Let $S = \{\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_k\}$ be a set of vectors from $\mathbb{R}^n$ and $t_1, ..., t_k \in \mathbb{R}$.

1. **Linear Combinations:**

The vector $\overrightarrow{w} \in \mathbb{R}^n$ is a linear combination of the vectors of S if $\overrightarrow{w}$ can be written as:

$$\overrightarrow{w} = t_1\overrightarrow{v}_1 + t_2\overrightarrow{v}_2 + \cdots + t_k\overrightarrow{v}_k$$

Notes:

(a) If the system $\left[\begin{array}{ccc|c} \overrightarrow{v}_1 & \overrightarrow{v}_2 \cdots & \overrightarrow{v}_k & \overrightarrow{w} \end{array}\right]$ has a non-trivial solution (i.e. consistent), then $\overrightarrow{w}$ can be written as a linear combination of the vectors $\overrightarrow{v}_1$, $\overrightarrow{v}_2$, and $\overrightarrow{v}_k$

(b) To find $t_1, ..., t_k$ in the linear combination, simply solve the system of equations $\left[\begin{array}{ccc|c} \overrightarrow{v}_1 & \overrightarrow{v}_2 \cdots & \overrightarrow{v}_k & \overrightarrow{w} \end{array}\right]$.

<u>Example 1</u>: Write $\vec{w} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$ as a linear combination of $\overrightarrow{v}_1 = \begin{bmatrix} 15 \\ 6 \end{bmatrix}$ and $\overrightarrow{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Must solve: $\left[\begin{array}{cc|c} 15 & 4 & -3 \\ 6 & 3 & -4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -2 \end{array}\right]$

So, $t_1 = \dfrac{1}{3}$ and $t_2 = -2 \Rightarrow \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 15 \\ 6 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ or $\overrightarrow{w} = \dfrac{1}{3}\overrightarrow{v}_1 - 2\overrightarrow{v}_2$

2. **Linear Independence**:

If the only solution to the system $\left[\begin{array}{ccc|c} \overrightarrow{v}_1 & \overrightarrow{v}_2 \cdots & \overrightarrow{v}_k & \overrightarrow{0} \end{array}\right]$ (i.e. $t_1\overrightarrow{v}_1 + t_2\overrightarrow{v}_2 + \cdots + t_k\overrightarrow{v}_k = \vec{0}$) is the trivial solution $t_1 = t_2 = ... = t_k = 0$, then the vectors in S are said to be linearly independent. Otherwise, if there are other solutions then the vectors are called linearly dependent (i.e. at least one vector can be written as a linear combination of the other vectors).

<u>Example 2</u>:

Consider the system $[\overrightarrow{v}_1 \quad \overrightarrow{v}_2 \quad \overrightarrow{w} \quad | \quad \overrightarrow{0}]$. Then $\left[\begin{array}{ccc|c} 15 & 4 & -3 & 0 \\ 6 & 3 & -4 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$

which implies the solution to the system is $t_1 = -\dfrac{1}{3}t_3$ and $t_2 = 2t_3$ (i.e. solution is non-trivial).

Therefore, the vectors in $\overrightarrow{v}_1, \overrightarrow{v}_2$, and $\overrightarrow{w}$ are linearly dependent

Note: Could have also just said they were linearly dependent since $\dfrac{1}{3}\overrightarrow{v}_1 - 2\overrightarrow{v}_2 - \overrightarrow{w} = \vec{0}$ (from Example 1)

Notes:

(a) If matrix $A = [\overrightarrow{v}_1 \quad \overrightarrow{v}_2 \cdots \overrightarrow{v}_k]$ (i.e. columns of A are the vectors in S), then the vectors in S are linearly independent if and only if $rank(A) = k$ (number of vectors in S)

Note: The rank of a matrix A, denoted by $rank(A)$, is the number of leading 1's in its row echelon form (REF)

<u>Example 3</u>: In example 2, $A = [\overrightarrow{v}_1 \quad \overrightarrow{v}_2 \quad \overrightarrow{w}] = \begin{bmatrix} 15 & 4 & -3 \\ 6 & 3 & -4 \end{bmatrix}$ has $rank(A) = 2 \neq 3 = k$.

This confirms that the vectors $\overrightarrow{v}_1, \overrightarrow{v}_2$, and $\overrightarrow{w}$ are linearly dependent.

(b) The number of free variables m in a solution to a system of equations is $m = k - rank(A)$

<u>Example 4</u>: In example 2, the system $[\overrightarrow{v}_1 \quad \overrightarrow{v}_2 \quad \overrightarrow{w} \quad | \quad \overrightarrow{0}]$ has $m = k - rank(A) = 3 - 2 = 1$ free variables (i.e. t_3).