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Efficiency-improved doubly robust estimation with non-confounding predictive covariates

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Abstract: The doubly robust (DR) estimators are widely utilized in observational studies to estimate the average treatment effect due to their consistency when either the outcome model or propensity score is correctly specified. However, in practice, the DR estimator may exhibit significant estimation variability due to misspecification of the outcome model. This paper introduces an alternative DR estimator to enhance efficiency when the outcome model may be misspecified. Specifically, our estimator achieves the smallest asymptotic variance among all DR estimators when the propensity score model is correctly specified, while retaining the double robustness property. To further improve efficiency, we incorporate additional covariates that have predictive power for the outcome variable but do not affect the treatment variable. Simulation studies demonstrate the superior performance of our proposed estimator compared to existing methods. Additionally, we provide two real data analyses to illustrate the proposed method.

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1. Introduction

Various methods have been developed to estimate average treatment effect (ATE) in the presence of potential confounders, including outcome regression, inverse probability weighting (IPW), matching, and others [33]. Among these, the augmented inverse probability weighted (AIPW) estimator, proposed by Robins, Rotnitzky and Zhao [32] and Bang and Robins [2], and also referred to as the doubly robust (DR) estimator, is widely employed in observational studies. The DR estimator is known for its double robustness, ensuring consistency when outcome model or propensity score is correctly specified [27, 2, 53, 47, 50, 54, 25, 22, 26, 11, 38, 48]. However, when the outcome regression model is misspecified, the usual DR estimator may exhibit suboptimal estimation efficiency. This inferior performance can be attributed to the substantial variance introduced by estimating coefficients within the incorrectly specified outcome regression model [39, 20, 44, 40, 5, 41, 46, 45, 30, 42, 49].

In practice, the propensity score may be known by design and is more likely to be correctly specified than the outcome model. From this perspective, some scholars focus on a class of DR estimators designed to be more efficient than usual DR estimators when the propensity score is correct. These estimators achieve improved performance relative to existing DR estimators by directly minimizing the asymptotic variance. In the context of outcome missing at random,

Cao, Tsiatis and Davidian [5] propose such a DR estimator to estimate the mean of the outcome. For optimal individual treatment regimes (ITRs), Pan and Zhao [30] introduce an enhanced DR estimator by directly maximizing the DR estimator of the expected outcome over a class of ITRs. However, both of these methods only focus on marginal mean outcomes. It remains unclear how their ideas can be applied to estimate the mean difference between two potential outcomes, namely the ATE. This challenge arises because minimizing the asymptotic variance of ATE estimators may introduce additional interaction terms through nuisance models, thus posing practical challenges to achieving similarly improved double robustness.

Another approach to further enhance estimation efficiency, without introducing additional risks of model misspecification, is incorporating additional predictive covariates, provided the propensity scores are correctly specified. These covariates, while not confounders themselves, possess predictive capability for the outcome variable without directly affecting the treatment. Theoretically, whenever the propensity score model regarding the true confounding is correctly specified, the practitioners can also correctly specify the propensity score model concerning both confounding and predictive covariates [16, 27, 4, 10, 13, 8, 43]. One of the most common scenarios where this idea is applied is in randomized experiments, where predictive covariates are frequently recommended for regression adjustment to improve efficiency [12, 23, 52]. In observational studies, Lunceford and Davidian [27] theoretically show that incorporating additional predictive covariates into the IPW estimator, under the assumption of correctly specified propensity scores, can enhance estimation efficiency compared to excluding them. Craycroft, Huang and Kong [8] reveal that the semiparametric efficiency bound evaluated with both true confounders and predictive covariates can be smaller than that evaluated with true confounders alone [15, 19].

While numerous studies have been on predictive covariates, it remains uncertain whether including them in DR estimators can improve estimation efficiency, particularly when dealing with potentially misspecified working models. This paper explores these issues and proposes an improved DR estimator for ATE. In addition to its double robustness properties, our proposed estimator achieves the smallest variance among a class of DR estimators. Our theoretical findings indicate that incorporating these predictive covariates into the proposed estimator substantially improves estimation efficiency, as long as the propensity scores are correctly specified, even if the outcome models are misspecified.

The subsequent sections of this paper follow this structure. Section 2 introduces the notation, framework, and assumptions. Section 3 presents a general strategy for the improved DR estimator when the propensity score is correctly specified. Section 4 conducts comparative analyses for different adjustment sets. Section 5 performs simulation studies designed to assess the finite-sample performance of the proposed estimators. Section 6 further illustrates the proposed method via two real datasets. The paper concludes with a brief discussion in Section 7. Technical details are available in the Supplementary Material.

2. Setup

Assume that there are n individuals who are independently and identically sampled from a superpopulation of interest. We consider evaluating the causal effect of a binary treatment Z on an outcome Y subject to confounding by the observed covariates X. The potential outcomes under treatment and control are denoted as Y_1 and Y_0, respectively. The observed outcome Y

can be expressed as a combination of these potential outcomes: $Y = ZY_1 + (1 - Z)Y_0$. We make the stable unit treatment value assumption (SUTVA), which states that only one version of potential outcomes exists for each individual, and there is no interference between units [34]. We are interested in estimating the ATE, formally defined as $\tau = \mathbb{E}(Y_1 - Y_0)$.

A fundamental problem of causal inference is that we can never simultaneously observe both potential outcomes for a unit. To ensure the identifiability of ATE, the strong unconfoundedness or ignorability assumption is commonly employed [33].

Assumption 1. *(i)* $Z \perp\!\!\!\perp (Y_0, Y_1) \mid X$, *(ii)* $0 < \mathrm{pr}(Z = 1 \mid X) < 1$.

Assumption 1(i) states that the treatment assignment Z is independent of the potential outcomes (Y_0, Y_1) given the covariates X, thereby eliminating any potential unobserved confounder between treatment and outcome. In a randomized experiment, the ignorability assumption holds naturally, because Z is independent of (Y_0, Y_1, X). Assumption 1(ii) is crucial to ensure adequate overlap between the covariate distributions of the treatment and control groups [33].

Define $e_0(X) = \mathrm{pr}(Z = 1 \mid X)$ and $Q_0(X, Z) = \mathbb{E}(Y \mid X, Z)$. We posit a parametric model $e_X(X; \alpha)$ for the propensity score $e_0(X)$, where α denotes the nuisance parameter. The subscript X is used to denote the propensity score model only using the covariates X, and other similar subscripts will be introduced in subsequent discussions. When propensity score $e_X(X; \alpha)$ is correctly specified, let α_0 denote the true parameter, and we have $e_0(X) = e_X(X; \alpha_0)$. Following Robins, Rotnitzky and Zhao [32], a DR estimator can be constructed as follows,

$$\tau_X(\hat{\alpha}, \hat{\beta}) = \mathbb{P}_n \left[\begin{array}{c} \dfrac{ZY - \{Z - e_X(X; \hat{\alpha})\} Q_X(X, 1; \hat{\beta})}{e_X(X; \hat{\alpha})} \\ - \dfrac{(1 - Z)Y - \{e_X(X; \hat{\alpha}) - Z\} Q_X(X, 0; \hat{\beta})}{1 - e_X(X; \hat{\alpha})} \end{array} \right], \tag{1}$$

where $\mathbb{P}_n(\cdot)$ denotes the empirical mean operator, $Q_X(X, Z; \beta)$ represents a specified working model for $Q_0(X, Z)$, $\hat{\alpha}$ and $\hat{\beta}$ are the estimated parameters. When the outcome model is correctly specified, there exists a constant β_0 such that $Q_0(X, Z) = Q_X(X, Z; \beta_0)$. Adding an augmentation term involving both propensity scores and outcome models, the DR estimator provides additional protection against model misspecification [32, 36, 2]. The proposed estimator $\tau_X(\hat{\alpha}, \hat{\beta})$ in (1) is doubly robust in the sense that it consistently estimates ATE as long as $e_X(X; \hat{\alpha}) \xrightarrow{p} e_0(X)$ or $Q_X(X, Z; \hat{\beta}) \xrightarrow{p} Q_0(X, Z)$.

3. Improved doubly robust estimation

A common approach for estimating the parameters β is to use the ordinary least square for $Q_X(X, Z; \beta)$, and then plug them into the usual DR estimator (1). However, this can lead to inefficiency and significant variability [39, 5, 41, 30], primarily because correctly specifying the outcome models can be challenging. In contrast, the propensity score is often more likely to be correctly specified since it may be known by design. This motivates us to develop improved DR estimators to achieve more desirable efficiency properties, leveraging propensity scores

that are more likely to be correctly specified. The maximum likelihood estimator $\hat{\alpha}$ is the solution to the following estimating equations,

$$\mathbb{P}_n\{S_\alpha(X,Z;\alpha)\} = \mathbb{P}_n\left\{\frac{\partial \log f(X,Z;\alpha)}{\partial \alpha}\right\} = 0, \tag{2}$$

where $\log f(X,Z;\alpha) = Z\log\{e_X(X;\alpha)\} + (1-Z)\log\{1-e_X(X;\alpha)\}$.

Lemma 3.1. *Let $\hat{\beta}$ be any root-n consistent estimator converging in probability to some β^*, that is, $\hat{\beta}-\beta^* = O_p(n^{-1/2})$. When the propensity score is correctly specified, but $Q_X(X,Z;\beta)$ may or may not be, the influence function corresponding to the estimator $\tau_X(\hat{\alpha},\hat{\beta})$ can be expressed as follows:*

$$\tilde{\varphi}_X(Y,X,Z;\alpha_0,\beta^*) = \varphi_X(Y,X,Z;\alpha_0,\beta^*) - \mathcal{D}_X(\beta^*)\mathcal{H}^{-1}_{\alpha\alpha,0}S_\alpha(X,Z;\alpha_0), \tag{3}$$

where $\varphi_X(Y,X,Z;\alpha,\beta)$ is defined as

$$\frac{ZY}{e_X(X;\alpha)} - \frac{(1-Z)Y}{1-e_X(X;\alpha)} - \frac{Z-e_X(X;\alpha)}{e_X(X;\alpha)}Q_X(X,1;\beta) + \frac{e_X(X;\alpha)-Z}{1-e_X(X;\alpha)}Q_X(X,0;\beta) - \tau,$$

$$\mathcal{D}_X(\beta) = -\mathbb{E}\left\{\partial\varphi_X(Y,X,Z;\alpha_0,\beta)/\partial\alpha^{\mathrm{T}}\right\} \text{ and } \mathcal{H}_{\alpha\alpha,0} = \mathbb{E}\left\{S_\alpha(X,Z;\alpha_0)S_\alpha^{\mathrm{T}}(X,Z;\alpha_0)\right\}.$$

The influence function (3) involves an additional term due to the estimation of the nuisance parameter α besides $\varphi_X(Y,X,Z;\alpha_0,\beta)$. Essentially, one can derive that $\varphi_X(Y,X,Z;\alpha_0,\beta)$ is the influence function for $\tau_X(\alpha_0,\hat{\beta})$. Details of the proof can be found in Section S1.1 of the Supplementary Material. The additional term $\mathcal{D}_X(\beta^*)\mathcal{H}^{-1}_{\alpha\alpha,0}S_\alpha(X,Z;\alpha_0)$ in (3) would disappear when both models are correctly specified. We can calculate the asymptotic variance of the influence function $\tilde{\varphi}_X(Y,X,Z;\alpha_0,\beta)$ in (3), denoted as $\Sigma_X(\beta)$ for any fixed β. Define $\mathcal{D}_{X\beta}(\beta) = \partial\mathcal{D}_X(\beta)/\partial\beta$, $Q_{X\beta}(X,Z;\beta) = \partial Q_X(X,Z;\beta)/\partial\beta$, $e_{X\alpha}(X;\alpha) = \partial e_X(X;\alpha)/\partial\alpha$, and $\tilde{e}_0(X,Z) = Ze_0(X) + (1-Z)\{1-e_0(X)\}$. The minimizer of the asymptotic variance $\Sigma_X(\beta)$ is denoted as β^{opt}, which is the solution to:

$$\mathbb{E}\left(\begin{array}{l} \dfrac{(-1)^{1-Z}}{\tilde{e}_0(X,Z)}\left\{Y - Q_X(X,Z;\beta) - \mathcal{D}_X(\beta)\mathcal{H}^{-1}_{\alpha\alpha,0}e_{X\alpha}(X;\alpha_0)\right\} \\ \times\left[\begin{array}{l} \dfrac{Z-e_0(X)}{e_0(X)}\left\{Q_{X\beta}(X,1;\beta) + \mathcal{D}_{X\beta}(\beta)\mathcal{H}^{-1}_{\alpha\alpha,0}e_{X\alpha}(X;\alpha_0)\right\} \\ -\dfrac{e_0(X)-Z}{1-e_0(X)}\left\{Q_{X\beta}(X,0;\beta) + \mathcal{D}_{X\beta}(\beta)\mathcal{H}^{-1}_{\alpha\alpha,0}e_{X\alpha}(X;\alpha_0)\right\} \end{array}\right] \end{array}\right) = 0. \tag{4}$$

We consider $\hat{\beta}^{\mathrm{opt}}$ as the solution to the sample version of (4),

$$\mathbb{P}_n\left(\begin{array}{l} \dfrac{(-1)^{1-Z}}{\tilde{e}_X(X,Z;\hat{\alpha})}\left\{Y - Q_X(X,Z;\beta) - \hat{\mathcal{D}}_X(\beta)\hat{\mathcal{H}}^{-1}_{\alpha\alpha}e_{X\alpha}(X;\hat{\alpha})\right\} \\ \times\left[\begin{array}{l} \dfrac{Z-e_X(X;\hat{\alpha})}{e_X(X;\hat{\alpha})}\left\{Q_{X\beta}(X,1;\beta) + \hat{\mathcal{D}}_{X\beta}(\beta)\hat{\mathcal{H}}^{-1}_{\alpha\alpha,0}e_{X\alpha}(X;\hat{\alpha})\right\} \\ -\dfrac{e_X(X;\hat{\alpha})-Z}{1-e_X(X;\hat{\alpha})}\left\{Q_{X\beta}(X,0;\beta) + \hat{\mathcal{D}}_{X\beta}(\beta)\hat{\mathcal{H}}^{-1}_{\alpha\alpha,0}e_{X\alpha}(X;\hat{\alpha})\right\} \end{array}\right] \end{array}\right) = 0, \tag{5}$$

where

$$\hat{\mathcal{D}}_X(\beta) = -\mathbb{P}_n\left\{\partial\varphi_X(Y,X,Z;\hat{\alpha},\beta)/\partial\alpha^{\mathrm{T}}\right\},$$
$$\hat{\mathcal{D}}_{X\beta}(\beta) = -\mathbb{P}_n\left\{\partial^2\varphi_X(Y,X,Z;\hat{\alpha},\beta)/\partial\alpha^{\mathrm{T}}\partial\beta\right\},$$
$$\hat{\mathcal{H}}_{\alpha\alpha} = \mathbb{P}_n\left\{S_\alpha(X,Z;\hat{\alpha})S_\alpha^{\mathrm{T}}(X,Z;\hat{\alpha})\right\},$$
$$\tilde{e}_X(X,Z;\hat{\alpha}) = Ze_X(X;\hat{\alpha}) + (1-Z)\{1-e_X(X;\hat{\alpha})\}.$$

The equations (4) and (5) using the baseline covariates X are similar in spirit to those in Cao, Tsiatis and Davidian [5] and Pan and Zhao [30]; see equation (16) in Cao, Tsiatis and Davidian [5] and equation (9) in Pan and Zhao [30]. Both previous studies in our context primarily focus on enhancing the estimation efficiency of the marginal expectations $E(Y_1)$ and $E(Y_0)$. In contrast, our study focuses on inferring the ATE, which cannot be regarded as a direct extension of the estimators proposed by Cao, Tsiatis and Davidian [5] and Pan and Zhao [30]. The formulation of equation (4) is challenging because calculating the minimizer of asymptotic variance $\Sigma_X(\beta)$ requires considering the interactions between two outcome models, complexities that are absent in the marginal case. For detailed derivations, please refer to Section S2.2 in the Supplementary Material.

When the propensity score is correct, $\hat{\beta}^{\mathrm{opt}} \xrightarrow{p} \beta^{\mathrm{opt}}$; and when the outcome model is correct, $\hat{\beta}^{\mathrm{opt}} \xrightarrow{p} \beta_0$. Therefore, the improved DR estimator $\tau_X(\hat{\alpha},\hat{\beta}^{\mathrm{opt}})$ achieves the double robustness property under the estimated propensity score. We summarize these conclusions as the following theorem.

Theorem 3.2. *We assume that (4) has a unique solution. Under Assumption 1, the proposed estimator $\tau_X(\hat{\alpha},\hat{\beta}^{\mathrm{opt}})$ has the following properties:*

(1) when either the propensity score or the outcome model is correctly specified, $\tau_X(\hat{\alpha},\hat{\beta}^{\mathrm{opt}})$ is a consistent estimator, and

$$\sqrt{n}\{\tau_X(\hat{\alpha},\hat{\beta}^{\mathrm{opt}}) - \tau\} \xrightarrow{d} N(0,\Lambda_X), \tag{6}$$

where the detailed expression of Λ_X can be found in Section S2.5 of Supplementary Material;

(2) when the propensity score model is correctly specified, $\tau_X(\hat{\alpha},\hat{\beta}^{\mathrm{opt}})$ achieves the smallest asymptotic variance among all estimators of the form in (1), and $\Lambda_X = \Sigma_X(\beta^{\mathrm{opt}})$.

Equation (6) also implies that the improved DR estimator $\tau_X(\hat{\alpha},\hat{\beta}^{\mathrm{opt}})$ is $\sqrt{n}$-consistent as the usual DR estimator. When the propensity score model is correctly specified, we know that β^{opt} minimizes the asymptotic variance of the candidate estimators in (1), and $\Lambda_X = \Sigma_X(\beta^{\mathrm{opt}})$. In such a scenario, for the potentially misspecified outcome working models and any nuisance parameter β, we have:

$$\Sigma_X(\beta) = \Sigma_X^{\mathrm{fps}}(\beta) - \mathcal{D}_X(\beta)\mathcal{H}_{\alpha\alpha,0}^{-1}\{\mathcal{D}_X(\beta)\}^{\mathrm{T}}, \tag{7}$$

where $\Sigma_X^{\mathrm{fps}}(\beta) = \mathbb{E}\left[\{\varphi_X(Y,X,Z;\alpha_0,\beta)\}^2\right]$. Essentially, the quantity $\Sigma_X^{\mathrm{fps}}(\beta)$ represents the asymptotic variance of the estimator $\tau_X(\alpha_0,\hat{\beta})$ in (1), which corresponds to the case where the propensity score is completely known. Here, $\hat{\beta}$ satisfies $\hat{\beta} - \beta = O_p(n^{-1/2})$. The detailed statements of $\Sigma_X^{\mathrm{fps}}(\beta)$ and the proof of (7) can be found in Section S1.1 and S2.3 of the Supplementary Material, respectively. This finding aligns with widely known results in causal inference, indicating that even when the propensity score is known, using the estimated propensity score can improve estimation efficiency [19].

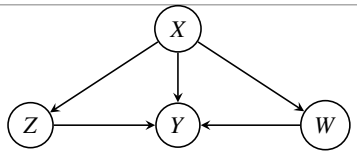

Fig 1. *Observational studies with additional predictive covariates W.*

In addition, we provide a discussion in Section S3 of the Supplementary Material on the symmetric case in which the outcome regression model is correctly specified while the propensity score model is misspecified. We present the corresponding influence functions, variance expressions, and the associated minimization problem. Our findings suggest that it is still possible to construct an estimator that attains the smallest asymptotic variance under this setting. However, some desirable doubly robust properties may no longer hold in this symmetric case, as the estimator is generally inconsistent when the propensity score model is correctly specified but the outcome regression model is misspecified.

4. Efficiency enhancement with predictive covariates

In practice, besides the true confounders X, some baseline covariates that have predictive power for the outcome may exist, which are denoted as W. These variables are correlated with the outcome but are not directly associated with the treatment, except through the true confounders X [27, 8]. The aforementioned conditional independence can be expressed as follows:

Assumption 2. $Z \perp\!\!\!\perp (W, Y_0, Y_1) \mid X$.

When combined with Assumption 1, Assumption 2 implies that genuine confounders X sufficiently address the potential confounding relationship between Z and (W, Y_0, Y_1). This condition also implies $Z \perp\!\!\!\perp (Y_0, Y_1) \mid (X, W)$. In practice, because W only predicts the outcome variable Y without directly affecting the treatment variable, it does not qualify as a genuine confounder [14]. Therefore, even in cases where covariates W are missing, it does not affect the identifiability of the ATE. In such cases, various methods can be applied to impute missing values for predictive covariates W [35]. This is especially relevant when enhancing the estimation efficiency of ATE in randomized experiments with missing covariates, as discussed by Zhao and Ding [52]. However, this paper assumes that covariates W are either fully observed or appropriately imputed. Refer to Figure 1 for a graphical representation of these concepts.

Therefore, we can also utilize the adjustment set (X, W) to construct the DR type estimators. In parallel with the previous discussion, we can specify a parametric model for $e_0(X, W) = \mathrm{pr}(Z = 1 \mid X, W)$ as $e_W(X, W; \alpha, \gamma)$. Here, α continues to denote the parameters only related to the terms with X, while γ is an additional parameter vector corresponding to W. The reason for this specification is that, given that W does not directly affect Z, it is clear that $e_0(X) = e_0(X, W)$, and the true parameters of $e_W(X, W; \alpha, \gamma)$ are $\alpha = \alpha_0$ and $\gamma = \gamma_0 = 0$. Therefore, when we can correctly parametrize $e_X(X; \alpha)$, we can also correctly model $e_W(X, W; \alpha, \gamma)$. For instance, we can propose the following pair of propensity score

models:

$$\begin{aligned} e_X(X;\alpha) &= \text{expit}(\alpha_0 + \alpha_1^{\mathrm{T}} X), \\ e_W(X,W;\alpha,\gamma) &= \text{expit}(\alpha_0 + \alpha_1^{\mathrm{T}} X + \gamma^{\mathrm{T}} W), \end{aligned} \tag{8}$$

where $\alpha = (\alpha_0, \alpha_1^{\mathrm{T}})^{\mathrm{T}}$. When γ takes its true value $\gamma_0 = 0$, the newly considered model $e_W(X,W;\alpha,\gamma)$ reduces to $e_X(X;\alpha)$, which depends solely on X and α. Based on this perspective, in our subsequent discussions, we will employ the same basic parametric model for both $e_X(X;\alpha)$ and $e_W(X,W;\alpha,\gamma)$. The only difference is that we will add an additional predictor to the model $e_W(X,W;\alpha,\gamma)$, thereby ensuring that when $e_X(X;\alpha)$ is correctly specified, $e_W(X,W;\alpha,\gamma)$ is also correctly specified. Similarly, we can also consider maximum likelihood estimation for the model $e_W(X,W;\alpha,\gamma)$. In addition to replacing $e_X(X;\alpha)$ in (2) with $e_W(X,W;\alpha,\gamma)$, we also need to account for the effect of including additional covariates W in the propensity score model $e_W(X,W;\alpha,\gamma)$. This can be jointly expressed as the following estimating equations:

$$\begin{aligned} \mathbb{P}_n\{S_\alpha(X,W,Z;\alpha,\gamma)\} &= 0, \\ \mathbb{P}_n\{S_\gamma(X,W,Z;\alpha,\gamma)\} &= 0, \end{aligned} \tag{9}$$

where $S_\alpha(X,W,Z;\alpha,\gamma)$ and $S_\gamma(X,W,Z;\alpha,\gamma)$ denote the score functions with respect to the parameters α and γ for the propensity score model $e_W(X,W;\alpha,\gamma)$, respectively. For simplicity, let $\eta = (\alpha^{\mathrm{T}}, \gamma^{\mathrm{T}})^{\mathrm{T}}$, $\eta_0 = (\alpha_0^{\mathrm{T}}, \gamma_0^{\mathrm{T}})^{\mathrm{T}}$, and $\hat{\eta} = (\hat{\alpha}^{\mathrm{T}}, \hat{\gamma}^{\mathrm{T}})^{\mathrm{T}}$, where $\hat{\alpha}$ and $\hat{\gamma}$ are the solutions to (9).

Besides the propensity score model, we also introduce an outcome working model $Q_W(X,W,Z;\delta)$ for the conditional expectation $Q_0(X,W,Z) = \mathbb{E}(Y \mid X,W,Z)$. When the outcome model is correctly specified, there exists a constant δ_0 such that $Q_0(X,W,Z) = Q_W(X,W,Z;\delta_0)$. Similar to (1), the improved DR estimator for the set (X,W) can be expressed as follows:

$$\tau_W(\hat{\eta},\hat{\delta}) = \mathbb{P}_n \left[\begin{array}{c} \dfrac{ZY - \{Z - e_W(X,W;\hat{\eta})\} Q_W(X,W,1;\hat{\delta})}{e_W(X,W;\hat{\eta})} \\ - \dfrac{(1-Z)Y - \{e_W(X,W;\hat{\eta}) - Z\} Q_W(X,W,0;\hat{\delta})}{1 - e_W(X,W;\hat{\eta})} \end{array} \right]. \tag{10}$$

The influence function $\tilde{\varphi}_W(Y,X,W,Z;\eta_0,\delta)$ of the estimator $\tau_W(\hat{\eta},\hat{\delta})$, with $\hat{\delta} - \delta = O_p(n^{-1/2})$, can be similarly constructed as Lemma 3.1 when the propensity score is correctly specified. We use $\Sigma_W(\delta)$ to characterize the asymptotic variance of the influence function $\tilde{\varphi}_W(Y,X,W,Z;\eta_0,\delta)$. Let $e_{W\eta}(X,W;\eta) = \partial e_W(X,W;\eta)/\partial\eta$. Similar to (5), we can derive an estimating equation based on the adjustment set (X,W) as follows:

$$\begin{aligned} &\mathbb{P}_n \left(\begin{array}{l} \dfrac{(-1)^{1-Z}}{\tilde{e}_W(X,W,Z;\hat{\eta})} \left\{ Y - Q_W(X,W,Z;\delta) - \hat{\mathcal{D}}_W(\delta) \hat{\mathcal{H}}_{W,0}^{-1} e_{W\eta}(X,W;\hat{\eta}) \right\} \times \\ \left[\begin{array}{l} \dfrac{Z - e_W(X,W;\hat{\eta})}{e_W(X,W;\hat{\eta})} \left\{ Q_{W\delta}(X,W,1;\delta) + \hat{\mathcal{D}}_{W\delta}(\delta) \hat{\mathcal{H}}_{W,0}^{-1} e_{W\eta}(X,W;\hat{\eta}) \right\} \\ - \dfrac{e_W(X,W;\hat{\eta}) - Z}{1 - e_W(X,W;\hat{\eta})} \left\{ Q_{W\delta}(X,W,0;\delta) + \hat{\mathcal{D}}_{W\delta}(\delta) \hat{\mathcal{H}}_{W,0}^{-1} e_{W\eta}(X,W;\hat{\alpha},\hat{\gamma}) \right\} \end{array} \right] \end{array} \right) \\ &= 0, \end{aligned} \tag{11}$$

where

$$
\begin{aligned}
Q_{W\delta}(X,W,Z;\delta) &= \partial Q_W(X,W,Z;\delta)/\partial\delta,\\
\hat{\mathcal{H}}_{W,0} &= \mathbb{P}_n\left\{S_\eta(X,W,Z;\hat\eta)S_\eta^{\mathrm{T}}(X,W,Z;\hat\eta)\right\},\\
\tilde e_W(X,W,Z;\hat\eta) &= Ze_W(X,W;\hat\eta)+(1-Z)\{1-e_W(X,W;\hat\eta)\},\\
\hat{\mathcal{D}}_W(\delta) &= -\mathbb{P}_n\left\{\partial\varphi_W(Y,X,Z;\hat\eta,\delta)/\partial\eta^{\mathrm{T}}\right\},\\
\hat{\mathcal{D}}_{W\delta}(\delta) &= -\mathbb{P}_n\left\{\partial^2\varphi_W(Y,X,Z;\hat\eta,\delta)/\partial\eta^{\mathrm{T}}\partial\delta\right\}.
\end{aligned}
$$

The corresponding solution, $\hat\delta^{\mathrm{opt}}$, would minimize the sample version of asymptotic variance $\Sigma_W(\delta)$, provided that the propensity score model $e_W(X,W;\eta)$ is correctly specified. The improved DR estimator $\tau_W(\hat\eta,\hat\delta^{\mathrm{opt}})$ also achieves a similar double robustness property as Theorem 3.2. Before presenting the comparative results for $\tau_X(\hat\alpha,\hat\beta^{\mathrm{opt}})$ and $\tau_W(\hat\eta,\hat\delta^{\mathrm{opt}})$, we need to impose certain restrictions on the outcome models $Q_X(X,Z;\beta)$ and $Q_W(X,W,Z;\delta)$. We formally characterize the model restrictions in the following assumption, which encompasses a wide range of models applicable in practice.

Assumption 3. *The working model $Q_X(X,Z;\beta)$ is covered by the working model $Q_W(X,W,Z;\delta)$.*

While we consider Assumption 3, it does not imply that our proposed improved DR estimator always relies on Assumption 3. We recommend employing additional covariates for enhancing dual efficiency only when we believe Assumption 3 holds. Even if Assumption 3 does not hold, as demonstrated in Theorem 3.2, our proposed improved DR estimator can still be more efficient than the usual DR estimator in some cases. Assumption 3 essentially requires that the outcome model employed for the set (X,W) should include the model corresponding to X. This common model specification, in practice, encompasses a broad class of generalized linear models. For instance, we can use linear or log-linear models for continuous outcomes as follows,

$$
\begin{aligned}
Q_X(X,Z;\beta) &= \beta_0+\beta_1 Z+\beta_2^{\mathrm{T}}X+\beta_3^{\mathrm{T}}ZX,\\
Q_W(X,W,Z;\delta) &= \delta_0+\delta_1 Z+\delta_2^{\mathrm{T}}X+\delta_3^{\mathrm{T}}ZX+\delta_4^{\mathrm{T}}W+\delta_5^{\mathrm{T}}ZW,
\end{aligned}
\tag{12}
$$

where $\beta=(\beta_0,\beta_1,\beta_2^{\mathrm{T}},\beta_3^{\mathrm{T}})$ and $\delta=(\delta_0,\delta_1,\delta_2^{\mathrm{T}},\delta_3^{\mathrm{T}},\delta_4^{\mathrm{T}},\delta_5^{\mathrm{T}})$. Logistic regression or Probit regression can be applied to binary outcomes. We do not require the working model in Assumption 3 to be correctly specified. In the context of randomized experiments, there has been considerable research on whether adding additional covariates in regression models can improve estimation efficiency [23, 28, 52, 51]. However, there is currently no discussion in observational studies on whether using regression models in the commonly-used DR estimators can improve estimation efficiency.

Theorem 4.1. *Assuming that Assumptions 1-3 hold, when the propensity scores are correctly specified, we have $\Sigma_W(\delta^{\mathrm{opt}})\le\Sigma_X(\beta^{\mathrm{opt}})$, where β^{opt} and δ^{opt} are the probability limits of $\hat\beta^{\mathrm{opt}}$ and $\hat\delta^{\mathrm{opt}}$, respectively.*

Theorem 4.1 suggests that the proposed DR estimators result in a dual efficiency enhancement when incorporating predictive covariates into both the outcome model and the propensity score model. This complements the findings of Theorem 3.2(ii), which only indicates that the proposed DR estimators lead to an efficiency improvement when the propensity score model

is correctly specified. It is noteworthy that we do not need to impose new modeling restrictions on the propensity score in practice, as the correct specification of the model $e_X(X;\alpha)$ always ensures the correct specification of $e_W(X,W;\alpha,\gamma)$. Hence, practitioners should consistently utilize propensity scores and adjust for predictive covariates to enhance estimation efficiency. From this perspective, the proposed estimator $\tau_W(\hat{\eta},\hat{\delta}^{\text{opt}})$ in Section 4 are more accurately characterized as outcome model-assisted rather than strictly outcome model-based. An estimation procedure that includes two improved estimators is provided in Algorithm 1.

Algorithm 1 Estimation procedure

1: **Input:** Data $\{Z_i, X_i, W_i, Y_i\}_{i=1}^n$
2: **Output:** Estimators $\tau_X(\hat{\alpha},\hat{\beta}^{\text{opt}})$ and $\tau_W(\hat{\eta},\hat{\delta}^{\text{opt}})$
3: Compute $\hat{\alpha}$ and $\hat{\gamma}$ using (2) and (9)
4: Compute $\hat{\beta}^{\text{opt}}$ and $\hat{\delta}^{\text{opt}}$ using (5) and (11)
5: Compute $\tau_X(\hat{\alpha},\hat{\beta}^{\text{opt}})$ and $\tau_W(\hat{\eta},\hat{\delta}^{\text{opt}})$ using (1) and (10)

When both models are misspecified, the performance of our proposed estimator, like the usual DR estimator, is no longer controllable. In such cases, the estimation results may suffer from substantial bias and lack of consistency [20], regardless of whether the predictive covariate W is included. This highlights the critical importance of selecting appropriate working models. In practical applications, incorporating prior knowledge about model selection, along with utilizing data-driven methods such as cross-validation, can help address the potential poor performance caused by the simultaneous misspecification of both models.

5. Simulation studies

We conduct several simulation studies to evaluate the finite-sample performance of our proposed method. Throughout this simulation, the true confounders $X = (X_1, X_2)^{\mathrm{T}}$ are drawn from a uniform distribution on $[-1,1]^2$. The predictive covariates $W = (W_1, W_2)^{\mathrm{T}}$ are also drawn from uniform distribution on $[-1,1]^2$. Let $\text{expit}(u) = \exp(u)/\{1+\exp(u)\}$. We consider the following four data-generating scenarios for Z and Y.

(CC): The treatment variable Z follows a Bernoulli distribution with $\text{pr}(Z = 1 \mid X) = \text{expit}(-0.2 + X_1 - X_2)$. The outcome variable Y is generated from $Y = 20 \times (2Z + 4X_1 - 5X_2 + 3W_1 - 4W_2) + \epsilon_y$, where $\epsilon_y \sim N(0,\sigma^2)$.

(IC): The treatment variable Z follows a Bernoulli distribution with $\text{pr}(Z = 1 \mid X) = \text{expit}(-0.2 + 4X_1^2 - X_2X_3)$, where X_3 follows a Bernoulli distribution with $\text{pr}(X_3 = -1) = \text{pr}(X_3 = 1) = 0.5$. The outcome variable Y is generated from $Y = 20 \times (2Z + 4X_1 - 5X_2 + 3W_1 - 4W_2) + \epsilon_y$, where $\epsilon_y \sim N(0,\sigma^2)$.

(CI) The treatment variable Z follows a Bernoulli distribution with $\text{pr}(Z = 1 \mid X) = \text{expit}(-0.2 + X_1 - X_2)$. The outcome variable Y is generated from $Y = 20 \times \{2Z + 2\log(2 + 3X_1^2W_2^2 + 3X_2^2W_1^2 + 3X_2) + 6X_1W_1 + 6X_2W_2\} + \epsilon_y$, where $\epsilon_y \sim N(0,\sigma^2)$.

(II) The treatment variable Z follows a Bernoulli distribution with $\text{pr}(Z = 1 \mid X) = \text{expit}(-0.2 + 4X_1^2 - X_2X_3)$, where X_3 follows a Bernoulli distribution with $\text{pr}(X_3 = -1) = \text{pr}(X_3 = 1) = 0.5$. The outcome variable Y is generated from $Y = 20 \times \{2Z + 2\log(2 + 3X_1^2W_2^2 + 3X_2^2W_1^2 + 3X_2) + 6X_1W_1 + 6X_2W_2\} + \epsilon_y$, where $\epsilon_y \sim N(0,\sigma^2)$.

We use the parameter σ to control the variance of Y, which allows us to assess the efficiency of the method under different levels of variability. The true ATE is 40 for all the

above scenarios. Algorithm 1 is used to obtain the improved DR estimators. We would like to compare $\tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$ and $\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$, and as we note in (5) below, although Cao, Tsiatis and Davidian [5] and Pan and Zhao [30] did not provide improved DR estimators for ATE, our estimator $\tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$, based solely on the covariates X, can be seen as an extension of their proposed marginal improved DR estimators. Thus, the comparison between $\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$ and $\tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$ can, to some extent, be interpreted as an additional utilization of predictive covariates, extending the methods proposed by Cao, Tsiatis and Davidian [5] and Pan and Zhao [30]. The proposed estimators are also compared with the usual DR estimator [32, 2], where the nuisance parameters $\hat{\beta}^{\text{ls}}$ and $\hat{\delta}^{\text{ls}}$ are obtained by ordinary least squares (OLS). Let $\tau_X(\hat{\alpha}, \hat{\beta}^{\text{ls}})$ and $\tau_W(\hat{\eta}, \hat{\delta}^{\text{ls}})$ be the corresponding usual DR estimators.

For all the settings, we consider a unified estimation procedure by using the Logistic model parameterizations in (8) for $e_X(X;\alpha)$ and $e_W(X, W;\alpha,\beta)$, and the linear model parameterizations in (12) for $Q_X(X, Z;\beta)$ and $Q_W(X, W, Z;\delta)$. Therefore, the four data-generating mechanisms satisfy the following conditions: (1) `CC`: correct propensity score and correct outcome model; (2) `IC`: incorrect propensity score and correct outcome model; (3) `CI`: correct propensity score and incorrect outcome model; and (4) `II`: incorrect propensity score and incorrect outcome model.

Tables 1 and 2 reports the bias, standard error (SE), and root mean squared error (RMSE) for estimating the average treatment effect. We consider two approaches for computing SE and RMSE. The first set of results, denoted as SE^1 and RMSE^1, is calculated using the analytical asymptotic variance formula presented in Theorem 3.2. Importantly, even under Setting `II`, where two working models are misspecified and the point estimator may be biased, the variance estimate derived from the M-estimation framework remains valid. In addition, we employ a bootstrap-based estimation method with 500 replications, yielding empirical estimates of SE and RMSE, denoted as SE^2 and RMSE^2, respectively. These results serve as an empirical validation of the theoretical findings established in Theorem 3.2. All evaluations are conducted under a sample size of 1000.

We observe that the two methods for computing SE and RMSE generally yield similar results across the different settings. As the noise level σ increases from 0.1 to 100, all estimators exhibit greater variability across different scenarios. Under low to moderate noise levels, the proposed estimators are comparable to or outperform existing methods. Our proposed estimators $\tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$ and $\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$ demonstrate good stability and relatively small bias when either working model is correctly specified (see scenarios `IC`, `CI`, and `CC`). In particular, under the `CC` scenario, the usual DR and improved DR estimators show similar asymptotic performance, with smaller estimated variance when adjustment is made using (X, W). Next, under the scenario where the propensity score model is correctly specified but the outcome model is misspecified (denoted as `CI`), the improved DR estimators exhibit smaller asymptotic variance than the usual DR estimators. Finally, when comparing the two sets of covariate adjustments, we find that the performance of the improved DR estimator aligns with the conclusion in Theorem 4.1. In particular, the improved DR estimator $\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$ yields a substantial reduction in asymptotic variance when adjusting for (X, W) instead of using X alone.

We also present boxplots of the point estimates for all scenarios derived from 500 replications, as shown in Figure 2. The results are as expected, clearly demonstrating that consistent estimates can be obtained for the first three scenarios (i.e., `IC`, `CI` and `CC`), while noticeable

Table 1

Simulation results for different scenarios. Results with superscript "1" indicate variance estimates derived from the analytical asymptotic variance formula, while results with superscript "2" indicate those based on the bootstrap method.

	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$
$\sigma = 0.1$	$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{ls}})$					$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$				
CC	0.08	3.97	3.98	4.13	4.13	0.14	3.95	3.95	4.35	4.35
IC	-0.07	3.96	3.96	3.97	3.97	-0.07	3.95	3.95	4.08	4.08
CI	-0.42	4.82	4.85	4.83	4.84	-0.38	4.77	4.79	4.82	4.83
II	4.66	4.59	6.54	4.65	6.59	3.86	5.54	6.75	5.81	6.97
$\sigma = 0.1$	$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{ls}})$					$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{opt}})$				
CC	-0.00	0.01	0.01	0.01	0.01	0.00	0.01	0.01	0.01	0.01
IC	0.00	0.01	0.01	0.01	0.01	-0.00	0.01	0.01	0.01	0.01
CI	-0.34	4.31	4.33	4.23	4.24	-0.20	3.99	4.00	4.14	4.15
II	4.67	3.92	6.10	3.99	6.14	5.40	8.30	9.90	8.61	10.16
	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$
$\sigma = 1$	$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{ls}})$					$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$				
CC	-0.17	3.98	3.98	3.94	3.94	-0.16	3.95	3.96	3.94	3.94
IC	-0.07	3.94	3.94	3.95	3.95	-0.06	3.94	3.94	4.06	4.06
CI	-0.43	4.85	4.87	4.63	4.64	-0.35	4.80	4.81	4.62	4.64
II	4.71	4.59	6.58	4.66	6.63	3.98	5.60	6.87	5.89	7.11
$\sigma = 1$	$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{ls}})$					$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{opt}})$				
CC	0.00	0.07	0.07	0.07	0.07	0.00	0.07	0.07	0.07	0.07
IC	0.00	0.07	0.07	0.07	0.07	0.01	0.07	0.07	0.07	0.07
CI	-0.40	4.33	4.35	4.16	4.18	-0.28	4.00	4.01	4.02	4.03
II	4.77	3.92	6.18	3.99	6.22	5.96	10.16	11.78	10.37	11.96
	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$
$\sigma = 10$	$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{ls}})$					$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$				
CC	-0.08	4.04	4.04	3.95	3.95	-0.07	4.01	4.01	3.96	3.96
IC	-0.05	4.00	4.00	4.03	4.03	-0.03	4.00	4.00	4.17	4.17
CI	-0.29	4.89	4.90	4.80	4.81	-0.21	4.83	4.84	4.75	4.75
II	4.79	4.65	6.68	4.72	6.73	3.87	5.66	6.85	5.75	6.93
$\sigma = 10$	$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{ls}})$					$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{opt}})$				
CC	0.03	0.69	0.69	0.65	0.65	0.04	0.68	0.68	0.66	0.66
IC	0.04	0.68	0.69	0.68	0.68	0.05	0.68	0.68	0.68	0.68
CI	-0.30	4.38	4.39	4.40	4.41	-0.17	4.05	4.06	4.14	4.14
II	4.80	3.98	6.24	4.05	6.28	4.85	7.30	8.76	7.51	8.94

bias appears in the II scenario. The performance observed in the CC and CI scenarios is well-aligned with the theoretical findings outlined in Theorem 4.1. In addition, Figure 2 indicates that all methods exhibit good robustness under low to moderate noise levels, though their performance deteriorates considerably when the noise level becomes extremely high ($\sigma = 100$).

Regarding the results in Tables 1, 2, and Figure 2, we would like to include further discussion on the performance of the proposed estimators under propensity score misspecification. We observe that when the propensity score is misspecified but the outcome regression model is correctly specified (IC scenario), the improved DR estimator may demonstrate lower efficiency compared to the usual DR estimator, particularly when only covariates X are used. However, with the inclusion of the additional predictor W, the proposed estimators perform comparably in the IC scenario, even without a theoretical guarantee. Furthermore, in the simulation

Table 2

Simulation results for different scenarios. Results with superscript "1" indicate variance estimates derived from the analytical asymptotic variance formula, while results with superscript "2" indicate those based on the bootstrap method.

	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$
$\sigma = 20$	$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{ls}})$					$X, \sigma = 20, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$				
CC	-0.03	4.21	4.21	4.05	4.05	-0.04	4.18	4.18	4.06	4.06
IC	-0.02	4.18	4.18	4.21	4.21	0.00	4.17	4.17	4.35	4.35
CI	-0.39	5.02	5.05	4.75	4.77	-0.32	4.97	5.00	4.73	4.74
II	4.75	4.80	6.68	4.89	6.82	3.89	5.76	6.95	5.91	7.08
$\sigma = 20$	$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{ls}})$					$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{opt}})$				
CC	0.03	1.38	1.38	1.31	1.31	0.04	1.37	1.37	1.34	1.34
IC	0.21	1.37	1.39	1.27	1.29	0.21	1.37	1.38	1.37	1.37
CI	-0.56	4.53	4.57	4.40	4.43	-0.36	4.23	4.25	4.14	4.15
II	4.68	4.15	6.26	4.39	6.42	4.71	6.92	8.37	7.16	8.57
	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$	Bias	SE^1	$RMSE^1$	SE^2	$RMSE^2$
$\sigma = 100$	$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{ls}})$					$X, \tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$				
CC	0.61	7.93	7.96	7.92	7.95	0.55	7.89	7.91	7.91	7.92
IC	0.02	7.90	7.90	7.74	7.74	0.04	7.89	7.90	7.83	7.83
CI	-0.54	8.38	8.40	8.34	8.36	-0.50	8.32	8.34	8.34	8.35
II	4.66	8.23	9.46	8.72	9.89	4.30	9.01	9.99	8.83	9.83
$\sigma = 100$	$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{ls}})$					$(X, W), \tau_W(\hat{\alpha}, \hat{\delta}^{\text{opt}})$				
CC	0.45	6.87	6.89	6.93	6.94	0.41	6.80	6.81	6.94	6.95
IC	0.08	6.83	6.83	7.03	7.03	0.02	6.84	6.85	7.50	7.50
CI	-0.52	8.10	8.12	8.12	8.13	-0.53	7.88	7.90	8.12	8.13
II	4.70	7.88	9.18	8.24	9.48	6.08	11.45	12.97	11.37	12.89

study, under the scenario where both models are misspecified (`II` scenario), the improved DR estimator with W may exhibit greater bias. This scenario suggests that in practical applications, it is important to leverage prior knowledge to introduce more appropriate working models and mitigate the risks associated with both working models being misspecified.

6. Real data analysis

6.1. Application to randomized trials

In this subsection, we first demonstrate our proposed method using the dataset from the AIDS Clinical Trials Group Protocol 175 (ACTG 175) study [17]. The ACTG 175 trial was conducted to evaluate the effectiveness of various antiretroviral therapy regimens for adult HIV-1-infected patients. It focused on those with CD4 T cell counts ranging from 200 to 500 cells/mm^3 and collected several variables to describe right-censored time-to-event observations. We conduct analysis using the `ACTG175` dataset provided in the R package `BART`.

We define patients who received didanosine treatment as $Z = 1$, and those who did not receive didanosine treatment as $Z = 0$. Our primary objective is to evaluate the effectiveness of didanosine in improving CD4 counts and extending survival time. Therefore, we consider two outcome variables of interest: Y_1, the mean CD4 count (cells/mm^3) measured at 20 ± 5 weeks; and Y_2, the time (in days) until the first occurrence of one of the following adverse events: (i) a decline in CD4 T cell count of at least 50 units; (ii) progression to AIDS; or (iii)

Table 3

Estimated treatment effects, asymptotic variances, and 95% confidence intervals for the ACTG175 data. The first part presents results for Y_1, while the second part focuses on Y_2.

	Difference-in-means estimators	Improved DR estimators	
	$\hat{\tau}^{\text{dif}}$	$\tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$	$\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$
Point estimate	34.28	34.28	33.23
Variance	38.59	38.98	**36.30**
95% CI	(22.10, 46.45)	(22.04, 46.51)	(21.42, 45.04)
	Difference-in-means estimators	Improved DR estimators	
	$\hat{\tau}^{\text{dif}}$	$\tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$	$\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$
Point estimate	51.35	51.35	52.42
Variance	158.61	160.12	**158.47**
95% CI	(26.67, 76.04)	(26.55, 76.15)	(27.74, 77.09)

death. Generally, higher CD4 counts reflect a stronger immune system, and a longer time until the occurrence of adverse events is clinically desirable. Since these variables were accurately recorded for all participants, our analysis does not involve missing values or censoring issues.

Note that this is a randomized experiment, and we denote X as the empty set. Additionally, the dataset includes several covariates, which we collectively denote as W. These covariates represent important baseline characteristics of the participants, such as demographic information (age, race, gender), clinical measurements (wtkg, hemo, homo), health status indicators (karnof, oprior, z30, preanti), and behavioral factors (drugs, str2). In total, we use 12 covariates in our analysis. To analyze the data, we use logistic regression models (8) to estimate the propensity scores $e_X(X;\alpha)$ and $e_W(X,W;\alpha,\gamma)$, and linear models (12) to estimate the outcomes $Q_X(X;\beta)$ and $Q_W(X,W;\delta)$. Algorithm 1 is used to obtain the improved DR estimators. Additionally, the difference-in-means estimator, denoted as $\hat{\tau}^{\text{dif}}$, is computed as the sample mean difference between treatment and control groups, serving as a reference under the randomized trial. Since the propensity score is always correctly specified under randomization, the proposed theoretical results in Section 4 are expected to hold.

Table 3 summarizes the treatment effect estimates, variance estimates, and corresponding 95% confidence intervals (CIs) for the ACTG175 dataset, where the asymptotic variance and CIs are computed using Theorem 3.2 under the correctly specified propensity score. The results are divided into two parts: the first corresponds to the outcome Y_1, and the second to Y_2. For CD4 count Y_1, the difference-in-means estimator $\hat{\tau}^{\text{dif}}$ yields a point estimate of 34.28, with a variance of 38.59 and a 95% CI of (22.10, 46.45). The improved DR estimator $\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$ provides a point estimate of 33.23 and a 95% CI of (21.42, 45.04). Notably, the improved estimator incorporating the covariate W achieves the smallest variance (36.30). For adverse events Y_2, the difference-in-means estimator produces a point estimate of 51.35, with a variance of 158.61 and a 95% CI of (26.67, 76.04). The improved DR estimator $\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$, incorporating W, yields a point estimate of 52.42, with the smallest variance of 158.47, resulting in narrower 95% CIs of (27.74, 77.09). These findings confirm the validity of Theorem 4.1, demonstrating that incorporating additional predictive covariates indeed improves efficiency. Overall, the results suggest that didanosine treatment helps improve CD4 levels, extend survival, or delay disease progression.

Table 4
Treatment effect estimates, asymptotic variance, and 95% confidence intervals for the RHC data.

	$\tau_X(\hat{\alpha}, \hat{\beta}^{\text{ls}})$	$\tau_W(\hat{\eta}, \hat{\delta}^{\text{ls}})$	$\tau_X(\hat{\alpha}, \hat{\beta}^{\text{opt}})$	$\tau_W(\hat{\eta}, \hat{\delta}^{\text{opt}})$
Point estimate	3.26	3.26	3.63	3.80
Variance	0.74	0.74	0.69	**0.66**
95% CI	(1.58, 4.94)	(1.59, 4.94)	(2.00, 5.26)	(2.19, 5.42)

6.2. Application to observational studies

In this subsection, we apply the proposed improved DR estimators to a dataset from the Study to Understand Prognoses and Preferences for Outcomes and Risks of Treatments (SUPPORT, Connors et al. [7]). SUPPORT examined the effectiveness and safety of the direct measurement of cardiac function by Right Heart Catheterization (RHC) for certain critically ill patients in intensive care units (ICUs). This dataset has been previously analyzed to estimate the average treatment effect of using RHC [24, 39, 9], and can be downloaded from `https://search.r-project.org/CRAN/refmans/ATbounds/html/RHC.html`. The dataset includes information on 5735 patients, with 2184 measured by RHC in the first 24 hours ($Z = 1$) and 3551 in the control group ($Z = 0$). Among these, 3817 patients survived, while 1918 patients passed away within the 30-day period. The outcome Y is the number of days between admission and death or censoring at day 30. Our objective is to evaluate whether the usage of RHC contributes to the survival time of critically ill patients from the day they are admitted or transferred to the ICU.

We collected a total of 51 covariates, covering diagnoses, comorbidities, vital signs, physiological indicators, as well as demographic details such as gender, race, age, education, and income. Due to relatively high missing rates, three covariates, namely `cat2` (secondary disease category, with a missing rate of 79%, denoted as $\tilde{W}_1$), `adld3p` (activities of daily living, with a missing rate of 75%, denoted as $\tilde{W}_2$), and `urin1` (urine output, with a missing rate of 53%, denoted as $\tilde{W}_3$), are typically excluded in previous literature [18]. Consequently, it is widely acknowledged that these covariates do not qualify as genuine confounders, although they may still have predictive power for the outcome. Following Assumption 2, we can utilize their observed values and impute the missing ones for analysis purposes. For binary variables $\tilde{W}_1$ and $\tilde{W}_2$, we use the Bernoulli distribution for imputation, relying on the observed sample mean. As for the continuous variable $\tilde{W}_3$, we apply the normal distribution for imputation, considering both the observed sample mean and variance. We denote the three imputed variables as W_1, W_2, and W_3. Additionally, the covariate set includes temperature variables, which are not expected to influence the choice of patient treatment directly but may impact the duration of a patient's hospital stay. Therefore, we propose incorporating the temperature variable `temp1` as W_4. The additional predictive covariates in this study are denoted as $W = (W_1, W_2, W_3, W_4)$. We employed a Logistic model to fit the propensity score model $e_X(X; \alpha)$ and empirically verified that W does not directly appear in the model. Furthermore, we explored alternative nonparametric methods such as classification and regression trees [3], Bayesian additive regression trees [6], and Causal Tree [1]. Importantly, almost all of these methods empirically confirmed that the covariate W does not directly affect treatment assignment. We denote all remaining covariates as true confounders X.

We use the model parameterizations in (8) for $e_X(X;\alpha)$ and $e_W(X,W;\alpha,\beta)$. Additionally, we use the model parameterizations in (12) for $Q_X(X,Z;\beta)$ and $Q_W(X,W,Z;\delta)$. We consider the estimators, $\tau_X(\hat{\alpha},\hat{\beta}^{\text{ls}})$, $\tau_W(\hat{\eta},\hat{\delta}^{\text{ls}})$, $\tau_X(\hat{\alpha},\hat{\beta}^{\text{opt}})$, and $\tau_W(\hat{\eta},\hat{\delta}^{\text{opt}})$ for data analysis, where the parameters $\hat{\beta}^{\text{opt}}$ and $\hat{\delta}^{\text{opt}}$ are obtained by solving (5) and (11), and $\hat{\beta}^{\text{ls}}$ and $\hat{\delta}^{\text{ls}}$ are obtained by the OLS. Table 4 shows the point estimates, variance estimates, and corresponding 95% CIs, where the variances of our proposed DR estimators are computed using Theorem 3.2, while those of the usual DR estimators are obtained from their respective estimating equations. In fact, all variance estimates are calculated using the sandwich variance estimator derived from the corresponding estimating equations. The point estimates of the improved DR estimators $\tau_X(\hat{\alpha},\hat{\beta}^{\text{opt}})$ and $\tau_W(\hat{\eta},\hat{\delta}^{\text{opt}})$ are slightly higher than $\tau_X(\hat{\alpha},\hat{\beta}^{\text{ls}})$ and $\tau_W(\hat{\eta},\hat{\delta}^{\text{ls}})$, but all results are still very close. Furthermore, all 95% confidence intervals indicate that our estimates are significant. These results are consistent with the findings in previous studies [21]. The consistency of these close estimates suggests that RHC treatment can extend the survival of critically ill patients by at least 3 days after admission to intensive care. We also observe that the use of the improved DR estimator $\tau_X(\hat{\alpha},\hat{\beta}^{\text{opt}})$ results in a lower estimated variance compared to the usual DR estimator $\tau_X(\hat{\alpha},\hat{\beta}^{\text{ls}})$. Besides, by integrating additional predictive covariates W, we find that $\tau_W(\hat{\eta},\hat{\delta}^{\text{opt}})$ further reduces the estimated variance.

Finally, we would like to emphasize that, for real data analysis, there is no need to partition the covariates X without missing values to define W. In fact, we strongly recommend incorporating additional covariates that may be missing in practice, as they are often not considered as genuine confounders, such as $(\tilde{W}_1,\tilde{W}_2,\tilde{W}_3)$ in this application. After appropriate imputation, we recommend including them in the propensity score for empirical validation, ensuring that these covariates do not directly impact the treatment variable. Subsequently, these variables can be employed in the outcome model, providing a highly attractive method for further enhancing efficiency in practical applications.

7. Discussion

In this paper, we introduce an enhanced DR estimator that directly minimizes the asymptotic variance within a defined class of working models. Our estimator is doubly robust and is designed to be more efficient than usual DR estimators when the propensity score model is correctly specified. For the usual DR estimator, whether similar variance orders exist when the propensity scores are correctly specified is uncertain. This uncertainty arises because the usual DR estimator typically relies on OLS regression and does not incorporate any propensity score information. Our method parallels covariate adjustment strategies in randomized trials, where incorporating additional predictive covariates improves estimation efficiency [12, 23, 52].

There are several potential directions for future research. First, while this paper demonstrates that incorporating predictive covariates with missing values can enhance estimation efficiency, exploring various imputation methods is also worth considering [52]. Second, this paper primarily focuses on covariates with predictive power but no direct effect on the treatment. In practice, exploring covariates that affect the treatment but have no direct effect on the outcome is also of significant interest [8]. Third, Figure 1 exhibits similarities to the proximal framework when X serves as the unobserved confounding [29, 37, 9, 31]. Hence, it is worth exploring whether estimation efficiency can be enhanced by incorporating an additional

outcome-inducing confounding. Fourth, incorporating machine learning techniques could significantly enhance the practical relevance and applicability of the proposed method. While the current framework performs well under parametric or semiparametric models, extending it to nonparametric or machine learning settings introduces additional challenges and represents an important direction for future research. Finally, a promising direction is to explore bias-reduced DR estimators for comparative analysis. This approach entails estimating nuisance parameters in a manner that minimizes bias in the DR estimator when working models are misspecified [46]. The study of these issues is beyond the scope of this paper, and we leave them as potential future research topics.

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Supplementary Material

The Supplementary Material provides discussions on known propensity score scenarios, theoretical derivations, and the symmetric case with a correctly specified outcome model.

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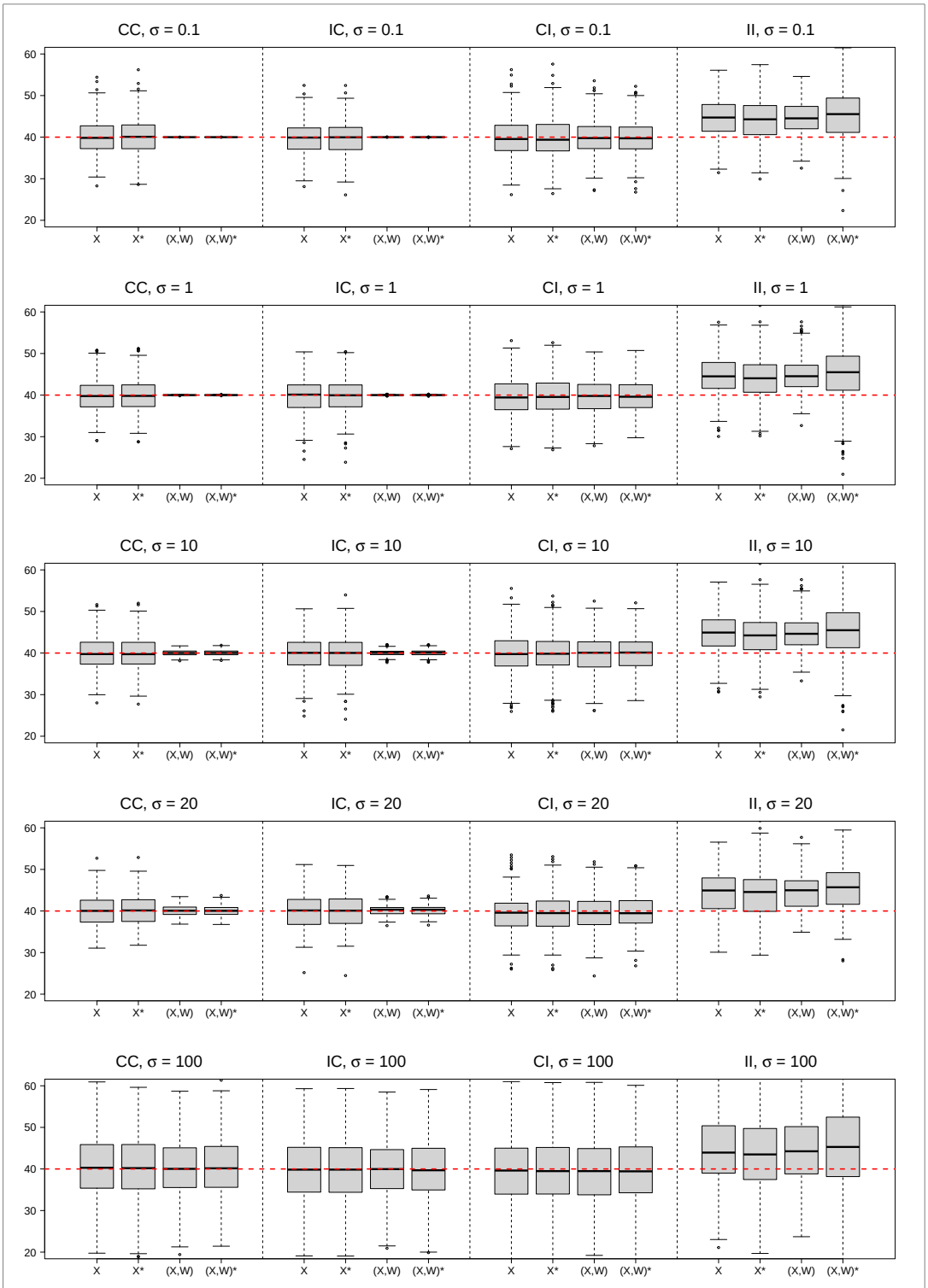

Fig 2. *Boxplot results for different scenarios. Symbols* X *and* X^* *denote the usual and improved DR estimators based on adjustment set* X*, respectively. Symbols* (X,W) *and* $(X,W)^*$ *are defined similarly for adjustment set* (X,W)*.*