Multivariable knot polynomials, the V_n-polynomials, and their patterns

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Joint work with Stavros Garoufalidis

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  - Rigidity: the *partial transposes* $\widetilde{R^{\pm 1}} := (\varepsilon \otimes I \otimes I) \circ (I \otimes R^{\pm 1} \otimes I) \circ (I \otimes I \otimes \eta)$ are invertible.
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- Reshetikhin–Turaev functor: a functor (determined by R) from the category of tangles to the category of vector spaces.

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Given a braided Hopf algebra with automorphisms, one can construct a rigid R-matrix.

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Hopf algebra

A *Hopf algebra over field* $\mathbb{F}$ is a unital algebra H equipped with coproduct $\Lambda\colon H \to H \otimes H$, counit $\varepsilon\colon H \to \mathbb{F}$ and invertible antipode $S\colon H \to H$.

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The procedure:

$$\left\{\begin{array}{l}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{array}\right\} \to \left\{\begin{array}{l}\text{Braided}\\ \text{Yetter–Drinfel'd}\\ \text{modules with autos}\end{array}\right\} \to \left\{\begin{array}{l}\text{Rigid}\\ R\text{-matrices}\end{array}\right\}$$

Summary: a systematic source of knot invariants:

$$\left\{\begin{matrix}\text{Braided} \\ \text{Hopf algebras} \\ \text{with autos}\end{matrix}\right\} \xrightarrow{\text{G \& K, 2023}} \left\{\begin{matrix}\text{Rigid} \\ R\text{-matrices}\end{matrix}\right\} \xrightarrow{\text{K, 2019}} \{\text{Knot invariants}\}$$

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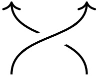
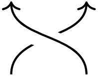
Reshetikhin–Turaev functor: tangles $\mapsto$ vector spaces

 $\mapsto \quad R,$  $\mapsto \quad R^{-1},$

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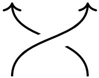
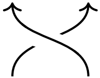
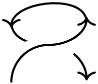
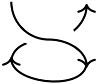
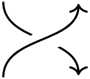
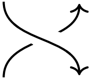
 $\mapsto \quad R,$  $\mapsto \quad R^{-1},$

 or  $\mapsto \quad I,$  $\mapsto \quad \varepsilon,$  $\mapsto \quad \eta.$

For local extrema going from left to right: (normalization)

 $\rightsquigarrow$ ,  $\rightsquigarrow$ ,

 $\mapsto \quad \left(\widetilde{R^{-1}}\right)^{-1},$  $\mapsto \quad \left(\widetilde{R}\right)^{-1}.$

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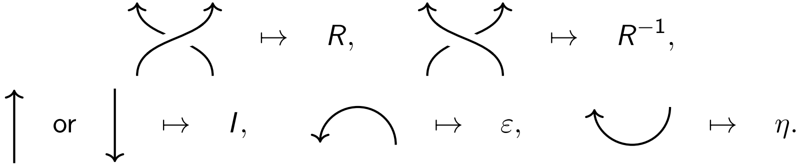

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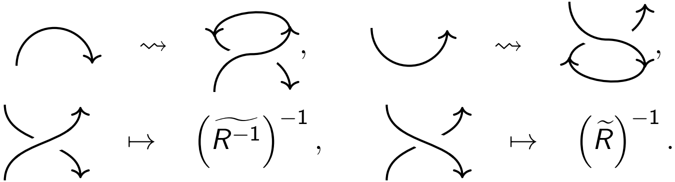
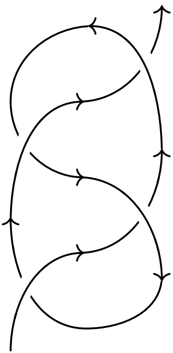


For V_n-polynomials, $\varepsilon \circ \left(\widetilde{R^{-1}}\right)^{-1} = \left(\widetilde{R^{-1}}\right)^{-1} \circ \eta = \varepsilon \circ \left(\widetilde{R}\right)^{-1} = \left(\widetilde{R}\right)^{-1} \circ \eta$ is a diagonalizable matrix with only ± 1's on the diagonal.

