# The complete census of orientable cusped hyperbolic 3-manifolds, up to 10 tetrahedra

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1 Introduction

- 2 Extending the census to 10 tetrahedra
- 3 Applications



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Knots		
31		
4 <sub>1</sub>		
51	52	
61	62	63
	3 <sub>1</sub> 4 <sub>1</sub> 5 <sub>1</sub>	3 <sub>1</sub> 4 <sub>1</sub> 5 <sub>1</sub> 5 <sub>2</sub>

Table: The census of knots up to 6 crossings

Crossings	Knots		
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Table: The census of knots up to 6 crossings

For orientable cusped hyperbolic manifolds: replace crossings with tetrahedra.

Tetrahedra	Orientable cusped hyperbolic manifolds				
2	m003	m004			
2	m006	m007	m009	m010	m011
3	m015	m016	m017	m019	

Table: The census of orientable cusped hyperbolic manifolds up to 3 tetrahedra

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### Burton, 2014

There are precisely 75,956 cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

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- 3 Deduplicate the eligible candidates



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Total	8,373,308	7,468,856	904,452

Deduplication: Group by canonical triangulations computed with SnapPea/SnapPy



Burton's method of seperation



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With verified computation, the canonical triangulation produced is a complete invariant of cusped hyperbolic 3-manifolds.



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#### L. 2025

There are precisely 150,730 orientable cusped hyperbolic 3-manifolds triangulable by a minimal of 10 tetrahedra.

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# Census of exceptional Dehn fillings

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# Corollary

There are only finitely many exceptional Dehn fillings on a 1-cusped hyperbolic 3-manifold.



#### Dunfield, 2019

There are precisely 205,822 exceptional Dehn fillings on 1-cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

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There are precisely 439,898 exceptional Dehn fillings on 1-cusped hyperbolic 3-manifolds triangulable by a minimal of 10 tetrahedra.

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(In progress) Classification of each filling & Verify conjectures





# Conjecture (Dunfield-Friedl-Jackson, 2024)

For any cusped hyperbolic 3-manifold M with  $b_1(M) = 1$ ,

$$x(M) = \frac{1}{2} \deg \tau_{2,\rho}(M)$$

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- The above equality holds for all the computed ones



Homological spheres



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200

The 10-tetrahedra census is available for installation on GitHub.



Figure: QR code to the GitHub repository

# Alternatively:

 $shana-y\text{-li.github.io} \rightarrow Research \rightarrow Projects \rightarrow snappy\_10\_tets$