Multivariable knot polynomials, the V_n-polynomials, and their patterns

Shana Y. Li

University of Illinois, Urbana-Champaign

April 2025

Joint work with Stavros Garoufalidis

Table of Contents

Table of Contents

Theorem (Kashaev, 2019)

Let R be a rigid R-matrix, then the corresponding Reshetikhin–Turaev functor gives an $\mathrm{End}(V)$-valued invariant of oriented knots.

Theorem (Kashaev, 2019)

Let R be a rigid R-matrix, then the corresponding Reshetikhin–Turaev functor gives an $\mathrm{End}(V)$-valued invariant of oriented knots.

- Rigid R-matrix: an element R in $\mathrm{Aut}(V \otimes V)$ satisfying:

Theorem (Kashaev, 2019)

Let R be a rigid R-matrix, then the corresponding Reshetikhin–Turaev functor gives an $\mathrm{End}(V)$-valued invariant of oriented knots.

- Rigid R-matrix: an element R in $\mathrm{Aut}(V \otimes V)$ satisfying:
  - Yang–Baxter equation: $(R \otimes I) \circ (I \otimes R) \circ (R \otimes I) = (I \otimes R) \circ (R \otimes I) \circ (I \otimes R)$.

Theorem (Kashaev, 2019)

Let R be a rigid R-matrix, then the corresponding Reshetikhin–Turaev functor gives an $\mathrm{End}(V)$-valued invariant of oriented knots.

- Rigid R-matrix: an element R in $\mathrm{Aut}(V \otimes V)$ satisfying:
  - Yang–Baxter equation: $(R \otimes I) \circ (I \otimes R) \circ (R \otimes I) = (I \otimes R) \circ (R \otimes I) \circ (I \otimes R)$.
  - Rigidity: the *partial transposes* $\widetilde{R^{\pm 1}} := (\varepsilon \otimes I \otimes I) \circ (I \otimes R^{\pm 1} \otimes I) \circ (I \otimes I \otimes \eta)$ are invertible.

    $\varepsilon \colon V \otimes V \to \mathbb{F}$ and $\eta \colon \mathbb{F} \to V \otimes V$: the evaluation and coevaluation maps.

Theorem (Kashaev, 2019)

Let R be a rigid R-matrix, then the corresponding Reshetikhin–Turaev functor gives an $\mathrm{End}(V)$-valued invariant of oriented knots.

- Rigid R-matrix: an element R in $\mathrm{Aut}(V \otimes V)$ satisfying:
  - Yang–Baxter equation: $(R \otimes I) \circ (I \otimes R) \circ (R \otimes I) = (I \otimes R) \circ (R \otimes I) \circ (I \otimes R)$.
  - Rigidity: the *partial transposes* $\widetilde{R^{\pm 1}} := (\varepsilon \otimes I \otimes I) \circ (I \otimes R^{\pm 1} \otimes I) \circ (I \otimes I \otimes \eta)$ are invertible.
    $\varepsilon\colon V \otimes V \to \mathbb{F}$ and $\eta\colon \mathbb{F} \to V \otimes V$: the evaluation and coevaluation maps.
- Reshetikhin–Turaev functor: a functor (determined by R) from the category of tangles to the category of vector spaces.

Theorem (Garoufalidis & Kashaev, 2023)

Given a braided Hopf algebra with automorphisms, one can construct a rigid R-matrix.

Theorem (Garoufalidis & Kashaev, 2023)

Given a braided Hopf algebra with automorphisms, one can construct a rigid R-matrix.

Hopf algebra

A *Hopf algebra over field* $\mathbb{F}$ is a unital algebra H equipped with coproduct $\Lambda\colon H \to H \otimes H$, counit $\varepsilon\colon H \to \mathbb{F}$ and invertible antipode $S\colon H \to H$.

Theorem (Garoufalidis & Kashaev, 2023)

Given a braided Hopf algebra with automorphisms, one can construct a rigid R-matrix.

Hopf algebra

A *Hopf algebra over field* $\mathbb{F}$ is a unital algebra H equipped with coproduct $\Lambda\colon H \to H \otimes H$, counit $\varepsilon\colon H \to \mathbb{F}$ and invertible antipode $S\colon H \to H$.

Braided Hopf algebra

A *braided Hopf algebra* is a Holf algebra with a braiding $\tau\colon H \otimes H \to H \otimes H$.

Theorem (Garoufalidis & Kashaev, 2023)

Given a braided Hopf algebra with automorphisms, one can construct a rigid R-matrix.

