements of L_t's Hessian with respect to $\boldsymbol{\beta}^{(t)}$ are:

$$\nabla^2_{\boldsymbol{\beta}_i^{(t)}} L_t = \sum_{i \neq j} (x_{ij}^{(t)} + x_{ji}^{(t)}) \frac{\exp \boldsymbol{\beta}_i^{(t)} \exp \boldsymbol{\beta}_j^{(t)}}{(\exp \beta_i^{(t)} + \exp \beta_j^{(t)})^2} \tag{15}$$

$$\nabla_{\boldsymbol{\beta}_j^{(t)}} \nabla_{\boldsymbol{\beta}_i^{(t)}} L_t = -\, (x_{ij}^{(t)} + x_{ji}^{(t)}) \frac{\exp \boldsymbol{\beta}_i^{(t)} \exp \boldsymbol{\beta}_j^{(t)}}{(\exp \beta_i^{(t)} + \exp \beta_j^{(t)})^2} \tag{16}$$

Also, R_i's Hessian with respect to $\boldsymbol{\beta}_i := (\boldsymbol{\beta}_i^{(1)}, \boldsymbol{\beta}_i^{(2)}, \ldots, \boldsymbol{\beta}_i^{(T)})$ is:

$$\begin{bmatrix} 2 & -2 & 0 & \cdots & 0 & 0 \\ -2 & 4 & -2 & \cdots & 0 & 0 \\ 0 & -2 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -2 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{bmatrix} \tag{17}$$

Hence, both kinds of Hessians have zero column sums and so does the sum of them, i.e., the Hessian of the loss function. Let H denote the Hessian of f and $H(\boldsymbol{\beta}_i^{(t)}, \boldsymbol{\beta}_j^{(s)})$ denote each element of H. Then, H has a positive diagonal, and

$$H(\boldsymbol{\beta}_i^{(t)}, \boldsymbol{\beta}_j^{(s)}) = \begin{cases} -(x_{ij}^{(t)} + x_{ji}^{(t)}) \frac{\exp \boldsymbol{\beta}_i^{(t)} \exp \boldsymbol{\beta}_j^{(t)}}{(\exp \beta_i^{(t)} + \exp \beta_j^{(t)})^2} & \text{if } t = s \\ -2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

As $H(\boldsymbol{\beta}_i^{(t)}, \boldsymbol{\beta}_j^{(t)}) < 0$ if $x_{ij}^{(t)} > 0$ or $x_{ji}^{(t)}$, Condition (1) implies that H can be regarded as a graph Laplacian for a connected graph. Following the classical proof of the property of graph Laplacian [18],

$$v^T H v = \sum_{i<j} |X_{ij}| (v_i - v_j)^2 \geq 0,$$

and Condition (1) guarantees that "=" is achieved if and only if $v = c\mathbf{1}$. This proves the uniqueness up to constant shifts.

8.1.2 Existence of solution

Because of its continuity, h attains its minimum in $\mathbb{R}^T$. Since we still get an equivalent optimization after constant shifting h, we can assume h has minimum value 0 without loss of generality. Also, note that $-\ell_t(\boldsymbol{\beta}_t)$ is non-negative:

$$-\ell_t(\boldsymbol{\beta}_t) = -\sum_{i \neq j} x_{ij}^{(t)} \left(\log \left(\frac{\exp \boldsymbol{\beta}_i^{(t)}}{\exp \boldsymbol{\beta}_i^{(t)} + \exp \boldsymbol{\beta}_j^{(t)}} \right) \right) \geq 0 \tag{19}$$

Plugging in $\boldsymbol{\beta} = \mathbf{0}$, we get an upperbound for the minimum loss function $f^\star$:

$$f^\star \leq (\log 2) \sum_{t=1}^{T} \sum_{i \neq j} x_{ij}^{(t)}. \tag{20}$$

As f is continuous, it suffices to show that the level set with $\sum_{i=1}^{N} \boldsymbol{\beta}_i^{(1)} = 0$ is bounded so that it is compact.

