

# Multivariable knot polynomials, the $V_n$ -polynomials, and their patterns

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April 2025

Joint work with Stavros Garoufalidis

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- 1 Garoufalidis & Kashaev's multivariable knot polynomials
- 2 Computation of the  $V_n$ -polynomials
- 3 Patterns of the  $V_n$ -polynomials

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  - Rigidity: the *partial transposes*  
$$\widetilde{R^{\pm 1}} := (\varepsilon \otimes I \otimes I) \circ (I \otimes R^{\pm 1} \otimes I) \circ (I \otimes I \otimes \eta)$$
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 $\varepsilon: V \otimes V \rightarrow \mathbb{F}$  and  $\eta: \mathbb{F} \rightarrow V \otimes V$ : the evaluation and coevaluation maps.
- Reshetikhin–Turaev functor: a functor (determined by  $R$ ) from the category of tangles to the category of vector spaces.



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### Braided Hopf algebra

A *braided Hopf algebra over field  $\mathbb{F}$*  is a unital algebra  $H$  with product  $\nabla: H \otimes H \rightarrow H$  and unit  $\eta: \mathbb{F} \rightarrow H$ , equipped with coproduct  $\Delta: H \rightarrow H \otimes H$ , counit  $\varepsilon: H \rightarrow \mathbb{F}$  and invertible antipode  $S: H \rightarrow H$ .

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The procedure:

$$\left\{ \begin{array}{l} \text{Braided} \\ \text{Hopf algebras} \\ \text{with autos} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Braided} \\ \text{Yetter-Drinfel'd} \\ \text{modules with autos} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Rigid} \\ R\text{-matrices} \end{array} \right\}$$

Summary: a systematic source of knot invariants:

$$\left\{ \begin{array}{l} \text{Braided} \\ \text{Hopf algebras} \\ \text{with autos} \end{array} \right\} \xrightarrow{\text{G \& K, 2023}} \left\{ \begin{array}{l} \text{Rigid} \\ R\text{-matrices} \end{array} \right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

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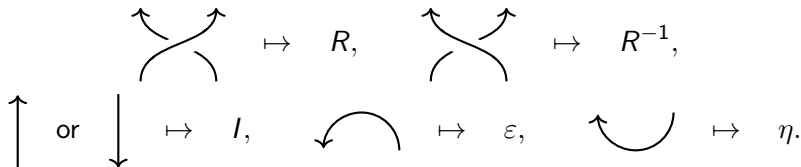
- Nichols algebras of rank 1: recovers the colored Jones polynomials and the ADO polynomials.
- Nichols algebras of rank 2: recovers the Links–Gould polynomial, and gives the  $V_n$ -polynomials.

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## Reshetikhin–Turaev functor: tangles $\mapsto$ vector spaces



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$$\begin{array}{ccc}
 \begin{array}{c} \nearrow \\ \searrow \end{array} & \mapsto R, & \begin{array}{c} \nearrow \\ \swarrow \end{array} & \mapsto R^{-1}, \\
 \uparrow \text{ or } \downarrow & \mapsto I, & \curvearrowright & \mapsto \varepsilon, & \curvearrowleft & \mapsto \eta.
 \end{array}$$

For local extrema going from left to right: (normalization)

$$\begin{array}{ccc}
 \curvearrowright & \rightsquigarrow & \begin{array}{c} \curvearrowright \\ \downarrow \end{array}, & \curvearrowleft & \rightsquigarrow & \begin{array}{c} \downarrow \\ \curvearrowleft \end{array}, \\
 \begin{array}{c} \searrow \\ \nearrow \end{array} & \mapsto & (R^{-1})^{-1}, & \begin{array}{c} \searrow \\ \swarrow \end{array} & \mapsto & (\tilde{R})^{-1}.
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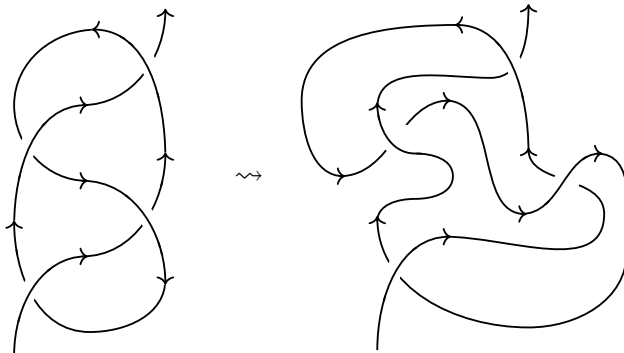
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 \uparrow \text{ or } \downarrow & \mapsto I, & \text{curved arrow} & \mapsto \varepsilon, & \text{cup} & \mapsto \eta.
 \end{array}$$

For local extrema going from left to right: (normalization)

$$\begin{array}{ccc}
 \text{cup} & \rightsquigarrow & \text{cup with arrows} \\
 \text{cross} & \mapsto & (\widetilde{R^{-1}})^{-1}, & \text{cross} & \mapsto & (\widetilde{R})^{-1}.
 \end{array}$$

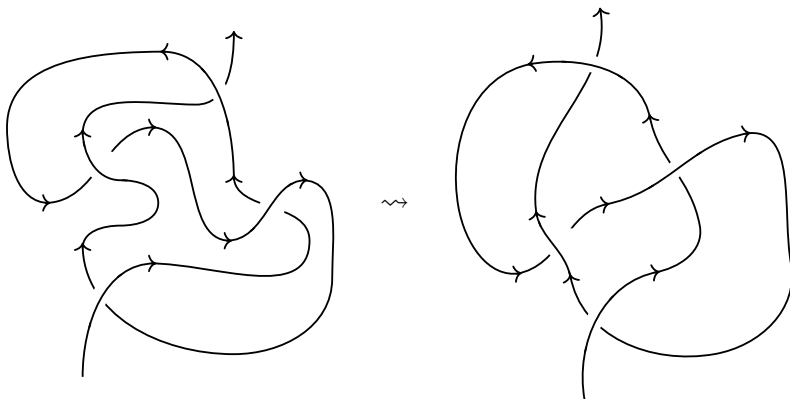
For  $V_n$ -polynomials,  $\varepsilon \circ (\widetilde{R^{-1}})^{-1} = (\widetilde{R^{-1}})^{-1} \circ \eta = \varepsilon \circ (\widetilde{R})^{-1} = (\widetilde{R})^{-1} \circ \eta$  is a diagonalizable matrix with only  $\pm 1$ 's on the diagonal.

Reshetikhin–Turaev: tangles  $\mapsto$  normalized tangles  $\mapsto$  vector spaces  
Example: the  $4_1$  knot



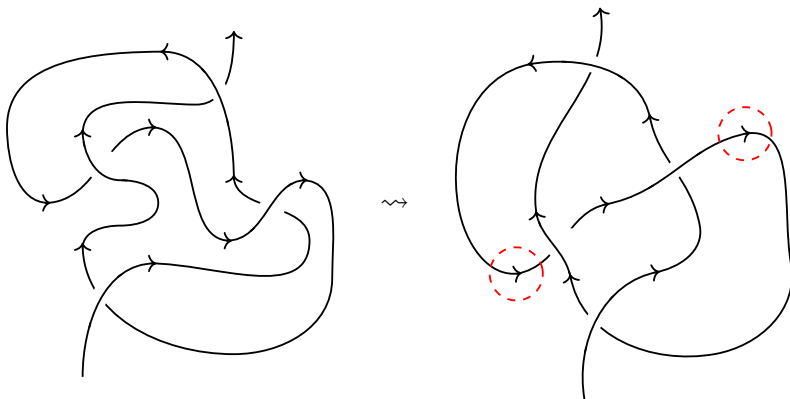
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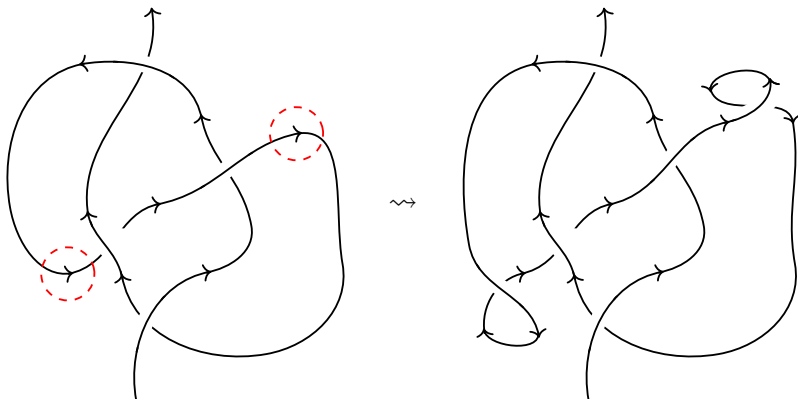
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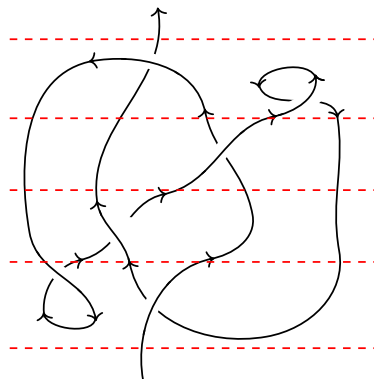
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$\mapsto$

$V$

$$\uparrow \quad \left( (\varepsilon \otimes I) \circ (I \otimes R^{-1}) \right) \otimes \left( \varepsilon \circ (\tilde{R})^{-1} \right)$$

$V^{\otimes 5}$

$\uparrow$

$$I \otimes I \otimes R \otimes I$$

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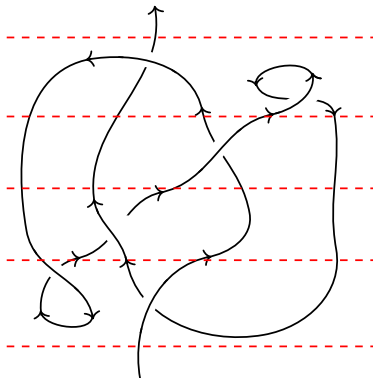
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$$\left( (\widetilde{R^{-1}})^{-1} \circ \eta \right) \otimes ((R \otimes I) \circ (I \otimes \eta))$$

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## Fact

For  $V_n$ -polynomials, the endomorphism on  $V$  we obtained is a scalar multiple of  $1_V$ .

Fix a basis  $\mathcal{B} := \{e_i\}$  of  $V$ ,  $R^{\pm 1}$  become matrices whose entries can be denoted by  $(R^{\pm 1})_{e_i \otimes e_j}^{e_k \otimes e_l}$ .

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To compute the eigenvalue of the  $\text{End}(V)$ -valued invariant is to evaluate a sum of the form

$$\sum_{\substack{a_1, \dots, a_{2c-1} \in \mathcal{B} \\ a_0 = a_{2c} = 1}} \pm \underbrace{(R^{\pm 1})_{a_0 \otimes a_1}^{a_2 \otimes a_3} \cdots (R^{\pm 1})_{a_{2c-3} \otimes a_{2c-2}}^{a_{2c-1} \otimes a_{2c}}}_{\text{a product of length } c},$$

where  $c$  is the number of crossings of the knot. This sum is called the *state sum*.

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Therefore, it requires

$$c \cdot (\dim V)^{2c-1}$$

times of computations to compute the eigenvalue.

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Worse, the entries  $(R^{\pm 1})_{e_i \otimes e_j}^{e_k \otimes e_l}$  are polynomials in two variables, instead of scalars. We computed the  $V_2$ -polynomials for all knots with  $\leq 15$  crossings, and more.

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- The  $R$ -matrix is sparse: a divide and conquer method sees the 0's at each step and eliminates a lot of terms.

n	Nonzero elements (%)	# $R$
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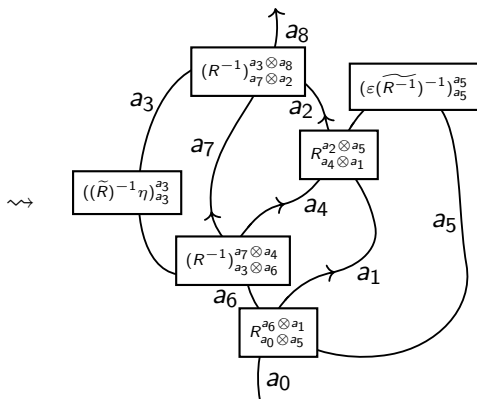
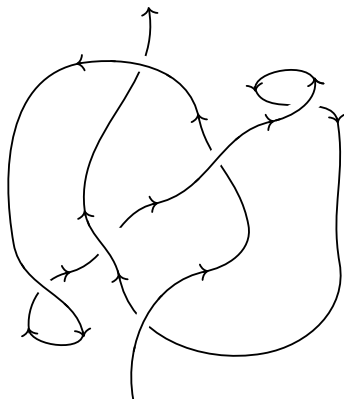
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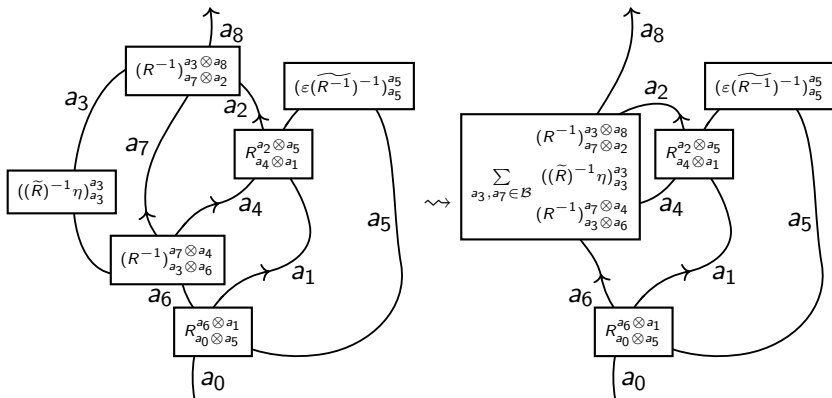
- Use optimized tensor contraction path.



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■ Specialization (conjecturally):

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■ Genus bound (conjecturally):

$$\deg_t V_{K,n}(t, q) \leq 4g(K)$$

where  $g(K)$  is Seifert genus of  $K$ .

## Theorems

- (GKKST) The  $V_1$ -polynomial is the Links–Gould polynomial.
- (KT) The Links–Gould polynomial satisfies both the specialization and the genus bound.
- $V_{K,2}$  is determined by

$$V_{K,2}(t^2, q^2) = c_{2,0}(t, q)V_{K(2,1),1}(t, q) + c_{2,-1}(t, q)V_{K,1}(t^2q^{-1}, q) + c_{2,1}(t, q)V_{K,1}(t^2q, q)$$

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Conjecturally,  $V_n$ -polynomials satisfy relations similar to the one above.

## Question

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We call knots satisfying eq. (1) *tight*, and others *loose*.

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crossings	11	12	13	14	15	16
Knots	552	2176	9988	46972	253293	1388705
Loose knots	7	29	208	1220	6319	48174

Table: Knot counts, up to mirror image



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Loose knots	7	29	208	1220	6319	48174

Table: Knot counts, up to mirror image

polynomial	$V_1$	$V_2$	$V_3$	$V_4$
Knots	$\leq 15$	$\leq 15$	$\leq 11$	$\leq 10$
Loose knots	$\leq 16$	$\leq 16$		

Table: Computed knots for each  $V_n$

There are no loose knots with  $\leq 10$  crossings.

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Table: Computed knots for each  $V_n$

crossings	11	12	13	14	15	16
$V_1$ genus bound $<$	7	20	173	974	5025	37205
$V_2$ genus bound $<$	0	0	0	0	0	0

Table: Non-sharp genus bound counts

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The genus bound inequality is an equality for  $V_2$ -polynomials for all 1,701,936 knots with  $\leq 16$  crossings.

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## Question

Does the  $V_2$ -polynomials *actually* detect the genus of knots? Why?

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When do two knots have equal  $V_2$  polynomial?

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pairs	0	3	25	187	2324
triples	0	0	0	1	38

**Table:** Number of  $V_2$ -equivalence classes of size more than 1 (up to mirror image).

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All knots with  $\leq 15$  crossings in the same  $V_2$ -equivalence classes

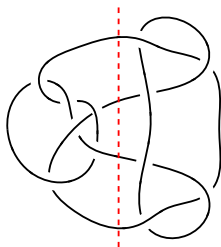
- have equal HFK and equal Khovanov Homology,
- are Conway mutant knots to each other, in particular they have equal volumes, trace field, colored Jones polynomials, ADO polynomials and HOMFLY polynomial.



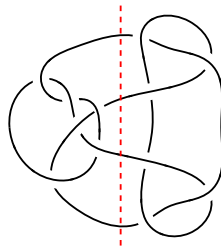
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12n364



$\overline{12n365}$

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crossings	11	12	13	14	15
$V_2$ -equiv classes	0	3	25	188	2362
mutant classes	16	75	774	4435	29049

**Table:** Number of nontrivial  $V_2$ -equiv classes versus Conway mutant classes.

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total	tight & thin	tight & thick	loose & thick
2578	1877	457	244

**Table:** Number of nontrivial  $V_2$ -equiv classes in each flavor, up to 15 crossings.

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### Proposition

For all alternating knots with  $\leq 15$  crossings, we have

$$V_1(t, -q), V_2(t, -q) \in \mathbb{Z}_{\geq 0}[t^{\pm 1}, q^{\pm 1}].$$

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### Question

Does this indicate a categorification of  $V_1$  and  $V_2$ ?