Hopf algebra

A *Hopf algebra over field* $\mathbb{F}$ is a unital algebra H equipped with coproduct $\Lambda\colon H \to H \otimes H$, counit $\varepsilon\colon H \to \mathbb{F}$ and invertible antipode $S\colon H \to H$.

Braided Hopf algebra

A *braided Hopf algebra* is a Holf algebra with a braiding $\tau\colon H \otimes H \to H \otimes H$.

The procedure:

$$\left\{\begin{matrix}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Braided}\\ \text{Yetter–Drinfel'd}\\ \text{modules with autos}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Rigid}\\ R\text{-matrices}\end{matrix}\right\}$$

Summary: a systematic source of knot invariants:

$$\left\{\begin{array}{l}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{array}\right\} \xrightarrow{\text{G \& K, 2023}} \left\{\begin{array}{l}\text{Rigid}\\ R\text{-matrices}\end{array}\right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

Summary: a systematic source of knot invariants:

$$\left\{\begin{matrix}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{matrix}\right\} \xrightarrow{\text{G \& K, 2023}} \left\{\begin{matrix}\text{Rigid}\\ R\text{-matrices}\end{matrix}\right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

One source of braided Hopf algebras: Nichols algebras.

Summary: a systematic source of knot invariants:

$$\left\{\begin{matrix}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{matrix}\right\} \xrightarrow{\text{G \& K, 2023}} \left\{\begin{matrix}\text{Rigid}\\ R\text{-matrices}\end{matrix}\right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

One source of braided Hopf algebras: Nichols algebras.

- Nichols algebras of rank 1: recovers the colored Jones polynomials and the ADO polynomials.

Summary: a systematic source of knot invariants:

$$\left\{\begin{matrix}\text{Braided} \\ \text{Hopf algebras} \\ \text{with autos}\end{matrix}\right\} \xrightarrow{\text{G \& K, 2023}} \left\{\begin{matrix}\text{Rigid} \\ R\text{-matrices}\end{matrix}\right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

One source of braided Hopf algebras: Nichols algebras.

- Nichols algebras of rank 1: recovers the colored Jones polynomials and the ADO polynomials.
- Nichols algebras of rank 2: recovers the Links–Gould polynomial, and gives the V_n-polynomials.

Summary: a systematic source of knot invariants:

$$\left\{\begin{array}{l}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{array}\right\} \xrightarrow{\text{G \& K, 2023}} \left\{\begin{array}{l}\text{Rigid}\\ R\text{-matrices}\end{array}\right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

One source of braided Hopf algebras: Nichols algebras.

- Nichols algebras of rank 1: recovers the colored Jones polynomials and the ADO polynomials.
- Nichols algebras of rank 2: recovers the Links–Gould polynomial, and gives the V_n-polynomials.
- Nichols algebras of rank 3: ...

Summary: a systematic source of knot invariants:

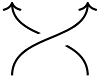
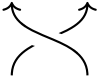
$$\left\{\begin{matrix}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{matrix}\right\} \xrightarrow{\text{G \& K, 2023}} \left\{\begin{matrix}\text{Rigid}\\ R\text{-matrices}\end{matrix}\right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

One source of braided Hopf algebras: Nichols algebras.

- Nichols algebras of rank 1: recovers the colored Jones polynomials and the ADO polynomials.
- Nichols algebras of rank 2: recovers the Links–Gould polynomial, and gives the V_n-polynomials.
- Nichols algebras of rank 3: ...
- ...

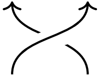
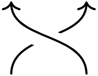
Table of Contents

Reshetikhin–Turaev functor: tangles $\mapsto$ vector spaces

 $\mapsto \quad R,$  $\mapsto \quad R^{-1},$

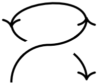
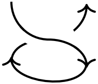
$\uparrow$ or $\downarrow \quad \mapsto \quad I,$  $\mapsto \quad \varepsilon,$  $\mapsto \quad \eta.$

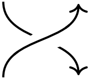
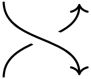
Reshetikhin–Turaev functor: tangles $\mapsto$ vector spaces

 $\mapsto \quad R,$  $\mapsto \quad R^{-1},$

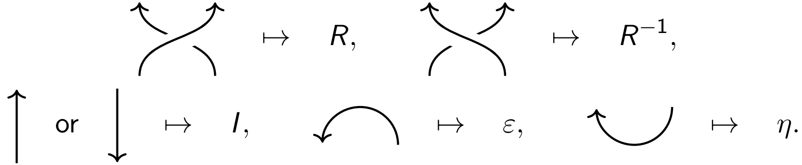
 or  $\mapsto \quad I,$  $\mapsto \quad \varepsilon,$  $\mapsto \quad \eta.$

For local extrema going from left to right: (normalization)

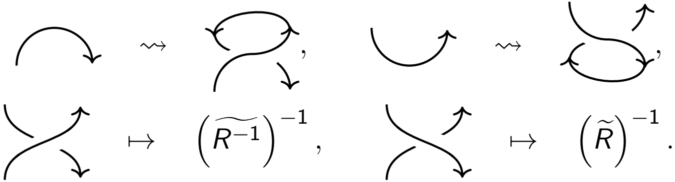
 $\rightsquigarrow$ ,  $\rightsquigarrow$ ,

 $\mapsto \quad \left(\widetilde{R^{-1}}\right)^{-1},$  $\mapsto \quad \left(\widetilde{R}\right)^{-1}.$

Reshetikhin–Turaev functor: tangles $\mapsto$ vector spaces



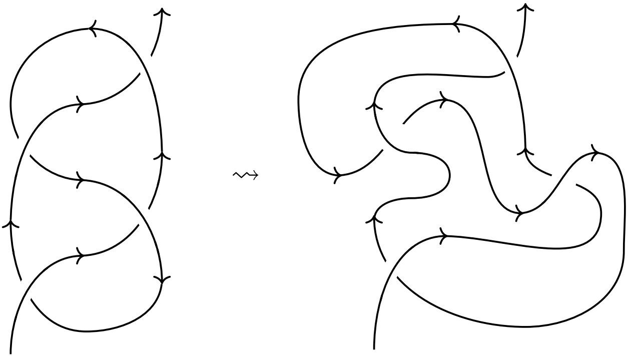
For local extrema going from left to right: (normalization)



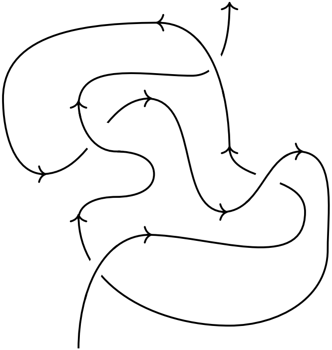
For V_n-polynomials, $\varepsilon \circ \left(\widetilde{R^{-1}}\right)^{-1} = \left(\widetilde{R^{-1}}\right)^{-1} \circ \eta = \varepsilon \circ \left(\widetilde{R}\right)^{-1} = \left(\widetilde{R}\right)^{-1} \circ \eta$ is a diagonalizable matrix with only ± 1's on the diagonal.

Reshetikhin–Turaev: tangles $\mapsto$ normalized tangles $\mapsto$ vector spaces

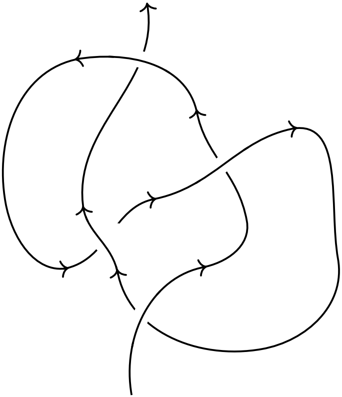
Example: the 4_1 knot



Reshetikhin–Turaev: tangles $\mapsto$ normalized tangles $\mapsto$ vector spaces
Example: the 4_1 knot

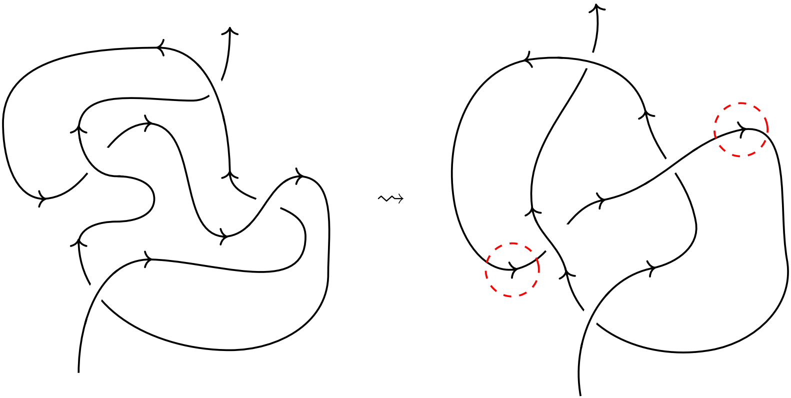


$\rightsquigarrow$

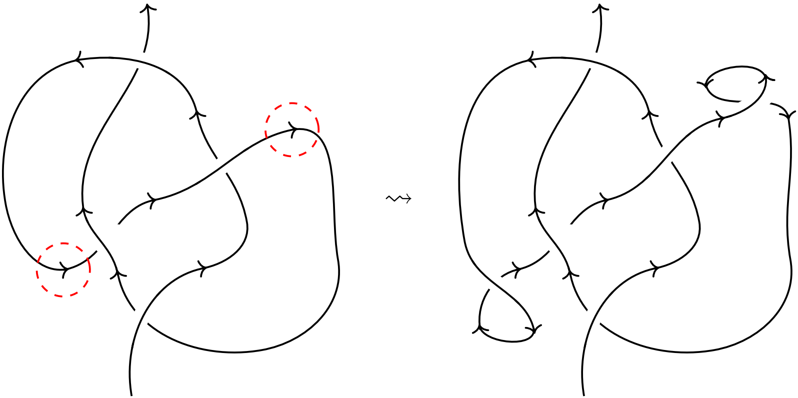


Reshetikhin–Turaev: tangles $\mapsto$ normalized tangles $\mapsto$ vector spaces

Example: the 4_1 knot

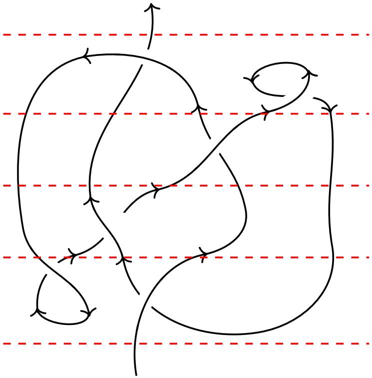


Reshetikhin–Turaev: tangles $\mapsto$ normalized tangles $\mapsto$ vector spaces
Example: the 4_1 knot



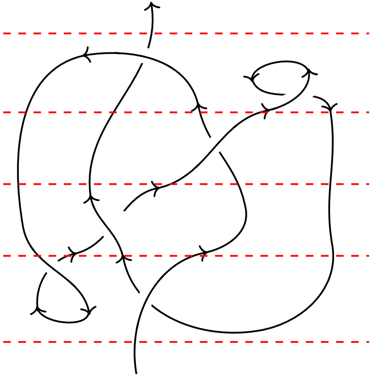
Reshetikhin–Turaev: tangles $\mapsto$ normalized tangles $\mapsto$ vector spaces

Example: the 4_1 knot



$$\mapsto \quad \begin{array}{lr}
V & \\
\uparrow & \left((\varepsilon \otimes I) \circ \left(I \otimes R^{-1}\right)\right) \otimes \left(\varepsilon \circ \left(\widetilde{R}\right)^{-1}\right) \\
V^{\otimes 5} & \\
\uparrow & I \otimes I \otimes R \otimes I \\
V^{\otimes 5} & \\
\uparrow & I \otimes R^{-1} \otimes I \otimes I \\
V^{\otimes 5} & \\
\uparrow & \left(\left(\widetilde{R^{-1}}\right)^{-1} \circ \eta\right) \otimes ((R \otimes I) \circ (I \otimes \eta)) \\
V &
\end{array}$$

Example: the 4_1 knot



$$\mapsto \quad \begin{array}{lr} V & \\ \uparrow & \left((\varepsilon\otimes I)\circ\left(I\otimes R^{-1}\right)\right)\otimes\left(\varepsilon\circ\left(\widetilde{R}\right)^{-1}\right) \\ V^{\otimes 5} & \\ \uparrow & I\otimes I\otimes R\otimes I \\ V^{\otimes 5} & \\ \uparrow & I\otimes R^{-1}\otimes I\otimes I \\ V^{\otimes 5} & \\ \uparrow & \left(\left(\widetilde{R^{-1}}\right)^{-1}\circ\eta\right)\otimes((R\otimes I)\circ(I\otimes\eta)) \\ V & \end{array}$$

Fact

For V_n-polynomials, the endomorphism on V we obtain is a scalar multiple of 1_V. The scalar gives our polynomial invariant.

Fix a basis $\mathcal{B} := \{e_i\}$ of V, $R^{\pm 1} \in \mathrm{Aut}(V \otimes V)$ become matrices whose entries can be denoted by $\left(R^{\pm 1}\right)^{e_k \otimes e_l}_{e_i \otimes e_j}$.

Fix a basis $\mathcal{B} := \{e_i\}$ of V, $R^{\pm 1} \in \mathrm{Aut}(V \otimes V)$ become matrices whose entries can be denoted by $\left(R^{\pm 1}\right)_{e_i \otimes e_j}^{e_k \otimes e_l}$.
To compute the eigenvalue of the $\mathrm{End}(V)$-valued invariant is to evaluate a sum of the form

$$\sum_{\substack{a_1, \cdots, a_{2c-1} \in \mathcal{B} \\ a_0 = a_{2c} = 1}} \pm \underbrace{\left(R^{\pm 1}\right)_{a_0 \otimes a_1}^{a_2 \otimes a_3} \cdots \left(R^{\pm 1}\right)_{a_{2c-3} \otimes a_{2c-2}}^{a_{2c-1} \otimes a_{2c}}}_{\text{a product of length } c},$$

where c is the number of crossings of the knot. This sum is the so called *state sum*.

Fix a basis $\mathcal{B} := \{e_i\}$ of V, $R^{\pm 1} \in \mathrm{Aut}(V \otimes V)$ become matrices whose entries can be denoted by $\left(R^{\pm 1}\right)^{e_k \otimes e_l}_{e_i \otimes e_j}$.
To compute the eigenvalue of the $\mathrm{End}(V)$-valued invariant is to evaluate a sum of the form

$$\sum_{\substack{a_1,\cdots,a_{2c-1}\in\mathcal{B} \\ a_0=a_{2c}=1}} \pm \underbrace{\left(R^{\pm 1}\right)^{a_2 \otimes a_3}_{a_0 \otimes a_1} \cdots \left(R^{\pm 1}\right)^{a_{2c-1} \otimes a_{2c}}_{a_{2c-3} \otimes a_{2c-2}}}_{\text{a product of length } c},$$

where c is the number of crossings of the knot. This sum is the so called *state sum*.
Therefore, it requires

$$c \cdot (\dim V)^{2c-1}$$

times of computations to compute the eigenvalue.

For V_n-polynomials, $\dim V = 4n$.

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

This is how you know that you wouldn't want to compute it by hand.

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

This is how you know that you wouldn't want to compute it by hand.
For $c = 12$,

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

This is how you know that you wouldn't want to compute it by hand.
For $c = 12$,

$$c \cdot (\dim V)^{2c-1} = 7{,}083{,}549{,}724{,}304{,}467{,}820{,}544,$$

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

This is how you know that you wouldn't want to compute it by hand.
For $c = 12$,

$$c \cdot (\dim V)^{2c-1} = 7{,}083{,}549{,}724{,}304{,}467{,}820{,}544,$$

and for $c = 15$,

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

This is how you know that you wouldn't want to compute it by hand.
For $c = 12$,

$$c \cdot (\dim V)^{2c-1} = 7{,}083{,}549{,}724{,}304{,}467{,}820{,}544,$$

and for $c = 15$,

$$c \cdot (\dim V)^{2c-1} = 2{,}321{,}137{,}573{,}660{,}088{,}015{,}435{,}857{,}920.$$

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

This is how you know that you wouldn't want to compute it by hand.
For $c = 12$,

$$c \cdot (\dim V)^{2c-1} = 7{,}083{,}549{,}724{,}304{,}467{,}820{,}544,$$

and for $c = 15$,

$$c \cdot (\dim V)^{2c-1} = 2{,}321{,}137{,}573{,}660{,}088{,}015{,}435{,}857{,}920.$$

Worse, the entries $\left(R^{\pm 1}\right)^{e_k \otimes e_l}_{e_i \otimes e_j}$ are polynomials in two variables, instead of scalars.

For V_n-polynomials, $\dim V = 4n$.
With $n = 2$, for the simplest knot 3_1, we have

$$c \cdot (\dim V)^{2c-1} = 98{,}304.$$

This is how you know that you wouldn't want to compute it by hand.
For $c = 12$,

$$c \cdot (\dim V)^{2c-1} = 7{,}083{,}549{,}724{,}304{,}467{,}820{,}544,$$

and for $c = 15$,

$$c \cdot (\dim V)^{2c-1} = 2{,}321{,}137{,}573{,}660{,}088{,}015{,}435{,}857{,}920.$$

Worse, the entries $(R^{\pm 1})_{e_i \otimes e_j}^{e_k \otimes e_l}$ are polynomials in two variables, instead of scalars.
We computed the V_2-polynomials for all knots with ≤ 15 crossings, and more.

