Multivariable knot polynomials, the V_n-polynomials, and their patterns

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Joint work with Stavros Garoufalidis

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- Reshetikhin–Turaev functor: a functor (determined by R) from the category of tangles to the category of vector spaces.

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Given a braided Hopf algebra with automorphisms, one can construct a rigid R-matrix.

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Hopf algebra

A *Hopf algebra over field* $\mathbb{F}$ is a unital algebra H equipped with coproduct $\Lambda\colon H \to H \otimes H$, counit $\varepsilon\colon H \to \mathbb{F}$ and invertible antipode $S\colon H \to H$.

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The procedure:

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Summary: a systematic source of knot invariants:

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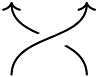
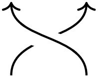
Reshetikhin–Turaev functor: tangles $\mapsto$ vector spaces

 $\mapsto \quad R,$  $\mapsto \quad R^{-1},$

$\uparrow$ or $\downarrow \quad \mapsto \quad I,$  $\mapsto \quad \varepsilon,$  $\mapsto \quad \eta.$

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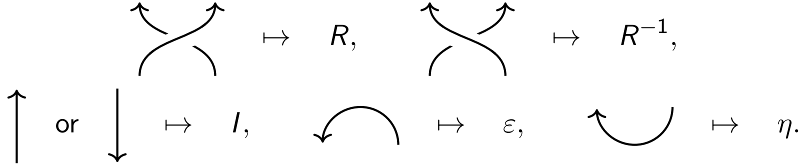
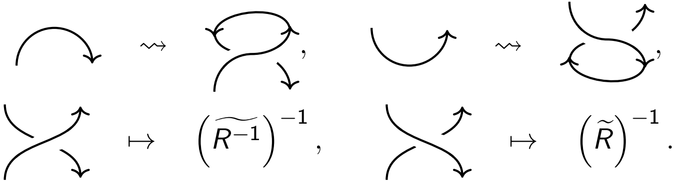


For local extrema going from left to right: (normalization)



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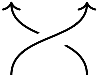
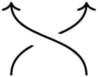
 $\mapsto$ R,  $\mapsto$ R^{-1},

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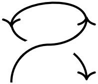
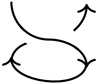
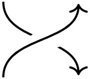
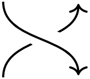
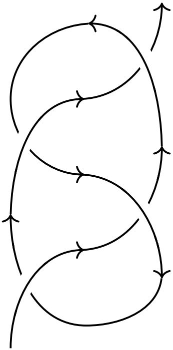
 $\rightsquigarrow$ ,  $\rightsquigarrow$ ,

 $\mapsto$ $\left(\widetilde{R^{-1}}\right)^{-1}$,  $\mapsto$ $\left(\widetilde{R}\right)^{-1}$.

For V_n-polynomials, $\varepsilon \circ \left(\widetilde{R^{-1}}\right)^{-1} = \left(\widetilde{R^{-1}}\right)^{-1} \circ \eta = \varepsilon \circ \left(\widetilde{R}\right)^{-1} = \left(\widetilde{R}\right)^{-1} \circ \eta$ is a diagonalizable matrix with only ±1's on the diagonal.

Reshetikhin–Turaev: tangles $\mapsto$ normalized tangles $\mapsto$ vector spaces
Example: the 4_1 knot



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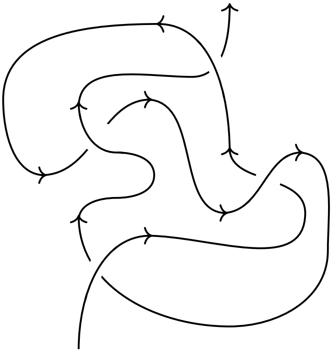

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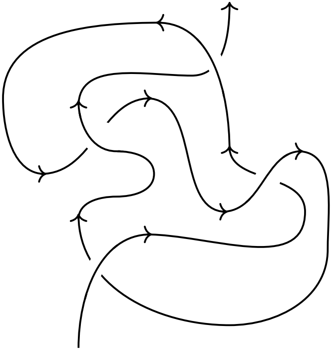

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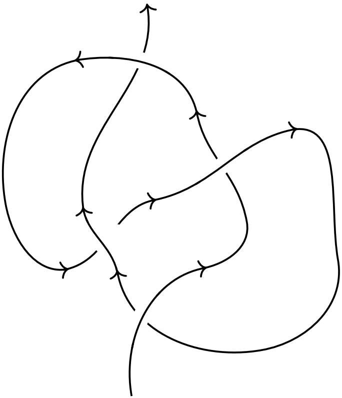

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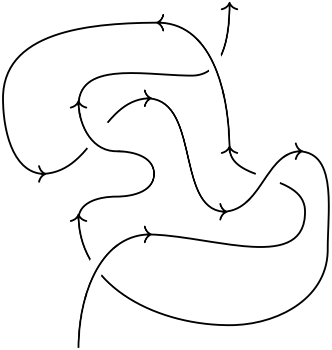

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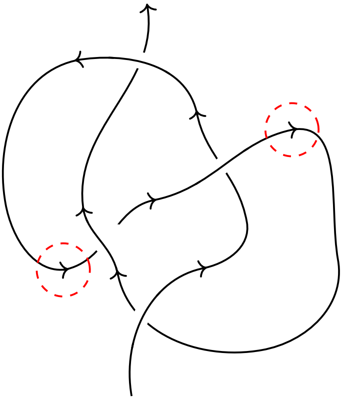

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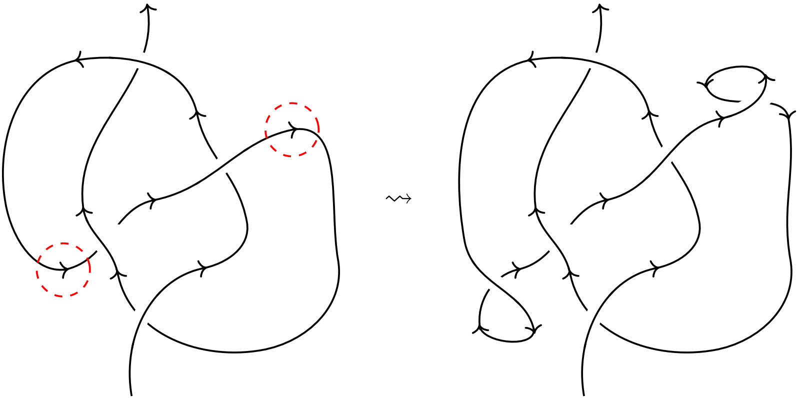

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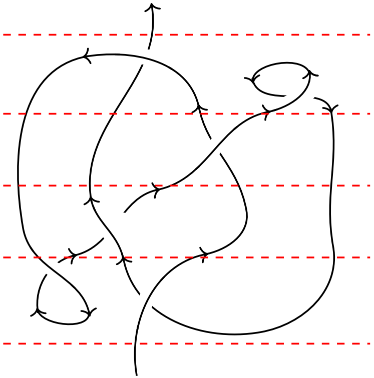

$$\mapsto \quad \begin{array}{ll} V & \\ \uparrow & \left((\varepsilon \otimes I) \circ \left(I \otimes R^{-1}\right)\right) \otimes \left(\varepsilon \circ \left(\tilde{R}\right)^{-1}\right) \\ V^{\otimes 5} & \\ \uparrow & I \otimes I \otimes R \otimes I \\ V^{\otimes 5} & \\ \uparrow & I \otimes R^{-1} \otimes I \otimes I \\ V^{\otimes 5} & \\ \uparrow & \left(\left(\widetilde{R^{-1}}\right)^{-1} \circ \eta\right) \otimes ((R \otimes I) \circ (I \otimes \eta)) \\ V & \end{array}$$

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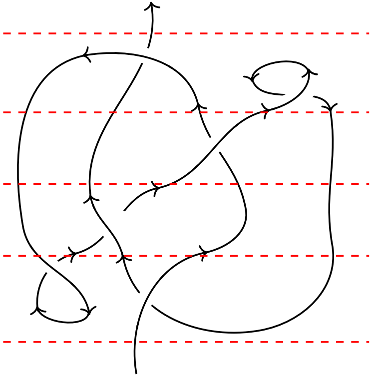

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Fact

For V_n-polynomials, the endomorphism on V we obtain is a scalar multiple of 1_V. The scalar gives our polynomial invariant.

