

Module 3

2.1.11

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA . Explain how the columns or rows of

A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B , not the identity matrix or the zero matrix such that $AB = BA$.

Solution:

$$AD = \begin{bmatrix} 1 * 2 + 1 * 0 + 1 * 0 & 1 * 0 + 1 * 3 + 1 * 0 & 1 * 0 + 1 * 0 + 1 * 5 \\ 1 * 2 + 2 * 0 + 3 * 0 & 1 * 0 + 2 * 3 + 3 * 0 & 1 * 0 + 2 * 0 + 3 * 5 \\ 1 * 2 + 4 * 0 + 5 * 0 & 1 * 0 + 4 * 3 + 5 * 0 & 1 * 0 + 4 * 0 + 5 * 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$$

$$DA = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$$

To find a 3×3 matrix B such that $AB = BA$

We chose B to any multiple of the identity matrix I_3

B

2.1.29

Prove theorem 2(b) and 2(c). Use the row-column rule. The (i,j)-entry in $A(B+C)$ can be written as:

$$a_{i1}b_{1j} + \dots + a_{in}(b_{nj} + c_{nj}) \text{ or } \sum_{k=1}^n a_{ik}(b_{kj} + c_{kj})$$

Solution:

Part 1

Theorem (2b): $A(B+C) = AB+AC$ the left distributive law

Multiplication of two matrices is possible when number of rows in the first matrix is equal to the number of columns in the second matrix

$$\text{the } (i, j)^{th} \text{ entry of } A(B + C) = \sum_{k=1}^n a_{ik}(b_{kj} + c_{kj})$$

$$= a_{i1}(b_{1j} + c_{1j}) + \dots + a_{in}(b_{nj} + c_{nj})$$

$$= a_{i1}b_{1j} + a_{i1}c_{1j} + \dots + a_{in}b_{nj} + a_{in}c_{nj}$$

$$= (a_{i1}b_{1j} + \dots + a_{in}b_{nj}) + (a_{i1}c_{1j} + \dots + a_{in}c_{nj})$$

there for

$$\sum_{k=1}^n a_{ik}b_{kj} + \sum_{k=1}^n a_{ik}c_{kj}$$

$$\text{and the } (i, j)^{th} \text{ entry in } (AB + AC)$$

Part 2

Theorem 2(c): $(B+C)A = BA + CA$ the right distributive law

$$\text{the } (i, j)^{th} \text{ entry of } (B + C)A = \sum_{k=1}^n (b_{kj} + c_{kj})a_{kj}$$

$$= (b_{i1} + c_{i1})a_{i1} + \dots + (b_{in} + c_{in})a_{in}$$

$$= b_{i1}a_{i1} + c_{i1}a_{i1} + \dots + b_{in}a_{in} + c_{in}a_{in}$$

$$= (b_{i1}a_{i1} + b_{in}a_{in}) + \dots + (c_{i1}a_{i1} + c_{in}a_{in})$$

there for

$$\sum_{k=0}^n b_{ik}a_{kj} + \sum_{k=0}^n c_{ik}a_{kj}$$

and the $(i, j)^{th}$ entry in $(BA + CA)$

In []:

2.1.30

Prove theorem 2(d): $r(AB) = r(A)B = A(rB)$ for any scalar r

solution:

Part 1

knowing $AB_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

the $(i, j)^{th}$ entry in $r(A)B$ is

$$= (ra_{i1})b_{1j} + \dots + (ra_{in})b_{nj}$$

$$= r(a_{i1})b_{1j} + \dots + r(a_{in})b_{nj}$$

$$= r((a_{i1}b_{1j}) + \dots + (a_{in}b_{nj}))$$

$$= (i, j)^{th} \text{ entry in } r(AB)$$

giving $(rA)B = r(AB)$

Part 2

$$= (ra_{i1})b_{1j} + \dots + (ra_{in})b_{nj}$$

$$= a_{i1}r(b_{1j}) + \dots + a_{in}r(b_{nj})$$

$$= a_{i1}r(b_{1j}) + \dots + a_{in}r(b_{nj})$$

$$= a_{i1}(rb_{1j}) + \dots + a_{in}(rb_{nj})$$

$$\text{gives } (rA)B = A(rB)$$

Part 3

$$\text{Therefore } r(AB) = (rA)B = A(rB)$$

2.1.33

$$\text{Prove theorem 3(d): } (AB^T) = B^T A^T$$

Solution:

Part 1

$$\text{knowing } AB_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

$$\text{the } (i, j)^{th} \text{ entry of } (AB)^T \text{ is the } (j, i)^{th} \text{ entry of } AB$$

$$\text{and is } a_{ji}b_{1i} + \dots + a_{jn}b_{ni}$$

$$\text{Therefore the } (i, j)^{th} \text{ entry in } (AB)^T \text{ is } a_{ji}b_{1i} + \dots + a_{jn}b_{ni}$$

Part 2

$$\text{entries in row } i \text{ of } B^T \text{ are the entries of column } i \text{ in } B$$

$$\text{so row } i \text{ of } B^T \text{ are } b_{1i}, \dots, b_{ni}$$

$$\text{The entries in column } j \text{ of } A^T \text{ are } a_{ji}, \dots, a_{jn}$$

$$\text{Therefore, the } (i, j)^{th} \text{ entry in } B^T A^T = a_{ji}b_{1i} + \dots + a_{jn}b_{ni}$$

$$\text{And thus } (AB)^T = B^T A^T$$

2.2.7

Find A^{-1} and use it to solve the four equations $Ax = b_1$, $Ax = b_2$, $Ax = b_3$, $Ax = b_4$

$$\text{Given: } A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}, b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, b_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Solution:

Part 1:

Using Theorem 4 and a Ti-84

$$A^{-1} = \begin{bmatrix} 6 & -1 \\ \frac{-5}{2} & \frac{1}{2} \end{bmatrix}$$

$$Ab_1 = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

$$Ab_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$Ab_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$$

Part 2

$$\left[A \quad b_1 \quad b_2 \quad b_3 \quad b_4 \right] = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 4 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$$

Row reduce the augmented matrix

$$1. \ R_2 \rightarrow -5R_1 + R_2$$

$$2. \ R_2 \rightarrow \frac{1}{2}R_2$$

$$3. \ R_1 \rightarrow -2R_2 + R_1$$

$$= \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

2.2.8

Use matrix Algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$

Solution:

If $AD = I$ we can left multiply both sides by A^{-1}

to get $ID = A^{-1}I$ (since $A^{-1}A = I$ and $A^{-1}I = A^{-1}$)

$$D = A^{-1}$$

2.2.15

Suppose A, B, C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a Matrix D such that $(ABC)D = I$ and $D(ABC) = I$.

Solution:

Part 1

$$\text{Set } D = (CBA)^{-1} = (C^{-1}B^{-1}A^{-1})$$

$$\text{then } (ABC)D = I$$

$$= (ABC)(C^{-1}B^{-1}A^{-1}) = I$$

By regrouping

$$= AB(CC^{-1})B^{-1}A^{-1} = I$$

$$= AB(I)B^{-1}A^{-1} = I$$

Continuing regrouping until

$$= I$$

$$\text{Therefore } (ABC)D = (ABC)(C^{-1}B^{-1}A^{-1}) = I$$

Part 2

Again, set $D = (ABC)^{-1} = (C^{-1}B^{-1}A^{-1})$

then $D(ABC) = I$

$$= (C^{-1}B^{-1}A^{-1})(ABC) = I$$

By regrouping

$$= C^{-1}B^{-1}(A^{-1}A)BC = I$$

$$= C^{-1}B^{-1}(I)BC = I$$

$$= C^{-1}B^{-1}BC = I$$

Continuing regrouping until

$$= I$$

Therefore $D(ABC) = (C^{-1}B^{-1}A^{-1})(ABC) = I$

2.2.25

Show that if $ad - bc = 0$, then the equation $Ax = 0$ has more than one solution. Why does this imply that A is not invertible?

Solution:

Step 1.

$$[A0] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The matrix can then be written as,

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a & b & 0 \\ ac & ad & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a & b & 0 \\ 0 & ad - bc & 0 \end{bmatrix}$$

Step 2.

It is now given that $ad - bc = 0$

and therefore

$$\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $Ax = 0$ and has a free variable

Since there is a free variable, A does not have a pivot position in every row and cannot be reduced to an Identity matrix.

