Binomial Distribution:

This deal with the random experiment which can be repeated a number of times resulting in two outcomes, success and failure and trials are independent of each other.

Let the number of trials in conducting the experiment be "n" of which "x" trials result in success so that "n-x" trials give a failures.

Let the probability of success is "p" and the probability of failure is "q"

The probability distribution function for above random experiment is given as

$$\boxed{P(X=x) = P(x) = nC_x \, p^x \, q^{n-x}}$$

Where $x = 0, 1, 2, \dots n$, $q = 1-p$.

where $p(x)$ is called Binomial distribution OR Bernoulli distribution.

Mean and variance of a Binomial distribution

for $P(x) = nC_x \, p^x q^{n-x}$

WKT $\mu = \sum x_i p_i$

$\mu = \sum x_i [nC_{x_i} p^{x_i} q^{n-x_i}]$ (after simplification & NO proof)

$$\boxed{\mu = np}$$

Also

$$\text{Variance} = \sigma^2 = \sum x_i^2 p_i - \mu^2$$

$$\sigma^2 = \sum x^2 \left[nC_x\, p \cdot q^{n-x} \right] - \mu^2$$

$$\boxed{\sigma^2 = npq}$$ (without proof)

Standard Results on Binomial distribution

1. $P(x) = nC_x\, p^x q^{n-x} \; : \; x = 0, 1, 2 \dots n.$

2. Mean $= \mu = np$

3. Variance $= \sigma^2 = npq, \quad q = 1 - p.$

4. $nC_r = \dfrac{n!}{r!(n-r)!}$ [combination of r items out of n]

5. Atleast $\longrightarrow$ Minimum $\longrightarrow \geq$

6. Atmost $\longrightarrow$ Maximum $\longrightarrow \leq$

Note: Binomial frequency distribution:

If "n" independent trials constitute one experiment and this experiment is repeated for "N" times then the frequency of "x" success is given as

$$\boxed{f = N\left[nC_x\, p^x q^{n-x} \right]}$$

The possible number of success together with these expected frequencies constitute Binomial frequency Distribution

Problems based on Binomial distribution:

1) If X is a Binomially distributed random variable with mean and variance of X are 2 and 3/2 respectively. (i) find the binomial distribution (ii) find $P(x \le 2)$ (iii) $P(x > 5)$ (iv) $P(x < 7)$.

Solution: Since X is binomially distributed random variable

$\therefore$ Probability distribution function of X is

$$\boxed{P(x) = nC_x\, p^x q^{n-x}}$$

with mean $= \mu = np$

Variance $= \sigma^2 = npq$.

Given $\mu = 2$ & $\sigma^2 = 3/2$

$\therefore$ $np = 2$, $npq = 3/2$

Since $np = 2 \Rightarrow 2q = 3/2$

$$\boxed{q = \frac{3}{4}}$$

WKT $q = 1 - p \Rightarrow p = 1 - q = 1 - \frac{3}{4}$

$$\boxed{p = \frac{1}{4}}$$

To find 'n',

WKT $np = 2$

$n\left(\frac{1}{4}\right) = 2 \Rightarrow \boxed{n = 8}$

using $n = 8$, $p = \frac{1}{4}$, $q = \frac{3}{4}$ In $P(x)$ we get

$P(x) = 8C_x \left[\frac{1}{4}\right]^x \left[\frac{3}{4}\right]^{8-x}$ which is required binomial distribution.

with $x = 0, 1, 2, 3, \dots 8$.

To find (ii) $P(x \leq 2)$

ie $P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$

$$P(x \leq 2) = 8C_0\left[\frac{1}{4}\right]^0\left[\frac{3}{4}\right]^{8-0} + 8C_1\left[\frac{1}{4}\right]^1\left[\frac{3}{4}\right]^{8-1}$$

$$+ \; 8C_2\left[\frac{1}{4}\right]^2\left[\frac{3}{4}\right]^{8-2}$$

$$P(x \leq 2) = 8C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^8 + 8C_1\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^7 + 8C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6$$

$$= 1\,[1]\,[0.1001] + 8\,[0.25]\,[0.1334]$$

$$+ \; 28\,[0.0625]\,[0.1779]$$

$$= 0.1001 + 0.2668 + 0.311325$$

(ii) $\boxed{P(x \leq 2) = 0.678225}$

(iii) $P(x > 5) = P(x=6) + P(x=7) + P(x=8)$

$$= 8C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{8-6} + 8C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^1 + 8C_8\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^0$$

$$= 8C_6\,(0.25)^6(0.75)^2 + 8C_7\,(0.25)^7(0.75)^1 +$$

$$8C_8\,(0.25)^8(0.75)^0$$

$P(x > 5) = 0.00385 + 0.000366 + 0.00015$

$\boxed{P(x > 5) = 0.004366}$

(iv) $P(x < 7) = 1 - P(x \geq 7)$

$1 - [P(x=7) + P(x=8)]$

$= 1 - (0.26680 + 0.311325) = 0.42188.$

(2) Six coins are tossed. find the probability of getting (i) exactly 3 heads (ii) atleast 2 heads (iii) atmost 4 heads (iv) No tail.

Solution: Let us take

X: Number of heads in a tossing a coin for six times distributed binomially

$\therefore X = \{0, 1, 2, 3, 4, 5, 6\}$

WKT binomial distribution

$$\boxed{P(x) = nC_x\, p^x q^{n-x}} \longrightarrow ①$$

where n: no of trials $= 6$

p: probability of success ie, probability of getting head $= 1/2$

q: probability of failure ie, probability of getting tail $= 1/2$. OR $q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$

$\therefore ① \Rightarrow$

$$P(x) = 6C_x \,(1/2)^x \cdot \left(\frac{1}{2}\right)^{6-x}$$

$$\boxed{P(x) = 6C_x\,(0.5)^x\,(0.5)^{6-x}} \longrightarrow ②$$

(i) $P(\text{exactly 3 heads}) = P(x=3) = 6C_3\,(0.5)^3(0.5)^3$

$$\boxed{P(x=3) = 0.3125}$$

(ii) $P(\text{atleast 2 heads}) = P(x \geq 2)$

$P(x \geq 2) = P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6)$

OR $P(x \geq 2) = 1 - [P(x < 2)]$

$= 1 - [P[(x=0)] + P(x=1)]$

$1 - [6C_0\,(0.5)^0(0.5)^6 + 6C_1\,(0.5)^1(0.5)^5]$

$P(x \geq 2) = 1 - [0.01563 + 0.09375]$

$= 1 - (0.10938)$

$P(x \geq 2) = 0.89062$

(iii) $P(\text{atmost 4 heads}) = P(x \leq 4)$

$= P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$
$= 0.015625 + 0.09375 + 0.234375 + 0.3125 + 0.234375$

$\boxed{P(x \leq 4) = 0.890625}$

(iv) $P(\text{no tail}) = P(\text{all the time head})$

$= P(x=6)$
$= {}^6C_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^{6-6}$

$\boxed{P(\text{no tail}) = 0.015625}$

3) If the consignment of bulbs manufactured by a factory are distributed binomially with the mean numbers of defectives 10 and standard deviation 3 in a consignment. find the probability that
(i) there is no defective bulb
(ii) there are two defective bulbs
(iii) Atmost 3 defective bulbs.

Solution: X: Number of defective bulbs in a consignment distributed binomially

$\therefore P(x) = {}^nC_x p^x q^{n-x}$

where n: no of bulbs.
p: probability of getting defective bulb

Also given mean no of defective bulbs $= 10$
and standard deviation $= \sigma = 3$

WKT $\quad$ mean $= np = 10$

$SD = \sqrt{npq} = 3$

$\Rightarrow npq = 9$

$\Rightarrow 10q = 9 \Rightarrow q = \frac{9}{10} = 0.9$

$\Rightarrow q = 1 - p$ OR

$p = 1 - q = 1 - 0.9 = 0.1$

$\boxed{p = 0.1}$

To find 'n'

$np = 10$ ~~[illegible]~~

$n(0.1) = 10 \Rightarrow \boxed{n = 100}$

Put $n = 100$, $p = 0.1$, $q = 0.9$ in $P(x) = nC_x p^x q^{n-x}$

$P(x) = 100C_x (0.1)^x (0.9)^{100-x}$

(i) $P(x=0) = 100C_0 (0.1)^0 (0.9)^{100-0}$

$= 2.656 \times 10^{-5} = 0.00002656$

(ii) $P(x=2) = 100C_2 (0.1)^2 (0.9)^{98}$

$= 0.001623$

(iii) $P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= 0.00002656 + 0.0002952 + 0.001623 + 0.00589$

$\boxed{P(x \leq 3) = 0.00783}$

4) The number of telephone lines busy at an instant of time is a binomial variate with probability 0.2. If at an instant 10 lines are chosen at random, what is the probability that
(i) 5 lines are busy (ii) at the most 2 lines are busy (iii) atleast one line is busy (iii) all the lines are busy.

Solution: Let X : Number of lines are busy distributed binomially

$$\therefore \boxed{P(x) = {}^nC_x p^x q^{n-x}} \quad ; n = 10. \quad x = 0, 1, 2 \ldots 10$$

Given : probability of line is busy $= p = 0.2$

$\Rightarrow q = 1 - 0.2 = 0.8$

$$\Rightarrow \boxed{P(x) = 10C_x (0.2)^x (0.8)^{10-x}}$$

(i) $P(5 \text{ lines are busy}) = P(x=5) = 10C_5 (0.2)^5 (0.8)^{10-5}$

$$P(x=5) = 10C_5 (0.2)^5 (0.8)^5 = 0.0264.$$

(ii) $P(\text{at the most 2 lines are busy}) = P(x \leq 2)$

$$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$$

$$P(x \leq 2) = 10C_0 (0.2)^0 (0.8)^{10-0} + 10C_1 (0.2)^1 (0.8)^{10-1} + 10C_2 (0.2)^2 (0.8)^{10-2}$$

$$P(x \leq 2) = 1 (0.2)^0 (0.8)^{10} + 10 (0.2) (0.8)^9 + 45 (0.2)^2 (0.8)^8$$

$$= 0.1074 + 0.2684 + 0.30199$$

$$P(x \leq 2) = 0.67779$$

(iii) $P(\text{atleast one line busy}) = P(x \geq 1)$

$$P(x \geq 1) = 1 - P(x < 1)$$

$$1 - P(x=0)$$

$$1 - [10C_0 (0.2)^0 (0.8)^{10-0}]$$

$$1 - (0.1074) = 0.8926$$

(5) (iv) P(all the lines are busy) = $P(x=10)$

$$P(x=10) = {}^{10}C_{10} (0.2)^{10} (0.8)^{10-10}$$

$$= {}^{10}C_{10} (0.2)^{10} (0.8)^{0}$$

$$= (0.2)^{10} = 0.000000102 \approx 0.$$

(5) An air line knows that 5% of the people making reservations on a certain flight will not turn up. consequently their policy is to sell 52 tickets for a flight that can only hold 50 passengers. What is the probability that there will be a seat for every passenger who turn up.

Solution: X : Number of passengers fails to turn up distributed binomially

$$\therefore P(x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x}, \quad n = 52$$

$$x = 0, 1, 2, 3 \ldots 52.$$

Given: probability of passenger fails to turn up $= p = 5\% = 0.05$

$$\Rightarrow \quad q = 1-p = 1-0.05 = 0.95 \ (95\%).$$

$$P(x) = {}^{52}C_{x} (0.05)^{x} (0.95)^{52-x}$$

A seat is assured for every passenger who turns up, if the number of passengers who fail to turn up is more than OR equal to 2.

$\therefore$ P(seat is assured for every passenger) = $P(x \geq 2)$

$$P(x \geq 2) = 1 - P(x < 2)$$

$$1 - [P(x=0) + P(x=1)]$$

$$P(x \geq 2) = 1 - \left[52C_0 (0.05)^0 (0.95)^{52} + 52C_1 (0.05)^1 (0.95)^{51} \right]$$

$$1 - [0.06944 + 0.19005]$$

$$\boxed{P(x \geq 2) = 0.74051}$$

6) In sampling a large number of parts manufactured by a company, the mean number of defectives in sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain (i) No defective (ii) atleast two defective (iii) at most three defective parts.