We get an upper-bound on the extent to which $\boldsymbol{\beta}^{(t)}$'s are dispersed in the level set:

$$\|\boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}^{(t+1)}\|_\infty \leq \sqrt{\frac{1}{\lambda}(\log 2) \sum_{i \neq j} x_{ij}^{(t)}} =: B \tag{21}$$

and

$$\|\boldsymbol{\beta}^{(t)} - \overline{\boldsymbol{\beta}}\|_\infty \leq BT \tag{22}$$

where $\overline{\boldsymbol{\beta}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{\beta}^{(t)}$.

Then,

$$-\ell_t(\boldsymbol{\beta}^{(t)}) \geq -\sum_{i \neq j} x_{ij}^{(t)} \log \left(\frac{\exp(\overline{\boldsymbol{\beta}}_i - B)}{\exp(\overline{\boldsymbol{\beta}}_i + B) + \exp(\overline{\boldsymbol{\beta}}_j + B)} \right) \tag{23}$$

$$\geq -\sum_{i \neq j} x_{ij}^{(t)} \log \left(\frac{\exp \overline{\boldsymbol{\beta}}_i}{\exp \overline{\boldsymbol{\beta}}_i + \exp \overline{\boldsymbol{\beta}}_j} \right) - 2BT \sum_{i \neq j} x_{ij}^{(t)} \tag{24}$$

$$\geq \sum_{i \neq j} x_i j^{(t)} \log(1 + \exp(\overline{\boldsymbol{\beta}}_j - \overline{\boldsymbol{\beta}}_i)) - 2BT \sum_{i \neq j} x_{ij}^{(t)} \tag{25}$$

Hence, under the level set,

$$(\log 2) \sum_{t=1}^{T} \sum_{i \neq j} x_{ij}^{(t)} \geq f(\boldsymbol{\beta}) \tag{26}$$

$$\geq \sum_{i \neq j} (\sum_{t=1}^{T} x_{ij}^{(t)}) \log(1 + \exp(\overline{\boldsymbol{\beta}}_j - \overline{\boldsymbol{\beta}}_i)) - 2BT \sum_{t=1}^{T} \sum_{i \neq j} x_{ij}^{(t)} \tag{27}$$

and if $\sum_{t=1}^{T} x_{ij}^{(t)} \neq 0$ then $\overline{\boldsymbol{\beta}}_j - \overline{\boldsymbol{\beta}}_i$ is upperbounded.

By Condition (2) and the constraint $\sum_{i=1}^{N} \boldsymbol{\beta}_i^{(1)} = 0$, now every elements of $\boldsymbol{\beta}$ in the level set are bounded. This proves the existence part of the theorem.

8.2 Optimization Methods

8.2.1 Backtracking Line Search

Algorithm 1 Backtracking Line Search for PGD

Given the variable $\boldsymbol{\beta}$ and the backtracking parameter $b \in (0, 1)$.
Set $s = s_{\text{init}}$.
loop
    $\boldsymbol{\beta}' = \text{prox}_{s,H}(\boldsymbol{\beta} - s\nabla g(\boldsymbol{\beta}))$
    if $g(\boldsymbol{\beta}') \leq g(\boldsymbol{\beta}) + (\nabla g(\boldsymbol{\beta}))^T(\boldsymbol{\beta}' - \boldsymbol{\beta}) + \frac{1}{2s}\|\boldsymbol{\beta}' - \boldsymbol{\beta}\|_2^2$ **then**
        Break the loop.
    end if
    $s \mathrel{*}= b$
end loop
Update $\boldsymbol{\beta} = \boldsymbol{\beta}'$.

8.2.2 general smoothing penalty case

On the other hand, getting the proximal operator for general smoothing penalty functions is nontrivial. In case of l_1-norm penalty, for instance, the marginal proximal optimization problem in 8,

$$\arg\max_{\boldsymbol{\beta}_i \in \mathbb{R}^T} \mathcal{F}_i = \arg\max_{\boldsymbol{\beta}_i \in \mathbb{R}^T} \Big[\frac{1}{s}\|\boldsymbol{\beta}_i - \boldsymbol{\beta}_i'\|_2^2 + \lambda \sum_{t=1}^{T-1} |\boldsymbol{\beta}_i^{(t+1)} - \boldsymbol{\beta}_i^{(t)}|\Big] \tag{28}$$

is the 1-dimensional fused-lasso problem, which does not have a closed form of the solution and requires nontrivial amount of computation to approximate the optimum.

8.2.3 Alternating Direction Method of Multipliers

Let $\boldsymbol{\beta} \in \mathbb{R}^{T\cdot N} = (\boldsymbol{\beta}_1^\top, \cdots, \boldsymbol{\beta}_T^\top)^\top$ be the vector of export scores of all journals in all years. Introduce a new variable $\boldsymbol{\theta} \in \mathbb{R}^{(T-1)N}$, whose entries are given by $\theta(t\cdot N + i) = \mu_i^{t+2} - \mu_i^{t+1}$ for $1 \le i \le N,\ 0 \le t \le T-2$. By introducing a matrix $A \in \mathbb{R}^{[(T-1)N]\times TN}$ we can express $\boldsymbol{\theta}$ by $\boldsymbol{\theta} = A\boldsymbol{\mu}$. For A we can give an explicit expression

$$\begin{aligned}
&A(t\cdot N + i, (t+1)\cdot N + i) = 1,\ 1 \le i \le N,\ 0 \le t \le T-2\\
&A(t\cdot N + i, t\cdot N + i) = -1,\ 1 \le i \le N,\ 0 \le t \le T-2\\
&A(t\cdot N + i, k) = 0, k \ne t\cdot N + i,\ 1 \le i \le N,\ 0 \le t \le T-2
\end{aligned}$$

Now we can write the optimization problem (3) as

$$\begin{aligned}
&\text{minimize} \quad -L_T(\boldsymbol{\beta}) + \lambda \tilde{h}(\boldsymbol{\theta}),\\
&\text{subject to } A\boldsymbol{\beta} = \boldsymbol{\theta},
\end{aligned} \tag{29}$$

where $L_T(\boldsymbol{\beta}) = \sum_{s=1}^T L_s(\boldsymbol{\beta}^{(s)})$, and $\tilde{h} = \sum_{t=1}^{T-1} h(\boldsymbol{\theta}^{I_t})$ with the index set $I_t = \{i : (t-1)\cdot N + 1 \le i \le t\cdot N\}$.

Thus the augmented Lagrangian function should be

$$-L_T(\boldsymbol{\beta}) + \lambda\tilde{h}(\boldsymbol{\theta}) + (A\boldsymbol{\beta} - \boldsymbol{\theta})^\top \boldsymbol{\mu} + \frac{\eta}{2}\|A\boldsymbol{\beta} - \boldsymbol{\theta}\|^2. \tag{30}$$

The alternating direction multiplier minimization scheme can be written as

$$\begin{aligned}
\boldsymbol{\beta}^{k+1} &= \arg\min_{\boldsymbol{\beta}} -L_T(\boldsymbol{\beta}) + (A\boldsymbol{\beta})^\top \boldsymbol{\mu}^k + \frac{\eta}{2}\|A\boldsymbol{\beta} - \boldsymbol{\theta}^k\|^2,\\
\boldsymbol{\theta}^{k+1} &= \arg\min \lambda\tilde{h}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \boldsymbol{\mu}^k + \frac{\eta}{2}\|A\boldsymbol{\beta}^{k+1} - \boldsymbol{\theta}\|^2,\\
\boldsymbol{\mu}^{k+1} &= \boldsymbol{\mu}^k + \eta(A\boldsymbol{\beta}^{k+1} - \boldsymbol{\theta}^{k+1}).
\end{aligned} \tag{31}$$

The update of $\boldsymbol{\theta}$ is simply a proximal operator of $\|\cdot\|_q$, and in the case of $q = 1$,

$$\begin{aligned}
\boldsymbol{\theta}^{k+1} &= \arg\min \lambda\|\boldsymbol{\theta}\|_1 - \boldsymbol{\theta}^\top \boldsymbol{\mu}^k + \frac{\eta}{2}\|A\boldsymbol{\beta}^{k+1} - \boldsymbol{\theta}\|^2\\
&= \arg\min \|\boldsymbol{\theta}\|_1 + \frac{\eta}{2\lambda}\|\boldsymbol{\theta} - A\boldsymbol{\beta}^{k+1} - \frac{\boldsymbol{\mu}^k}{\eta}\|^2\\
&= \text{shrink}(A\boldsymbol{\beta}^{k+1} + \frac{\boldsymbol{\mu}^k}{\eta}, \frac{\lambda}{\eta}).
\end{aligned}$$

The update of $\boldsymbol{\beta}$ is a convex optimization problem of a well-behaved function with no constraints and can be solved by some basic methods, like we do in the squared-ℓ_2 norm case.