Reshetikhin–Turaev: tangles $\mapsto$ normalized tangles $\mapsto$ vector spaces
Example: the 4_1 knot



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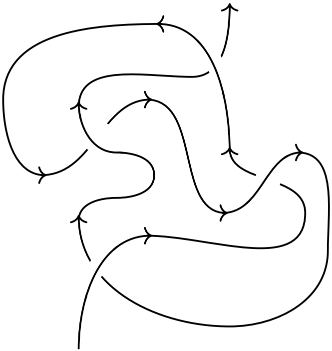

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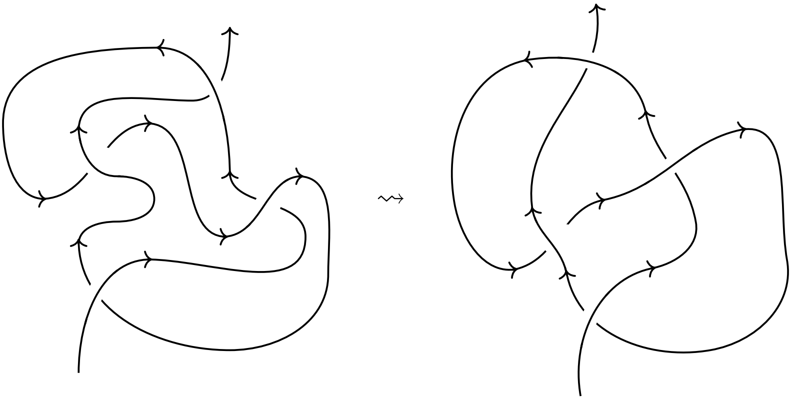

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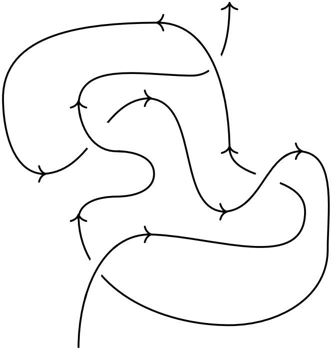

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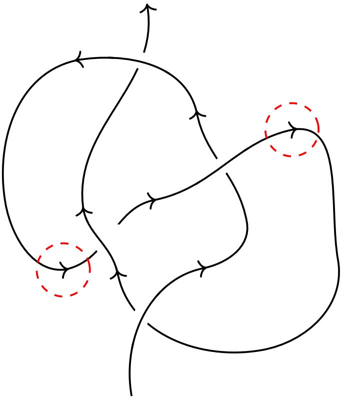

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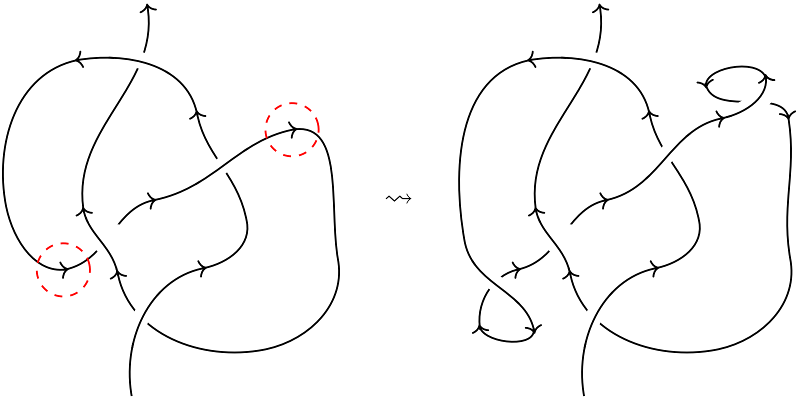

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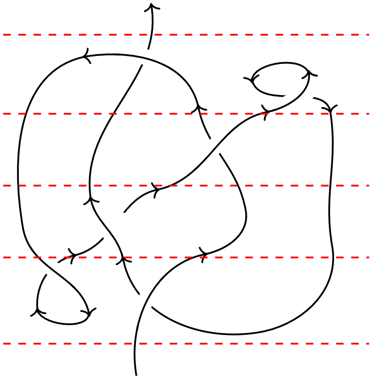

$$\mapsto \quad \begin{array}{lr} V & \\ \uparrow & \left((\varepsilon \otimes I) \circ \left(I \otimes R^{-1}\right)\right) \otimes \left(\varepsilon \circ \left(\widetilde{R}\right)^{-1}\right) \\ V^{\otimes 5} & \\ \uparrow & I \otimes I \otimes R \otimes I \\ V^{\otimes 5} & \\ \uparrow & I \otimes R^{-1} \otimes I \otimes I \\ V^{\otimes 5} & \\ \uparrow & \left(\left(\widetilde{R^{-1}}\right)^{-1} \circ \eta\right) \otimes ((R \otimes I) \circ (I \otimes \eta)) \\ V & \end{array}$$

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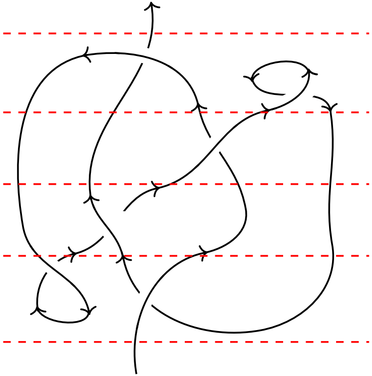

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Fact

For V_n-polynomials, the endomorphism on V we obtain is a scalar multiple of 1_V. The scalar gives our polynomial invariant.

Fix a basis $\mathcal{B} := \{e_i\}$ of V, $R^{\pm 1} \in \mathrm{Aut}(V \otimes V)$ become matrices whose entries can be denoted by $\left(R^{\pm 1}\right)_{e_i \otimes e_j}^{e_k \otimes e_l}$.

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To compute the eigenvalue of the End(V)-valued invariant is to evaluate a sum of the form

$$\sum_{\substack{a_1, \cdots, a_{2c-1} \in \mathcal{B} \\ a_0 = a_{2c} = 1}} \pm \underbrace{\left(R^{\pm 1}\right)_{a_0 \otimes a_1}^{a_2 \otimes a_3} \cdots \left(R^{\pm 1}\right)_{a_{2c-3} \otimes a_{2c-2}}^{a_{2c-1} \otimes a_{2c}}}_{\text{a product of length } c},$$

where c is the number of crossings of the knot. This sum is the so called *state sum*.

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Therefore, it requires

$$c \cdot (\dim V)^{2c-1}$$

times of computations to compute the eigenvalue.

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We computed the V_2-polynomials for all knots with ≤ 16 crossings.

polynomial	V_1	V_2	V_3	V_4
Knots	≤ 16	≤ 16	≤ 14	≤ 13

Table: Computed knots for each V_n (2025 ver.)

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- Use optimized tensor contraction path.

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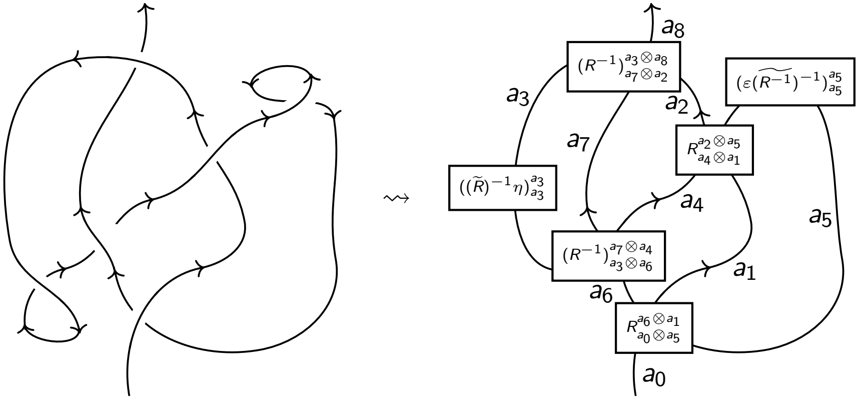

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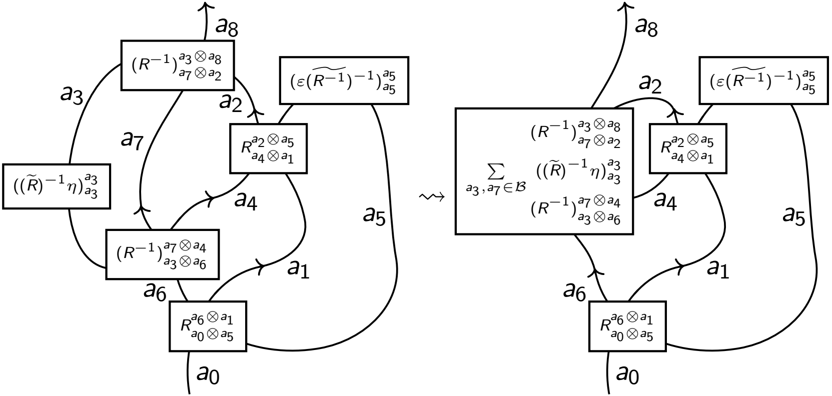

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$$V_{K,n}(q^{n/2},q) = 1, \quad V_{K,n}(t,1) = \Delta_K(t)^2$$

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- Genus bound (conjecturally):

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

  where $g(K)$ is Seifert genus of K.

Theorems

- (GKKST) The V_1-polynomial is the Links–Gould polynomial.
- (KT) The Links–Gould polynomial satisfies both the specialization and the genus bound.
- $V_{K,2}$ is determined by

$$V_{K,2}(t^2, q^2) = c_{2,0}(t,q)V_{K(2,1),1}(t,q) + c_{2,-1}(t,q)V_{K,1}(t^2q^{-1}, q) + c_{2,1}(t,q)V_{K,1}(t^2q, q)$$

  where $K(2,1)$ is the $(2,1)$-parallel of K.

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Theorems

- (GKKST) The V_1-polynomial is the Links–Gould polynomial.
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Conjecturally, V_n-polynomials satisfy relations similar to the one above.

Question

When is the equality achieved in the genus bound inequality?

$$\deg_t V_{K,n}(t,q) \leq 4g(K)$$

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We call knots satisfying eq. (1) *tight*, and others *loose*.

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crossings	11	12	13	14	15	16
Knots	552	2176	9988	46972	253293	1388705
Loose knots	7	29	208	1220	6319	48174

Table: Knot counts, up to mirror image

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polynomial	V_1	V_2	V_3	V_4
Knots	≤ 15	≤ 15	≤ 11	≤ 10
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Table: Computed knots for each V_n (2024 ver.)

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V_1 genus bound $<$	7	20	173	974	5025	37205
V_2 genus bound $<$	0	0	0	0	0	0

Table: Non-sharp genus bound counts

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The genus bound inequality is an equality for V_2-polynomials for all 1,701,935 knots with ≤ 16 crossings.

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Question

Does the V_2-polynomials *actually* detect the genus of knots? Why?

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When do two knots have equal V_2 polynomial?

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All knots with ≤ 16 crossings in the same V_2-equivalence classes

- have equal HFK and equal Khovanov Homology,
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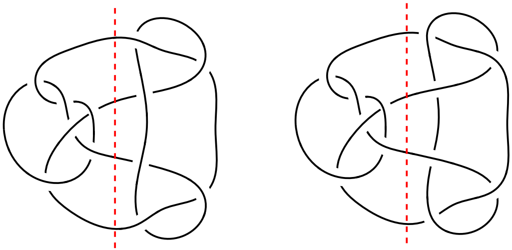

$12n364$ $\overline{12n365}$

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crossings	11	12	13	14	15	16
V_2-equiv classes	0	3	50	334	2362	14626
mutant classes	16	75	774	4435	29049	

Table: Number of nontrivial V_2-equiv classes versus Conway mutant classes.

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total	tight & thin	tight & thick	loose & thick
2578	1877	457	244

Table: Number of nontrivial V_2-equiv classes in each flavor, up to 15 crossings.

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Proposition

For all alternating knots with ≤ 16 crossings, we have

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Question

Does this indicate a categorification of V_1 and V_2?

(Ongoing)

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Upon closer look at the mutations within V_2-equivalence classes, we found 7 tangles (non-exhaustive), the mutation of which has been proved to preserve the V_2-polynomial (and also V_3 and V_4 for most of them).

(Ongoing)

Here are three of them:

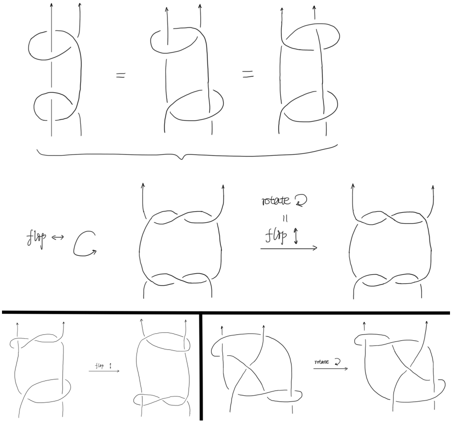
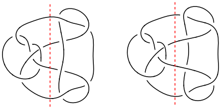


Figure: 3 special tangles found.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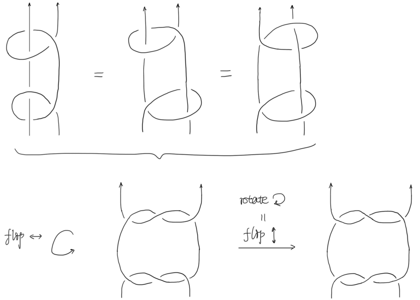
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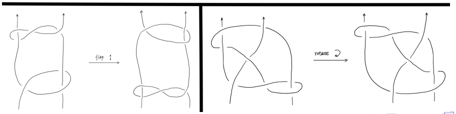
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