Derek Haynes Probability Theory HW #12

Section 7.1

Exercise 7.4

Given: Suppose that X and Y are independent exponential random variables with parameters $\lambda \neq \mu$. Find the density functions of X+Y.

Solution:

$f_{x+y} = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_0^z f_X(x) f_Y(z-x)dx$

$= \int_0^z \lambda e^{-\lambda x} \mu e^{-\mu(z-x)}$

$= \lambda\mu \int_0^z e^{-\lambda x} e^{-\mu(z-x)}$

$= \lambda\mu \int_0^z e^{-\lambda x - \mu z + \mu x} dx$

$= \lambda\mu e^{-\mu z} \int_0^z e^{-\lambda x + \mu x} dx$

$= \lambda\mu e^{-\mu z} \int_0^z e^{(-\lambda+\mu)x} dx$

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using u substitution
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$= \lambda\mu e^{-\mu z} \left[\frac{e^{(-\lambda+\mu)x}}{-\lambda+\mu}\right]_0^z$

$= \lambda\mu e^{-\mu z} \left(\left[\frac{e^{(-\lambda+\mu)z}}{-\lambda+\mu}\right] - \left[\frac{e^0}{-\lambda+\mu}\right]\right)$

$= \frac{\lambda\mu e^{-\mu z}}{-\lambda+\mu} \left[e^{(-\lambda+\mu)z} - 1\right]$

Exercise 7.16

Given Let X be a Poisson random variable with parameter λ and Y an independent Bernoulli random variable with parameter p. Find the probability mass function of X+Y.

Solution:

$X \sim Poisson(\lambda) = e^{-\lambda} \frac{\lambda^k}{k!}$

$Y \sim Ber(p)$ $P(Y=1) = p$ and $P(Y=0) = (1-p)$

$P_{X+Y}(n) = \Sigma_{k=0}^{\infty} P_X(k) P_y(n-k)$

Where

$P_Y(n-k)$ is p if $n-k = 1 \rightarrow k = n-1$

or

$1-p$ if $n-k=0 \rightarrow k=n$

Therefore ...

$P_{X+Y}(n) = P_X(n-1) \cdot p + P_x(n) \cdot (1-p) + 0$

$= e^{-\lambda} \frac{\lambda^{n-1}}{(n-1)!} p + e^{-\lambda} \frac{\lambda^n}{(n)!} (1-p)$

7.18

Suppose that X and Y are independent random variables with density functions

$f_X(x) = \begin{cases} 2e^{-2x}, x \geq 0 \\ 0, x < 0 \end{cases}$

$f_Y(x) = \begin{cases} 4xe^{-2x}, x \geq 0 \\ 0, x < 0 \end{cases}$

Find the density function of X + Y

Solution:

$z = x + y$

where ...

$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_0^z f_X(x) f_Y(z-x)dx$

$= \int_0^z 2e^{-2x} 4(z-x)e^{-2(z-x)} dx$

$= 8\int_0^z (z-x)e^{-2z}\,dx$

$= 8\left[\frac{x^2}{2}e^{-2z}\right]_0^z$

$= 8e^{-2z}\int_0^z (z-x)dx$

$= 8e^{-2z}(xz - \frac{x^2}{2})$

7.21

Let X and Y be independent normal random variables with distributions $X \sim N(6,3)$ and $Y \sim N(-2,2)$. Let W=3X+4Y.

(a) Identify the distribution of W.

(b) Find the probability P(W > 15).

solution a):

$X + Y \sim N(\mu_1^2 + \mu_2^2, \sigma_1^2 + \sigma_2^2)$ `sum of independent normals`

Therefore ...

$W = 3X + Y \sim N((3\cdot 6) + (4\cdot -2), (3)^2\cdot 3 + (4)^2\cdot 2) = N(10, 59)$

solution b):

$P(W > 15) = P(\frac{W-E[W]}{\sqrt{Var(W)}} > \frac{15-10}{\sqrt{59}})$

= $P(Z > 0.65)$

$= 1 - P(Z < 0.65)$

$= 1 - \Phi(0.65)$

$= 1 - 0.7422$

$= 0.2578$

Section 7.2

7.6

We deal 5 cards from a standard deck of 52, one by one. What is the probability that the third is a king, and the fifth is the ace of spades?

Solution:

`sampling without replacement`

$P(X_3 = king, X_5 = aceofspades)$

$= P(X_1 = king, X_2 = aceofspades) = \frac{4\cdot 1}{52\cdot 51} = \frac{4}{2652}$

7.8

An urn contains 7 red balls, 6 green balls, and 9 yellow balls. Ten balls are chones without replacement. What is the probabiltiy that the 3rd ball chosen was green, given the the 5th ball chosen was yellow.

$P(X_3 = Green|X_5 = Yellow)$

$= P(X_2 = Green|X_1 = Yellow)$

$= \frac{P(X_1=Yellow \cap X_2=Green)}{P(X_1=Yellow)}$

$= \frac{\frac{6}{21}\cdot\frac{9}{22}}{\frac{9}{22}}$

$= \frac{6}{21}$

7.29

Let $X_1, X_2, \cdots, X_{100}$ be independent standard normal random variables. Find the probability that X_{20} is the 50th largest number among these 100 numbers.

Solution:

The probability that X_1 and X_{20} are the 50th largest are the same because they are (i.i.d), therefore the probability that X_{20} is the 50th largest is equal to $\frac{1}{n}$ or $\frac{1}{100}$

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