$Derek\ Haynes$
$Bridge\ To\ Abstract\ Math$
$Week\ 8$

Section 3.1 Relations

2.

Let T be the relation $\{(1,3),(2,3),(3,5),(2,2),(1,6),(2,6),(1,2)\}$

(d) Find $(T^{-1})^{-1}$

Solution:

$T^{-1} is B to A$

$= \{(3,1),(3,2),(5,3),(2,2),(6,1),(6,2),(2,1)\}$

so $(T^{-1})^{-1} = T$

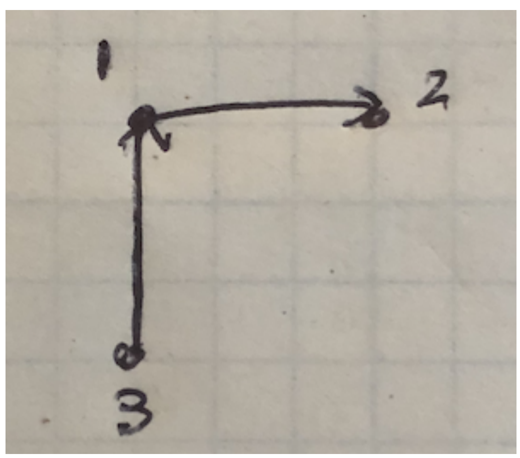
9.

Give the digraphs for these relations on the set $\{1,2,3\}$ where $S = \{(1,3),(2,1)\}$

(d) S^{-1}

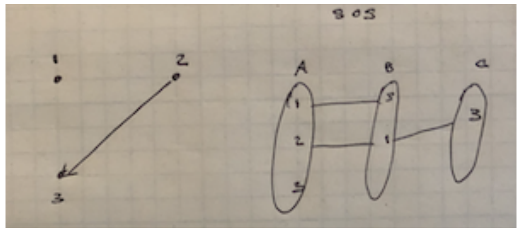
Solution:

$S^{-1} = \{(3,1),(2,1)\}$



(f) $S \circ S$

Solution:



11.

(b) Let R be a relation from $A\ to\ B$ and S be a relation from $B\ to\ C$.

Prove that $Dom(S \circ R) \subseteq Dom(R)$

Solution:

$S \circ R = \{(a, c)\ :\ there\ exists\ b \in B\ such\ that (a, b) \in R\ and\ (b, c) \in S\}$

then

$Dom(S \circ R) = \{a \in A : (a, b) \in R\ for\ some\ b \in B\}$

And

$Dom(R) = \{a \in A : (a, b) \in R\ for\ some\ b \in B\}$

then if $x \in Dom(S \circ R)$ then $x \in Dom(R)$

then by the definition of subsets

$Dom(S \circ R) \subseteq Dom(R)$.

12.

Complete the proof of Theorem 3.1.2 by proving that if R is a relation from A to B and S is a relation from B to C, then :$

(b) $R \circ I_A = R$

Solution:

Suppose

$a \in A$ and $b \in B$

Let R be the relations from A to B and I_A is the identity relation on the set A.

Then

$R = \{(a, b) : a \in A \; and \; b \in B)\}$

$I_A = \{(a, a) : a \in A\}$

$R \circ I_A = \{(a, b) : \; there \; exists \; an \; a \in A \; such \; that \; (a, a) \in I_A, and (a, b) \in R\}$

Since R is a relation from $A to B

this is true for ever $(a, b) \in R$

Therefore,

$R \circ I_A = R$

(c) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Let R be the relations from A to B and S is the relation on the set B to C.

Then

$R = \{(a, b) : a \in A \; and \; b \in B)\}$

$S = \{(b, c) : b \in B \; and \; c \in C)\}$

Then the composite of R and S is,

$S \circ R = \{(a, c) : \; there \; exists \; b \in B \; such \; that \; (a, b) \in R \; and \; (b, c) \in S\}$

Then

$(S \circ R)^{-1} = \{(c, a) : (a, c) \in S \circ R\}$

furthermore

$R^{-1} = \{(b, a) : (a, b) \in R\}$

$S^{-1} = \{(c, b) : (a, c) \in R\}$

Then the composite relation R^{-1} and S^{-1} is,

$R^{-1} \circ S^{-1} = \{(c, a) : \; there \;\; exists \; b \in B \; such \; that \; (c, b) \in S^{-1} \; and \; (b, a) \in R^{-1}\}$

So

$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Section 3.2 Equivalence Relation

1.

Indicate which of the following releations on the given sets are reflexive, which are symmetric, and which are transitive.

(h) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + y = 10\}$

Solution:

The given relationship is not reflexive because $7 + 7 \neq 10$

The relationship is symmetric because $(\forall_{x,y} \in \mathbb{Z})(x + y = 10 \Rightarrow y + x = 10)$.

The relationship is not transitive because $(3 + 7 = 10 \wedge 7 + 3 = 10)$, but $3 + 3 \neq 10$

4.

The properties of reflexivity, symmetry, and transitivity are related to the identity relation and the opertions of inversion and composition. Prove that :

(c) R is transitive if and only iff $R \circ R \subseteq R$.

i) Suppose R is transitive

Then that is equivalent to saying

(x, y) \in R and $(y, z) \in R$ therefore $(x, z) \in R$

Then

$R \circ R = \{(x, z) : there\ exists\ an\ (x, y) \in R\ and\ (y, z) \in R\}$

and by the property of transitivity, $(x, z) \in R$

That is for each

$(x, z) \in R \circ R \Rightarrow \in R$

Therefore

$R \circ R \subseteq R$

ii) Suppose $R \circ R \subseteq R$

then $R \circ R = \{(x, z) : (x, y) \in R\ and\ (y, z) \in R\}$

then $(x, y) \in R\ and\ (y, z) \in R \Rightarrow (x, z) \in RoR$

so

$R \circ R \subseteq R$

Therefor $(x, z) \in R$ when $(x, y) \in R$ and $(y, z) \in R$

Proving, R is transitive.

6.

For each of the following, prove that the relation is an equivalence relation. Then give infomration aboout the equivalence classes as specified.

a)The relations S on $\mathscr{R}$ given by xSy iff $x - y \in \mathscr{Q}$

i) Check if the relation is reflexive:

$(\forall x \in R)(x - x = 0 \in \mathbb{Q} \Leftrightarrow x\mathbb{S}x)$

Check if the relation is symmetric:

$(\forall x, y \in R)(xSy \Leftrightarrow x - y \in \mathbb{Q} \Leftrightarrow y - x \in \mathbb{Q} \Leftrightarrow ySx)$

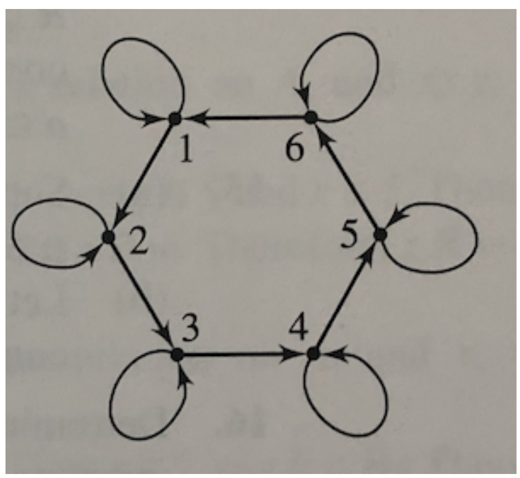
Check if the relation is transitive:

$(\forall x, y, z \in \mathbb{R})((xSy \wedge ySz) \Leftrightarrow (x - z = (x - y) - (z - y) \in \mathbb{Q} \Leftrightarrow (xSz))$

Since the relationship is reflexive, symmetric, and transitive

8b

Does the following digraph represent a relation that is (iii) transitive on the given set?



Solution:

The relationship is not transitive on the Set because xRy, and yRz, but $x\not{R}z$

or $1R2$, and $2R3$, but $1\not{R}3$

9.

Determine the equivalence class for the relation:

b) Congruence Modulo 8

Solution:

$\bar{0} = \{x : x = 0(mod\ 8)\}$

$= \{x : 8\ divides\ (x - 0)\}$

$\bar{0} = \{\ldots, 24, -16, -8, 0, 8, 16, 24\ldots\}$

Continuing this to

$\bar{1} = \{x : x = 1(mod\ 8)\}$

$= \{x : 8\ divides\ (x - 1)\}$

$= \{8x + 1 : x \in \mathbb{Z}\}$ $

$\bar{1} = \{\ldots, -31, -23, -15, -7, 1, 9, 17, 25\ldots\}$

$\vdots$

$\bar{2} = \{\ldots, -30, -22, -14, -6, 2, 10, 18, 26\ldots\}$

$\bar{3} = \{\ldots, -29, -21, -13, -5, 3, 11, 19, 27\ldots\}$

$\bar{4} = \{\ldots, -28, -20, -12, -4, 4, 12, 20, 28\ldots\}$

$\bar{5} = \{\ldots, -27, -19, -11, -3, 5, 13, 21, 29\ldots\}$

$\bar{6} = \{\ldots, -26, -18, -10, -2, 6, 14, 22, 30\ldots\}$

$\bar{7} = \{\ldots, -25, -17, -9, -1, 7, 15, 23, 31, \ldots\}$

12.

Using the fact that congruence modulo m is an equivalence relation on $\mathbb{Z}$ and without reference to Theorems 3.2.2 and 3.2.4, prove that for all $x, y \in \mathbb{Z}$:

(e) if $\bar{x} \cap \bar{y} \neq \emptyset,\ then\ \bar{x} = \bar{y}$

Solution:

Suppose if $\bar{x} \cap \bar{x} \neq \emptyset$

then $a \in \bar{x}$ and $a \in \bar{y}$

which means that

xRa and yRa

and by defintion of an equivalence, the relationship R is transitive, and

xRy

And R is is symmetric

yRx

which implies

$\bar{y} = \{x\}$ and $\bar{x} = \{y\}$

Thus $\bar{y} = \bar{x}$

(f)

Assume $\bar{x} = \bar{y}$

then

15.

Suppose that R and S are equivalence relations on a Set A. Prove that $R \cap S$ is an equivalence relation on A.

Solution:

i) Suppose and S are equivalence relations on Set A.

Then

R and S are Reflexive, Symmetric, and Transitive.

Which means for all $x \in A, xRx$ and for all $x \in A, xSx$

So $(x, x) \in R \cap S$ for all $x \in A$ proving $R \cap S$ is Reflexive on the set A

ii) Furthermore if we assume that $(x, y) \in R \cap S$

it means that

$(x, y) \in R$ and $(x, y) \in S$

since R and S are symmetric,

$(y, x) \in R$ and $(y, x) \in S$

so then $(y, x) \in R \cap S$

Thus $(x, y) \in R \cap S$ and $(y, x) \in R \cap S$ proving the relation is symmetric.

iii) Finally, if we assume that $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$

it means

$(x, y) \in R$

$(y, z) \in R$

$(x, y) \in S$

$(y, z) \in S$

and since R and S are transitive, it implies

$(x, z) \in R$ and $(x, z) \in S$

Therefore

$(x, z) \in R \cap S$

Hence, if $(x, y) and (y, z) \in R \cap S, \ then \ (x, z) \in R \cap S$ and $R \cap S$ is a transitive relationship.

Arriving at the conclusion that $R \cap S$ is an equivalance relationship

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