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Unit - 3

Ordinary differential Equation of higher order.

Linear differential equations of Second & heighre order with Constant Coefficient:

General form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots a_{n-1} \frac{dy}{dx} + a_n y = f(x)$$

$$(a_0 D^n + a_n D^{n-1} + a_2 D^{n-2} + \dots a_{n-1} D + a_n)\, y = f(x) \rightarrow ①$$

where $D = \frac{d}{dx}$

and $a_0, a_1, a_2, \dots a_{n-1}, a_n$ are Constant.

The general Soln of Equ ① Consists of

1. Complementary function (CF)
2. Particular Integral (PI)

Hence, the Soln is $y = CF + PI$

To find Complementary functions:

Replace D by m and RHS = 0 in Equ ①. Then we get accsilhrry equation (AE)

Solve the auxillary Equ we get the root.

i) If the roots are real and distinct

i.e $m_1, m_2, m_3 \Rightarrow m_1 \neq m_2 \neq m_3$

$CF = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + \dots$

ii) If the roots are real & Equal.

i.e $m_1 = m_2 = m_3 = m$

$CF = (Ax^2 + Bx + C)\, e^{mx}$

$m_1 = m_2 = m$

$CF = (Ax + B)\, e^{mx}$

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iii) If the roots are Complex
i.e $\alpha \pm i\beta$

$$CF = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

1) Type - 1

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

$$(D^3 - 6D^2 + 11D - 6)y = 0$$

D by m

$$(m^3 - 6m^2 + 11m - 6)y = 0$$

$$m^3 - 6m^2 + 11m - 6 = 0$$

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$$m^2 - 5m + 6 = 0$$

$6m^2$

$\cancel{-3}$ -2

$$\cancel{m^2 + m - 6m + 6 = 0}$$

$$\cancel{m(m+1) - 6(m+1)} =$$

$$m^2 - 3m - 2m + 6 = 0$$

$$m(m-3) - 2(m-3) = 0$$

$$m = 1, 2, 3$$

$\therefore m_1 = 1 \quad m_2 = 2 \quad m_3 = 3$

$$CF = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x}$$

$$= Ae^{x} + Be^{2x} + Ce^{3x}$$

The Soln is

$$y = CF$$

$$y = Ae^{x} + Be^{2x} + Ce^{3x}$$

2. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{5dy}{dx} + 3y = 0$

$(D^3 + D^2 - 5D + 3)y = 0$

D by m

$(m^3 + m^2 - 5m + 3) = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -5 & 3 \\ & 0 & 1 & 2 & -3 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$

$m^2 + 2m - 3 = 0$

$-3m^2$

$\wedge$

$-1 \quad +3$

$m^2 - m + 3m - 3 = 0$

$m(m-1)\ 3(m-1) = 0$

~~$m = 1, 3, -3\ 1$~~

$m = 1, 1, -3$

$m_1 = m_2 = m = 1$

$CF = (Ax+B)e^{mx} + Ce^{m_3 x}$

$= (Ax+B)e^{x} + Ce^{-3x}$

The Soln is

$y = CF$

$y = (Ax+B)e^{x} + Ce^{-3x}$

2. $D^2 + 8D + 12y = 0$

$m^2 + 8m + 12 = 0$

$12m^2$

$\wedge$

$6 \quad 2$

$m^2 + 6m + 2m + 12 = 0$

$m(m+6)\ 2(m+6) = 0$

$m = -6, -2 \qquad m_1 = -6 \quad m_2 = -2$

$CF = Ae^{m_1 x} + Be^{m_2 x}$

$= Ae^{-6x} + Be^{-2x}$

The Soln is $y = CF$

$y = Ae^{-6x} + Be^{-2x}$

4. $(D^2 - 6D + 25)y = 0$

$a = m$
$2ab = -6m$
$2mb = -6m$
$b = -3$

$m^2 - 6m + 25 = 0$

$25m^2$

$a = 1 \quad b = -6 \quad c = 25$

$$\boxed{m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$m = \frac{6 \pm \sqrt{36 - 100}}{2}$$

$$m = \frac{6 \pm \sqrt{-64}}{2}$$

$$m = \frac{6 \pm 8i}{2}$$

$m = 6 \pm 8i$

$m = 3 \pm 4i$ Complex root $\sqrt{-1} = i$

where $\alpha = 3 \quad \beta = 4$

$CF = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

$CF = e^{3x}(A \cos 4x + B \sin 4x)$

5. $(D^2 + 3D - 4)y = 0$

$m^2 + 3m - 4 = 0$

$-4m^2$

$-1 \quad +4$

$m^2 - m + 4m - 4 = 0$

$m(m-1) \quad 4(m-1) = 0$

$m = -4, 1$

$CF = Ae^{m_1 x} + Be^{m_2 x}$

$CF = Ae^{-4x} + Be^{x}$

Type: 2
RHS of eqn (1), ie $f(x) = e^{ax}$

$$PI = \frac{1}{f(D)} e^{ax}$$
$$= \frac{1}{f(a)} e^{ax}$$

if $f(a) = 0$, then
$$PI = \frac{x}{f'(D)} e^{ax}$$
$$PI = x$$
$$PI = \frac{x}{f'(a)} e^{ax}$$

if $f'(a) = 0$ then
$$PI = \frac{x \cdot x}{f''(D)} e^{ax}$$
$$PI = \frac{x^2}{f''(a)} e^{ax}$$

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1) $(D^2+6D+5)y = e^{2x}$

$m^2+6m+5=0$

$5m^2$

1 5

$m^2+m+5m+5=0$

$m(m+1)\ 5(m+1)=0$

$m = -1, -5$

$CF = Ae^{m_1x} + Be^{m_2x}$

$CF = Ae^{-x} + Be^{-5x}$

To find PI.

$$PI = \frac{1}{D^2+6D+5} e^{2x}$$
$$= \frac{1}{4+12+5} e^{2x} \Rightarrow \frac{1}{21} e^{2x}$$

The Soln

$$y = Ae^{-x} + Be^{-5x} + \frac{1}{21} e^{2x}$$

2. $(D^2+2D+1)y = e^{-x}+5$

$(D^2+2D+1)y = e^{-x}+5e^{0x}$

$m^2+2m+1=0$

$m^2+m+m+1=0$

$m(m+1)+1(m+1)$

$m=-1,-1$

$CF = (Ax+B)e^{mx}$

$= (Ax+B)e^{-x}$

$(PT)_1 = \frac{1}{D^2+2D+1} \cdot e^{-x}$

$$\boxed{\begin{matrix} D=a \\ D=-1 \end{matrix}}$$

$= \frac{1}{1-2+1} e^{-x}$

$(PT_1) = \frac{1}{0} e^{-x}$

$(PT_1) = \frac{x}{2D+2} \cdot e^{-x}$

$P(I_1) = \frac{x}{-2+2} \cdot e^{-x}$

$P(I_1) = \frac{x}{0} e^{-x}$

$$\boxed{P(I_1) = \frac{x^2}{2} \cdot e^{-x}}$$

$P(I_2) = \frac{1}{D^2+2D+1} \cdot 5e^{0x}$

$= \frac{1}{1} 5e^{0x}$

$= 5e^{0x}$

$= 5$

$(D=0)$

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$y = CF + PI$

$y = (Ax+B)e^{-x} + \frac{x^2}{2} \cdot e^{-x} + 5$.

