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Homework 7

Probability Theory

Exercise 3.8

Let X have the possilbe values {1,2,3,4,5} and probability mass function:

$p_X(1) = \frac{1}{7}$

$p_X(2) = \frac{1}{14}$

$p_X(3) = \frac{3}{14}$

$p_X(4) = \frac{2}{7}$

$p_X(5) = \frac{2}{7}$

a) Compute the mean of X

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By definition 3.21 The expectation of a discrete random variable X is
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$E(X) = \Sigma_k kP(X = k)$

Therefore...

$E(X) = (\frac{1}{7} \cdot 1) + (\frac{1}{14} \cdot 2) + (\frac{2}{14} \cdot 3) + (\frac{2}{7} \cdot 4) + (\frac{2}{7} \cdot 5)$

$= \frac{7}{2}$

b) Compute $E[[X - 2]]$

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by Fact 3.33 Let g be a real-valued function defined on the range of an random
variable X. If X is a discrete random variable then
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$E(g(X)) = \Sigma_k g(k)P(X = k)$

Therefore...

$E[[X - 2]] = \Sigma_k (X - 2)P(X = k)$

$= (1 - 2)(\frac{1}{7}) + (2 - 2)(\frac{1}{14)} + (3 - 2)(\frac{3}{14}) + (4 - 2)(\frac{2}{7}) + (5 - 2)(\frac{2}{7})$

$= \frac{3}{2}$

3.11

Let Y be a random variable with density function $f(x) = \frac{2}{3}x$ for $x \in [1, 2]$ and $f(x) = 0$ otherwise.

Compute $E[(Y-1)^2]$

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 by Fact 3.33 Let g be a real-valued function defined on the range of an random
variable X. If X is a discrete random variable then
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$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)$

Therefore...

$E[(Y-1)^2] = \int_1^2 (x-1)^2 \frac{2}{3} x dx$

$E[(Y-1)^2] = \int_1^2 (x^2 - 2x + 1)\frac{2}{3} x dx$

$= \frac{2}{3}[\int_1^2 x^3 dx - \int_1^2 (2x^2 + x) dx]$

$= \frac{2}{3}[\frac{x^4}{4}|_1^2 - 2\frac{x^3}{3}|_1^2 + \frac{x^2}{2}|_1^2]$

$= \frac{2}{3}[\frac{16-1}{4} - 2\frac{8-1}{3} + \frac{4-1}{2}]$

$= \frac{7}{18}$

3.12

Suppose that X is a random variable taking values in {1,2,3,...} with probability mass function

$p_X(n) = \frac{6}{\pi^2} \cdot \frac{1}{n^2}$

Show that $E[X] = \infty$

Solution:

$E[X] = \Sigma_{n=1}^{\infty} n(\frac{6}{\pi^2} \cdot \frac{1}{n^2})$

$= \frac{6}{\pi^2} \Sigma_{n=1}^{\infty} \frac{1}{n}$

$= \frac{6}{\pi^2}[1 + \frac{1}{2} + \frac{1}{3} + \ldots]$

$= E[X] = \infty$

3.23

Ten thousand people each buy a lottery ticket. Each lottery ticket costs 1 dollar. 100 people are chosen as winners. Of those 100 people, 80 will win 2 dollars, 19 will win 100 dollars, and one lucky winner will win 7000 dollars. Let X denote the profit(profit = winnings -cost) of a randomly chosen play of this game.

a) Give both the possible values and probability mass function for X.

Solution:

$P(X = -1) = \frac{10000-100}{10000} = \frac{99}{100}$

$P(X = 1) = \frac{80}{10000}$

$P(X = 99) = \frac{19}{10000}$

$P(X = 6999) = \frac{1}{10000}$

b) Find $P(X \geq 100)$

Solution:

$P(X \geq 100) = P(X = 6999) = \frac{1}{10000}$

c) Compute $E[X]$

Solution:

$E[X] = \Sigma_k kP(X = k)$

$= \frac{99}{100}(-1) + (1)\frac{8}{1000} + (99)\frac{19}{10000} + (6999)\frac{1}{10000}$

$= E[X] = 4918.21$

Exercise 3.26

Suppose that X is a discrete random variable with possible values {1,2,3, ...}, and probability mass function

$p_X(k) = \frac{c}{k(k+1)}$

with some constant c > 0.

a) What is the value of c?

Solution:

$\Sigma_{k=1}^{\infty} p_X(k) = 1$

Therefore...

$\Sigma_{k=1}^{\infty} \frac{c}{k(k+1)} = 1$

Note: $\frac{1}{(k+1)k} = \frac{1}{k} - \frac{1}{k+1}$ by partial fraction decomposition

$c\Sigma_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

$c\Sigma_{k=1}^{\infty} \frac{1}{k} - \frac{1}{(k+1)} = 1$

$c[(1 - \frac{1}{2} + (\frac{1}{2} - \frac{1}{3})+ \ldots] = 1$

$c \cdot 1 = 1$

$c = 1$

b) Find E(X)

Solution:

$E(X) = \Sigma_{k=1}^{\infty} k \frac{1}{k+1} = \Sigma_{k=1}^{\infty} \frac{1}{k+1}$

$E(X) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$

The series doesn't converge, so E(X) is non-existing

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In [ ]:
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