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Math 303

Module 3

Module 3

2.1.11

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of Achange when A is multiplied by D on the right or on the left. Find a 3x3 matrix B, not the identity matrx or the zero matrix such that AB = BA.

Solution:

$$AD = \begin{bmatrix} 1*2+1*0+1*0 & 1*0+1*3+1*0 & 1*0+1*0+1*5 \\ 1*2+2*0+3*0 & 1*0+2*3+3*0 & 1*0+2*0+3*5 \\ 1*2+4*0+5*0 & 1*0+4*3+5*0 & 1*0+4*0+5*5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$$

$$AD = \begin{bmatrix} 1*2+1*0+1*0 & 1*0+1*3+1*0 & 1*0+1*0+1*5 \\ 1*2+2*0+3*0 & 1*0+2*3+3*0 & 1*0+2*0+3*5 \\ 1*2+4*0+5*0 & 1*0+4*3+5*0 & 1*0+4*0+5*5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$$

$$DA = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$$$$DA = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$$

To find a 3x3 matrix $B$$B$ such that $AB = BA$$AB = BA$

We chose $B$$B$ to any multiple of the identity matrix $I_3$$I_3$

B

2.1.29

Prove theorem 2(b) and 2(c). Use the row-column rule. The (i,j)-entry in A(B+C) can be written as:

$a_{i1}b_{ij}+...+a_{in}(b_{nj}+c_{nj})$ $a_{i1}b_{ij}+...+a_{in}(b_{nj}+c_{nj})$ or $\sum_{k=0}^{n} a_{ik}(b_{kj}+c_{kj})$ $\sum_{k=0}^{n} a_{ik}(b_{kj}+c_{kj})$

Solution:

Part 1

Theorem (2b): A(B+C) = AB+AC `the left distributive law`

Multiplication of two matrices is possible when number of rows in the first matrix is equal to the number of columns in the second matrix

the $(i,j)^{th}$ $(i,j)^{th}$ entry of $A(B+C) = \sum_{k=0}^{n} a_{ik}(b_{kj}+c_{kj})$ $A(B+C) = \sum_{k=0}^{n} a_{ik}(b_{kj}+c_{kj})$

$= a_{i1}(b_{ij}+c_{ij})...+a_{in}(b_{nj}+c_{nj})$ $= a_{i1}(b_{ij}+c_{ij})...+a_{in}(b_{nj}+c_{nj})$

$= a_{i1}b_{ij}+a_{ij}c_{ij}...+a_{in}b_{nj}+a_{in}c_{nj}$ $= a_{i1}b_{ij}+a_{ij}c_{ij}...+a_{in}b_{nj}+a_{in}c_{nj}$

$= (a_{i1}b_{1j}+...+a_{in}b_{nj})+(a_{i1}c_{1j}+...+a_{in}c_{nj})$ $= (a_{i1}b_{1j}+...+a_{in}b_{nj})+(a_{i1}c_{1j}+...+a_{in}c_{nj})$

there for

$\sum_{k=0}^{n} a_{ik}b_{kj} + \sum_{k=0}^{n} a_{ik}c_{kj}$ $\sum_{k=0}^{n} a_{ik}b_{kj} + \sum_{k=0}^{n} a_{ik}c_{kj}$

and the $(i,j)^{th}$ $(i,j)^{th}$ entry in $(AB+AC)$ $(AB+AC)$

Part 2

Theorem 2(c): (B+C)A = BA + CA `the right distributive law`

the $(i,j)^{th}$ $(i,j)^{th}$ entry of $(B+C))A = \sum_{k=0}^{n}(b_{kj}+c_{kj})a_{kj}$ $(B+C))A = \sum_{k=0}^{n}(b_{kj}+c_{kj})a_{kj}$

= $(b_{i1} + c_{i1})a_{i1} + \ldots + (b_{in} + c_{in})a_{in}$

= $b_{i1}a_{i1} + c_{i1}a_{i1} + \ldots + b_{in}a_{in} + c_{in}a_{in}$

= $(b_{i1}a_{i1} + b_{in}a_{in}) + \ldots + (c_{i1}a_{i1} + c_{in}a_{in})$

there for

$\sum_{k=0}^{n} b_{ik}a_{kj} + \sum_{k=0}^{n} c_{ik}a_{kj}$

and the $(i, j)^{th}$ entry in $(BA + CA)$

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In [ ]:
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2.1.30

Prove theorem 2(d): $r(AB) = r(A)B = A(rB)$ for any scalar r

solution:

Part 1

knowing $AB_{ij} = a_{i1}b_{1j} + a_{i2}b_2j + \ldots + a_{in}b_{nj}$

the $(i, j)^{th}$ entry in $r(A)B$ is

= $(ra_{i1})b_{1j} + \ldots + (ra_{in})b_{nj}$

= $r(a_{i1})b_{1j} + \ldots + r(a_{in})b_{nj}$

= $r((a_{i1}b_{1j}) + \ldots + (a_{in}b_{nj}))$

= $(i, j)^{th}$ entry in $r(AB)$

giving $(rA)B = r(AB)$

Part 2

= $(ra_{i1})b_{1j}+...+(ra_{in})b_{nj}$ $(ra_{i1})b_{1j}+...+(ra_{in})b_{nj}$

= $a_{i1}r(b_{1j})+...+a_{in}r(b_{nj})$ $a_{i1}r(b_{1j})+...+a_{in}r(b_{nj})$

= $a_{i1}r(b_{1j})+...+a_{in}r(b_{nj})$ $a_{i1}r(b_{1j})+...+a_{in}r(b_{nj})$

= $a_{i1}(rb_{1j})+...+a_{in}(rb_{nj})$ $a_{i1}(rb_{1j})+...+a_{in}(rb_{nj})$

gives $(rA)B = A(rB)$ $(rA)B = A(rB)$

Part 3

Therefore $r(AB) = (rA)B = A(rB)$ $r(AB) = (rA)B = A(rB)$

2.1.33

Prove theorem 3(d): $(AB^T) = B^T A^T$ $(AB^T) = B^T A^T$

Solution:

Part 1

knowing $AB_{ij} = a_{i1}b_{1j} + a_{i2}b_2j+...+a_{in}b_{nj}$ $AB_{ij} = a_{i1}b_{1j} + a_{i2}b_2j + ... + a_{in}b_{nj}$

the $(i,j)^{th}$ $(i,j)^{th}$ entry of $(AB)^T$ $(AB)^T$ is the $(j,i)^{th}$ $(j,i)^{th}$ entry of AB AB

and is $a_{ji}b_{1i}+...+a_{jn}b_{ni}$ $a_{ji}b_{1i}+...+a_{jn}b_{ni}$

Therefore the $(i,j)^{th}$ $(i,j)^{th}$ entry in $(AB)^T$ $(AB)^T$ is $a_{ji}b_{1i}+...+a_{jn}b_{ni}$ $a_{ji}b_{1i}+...+a_{jn}b_{ni}$

Part 2

entries in row i i of B^T B^T are the entries of column i i in B B

so row i i of B^T B^T are $b_{1j},....,b_{ni}$ $b_{1j},....,b_{ni}$

The entries in column j j A^T A^T are $a_{ji},....a_{jn}$ $a_{ji},....a_{jn}$

Therefore, the $(i,j)^{th}$ $(i,j)^{th}$ entry in $B^T A^T$ $B^T A^T$ = $a_{ji}b_{1i}+...+a_{jn}b_{ni}$ $a_{ji}b_{1i}+...+a_{jn}b_{ni}$

And thus $(AB)^T = B^T A^T$ $(AB)^T = B^T A^T$

2.2.7

Find A^{-1} A^{-1} and use it to solve the four equations $Ax = b_1$ $Ax = b_1$, $Ax = b_2$ $Ax = b_2$, $Ax = b_3$ $Ax = b_3$, $Ax = b_4$ $Ax = b_4$

Given: $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ $b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $b_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $b_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Solution:

Part 1:

Using Theorem 4 and a Ti-84

$$A^{-1} = \begin{bmatrix} 6 & -1 \\ \frac{-5}{2} & \frac{1}{2} \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 6 & -1 \\ \frac{-5}{2} & \frac{1}{2} \end{bmatrix}$$

$$Ab_1 = \begin{bmatrix} -9 \\ 4 \end{bmatrix} \quad Ab_1 = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix} \quad Ab_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

$$Ab_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \quad Ab_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$Ab_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix} \quad Ab_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$$

Part 2

$$\begin{bmatrix} A & b_1 & b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} A & b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 4 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 4 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$$

Row reduce the augmented matrix

1. $R_2 \to -5R_1 + R_2$
2. $R_2 \to \frac{1}{2}R_2$
3. $R_1 \to -2R_2 + R_1$

$$= \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

2.2.8

Use matrix Algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$

Solution:

If $AD = I$ we can left multiply both sides by A^{-1}

to get $ID = A^{-1}I$ (since $A^{-1}A = I$ and $A^{-1}I = A^{-1}$)

$D = A^{-1}$

2.2.15

Suppose A, B, C are invertible nxn matrices. Show that ABC is also invertible by producing a Matrix D such that $(ABC)D = I$ and $D(ABC) = I$.

Solution:

Part 1

Set $D = (CBA)^{-1} = (C^{-1}B^{-1}A^{-1})$$D = (CBA)^{-1} = (C^{-1}B^{-1}A^{-1})$

then $(ABC)D = I$$(ABC)D = I$

$= (ABC)(C^{-1}B^{-1}A^{-1}) = I$ $= (ABC)(C^{-1}B^{-1}A^{-1}) = I$

By regrouping

$= AB(CC^{-1})B^{-1}A^{-1} = I$ $= AB(CC^{-1})B^{-1}A^{-1} = I$

$= AB(I)B^{-1}A^{-1} = I$ $= AB(I)B^{-1}A^{-1} = I$

Continuing regrouping until

= $I$$I$

Therefore $(ABC)D = (ABC)(C^{-1}B^{-1}A^{-1}) = I$$(ABC)D = (ABC)(C^{-1}B^{-1}A^{-1}) = I$

Part 2

Again, set $D = (ABC)^{-1}= (C^{-1}B^{-1}A^{-1})

then $D(ABC) = I$$D(ABC) = I$

$= (C^{-1}B^{-1}A^{-1})(ABC) = I$ $= (C^{-1}B^{-1}A^{-1})(ABC) = I$

By regrouping

$= C^{-1}B^{-1}(A^{-1}A)BC = I$ $= C^{-1}B^{-1}(A^{-1}A)BC = I$

$= C^{-1}B^{-1}(I)BC = I$ $= C^{-1}B^{-1}(I)BC = I$

= $= C^{-1}B^{-1}BC = I$ $= C^{-1}B^{-1}BC = I$

Continuing regrouping until

= $I$$I$

Therefore $D(ABC) = (C^{-1}B^{-1}A^{-1})(ABC) = I$$D(ABC) = (C^{-1}B^{-1}A^{-1})(ABC) = I$

2.2.25

Show that if $ad - bc = 0$$ad - bc = 0$, then the equation $Ax = 0$$Ax = 0$ has more than one solution. Why does this imply that A is not invertible?

Solution:

Step 1.

$[A0] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $[A0] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The matrix can then be written as,

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a & b & 0 \\ ac & ad & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a & b & 0 \\ 0 & ad - bc & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a & b & 0 \\ ac & ad & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a & b & 0 \\ 0 & ad - bc & 0 \end{bmatrix}$$

Step 2.

It is now given that ad-bc = 0

and therefore

$$\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $Ax = 0$ $Ax = 0$ and has a free variable

Since there is a free variable, A does not have a pivot position in every row and cannot be reduced to and Identity matrix.

A matrix is invertible, if and only if it can be reduced to a row equivalent identity matrix (I_n I_n)

2.2.32

Find the inverse of the matrix (if it exists)

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

Solution

$$[AI] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$[AI] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}$$

step1 $R_2 \to -4R_1 + R_2$ $R_2 \to -4R_1 + R_2$

$$= \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}$$

step 2 $R_3 \to 2R_1 + R_3$ $R_3 \to 2R_1 + R_3$

$$= \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$$

step 3 $R_3 \to -2R_2 + R_3$ $R_3 \to -2R_2 + R_3$

$$= \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{-1}{5} & \frac{1}{10} \end{bmatrix}$$
$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{-1}{5} & \frac{1}{10} \end{bmatrix}$$

Since all the entries in row 3 of the reduced echelon form of A A implies that A A is not equivalent to I_3 I_3 and the matrix A A is `not invertible`.

2.2.34

Guess the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ .. & & & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ .. & & & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{bmatrix}$$

Solution:

Guess

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & & 0 \\ \dots & & & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & & 0 \\ \dots & & & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{bmatrix}$$

Prove Guesss is correct

$AA^{-1} = ?$ $AA^{-1} = ?$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ .. & & & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & & 0 \\ \dots & & & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ .. & & & \dots & \dots \\ 1 & 2 & 3 & \dots & n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 & & 0 \\ 0 & \frac{-1}{3} & \frac{1}{3} & & 0 \\ \dots & & & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{bmatrix} = I$$

Therefore the guess was correct for the A^{-1} A^{-1}

2.3.1

Determine if the matrix is invertible

Given: $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$

Solution:

A set of two vectors are `linearly dependent` if at least one of the vectors is a multiple of the the other. Vector a_1 a_1 is not a multiple of vector a_2 a_2 so by thereom 8(e), the matrix is `invertible` .

2.3.2

Determine if the matrix is invertible

Given: $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

Solution:

A set of two vectors are linearly dependent if at least one of the vectors is a multiple of the the other. Vector a_1 a_1 is a multiple of vector a_2 a_2 (using -1.5 as the scalar) so by thereom 8(e), the matrix is `not invertible` .

2.3.3

Determine if the matrix is invertible

Given: $\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$

Solution:

step 1. $R_2 \rightarrow \frac{3}{5} R_1 + R_2$ $R_2 \rightarrow \frac{3}{5} R_1 + R_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

Step 2. $R_2 \rightarrow \frac{-8}{5}R_1 + R_3$ $R_2 \rightarrow \frac{-8}{5}R_1 + R_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$ $$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

Step 3. $R_2 \rightarrow \frac{-1}{7}R_2$ $R_2 \rightarrow \frac{-1}{7}R_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$ $$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

Step 4. $R_3 \rightarrow -5R_2 + R_3$ $R_3 \rightarrow -5R_2 + R_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ $$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

By Theorem 8c, there are n pivot positions, therefore the matrix is `invertible`

2.3.4

Determine if the matrix is invertible

Given: $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$

Since there is a zero vector in the set, the set is linearly dependent and therfore, `not invertible` .

2.3.5

Determine if the matrix is invertible

Given: $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$ $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$

Step 1. Swap R_1 R_1 and R_2 R_2

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix}$

Step 2. $R_3 \to 4R_1 + R_3$ $R_3 \to 4R_1 + R_3$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix}$

Step 3. $R_3 \to 3R_2 + R_3$ $R_3 \to 3R_2 + R_3$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

The Matrix is does not have an element in every pivot postion and is therefore `not invertible` .

2.3.6

Determine if the matrix is invertible

Given: $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ -3 & 6 & 0 \\ 0 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & -9 & -12 \\ 0 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & -9 & -12 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is `not invertible` since its row equivalent is not equal to the identity matrix.

2.3.7

Determine if the matrix is invertible

Given: $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$

Solution:

Step 1: $R_2 \rightarrow 3R_1 + R2$

Step 2: $R_3 \rightarrow -2R_1 + R_3$

Step 3. Swap R_2 with R_4

Step 4. $R_4 \rightarrow -4R_2 + R_4$

$$\begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

The equivalent matrix has four pivot positions, and according to Theorem 8c, the matrix is `invertible`

2.3.8

Determine if the matrix is invertible Given: $\begin{bmatrix} 1 & -3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ $\begin{bmatrix} 1 & -3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

Solution:

The matrix has four pivot positions, and according to Theorem 8c, the matrix is `invertible`

2.3.27

Show that if AB is invertible, so is A.

Solution:

If both A and B are $n$$n$ x $n$$n$ matrices then the product $AB$$AB$ is a $2x2$$2x2$ matrix.

If the product of A and B is invertible then there has to be a matrix $W$$W$ so that

$(AB)W = I$$(AB)W = I$

Using the associative law of multiplication

$A(BW) = I$$A(BW) = I$

In this case BW BW is an inverse of $A$$A$ and thus implies $A$$A$ is invertible

2.8.2

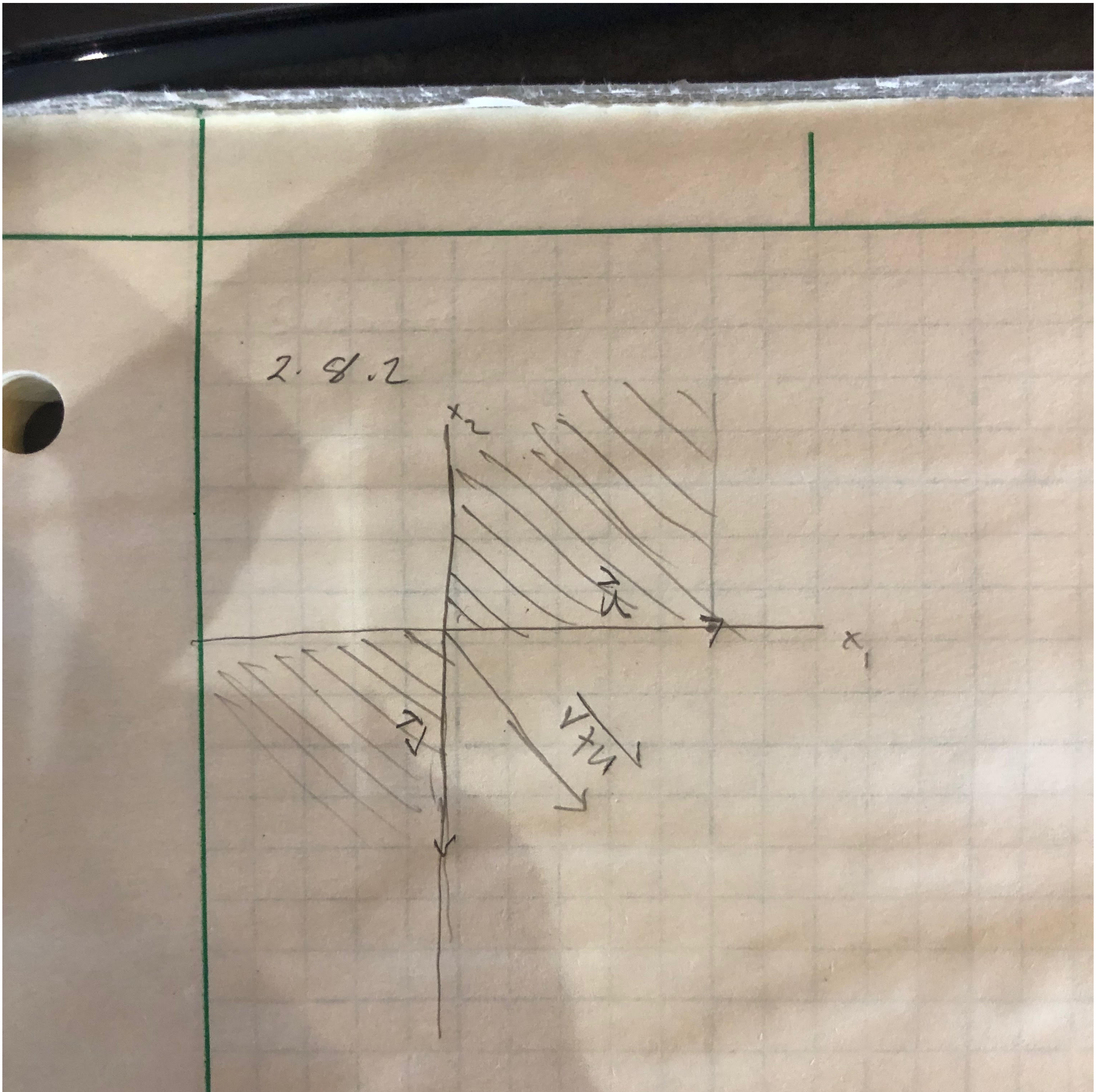
Given: Display sets in $\mathbb{R}^2$$R^2$. Asumme the sets include bounding lines. In each case give a specific reason why the set $H$$H$ is not a subspace of $\mathbb{R}^2$$R^2$

Solution:

The vectors $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are elements in H H

The sum of these two vectors:

$v + u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $v + u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are not in the given region and the set is not closed under addtion.



2.8.4

Given: Display sets in $\mathbb{R}^2 \mathrm{R}^2$. Asumme the sets include bounding lines. In each case give a specific reason why the set $H H$ is not a subspace of $\mathbb{R}^2 \mathrm{R}^2$

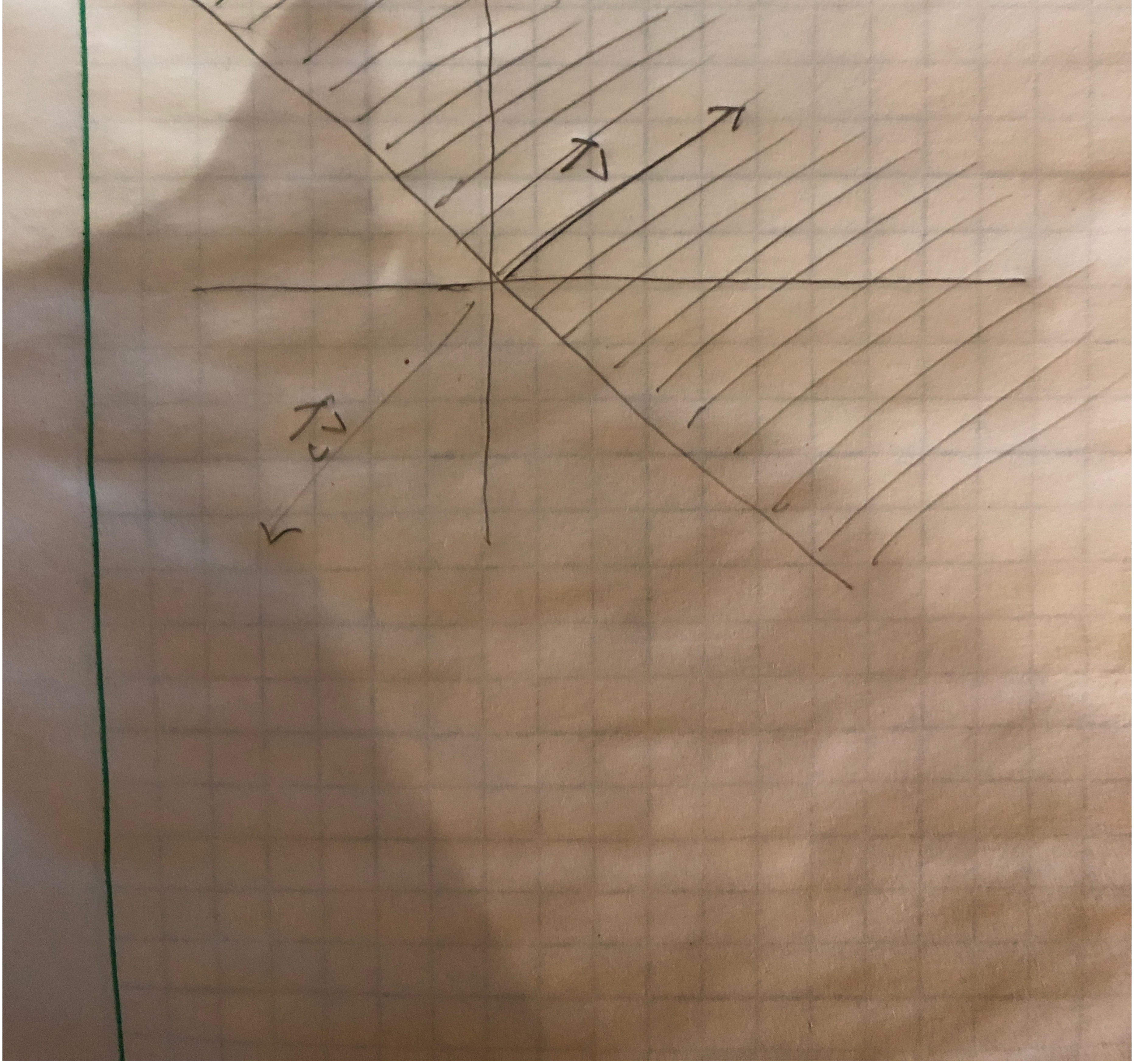
Solution:

Vector $u = \begin{bmatrix} a \\ a \end{bmatrix} u = \begin{bmatrix} a \\ a \end{bmatrix}$ is in the shaded region.

if $c = -1 c = -1$ and

$$cv = c \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} -a \\ -a \end{bmatrix} cv = c \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} -a \\ -a \end{bmatrix}$$

This does not lie in the shaded region and the set H is not in the subspace of $\mathbb{R}^2 \mathrm{R}^2$ because it is not closed under scalar multiplication.



2.8.6

Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -7 \\ 9 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -8 \\ 6 \\ 5 \end{bmatrix}, u = \begin{bmatrix} -4 \\ 10 \\ -7 \\ -5 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -7 \\ 9 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -8 \\ 6 \\ 5 \end{bmatrix}, u = \begin{bmatrix} -4 \\ 10 \\ -7 \\ -5 \end{bmatrix}$

Determine if u u is in the subspace $\mathbb{R}^4$ R^4 genereated by $\{v_1\, v_1, v_2\, v_2, v_3\, v_3\}$

Solution:

Step 1. Row Reduce the Augmented Matrix

$$[vu] = \begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{bmatrix} [vu] = \begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Vector u is a linear combination of the columns of V V since u can be written as Vx Vx for some x, that is, if and only if the equation $Vx = u$ $Vx = u$ has a solution.

Since the third row implies that there is no solution $0 \neq 1$ $0 \neq 1$, the system is incosistent. And u u is not in the subspace of $\mathbb{R}^4$ R^4 generated by $\{v_1$ v_1, v_2 v_2, v_3 $v_3\}$

2.8.8

Let $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}, p = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$

Determine if p is in the subspace $\mathbb{R}^3$ genereated by $\{v_1, v_2, v_3\}$

Solution:

Step 1. Row Reduce the Augmented Matrix

$$[Vp] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is cosistent and p is in the subspace of $\mathbb{R}^3$ generated by $\{v_1, v_2, v_3\}$

2.8.12

Given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$$

Give integers p and q such that Nul A is a subspace of $\mathbb{R}^p$ and Col A is a subspace of $\mathbb{R}^q$

Solution:

Matrix A hs 4 rows and 3 columns. The null space of an mxn matrix is a subspace of $\mathbb{R}^n$. NulA is a subspace of $\mathbb{R}^3$ because solutions of $Ax = 0$ must have 3 entries to match the number of columns in A.

$p = 3$

The column space of an mxn matri A is a subspace of $\mathbb{R}^m$ so Col$A$ is a subspace of $\mathbb{R}^4 because each column vector has 4 entries$

$q = 4$

2.8.14

Given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$$

Find a non-zero vector in Nul$A$ and a non-zero vector in Col$A$

Solution: Step 1.

Solve $Ax = 0$ by writing the Augmented Matrix [A 0] and row reducing

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2.

The system of equations becomes

$$x_1 + \frac{-1}{3}x_3 = 0$$
$$x_2 + \frac{5}{3} = 0$$

$x_1 = \frac{1}{3}x_3$

$x_2 = \frac{-5}{3}x_3$

x_3 is free

thus the solution set is $= x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{-5}{3} \\ 1 \end{bmatrix}$

To find the nonzer vector in column space of A using the row reduced form from above

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In the resultant matrix, we can see colums a_1 and a_2 are pivot columns and therefore know theses coliumns from the original matrix define the column space of the matrix (Theorem 13).

so $\begin{bmatrix} 1 \\ 4 \\ -5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ -1 \\ 7 \end{bmatrix}$ are non-zero vectors in $Col A$

2.8.18

given:

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

Determine which sets are bases for $\mathbb{R}^3$

Solution:

Step 1. We verify the 3 vector are linearly independent because they are not scalar multiples of each other.

Step 2. Write the vectors in matrix form and row reduce into echelon form.

$$\begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see the matrix has three pivots so, by the inverse matrix transform rule, the matrix is invertible and its columns form the basis for $\mathbb{R}^3$

2.8.20

given:

$$\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$$

Determine which sets are bases for $\mathbb{R}^3$

Step 1. We verify the 4 vector are linearly independent because they are not scalar multiples of each other.

However since the set contains more vectors than number of entries in each vector, the set is `linearly dependent.`

Since a basis for subspace H in $\mathbb{R}^3$ is a set in linearly independent set in H in H that spans H, the 4 vectors cannot be a basis for $\mathbb{R}^3$

2.8.26

given:

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for $Col A$ and $Nul A$

Solution:

From the echelon form of a, we can see that a_1 , a_2 , and a_4 are pivot columns and form the basis for $Col A$

Solving for $Nul A$

Step 1:

$$[A0] = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 & 0 \\ -2 & 2 & -2 & 7 & 5 & 0 \\ -5 & 9 & 3 & 3 & 4 & 0 \\ -2 & 6 & 6 & 3 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} & 0 \\ 0 & 1 & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_3 + \tfrac{5}{2}x_5 = 0$$
$$x_2 + 2x_3 + \tfrac{3}{2}x_5 = 0$$
$$x_4 + x_5 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 - \frac{5}{2}x_5 \\ -2x_3 - \frac{3}{2}x_5 \\ x_4 = -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{5}{2} \\ \frac{-3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ \frac{-3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}$ are the basis for $Nul A$

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Derek Haynes

Math 303

Module 3

Module 3

2.1.11

Let $A = \begin{bmatrix}1&1&1\\1&2&3\\1&4&5\end{bmatrix}$ and $D = \begin{bmatrix}2&0&0\\0&3&0\\0&0&5\end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of Achange when A is multiplied by D on the right or on the left. Find a 3x3 matrix B, not the identity matrx or the zero matrix such that AB = BA.

Solution:

$$AD = \begin{bmatrix} 1*2+1*0+1*0 & 1*0+1*3+1*0 & 1*0+1*0+1*5\\ 1*2+2*0+3*0 & 1*0 + 2*3+3*0 & 1*0+2*0+3*5 \\ 1*2+4*0+5*0 & 1*0+4*3+5*0 & 1*0 +4*0+5*5 \end{bmatrix} = \begin{bmatrix} 2 & 3 &5 \\ 2& 6 &15 \\ 2&12&25\end{bmatrix}$$

$$DA = \begin{bmatrix}2&2&2\\3&6&9\\5&20&25\end{bmatrix}$$

To find a 3x3 matrix B such that $AB = BA$

We chose B to any multiple of the identity matrix I_3

B

2.1.29

Prove theorem 2(b) and 2(c). Use the row-column rule. The (i,j)-entry in A(B+C) can be written as:

$a_{i1}b_{ij}+...+a_{in}(b_{nj}+c_{nj})$ or $\sum\limits^{n}_{k=0}a_{ik}(b_{kj}+c_{kj})$

Solution:

Part 1

Theorem (2b): A(B+C) = AB+AC `the left distributive law`

Multiplication of two matrices is possible when number of rows in the first matrix is equal to the number of columns in the second matrix

the $(i,j)^{th}$ entry of $A(B+C) = \sum\limits^{n}_{k=0}a_{ik}(b_{kj}+c_{kj})$

$= a_{i1}(b_{ij}+c_{ij})...+a_{in}(b_{nj}+c_{nj})$

$= a_{i1}b_{ij}+a_{ij}c_{ij}...+a_{in}b_{nj}+a_{in}c_{nj}$

$= (a_{i1}b_{1j}+...+a_{in}b_{nj})+(a_{i1}c_{1j}+...+a_{in}c_{nj})$

there for

$\sum\limits^{n}_{k=0}a_{ik}b_{kj} + \sum\limits^{n}_{k=0}a_{ik}c_{kj}$

and the $(i,j)^{th}$ entry in $(AB+AC)$

Part 2

Theorem 2(c): (B+C)A = BA + CA `the right distributive law`

the $(i,j)^{th}$ entry of $(B+C))A = \sum\limits^{n}_{k=0}(b_{kj}+c_{kj})a_{kj}$

$= (b_{i1}+c_{i1})a_{i1}+...+(b_{in}+c_{in})a_{in}$

$= b_{i1}a_{i1}+c_{i1}a_{i1}+...+b_{in}a_{in}+c_{in}a_{in}$

$= (b_{i1}a_{i1}+b_{in}a_{in})+...+(c_{i1}a_{i1}+c_{in}a_{in})$

there for

$\sum\limits^{n}_{k=0}b_{ik}a_{kj} + \sum\limits^{n}_{k=0}c_{ik}a_{kj}$

and the $(i,j)^{th}$ entry in $(BA+CA)$

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2.1.30

Prove theorem 2(d): $r(AB) = r(A)B = A(rB)$ for any scalar r

solution:

Part 1

knowing $AB_{ij} = a_{i1}b_{1j}+a_{i2}{b_2j}+...+a_{in}b_{nj}$

the $(i,j)^{th}$ entry in $r(A)B$ is

= $(ra_{i1})b_{1j}+ ...+(ra_{in})b_{nj}$

= $r(a_{i1})b_{1j}+ ...+r(a_{in})b_{nj}$

= $r((a_{i1}b_{1j})+ ...+(a_{in}b_{nj)}})$

= $(i,j)^{th}$ entry in $r(AB)$

giving $(rA)B = r(AB)$

Part 2

= $(ra_{i1})b_{1j}+ ...+(ra_{in})b_{nj}$

= $a_{i1}r(b_{1j})+ ...+a_{in}r(b_{nj})$

= $a_{i1}r(b_{1j})+ ...+a_{in}r(b_{nj})$

= $a_{i1}(rb_{1j})+ ...+a_{in}(rb_{nj})$

gives $(rA)B = A(rB)$

Part 3

Therefore $r(AB) = (rA)B = A(rB)$

2.1.33

Prove theorem 3(d): $(AB^T) = B^TA^T$

Solution:

Part 1

knowing $AB_{ij} = a_{i1}b_{1j}+a_{i2}{b_2j}+...+a_{in}b_{nj}$

the $(i,j)^{th}$ entry of $(AB)^T$ is the $(j,i)^{th}$ entry of AB

and is $a_{ji}b_{1i}+...+a_{jn}b_{ni}$

Therefore the $(i,j)^{th}$ entry in $(AB)^T$ is $a_{ji}b_{1i}+...+a_{jn}b_{ni}$

Part 2

entries in row i of B^T are the entries of column i in B

so row i of B^T are $b_{1j},....,b_{ni}$

The entries in column j A^T are $a_{ji},....a_{jn}$

Therefore, the $(i,j)^{th}$ entry in $B^TA^T = a_{ji}b_{1i}+...+a_{jn}b_{ni}$

And thus $(AB)^T = B^TA^T$

2.2.7

Find A^-1 and use it to solve the four equations $Ax = b_{1}$, $Ax = b_{2}$, $Ax = b_{3}$, $Ax = b_{4}$

Given: $A = \begin{bmatrix}1&2\\5&12\end{bmatrix}$, $b_1 = \begin{bmatrix}-1\\3\end{bmatrix}$, $b_2 = \begin{bmatrix}1\\-5\end{bmatrix}$, $b_3 = \begin{bmatrix}2\\6\end{bmatrix}$, $b_4 = \begin{bmatrix}3\\5\end{bmatrix}$

Solution:

Part 1:

Using Theorem 4 and a Ti-84

$A^-1 = \begin{bmatrix} 6&-1\\\frac{-5}{2}&\frac{1}{2}\end{bmatrix}$

$Ab_1 = \begin{bmatrix}-9\\4\end{bmatrix}$

$Ab_2 = \begin{bmatrix}11\\-5\end{bmatrix}$

$Ab_3 = \begin{bmatrix}6\\-2\end{bmatrix}$

$Ab_4 = \begin{bmatrix}13\\-5\end{bmatrix}$

Part 2

$\begin{bmatrix}A&b_1&b_2&b_3&b_4\end{bmatrix} = \begin{bmatrix}1&2&-1&1&2&4\\5&12&3&-5&6&5\end{bmatrix}$

Row reduce the augmented matrix

1. $R_2 \rightarrow -5R_1+R_2$
2. $R_2 \rightarrow \frac{1}{2}R_2$
3. $R_1 \rightarrow -2R_2+R_1$

$= \begin{bmatrix}1&0&-9&11&6&13\\0&1&4&-5&-2&-5\end{bmatrix}$

2.2.8

Use matrix Algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$

Solution:

If $AD = I$ we can left multiply both sides by A^{-1}

to get $ID = A^{-1}I$ (since $A^{-1}A = I$ and $A^{-1}I = A^{-1}$)

$D = A^{-1}$

2.2.15

Suppose A, B, C are invertible nxn matrices. Show that ABC is also invertible by producing a Matrix D such that $(ABC)D = I$ and $D(ABC) = I$.

Solution:

Part 1

Set $D = (CBA)^{-1}= (C^{-1}B^{-1}A^{-1})$

then $(ABC)D = I$

$= (ABC)(C^{-1}B^{-1}A^{-1}) =I$

By regrouping

$= AB(CC^{-1})B^{-1}A^{-1} =I$

$= AB(I)B^{-1}A^{-1} =I$

Continuing regrouping until

$= I$

Therefore $(ABC)D = (ABC)(C^{-1}B^{-1}A^{-1}) = I$

Part 2

Again, set $D = (ABC)^{-1}= (C^{-1}B^{-1}A^{-1})$

then $D(ABC) = I$

$= (C^{-1}B^{-1}A^{-1})(ABC) =I$

By regrouping

$= C^{-1}B^{-1}(A^{-1}A)BC=I$

$= C^{-1}B^{-1}(I)BC=I$

$= = C^{-1}B^{-1}BC=I$

Continuing regrouping until

$= I$

Therefore $D(ABC) = (C^{-1}B^{-1}A^{-1})(ABC) = I$

2.2.25

Show that if $ad - bc = 0$, then the equation $Ax = 0$ has more than one solution. Why does this imply that A is not invertible?

Solution:

Step 1.

$[A0] = \begin{bmatrix}a&b\\c&d\end{bmatrix}$

The matrix can then be written as,

$\begin{bmatrix}a&b&0\\c&d&0\end{bmatrix} \rightarrow \begin{bmatrix}a&b&0\\ac&ad&0\end{bmatrix} \rightarrow \begin{bmatrix} a&b&0\\0&ad-bc&0\end{bmatrix}$

Step 2.

It is now given that ad-bc = 0

and therefore

$\begin{bmatrix}a&b&0\\0&0&0\end{bmatrix}$

and $Ax = 0$ and has a free variable

Since there is a free variable, A does not have a pivot position in every row and cannot be reduced to and Identity matrix.

A matrix is invertible, if and only if it can be reduced to a row equivalent identity matrix (I_n)

2.2.32

Find the inverse of the matrix (if it exists)

$\begin{bmatrix}1&-2&1\\4&-7&3\\-2&6&-4\end{bmatrix}$

Solution

$[A\ I] = \begin{bmatrix}1&-2&1&1&0&0\\4&-7&3&0&1&0\\-2&6&-4&0&0&1\end{bmatrix}$

step1 $R_2 \rightarrow -4R_1+R_2$

= $\begin{bmatrix}1&-2&1&1&0&0\\0&1&-1&-4&1&0\\-2&6&-4&0&0&1\end{bmatrix}$

step 2 $R_3 \rightarrow 2R_1 +R_3$

= $\begin{bmatrix}1&-2&1&1&0&0\\0&1&-1&-4&1&0\\0&2&-2&2&0&1\end{bmatrix}$

step 3 $R_3 \rightarrow -2R_2 +R_3$

$= \begin{bmatrix}1&-2&1&1&0&0\\0&1&-1&-4&1&0\\0&0&0&1&\frac{-1}{5}&\frac{1}{10}\end{bmatrix}$

Since all the entries in row 3 of the reduced echelon form of A implies that A is not equivalent to I_3 and the matrix A is `not invertible`.

2.2.34

Guess the inverse of

$A =\begin{bmatrix}1&0&0&...&0\\1&2&0&{}&0\\1&2&3&{}&0\\..&{}&{}&...&...\\1&2&3&...&n\end{bmatrix}$

Solution:

Guess

$A^{-1} = \begin{bmatrix}1&0&0&...&0\\\frac{-1}{2}&\frac{1}{2}&0&{}&0\\0&\frac{-1}{3}&\frac{1}{3}&{}&0\\...&{}&{}&...&...\\0&0&0&...&\frac{1}{n}\end{bmatrix}$

Prove Guesss is correct

$A A^{-1} = ?$

$\begin{bmatrix}1&0&0&...&0\\1&2&0&{}&0\\1&2&3&{}&0\\..&{}&{}&...&...\\1&2&3&...&n\end{bmatrix}\begin{bmatrix}1&0&0&...&0\\\frac{-1}{2}&\frac{1}{2}&0&{}&0\\0&\frac{-1}{3}&\frac{1}{3}&{}&0\\...&{}&{}&...&...\\0&0&0&...&\frac{1}{n}\end{bmatrix} = I$

Therefore the guess was correct for the A^{-1}

2.3.1

Determine if the matrix is invertible

Given: $\begin{bmatrix}5&7\\-3&-6\end{bmatrix}$

Solution:

A set of two vectors are `linearly dependent` if at least one of the vectors is a multiple of the the other. Vector a_1 is not a multiple of vector a_2 so by thereom 8(e), the matrix is `invertible`.

2.3.2

Determine if the matrix is invertible

Given: $\begin{bmatrix}-4&6\\6&-9\end{bmatrix}$

Solution:

A set of two vectors are linearly dependent if at least one of the vectors is a multiple of the the other. Vector a_1 is a multiple of vector a_2 (using -1.5 as the scalar) so by thereom 8(e), the matrix is `not invertible`.

2.3.3

Determine if the matrix is invertible

Given: $\begin{bmatrix}5&0&0\\-3&-7&0\\8&5&-1\end{bmatrix}$

Solution:

step 1. $R_2 \rightarrow \frac{3}{5}R_1 + R_2$

$$\begin{bmatrix}5&0&0\\0&-7&0\\8&5&-1\end{bmatrix}$$

Step 2. $R_2 \rightarrow \frac{-8}{5}R_1 +R_3$

$$\begin{bmatrix}5&0&0\\0&-7&0\\0&5&-1\end{bmatrix}$$

Step 3. $R_2 \rightarrow \frac{-1}{7}R_2$

$$\begin{bmatrix}5&0&0\\0&1&0\\0&5&-1\end{bmatrix}$$

Step 4. $R_3 \rightarrow -5R_2 + R_3$

$$\begin{bmatrix}5&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$

By Theorem 8c, there are n pivot positions, therefore the matrix is `invertible`

2.3.4

Determine if the matrix is invertible

Given: $\begin{bmatrix}-7&0&4\\3&0&-1\\2&0&9\end{bmatrix}$

Since there is a zero vector in the set, the set is linearly dependent and therfore, `not invertible` .

2.3.5

Determine if the matrix is invertible

Given: $\begin{bmatrix}0&3&-5\\1&0&2\\-4&-9&7\end{bmatrix}$

Step 1. Swap R_1 and R_2

$\begin{bmatrix}1&0&2\\0&3&-5\\-4&-9&7\end{bmatrix}$

Step 2. $R_3 \rightarrow 4R_1 + R_3$

$\begin{bmatrix}1&0&2\\0&3&-5\\0&-9&15\end{bmatrix}$

Step 3. $R_3 \rightarrow 3R_2 + R_3$

$\begin{bmatrix}1&0&2\\0&3&-5\\0&0&0\end{bmatrix}$

The Matrix is does not have an element in every pivot postion and is therefore `not invertible` .

2.3.6

Determine if the matrix is invertible

Given: $\begin{bmatrix}1&-5&-4\\0&3&4\\-3&6&0\end{bmatrix}$

Solution:

$\begin{bmatrix}1&-5&-4\\0&3&4\\-3&6&0\end{bmatrix}$ ~ $\begin{bmatrix}1&-5&-4\\-3&6&0\\0&3&4\end{bmatrix}$ ~ $\begin{bmatrix}1&-5&-4\\0&-9&-12\\0&3&4\end{bmatrix}$ ~ $\begin{bmatrix}1&-5&-4\\0&-9&-12\\0&0&0\end{bmatrix}$

The matrix is `not invertible` since its row equivalent is not equal to the identity matrix.

2.3.7

Determine if the matrix is invertible

Given: $\begin{bmatrix}-1&-3&0&1\\3&5&8&-3\\-2&-6&3&2\\0&-1&2&1\end{bmatrix}$

Solution:

Step 1: $R_2 \rightarrow 3R_1 + R2$

Step 2: $R_3 \rightarrow -2R_1+R_3$

Step 3. Swap R_2 with R_4

Step 4. $R_4 \rightarrow -4R_2+R_4$

$$\begin{bmatrix}-1&-3&0&1\\0&-1&2&1\\0&0&3&0\\0&0&0&-4\end{bmatrix}$$

The equivalent matrix has four pivot positions, and according to Theorem 8c, the matrix is `invertible`

2.3.8

Determine if the matrix is invertible Given:
$$\begin{bmatrix}1&-3&7&4\\0&5&9&6\\0&0&2&8\\0&0&0&10\end{bmatrix}$$

Solution:

The matrix has four pivot positions, and according to Theorem 8c, the matrix is `invertible`

2.3.27

Show that if AB is invertible, so is A.

Solution:

If both A and B are n x n matrices then the product AB is a $2 \text{ x } 2$ matrix.

If the product of A and B is invertible then there has to be a matrix W so that

$$(AB)W = I$$

Using the associative law of multiplication

$$A(BW) = I$$

In this case BW is an inverse of A and thus implies A is invertible

2.8.2

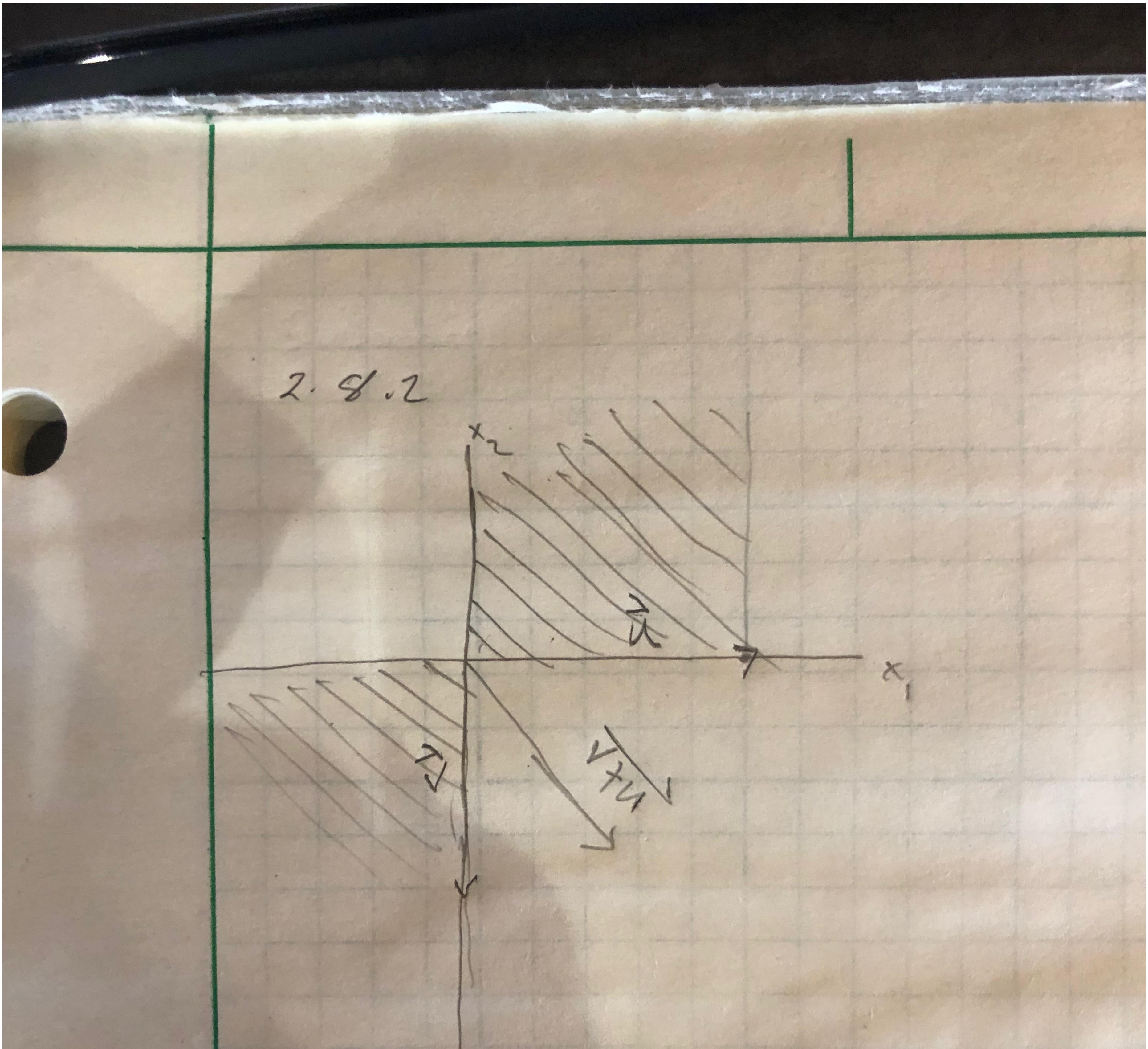
Given: Display sets in $\mathbb{R^2}$. Asumme the sets include bounding lines. In each case give a specific reason why the set H is not a subspace of $\mathbb{R^2}$

Solution:

The vectors $u = \begin{bmatrix}1\\0\end{bmatrix}$ and $v = \begin{bmatrix}0\\-1\end{bmatrix}$ are elements in H

The sum of these two vectors:

$v+u = \begin{bmatrix}1\\-1\end{bmatrix}$ are not in the given region and the set is not closed under addtion.



2.8.4

Given: Display sets in $\mathbb{R^2}$. Asumme the sets include bounding lines. In each case give a specific reason why the set H is not a subspace of $\mathbb{R^2}$

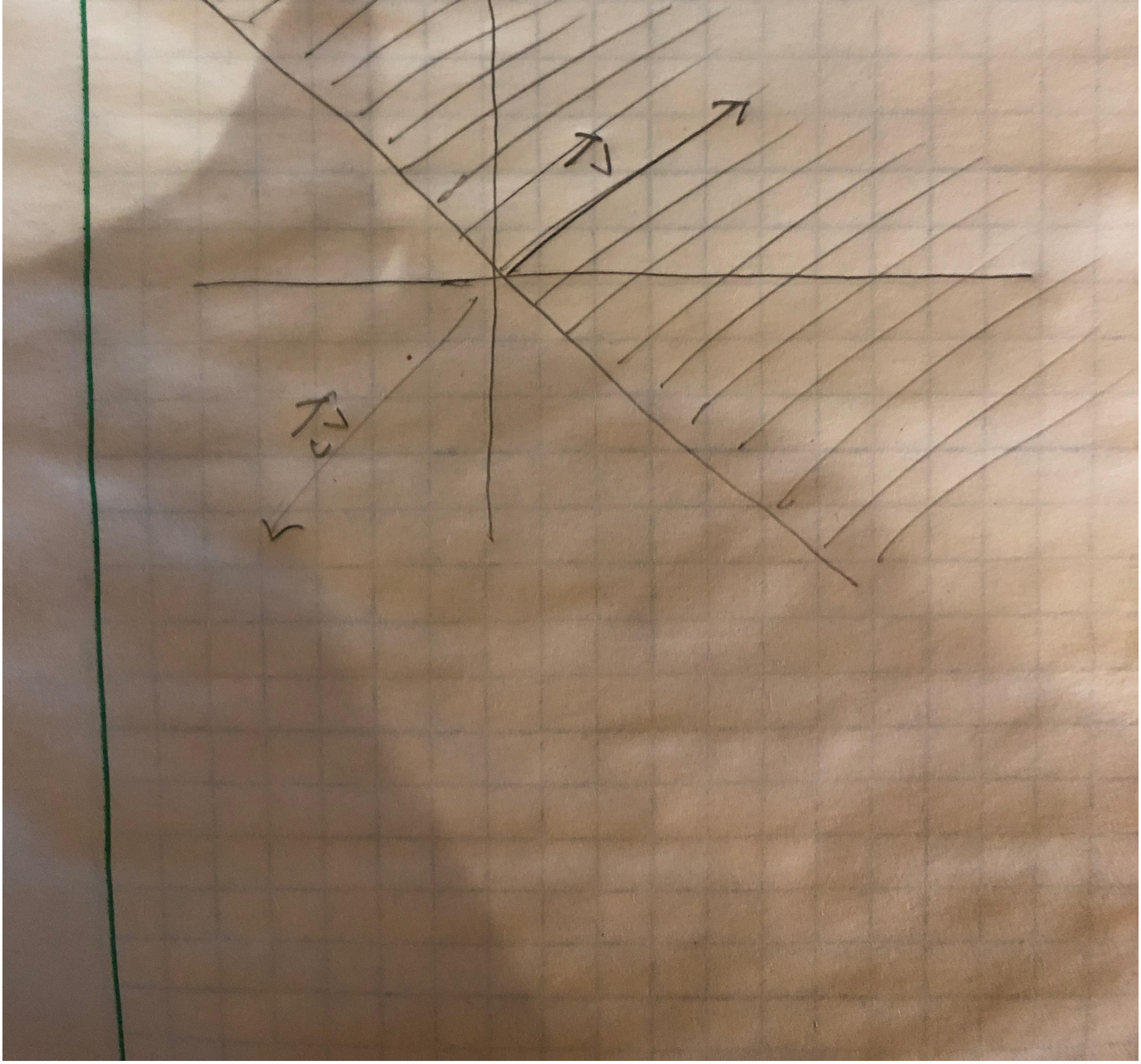
Solution:

Vector $u = \begin{bmatrix} a\\a\end{bmatrix}$ is in the shaded region.

if $c = -1$ and

$cv = c\begin{bmatrix}a\\a\end{bmatrix} = \begin{bmatrix}-a\\-a\end{bmatrix}$

This does not lie in the shaded region and the set H is not in the subspace of $\mathbb{R^2}$ because it is not closed under scalar multiplication.