8.2.4 Newton methods

We can directly apply Newton method when $h = \|\cdot\|_2^2$ is continuously twice differentiable. In case of $h = \|\cdot\|_1$ and $h = \|\cdot\|_2$, we can use some common tricks to transform the original function into a smooth function with constraints and use some methods that use Newton-type update.

8.3 Synthetic Data

We have conducted a number of simulation experiments in which we know the underlying global ranks of all entries by generating data from a synthetic process. We used the synthetic data to implement our proposed time varying Bradley Terry model with the variety of convex optimization algorithms and to compare the output with the ground truth.

The overall process of the simulation is as follows:

1. The number of entries and the number of time points are given as N and T, respectively.
2. For each $i \in \{1, \ldots, N\}$, simulate time-varying parameter $\boldsymbol{\beta}_i^* \in \mathbb{R}^T$ from a distribution of time-series.
3. For each $i \neq j$ and $t = 1, \ldots, T$, given that entry i and j has $n_{ij}^{(t)}$ matches at the time point t, simulate the number of matches in which entry i out performs entry j according to $\boldsymbol{\beta}^*$ simulated above.

For the choice of distribution in the step 2, we can choose any random time-series to simulate the ground-truth $\boldsymbol{\beta}_i^*$'s. For instance, we could assume the parameter $\boldsymbol{\beta}_i^*$ follows Gaussian process. In our experiment, we simulated the parameter under a Markov chain from which was assumed that $\boldsymbol{\beta}_i^*$ is generated. For each $i = 1, \ldots, N$ the Markov Chain is specified as follows:

$$\begin{aligned} \boldsymbol{\beta}_i^{*(0)} &\sim \mathcal{N}(0, \sigma_{i,0}^2) \\ \boldsymbol{\beta}_i^{*(t+1)} | \boldsymbol{\beta}_i^{*(t)} &\sim \mathcal{N}(\gamma \boldsymbol{\beta}_i^{*(t)}, \sigma_{i,t+1}^2), \, t = 1, \ldots, T-1. \end{aligned} \tag{32}$$

where γ is the decay parameter and $\sigma_{i,t}^2$'s are the conditional variances of $\boldsymbol{\beta}_i^{*(t)}$ with respect to the others. An example of simulated $\boldsymbol{\beta}$ is the Figure 1, (left). In this example, there are 20 entries and 10 time points. The latent parameter $\boldsymbol{\beta}_i^{*(t)}$ is set not to decay (i.e., $\gamma = 1$), and the conditional variance $\sigma_{i,t}^2$ were set to 1 for every $i = 1, \ldots, N$ and $t = 1, \ldots, T$.

In turn, the matches between i and j for $i \neq j$ was simulated on the same probabilistic model as the Bradley-Terry Model:

$$x_{ij}^{(t)} \sim \text{Binom}\Big(n_{ij}^{(t)}, \frac{1}{1 + \exp\{\beta_j^{(t)} - \beta_i^{(t)}\}}\Big), \; x_{ji}^{(t)} = n_{ij}^{(t)} - x_{ij}^{(t)} \tag{33}$$

We generated the pairwise comparison data from the latent parameter $\boldsymbol{\beta}^*$ above with $n_{ij}^{(t)} = 10$ for every $i \neq j$ and $t = 1, \ldots, T$ and solved (2) by GD, PGD, and Newton's method. The resulted maximum likelihood estimator for $\boldsymbol{\beta}$ was gotten as Figure 1, (right). We can compare between two figures of $\boldsymbol{\beta}^*$ and $\hat{\boldsymbol{\beta}}$ to see how our formulation of time-varying Bradley-Terry Model works. First, our model could recover the comprehensive global ranking of each entry at every time point. In both figures, we can check that entry 20 is dominating and entry 2 is maintaining the lowest position at most of the time points. However, we can also find some differences of $\hat{\boldsymbol{\beta}}$ from $\boldsymbol{\beta}^*$ which come from the smoothing property of our model. For instance, the estimated parameter of entry 20 at the first two time points seem overestimated than the ground-truth. These two estimates could be influenced by the following high estimates, and entry 20 is ranked higher than indeed it is for those time points.