P_1

GR5241
HW1
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1. $P(X|\theta) = P(X|,\mu,\nu) = \left(\frac{\nu}{\mu}\right)^{\nu} \frac{x^{\nu-1}}{\Gamma(\nu)} \exp\left(-\frac{\nu x}{\mu}\right)$

$$L(\theta) = \prod_{i=1}^{n} P(X_i|\theta) = \left(\frac{\nu}{\mu}\right)^{n\nu} \frac{\prod x_i^{\nu-1}}{\Gamma^{(n)}(\nu)} \exp\left(-\frac{\nu}{\mu}\sum_{i=1}^{n} x_i\right)$$

suppose $ll(\theta)$ to be log-likelihood.

$$ll(\theta) = \log L(\theta) = n\nu(\log\nu - \log\mu) - n\log\Gamma(\nu) + \sum_{i=1}^{n} (\nu-1)\log(x_i) - \frac{\nu}{\mu}\sum_{i=1}^{n} x_i$$

2. ① $\frac{\partial ll(\theta)}{\partial\mu} = -\frac{1}{\mu}n\nu + \frac{\nu}{\mu^2}\sum_{i=1}^{n} x_i = 0 \Rightarrow \frac{\sum_{i=1}^{n} x_i}{\mu} = n \quad \hat{\mu} = \frac{\sum x_i}{n}$

3. ② $\frac{\partial ll(\theta)}{\partial\nu} = n(\log\nu - \log\mu) + n - n\frac{\Gamma'(\nu)}{\Gamma(\nu)} + \sum_{i=1}^{n}\log(x_i) - \frac{\sum_{i=1}^{n} x_i}{\mu} = 0$

$$\Rightarrow n(\log\nu - \log\mu) + \sum\log x_i = \sum\log\left(\frac{x_i\hat{\nu}}{\mu}\right)$$

$$-n\frac{\Gamma'(\nu)}{\Gamma(\nu)} = -\sum\phi(\hat{\nu})$$

So ② $= \sum\log\left(\frac{x_i\hat{\nu}}{\mu}\right) - \frac{\sum x_i}{\mu} + n - \sum\phi(\hat{\nu}) = \sum\left(\log\left(\frac{x_i\hat{\nu}}{\mu}\right) - \left(\frac{x_i}{\mu} - 1\right) - \phi(\hat{\nu})\right) = 0$

P_2. our target is to get $\min R(f)$, due to monotonicity of integral, $\min R(f)$ is to get $\min R(f|x)$ $\left(R(f) = \int_{\mathbb{R}^d} R(f(x))\,p(x)\,dx\right)$

So we need to find f to get $\min R(f|x) = \sum_{y\in[k]} L^{0-1}(y, f(x))\,P(y|x)$

$$f(x) = \underset{y\in[k]}{\arg\min} \sum_{y\in[k]} L^{0-1}(y, f(x))\,P(y|x)$$

$$= \underset{y\in[k]}{\arg\min}\,(1 - P(y|x))$$

(explain: mismatch loss is 1 and match Loss 0
$0\cdot P(\text{match}) + 1\cdot P(\text{mismatch})$
$= 1\cdot(1 - P(\text{match})) = 1 - P(y|x)$)

So $f(x) = \underset{y \in [K]}{\arg\max}\, P(y|X)$

this is exactly $f_0(X) = \underset{y \in [K]}{\arg\max}\, P(y|X)$

So $f_0(x)$ is the classifier with minimum $R(f|X)$, i.e. $\min R(f)$.