2.8.6

Let $v_1 = \begin{bmatrix}1\\-2\\-4\\3\end{bmatrix}$, $v_2 =\begin{bmatrix}4\\-7\\9\\7\end{bmatrix}$, $v_3 = \begin{bmatrix}5\\-8\\6\\5\end{bmatrix}$, $u = \begin{bmatrix}-4\\10\\-7\\-5\end{bmatrix}$

Determine if u is in the subspace $\mathbb{R^4}$ genereated by {v_1, v_2, v_3}

Solution:

Step 1. Row Reduce the Augmented Matrix

$$[vu]= \begin{bmatrix}1&4&5&-4\\-2&-7&-8&10\\4&9&6&-7\\3&7&5&-5\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Vector u is a linear combination of the columns of V since u can be written as Vx for some x, that is, if and only if the equation $Vx = u$ has a solution.

Since the third row implies that there is no solution $0\neq1$, the system is incosistent. And u is not in the subspace of $\mathbb{R^4}$ generated by $\{v_1, v_2, v_3\}$

2.8.8

Let $v_1 = \begin{bmatrix}-3\\0\\6\end{bmatrix}$, $v_2 =\begin{bmatrix}-2\\2\\3\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\-6\\3\end{bmatrix}$, $p = \begin{bmatrix}1\\14\\-9\end{bmatrix}$

Determine if p is in the subspace $\mathbb{R^3}$ genereated by $\{v_1, v_2, v_3\}$

Solution:

Step 1. Row Reduce the Augmented Matrix

$$[Vp]= \begin{bmatrix}-3&-2&0&1\\0&2&-6&14\\6&3&3&-9\end{bmatrix} \sim \begin{bmatrix}1&0&2&5\\0&1&-3&7\\0&0&0&0\end{bmatrix}$$

The system is cosistent and p is in the subspace of $\mathbb{R^3}$ generated by $\{v_1, v_2, v_3\}$

2.8.12

Given:

$$A = \begin{bmatrix}1&2&3\\4&5&7\\-5&-1&0\\2&7&11\end{bmatrix}$$

Give integers p and q such that Nul A is a subspace of $\mathbb{R^p}$ and Col A is a subspace of $\mathbb{R^q}$

Solution:

Matrix A hs 4 rows and 3 columns. The null space of an mxn matrix is a subspace of $\mathbb{R^n}$. NulA is a subspace of $\mathbb{R^3}$ because solutions of $Ax =0$ must have 3 entries to match the number of columns in A.

$p = 3$

The column space of an mxn matri A is a subspace of $\mathbb{R^m}$ so Col$A$ is a subspace of $\mathbb{R^4}$ because each column vector has 4 entries

$q = 4$

2.8.14

Given:

$$A = \begin{bmatrix}1&2&3\\4&5&7\\-5&-1&0\\2&7&11\end{bmatrix}$$

Find a non-zero vector in Nul$A$ and a non-zero vector in Col$A$

Solution: Step 1.

Solve $Ax = 0$ by writing the Augmented Matrix [A 0] and row reducing

$$A = \begin{bmatrix}1&2&3&0\\4&5&7&0\\-5&-1&0&0\\2&7&11&0\end{bmatrix} \sim\begin{bmatrix}1&0&\frac{-1}{3}&0\\0&1&\frac{5}{3}&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Step 2.

The system of equations becomes

$$\begin{matrix}x_1+\frac{-1}{3}x_3 = 0\\ x_2+\frac{5}{3} = 0\end{matrix}$$

$x_1 = \frac{1}{3}x_3$

$x_2 = \frac{-5}{3}x_3$

x_3 is free

thus the solution set is $= x_3\begin{bmatrix}\frac{1}{3}\\\frac{-5}{3}\\1\end{bmatrix}$

To find the nonzer vector in column space of A using the row reduced form from above

$A = \begin{bmatrix}1&2&3&0\\4&5&7&0\\-5&-1&0&0\\2&7&11&0\end{bmatrix} \sim \begin{bmatrix}1&0&\frac{-1}{3}&0\\0&1&\frac{5}{3}&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

In the resultant matrix, we can see colums a_1 and a_2 are pivot columns and therefore know theses coliumns from the original matrix define the column space of the matrix (Theorem 13).

so $\begin{bmatrix}1\\4\\-5\\2\end{bmatrix}$ and $\begin{bmatrix}2\\5\\-1\\7\end{bmatrix}$ are non-zero vectors in $ColA$

2.8.18

given:

$\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}-5\\-1\\2\end{bmatrix}, \begin{bmatrix}7\\0\\-5\end{bmatrix}$

Determine which sets are bases for $\mathbb{R^3}$

Solution:

Step 1. We verify the 3 vector are linearly independent because they are not scalar multiples of each other.

Step 2. Write the vectors in matrix form and row reduce into echelon form.

$\begin{bmatrix}1&-5&7\\1&-1&0\\-2&2&-5\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

We can see the matrix has three pivots so, by the inverse matrix transform rule, the matrix is invertible and its columns form the basis for $\mathbb{R^3}$

2.8.20

given:

$\begin{bmatrix}1\\-6\\-7\end{bmatrix}, \begin{bmatrix}3\\-4\\7\end{bmatrix}, \begin{bmatrix}-2\\7\\5\end{bmatrix}, \begin{bmatrix}0\\8\\9\end{bmatrix}$

Determine which sets are bases for $\mathbb{R^3}$

Step 1. We verify the 4 vector are linearly independent because they are not scalar multiples of each other.

However since the set contains more vectors than number of entries in each vector, the set is `linearly dependent.`

Since a basis for subspace H in $\mathbb{R^3}$ is a set in linearly independent set in H in H that spans H, the 4 vectors cannot be a basis for $\mathbb{R^3}$

2.8.26

given:

$A = \begin{bmatrix}3&-1&7&3&9\\-2&2&-2&7&5\\-5&9&3&3&4\\-2&6&6&3&7\end{bmatrix}$ ~ $\begin{bmatrix}3&-1&7&0&6\\0&2&4&0&3\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}$

Find a basis for $ColA$ and $NulA$

Solution:

From the echelon form of a, we can see that a_1, a_2, and a_4 are pivot columns and form the basis for $ColA$

Solving for $Nul A$

Step 1:

$[A 0] = \begin{bmatrix}3&-1&7&3&9&0\\-2&2&-2&7&5&0\\-5&9&3&3&4&0\\-2&6&6&3&7&0\end{bmatrix} = \begin{bmatrix}1&0&3&0&\frac{5}{2}&0\\0&1&2&0&\frac{3}{2}&0\\0&0&0&1&1&0\\0&0&0&0&0&0\end{bmatrix}$

$\begin{matrix}x_1+3x_3+\frac{5}{2}x_5 = 0\\ x_2+2x_3+\frac{3}{2}x_5 = 0\\x_4+x_5 = 0\end{matrix}$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix} -3x_3-\frac{5}{2}x_5\\-2x_3-\frac{3}{2}x_5\\x_4 = -x_5\\ x_5 \end{bmatrix} = x_3\begin{bmatrix}-3\\-2\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-\frac{5}{2}\\\frac{-3}{2}\\0\\-1\\1\end{bmatrix}$

$\begin{bmatrix}-3\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-\frac{5}{2}\\\frac{-3}{2}\\0\\-1\\1\end{bmatrix}$ are the basis for $NulA$

In [ ]: