Two events A & B are independent.

$\iff P(A \cap B) = P(A) \cdot P(B)$.

Section 7.1
Sums of Independent Random Variables

① Two R.Vs X & Y are independent.

② What is the distribution of $X+Y$?

Section 6.3 Joint Distributions and Independence

Fact 6.22. Let $p(k_1, \ldots, k_n)$ be the joint probability mass function of the discrete random variables $X_1, \ldots, X_n$. Let $p_{X_j}(k) = P(X_j = k)$ be the marginal probability mass function of the random variable X_j. Then $X_1, \ldots, X_n$ are independent if and only if

$$p(k_1, \ldots, k_n) = p_{X_1}(k_1) \cdots p_{X_n}(k_n) \tag{6.21}$$

for all possible values $k_1, \ldots, k_n$.

X, Y : Discrete R.Vs. $\Rightarrow P_{X,Y}(k_1, k_2)$: Joint Probability Mass Function

$P_X(k_1)$: Marginal p.m.f of X

$P_Y(k_2)$: Marginal p.m.f of Y

$P_{X,Y}(k_1, k_2) = P_X(k_1) \cdot P_Y(k_2) \iff X$ & Y are independent. R.Vs.

Fact 6.25. Let $X_1, \ldots, X_n$ be random variables on the same sample space. Assume that for each $j = 1, 2, \ldots, n$, X_j has density function f_{X_j}.

(a) If $X_1, \ldots, X_n$ have joint density function

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n) \tag{6.22}$$

then $X_1, \ldots, X_n$ are independent.

(b) Conversely, if $X_1, \ldots, X_n$ are independent, then they are jointly continuous with joint density function

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n).$$

$\underbrace{f(x,y)}_{\text{Joint Density ft. of } X, Y} = f_X(x) \cdot f_Y(y) \iff X \,\&\, Y$ are independent. R.Vs.

Fact 7.1. (Convolution of distributions) If ① X and Y are independent discrete random variables with probability mass functions p_X and p_Y, then the probability mass function of $\boxed{X+Y}$ is

$$p_{X+Y}(n) = p_X \underset{\text{convolution.}}{*} p_Y(n) = \sum_k p_X(k)\, p_Y(n-k) \underset{\ell = n-k}{=} \sum_\ell p_X(n-\ell)\, p_Y(\ell). \tag{7.2}$$

If ② X and Y are independent continuous random variables with density functions f_X and f_Y then the density function of $\boxed{X+Y}$ is

$$f_{X+Y}(z) = f_X \underset{\text{convolution.}}{*} f_Y(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_{-\infty}^{\infty} f_X(z-x) f_Y(x)\, dx. \tag{7.3}$$

$$\underset{z-x=y}{=} \int_{-\infty}^{\infty} f_X(z-y) \cdot f_Y(y)\, dy \underset{y=x}{=}$$

Example 7.2. (Convolution of Poisson random variables) Suppose that $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ and these are independent. Find the distribution of $X + Y$.

$$P_X(k_1) = P(X = k_1) = e^{-\lambda} \cdot \frac{\lambda^{k_1}}{k_1!} \qquad P_Y(k_2) = P(Y = k_2) = e^{-\mu} \cdot \frac{\mu^{k_2}}{k_2!}$$

$$P_{X+Y}(n) = \sum_k P_X(k) P_Y(n-k) = \sum_k \left(e^{-\lambda} \cdot \frac{\lambda^k}{k!} \right) \cdot \left(e^{-\mu} \cdot \frac{\mu^{n-k}}{(n-k)!} \right)$$

$$= e^{-\lambda} \cdot e^{-\mu} \sum_k \frac{1}{n!} \frac{n!}{k!\,(n-k)!} \lambda^k \cdot \mu^{n-k}.$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \underbrace{\sum_k \boxed{\frac{n!}{k!\,(n-k)!}} \lambda^k \cdot \mu^{(n-k)}}_{\underset{\text{binomial thm.}}{=} (\lambda+\mu)^n.} = e^{-(\lambda+\mu)} \cdot \frac{(\lambda+\mu)^n}{n!}$$

$\therefore \; X + Y \sim \text{Poisson}(\lambda + \mu).$

Example 7.4. (Convolution of binomials with the same success probability) Let $X \sim \text{Bin}(m_1, p)$ and $Y \sim \text{Bin}(m_2, p)$ be independent. Find the distribution of $X + Y$.

$P_X(k_1) = P(X=k_1) = \binom{m_1}{k_1} p^{k_1} \cdot (1-p)^{m_1-k_1}$, $P_Y(k_2) = P(Y=k_2) = \binom{m_2}{k_2} p^{k_2} \cdot (1-p)^{m_2-k_2}$.

$$P_{X+Y}(n) = \sum_k P_X(k) \cdot P_Y(n-k) = \sum_k \left(\binom{m_1}{k} p^k (1-p)^{m_1-k}\right)\left(\binom{m_2}{n-k} p^{n-k} \cdot (1-p)^{m_2-(n-k)}\right)$$

$$= \sum_k \binom{m_1}{k}\binom{m_2}{n-k} p^{\cancel{k}+n-\cancel{k}} (1-p)^{m_1-\cancel{k}+m_2-n+\cancel{k}}$$

$$= \sum_k \binom{m_1}{k}\binom{m_2}{n-k} \boxed{p^n (1-p)^{m_1+m_2-n}}$$

$$= p^n (1-p)^{m_1+m_2-n} \underbrace{\sum_k \binom{m_1}{k}\binom{m_2}{n-k}}_{\text{Vandermonde's Indentity} = \binom{m_1+m_2}{n}} = \binom{m_1+m_2}{n} p^n \cdot (1-p)^{m_1+m_2-n}$$

$\therefore X+Y \sim \text{Bin}(m_1+m_2, p)$.

Example 7.5. (Convolution of geometric random variables) Let X and Y be independent geometric random variables with the same success parameter $p < 1$. Find the distribution of $X + Y$.

$$P_X(k_1) = P(X = k_1) = p \cdot (1-p)^{k_1 - 1} \qquad P_Y(k_2) = P(Y = k_2) = p \cdot (1-p)^{k_2 - 1}.$$

$$P(X+Y=n) = P_{X+Y}(n) = \sum_{k=1}^{n-1} P_X(k) \cdot P_Y(n-k).$$

$\boxed{k,\ n-k < n}$

$$= \sum_{k=1}^{n-1} p(1-p)^{k-1} \cdot p(1-p)^{n-k-1}$$

$$= \sum_{k=1}^{n-1} p^2 (1-p)^{n-2}. = p^2 \cdot \sum_{k=1}^{n-1} (1-p)^{n-2}. = p^2 (1-p)^{n-2} (n-1).$$

$$= \binom{n-1}{1} p^2 (1-p)^{n-2}.$$

$X+Y \not\sim$ Geom / $\exists$ $X+Y$ = # of trials to see exactly 2 successes.

The distribution of the number of trials needed to get exactly k successes is called the *negative binomial* distribution.

Definition 7.6. Let k be a positive integer and $0 < p < 1$. A random variable X has the **negative binomial distribution** with parameters (k, p) if the set of possible values of X is the set of integers $\{k, k+1, k+2, \ldots\}$ and the probability mass function is

$$P(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad \text{for } n \geq k.$$

Abbreviate this by $X \sim \text{Negbin}(k, p)$.

Example 7.8. (Convolution of normal random variables) Suppose that $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and X and Y are independent. Find the distribution of $X + Y$.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-(x-\mu_1)^2/2\sigma_1^2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-(x-\mu_2)^2/2\sigma_2^2}$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx. = \int_{-\infty}^{\infty} \boxed{\frac{1}{\sqrt{2\pi}\,\sigma_1}} e^{(-x-\mu_1)^2/2\sigma_1^2} \cdot \boxed{\frac{1}{\sqrt{2\pi}\,\sigma_2}} e^{-(z-x-\mu_2)^2/2\sigma_2^2} dx$$

$$= \frac{1}{2\pi \cdot \sigma_1 \sigma_2} \int_{-\infty}^{\infty} e^{-(x-\mu_1)^2/2\sigma_1^2} \cdot e^{-(z-x-\mu_2)^2/2\sigma_2^2} dx.$$

$$\underset{\substack{\uparrow \\ \text{Algebra}}}{=} \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \cdot e^{-(z-\mu_1-\mu_2)^2/2(\sigma_1^2+\sigma_2^2)}.$$

$$X + Y \sim \mathcal{N}(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2).$$

$X + Y + Z \sim \mathcal{N}$ (sum of means, sum of variances).

Fact 7.9. Assume $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $a_i \neq 0$, and $b \in \mathbb{R}$. Let $X = a_1X_1 + \cdots + a_nX_n + b$. Then $X \sim \mathcal{N}(\mu, \sigma^2)$ where

$$\mu = a_1\mu_1 + \cdots + a_n\mu_n + b \quad \text{and} \quad \sigma^2 = a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2. \cancel{+b}$$

Example 7.10. Let $X \sim \mathcal{N}(1, 3)$ and $Y \sim \mathcal{N}(0, 4)$ be independent and let $W = \frac{1}{2}X - Y + 6$. Identify the distribution of W.

$W = \frac{1}{2}X - Y + 6 \Rightarrow E[W] = \frac{1}{2}E[X] - E[Y] + 6$

$= \frac{1}{2}(1) - (0) + 6 = \frac{13}{2}$

$Var(W) = (\frac{1}{2})^2 \cdot Var(X) + (-1)^2 \cdot Var(Y)$

$= \frac{1}{4} \cdot 3 + 1 \cdot 4 = \frac{19}{4}.$

$\Rightarrow W \sim \mathcal{N}(\frac{13}{2}, \frac{19}{4}).$

Example 7.11. Adult border collies have mean weight $\mu=40$ 40 pounds with standard deviation $\sigma=4$ 4 pounds, while adult pugs have mean weight $\mu=15$ 15 pounds with standard deviation $\sigma=2$ 2 pounds. (Border collies and pugs are dog breeds.) What is the probability that a border collie weighs at least 12 pounds more than two pugs? Assume that the weights of dogs within a breed are well modeled by a normal distribution, and that the weights are independent of each other.

$X_1 =$ weights of border collies
$X_2 =$ weight of the 1^{st} pug
$X_3 =$ 〃 2^{n} pug.

$\Rightarrow$ $X_1 \sim N(40, 4^2)$
$X_2 \sim N(15, 2^2)$
$X_3 \sim N(15, 2^2)$

$$\Rightarrow P(X_1 \geq X_2 + X_3 + 12) \underset{Y = X_1 - X_2 - X_3}{=\!=\!=} P(Y \geq 12)$$

Example 7.11. Adult border collies have mean weight 40 pounds with standard deviation 4 pounds, while adult pugs have mean weight 15 pounds with standard deviation 2 pounds. (Border collies and pugs are dog breeds.) What is the probability that a border collie weighs at least 12 pounds more than two pugs? Assume that the weights of dogs within a breed are well modeled by a normal distribution, and that the weights are independent of each other.

$$Y = X_1 - X_2 - X_3 \sim N(\mu, \sigma^2)$$

$$\mu = E[X_1] - E[X_2] - E[X_3] = 40 - 15 - 15 = 10.$$

$$\sigma^2 = 1\,Var(X_1) + (-1)^2 Var(X_2) + (-1)^2 Var(X_3) = 16 + 4 + 4 = 24.$$

$$P(Y \geq 12) = P\left(\frac{Y-\mu}{\sigma} \geq \frac{12-10}{\sqrt{24}}\right) \underset{C.L.T}{\approx} P(Z \geq 0.41)$$

$$= 1 - \Phi(0.41) = 1 - 0.6591$$

$$= \boxed{0.3409}$$

$X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\lambda)$

Example 7.12. (Convolution of exponential random variables) Suppose that X and Y are independent Exp(λ) random variables. Find the density of $X + Y$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y \ge 0 \\ 0 & \text{if } y < 0. \end{cases}$$

If $z < 0$.

$$f_{X+Y}(z) = P(X+Y \le z < 0) = 0$$

($\Rightarrow$ At least one of X & Y must be negative.
$\Rightarrow f_X(x) \cdot f_Y(z-x) = 0$)

If $z \ge 0$,

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} \underbrace{f_X(x) \cdot f_Y(z-x)}_{\ne 0} \, dx.$$

(x & $z - x \ge 0 \Rightarrow x \ge 0,\ x \le z$.)

$$= \int_0^z f_X(x) \cdot f_Y(z-x)\, dx$$

$$= \int_0^z \lambda e^{-\lambda x} \cdot \lambda \cdot e^{-\lambda(z-x)}\, dx$$

$$= \int_0^z \boxed{\lambda^2 \cdot e^{-\lambda z}}\, dx. = \lambda^2 e^{-\lambda z} \int_0^z 1\, dx$$

$$= \lambda^2 \cdot z \cdot e^{-\lambda z}$$ ∴ Gamma Distribution

$\Rightarrow X+Y \nsim \exp(\cdot)$ $\sim \text{Gamma}(2, \lambda)$

Example 7.13. (Convolution of uniform random variables) Suppose that X and Y are independent and distributed as Unif[0, 1]. Find the distribution of $X + Y$.

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases}, \qquad f_Y(y) = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{if } y \notin [0,1] \end{cases}.$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx.$$

To $f_X(x) \cdot f_Y(z-x) = 1$, $\quad x,\ \underline{z-x \in [0,1]}$.

If $0 \le z-x \le 1 \Rightarrow -z \le -x \le -z+1$

$\Rightarrow \underline{z-1 \le x \le z}$.

If $\underline{0 \le x \le 1}$ $\Rightarrow$ $\boxed{\max(0, z-1) \le x \le \min(1, z)}$.

Example 7.13. (Convolution of uniform random variables) Suppose that X and Y are independent and distributed as Unif[0, 1]. Find the distribution of $X + Y$.

$$f_{X+Y}(z) = \int_{\max(0,z-1)}^{\min(1,z)} 1 \, dx = \min(1,z) - \max(0,z-1).$$

If $0 \le z \le 1$, then $\min(1,z) = z$, $\max(0,z-1) = 0$.

If $1 < z \le 2$, then $\min(1,z) = 1$, $\max(0,z-1) = z-1$.

$$f_{X+Y}(z) = \begin{cases} z & \text{if } 0 \le z \le 1 \\ 2-z & \text{if } 1 < z \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$X+Y \not\sim$ Unif.