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Probability Theory

Homework 8

Exercise 3.15

Given: Suppose that the random variable X has expected value $E[X] = 3$ and $Var(X) = 4$. Compute the following quantities.

a) $E[3X + 2]$

Solution:

$E[3x + 2] = 3E[x] + 2 = 11$ `by Fact 3.52`

b) $E[X^2]$

Solution:

Var(X) = $E(X^2) - (E[X])^2$

Therefore...

$4 = E(X^2) - 9$ `by fact 3.48 alternative formula for the variance`

$E(X^2) = 13$

c) $E[(2x + 3)^2]$

Solution:

$E[4x^2 + 12x + 9]$

$= 4E[X^2] + 12E[X] + 9$ `by linearity property`

$= 52 + 36 + 9 = 97$

d) $Var(4X - 2)$

Solution:

`by fact 3.52:` $Var(aX + b) = a^2 Var(X)$

$= 4^2 Var(X) = 64$ `

Exercise 3.16

Given: Let Z have the following density function

$$f_Z(z) = \begin{cases} \frac{1}{7}, 1 \le z \le 2 \\ \frac{3}{7}, 5 \le z \le 7 \\ 0, other \end{cases}$$

Compute both $E[Z]$ and $Var(Z)$

$E[Z] = \int_1^2 \frac{1}{7} z dz + \int_5^7 \frac{3}{7} z dz$

$= \frac{1}{7} \frac{z^2}{2} |_1^2 + \frac{3}{7} \frac{z^2}{2} |_5^7$

$= \frac{1}{7} [\frac{3}{2}] + \frac{3}{7} [12]$

$= \frac{75}{14}$

$E[Z^2] = \int_1^2 \frac{1}{7} z^2 dz + \int_5^7 \frac{3}{7} z^2 dz$

$= \frac{1}{7} \frac{z^3}{3} |_1^2 + \frac{3}{7} \frac{z^3}{3} |_5^7$

$= \frac{1}{7} (\frac{8-1}{3}) + \frac{3}{7} (\frac{343-125}{3})$

$= \frac{7}{21} + \frac{654}{21} = \frac{661}{21}$

$Var(Z) = E[Z^2] - (E[Z])^2$

$Var(Z) = \frac{661}{21} - (\frac{75}{14})^2$

$Var(Z) = 2.77721$

Exercise 3.17

Let $X \sim N(-2, 7)$. Find the following probabilities using the table in Appendix E.

a) P(X > 3.5)

Solution:

Since Z = $\frac{X-(-2)}{\sqrt{7}}$

$$P(X > 3.5) = P(Z > \frac{3.5-(-2)}{\sqrt{7}})$$

$$= 1 - \Phi(2.078)$$

$$= 1 - .9812$$

$$= 0.0188$$

b) P(-2.1 < X < -1.9)

Solution:

$$P(-2.1 < X < -1.9)$$

= $P(\frac{-2.1-(-2)}{\sqrt{7}}) < z < \frac{-1.9-(-2)}{\sqrt{7}})$

= P(-0.04 < z < 0.04)

= P(z < 0.04) - (1 - (P(z < 0.04))

= 0.516 - 0.484 = 0.0302

c) P(X < 2)

Solution:

$$P(z < \frac{2-(-2)}{\sqrt{7}})$$

$$= P(z < 1.511)$$

$$= 0.9345$$

d) $P(X < -10)$

Solution:

$P(X < -10)$

$= P(z < \frac{-10-(-2)}{\sqrt{7}})$

$= P(z < -3.023)$

$= 1 - P(z < 3.023)$

$= 1 - 0.9987$

$= 0.0013$

e) $P(X > 4)$

Solution:

$P(z > \frac{4-(-2)}{\sqrt{7}})$

$P(z > 2.27)$

$= 1 - P(z < 2.27)$

$= 0.0116$

Exercise 3.18

Let X ~ N(3,4)

a) Find the probability P(2 < X < 6)

solution:

$P(\frac{2-3}{2} < z < \frac{6-3}{2})$

$= P(-0.5 < z < 1.5)$

$= P(z < 1.5) - (1 - P(z < 0.5))$

$= .9332 - 0.3085$

$= 0.6247$

b) Find the value c such that $P(X > c) = 0.33$

Solution:

$P(z > \frac{c-3}{2})$

$1 - P(z < \frac{c-3}{2}) = 0.33$

$P(z < \frac{c-3}{2}) = 0.67$

$\Phi(0.444) = 0.67$

Therefore ...

$\frac{c-3}{2} = 0.444$

and ...

$c = 3.888$

c) Find $E[X^2]$

$Var(X) = E[X^2] - (E[X])^2$

$4 = E[X^2] - (3)^2$

$13 = E[X^2]$

Exercise 3.67 and 3.68

Let Z ~ N(0,1).

a) Calculate $E[Z^3]$

Solution:

$f(z) = z^3 e^{(\frac{-z^2}{2})}$ `is an odd function f(-x) = -f(x)`

Therefore...

$\int_{-a}^{a} f(z)dz = 0$

then $E[Z^3] = 0$

b) Calculate $E[Z^4]$

Solution:

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using the moment generating function
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$E[Z^4]= z^4e{}z

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In [ ]:
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