Fix a basis $\mathcal{B} := \{e_i\}$ of V, $R^{\pm 1} \in \mathrm{Aut}(V \otimes V)$ become matrices whose entries can be denoted by $\left(R^{\pm 1}\right)^{e_k \otimes e_l}_{e_i \otimes e_j}$.

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To compute the eigenvalue of the End(V)-valued invariant is to evaluate a sum of the form

$$\sum_{\substack{a_1,\cdots,a_{2c-1} \in \mathcal{B} \\ a_0 = a_{2c} = 1}} \pm \underbrace{\left(R^{\pm 1}\right)_{a_0 \otimes a_1}^{a_2 \otimes a_3} \cdots \left(R^{\pm 1}\right)_{a_{2c-3} \otimes a_{2c-2}}^{a_{2c-1} \otimes a_{2c}}}_{\text{a product of length } c},$$

where c is the number of crossings of the knot. This sum is the so called *state sum*.

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Therefore, it requires

$$c \cdot (\dim V)^{2c-1}$$

times of computations to compute the eigenvalue.

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We computed the V_2-polynomials for all knots with ≤ 16 crossings.

polynomial	V_1	V_2	V_3	V_4
Knots	≤ 16	≤ 16	≤ 14	≤ 14
Loose Knots	≤ 18	≤ 17		

Table: Computed knots for each V_n

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- The R-matrix is sparse: a divide and conquer method sees the 0's at each step and eliminates a lot of terms.

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- Use optimized tensor contraction path.

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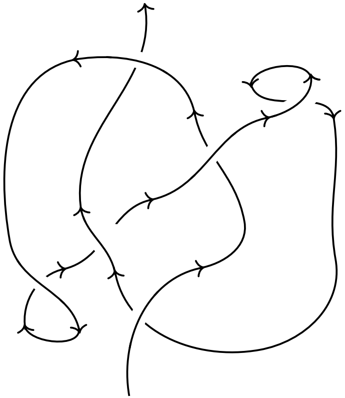

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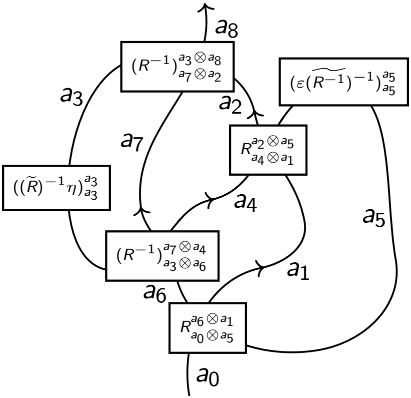

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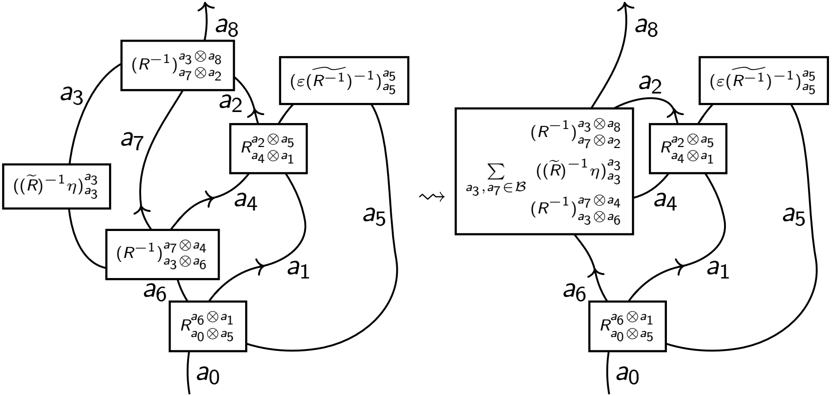

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$$V_{K,n}(q^{n/2},q) = 1, \quad V_{K,n}(t,1) = \Delta_K(t)^2$$

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- Genus bound (conjecturally):

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

  where $g(K)$ is Seifert genus of K.

Theorems

- (GKKST) The V_1-polynomial is the Links–Gould polynomial.
- (KT) The Links–Gould polynomial satisfies both the specialization and the genus bound.
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- $V_{K,2}$ is the 2-colored Links–Gould polynomial, for which the specialization and the genus bound both hold.

Conjecturally, $V_{K,n}$ is the n-colored Links–Gould polynomial in general.

Question

When is the equality achieved in the genus bound inequality?

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

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We call knots satisfying eq. (1) *tight*, and others *loose*.

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crossings	11	12	13	14	15	16	17	18
Knots	552	2176	9988	46972	253293	1388705	8053393	48266466
Loose knots	7	29	208	1220	6319	48174	303823	2001954

Table: Knot counts, up to mirror image

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Table: Number of knots where detection of genus fails

Theorem (Garoufalidis & L., 2025)

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Question

Does the V_2-polynomials *actually* detect the genus of knots? Why?

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All knots with ≤ 16 crossings in the same V_2-equivalence classes

- have equal HFK and equal Khovanov Homology,
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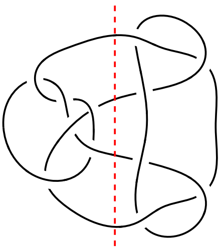
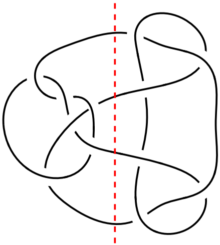


$12n364$



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crossings	11	12	13	14	15	16
V_2-equiv classes	0	3	50	334	2362	14626
mutant classes	16	75	774	4435	29049	

Table: Number of nontrivial V_2-equiv classes versus Conway mutant classes.

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total	tight & thin	tight & thick	loose & thick
2578	1877	457	244

Table: Number of nontrivial V_2-equiv classes in each flavor, up to 15 crossings.

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Proposition

For all alternating knots with ≤ 16 crossings, we have

$$V_1(t,-q), V_2(t,-q) \in \mathbb{Z}_{\geq 0}[t^{\pm 1}, q^{\pm 1}].$$

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Question

Does this indicate a categorification of V_1 and V_2?

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7 tangles whose mutations have been proved to preserve the V_2-polynomial (and also V_3 and V_4 for most of them).

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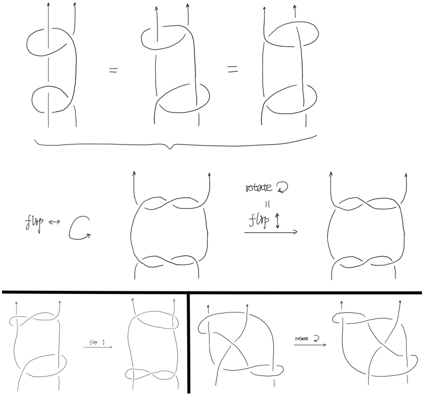

Figure: 3 of the 7 special tangles.

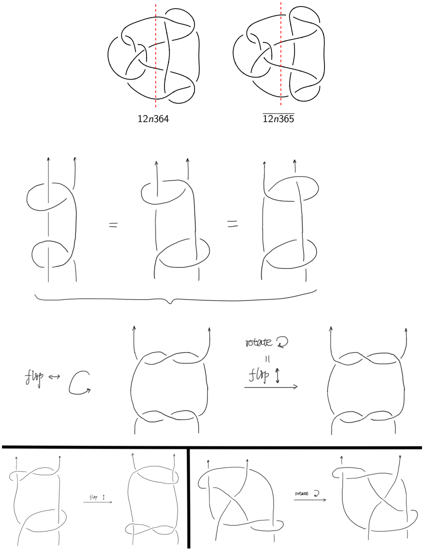

Figure: 3 of the 7 special tangles