A matrix is invertible, if and only if it can be reduced to a row equivalent identity matrix (I_n)

2.2.32

Find the inverse of the matrix (if it exists)

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

Solution

$$[AI] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}$$

step1 $R_2 \rightarrow -4R_1 + R_2$

$$= \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}$$

step 2 $R_3 \rightarrow 2R_1 + R_3$

$$= \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$$

step 3 $R_3 \rightarrow -2R_2 + R_3$

$$= \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{-1}{5} & \frac{1}{10} \end{bmatrix}$$

Since all the entries in row 3 of the reduced echelon form of A implies that A is not equivalent to I_3 and the matrix A is not invertible .

2.2.34

Guess the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \dots & & & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{bmatrix}$$

Solution:

Guess

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & & 0 \\ \dots & & & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{bmatrix}$$

Prove Guesss is correct

$$AA^{-1} = ?$$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ .. & & & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & & 0 \\ \dots & & & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{bmatrix} = I$$

Therefore the guess was correct for the A^{-1}

2.3.1

Determine if the matrix is invertible

$$\text{Given: } \begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$$

Solution:

A set of two vectors are linearly dependent if at least one of the vectors is a multiple of the the other. Vector a_1 is not a multiple of vector a_2 so by thereom 8(e), the matrix is invertible .

2.3.2

Determine if the matrix is invertible

$$\text{Given: } \begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$$

Solution:

A set of two vectors are linearly dependent if at least one of the vectors is a multiple of the the other. Vector a_1 is a multiple of vector a_2 (using -1.5 as the scalar) so by thereom 8(e), the matrix is not invertible .

2.3.3

Determine if the matrix is invertible

$$\text{Given: } \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

Solution:

$$\text{step 1. } R_2 \rightarrow \frac{3}{5}R_1 + R_2$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

$$\text{Step 2. } R_2 \rightarrow \frac{-8}{5}R_1 + R_3$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

$$\text{Step 3. } R_2 \rightarrow \frac{-1}{7}R_2$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

Step 4. $R_3 \rightarrow -5R_2 + R_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

By Theorem 8c, there are n pivot positions, therefore the matrix is `invertible`

2.3.4

Determine if the matrix is invertible

Given: $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$

Since there is a zero vector in the set, the set is linearly dependent and therefore, `not invertible` .

2.3.5

Determine if the matrix is invertible

Given: $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$

Step 1. Swap R_1 and R_2

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix}$$

Step 2. $R_3 \rightarrow 4R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix}$$

Step 3. $R_3 \rightarrow 3R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

The Matrix is does not have an element in every pivot postion and is therefore `not invertible` .

2.3.6

Determine if the matrix is invertible

Given: $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ -3 & 6 & 0 \\ 0 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & -9 & -12 \\ 0 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & -9 & -12 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is `not invertible` since its row equivalent is not equal to the identity matrix.

2.3.7

Determine if the matrix is invertible

Given: $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$

Solution:

$$\text{Step 1: } R_2 \rightarrow 3R_1 + R_2$$

$$\text{Step 2: } R_3 \rightarrow -2R_1 + R_3$$

$$\text{Step 3. Swap } R_2 \text{ with } R_4$$

$$\text{Step 4. } R_4 \rightarrow -4R_2 + R_4$$

$$\begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

The equivalent matrix has four pivot positions, and according to Theorem 8c, the matrix is invertible

2.3.8

Determine if the matrix is invertible Given:

$$\begin{bmatrix} 1 & -3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Solution:

The matrix has four pivot positions, and according to Theorem 8c, the matrix is invertible

2.3.27

Show that if AB is invertible, so is A .

Solution:

If both A and B are $n \times n$ matrices then the product AB is a 2×2 matrix.

