Chapter 14 solutions

1. Let H represent heads, T represent tails, and the integers 1-6 represent the numbers on the die. Then the seven possible outcomes in the sample space are

$$\Omega = \{T, (H,1), (H,2), (H,3), (H,4), (H,5), (H,6)\}$$

2. (a) $\Omega = \{(O,O), (O,A), (O,B), (O,AB), (A,O), (A,A), (A,B), (A,AB), (B,O), (B,A), (B,B), (B,AB), (AB,O), (AB,A), (AB,B), (AB,AB)\}$

   (b) $E = \{(O,O), (O,A), (O,B), (O,AB), (A,A), (A,AB), (B,B), (B,AB), (AB,AB)\}$

   (c) $E = \{(O,O), (A,A), (B,B), (AB,AB)\}$

3. (a) $E = \{0, 1, 2, ...\}$

   (b) $E = \{$ non-negative real numbers $\}$

6. (a) $B - (A \cup B)$

   (b) $(B \cap C) - A$

   (c) $A \cap B \cap C$

   (d) $A \cup B \cup C$

   (e) $\Omega - (A \cup B \cup C) = \overline{A \cup B \cup C}$

7. First note (e.g., from a Venn diagram) that $\overline{A} \cap \overline{B} = \overline{A \cup B}$. Then $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$.

8. No, this is not possible. $A \cap B$ is a subset of B, so it must be that $P(A \cap B) \leq P(B)$.

11. (a) $P(\overline{A} \cap B) = P(B - A) = \beta - \gamma$

    (b) $P(A \cap \overline{B}) = P(A - B) = \alpha - \gamma$

    (c) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - (\alpha + \beta - \gamma)$

13. (a) $P(A|B) = 0$

    (b) $P(A|B) \geq 0.6$

    (c) $P(A|B) = 1$

14. (a) In this case $P(A \cup B) = P(A) + P(B)$, so $P(B) = 0.3$.

    (b) In this case $P(A|B) = P(A \cap B)/P(B) = P(A)$, so $P(A \cap B) = 0.4 \times P(B)$. Then,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [addition rule]
$P(A \cup B) = P(A) + P(B) - 0.4 \times P(B)$
$0.7 = 0.4 + P(B) - 0.4 \times P(B)$
$P(B) = 0.5$

16. $1 - 0.5^n \geq 0.99$
$0.5^n \leq 0.01$
$\log 0.5^n \leq \log 0.01$
$n \geq \log 0.01/\log 0.5 = 6.64$

So we need at least seven coin tosses.

19. (a) The sample space is $\Omega = \{$ black, (red, black), (red, red) $\}$

(b) The probability of winning on the first roll is 0.5. The probability of not winning on the first roll and then winning on the second roll is 0.25. These are mutually exclusive events, so the total probability of winning is 0.75.

21. The sample space is $\Omega = \{(M,M),(M,F),(F,M),(F,F)\}$. Each outcome has probability 0.25.

(a)

$$\begin{aligned} P(\text{ both boys } \mid \text{ eldest is boy }) &= P(\{(M,M)\}|\{(M,M),(M,F)\}) \\ &= \frac{P(\{(M,M)\} \cap \{(M,M),(M,F)\})}{P(\{(M,M),(M,F)\})} \\ &= \frac{P(\{(M,M)\})}{P(\{(M,M),(M,F)\})} \\ &= 0.25/0.50 = 0.50 \end{aligned}$$

(b)

$$\begin{aligned} P(\text{ both boys } \mid \text{ at least one boy }) &= P(\{(M,M)\}|\{(M,M),(M,F),(F,M)\}) \\ &= \frac{P(\{(M,M)\} \cap \{(M,M),(M,F),(F,M)\})}{P(\{(M,M),(M,F),(F,M)\})} \\ &= \frac{P(\{(M,M)\})}{P(\{(M,M),(M,F),(F,M)\})} \\ &= 0.25/0.75 = 0.33 \end{aligned}$$

22. (a) $P(A|A \text{ or } B \text{ but not both })$
$= P(A \text{ and } (A \text{ or } B \text{ but not both }))/P(A \text{ or } B \text{ but not both })$
$= P(A \text{ and not } B)/P(A \text{ or } B \text{ but not both })$
$= P(A)P(\text{ not } B)/(P(A) + P(B) - 2P(A \text{ and } B))$
$= 0.65 \times (1 - 0.50)/(0.65 + 0.50 - 2 \times 0.65 \times 0.50) = 0.65$

(b) $P(B|A \text{ or } B) = P(B \text{ and } (A \text{ or } B))/P(A \text{ or } B) = P(B)/P(A \text{ or } B)$
$= P(B)/(P(A) + P(B) - P(A \text{ and } B)) = 0.50/(0.65 + 0.50 - 0.65 \times 0.50) = 0.61$

26. Let R_k be a random variable that represents the number we roll on roll number k. That is, R_1 is the result of the first roll, R_2 is the result of the second roll, and so on. When we want to refer to a roll of the dice without specifying which roll number it is, we will use the random variable R. Thus $P(R=2)=1/36$, $P(R=3)=2/36$, and so on.

We can express the probability of winning the game as a sum over disjoint events.

$$P(\text{ win }) = P(\text{ win on roll 1 }) + P(\text{ win on roll 2 }) + ...$$
$$= P(R_1 = 7 \text{ or } 11) + \textstyle\sum_{k=2}^{\infty} P(\text{ win on roll } k\text{ })$$

The probability of winning on rolls 2 and up depends on the result on roll 1, so we partition over results on roll 1. We only need to consider results 4, 5, 6, 8, 9, and 10 on roll 1, because any other number results in a win or a lose, and in those cases the probability of a win on a later roll is zero.

$$= P(R_1 = 7 \text{ or } 11) + \textstyle\sum_{r=4,5,6,8,9,10} \sum_{k=2}^{\infty} P(\text{ win on roll } k \mid R_1 = r\text{ })P(R_1 = r)$$
$$= P(R_1 = 7 \text{ or } 11) + \textstyle\sum_{r=4,5,6,8,9,10} P(R_1 = r) \sum_{k=2}^{\infty} P(\text{ win on roll } k \mid R_1 = r\text{ })$$

Now we evaluate $P(\text{ win on roll } k \mid R_1 = r\text{ })$. This equals the probability of not winning or losing on trials $2, ..., k-1$, and then winning on trial k.

$$= P(R_1 = 7 \text{ or } 11) + \textstyle\sum_{r=4,5,6,8,9,10} P(R_1 = r) \sum_{k=2}^{\infty} P(R \neq 7 \text{ or } r)^{k-2} P(R = r)$$
$$= P(R_1 = 7 \text{ or } 11) + \textstyle\sum_{r=4,5,6,8,9,10} P(R_1 = r) \sum_{k=0}^{\infty} P(R \neq 7 \text{ or } r)^{k} P(R = r)$$

Recall that an infinite geometric series can be evaluated as $\sum_{k=0}^{\infty} ar^k = a/(1-r)$. This gives

$$= P(R_1 = 7 \text{ or } 11) + \textstyle\sum_{r=4,5,6,8,9,10} P(R_1 = r)P(R = r)/(1 - P(R \neq 7 \text{ or } r))$$
$$= P(R_1 = 7 \text{ or } 11) + \textstyle\sum_{r=4,5,6,8,9,10} P(R = r)^2/(1 - P(R \neq 7 \text{ or } r))$$

See `solutions.R` for an illustration of how to compute this probability in R. It evaluates to 0.4929, which is just slightly below chance.