Solution: X: Number of defective parts in a sample of 20 is distributed binomially

$\therefore P(x) = nC_x \, p^x q^{n-x}$; $n = 20$

Given Mean $= 2 = np$ $\quad x = 0, 1, 2, 3 \ldots 20$

$2 = 20p \Rightarrow p = 0.1, \; q = 0.9.$

$$\boxed{P(x) = 20C_x \, (0.1)^x (0.9)^{20-x}}$$

(i) P(No defective part) = $P(x=0)$

$P(x=0) = 20C_0 \, (0.1)^0 (0.9)^{20-0}$

$P(x=0) = 0.12158.$

Hence,

Out of 1000 such samples, number of samples with no defective parts is equal to

$f = 1000 \cdot P(x=0) = 1000 \times [0.12158]$

$$\boxed{f = 12.158 \approx 12}$$

(ii) $P(\text{atleast two defective}) = P(x \geq 2)$

$$P(x \geq 2) = 1 - P(x < 2)$$
$$= 1 - \{P(x=0) + P(x=1)\}$$
$$1 - \left[20C_0 (0.1)^0 (0.9)^{20} + 20C_1 (0.1)^1 (0.9)^{19} \right]$$
$$1 - [0.12158 + 0.27017]$$

$$\boxed{P(x \geq 2) = 0.608403}$$

Hence,

Out of 1000 such samples, number of samples with atleast two defective parts is equal to

$$f = 1000 \times P(x \geq 2) = 1000 [0.608403]$$

$$\boxed{f = 608.403 \simeq 608}$$

(iii) $P(\text{atmost three defective parts}) = P(x \leq 3)$

$$P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$
$$= 0.12158 + 0.27017 + 0.28518$$
$$= 0.67693$$

Hence,

Out of 1000 such samples, number of samples with atmost three defective parts is equal to

$$f = 1000 \times P(x \leq 3)$$

$$f = 1000 \times (0.67693)$$

$$\boxed{f = 676.93 \simeq 677}$$

(8) Out of 800 families with four children each, how many families would be expected to have (i) Two boys (ii) atleast one boy (iii) atmost three boys (iv) atmost two girls. Assume equal probabilities for boys and girls.

Solution: Let X : Number of boys in a family of four children distributed Binomially.

$\therefore \quad X = \{0, 1, 2, 3, 4\}$.

Let p = probability of having boy = 1/2
q = probability of having girl = 1/2

$$P(x) = nC_x \, p^x q^{n-x} \quad ; \quad n = 4, \; p = \frac{1}{2}, \; q = \frac{1}{2}$$

$$P(x) = 4C_x \left[\frac{1}{2}\right]^x \left[\frac{1}{2}\right]^{4-x}$$

$$P(x) = 4C_x \left(\frac{1}{2}\right)^4 = \frac{4C_x}{2^4} = \frac{4C_x}{16}$$

(i) $P[\text{Two boys}] = P(x=2) = \frac{4C_2}{16} = \frac{6}{16}$

hence,
The expected number of families out of 800, having two boys is equal to

$$f = 800 \times P(x=2) = 800 \times \frac{6}{16} = 300.$$

(ii) $P[\text{atleast one boy}] = P[x \geq 1] = 1 - P(x < 0)$

$$P(x \geq 1) = 1 - \left[\frac{4C_0}{16}\right] = 1 - \frac{1}{16} = \frac{15}{16}$$

hence, The expected number of families out of 800, having atleast one boy is equal to

$$f = 800 \times P(x \geq 1) = 800 \times \frac{15}{16} = 750$$

iii) P[atmost three boys] = $P(x \leq 3) = 1 - P(x > 3)$

$$P(x \leq 3) = 1 - [P(x=4)]$$

$$1 - \left[\frac{{}^4C_4}{16}\right] = 1 - \frac{1}{16} = \frac{15}{16}$$

hence, The expected number of families out of 800, having atmost three boys is equal to

$$f = 800 \times [P(x \leq 3)] = 800 \times \left[\frac{15}{16}\right] = 750.$$

iv) P(Atmost two girls) = P(Atleast two boys)

$$= P(x \geq 2) = P(x=2) + P(x=3) + P(x=4)$$

i.e family with
0 G → 4 B
1 G → 3 B
2 G → 2 B

$$= \frac{{}^4C_2}{16} + \frac{{}^4C_3}{16} + \frac{{}^4C_4}{16}$$

$$= \frac{6+4+1}{16} = \frac{11}{16}$$

hence, The expected number of families out ~~number~~ of 800, having atmost two girls is equal to

$$f = 800 \times [P(x \geq 2)] = 800 \times \left[\frac{11}{16}\right]$$

$$f = 550.$$

(9) In a binomial distribution, consisting of 5 independent trials, probabilities of 1 and 2 success are 0.4096 and 0.2048 respectively. find the parameter 'p' of the distribution and also write binomial distribution.

Solution: Let X is a random variable distributed binomially.

