Derek Haynes

Homework 5

Probability Theory

1.17

Given: An urn contains 4 red balls and 3 green balls. Two balls are drawn randomly.

a) Let Z denote the number of green balls in the sample when the draws are done without replacement. Give the possible values and the probability mass fucntion of Z.

b) Let W denote the number of green balls in the sample when the draws are done with replacement. Give the possible values and the probability mass function of W.

Solution:

a) Probability Mass function without replacement (Type 3 counting)

Z = {0 green, 1 green, 2 green}

PMF Function

$$P_Z(2) = \frac{\binom{3}{2} \cdot \binom{4}{0}}{\binom{7}{2}} = \frac{1}{7}$$

$$P_Z(1) = \frac{\binom{3}{1} \cdot \binom{4}{1}}{\binom{7}{2}} = \frac{4}{7}$$

$$P_Z(0) = \frac{\binom{3}{0} \cdot \binom{4}{2}}{\binom{7}{2}} = \frac{2}{7}$$

b) Probability of drawing two green with replacement modeled with binomial distribution

W = {0 green, 1 green, 2 green}

$$P_W(0) = \binom{2}{0} \cdot (3/7)^0 \cdot (1 - (3/7))^2 = \frac{16}{49}$$

$$P_W(1) = \binom{2}{1} \cdot (3/7)^1 \cdot (1 - (3/7))^1 = \frac{24}{49}$$

$$P_W(2) = \binom{2}{2} \cdot (3/7)^2 \cdot (1 - (3/7))^0 = \frac{9}{49}$$

1.48

Given: Consider the experiment of drawing a point uniformly at random from the unit interval [0,1] Let Y be the first digit after the decimal point of the chosen number. Explain why Y is discrete and find its probability mass function.

Solution:

Possible values of Y are ...

$Y = \{.0, .1, .2, .3, .4, .5, .6, .7, .8, .9\}$

and are in a `finite` countable set which makes Y a discrete random variable

$P(Y = .0) = P(Y = 0.1) = P(Y = 0.2) = \ldots P(Y = 0.9) = \frac{1}{10}$

The `p.m.f.` of the variable Y is ...

Probability mass FUNCTION OF Y

K	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
$P_Y(K) = P(Y = K)$	1/10	1/10	1/10	1/10	1/10	1/10	1/10	1/10	1/10	1/10

$\sum_i P(Y = K_i) = 1$

3.3

Given: Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 3e^{-3x}, & x \le 1 \\ 0, & else. \end{cases}$$

a) Verify that f is a density function.

Solution:

$f(x) = \int_0^\infty 3e^{-3x}dx$

$= 3\int_0^\infty e^u du\frac{-1}{3}$, where $\frac{-1}{3}du = dx$

$= \int_0^\infty e^u du$

$= [\lim_{x\to\infty} -e^-3x] - (-1)$

$= 1$

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 X is continuous for all X greater than zero according to
condition 3.7
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b) Calculate $P(-1 < X < 1)$

Solution:

$-e^{-3x}|_0^1 = -\frac{1}{e^3} + 1$

c) Calculate P(x <5)

Solution:

$-e^{-3x}|_0^5 = -\frac{1}{e^{15}} + 1$

d) Calculate P(2 < X < 4 | X <5)

Solution:

$= \frac{P(2<X<4)}{P(X<5)}$

$= \frac{-e^{-3x}|_2^4}{-e^{-3x}|_0^5}$

$= \frac{-e^{-12}-e^{-6}}{1-e^{-15}}$

3.4

Given: Let X $Unif[4, 10]$

a) Calculate P(X < 6)

Solution:

$f(x) = \frac{1}{b-a}$

$= \frac{1}{10-4} = \frac{1}{6}$

$\int_4^6 \frac{1}{6} dx$

$= \frac{1}{6}|_4^6$

$= \frac{1}{3}$

b) Calculate P(|X-7| >1)

Solution:

$$P(|X-7| > 1) = P(1 < X - 7 \cup X - 7 < -1)$$

$$= P(8 < X \cup X < 6) = P(8 < X) + P(X < 6)$$

Thus ...

$$P(X > 8) = f(x) = \int_8^{10} \frac{1}{6} dx$$

$\frac{1}{6}x|_8^{10}$

$= \frac{1}{3}$

Therefore ...

$$P(X > 8) + P(X < 6) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

c) For $4 \leq t \leq 6$, calculate $P(X < t|X < 6)$

Solution:

$$P(X < t|X < 6) = \frac{P(X<t \cap P(X<6)}{P(X<6)}$$

$$= \frac{P(4 \leq X<6)}{P(X<6)}$$

Thus...

$P(4 \leq X < 6) = \int_4^6 \frac{1}{6} dx$

$\frac{1}{6} x|_4^6 = \frac{1}{6}[6 - 4]$

$= \frac{1}{3}$

And ...

$P(X < t|X < 6) = \frac{P(4 \leq X < 6)}{P(X < 6)}$

$\frac{\frac{1}{3}}{\frac{1}{3}}$

$= 1$

3.20

Given: Let $c > 0$ and $X \sim Unif[0, c]$. Show that the random variable $Y = c - x$ has the same density function.

Solution:

`PDF of X` $0 \le x \le c\ , c > 0$

$f_X(x) = \frac{1}{c-0} = \frac{1}{c}$

`CDF of X`

$F_X(x) = \int \frac{1}{c} dx$

$F_X(x) = \frac{1}{c} x$

Y = C - x

`CDF of Y`

$F_Y(y) = P(y \le Y)$

$= P(c - x \le Y)$

$= P(-x \le Y - c)$

$= P(x > -(Y - c))$

$= 1 - P(x \le c - Y)$

$= 1 - F_x(c - Y)$

$= 1 - \frac{(c-Y)}{c}$

$F_Y(y) = \frac{y}{c}$

Since the cdf of X and Y are on the same probability density fucntion of y, the pdf and cdf of X and Y are the same.

In [ ]: