The complete census of orientable cusped hyperbolic 3-manifolds, up to 10 tetrahedra

Shana Y. Li

University of Illinois, Urbana-Champaign

October 2025



Table of Contents

1 Introduction

- 2 Extending the census to 10 tetrahedra
- 3 Applications



Table of Contents

1 Introduction

- 2 Extending the census to 10 tetrahedra
- 3 Applications



Crossings	Knots		
3	31		
4	4 ₁		
5	51	52	
6	61	62	63

Table: The census of knots up to 6 crossings

Crossings	Knots		
3	31		
4	41		
5	51	52	
6	61	62	63

Table: The census of knots up to 6 crossings

The left column:

minimal number of crossings in planar diagrams of knots

Crossings	Knots		
3	31		
4	4 ₁		
5	51	52	
6	61	62	63

Table: The census of knots up to 6 crossings

The left column:

minimal number of crossings in planar diagrams of knots



minimal number of tetrahedra in triangulations of manifolds



Tetrahedra	Orientable cusped hyperbolic 3-manifolds				
2	m003	m004			
2	m006	m007	m009	m010	m011
3	m015	m016	m017	m019	

Table: The census of orientable cusped hyperbolic manifolds up to 3 tetrahedra

The left column:

minimal number of crossings in planar diagrams of knots



minimal number of tetrahedra in triangulations of manifolds

Tetrahedra	Name convention	Year	Contributer(s)
2 - 5	$m003 \sim m412$	1989	Hildebrand & Weeks

Tetrahedra	Name convention	Year	Contributer(s)
2 - 5	m003 \sim m412	1989	Hildebrand & Weeks
6	$s000\sims961$	1999	Callahan,
7	v0000 \sim v3551	1999	Hildebrand & Weeks

Tetrahedra	Name convention	Year	Contributer(s)
2 - 5	m003 \sim m412	1989	Hildebrand & Weeks
6	$s000\sims961$	1999	Callahan,
7	$v0000\sim v3551$	1999	Hildebrand & Weeks
8	t00000 ~ t12845	2010	Thistlethwaite

Tetrahedra	Name convention	Year	Contributer(s)
2 - 5	m003 \sim m412	1989	Hildebrand & Weeks
6	$s000\sims961$	1999	Callahan,
7	v0000 \sim v3551	1999	Hildebrand & Weeks
8	t00000 \sim t12845	2010	Thistlethwaite
9	o9_00000 ~ o9_44249	2014	Burton

Tetrahedra	Name convention	Year	Contributer(s)
2 - 5	m003 \sim m412	1989	Hildebrand & Weeks
6	$s000\sims961$	1999	Callahan,
7	v0000 \sim v3551	1999	Hildebrand & Weeks
8	t00000 ~ t12845	2010	Thistlethwaite
9	o9_00000 ~ o9_44249	2014	Burton
10	o $10_{-}000000 \sim o10_{-}150729$	2025	L.

Table: Timeline of census of orientable cusped hyperbolic 3-manifolds

Tetrahedra	Name convention	Year	Contributer(s)
2 - 5	m003 \sim m412	1989	Hildebrand & Weeks
6	$s000\sims961$	1999	Callahan,
7	v0000 \sim v3551	1999	Hildebrand & Weeks
8	t00000 ~ t12845	2010	Thistlethwaite
9	o9_00000 ~ o9_44249	2014	Burton
10	o10 $_$ 0000000 \sim o10 $_$ 150729	2025	L.

The completeness of the census was not confirmed until Burton's work in 2014.

Tetrahedra	Name convention	Year	Contributer(s)
2 - 5	m003 \sim m412	1989	Hildebrand & Weeks
6	$s000\sims961$	1999	Callahan,
7	v0000 \sim v3551	1999	Hildebrand & Weeks
8	t00000 \sim t12845	2010	Thistlethwaite
9	o9_00000 ~ o9_44249	2014	Burton
10	o10_000000 \sim o10_150729	2025	L.

The completeness of the census was not confirmed until Burton's work in 2014.

Burton, 2014

There are precisely 75,956 cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

) Q (P

Table of Contents

1 Introduction

- 2 Extending the census to 10 tetrahedra
- 3 Applications





- 3-step outline of creating a complete census:
 - Generate all candidates



- 3-step outline of creating a complete census:
 - Generate all candidates
 - for hyperbolic 3-manifolds: triangulations of 3-manifolds

- Generate all candidates
 - for hyperbolic 3-manifolds: triangulations of 3-manifolds
- Decide the eligibility of each candidate



- Generate all candidates
 - for hyperbolic 3-manifolds: triangulations of 3-manifolds
- Decide the eligibility of each candidate
 - for hyperbolic 3-manifolds: minimality and hyperbolicity of the triangulations