$\therefore \quad p(x) = {}^nC_x\, p^x q^{n-x}, \quad \boxed{n=5}$ (given)

$\Rightarrow \quad P(x) = {}^5C_x\, p^x q^{5-x}$ —— ①

given $p(x) = P(1) = {}^5C_1\, p^1 q^{5-1} = 5pq^4$

$$\boxed{0.4096 = 5pq^4} \text{ —— ②}$$

Also: $P(x=2) = {}^5C_2\, p^2 q^{5-2}$

$$\boxed{0.2048 = 10p^2q^3} \text{ —— ③}$$

To find p OR q, let us take ② ÷ ③

$$\frac{0.4096}{0.2048} = \frac{5pq^4}{10p^2q^3} = \frac{q}{2p}$$

$$2 = \frac{q}{2p} \Rightarrow 2(2p) = q \Rightarrow 4p = q$$

$4p = 1-p \qquad \boxed{\because q = 1-p}$

$5p = 1 \Rightarrow p = \frac{1}{5} = 0.2$

$$\boxed{q = 0.8}$$

To write binomial distribution, Put $p = 0.2, q = 0.8$ in ① we get

$$P(x) = {}^5C_x\, [0.2]^x\, [0.8]^{5-x}$$

(10) The screws produced by a certain machine were checked by examining samples of 12. The following table shows the distribution of 128 samples according to the number of defective items they obtained.

No. of defective items	0	1	2	3	4	5	6	7
No. of samples	7	6	19	35	30	23	7	1

Fit a binomial distribution for the above data and also calculate theoretical frequencies.

Solution: Let X: Number of defective items in a sample of 12.

$\therefore X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$$P(x) = {}^nC_x \, p^x q^{n-x}, \quad \boxed{n = 12}$$

Since p is not given, let us calculate p using mean of the distribution statistically.

WKT $\quad$ mean $= np$

$$\frac{\sum x_i f_i}{\sum f_i} = np = 12p$$

$$\therefore 12p = \frac{\left[\begin{array}{l}(0\times7)+(1\times6)+(2\times19)+(3\times35)+ \\ (4\times30)+(5\times23)+(6\times7)+(7\times1)\end{array}\right]}{128}$$

$$12p = \frac{433}{128} = 3.3829$$

$$\boxed{p = \frac{3.3829}{12} = 0.2819} \;\&\; q = 1-p = 1-0.2819$$

$$q = 0.7181.$$

using $n = 12$, $p = 0.2819$ & $q = 0.7181$ in

$P(x) = nC_x p^x q^{n-x}$ we get binomial distribution for the data given as

$$\boxed{P(x) = 12C_x [0.2819]^x [0.7181]^{12-x}}$$

To find theoretical frequencies put $x = 0, 1, 2, 3, 4, 5, 6, 7$.

x	$P(x)$	$f = 128 \times P(x)$
0	0.0188	$2.4067 \approx 2.00 \approx 3$
1	0.0885	$11.3375 \approx 11.00$
2	0.19124	$24.47 \approx 24.00 \approx 25$
3	0.25025	$32.031 \approx 32.00$
4	0.2210	$28.29 \approx 28.00$
5	0.1388	$17.77 \approx 18.00$
6	0.0635	$8.13 \approx 8.00$
7	0.0213	$2.73 \approx 3.00$
		$\approx 128.$

Test Yourself

11) The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that (i) Exactly two will be defective.
(ii) atleast two will be defective.
(iii) atmost 10 will be defective
(iv) none will be defective.

12) When a coin is tossed 5 times, find the probability of getting (i) exactly two heads (ii) atmost 3 heads (iii) atleast one head.

13) Given that 2% of the fuses manufactured by a firm are defective. find the probability that a box containing 200 fuses are (i) atleast one defective (ii) three or more defective (iv) atleast one defective.

14) fit a binomial distribution for the data given below, and also calculate theoretical frequencies.

(i)

x	0	1	2	3	4	5
f	4	14	20	34	22	6

(ii)

x	0	1	2	3	4
f	5	29	36	25	5

= x =

MQP - 1: (7 Marks)
(15) The probability of germination of a seed in a packet of seeds is found to be 0.7. If 10 seeds are taken for experimenting on germination in a laboratory find the probability that (i) 8 seeds germinate (ii) atleast 8 seeds germinate (iii) atmost 8 seeds germinate.