**1.13**

Let $X_1, \ldots, X_n$ be independent random variables, all having the same distribution with expected values $\mu$ and variance $\sigma^2$ The Random Variable $\bar{X}$, defined as the arithmetic average of these variables is called the sample mean. That is, the sample mean is given by:

$\bar{X} = \frac{\Sigma_{i=1}^{n} X_i}{n}$

(a) Show that $E[\bar{X}] = \mu$

(b) Show that $Var(\bar{X}) = \frac{\sigma^2}{n}$

The random variable $S^2$ is called the sample variance, defined by:

$S^2 = \frac{\Sigma_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$

(c) Show that $\Sigma_{i=1}^{n}(X_i - \bar{X})^2 = \Sigma_{i=1}^{n} X_i^2 - n\bar{X}^2$

(d) Show that $E[S^2] = \sigma^2$

***1.13a Solution***

$E[\bar{X}]$

$= E[\frac{\Sigma_{i=1}^{n} X_i}{n}] = \frac{1}{n} E[\Sigma_{i=1}^{n} X_i]$ `by proposition 1.3.1`

given that $\frac{1}{n}(E[X_1] + E[X_2] + \ldots + E[X_n]) = \mu$

then $\frac{1}{n}(\mu_1 + \mu_2 + \ldots \mu_n) = \mu$

***1.13b Solution***

$Var(\bar{X}) = Var(\frac{\Sigma_{i=1}^{n} X_i}{n})$

$= (\frac{1}{n}^2 Var(X_1 + X_2 + \ldots + X_n)$. `from equation 1.8 and expanding the summation`

$= (\frac{1}{n}^2(\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2)$ `given prop 1.3.2 and the stated variance`

$= (\frac{1}{n})^2 \cdot n\sigma^2$

$= \frac{\sigma^2}{n}$

***1.13c Solution***

Expand the binomial

$\Sigma_{i=1}^{n}(X_i - \bar{X})^2 = \Sigma_{i=1}^{n}(X_i^2 - 2X\bar{X} + \bar{X}^2)$

$= \Sigma_{i=1}^{n}X_i^2 + \Sigma_{i=1}^{n}2X_i\bar{X} + \Sigma_{i=1}^{n}\bar{X}^2$

using the useful identity

$\Sigma X_i = n\bar{X}$

we substitue in and get

$\Sigma_{i=1}^{n}X_i^2 - 2n\bar{X}^2 + n\bar{X}^2$

and combine like terms

$\Sigma_{i=1}^{n}X_i^2 - n\bar{X}^2$

#### ***1.13d Solution***

L.H.S. $E[S^2] = E[\frac{\Sigma(X_i - \bar{X})^2}{n-1}]$

$(n-1)E(S^2) = E[\Sigma(X_i - \bar{X})^2]$

Then for R.H.S we expand out...

$E[\Sigma(X_i - 2X_i\bar{X} + \bar{X}^2)]$

$E[\Sigma X_i^2 - 2\bar{X}\Sigma X_i + \Sigma\bar{X}^2]$

$E[\Sigma X_i^2 - 2n\bar{X} + n\bar{X}^2]$. Note that $\Sigma\bar{X}^2 = n\bar{X}^2$ and $\Sigma X_i = n\bar{X}$

$= E[\Sigma X_i^2 - n\bar{X}^2] = \Sigma E[X_i]^2 - E[n\bar{X}^2]$ using proposition 1.3.1

$= \Sigma E[X_i]^2 - nE[\bar{X}^2]$

`Special substitution notes`

1. $Var(\bar{X}) = E[\bar{X}^2] - (E[\bar{X}])^2$

therefore $E[\bar{X}^2] = \frac{\sigma^2}{n} + \mu^2$

2. $Var(X) = E[X^2] - (E[X])^2$

therefore $E[X^2] = \sigma^2 + \mu^2$

Then

$= \Sigma(\sigma^2 + \mu^2) - n(\frac{\sigma^2}{n} + \mu^2)$

$= n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2$

$= (n-1)\sigma^2$

Going back to the top

$(n-1)E[S^2] = (n-1)\sigma^2$

$E[S^2] = \sigma^2$

#### 1.14

Verify that $Cov(X, Y) = E[XY] - E[X][Y]$

#### *1.14 Solution*

Given $E[X] = \mu$

$Cov(X, Y) = E[(X - E[X)(Y - E[Y])]$

$= E[(XY - XE[Y] - E[X]Y + \mu_X \mu_Y)]$

$= E[(XY - X\mu_Y - Y\mu_X + \mu_X \mu_Y)]$

$= E[XY] - \mu_Y E[X] - \mu_X E[Y] + \mu_x \mu_Y$ `by 1.7`

$= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y]$

$= E[XY] - E[X][Y]$

#### 1.15a

Prove $Cov(XY) = Cov(YX)$

#### *1.15a solution*

$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$

$= E[(Y - E[Y])(X - E[X]]$ `by commutative property`

$= Cov(Y, X)$

#### 1.15c

Prove $Cov(cX, Y) = cCov(X, Y)$

#### *1.15c Solution*

$Cov(cX, Y) = E[c(X - E[x])(Y - E[Y])]$

$= E[cXY - cX\mu_Y - cY\mu_X + c\mu_X \mu_Y]$

$= cE[XY] - c\mu_Y \mu_X - cY\mu_X + c\mu_X \mu_Y$

$= c(E[XY] - E[X]E[Y])$

### 1.16

If U and V are independent random variables, both having variance 1, find $Cov(X, Y)$ when:

$X = aU + bV$ and $Y = cU + dV$

#### *1.16 Solution*

$Cov(X, Y) = Cov(aU + bV, cU = dV)$

$= Cov(aU, cU) + Cov(aU, dV) + Cov(bV, cU) + Cov(bV, dV)$

$= acCov(U, U) + ad(U, V) + bcCov(V, U) + bdCov(bd)$

$= acVar(U, U) + adCov(U, V) + bcCov(U, V) + bdVar(V, V)$

$= ac + adCov(U, V) + bcCov(U, V) + bd$ `because Cov(U,V) is the covariance of independent vars,  it is zero`

$= ac + bd$

### 1.19

Can you construct a pair of random variables such that $Var(X) = Var(Y) = 1$, and $Cov(X, Y) = 2$?

#### *1.19 Solution*

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Answer is no
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Because

$-1 <\leq \rho(X, Y) \leq 1$

and $\rho(X, Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

thus $-1 \leq \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \leq 1$

$= -1 \leq \frac{2}{\sqrt{1 \cdot 1}} \leq 1$

$-1 \leq 2 \leq 1$ whichi is an impossible inequality

### 1.21

The distribution function $F(x)$ of the random variable X is defined by

$F(x) = P(X \leq x)$

If $X$ takes on one of the values $1, 2, \ldots,$ and $F$ is a known function, how would you obtain $P(X = i)$?

#### *1.21 Solution*

$P(X = i) = F(1) - F(1 - i)$

In [ ]: