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Math 303

Linear Algrebra

Module 4 HW

3.1.14

given: Compute the determinants by cofactor expansions. At each step, choose are row or column that involves the least amount of computation

$$\begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$$

Solution:

Step 1: choose row 4, column 1

$$\begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix} = (-1)^{4+1}(2)\begin{bmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{bmatrix}$$

Step 2: choose column 4, row 3

$$(-1)^{4+1}(2)\begin{bmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{bmatrix} = -2(-1)^{3+4}(1)\begin{bmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Step 3: chose column 1, row 1

= $2\{(-1)^{1+1}(3)\begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix} + (-1)^{3+1}(2)\begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix}\}$

= $2\{(3)\begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix} + (2)\begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix}\}$

= $2\{(3)(-8-3)+(2)(2+16)\}$

= 6

3.1.18

Given: Add the downward diagonal producs and subtract the upward products to determine the determinants. Warning: this trick does not generalize to 4x4 and larger matrices

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 3 & 3 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 3 & 3 & 2 \end{bmatrix}\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 3 \end{matrix}$$

$$(1x3x2)+(3x1x3)+(4x2x3)-(3x3x4)-(3x1x1)-(2x2x3)$$

= -12

3.2.10

given: Find the determinant by row reduction to echelon form

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{bmatrix}$$

Solution:

step 1 $R3 \to 2R1 + R3$

step 2. $R4 \to -R1 + R4$

step 3. $R3 \leftrightarrow R4$

step 4. $R5 \to -2R + R5$

step 5. $R3 \leftrightarrow R4$

step 6. $R4 \leftrightarrow R5$

step 7. $R5 \to -\frac{3}{7}R4 + R5$

$$= \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 7 & 15 \\ 0 & 0 & 0 & 0 & -\frac{3}{7} \end{bmatrix}$$

Multiply down the diagnal

= 6

3.2.14

Given: Combine the methods of row reduction with cofactor expansion to compute the determinant

$$A = \begin{bmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{bmatrix}$$

Solution:

step 1. $R3 \rightarrow -R1 + R3$

$$A = \begin{bmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 2 & 0 & 0 & 0 \\ -6 & 5 & 5 & 0 \end{bmatrix}$$

step 2. Apply cofactor expansion along the 4th column

$$det A = 1(-1)^{4+1} \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 0 \\ -6 & 5 & 5 \end{bmatrix}$$

step 3. Apply cofactor expansion along second row

$$det A = -1(2)(-1)^{2+1} \begin{bmatrix} -2 & -4 \\ 5 & 5 \end{bmatrix}$$

$det A = 2\{(-10 + 20)\}$

$det A = 20$

3.2.22

given: use determinants to find out if the matrix is invertible

$$\begin{bmatrix} 5 & 1 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$$

solution: apply coffactor expansion on the first first column

$$5(-1)^{1+1} \cdot \begin{bmatrix} -3 & -2 \\ 5 & 3 \end{bmatrix} + 1(-1)^{2+1} \cdot \begin{bmatrix} 1 & -1 \\ 5 & 3 \end{bmatrix}$$

$= 5(1) - 8$

$det A = -3$

3.2.24

given: use determinants to decide if the set of vectors is linearly independent

$$\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}$$

solution:

$$\begin{bmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{bmatrix} = 2\begin{bmatrix} 1 & \frac{7}{2} & -1 \\ 6 & 0 & -5 \\ 4 & -7 & -3 \end{bmatrix} = 2\begin{bmatrix} 1 & \frac{7}{2} & -1 \\ 4 & -7 & -3 \\ 6 & 0 & -5 \end{bmatrix} = 2\begin{bmatrix} 1 & \frac{7}{2} & -1 \\ 0 & -21 & 1 \\ 6 & 0 & -5 \end{bmatrix}$$

apply cofactor expansion on the first column

$$2\{(1)(-1)^{1+1} \cdot \begin{bmatrix} -21 & 1 \\ 0 & -5 \end{bmatrix} + 0 + (6)(-1)^{3+1} \begin{bmatrix} \frac{7}{2} & -1 \\ 0 & -5 \end{bmatrix}\}$$

= $2\{105 - 105\}$ = 0

$det A \neq 0$

therefore the columns of the matrix for a linearly dependent set

3.2.29

Compute $det B^4$ where :

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

step 1. reduce matrix B to reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

a. $R2 \rightarrow -R1 + R2$

b. $R3 \rightarrow -R1 + R3$

c. $R3 \rightarrow -R2 + R3$

thus the reduced row echelon form of B is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

letting $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

means B is row equivalent to matrix A

using cofactor expansion down the first column

$det A = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

$= (-1)^{1+1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} + 0 + 0$

$= -2$

$det A = det B$

therefor $det A^4 = det B^4$

$det B^4 = 16$

3.2.32

Suppose that A is a square matrix such that $det A^3 = 0$.