If the product of A and B is invertible then there has to be a matrix W so that

$$(AB)W = I$$

Using the associative law of multiplication

$$A(BW) = I$$

In this case BW is an inverse of A and thus implies A is invertible

2.8.2

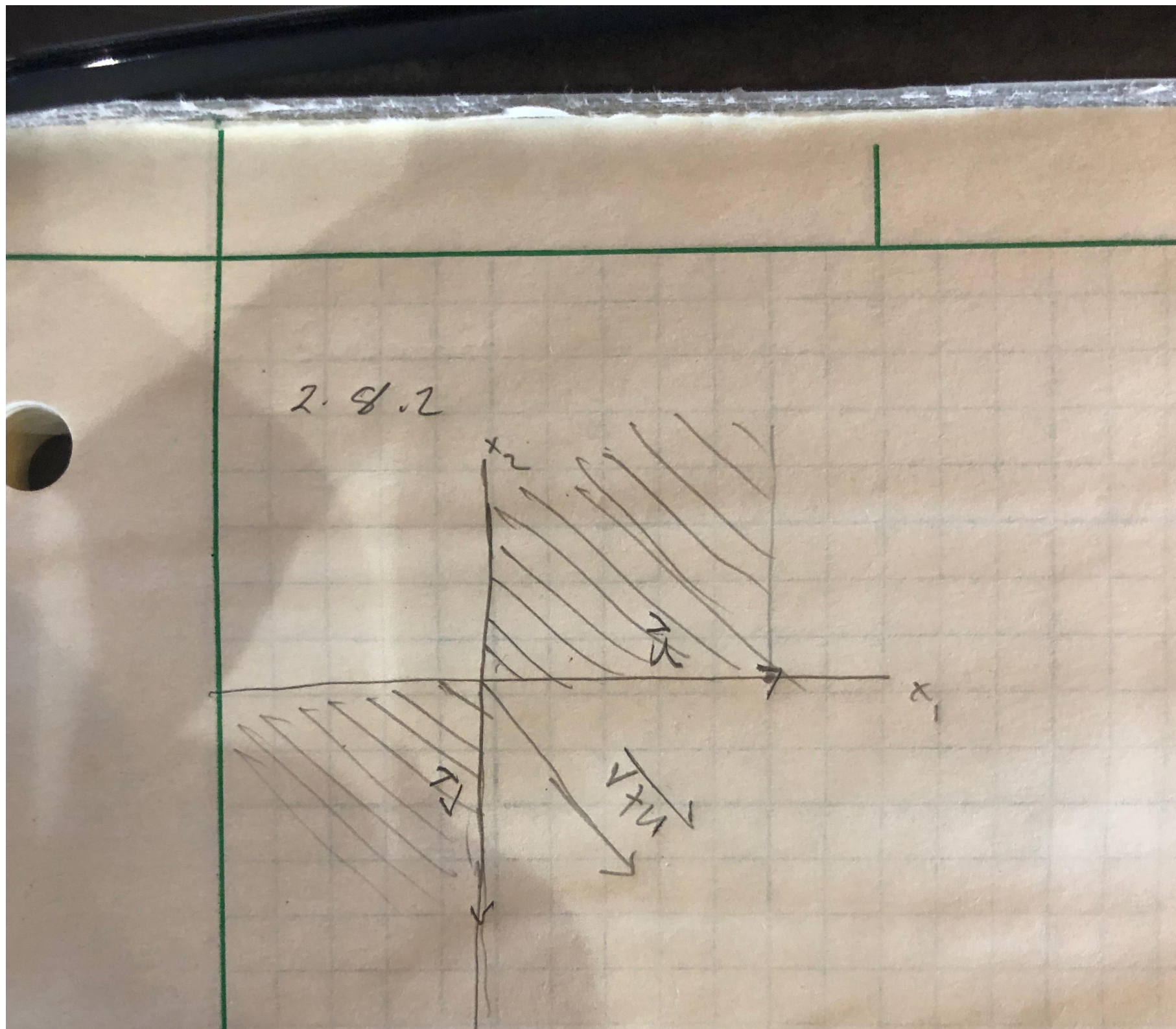
Given: Display sets in \mathbb{R}^2 . Assume the sets include bounding lines. In each case give a specific reason why the set H is not a subspace of \mathbb{R}^2

Solution:

The vectors $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are elements in H

The sum of these two vectors:

$v + u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are not in the given region and the set is not closed under addition.



2.8.4

Given: Display sets in \mathbb{R}^2 . Assume the sets include bounding lines. In each case give a specific reason why the set H is not a subspace of \mathbb{R}^2

Solution:

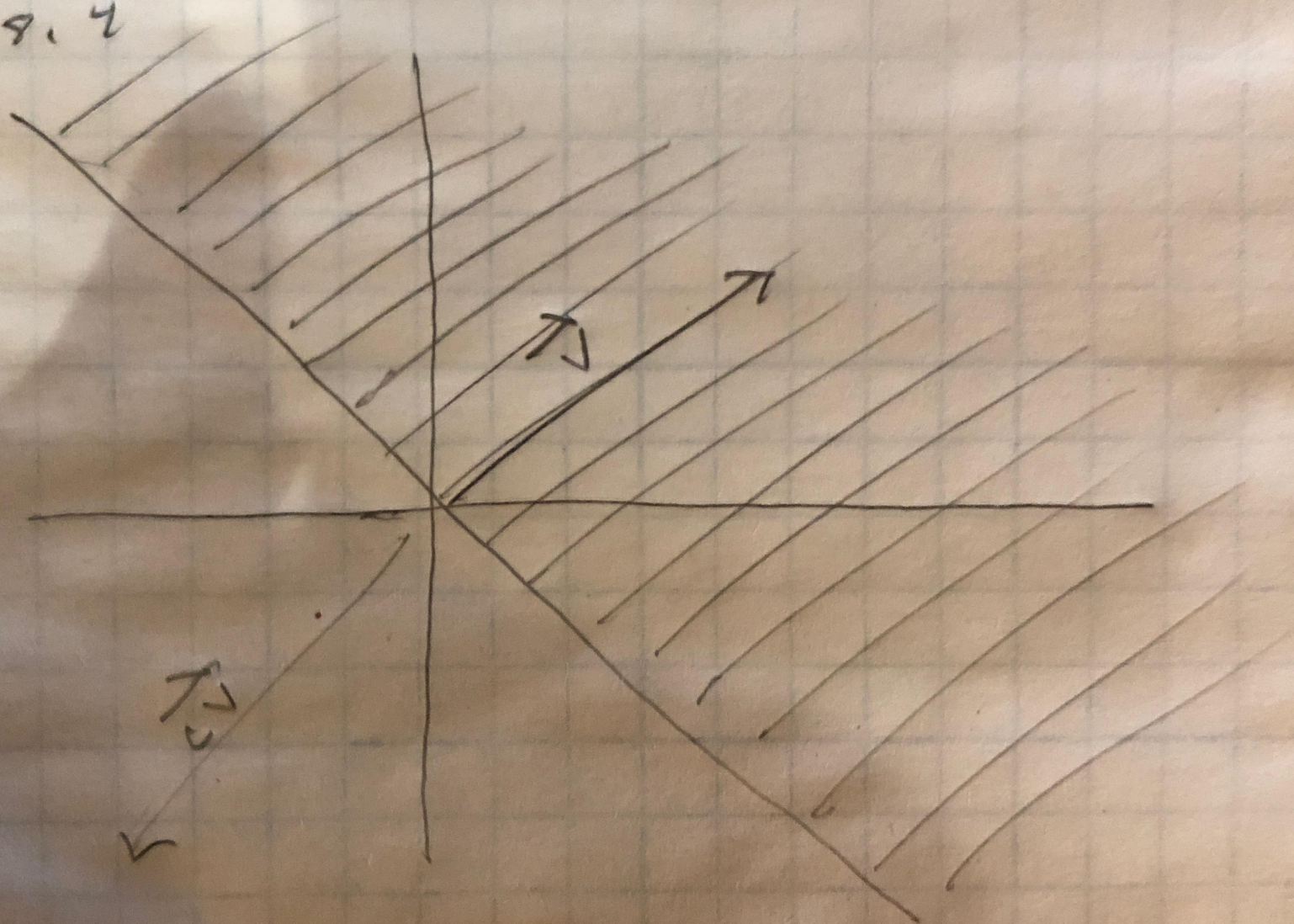
Vector $u = \begin{bmatrix} a \\ a \end{bmatrix}$ is in the shaded region.

if $c = -1$ and

$$cv = c \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} -a \\ -a \end{bmatrix}$$

This does not lie in the shaded region and the set H is not in the subspace of \mathbb{R}^2 because it is not closed under scalar multiplication.

