The complete census of orientable cusped hyperbolic 3-manifolds, up to 10 tetrahedra

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Tetrahedra	Orientable cusped hyperbolic 3-manifolds				
2	m003	m004			
3	m006	m007	m009	m010	m011
	m015	m016	m017	m019	

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3. Deduplicate the eligible candidates

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Total	8,373,308	7,468,856	904,452

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With verified computation, the canonical triangulation produced is a complete invariant of cusped hyperbolic 3-manifolds.

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L. 2025

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Corollary

There are only finitely many exceptional Dehn fillings on a 1-cusped hyperbolic 3-manifold.

Dunfield, 2019

There are precisely 205,822 exceptional Dehn fillings on 1-cusped hyperbolic 3-manifolds triangulable by no more than 9 tetrahedra.

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(In progress) Classification of each filling & Verify conjectures

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Conjecture (Dunfield-Friedl-Jackson, 2024)

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- Equality holds for all the computed ones

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In particular, it is the smallest among all.

The 10-tetrahedra census is available for installation on GitHub.

Figure: QR code to the GitHub repository

Alternatively:

shana-y-li.github.io → Code → snappy_10_tets