Explain why A cannot be invertible

Solution:

By Theorem 4 a square matrix is invertible if anly only if $det A \neq 0$

so if $det A^3 = 0$

then $(det A)^3 = 0$

and $det A = 0$

since $det A = 0$ the matrix A is not invertible

3.2.34

Let A and P be square matrices, with P invertible. Show that $det(PAP^{-1}) = det A$

Solution:

By the associaive property $PAP^{-1} = P(AP^{-}1)$

finding the $det(PAP^{-}1) = (det P)(det A)(det P^{-}1)$

$= (det P)(det P^{-1})det(A)$

$= det(A)$

Therefore $det(PAP^{-1}) = det A$

3.2.43

Verify that $det A = det B + det C$

given : A = $\begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix}$

$$B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

Solution

first solve the $det A$

coefficient expansion down the third column

$$= (u_1 + v_1)\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} - (u_2 + v_2)\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} + (u_3 + v_3)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$=(u_1 + v_1)(a_{21}a_{32} - a_{31}a_{22}) - (u_2 + v_2)(a_{11}a_{32} - a_{12}a_{31}) + (u_3 + v_3)(a_{11}a_{22} - a_{12}a_{21})$

$det A = u_1(a_{21}a_{32} - a_{31}a_{22}) - u_2(a_{11}a_{32} - a_{12}a_{31})$
$+ u_3(a_{11}a_{22} - a_{12}a_{21} + v_1(a_{21}a_{32} - a_{31}a_{22}) - v_2(a_{11}a_{32} - a_{12}a_{31}) + v_3(a_{11}a_{22} - a_{12}a_{21})$

$det B = u_1(a_{21}a_{32} - a_{31}a_{22}) - u_2(a_{11}a_{32} - a_{12}a_{31}) + u_3(a_{11}a_{22} - a_{12}a_{21})$

$detC = v_1(a_{21}a_{32} - a_{31}a_{22}) - v_2(a_{11}a_{32} - a_{12}a_{31}) + v_3(a_{11}a_{22} - a_{12}a_{21})$

$det B + detC = u_1(a_{21}a_{32} - a_{31}a_{22}) - u_2(a_{11}a_{32} - a_{12}a_{31}) + u_3(a_{11}a_{22} - a_{12}a_{21})$
$+ v_1(a_{21}a_{32} - a_{31}a_{22}) - v_2(a_{11}a_{32} - a_{12}a_{31}) + v_3(a_{11}a_{22} - a_{12}a_{21})$

Therefore $det A = det B + detC$

3.3.6

Given: Use Cramer's rule to compute the solutions of the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ -x_1 + 2x_3 &= 2 \\ 3x_1 + x_2 &= 2 \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}$$ and

$$b = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A_2(b) = \begin{bmatrix} 1 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

$$A_3(b) = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

Using the T-84

$$det A = 15$$

$$det A_1(b) = 6$$

$$det A_2(b) = 12$$

$$det A_3(b) = 18$$

$$x_1 = \frac{det A_1(b)}{det A} = \frac{6}{15} = \frac{2}{5}$$

$$x_2 = \frac{det A_2(b)}{det A} = \frac{12}{15} = \frac{4}{5}$$

$$x_3 = \frac{det A_3(b)}{det A} = \frac{18}{15} = \frac{6}{5}$$

$$x = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ \frac{6}{5} \end{bmatrix}$$

3.3.14

Given: Compute the adjugate of given matrix and use Theorem 8 to give the inverse

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

Solution:

$$c_{11} = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} = 8, c_{12} = -\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = 2, c_{13} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = -4$$

$$c_{21} = -\begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix} = 4, c_{22} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0, c_{23} = -\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = -2$$

$$c_{31} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = -5, c_{32} = -\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = -1, c_{33} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 2$$

$$adjA = \begin{bmatrix} 8 & 4 & -5 \\ 2 & 0 & -1 \\ 4 & -2 & 2 \end{bmatrix}$$

Using Ti-84

$$detA = -2$$

$$A^{-1} = \frac{1}{detA} \cdot adjA$$

$$A^{-1} = \frac{1}{-2} \cdot \begin{bmatrix} 8 & 4 & -5 \\ 2 & 0 & -1 \\ 4 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -4 & -2 & \frac{5}{2} \\ -1 & 0 & \frac{1}{2} \\ -2 & 1 & -1 \end{bmatrix}$$

3.3.17

Given: Show that if A is 2x2, then Theorm 8 gives the same formula for $A^{-}1$ as that given by Theorem 4 in section 2.2

`Theorem 8:` Let A be an invertible nxn matrix. then

$$A^{-1} = \frac{1}{det A} adj A$$

`Theorem 4:` Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ If $ad - bc \neq 0$ then A is invertible.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solution:

`Theorem 8`

$c_{11} = d, c_{12} = -c$

$c_{21} = -b, c_{22} = a$

$adj A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$adj A \cdot A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \begin{bmatrix} ad - bc & db - db \\ -ac + ac & -bc + ad \end{bmatrix}$

$= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= (ad - bc)\boldsymbol{I}$

$det A = (ad - bc)$ and the

$adj A \cdot A = (ad - bc)\boldsymbol{I}$

The inverse of the nxn matrix is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3.3.22

Find the area of the parallelogram whose vertices are listed:

$(0, -2), (5, -2), (-3, 1), (2, 1)$

First Translate the parallelogram to one having the origin as a vertex by adding (0, 2) to each one of the vertices.

$(0, 0), (5, 0), (-3, 3), (2, 3)$ are the new vertices

The parallelogram is determined by the columns

$$A = \begin{bmatrix} 0 & -3 \\ 5 & 3 \end{bmatrix}$$

the $det A = 15$ so the area of the parallelogram is 15

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