- Generate all candidates
 - for hyperbolic 3-manifolds: triangulations of 3-manifolds
- Decide the eligibility of each candidate
 - for hyperbolic 3-manifolds: minimality and hyperbolicity of the triangulations
- 3 Deduplicate the eligible candidates



Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0

Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0

Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0
Certify hyperbolicity $(r=1)$	1,072,874	0	873,908

Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0
Certify hyperbolicity $(r=1)$	1,072,874	0	873,908
Exhaustive nonminimality $(h = 2)$	698,650	374,224	0

Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0
Certify hyperbolicity $(r=1)$	1,072,874	0	873,908
Exhaustive nonminimality $(h=2)$	698,650	374,224	0
Special planes (non-hyperbolicity)	33,807	664,843	0

Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0
Certify hyperbolicity $(r=1)$	1,072,874	0	873,908
Exhaustive nonminimality $(h = 2)$	698,650	374,224	0
Special planes (non-hyperbolicity)	33,807	664,843	0
Certify hyperbolicity $(r = 60)$	3269	0	30,538

Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0
Certify hyperbolicity $(r=1)$	1,072,874	0	873,908
Exhaustive nonminimality $(h = 2)$	698,650	374,224	0
Special planes (non-hyperbolicity)	33,807	664,843	0
Certify hyperbolicity $(r = 60)$	3269	0	30,538
Exhaustive nonminimality $(h = 5 \text{ and } 6)$	6	3263	0

Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0
Certify hyperbolicity $(r=1)$	1,072,874	0	873,908
Exhaustive nonminimality $(h = 2)$	698,650	374,224	0
Special planes (non-hyperbolicity)	33,807	664,843	0
Certify hyperbolicity $(r = 60)$	3269	0	30,538
Exhaustive nonminimality $(h = 5 \text{ and } 6)$	6	3263	0
Certify hyperbolicity $(r=1000)$	0	0	6



Step	Candidates	Discarded	Eligible
Candidate generation	8,373,308	0	0
Greedy nonminimality	1,946,782	6,426,526	0
Certify hyperbolicity $(r=1)$	1,072,874	0	873,908
Exhaustive nonminimality $(h = 2)$	698,650	374,224	0
Special planes (non-hyperbolicity)	33,807	664,843	0
Certify hyperbolicity $(r = 60)$	3269	0	30,538
Exhaustive nonminimality $(h = 5 \text{ and } 6)$	6	3263	0
Certify hyperbolicity $(r=1000)$	0	0	6
Total	8,373,308	7,468,856	904,452

Deduplication: Group by canonical triangulations computed with SnapPea/SnapPy



Burton's method of seperation



Burton's method of seperation

By algebraic invariants



Burton's method of seperation

- By algebraic invariants
 - {orientability, $H_1 = \pi_1^{\mathsf{ab}}, S_2^{\mathsf{ab}}(\pi_1), \dots, S_{11}^{\mathsf{ab}}(\pi_1)$ }

Burton's method of seperation

- By algebraic invariants
 - {orientability, $H_1 = \pi_1^{\mathsf{ab}}, S_2^{\mathsf{ab}}(\pi_1), \dots, S_{11}^{\mathsf{ab}}(\pi_1)$ }

Burton's method of seperation

- By algebraic invariants
 - {orientability, $H_1 = \pi_1^{\mathsf{ab}}, S_2^{\mathsf{ab}}(\pi_1), \dots, S_{11}^{\mathsf{ab}}(\pi_1)$ }

 $S_i^{ab}(G)$: the multi-set of abelianisations of index i subgroups of G, one subgroup for each conjugacy class.

Burton's method of seperation

- By algebraic invariants
 - {orientability, $H_1 = \pi_1^{\mathsf{ab}}, S_2^{\mathsf{ab}}(\pi_1), \dots, S_{11}^{\mathsf{ab}}(\pi_1)$ }

 $S_i^{ab}(G)$: the multi-set of abelianisations of index *i* subgroups of *G*, one subgroup for each conjugacy class.

Our method of seperation

Burton's method of seperation

- By algebraic invariants
 - {orientability, $H_1 = \pi_1^{\mathsf{ab}}, S_2^{\mathsf{ab}}(\pi_1), \dots, S_{11}^{\mathsf{ab}}(\pi_1)$ }

 $S_i^{ab}(G)$: the multi-set of abelianisations of index *i* subgroups of *G*, one subgroup for each conjugacy class.

Our method of seperation

■ By verified computations (implemented in SnapPy by M. Goerner)

Burton's method of seperation

- By algebraic invariants
 - {orientability, $H_1 = \pi_1^{\mathsf{ab}}, S_2^{\mathsf{ab}}(\pi_1), \dots, S_{11}^{\mathsf{ab}}(\pi_1)$ }

 $S_i^{ab}(G)$: the multi-set of abelianisations of index *i* subgroups of *G*, one subgroup for each conjugacy class.