2.8.6



2.8.6

$$\text{Let } v_1 = \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -7 \\ 9 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -8 \\ 6 \\ 5 \end{bmatrix}, u = \begin{bmatrix} -4 \\ 10 \\ -7 \\ -5 \end{bmatrix}$$

Determine if u is in the subspace \mathbb{R}^4 generated by $\{v_1, v_2, v_3\}$

Solution:

Step 1. Row Reduce the Augmented Matrix

$$[vu] = \begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Vector u is a linear combination of the columns of V since u can be written as Vx for some x , that is, if and only if the equation $Vx = u$ has a solution.

Since the third row implies that there is no solution $0 \neq 1$, the system is inconsistent. And u is not in the subspace of \mathbb{R}^4 generated by $\{v_1, v_2, v_3\}$

2.8.8

$$\text{Let } v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}, p = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$$

Determine if p is in the subspace \mathbb{R}^3 generated by $\{v_1, v_2, v_3\}$

Solution:

Step 1. Row Reduce the Augmented Matrix

$$[Vp] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent and p is in the subspace of \mathbb{R}^3 generated by $\{v_1, v_2, v_3\}$

2.8.12

Given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$$

Give integers p and q such that $\text{Nul } A$ is a subspace of \mathbb{R}^p and $\text{Col } A$ is a subspace of \mathbb{R}^q

Solution:

Matrix A has 4 rows and 3 columns. The null space of an $m \times n$ matrix is a subspace of \mathbb{R}^n . $\text{Nul}A$ is a subspace of \mathbb{R}^3 because solutions of $Ax = 0$ must have 3 entries to match the number of columns in A .

$$p = 3$$

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m so $\text{Col}A$ is a subspace of \mathbb{R}^4 because each column vector has 4 entries

$$q = 4$$

2.8.14

Given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$$

Find a non-zero vector in $\text{Nul}A$ and a non-zero vector in $\text{Col}A$

Solution: Step 1.

Solve $Ax = 0$ by writing the Augmented Matrix $[A \ 0]$ and row reducing

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2.

The system of equations becomes

$$x_1 + \frac{-1}{3}x_3 = 0$$

$$x_2 + \frac{5}{3} = 0$$

$$x_1 = \frac{1}{3}x_3$$

$$x_2 = \frac{-5}{3}x_3$$

x_3 is free

$$\text{thus the solution set is } = x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{-5}{3} \\ 1 \end{bmatrix}$$

To find the nonzer vector in column space of A using the row reduced form from above

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In the resultant matrix, we can see columns a_1 and a_2 are pivot columns and therefore know theses columns from the original matrix define the column space of the matrix (Theorem 13).

$$\text{so } \begin{bmatrix} 1 \\ 4 \\ -5 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 5 \\ -1 \\ 7 \end{bmatrix} \text{ are non-zero vectors in } \text{Col } A$$

2.8.18

given:

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

Determine which sets are bases for \mathbb{R}^3

Solution:

Step 1. We verify the 3 vector are linearly independent because they are not scalar multiples of each other.

Step 2. Write the vectors in matrix form and row reduce into echelon form.

$$\begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see the matrix has three pivots so, by the inverse matrix transform rule, the matrix is invertible and its columns form the basis for \mathbb{R}^3

2.8.20

given:

$$\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$$

Determine which sets are bases for \mathbb{R}^3

Step 1. We verify the 4 vector are linearly independent because they are not scalar multiples of each other.

However since the set contains more vectors than number of entries in each vector, the set is linearly dependent.

Since a basis for subspace H in \mathbb{R}^3 is a set in linearly independent set in H in H that spans H, the 4 vectors cannot be a basis for \mathbb{R}^3

2.8.26

given:

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for $ColA$ and $NulA$

Solution:

From the echelon form of a, we can see that a_1 , a_2 , and a_4 are pivot columns and form the basis for $ColA$

Solving for $Nul A$

Step 1:

$$[A0] = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 & 0 \\ -2 & 2 & -2 & 7 & 5 & 0 \\ -5 & 9 & 3 & 3 & 4 & 0 \\ -2 & 6 & 6 & 3 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} & 0 \\ 0 & 1 & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_3 + \frac{5}{2}x_5 = 0$$

$$x_2 + 2x_3 + \frac{3}{2}x_5 = 0$$

$$x_4 + x_5 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 - \frac{5}{2}x_5 \\ -2x_3 - \frac{3}{2}x_5 \\ x_4 = -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{5}{2} \\ \frac{-3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ \frac{-3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ are the basis for } Nul A$$

In []: