1. (General) Matrix Transformations:

A transformation T_A is defined by $T_A(\vec{x}) = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector and A is called the standard matrix.

Notes:

(a) T_A essentially transforms a vector into another vector through the matrix A.

(b) The initial (or starting) vector is the called the "pre-image" and the resulting (or final) vector is called the image.

Example: Suppose transformation T_A has standard matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the image of vector $\vec{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

$$T_A\left(\begin{bmatrix} 5 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1(5) + 2(-1) \\ 3(5) + 4(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

(c) To find the standard matrix A given a specific function for $T_A(\vec{x})$:

i. Evaluate $T_A(\vec{e}_1), T_A(\vec{e}_2), ..., T_A(\vec{e}_n)$ where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, ..., $\vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ are each of length n.

ii. The standard matrix $A = [T_A(\vec{e}_1) \ \ T_A(\vec{e}_2) \ \ \cdots \ \ T_A(\vec{e}_n)]$ (i.e. columns are transformed vectors found in (a)).

Example: Consider the transformation $T_B(\vec{x}) = \begin{bmatrix} -5x_1 + 7x_2 \\ -9x_1 - 6x_2 \end{bmatrix}$ for any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

Then: $T_B(\vec{e}_1) = T_B\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -5(1) + 7(0) \\ -9(1) - 6(0) \end{bmatrix} = \begin{bmatrix} -5 \\ -9 \end{bmatrix}$

$T_B(\vec{e}_2) = T_B\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5(0) + 7(1) \\ -9(0) - 6(1) \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$

and the standard matrix $B = [T_A(\vec{e}_1) \ \ T_A(\vec{e}_2)] = \begin{bmatrix} -5 & 7 \\ -9 & -6 \end{bmatrix}$

Note: Example above results in same standard matrix as the first example.

2. Composition of Transformations:

Consider two transformations T_A and T_B with standard matrices A and B, respectively. To perform multiple transformations on a vector, determine the composition of the matrix transformations.

Noting that order is important, if transformation T_B is first applied to a vector followed by T_A, then this is denoted by $(T_A \circ T_B)(\vec{x}) = T_A(T_B(\vec{x}))$ and the standard matrix representing the composition of transformations is AB.

Example: Suppose T_A and T_B are the transformations from the examples above. Then:

The standard matrix for "T_B followed by T_A" is $(T_A \circ T_B)(\vec{x})$ with standard matrix $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -5 & 7 \\ -9 & -6 \end{bmatrix}$

$= \begin{bmatrix} -23 & -5 \\ -51 & -3 \end{bmatrix}$

The standard matrix for "T_A followed by T_B" is $(T_B \circ T_A)(\vec{x})$ with standard matrix $BA = \begin{bmatrix} -5 & 7 \\ -9 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 16 & 18 \\ -27 & -42 \end{bmatrix}$

Note that the results are not the same. That is, T_A and T_B don't necessarily commute (i.e. order is important).

3. Geometrical Matrix Transformations:

   Consider transformation $T_A(\vec{x})$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

   (a) T_A is counter-clockwise rotation about origin at angle θ has standard matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

   Note: Use $-\theta$ for a clockwise rotation.

   (b) T_A is horizontal stretch by a factor of $t > 0$ has standard matrix $A = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$

   (c) T_A is vertical stretch by a factor of $t > 0$ has standard matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$

   (d) T_A is scaling (or stretch in both directions) by a factor of t has standard matrix $A = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$

   Note: Also called a dilation if $t > 1$ and a contraction if $0 < t < 1$

   (e) T_A is horizontal shear by s has standard matrix $A = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$

   (f) T_A is vertical shear by s has standard matrix $A = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$

   (g) T_A is reflection over x-axis has standard matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   (h) T_A is ref lection over y-axis has standard matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   (i) T_A is ref lection over $y = x$ has standard matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$