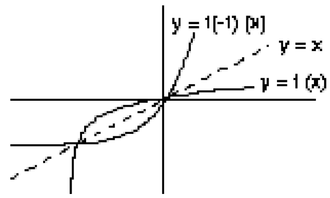
MA102 Lab 5

Inverse Functions (Text: 1.5)

A function f is *one-to-one* if, for all x_1, x_2 in the domain of f, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$; that is, no two different domain values can produce the same function value. A one-to-one function f with domain D and range R then has an inverse function, denoted f^{-1}, having domain R and range D with $f(a) = b$ if and only if $f^{-1}(b) = a$ for all $a \in D$. Thus, two functions f and g are inverses if and only if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for all x.

Graphically, f and f^{-1} are reflections of each other through the line $y = x$.



Be careful! $f^{-1}(x) \neq \dfrac{1}{f(x)}$. The inverse function is denoted by $f^{-1}(x)$ while $\dfrac{1}{f(x)} = [f(x)]^{-1}$ refers to the reciprocal function of f.

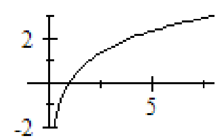
There are various ways of showing that a function is or is not one-to-one. If an accurate graph of the function is obtainable, the *Horizontal Line Test* may be used: if every horizontal line would only intersect the curve in, at most, one point then the function is 1–1.

Algebraically, a function $y = f(x)$ may be shown to be not 1–1 by stating a counterexample; i.e., by giving an example of two different x-values producing the same function value. To conclude that a function is one-to-one, we could prove the contrapositive to be true: we would show that $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ for all x_1, $x_2 \in Dom_f$. MA121 studies such proofs in more detail.

Logarithmic Functions (Text: 1.5)

By its definition, an exponential function is one-to-one and thus has an inverse function, which we call the *logarithmic function*. For example, if $f(x) = a^x$ then $f^{-1}(x) = \log_a x$, or expressed another way: $\log_a x = y$ if and only if $a^y = x$.

Considering $a > 1$, $f(x) = \log_a x$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.



y = $\log_a$x, a >1

Properties of Logarithms include:

$$\log_a(a^x) = x, \quad x \in \mathbb{R} \qquad a^{\log_a x} = x \qquad \log_a(x^r) = r\log_a x, \quad r \in \mathbb{R}$$

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y \qquad \log_a x = \frac{\log_b x}{\log_b a}$$

These relations hold whenever all the logarithms are defined [i.e., we assume a, b, x and y are positive real numbers unless otherwise indicated].

In the special case of $f(x) = e^x$, we write $f^{-1}(x) = \ln x$, the *natural logarithm* of x [$\log_e x = \ln x$]. Note that $\ln e = 1$ and $\ln 1 = 0$. Also, be sure that you know how to evaluate expressions involving e and $\ln$ on your own calculator!

An additional warning when working with logarithms: $\ln^2 x \neq \ln x^2$. We can apply the relevant property to write $\ln x^2 = \ln(x^2) = 2\ln x$, but $\ln^2 x$ means $(\ln x)^2$ which is generally not equal to $2\ln x$.

Properties can be applied to the logarithm of a complicated expression in order to expand it as the sum/difference of simpler expressions.

Example: $\ln\left(\dfrac{2^x\sqrt{3-4x}}{x^3\sin x}\right) = \ln\left(2^x\right) + \ln\left(3-4x\right)^{1/2} - \left[\ln\left|x^3\right| + \ln\left|\sin x\right|\right]$
$\qquad = x\ln 2 + \frac{1}{2}\ln|3-4x| - 3\ln|x| - \ln|\sin x|$

Take note of the absolute value bars on the right hand side of the equal sign. When we use logarithmic properties to "break down" an expression, we need to add absolute value bars to any argument of a logarithm that could possibly have negative values, to ensure that the new expression is defined for all values for which the original expression was defined. In our above example, without the absolute value bars the expression on the left would not be equivalent to the expression on the right as, for instance, $x = -\frac{\pi}{2}$ would make the left side defined while the right side would be undefined.

Inverse Trigonometric Functions

To have an inverse, a function must be one-to-one. Thus to define inverse trigonometric functions, we restrict the domain of the trigonometric functions so that they become one-to-one. The inverse trig functions and their derivatives are summarized by the following chart:

Function	**Domain**	**Range**
$y = \sin^{-1} x \Leftrightarrow x = \sin y$	$-1 \leq x \leq 1$	$-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$
$y = \cos^{-1} x \Leftrightarrow x = \cos y$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \tan^{-1} x \Leftrightarrow x = \tan y$	$x \in \mathbb{R}$	$-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$
$y = \csc^{-1} x \Leftrightarrow x = \csc y$	$\lvert x\rvert \geq 1$	$0 < y \leq \dfrac{\pi}{2}, \quad \pi < y \leq \dfrac{3\pi}{2}$
$y = \sec^{-1} x \Leftrightarrow x = \sec y$	$\lvert x\rvert \geq 1$	$0 \leq y < \dfrac{\pi}{2}, \quad \pi \leq y < \dfrac{3\pi}{2}$
$y = \cot^{-1} x \Leftrightarrow x = \cot y$	$x \in \mathbb{R}$	$0 < y < \pi$

Inverse trig functions can also be denoted using the descriptive "arc": $\sin^{-1}(x) \equiv \arcsin(x)$, etc.

Note, for instance, $\sin\left(\sin^{-1} x\right) = x$ for all $x \in [-1, 1]$, the domain of $\arcsin$; but, $\sin^{-1}(\sin x) = x$ if and only if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ while the domain of sine is all reals. Thus, for values of $x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the angle that is within $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which has an equivalent sine ratio must first be determined [using the CAST rule, etc.]. For example:

$\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$ as $\sin\left(\frac{3\pi}{4}\right) \equiv \sin\left(\frac{\pi}{4}\right)$.

Lab Preparation

1. Complete textbook practice problems.
2. Complete WeBWorK homework assignment (link found on MyLearningSpace).
3. Go through posted Maple worksheet.