Note:

1) $\cos hx = \frac{e^{x}+e^{-x}}{2}$

2) $\sin hx = \frac{e^{x}-e^{-x}}{2}$

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1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cos h2x$

$(D^2 + 2D+1)y = \cos h2x$

$(D^2+2D+1)\, y = \frac{e^{2x}+e^{-2x}}{2}$

$m^2+2m+1=0$

$(D^2+2D+1)y = \frac{e^{2x}}{2} + \frac{e^{-2x}}{2}$

m^2

1 1

$m^2+m+m+1=0$

$m=-1,-1$

$CF = (Ax+B)e^{-x}$

To find PI

$(PI)_1 = \frac{1}{D^2+2D+1} \cdot \frac{e^{2x}}{2}$

$(PI)_1 = \frac{1}{4+4+1} \cdot e^{2x}/2$

$= \frac{1}{9} \cdot e^{2x}/2$

$= \frac{e^{2x}}{18}$

$(PI)_2 = \frac{1}{D^2+2D+1} \cdot \frac{e^{-2x}}{2}$

$= \frac{1}{4-4+1} \cdot \frac{e^{-2x}}{2} \Rightarrow \frac{e^{-2x}}{2} = (PI)_2$

The Soln is

$y = (Ax+B)e^{-x} + \frac{e^{2x}}{18} + \frac{e^{-2x}}{2}$

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2. $(D^3-3D^2+4D-2)y=e^x$

3 $(D^2+6D+9)y=5e^{3x}$

2. $(D^3-3D^2+4D-2)y=e^x$

$(m^3-3m^2+4m-2)\cancel{y}=0$

1	1	-3	4	-2
	0	1	-2	2
	1	-2	2	0

$m^2-2m+2=0$

$2m^2$

$a=1 \quad b=-2 \quad c=2$

$m = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \boxed{e^{2x}+1e^{0x}}$

$m = \frac{2\pm\sqrt{4-8}}{2a}$

$m = \frac{2\pm\sqrt{-4}}{2a}$ $\quad i^2=-1$

$m = \frac{2\pm 2i}{2}$

$m = 1\pm i$ Complex root. $\sqrt{-1}=i$

where $\alpha=1 \quad \beta=1$

$CF = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$CF = e^{x}(A\cos x + B\sin x)$

$PI = \frac{1}{\cancel{D^3+61}} \quad \frac{1}{D^3-3D^2+4D-2} e^x$

$D=1$

$PI = \frac{1}{1-3+4-2} e^x$

$PI = \frac{1}{0} e^x$

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$$PI = \frac{x}{3D^2 - 6D + 4} e^x$$

$$PI = \frac{x}{3-6+4} e^x$$

$$PI = \frac{x}{1} e^x$$

$$PI = x e^x$$

$$y = CF + PI$$

$$y = e^x (A \cos x + B \sin x) + x e^x$$

3. $(D^2 + 6D + 9) y = 5e^{3x}$

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Solve $(D^2 - 4D + 3)y = \cos 2x$

$m^2 - 4m + 3 = 0$

$3m^2$

$-1 \quad -3$

$m^2 - m - 3m + 3 = 0$

$m(m-1) - 3(m-1) = 0$

$m = 1, 3$

$CF = Ae^{m_1 x} + Be^{m_2 x}$

$CF = Ae^{x} + Be^{3x}$

To find PI

$a = 2$

$D^2 = -a^2$

$D^2 = -4$

$$PI = \frac{1}{D^2 - 4D + 3} \cos 2x$$

$$= \frac{1}{-4 - 4D + 3} \cos 2x$$

$$= \frac{1}{-1 - 4D} \cos 2x$$

$$= \frac{(1-4D)}{(1-4D)(-1-4D)} \cos 2x$$

$$= \frac{(1-4D)}{(-4D+1)(-4D-1)} \cos 2x$$

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$$= \frac{1-4D}{(-4D)^2-1^2} \cos 2x$$

$D^2 = -a^2$

$$= \frac{1-4D}{16D^2-1} \cos 2x$$

$D^2 = -4$

$$= \frac{1-4D}{-64-1} \cos 2x$$

$$= \frac{-1}{65}\left(\cos 2x - 4D(\cos 2x)\right)$$

$$= \frac{-1}{65}\left(\cos 2x + 8 \sin 2x\right)$$

The Soln is

$$y = Ae^{x} + Be^{3x} + \frac{-1}{65}\left(\cos 2x + 8 \sin 2x\right)$$

3. $(D^2+6D+8)y = \cos^2 x$

$m^2+6m+8=0$

$8m^2$

4 2

$m^2+4m+2m+8=0$

$m(m+4)\ 2(m+4)=0$

$m = -4, -2$

$CF = Ae^{-2x} + Be^{-4x}$

$$PI = \frac{1}{D^2+6D+8} \cdot \cos^2 x$$

$$= \frac{1}{D^2+6D+8} \frac{1+\cos 2x}{2}$$

$$= \frac{1}{D^2+6D+8} \cdot \frac{1}{2} + \frac{1}{D^2+6D+8} \frac{\cos 2x}{2}$$

$$= \frac{1}{D^2+6D+8} \cdot \frac{1}{2} e^{0x} + \frac{1}{D^2+6D+8} \frac{\cos 2x}{2}$$

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① $\Rightarrow \dfrac{1}{D^2+6D+8} \cdot \dfrac{1}{2} e^{0x}$ $D=0$

$= \dfrac{1}{8} \cdot \dfrac{1}{2} e^{0x}$

$= \dfrac{1}{16}$

② $\rightarrow \dfrac{1}{D^2+6D+8} \cdot \dfrac{\cos 2x}{2}$ $D^2 = -a^2$, $D^2 = -4$

$\dfrac{1}{2}\left[\dfrac{1}{-4+6D+8} \cos 2x\right]$

$\dfrac{1}{2}\left[\dfrac{1}{4+6D} \cos 2x\right]$

$\dfrac{1}{2}\left[\dfrac{4-6D}{(4-6D)(4+6D)} \cos 2x\right]$

$\dfrac{1}{2}\left[\dfrac{4-6D}{4^2-(6D)^2} \cos 2x\right]$ $\dfrac{36\times4}{144}$

$\dfrac{1}{2}\left[\dfrac{4-6D}{16-36D^2} \cos 2x\right]$ $D^2=-a^2$, $D^2=-4$

$\dfrac{1}{2}\left[\dfrac{4-6D}{16+144} \cos 2x\right]$ $\dfrac{36\times4}{140\times2}$ 296

$\dfrac{1}{2}\left[\dfrac{4-6D}{160} \cos 2x\right]$

~~$\dfrac{1}{296}\left[4-6D(\cos 2x)\right]$~~

~~$\dfrac{1}{296}\left[4+12 \sin 2x\right]$~~

$\dfrac{1}{320\ \cancel{280}}\left[4\cos 2x - 6D(\cos 2x)\right]$

$\dfrac{1}{320\ \cancel{280}}\left[4\cos 2x + 12 \sin 2x\right] \Rightarrow \dfrac{4}{320\ \cancel{280}}\left[\cos 2x + 3\sin 2x\right]$ 80

① + ② $\dfrac{1}{16} + \dfrac{1}{80}\left[\cos 2x + 3 \sin 2x\right]$ //

Note:

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\sin(A+B) = \sin A \cos B + \cancel{\cos A \sin B}$
$\sin(A-B) = \sin A \cos B - \cancel{\cos A \sin B}$

$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

$m^2 - 4m + 3$
$3m^2$
1 · 3

$m^2 - m - 3m + 3 = 0$
$m(m-1) - 3(m-1) = 0$
$m = 3, 1$
$CF = Ae^x + Be^{3x}$

$PI = \frac{1}{D^2 - 4D + 3} (\sin 3x \cos 2x)$

$= \frac{1}{D^2 - 4D + 3} \cdot \frac{1}{2} (\sin(3x+2x) + \sin(3x-2x))$

$= \frac{1}{D^2 - 4D + 3} \cdot \frac{1}{2} (\sin 5x + \sin x)$

$= \frac{1}{2} \left[\frac{1}{D^2 - 4D + 3} \cdot \sin 5x + \sin x \right]$

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$$\frac{1}{2}\left[\frac{1}{D^2-4D+3}\cdot \sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

$\downarrow$

①

$$PI = \frac{1}{D^2-4D+3}\sin 5x$$

$D^2 = -a^2$

$D^2 = -25$

$$PI = \frac{1}{-25-4D+3}\sin 5x$$

$$= \frac{1}{-22-4D}\sin 5x$$

$$= \frac{-22+4D}{(-22-4D)(-22+4D)}\sin 5x$$

$$= \frac{-22+4D}{(-22)^2-(4D)^2}\sin 5x$$

$$= \frac{-22+4D}{484-16D^2}\sin 5x$$

$D^2 = -a^2$

$D^2 = -25$

$$= \frac{-22+4D}{484+400}\sin 5x$$

$$= \frac{-22+4D}{884}\sin 5x$$

$$= \frac{1}{884}\left[-22\sin 5x + 4D(\sin 5x)\right]$$

$$= \frac{1}{884}\left[-22\sin 5x + 20\cos 5x\right]$$

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Type - 4

$f(x) = x^n$

$PI = \frac{1}{f(D)} x^n$

Take the least degree term from $f(D)$ to make the first term unity,
The remaining factor will be of the form $1+g(D)$ or $1-g(D)$
Expand this in ascending powers of D by using the binomial Series.

Formula [Binomial Series]

$(1-x)^{-1} = 1+x+x^2+x^3+x^4+\cdots$
$(1+x)^{-1} = 1-x+x^2-x^3+x^4+\cdots$
$(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$
$(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$

Solve $(D^2+5D+6)y = x^2 \longrightarrow$ Polynomial

$m^2+5m+6=0$
$6m^2$
3 2

$m^2+3m+2m+6=0$
$m(m+3)\ 2(m+3)=0$
$m = -2, -3$
$CF = Ae^{m_1 x} + Be^{m_2 x}$
$= Ae^{-2x} + Be^{-3x}$

$PI = \frac{1}{D^2+5D+6} x^2$

classmate

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Take the lowest degree term outside

$$PI = \frac{1}{6\left(1+\frac{D^2+5D}{6}\right)} \cdot x^2$$

$$= \frac{1}{6}\left[1+\left(\frac{D^2+5D}{6}\right)\right]^{-1} \cdot x^2$$

$$(1+x)^{-1} = 1-x+x^2-x^3+x^4+\cdots$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2+5D}{6}\right)+\left(\frac{D^2+5D}{6}\right)^2 - \cdots\right]x^2$$

$$= \frac{1}{6}\left[1-\frac{1}{6}\left(D^2+5D\right)+\frac{1}{36}\left(D^4+25D^2+10D^3\right)\right]x^2$$

$$= \frac{1}{6}\left[x^2-\frac{x^2}{6}\left(D^2+5D\right)+\frac{x^2}{36}\left(D^4+25D^2+10D^3\right)\right]$$

$$= \frac{1}{6}\left[x^2-\frac{1}{6}\left[D^2x^2+5Dx^2\right]+\frac{1}{36}\left[D^4x^2+25D^2x^2+10D^3x^2\right]\right]$$

$$= \frac{1}{6}\left[x^2-\frac{1}{6}\left(2+5(2x)\right)+\frac{1}{36}\left(0+25(2)+10(0)\right)\right]$$

$$= \frac{1}{6}\left[x^2-\frac{1}{6}\left(2+10x\right)+\frac{1}{\cancel{36}_{18}}\left(\cancel{50}^{25}\right)\right]$$

$$= \frac{1}{6}\left[x^2-\frac{1}{3}-\frac{5x}{3}+\frac{25}{18}\right]$$

$$= \frac{1}{6}\left[x^2-\frac{5x}{3}+\frac{19}{18}\right]$$

The Gum is $y = Ae^{-2x}+Be^{-3x}+\frac{1}{6}\left[x^2-\frac{5x}{3}+\frac{19}{18}\right]$

Solve $(D^3+3D^2+2D)y = x^2$

$\cancel{m^3+3D}$

$m^3+3m^2+2m=0$

$m(m^2+3m+2)=0$

$m=0$

$2m^2$

1 2

$m^2+m+2m+2=0$

$m(m+1)+2(m+1)=0$

$m=-2, -1$

$CF = Ae^{m_1x}+Be^{m_2x}+Ce^{m_3x}$

$= Ae^{0x}+Be^{-1x}+Ce^{-2x}$

$$PI = \frac{1}{D^3+3D^2+2D}x^2$$

$$PI = \frac{1}{2D}\left(\frac{1}{\frac{D^3+3D^2+1}{2D}}\right)x^2$$

$$= \frac{1}{2D}\left(\frac{1}{1+\frac{D^3+3D^2}{2D}}\right)x^2$$

$$= \frac{1}{2D}\left(1+\frac{D^3+3D^2}{2D}\right)^{-1}\cdot x^2$$

$$= \frac{1}{2D}\left(1+\frac{D^2+3D}{2}\right)^{-1}x^2$$

$2D^2 3D$

$(1+x)^{-1} = 1-x+x^2-x^3+x^4+\dots$

$$= \frac{1}{2D}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\dots\right]x^2$$

$$= \frac{1}{2D}\left[1-\frac{1}{2}(D^2+3D)+\frac{1}{4}(D^4+9D^2+6D^3)\right]x^2$$

classmate

$q^{-1}\ 3(2x)$ $D^4 x^2$ q' 6

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$$= \frac{1}{2D}\left[x^2 - \frac{x^2}{2}(D^2+3D) + \frac{x^2}{4}(D^4+9D^2+6D^3)\right]$$

$$= \frac{1}{2D}\left[x^2 - \frac{1}{2}(D^2x^2+3Dx^2) + \frac{1}{4}(D^4x^2+9D^2x^2+6D^3x^2)\right]$$

$$= \frac{1}{2D}\left[x^2 - \frac{1}{2}(2+6x) + \frac{1}{4}(0+18+0)\right]$$

$$= \frac{1}{2D}\left[x^2 - 1 - 3x + \frac{9}{2}\right]$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{7}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7x}{2}\right]$$

$$= \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4}$$

Solve $(D^3 - D^2 - 6D)y = 1+x^2$

$m^3 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m = 0$

$-6m^2$

$-3 \quad +2$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$m = 0, -2, 3$

$$CF = Ae^{0x} + Be^{-2x} + Ce^{3x}$$

$$PI = \frac{1}{D^3 - D^2 - 6D}(1+x^2)$$

$$= \frac{1}{-6D}\left(\frac{1}{\frac{D^3 - D^2}{-6D} + 1}\right)(1+x^2)$$

$$= \frac{1}{-6D}\left(\frac{1}{1 + \frac{D^2 - D}{-6}}\right)(1+x^2)$$

$2D$

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$$= \frac{1}{-6D}\left(1+\left(\frac{D^2-D}{-6}\right)\right)^{-1}(1+x^2)$$

$$= \frac{1}{-6D}\left[1-\left(\frac{D^2-D}{-6}\right)+\left(\frac{D^2-D}{-6}\right)^2\right](1+x^2)$$

$$= \frac{1}{-6D}\left[1+\frac{1}{6}(D^2-D)+\frac{1}{36}(D^2-D)^2\right](1+x^2)$$

$$= \frac{1}{-6D}\left[1+\frac{1}{6}(D^2\right.$$

$$= \frac{1}{-6D}\left[1+x^2+\frac{1}{6}(D^2-D+x^2D^2-x^2D)+\frac{1+x^2}{36}(D^4+D^2-2D^3)\right.$$

$$= \frac{1}{-6D}\left[1+x^2+\frac{1}{6}(\right.$$

25, $\frac{75}{54}$, 18, $\frac{54}{21}$, 75

$2($, $\frac{1}{18}+\frac{1}{3}$, $\frac{18\times 3}{54}$, $\frac{3+18}{54}$, $\frac{21+54}{54}$, $\frac{21}{54}+1$

$$= \frac{1}{-6D}\left[(1+x^2)+\frac{1}{6}\left[(1+x^2)D^2-(1+x^2)D\right]+\frac{1}{36}\left[D^4(1+x^2)+D^2(1+x^2)-2D^3(1+x^2)\right]\right.$$

$$= \frac{1}{-6D}\left[1+x^2+\frac{1}{6}\left[2-2x\right]+\frac{1}{36}\left[0+2-0\right]\right]$$

$$= \frac{1}{-6D}\left[1+x^2+\frac{1}{3}-\frac{x}{3}+\frac{1}{18}\right]$$

$$= \frac{1}{-6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

$=$

$i^2 = -1$

DATE

1. $y'' + y = e^{-x} + x^3$

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + x^3$

$D^2 y + Dy = e^{-x} + x^3$

$(D^2 + D) y = e^{-x} + x^3$

$m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

$CF = Ae^{0x} + Be^{-x}$

$\boxed{m = \alpha \pm \beta I}$

$\frac{d^2y}{dx^2} + y = e^{-x} + x^3$

$D^2 y + y = e^{-x} + x^3$

$(D^2+1)(y) = e^{-x} + x^3$

$D^2 + 1 = 0$

$D^2 = -1$

D

$(m^2+1) = 0$

$m^2 = -1$

$m^2 = i^2$

$m = \pm i$

$m = 0 \pm i$

$\alpha = 0 \qquad \beta = 1$

$CF = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

$CF = e^{0x}(A \cos x + B \sin x)$

$PI = \frac{1}{D^2+1} e^{-x} + x^3$

$PI_1 = \frac{1}{D^2+1} e^{-x} \qquad D = -1$

$= \frac{1}{1+1} e^{-x}$

$= \frac{1}{2} e^{-x}$

$= \frac{e^{-x}}{2}$

$PI_2 = \frac{1}{D^2+1} x^3$

$(1+x)^{-1} = +1 - x + x^2 - x^3$

$1 - D^2 + D^4 - D^6$

$= \frac{1}{1}\left(\frac{1}{D^2+1}\right) x^3$

$= \left[(D^2+1)^{-1} \cdot x^3\right]$

$= \cancel{1 - (D^2+1) + (}$

$1 - (D^2+1)^1 + (D^2+1)^2$

$= [1 - D^2 + D^4 - D^6] x^3$

$1 - D^2 - 1 + D^4 +$

$= x^3 - D^2 x^3 + D^4 x^3 - D^6 x^3$

$= x^3 - 6x + 0$

$y = CF + PI_1 + PI_2$

$= A \cos x + B \sin x + \frac{e^{-x}}{2} + x^3 - 6x$

DATE

4. $y'' - 4y = x^2$

$$\frac{d^2y}{dx^2} - 4y = x^2$$

$$(D^2 - 4)y = x^2$$

$$m^2 - 4 = 0$$

$$m^2 = 4$$

$$m = \pm 2$$

$$\alpha = 0 \qquad \beta = 2$$

$$CF = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

$$CF = e^{0x}(A \cos 2x + B \sin 2x)$$

$$CF = (A \cos 2x + B \sin 2x)$$

$$PI = \frac{1}{D^2 - 4} \cdot x^2$$

$$= \frac{1}{-4}\left[\frac{1}{\left(\frac{D^2}{-4} + 1\right)}\right] x^2$$

$$= -\frac{1}{4}\left[\frac{D^2}{-4} + 1\right]^{-1} x^2$$

$$(x + 1)^{-1} = 1 - x + x^2$$

$$= -\frac{1}{4}\left[1 - \left(\frac{D^2}{-4}\right) + \left(\frac{D^2}{-4}\right)^2\right] x^2$$

$$= \frac{-1}{4}\left[1 + \frac{D^2}{4} + \frac{1}{16} D^4\right] x^2$$

$$= \frac{-1}{4}\left[x^2 + \frac{D^2 x^2}{4} + \frac{D^4 x^2}{16}\right]$$

$$= \frac{-1}{4}\left[x^2 + \frac{2}{4} + 0\right]$$

$$= \frac{-1}{4}\left[x^2 + \frac{1}{2}\right]$$

$$= \frac{-x^2}{4} - \frac{1}{8}$$

$$y = A \cos 2x + B \sin 2x - \frac{x^2}{4} - \frac{1}{8}$$

When $f(x) = e^{ax} (\cos bx)$ or
$e^{ax} (\sin bx)$ or
$e^{ax} (x^n)$

$$PI = \frac{1}{f(D)} e^{ax} (\cos bx) \text{ (or) } e^{ax} \sin bx \text{ (or) } e^{ax} x^n$$

$$= e^{ax} \frac{1}{f(D+a)} \cos bx$$

1) Solve $(D^2+5D+4)y = e^{-x} \sin 2x$

$m^2+5m+4=0$

$4m^2$

$1 \quad 4$

$m^2+m+4m+4=0$

$m(m+1)\ 4(m+1)=0$

$m = -1, -4$

$CF = Ae^{-x} + Be^{-4x}$

$$PI = \frac{1}{D^2+5D+4} e^{-x} \sin 2x \qquad \boxed{D = D+a}$$

$$PI = e^{-x} \frac{1}{(D-1)^2 + 5(D-1)+4} \sin 2x$$

$$= e^{-x} \frac{1}{D^2+1-2D+5D-5+4} \sin 2x$$

$$= e^{-x} \frac{1}{D^2+3D} \sin 2x$$

$D^2 = -a^2$

$D^2 = -4$

$$= e^{-x} \frac{1}{-4+3D} \sin 2x$$

$$= e^{-x} \frac{3D+4}{(3D-4)(3D+4)} \sin 2x$$

$$= e^{-x} \frac{3D+4}{(3D)^2-(4)^2} \sin 2x$$

$$= e^{-x} \frac{3D+4}{9D^2-16} \sin 2x$$

$D^2 = -a^2$

$D^2 = -4$

$$= e^{-x} \frac{3D+4}{9(-4)-16} \sin 2x$$

36
16
52

$$= e^{-x} \frac{3D+4}{-36-16} \sin 2x$$

$$= e^{-x} \frac{3D+4}{-52} \sin 2x$$

$$= \frac{e^{-x}}{-52}\left[3D(\sin 2x) + 4 \sin 2x\right]$$

$$= \frac{e^{-x}}{-52}\left[6 \cos 2x + 4 \sin 2x\right]$$

$$= \frac{\cancel{2} e^{-x}}{-\cancel{52}\ 26}\left[3 \cos 2x + 2 \sin 2x\right]$$

$$= -\frac{e^{-x}}{26}\left[3 \cos 2x + 2 \sin 2x\right]$$

The Soln is $y = CF + PI$

$$y = Ae^{-x} + Be^{-4x} - \frac{e^{-x}}{26}\left[3 \cos 2x + 2 \sin 2x\right]$$

2) Solve $(D^3 - 7D - 6)y = (1+x)e^{2x}$

$m^3 - 7D - 6 = 0$

$m = -1$

-1	1	0	-7	-6
	0	-1	1	6
	1	-1	-6	$\boxed{0}$

$m^2 - m - 6 = 0$

$-6m^2$

$-3 \quad +2$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) \quad 2(m-3) = 0$