Our method of seperation

■ By verified computations (implemented in SnapPy by M. Goerner)

Burton's method of seperation

- By algebraic invariants
 - {orientability, $H_1 = \pi_1^{\mathsf{ab}}, S_2^{\mathsf{ab}}(\pi_1), \dots, S_{11}^{\mathsf{ab}}(\pi_1)$ }

 $S_i^{ab}(G)$: the multi-set of abelianisations of index *i* subgroups of *G*, one subgroup for each conjugacy class.

Our method of seperation

■ By verified computations (implemented in SnapPy by M. Goerner)

With verified computation, the canonical triangulation produced is a complete invariant of cusped hyperbolic 3-manifolds.



■ Need to do algebra on the number field of shapes.



Need to do algebra on the number field of shapes.

BoGwang Jeon, 2014

$$\deg\{ {\sf shape field} \} \propto rac{1}{\min\{ {\sf length of geodesics} \}}$$

■ Need to do algebra on the number field of shapes.

BoGwang Jeon, 2014

$$\deg\{ {\sf shape field} \} \propto \frac{1}{\min\{ {\sf length of geodesics} \}}$$

The verified computation of 16/608,918 of orientable eligible triangulations ended up OOM (with 60GB RAM).

■ Need to do algebra on the number field of shapes.

BoGwang Jeon, 2014

$$\deg\{\text{shape field}\} \propto \frac{1}{\min\{\text{length of geodesics}\}}$$

The verified computation of 16/608,918 of orientable eligible triangulations ended up OOM (with 60GB RAM).

■ Using (volume[:60], H_1), the 16 triangulations were separated from other triangulations (≤ 10 tet.) and into 6 groups;

■ Need to do algebra on the number field of shapes.

BoGwang Jeon, 2014

$$\deg\{ {\sf shape field} \} \propto \frac{1}{\min\{ {\sf length of geodesics} \}}$$

The verified computation of 16/608,918 of orientable eligible triangulations ended up OOM (with 60GB RAM).

- Using (volume[:60], H_1), the 16 triangulations were separated from other triangulations (≤ 10 tet.) and into 6 groups;
- Within each group: all reprsent the same manifold via unverified canonical triangulations.

■ Need to do algebra on the number field of shapes.

BoGwang Jeon, 2014

$$\deg\{ {\sf shape field} \} \propto \frac{1}{\min\{ {\sf length of geodesics} \}}$$

The verified computation of 16/608,918 of orientable eligible triangulations ended up OOM (with 60GB RAM).

- Using (volume[:60], H_1), the 16 triangulations were separated from other triangulations (≤ 10 tet.) and into 6 groups;
- Within each group: all reprsent the same manifold via unverified canonical triangulations.

Need to do algebra on the number field of shapes.

BoGwang Jeon, 2014

$$\deg\{ {\sf shape field} \} \propto \frac{1}{\min\{ {\sf length of geodesics} \}}$$

The verified computation of 16/608,918 of orientable eligible triangulations ended up OOM (with 60GB RAM).

- Using (volume[:60], H_1), the 16 triangulations were separated from other triangulations (≤ 10 tet.) and into 6 groups;
- Within each group: all reprsent the same manifold via unverified canonical triangulations.

L. 2025

There are precisely 150,730 orientable cusped hyperbolic 3-manifolds triangulable by a minimal of 10 tetrahedra.

Table of Contents

1 Introduction

- 2 Extending the census to 10 tetrahedra
- 3 Applications



Census of exceptional Dehn fillings

Exceptional Dehn filling

A Dehn filling on a cusped hyperbolic 3-manifold is *exceptional* if the resulting manifold is not hyperbolic.

Census of exceptional Dehn fillings

Exceptional Dehn filling

A Dehn filling on a cusped hyperbolic 3-manifold is *exceptional* if the resulting manifold is not hyperbolic.

6-Theorem (Agol, 2000)

Let M be a 1-cusped hyperbolic 3-manifold. If a slope (p,q) has length larger than 6, then M(p,q) is hyperbolic.

Census of exceptional Dehn fillings

Exceptional Dehn filling

A Dehn filling on a cusped hyperbolic 3-manifold is *exceptional* if the resulting manifold is not hyperbolic.

6-Theorem (Agol, 2000)

Let M be a 1-cusped hyperbolic 3-manifold. If a slope (p,q) has length larger than 6, then M(p,q) is hyperbolic.

Corollary

There are only finitely many exceptional Dehn fillings on a 1-cusped hyperbolic 3-manifold.