To optimize the computation:

To optimize the computation:

- The R-matrix is sparse: a divide and conquer method sees the 0's at each step and eliminates a lot of terms.

n	Nonzero elements (%)	$\#R$
2	177 (4.3%)	4096
3	585 (2.8%)	20,736
4	1377 (2.1%)	65,536

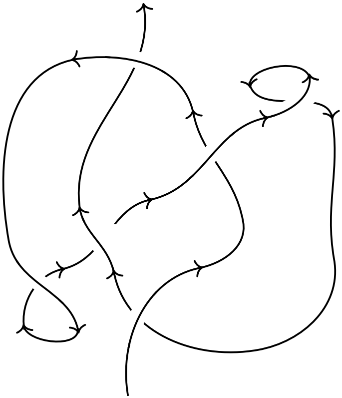
To optimize the computation:

- The R-matrix is sparse: a divide and conquer method sees the 0's at each step and eliminates a lot of terms.

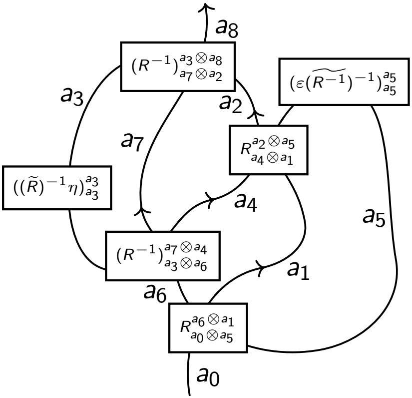
n	Nonzero elements (%)	$\#R$
2	177 (4.3%)	4096
3	585 (2.8%)	20,736
4	1377 (2.1%)	65,536

- Use optimized tensor contraction path.

Example: the 4_1 knot again



$\rightsquigarrow$



Example: the 4_1 knot again

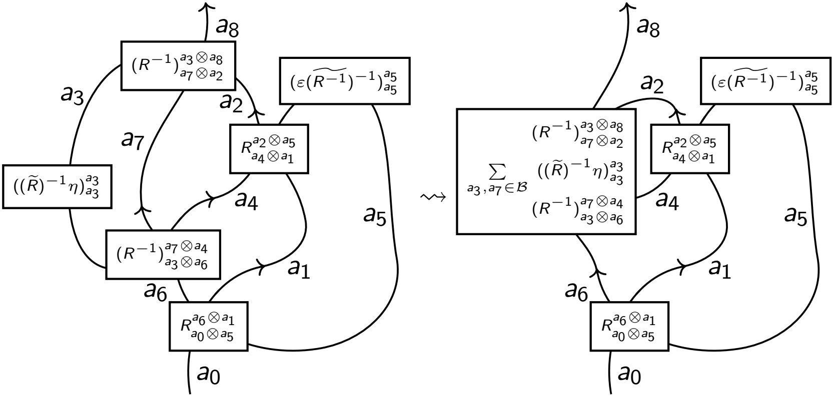


Table of Contents

Let $V_{K,n}(t,q) \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ be the V_n-polynomial of knot K in variables t and q.

Let $V_{K,n}(t,q) \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ be the V_n-polynomial of knot K in variables t and q.

- Symmetry:

$$V_{K,n}(t,q) = V_{K,n}(t^{-1},q), \quad V_{\overline{K},n}(t,q) = V_{K,n}(t,q^{-1})$$

Let $V_{K,n}(t,q) \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ be the V_n-polynomial of knot K in variables t and q.

- Symmetry:

$$V_{K,n}(t,q) = V_{K,n}(t^{-1},q), \quad V_{\overline{K},n}(t,q) = V_{K,n}(t,q^{-1})$$

- Specialization (conjecturally):

$$V_{K,n}(q^{n/2},q) = 1, \quad V_{K,n}(t,1) = \Delta_K(t)^2$$

where $\Delta_K(t)$ is the Alexander polynomial.

Let $V_{K,n}(t,q) \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ be the V_n-polynomial of knot K in variables t and q.

- Symmetry:

$$V_{K,n}(t,q) = V_{K,n}(t^{-1},q), \quad V_{\overline{K},n}(t,q) = V_{K,n}(t,q^{-1})$$

- Specialization (conjecturally):

$$V_{K,n}(q^{n/2},q) = 1, \quad V_{K,n}(t,1) = \Delta_K(t)^2$$

where $\Delta_K(t)$ is the Alexander polynomial.

- Genus bound (conjecturally):

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

where $g(K)$ is Seifert genus of K.

Theorems

- (GKKST) The V_1-polynomial is the Links–Gould polynomial.
- (KT) The Links–Gould polynomial satisfies both the specialization and the genus bound.
- $V_{K,2}$ is determined by

$$V_{K,2}(t^2, q^2) = c_{2,0}(t,q)V_{K(2,1),1}(t,q) + c_{2,-1}(t,q)V_{K,1}(t^2q^{-1}, q) + c_{2,1}(t,q)V_{K,1}(t^2q, q)$$

where $K(2,1)$ is the $(2,1)$-parallel of K.

Theorems

- (GKKST) The V_1-polynomial is the Links–Gould polynomial.
- (KT) The Links–Gould polynomial satisfies both the specialization and the genus bound.
- $V_{K,2}$ is determined by

$$V_{K,2}(t^2,q^2) = c_{2,0}(t,q)V_{K(2,1),1}(t,q) + c_{2,-1}(t,q)V_{K,1}(t^2q^{-1},q) + c_{2,1}(t,q)V_{K,1}(t^2q,q)$$

where $K(2,1)$ is the $(2,1)$-parallel of K.

Since $g(K(2,1)) = 2g(K)$, the last statement implies that V_2 also satisfies both the specialization and the genus bound.

Theorems

- (GKKST) The V_1-polynomial is the Links–Gould polynomial.
- (KT) The Links–Gould polynomial satisfies both the specialization and the genus bound.
- $V_{K,2}$ is determined by

$$V_{K,2}(t^2,q^2) = c_{2,0}(t,q)V_{K(2,1),1}(t,q) + c_{2,-1}(t,q)V_{K,1}(t^2q^{-1},q) + c_{2,1}(t,q)V_{K,1}(t^2q,q)$$

where $K(2,1)$ is the $(2,1)$-parallel of K.

Since $g(K(2,1)) = 2g(K)$, the last statement implies that V_2 also satisfies both the specialization and the genus bound.

Conjecturally, V_n-polynomials satisfy relations similar to the one above.

Question

When is the equality achieved in the genus bound inequality?

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

Question

When is the equality achieved in the genus bound inequality?

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

With the specialization, we have

$$2\deg_t \Delta_K(t) \leq \deg_t V_{K,n}(t,q) \leq 4g(K).$$

Question

When is the equality achieved in the genus bound inequality?

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

With the specialization, we have

$$2\deg_t \Delta_K(t) \leq \deg_t V_{K,n}(t,q) \leq 4g(K).$$

Since Alexander polynomials satisfy $\deg_t \Delta_K(t) \leq 2g(K)$, a sufficient condition:

$$\deg_t \Delta_K(t) = 2g(K). \tag{1}$$

Question

When is the equality achieved in the genus bound inequality?

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

With the specialization, we have

$$2\deg_t \Delta_K(t) \leq \deg_t V_{K,n}(t,q) \leq 4g(K).$$

Since Alexander polynomials satisfy $\deg_t \Delta_K(t) \leq 2g(K)$, a sufficient condition:

$$\deg_t \Delta_K(t) = 2g(K). \tag{1}$$

We call knots satisfying eq. (1) *tight*, and others *loose*.

There are no loose knots with ≤ 10 crossings.

There are no loose knots with ≤ 10 crossings.

crossings	11	12	13	14	15	16
Knots	552	2176	9988	46972	253293	1388705
Loose knots	7	29	208	1220	6319	48174

Table: Knot counts, up to mirror image

There are no loose knots with ≤ 10 crossings.

crossings	11	12	13	14	15	16
Knots	552	2176	9988	46972	253293	1388705
Loose knots	7	29	208	1220	6319	48174

Table: Knot counts, up to mirror image

polynomial	V_1	V_2	V_3	V_4
Knots	≤ 15	≤ 15	≤ 11	≤ 10
Loose knots	≤ 16	≤ 16		

Table: Computed knots for each V_n

There are no loose knots with ≤ 10 crossings.

crossings	11	12	13	14	15	16
Knots	552	2176	9988	46972	253293	1388705
Loose knots	7	29	208	1220	6319	48174

Table: Knot counts, up to mirror image

polynomial	V_1	V_2	V_3	V_4
Knots	≤ 15	≤ 15	≤ 11	≤ 10
Loose knots	≤ 16	≤ 16		

Table: Computed knots for each V_n

crossings	11	12	13	14	15	16
V_1 genus bound $<$	7	20	173	974	5025	37205
V_2 genus bound $<$	0	0	0	0	0	0

Table: Non-sharp genus bound counts

Theorem (Garoufalidis & L., 2024)

The genus bound inequality is an equality for V_2-polynomials for all 1,701,936 knots with ≤ 16 crossings.

Theorem (Garoufalidis & L., 2024)

The genus bound inequality is an equality for V_2-polynomials for all 1,701,936 knots with ≤ 16 crossings.

In other words, the V_2-polynomials (conjecturally) detect the genus.

Theorem (Garoufalidis & L., 2024)

The genus bound inequality is an equality for V_2-polynomials for all 1,701,936 knots with ≤ 16 crossings.

In other words, the V_2-polynomials (conjecturally) detect the genus.

Question

Does the V_2-polynomials *actually* detect the genus of knots? Why?

Question

When do two knots have equal V_2 polynomial?

Question

When do two knots have equal V_2 polynomial?

crossings	≤ 11	12	13	14	15
pairs	0	3	25	187	2324
triples	0	0	0	1	38

Table: Number of V_2-equivalence classes of size more than 1 (up to mirror image).

Question

When do two knots have equal V_2 polynomial?

crossings	≤ 11	12	13	14	15
pairs	0	3	25	187	2324
triples	0	0	0	1	38

Table: Number of V_2-equivalence classes of size more than 1 (up to mirror image).

Theorem (Garoufalidis & L., 2024)

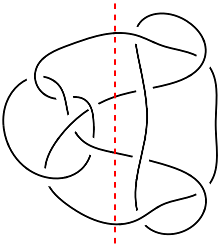
All knots with ≤ 15 crossings in the same V_2-equivalence classes

- have equal HFK and equal Khovanov Homology,
- are Conway mutant knots to each other, in particular they have equal volumes, trace field, colored Jones polynomials, ADO polynomials and HOMFLY polynomial.

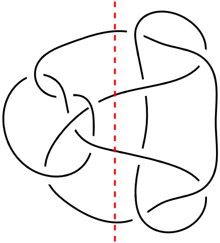
Theorem (Garoufalidis & L., 2024)

All knots with ≤ 15 crossings in the same V_2-equivalence classes

- have equal HFK and equal Khovanov Homology,
- are Conway mutant knots to each other, in particular they have equal volumes, trace field, colored Jones polynomials, ADO polynomials and HOMFLY polynomial.



$12n364$



$\overline{12n365}$

Theorem (Garoufalidis & L., 2024)

All knots with ≤ 15 crossings in the same V_2-equivalence classes

- have equal HFK and equal Khovanov Homology,
- are Conway mutant knots to each other, in particular they have equal volumes, trace field, colored Jones polynomials, ADO polynomials and HOMFLY polynomial.

crossings	11	12	13	14	15
V_2-equiv classes	0	3	25	188	2362
mutant classes	16	75	774	4435	29049

Table: Number of nontrivial V_2-equiv classes versus Conway mutant classes.

Theorem (Garoufalidis & L., 2024)

All knots with ≤ 15 crossings in the same V_2-equivalence classes

- have equal HFK and equal Khovanov Homology,
- are Conway mutant knots to each other, in particular they have equal volumes, trace field, colored Jones polynomials, ADO polynomials and HOMFLY polynomial.

Question

Are V_2-equivalent knots always Conway mutant? Do they always have equal HFK and equal Khovanov Homology? Why?

Question

Are V_2-equivalent knots always Conway mutant? Do they always have equal HFK and equal Khovanov Homology? Why?

A partial answer: most of them are HFK-thin and Khovanov-thin, for which equal HFK and equal Khovanov Homology follows given the mutant condition.

Question

Are V_2-equivalent knots always Conway mutant? Do they always have equal HFK and equal Khovanov Homology? Why?

A partial answer: most of them are HFK-thin and Khovanov-thin, for which equal HFK and equal Khovanov Homology follows given the mutant condition.

total	tight & thin	tight & thick	loose & thick
2578	1877	457	244

Table: Number of nontrivial V_2-equiv classes in each flavor, up to 15 crossings.

A Conspiracy Theory:

A Conspiracy Theory:

Proposition

For all alternating knots with ≤ 15 crossings, we have

$$V_1(t,-q), V_2(t,-q) \in \mathbb{Z}_{\geq 0}[t^{\pm 1}, q^{\pm 1}].$$

A Conspiracy Theory:

Proposition

For all alternating knots with ≤ 15 crossings, we have

$$V_1(t,-q), V_2(t,-q) \in \mathbb{Z}_{\geq 0}[t^{\pm 1}, q^{\pm 1}].$$

Question

Does this indicate a categorification of V_1 and V_2?