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Bridge to Abstract Math
Module 6

1. (c)

Every natural number greater than 33 can be written in the form $4s + 5t$, where s and t are integers, $s \geq 3$ and $t \geq 2$

Solution:

Suppose $S = \{n \in \mathbb{N} : n > 33 \ and \ n = 4s + 5t, \ for \ some \ integers \ s \geq 3 \ and \ t \geq 2\}$

Base case:

$34 = 4(6) + 5(2)$

$35 = 4(5) + 5(3)$

$36 = 4(4) + 5(4)$

Thus $34, 35, 36, \in S$

and the base case holds true.

Induction hypothesis:

Assume $m > 33$

and that $k \in \{34, 35, \ldots, m - 1\}, k \in S$

If $m = 34, 35, 36$ then clearly $m \in S$ otherwise $m \geq 37$

and so $m - 3 \geq 34$

Therefore $m - 3 \in S$. `By the induction hypothesis`

So, $m - 3 = 4s + 5t$, for some integers s and t, where $s \geq 3$ and $t \geq 2$

Therefore by PCI, Every natural number greater than 33 can be written in the form $4s + 5t$, where s and t are integers, $s \geq 3$ and $t \geq 2$ is true for all $n \in \mathbb{N}$ such that $n > 33$.

3.

Let $a_1 = 2, \ a_2 = 4$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$. Prove that $a_n = 2^n$ for all natural numbers n .

Solution:

Base case:

$a_1 = 2^1$

$a_2 = 2^2$

Suppose that $a_k = 2^k$, for $n = k, k + 1$

then for $n = k + 2$

$a_{k+2} = 5a_{k+1} - 6a_k$

$= 5(2^{k+1}) - 6(2^k)$

$= 10 \cdot 2^{k+1} - 5 \cdot 2^k$

Furthermore,

$a_{k+2} = 10 \cdot^2 k - 6 \cdot 2^k$

$= 4 \cdot 2^k$

$= 2^2 \cdot 2^k$

$= 2^{k+2}$

Therefore by PCI, the above statement is true.

5 b)

Calculate the first 10 Fibonaci numbers $f_1, f_2, \ldots, f_{10}$

Solution:

$f_1 = 1$

$f_2 = 1$

$f_3 = f_2 + f_1 = 2$

$f_4 = f_3 + f_2 = 3$

$f_5 = f_4 + f_3 = 5$

$f_6 = f_5 + f_4 = 8$

$f_7 = f_6 + f_5 = 13$

$f_8 = f_7 + f_6 = 21$

$f_9 = f_8 + f_7 = 34$

$f_{10} = f_9 + f_8 = 55$

7 (d).

Use the PCI to prove the following properties of Fibonancci numbers:

(Binet's formula) ϕ be the positive solution and rho be the negative solution to the equation $x^2 = x + 1$. (The values are $\phi = \frac{1+\sqrt{5}}{2}$ and $\rho = \frac{1-\sqrt{5}}{2}$) Show for all natural numbers n that $f_n = \frac{\phi^n - \rho^n}{\phi - \rho}$

Solution:

Base case for P(n) where n = 1, 2:

$f_1 = \frac{\phi^1 - \rho^1}{\phi - \rho}$

= 1

$f_2 = \frac{\phi^2 - \rho^2}{\phi - \rho}$

$= \phi + \rho$

$= \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}$

$= 1$

Consider S to be a set of natural numbers for which the statment P(k) is true. With that being said, P(k+1) is true.

$F_{k+1} = F_k + F_{k-1}$

$= \frac{\phi^k - \rho^k}{\phi - \rho} + \frac{\phi^{k-1} - \rho^{k-1}}{\phi - \rho}$

$= \frac{(\phi^k + \phi^{k-1}) - (\rho^k + \rho^{k-1})}{\phi - \rho}$

$= \frac{\phi^k - \rho^k}{\phi - \rho}$

thus, k belongs to S and by PCI, $\frac{\phi^n - \rho^n}{\phi - \rho}$ for all natural numbers of n.

12.

Let the $Fibonacci - 2$ numbers g_n be defined as follows:

$g_1 = 2, \ g_2 = 2$ and $g_{n+2} = g_{n+1} g_n$ for all $n \geq 1$.

(a) calculate the first five "Fibonnaci-2" numbers.

Solution:

$g_1 = 2$

$g_2 = 2$

$g_3 = g_2 g_1 = 4$

$g_4 = g_3 g_2 = 8$

$g_5 = g_4 g_3 = 32$

(b) Show that for all $n \in \mathbb{N}$, $g_n = 2^{f_n}$

Solution:

base case:

$g_1 = 2 = 2^1 = 2^{f_1}$

$g_2 = 2 = 2^1 = 2^{f_2}$

Assume $m > 2$ and $g_m = 2^{f_m}$ for all $m \in \mathbb{N}$

Then

$g_{m+2} = g_{m+1} g_m = 2^{f_{m+1}} \cdot 2^{f_m} = 2^{f_{m+2}}$

So by PCI, the result is true for m+2.

So, by PCI the resul is also trun for n,

Therefore it is true $g_n = 2^{f_n}$ for all $n \in \mathbb{N}$

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