Dunfield, 2019

There are precisely 205,822 exceptional Dehn fillings on 1-cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

Dunfield, 2019

There are precisely 205,822 exceptional Dehn fillings on 1-cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

L., 2025

Step	Candidates	Discarded	Confirmed
Candidate generation	800,447	0	0

Step	Candidates	Discarded	Confirmed
Candidate generation	800,447	0	0
Certify hyperbolicity	444,549	355,898	0

Step	Candidates	Discarded	Confirmed
Candidate generation	800,447	0	0
Certify hyperbolicity	444,549	355,898	0
Combinatorial recognition	12,280	4647	427,622

Step	Candidates	Discarded	Confirmed
Candidate generation	800,447	0	0
Certify hyperbolicity	444,549	355,898	0
Combinatorial recognition	12,280	4647	427,622
Essential S^2 or torus	4	0	12,276

Step	Candidates	Discarded	Confirmed
Candidate generation	800,447	0	0
Certify hyperbolicity	444,549	355,898	0
Combinatorial recognition	12,280	4647	427,622
Essential S^2 or torus	4	0	12,276
Identified to be m135(1,3)	0	4	0

Step	Candidates	Discarded	Confirmed
Candidate generation	800,447	0	0
Certify hyperbolicity	444,549	355,898	0
Combinatorial recognition	12,280	4647	427,622
Essential S^2 or torus	4	0	12,276
Identified to be m135(1,3)	0	4	0
Total	800,447	360,549	439,898

There are precisely 439,898 exceptional Dehn fillings on 1-cusped hyperbolic 3-manifolds triangulable by a minimal of 10 tetrahedra.

Step	Candidates	Discarded	Confirmed
Candidate generation	800,447	0	0
Certify hyperbolicity	444,549	355,898	0
Combinatorial recognition	12,280	4647	427,622
Essential S^2 or torus	4	0	12,276
Identified to be m135(1,3)	0	4	0
Total	800,447	360,549	439,898

(In progress) Classification of each filling & Verify conjectures





Conjecture (Dunfield-Friedl-Jackson, 2024)

For any cusped hyperbolic 3-manifold M with $b_1(M) = 1$,

$$x(M) = \frac{1}{2} \deg \tau_{2,\rho}(M)$$

Conjecture (Dunfield-Friedl-Jackson, 2024)

For any cusped hyperbolic 3-manifold M with $b_1(M) = 1$,

$$x(M) = \frac{1}{2} \deg \tau_{2,\rho}(M)$$

■ In the 10-tet census: 143,919 manifolds with $b_1(M) = 1$

Conjecture (Dunfield-Friedl-Jackson, 2024)

For any cusped hyperbolic 3-manifold M with $b_1(M) = 1$,

$$x(M) = \frac{1}{2} \operatorname{deg} \tau_{2,\rho}(M)$$

- In the 10-tet census: 143,919 manifolds with $b_1(M) = 1$
- Successfully computed x(M) for all but 21 of them

Conjecture (Dunfield-Friedl-Jackson, 2024)

For any cusped hyperbolic 3-manifold M with $b_1(M) = 1$,

$$x(M) = \frac{1}{2} \deg \tau_{2,\rho}(M)$$

- In the 10-tet census: 143,919 manifolds with $b_1(M) = 1$
- Successfully computed x(M) for all but 21 of them
- Equality holds for all the computed ones



Hyperbolic homology 3-spheres



Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census



Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729



Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

■ Rational: 1,899,926

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

■ Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

They can be used to test the *L*-space conjecture

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

■ Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

They can be used to test the *L*-space conjecture

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

■ Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

They can be used to test the *L*-space conjecture

The one cusped manifold with totally geodesic surfaces

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

They can be used to test the *L*-space conjecture

The one cusped manifold with totally geodesic surfaces

o 10_14425: kLwLLAQkccgghihijjjdegwtplldjo_aBBaaBBa

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

They can be used to test the *L*-space conjecture

The one cusped manifold with totally geodesic surfaces

o10_14425: kLwLLAQkccgghihijjjdegwtplldjo_aBBaaBBa

The only one in the orientable cusped census up to 10-tet.

200

Hyperbolic homology 3-spheres

By Dehn fillings on 1-cusped manifolds in the 10-tetrahedra census

■ Integral: 10,729

Rational: 1,899,926

■ The above include almost all hyperbolic homology 3-spheres with systole ≥ 0.163 obtainable in this way (did not deduplicate)

They can be used to test the *L*-space conjecture

The one cusped manifold with totally geodesic surfaces

o10_14425: kLwLLAQkccgghihijjjdegwtplldjo_aBBaaBBa

The only one in the orientable cusped census up to 10-tet. In particular, it is the smallest among all.



The 10-tetrahedra census is available for installation on GitHub.



Figure: QR code to the GitHub repository

Alternatively:

 $shana-y\text{-li.github.io} \rightarrow Research \rightarrow Projects \rightarrow snappy_10_tets$