$m = -2, 3$

$CF = Ae^{-x} + Be^{-2x} + Ce^{-3x}$

DATE

$$PI = \frac{1}{D^3-7D-6}(1+x)e^{2x}$$

$$\boxed{D = D+a}$$

$$PI = e^{2x}\frac{1}{(D+2)^3-7(D+2)-6}(1+x)$$

$$= e^{2x}\frac{1}{D^3+8+3(D^2 2)+3D(4)-7D-14-6}(1+x)$$

$$= e^{2x}\frac{1}{D^3+8+6D^2+12D-7D-14-6}(1+x)$$

$$= e^{2x}\frac{1}{D^3+6D^2+5D-12}(1+x)$$

$$= e^{2x}\frac{1}{-12\left(\frac{D^3+6D^2+5D}{-12}+1\right)}(1+x)$$

$$= \frac{e^{2x}}{-12}\left[1+\frac{D^3+6D^2+5D}{-12}\right]^{-1}(1+x)$$

$$= \frac{e^{2x}}{-12}\left[1-\frac{D^3+6D^2+5D}{12}\right]^{-1}(1+x)$$

$$= \frac{e^{2x}}{-12}\left[1+\frac{D^3+6D^2+5D}{12}+\left(\frac{D^3+6D^2+5D}{12}\right)^2\right](1+x)$$

$$= \frac{e^{2x}}{-12}\left[x+1+\frac{1}{12}\left[D^3(1+x)+6D^2(1+x)+5D(1+x)+0\right]\right]$$

$$= \frac{e^{2x}}{-12}\left[x+1+\frac{1}{12}\left[(0+0)+6\cancel{D}(0)+5(1)\right]\right]$$

$$= \frac{e^{2x}}{-12}\left[1+x+\frac{5}{12}\right]$$

$$= \frac{e^{2x}}{-12}\left[\frac{17}{12}+x\right]$$

DATE

3. Solve $(D^2+4D+3)y = e^x \cos 2x$

$m^2+4m+3=0$

$3m^2$

$1 \quad 3$

$m^2+m+3m+3=0$

$m(m+1) \; 3(m+1)=0$

$m=-3, -1$

$CF = Ae^{-3x} + Be^{-x}$

$D = D+a$

$PI = \frac{1}{D^2+4D+3} e^x \cos 2x$

$= e^x \frac{1}{(D+1)^2+4(D+1)+3} \cos 2x$

$= e^x \frac{1}{D^2+1+2D+4D+4+3} \cos 2x$

$D^2 = -a^2$

$= e^x \frac{1}{D^2+6D+8} \cos 2x$

$D^2 = -4$

$= e^x \frac{1}{-4+6D+8} \cos 2x$

$\frac{36 \times 4}{144} \quad 2$

$= e^x \frac{1}{6D+4} \cos 2x$

$= e^x \frac{6D-4}{(6D+4)(6D-4)} \cos 2x$

$= e^x \frac{6D-4}{(6D)^2-(4)^2} \cos 2x \Rightarrow e^x \frac{6D-4}{36D^2-16} \cos 2x$

$= e^x \frac{6D-4}{36(-4)-16} \cos 2x$

$= e^x \frac{6D-4}{-144-16} \cos 2x$

$= e^x \frac{6D-4}{-160} \cos 2x \Longrightarrow \frac{e^x}{-160}\left[6D(\cos 2x) - 4\cos 2x\right]$

$PI = -\frac{e^x}{160}\left[-12\sin 2x - 4\cos 2x\right]$

$PI =$

classmate $PI = \frac{4e^x}{160}\left[3\sin 2x + \cos 2x\right] \Longrightarrow$ PAGE $\frac{e^x}{40}\left[3\sin 2x + \cos 2x\right]$

Type-6

When $f(x) = x^n \cos ax$ (or) $x^n \sin ax$

$f(x) = x^n \cos ax$

if $PI = \frac{1}{f(D)} x^n \cos ax$

Note

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = R.P \text{ of } e^{i\theta}$

$\sin\theta = I.P \text{ of } e^{i\theta}$

$e^{iax} = \cos ax + i\sin ax$

$\cos ax = R.P \text{ of } e^{iax}$

$\sin ax = IP \text{ of } e^{iax}$

$$PI = R.P \text{ of } \frac{1}{f(D)} x^n e^{iax}$$

$$PI = R.P \text{ of } e^{iax} \frac{1}{f(D+ia)} x^n$$

if $PI = \frac{1}{f(D)} x^n \sin ax$

$$= IP \text{ of } \frac{1}{f(D)} x^n e^{iax}$$

$$PI = IP \text{ of } e^{iax} \frac{1}{f(D+ia)} x^n$$

Solve $(D^2-1)y = x^2 \cos x$

$D^2 - 1 = 0$

$(D+1)(D-1) = 0$

$D = -1, +1$

$CF = Ae^{-x} + Be^{x}$

$D = D + a$

$$PI = \frac{1}{D^2-1} x^2 \cos x$$

$$PI = R.P \text{ of } \frac{1}{D^2-1} x^2 e^{ix}$$

$$PI = R.P \text{ of } e^{ix} \frac{1}{(D+i)^2-1} x^2$$

$$= R.P \text{ of } e^{ix} \frac{1}{D^2+i^2+2Di-1} x^2$$

$$= R.P \text{ of } e^{ix} \frac{1}{D^2-1+2Di-1} x^2$$

$$= R.P \text{ of } e^{ix} \frac{1}{D^2+2Di-2} x^2$$

$$= R.P \text{ of } e^{ix} \frac{1}{2\left(\frac{D^2+2Di}{2}-1\right)} x^2$$

$$= R.P \text{ of } e^{ix} \frac{1}{2}\left(\frac{D^2+2Di}{2}-1\right)^{-1} x^2$$

$$= R.P \text{ of } \frac{e^{ix}}{-2}\left(1 - \frac{D^2+2Di}{2}\right)^{-1} x^2$$

$$PI = R.P \text{ of } \frac{e^{ix}}{-2}\left[1 + \frac{D^2+2Di}{2} + \left(\frac{D^2+2Di}{2}\right)^2 \cdot\right] x^2 \quad \boxed{i^2=-1}$$

$$PI = R.P \text{ of } \frac{e^{ix}}{-2}\left[x^2 + \frac{1}{2}\left(D^2x^2 + 2Di\,x^2\right) + \frac{1}{4}\left(D^4x^2 + 4iD^3x^2 - 4D^2x^2\right)\right.$$

$$PI = R.P \text{ of } \frac{e^{+ix}}{-2}\left[x^2 + \frac{1}{2}(2+4ix) + \frac{1}{4}(0+0-8)\right]$$

$$PI = R.P \text{ of } \frac{e^{+ix}}{-2}\left[x^2 + 1 + 2ix - 2\right]$$

$PI = R.P \text{ of } \frac{e^{ix}}{-2}\left[x^2+2ix-1\right]$

$PI = RP\left(\frac{\cos x + i\sin x}{-2}\right)(x^2+2ix-1)$

$PI = RP\left[\frac{\cos x}{-2} + \frac{i\sin x}{-2}\right](x^2+2ix-1)$

$PI = \frac{-x^2\cos x}{2} - \frac{ix\cos x + \cos x}{2} + \frac{ix^2 \sin x}{2} + \frac{2ix\sin x}{-2} + \frac{i\sin x}{2}$

$x \sin x$

$PI = \frac{-x^2\cos x}{2} + \frac{\cos x}{2} + x\sin x$ //

$y = CF + PI$

$y = Ae^{-x} + Be^{x} - \frac{x^2\cos x}{2} + \frac{\cos x}{2} + x\sin x$

Solve

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = t\sin t$

$(D^2+2D+3)x = 0$

$m^2+2D+3=0 \qquad a=1 \quad b=2 \quad c=3$

$3m^2$

2

$\cancel{m^2+m+2m+3=0}$

$\cancel{m(m+1)}\ \cancel{(m+1)}$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$m = \frac{-2 \pm \sqrt{4-12}}{2}$

$m = \frac{-2 \pm \sqrt{8}}{2}$

$m = \frac{-2 \pm 2\sqrt{2}}{2}$

$m = \frac{2(-1 \pm \sqrt{2})}{2}$

$m = -1 \pm \sqrt{2}$

$\alpha = -1 \qquad \beta = \sqrt{2}$

$CF = e^{\alpha t}(A\cos \beta t + B\sin \beta t)$

$CF = e^{-t}(A\cos\sqrt{2}t + B\sin\sqrt{2}t)$

$PI = \frac{1}{D^2+2D+3}\, t\sin t$

$= I.P \text{ of } \frac{1}{D^2+2D+3}\, t\, e^{it} \qquad D = D+a$

$= IP \text{ of } e^{it}\frac{1}{D^2+2D+3}\, t$

$= IP \text{ of } e^{it}\frac{1}{(D+i)^2+2(D+i)+3}\, t$

$= IP \text{ of } e^{it}\frac{1}{D^2+i^2+2Di+2D+2i+3}\, t$

$= IP \text{ of } e^{it}\frac{1}{D^2-1+2Di+2D+2i+3}\, t$

$= IP \text{ of } e^{it}\frac{1}{D^2+2Di+2D+2i+2}\, t$

$= IP \text{ of } e^{it}\frac{1}{D^2+2D(i+1)+(2i+2)}\, t$

$= IP \text{ of } \frac{e^{it}}{2i+2}\frac{1}{\left(\frac{D^2+2D(i+1)}{2i+2}+1\right)}\, t$

$= IP \text{ of } \frac{e^{it}}{2i+2}\left[1+\frac{D^2+2D(i+1)}{2i+2}\right]^{-1} t$

$= IP \text{ of } \frac{e^{it}}{2i+2}\left[1-\frac{D^2+2D(i+1)}{2i+2}+\left(\frac{D^2+2D(i+1)}{2i+2}\right)^2\right] t$

classmate

$= IP \text{ of } \frac{e^{it}}{2i+2}\left[1 - \frac{D^2+2D(i+1)}{2i+2}\right]t$

$= IP \text{ of } \frac{e^{it}}{2i+2}\left[t - \frac{D^2 t}{2i+2}\left(D^2+2D(i+1)\right)\right]$

$= IP \text{ of } \frac{e^{it}}{2i+2}\left[t+0 - \frac{2(i+1)}{2i+2}\right]$

$= IP \text{ of } \frac{e^{it}}{2i+2}(t-1)$

$= IP \left(\frac{\cos t + i\sin t}{2i+2}\right)(t-1)$

$\Rightarrow = IP \frac{(\cos t + i\sin t)(2i-2)}{(2i+2)(2i-2)}(t-1)$

$= IP \frac{(\cos t + i\sin t)(2i-2)}{-8}(t-1)$

$(2i)^2 - (2)^2$

$4i^2 - 4$

$= IP \frac{t-1}{-8}\; 2i\cos t + 2i^2\sin t - 2\cos t - 2i\sin t$

$= IP \frac{t-1}{-8}\; 2i\cos t - 2\sin t - 2\cos t - 2i\sin t$

$= IP \frac{t-1}{-8}\left(\mp 2i\sin t + 2i\cos t\right)$

$= IP = \frac{-2(t-1)}{-8}\left(i\sin t \mp i\cos t\right)$

$PI = IP = \frac{(t-1)}{4}\left(\sin t \mp \cos t\right)$ //

$y = CF + PI$ //

$i^2 = -1$
$i = \sqrt{-1}$

1) $(D^2+1)y = x^2 \sin 2x$

$(D^2+1)y = 0$

$m^2+1=0$

$m^2 = -1$

$m = \sqrt{-1}$

$m = 0 \pm i$

$\alpha = 0 \qquad \beta = 1$

$C.F = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

$PI = \frac{1}{D^2+1} x^2 \sin 2x \qquad \Rightarrow e^{iax}$

$PI = I.P \text{ of } \frac{1}{D^2+1} x^2 e^{2ix} \qquad D = D+a$

$= IP \text{ of } e^{2ix} \frac{1}{(D+2i)^2+1} x^2$

$= IP \text{ of } e^{2ix} \frac{1}{D^2+4i^2+4Di+1} x^2$

$= IP \text{ of } e^{2ix} \frac{1}{D^2-4+4Di+1} x^2$

$= IP \text{ of } e^{2ix} \frac{1}{D^2+4Di-3} x^2$

$= IP \text{ of } e^{2ix} \frac{1}{-3\left(\frac{D^2+4Di+1}{-3}\right)} x^2$

$= IP \text{ of } \frac{e^{2ix}}{-3}\left[\frac{D^2+4Di+1}{-3}\right]^{-1} x^2$

$= IP \text{ of } \frac{e^{2ix}}{-3}\left[1 - \left(\frac{D^2+4Di}{-3}\right) + \left(\frac{D^2+4Di}{-3}\right)^2\right] x^2$

$= IP \text{ of } \frac{e^{2ix}}{-3}\left[1 + \frac{1}{3}(D^2+4Di) + \frac{1}{9}(D^2+4Di)^2\right] x^2$

$PI = IP \text{ of } \frac{e^{2ix}}{-3}\left[1 + \frac{1}{3}(D^2+4Di) + \frac{1}{9}(D^4 - 16D^2 + 8D^3 i)\right] x^2$

$PI = IP \text{ of } \frac{e^{2ix}}{-3}\left[x^2 + \frac{1}{3}(2 + 8ix) + \frac{1}{9}(0 - 32 + 0)\right]$

$+2(2D^2)(4Di)$
$16D^3 i$

$(4Di)^2$
$16D^2 i^2$
-16

$(-3)^2$

$2D^2 \, 4Di$
$8D^3$

$\frac{-78}{27}$ $\frac{26}{9}$ $\quad$ $\frac{96}{18}\,\frac{}{78}$ $\quad$ $18-96$ $\quad$ $\frac{32\times 3}{96}$ $\quad$ $36-96$ $\quad$ $\frac{32\times 3}{96}$ $\quad$ 20 $\quad$ $27\ 9$

DATE

$$PI = IP \text{ of } \frac{e^{2ix}}{-3}\left[x^2 + \frac{1}{3}(2 + 8ix) + \frac{1}{9}(-32)\right]$$

$$PI = IP \text{ of } \frac{e^{2ix}}{-3}\left[x^2 + \frac{2}{3} + \frac{8ix}{3} - \frac{32}{9}\right]$$

$$PI = IP \text{ of } \frac{\cos 2x + i\sin 2x}{-3}\left[x^2 + \frac{2}{3} - \frac{32}{9} + \frac{8ix}{3}\right]$$

$$PI = IP \text{ of } \left(\frac{\cos 2x + i\sin 2x}{-3}\right)\left[x^2 - \frac{26}{9} + \frac{8ix}{3}\right]$$

$$PI = IP \text{ of } \frac{\cos 2x}{-3} + \frac{i\sin 2x}{-3}\left[x^2 - \frac{26}{9} + \frac{8ix}{3}\right]$$

$$PI = IP \text{ of } \frac{x^2\cos 2x}{-3} - \frac{26\cos 2x}{-27}$$

$$PI = IP \text{ of } \frac{-x^2\cos 2x}{3} + \frac{26\cos 2x}{27} \quad //$$

DATE

Differential Equ with Variable Coefficient
(Cauchy's Equation form)

The Equ of the form $(a_n x^n D^n + a_{n-1} x^n D^{n-1} + \dots + a_1 x D + a_0) y = f(x)$

wher $D = \frac{d}{dx}$

where $a_0, a_1, a_2 \dots a_n$ are Constant and $f(x)$ is a fn of x is Called differential Equ with Variable Coefficient this Equ is also Called as Cauchy's Equation or Euler's equation.

Euler - Cauchy's Equation:

Solution Euler - Cauchy's Equ is

Put $x = e^z$

$\log x = z$

we get,

$xD = \theta$

$x^2 D^2 = \theta(\theta - 1)$

$x^3 D^3 = \theta(\theta-1)(\theta-2)$

i) Convert the Equ $xy'' - 3y' + \frac{1}{x} y = x^2$ as a linear equ with Constant Coefficient

Soln $xy'' - 3y' + \frac{1}{x} y = x^2$

Multiply by x

$x^2 y'' - \cancel{3y}\ 3xy' + y = x^3$

which is in Cacchys form

Put $x = e^z$ (or) $\log x = z$

$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + y = x^3$

$(x^2 D^2 - 3xD + 1) y = x^3 \longrightarrow (1)$

$$x D = \theta$$
$$x^2 D^2 = \theta(\theta - 1)$$
$$x^2 D^2 = \theta^2 - \theta$$

Sub 2 in ①

$$① \Rightarrow (\theta^2 - \theta - 3\theta + 1) y = (e^z)^3$$
$$(\theta^2 - \theta\,4\theta + 1) y = e^{3z}$$

2) Solve

$$x^2 y'' - x y' + y = x^2$$

This Equ is in Cauchy's form

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = x^2$$
$$(x^2 D^2 - x D + 1) y = x^2 \rightarrow ①$$

$$\left.\begin{aligned} &\text{Put } x = e^z \quad \text{or} \quad \log x = z \\ &x D = \theta \qquad \text{where } \theta = d/dz \\ &x^2 D^2 = \theta(\theta - 1) \\ &\qquad = \theta^2 - \theta \end{aligned}\right\} \rightarrow ②$$

$$(\theta^2 - \theta - \theta + 1) y = (e^z)^2$$
$$(\theta^2 - 2\theta + 1) y = e^{2z}$$

which is a differential Equ with Constant Coefficient.

$$m^2 - 2m + 1 = 0$$

$$m^2$$
$$\wedge$$
$$-1 \quad -1$$

$$m^2 - m - m + 1 = 0$$
$$m(m-1) - (m-1) = 0$$
$$m = 1, 1$$

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$CF = (Ax+B)e^{mx}$
$= (Az+B)e^{z}$

To find PI

$\theta = a$
$\theta = 2$

$PI = \frac{1}{\theta^2 - 2\theta + 1} e^{2z}$

$= \frac{1}{4-4+1} e^{2z}$

$e^{2\log x}$

$PI = e^{2z}$

The Soln is $(Az+B)e^{z} + e^{2z}$
$(A(\log x)+B)x + x^2$

3. Solve $(x^2D^2 - 3xD + 4)y = x^2 \cos(\cancel{\cos} \log x) \rightarrow (1)$

This is in Cauchy's form

Put $x = e^{z}$ (or) $\log x = z$
where $\theta = d/dz$

$xD = \theta$
$x^2D^2 = \theta(\theta-1)$
$x^2D^2 = \theta^2 - \theta$

$(\theta^2 - \theta - 3\theta + 4)y = (e^{z})^2 \cos(\log x)$
$(\theta^2 - 4\theta + 4)y = (e^{z})^2 \cos(\log x)$
$(\theta^2 - 4\theta + 4)y = e^{2z} \cos z$

CF =

$m^2 - 4m + 4 = 0$
$4m^2$
$2 \quad 2$

$m^2 - 2m - 2m + 4 = 0$
$m(m-2) - 2(m-2) = 0$
$m = 2, 2$

$CF = (Az+B)e^{2z}$

$\theta = \theta + a$

To find PI

$PI = \frac{1}{\theta^2 - 4\theta + 4} e^{2z} \cos z$

$PI = e^{2z} \frac{1}{(\theta+2)^2 - 4(\theta+2) + 4} \cos z$

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$$PI = e^{2z} \frac{1}{\theta^2+4+4\theta-4\theta-8+4} \cos z$$

$$PI = e^{2z} \frac{1}{\theta^2} \cos z$$

$\theta^2 = -a^2$

$\theta^2 = -1$

$$PI = e^{2z} \frac{1}{-1} \cos z$$

$$PI = -e^{2z} \cos z$$

$$y = (Az+B)e^{2z} - e^{2z} \cos z$$

$$y = [A(\log x)+B] x^2 - x^2 \cos(\log x)$$

Question: Convert $x^2y'' - xy' + y = \log x + 11$ into differential equ with Constant Coefficient

$$(x^2D^2 - xD + 1)\, y = \log x + 11$$

This is in Cauchy's form

Put $x = e^z$ $\quad \log x = z$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$x^2D^2 = \theta^2 - \theta$

$$(\theta^2 - \theta - \theta + 1)\, y = z + 11$$

$$(\theta^2 - 2\theta + 1)\, y = z + 11$$

which is a differential equation with Constant Coefficient.

Solve the equation $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \sin(\log x)$

$$(x^2D^2 + 4xD + 2)\, y = \sin(\log x)$$

This is Cauchy's form

$x = e^z$ $\quad \log x = z$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

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$x^2D^2 = D^2 - D$

$(D^2 - D + 4D + 2)y = \sin(\log z)$

$(D^2 + 3D + 2)\, y = \sin z$

$CF = Ae^{-x} + Be^{-2x}$

$m^2 + 3m + 2 = 0$

$m = -1, -2$

$D^2 = -a^2$

$D^2 = -a^2$

$D^2 = -1$

$D = \frac{d}{dz}$

$PI = \frac{1}{D^2 + 3D + 2} \sin z$

$= \frac{1}{-1 + 3D + 2} \sin z$

$= \frac{1}{3D + 1} \sin z$

$= \frac{3D - 1}{(3D+1)(3D-1)} \sin z$

$= \frac{3D - 1}{(3D)^2 - (1)^2} \sin z$

$= \frac{3D - 1}{9D^2 - 1} \sin z$

$= \frac{3D - 1}{-9 - 1} \sin z$

$= \frac{3D - 1}{-10} \sin z$

$= \frac{1}{-10}\left[3D[\sin z] - \sin z\right]$

$= \frac{1}{-10}\left[3\cos z - \sin z\right]$

$= -\frac{3\cos z}{10} + \frac{\sin z}{10}$

$y = Ae^{-z} + Be^{-2z} - \frac{3\cdot\cos z}{10} + \frac{\sin z}{10}$

$y = Ae^{-\log x} + Be^{-2\log x} - \frac{3\cos(\log x)}{10} + \frac{\sin(\log x)}{10}$

$y = Ax^{-1} = Bx^{-2} - \frac{1}{10}\left(3\cos(\log x) - \sin(\log x)\right)$

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Q : 2 ⇒ $x^3 y'' + 5x^2 y' + 4xy = \log x$

$\div x$ on b/s

$$x^2 y'' + 5xy' + 4y = \frac{1}{x}\log x$$
$$(x^2D^2 + 5xD + 4)y = \frac{1}{x}\log x$$

This is in Cauchy's form

$x = e^z$ $\qquad$ $\log x = z$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$x^2D^2 = \theta^2 - \theta$

$$(\theta^2 - \theta + 5\theta + 4)y = \frac{1}{x}\log x$$

$$(\theta^2 + 4\theta + 4)y = \frac{1}{x}\log x$$

~~CF~~ = $(\theta^2 + 4\theta + 4)y = \frac{1}{e^z} z$

CF = $m^2 + 4m + 4 = 0$

$4m^2$

2 2

$m^2 - 2m - 2m + 4 = 0$

$m(m-2) - 2(m-2) = 0$

$m = 2, 2$

$CF = (Az + B)e^{2z}$

$$PI = \frac{1}{\theta^2 + 4\theta + 4} \frac{z}{e^z}$$

$D = D + a$

$\theta = \theta + a$

$\theta = \theta - 1$

$$= \frac{1}{\theta^2 + 4\theta + 4} e^{-z} \cdot z$$

$$= e^{-z} \frac{1}{\theta^2 + 4\theta + 4} \cdot z$$

$$= e^{-z} \frac{1}{(\theta-1)^2 + 4(\theta-1) + 4} z$$

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$$= e^{-z} \frac{1}{\theta^2+1-2\theta+4\theta-4+4} z$$

$$= e^{-z} \frac{1}{\theta^2+2\theta+1} z$$

$$= \frac{1}{e^z}\left[\frac{1}{1\left(\frac{\theta^2+2\theta+1}{1}\right)} z\right]$$

$$= \frac{1}{e^z}\left[\left[[\theta^2+2\theta]+1\right]^{-1}\right] z$$

$2\theta^2 2\theta$

$$= \frac{1}{e^z}\left[1-(\theta^2+2\theta)+(\theta^2+2\theta)^2\right] z$$

$$= \frac{1}{e^z}\left[1-(\theta^2+2\theta)+(\theta^4+4\theta^2+4\theta^3)\right] z$$

$$= \frac{1}{e^z}\left[z-2\right]$$

$$= \frac{1}{e^z}\left[\log x-2\right]$$

$$y = \frac{1}{x}\left[\log x-2\right]+\left[A(\log x)+B\right]e^{2\log x}$$

$$y = \left[A(\log x)+B\right]x^2+\frac{1}{x}\left[\log x-2\right] //$$

3. Convert $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 5x^2$

$$(xD^2 + D)y = 5x^2$$

x = $\times x$

$$(x^2D^2 + xD)y = 5x^3$$

$x = e^z$ $\quad \log x = z$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$x^2D^2 = \theta^2 - \theta$

$$(\theta^2 - \theta + \theta)y = 5(e^z)^3$$

$$\theta^2 y = 5e^{3z}$$

Legendre's linear equation.

$$a_n(ax+b)^n \frac{d^ny}{dx^n} + a_{n-1}(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(ax+b)\frac{dy}{dx} + a_0 y = f(x)$$

To Convert legendre's linear differential equation into equ with Constant

$ax+b = e^z$ $\quad \log(ax+b) = z$

$(ax+b)D = a\theta$ $\quad \theta = \frac{d}{dz}$

$(ax+b)^2D^2 = a^2\theta(\theta-1)$

$(ax+b)^3D^3 = a^3\theta(\theta-1)(\theta-2)\dots$

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1) Solve $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$

$$\left[(2x+3)^2 D^2 - (2x+3)D - 12\right] y = 6x$$

$(ax+b) = e^z$ $\quad\quad \log(ax+b) = z$

$(ax+b)D = a\theta$

$(ax+b)^2 D^2 = a^2 \cdot \theta(\theta - 1)$

$(ax+b)^2 D^2 = a^2 \cdot \theta^2 - \theta$

$$\left[4[\theta^2 - \theta) - 2\theta - 12\right] y = 6\left(\frac{e^z - 3}{2}\right)$$

$$\left[4\theta^2 - 4\theta - 2\theta - 12\right] y = 3\left[e^z - 3\right]$$

$$(4\theta^2 - 6\theta - 12) y = 3\left[e^z - 3\right]$$

which is a differential equ in Constant Coefficient

$$4m^2 - 6m - 12 = 0$$

$$2m^2 - 3m - 6 = 0$$

$\cancel{12m^2}$

$4(2)(-6)$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{3 \pm \sqrt{9 + 48}}{4}$$

$$m = \frac{3 \pm \sqrt{57}}{4} \Rightarrow$$

$CF = \cancel{e^{\alpha x}\left[A \cos \beta x + B \sin \beta x\right]}$

$CF = \cancel{e^{3/4 z}\left[A \cos \frac{\sqrt{57}}{4} z + B \sin \frac{\sqrt{57}}{4} z\right]}$

$CF = Ae^{\frac{3+\sqrt{57}}{4}} + Be^{\frac{3-\sqrt{57}}{4}}$

$(4\theta^2 - 6\theta - 12)y = 3e^z - 9e^{0z}$

$$(PI_1) = \frac{1}{4\theta^2 - 6\theta - 12} 3e^z$$

$D = a$

$\theta = 1$

$$= \frac{1}{4 - 6 - 12} 3e^z$$

$$= \frac{1}{-14} 3e^z \rightarrow \frac{-3(2x+3)}{14}$$

$$\Rightarrow \frac{-6x - 9}{14}$$

$(PI)_2 = \frac{1}{4\theta^2 - 6\theta - 12} \cdot 9e^{0z}$ $\qquad D = 0$, $\theta = 0$

$= \frac{1}{-12} \cdot -9e^{0z}$

$= \frac{9}{12}$

$= \frac{3}{4}$

$y = CF + \frac{3}{4} - \frac{3}{14}(2x+3)$

2) $[(x+3)^2 D^2 - 4(x+3)D + 6]y = \log(x+3)$

$ax+b = e^z \qquad \log z = \log(ax+b)$

$(ax+b)D = a\theta$

$(ax+b)^2 D^2 = a^2 \cdot \theta(\theta-1)$

$= a^2 \cdot \theta^2 - \theta$

$[\theta^2 - \theta - 4\theta + 6]y = \cancel{\log (ax+b) e^z}\ z$

$(\theta^2 - 5\theta + 6)y = \cancel{\log e^z}\ z$

$m^2 - 5m + 6 = 0$

$6m^2$

$-3\ \ -2$

$\cancel{m^2 + m - 6m + 6 = 0}$ $\qquad m^2 - 3m - 2m + 6 = 0$

$\cancel{m(m+1) - 6(m+1) =}$ $\qquad m(m-3) - 2(m-3) = 0$

$m = 2, 3$

$CF = Ae^{2z} + Be^{3z}$

$PI = \frac{1}{\theta^2 - 5\theta + 6} z$

$= \frac{1}{6\left(\frac{\theta^2 - 5\theta}{6} + 1\right)} z$

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$$= \frac{1}{6}\left[\frac{D^2-5D}{6}+1\right]^{-1} z$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2-5D}{6}\right)\right] z$$

$$= \frac{1}{6}\left[z-\frac{1}{6}\left[-5\right]\right]$$

$$= \frac{1}{6}\left[z+\frac{5}{6}\right]$$

$$= \frac{z}{6}+\frac{5}{36}$$

$$y = Ae^{2z}+Be^{3z}+\frac{z}{6}+\frac{5}{36}$$

$$y = Ae^{2\log(ax+b)}+Be^{3\log(ax+b)}+\frac{\log(ax+b)}{6}+\frac{5}{36}$$

$$y = A\,\cancel{ax+1}(x+3)^2+B\,\cancel{a}(x+3)^3+\frac{\log(x+3)}{6}+\frac{5}{36}$$