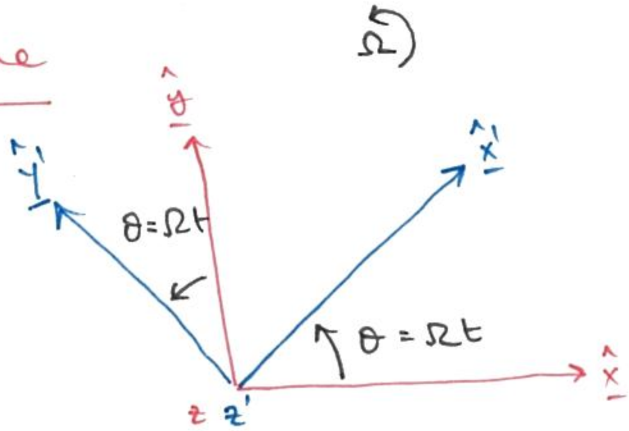
Rotating Reference Frame

fixed axis (x, y, z)

rotating axis (x', y', z')



Express rotating basis vectors in terms of the fixed basis vectors

$$\hat{\underline{x}}' = \cos\Omega t \, \hat{\underline{x}} + \sin\Omega t \, \hat{\underline{y}}$$

$$\hat{\underline{y}}' = -\sin\Omega t \, \hat{\underline{x}} + \cos\Omega t \, \hat{\underline{y}}$$

$$\hat{\underline{z}}' = \hat{\underline{z}}$$

Express rates of change of $\hat{\underline{x}}', \hat{\underline{y}}'$ as measured in fixed frame

$$\left.\frac{d}{dt}\hat{\underline{x}}'\right|_0 = -\Omega \sin\Omega t \, \hat{\underline{x}} + \Omega \cos\Omega t \, \hat{\underline{y}}$$

$$\left.\frac{d}{dt}\hat{\underline{y}}'\right|_0 = -\Omega \cos\Omega t \, \hat{\underline{x}} - \Omega \sin\Omega t \, \hat{\underline{y}}$$

$$\left.\frac{d}{dt}\hat{\underline{z}}'\right|_0 = 0$$

"as measured in the fixed frame"

This gives

$$\left.\frac{d}{dt}\hat{\underline{x}}'\right|_0 = \Omega \hat{\underline{y}}' = \underline{\Omega} \times \hat{\underline{x}}'$$

$$\left.\frac{d}{dt}\hat{\underline{y}}'\right|_0 = -\Omega \hat{\underline{x}}' = \underline{\Omega} \times \hat{\underline{y}}'$$

simple circular motion.

Using $\underline{\Omega} = \Omega \hat{\underline{z}} = \Omega \hat{\underline{z}}'$ & $\hat{\underline{z}}' \times \hat{\underline{x}}' = \hat{\underline{y}}'$, $\hat{\underline{z}}' \times \hat{\underline{y}}' = -\hat{\underline{x}}'$

Kinematics in a rotating frame

Let $\underline{a} = \underline{r}(t) = r_1(t)\,\hat{\underline{x}}' + r_2(t)\,\hat{\underline{y}}' + r_3(t)\,\hat{\underline{z}}'$

$=$ particle trajectory in a rotating frame

Def : relative velocity = velocity as measured in rotating frame.

$$\underline{v}_R = \left.\frac{d\underline{r}}{dt}\right|_R = \dot{r}_1\,\hat{\underline{x}}' + \dot{r}_2\,\hat{\underline{y}}' + \dot{r}_3\,\hat{\underline{z}}'$$

Def : absolute velocity = velocity measured in fixed frame.

$$\underline{v}_0 = \left.\frac{d\underline{r}}{dt}\right|_0 = \left.\frac{d\underline{r}}{dt}\right|_R + \underline{\Omega} \times \underline{r}$$

$$\Rightarrow \boxed{\underline{v}_0 = \underline{v}_R + \underbrace{\underline{\Omega} \times \underline{r}}_{\text{apparent extra velocity due to rotation.}}}$$

Acceleration

$$\left.\frac{d\underline{v}_0}{dt}\right|_0 = \left.\frac{d\underline{v}_0}{dt}\right|_R + \underline{\Omega} \times \underline{v}_0$$

$$= \left.\frac{d}{dt}\left(\underline{v}_R + \underline{\Omega} \times \underline{r}\right)\right|_R + \underline{\Omega} \times \left(\underline{v}_R + \underline{\Omega} \times \underline{r}\right)$$

$$= \left.\frac{d\underline{v}_R}{dt}\right|_R + 2\,\underline{\Omega} \times \underline{v}_R + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) .$$

Rearrange:

$$\underbrace{\left.\frac{d\underline{v}_R}{dt}\right|_R}_{\substack{\text{acceleration in}\\ \text{rotating frame}}} = \underbrace{\left.\frac{d\underline{v}_0}{dt}\right|_0}_{\substack{\text{force/mass}\\ \text{in fixed frame}}} \underbrace{- 2\,\underline{\Omega} \times \underline{v}_R}_{\substack{\text{Coriolis}\\ \text{force}}} \underbrace{- \underline{\Omega} \times (\underline{\Omega} \times \underline{r})}_{\substack{\text{Centrifugal}\\ \text{force}}} .$$

Coriolis force (per unit mass) $\quad -2\,\underline{\Omega} \times \underline{v}_R$

* only occurs when $\underline{v}_R \neq 0$
* $\perp^{ar}$ to both $\underline{\Omega}$ and $\underline{v}_R$.
* Work $= \int \underline{F} \cdot d\underline{r} = \int \underline{F} \cdot \underline{v}\, dt \sim \int \underline{\Omega} \times \underline{v} \cdot \underline{v}\, dt = 0$

Centrifugal force (per unit mass) $\quad -\underline{\Omega} \times (\underline{\Omega} \times \underline{r})$

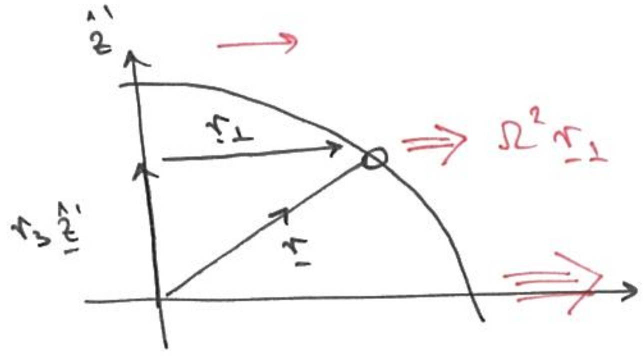
$$= -\Omega^2\, \hat{\underline{z}}' \times (\hat{\underline{z}}' \times \underline{r}).$$

$$= -\Omega^2\, \hat{\underline{z}}' \times \left(\hat{\underline{z}}' \times \left\{ \underbrace{r_1 \hat{\underline{x}}' + r_2 \hat{\underline{y}}'}_{\underline{r}_\perp} + r_3 \hat{\underline{z}}' \right\}\right)$$

$$= -\Omega^2\, \hat{\underline{z}} \times (\hat{\underline{z}}' \times \underline{r}_\perp)$$

$$= -\Omega^2 (-r_1 \hat{\underline{x}}' - r_2 \hat{\underline{y}}') = \Omega^2 \underline{r}_\perp$$

* Centrifugal force is always OUTWARDS from rotation axis
* increases with distance from axis.



* Think of Centrifugal force as an "anti-gravity".

$$\underline{f}_{grav} = -g\hat{\underline{r}} = -\underline{\nabla}(gr)$$

$$\underline{f}_{cent} = \Omega^2 \underline{r}_\perp = \underline{\nabla}\left(\tfrac{1}{2}\Omega^2 r_\perp^2\right)$$

(Ex: prove these)

$$\underline{f}_{grav} + \underline{f}_{cent} = -\underline{\nabla}\left(gr - \tfrac{1}{2}\Omega^2 r_\perp^2\right)$$

Tangent plane approximation.

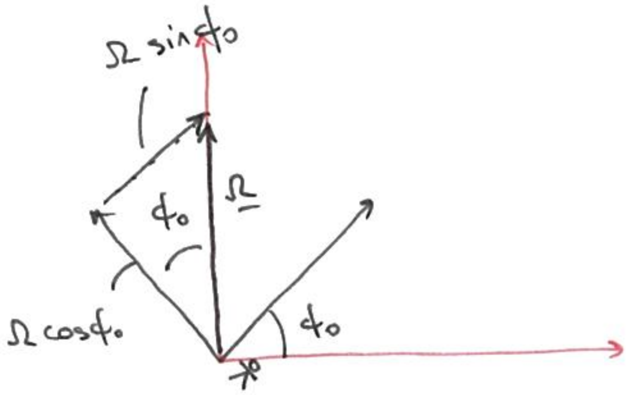
$\underline{v} = u\,\hat{\underline{x}}' + v\,\hat{\underline{y}}' + w\,\hat{\underline{z}}'$

$\underline{\Omega} = \Omega\cos\phi_0 \cdot \hat{\underline{y}}' + \Omega\sin\phi_0\,\hat{\underline{z}}'$

Coriolis force / mass

$$-2\,\underline{\Omega} \times \underline{v}$$

$$= \begin{vmatrix} \hat{\underline{x}}' & \hat{\underline{y}}' & \hat{\underline{z}}' \\ 0 & -2\Omega\cos\phi_0 & -2\Omega\sin\phi_0 \\ u & v & w \end{vmatrix}$$

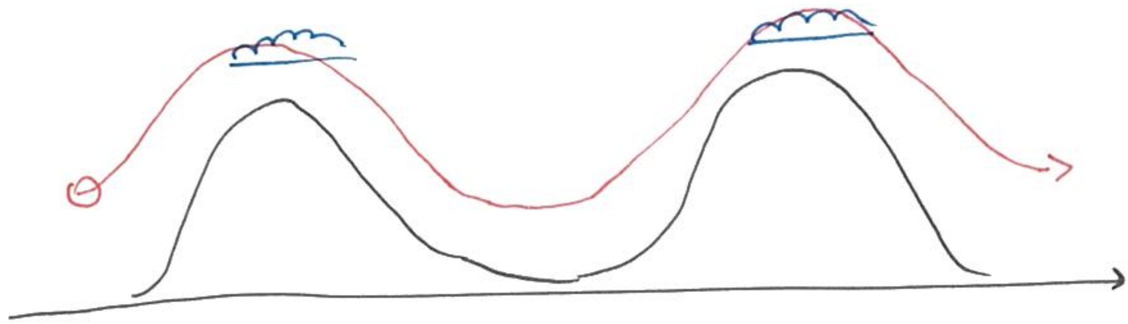


$$= \begin{bmatrix} -2\Omega w\cos\phi_0 + 2\Omega v\sin\phi_0 \\ 2\Omega u\sin\phi_0 \\ -2\Omega u\cos\phi_0 \end{bmatrix} = \begin{bmatrix} -w f_* + v f \\ u f \\ -u f_* \end{bmatrix}$$

$$f = 2\Omega\sin\phi_0 \qquad f_* = 2\Omega\cos\phi_0.$$

ADVECTIVE DERIVATIVE

(a.k.a. material derivative, Lagrangian derivative ...).



$C(x,y,z)$ = "cloudiness"

Even though $\left.\frac{\partial C}{\partial t}\right|_{\text{fixed point}} = 0$ everywhere, $\left.\frac{\partial C}{\partial t}\right|_{\text{particle}} \neq 0$

Change in cloudiness:

$$\delta C = \underbrace{\frac{\partial C}{\partial t}\delta t}_{=0\text{ here}} + \frac{\partial C}{\partial x}\delta x + \frac{\partial C}{\partial y}\delta y + \frac{\partial C}{\partial z}\delta z$$

In the limit $\delta t \to 0$

$$\lim_{\delta t \to 0} \frac{\delta C}{\delta t} = \left.\frac{\partial C}{\partial t}\right|_{\text{particle}} = \frac{\partial C}{\partial t} + \underbrace{\frac{\partial C}{\partial x}\frac{dx}{dt} + \frac{\partial C}{\partial y}\frac{dy}{dt} + \frac{\partial C}{\partial z}\frac{dz}{dt}}_{\text{evaluated at fixed } \underline{x}}$$

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}u + \frac{\partial C}{\partial y}v + \frac{\partial C}{\partial z}w$$

$$\boxed{\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \underline{v} \cdot \underline{\nabla} C}$$

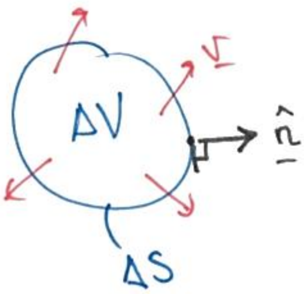
Lagrangian derivative

Eulerian derivative

"Parcels of fluid" have infinitessimal volume but can be squeezed and deformed

ΔV = volume of fluid parcel (limit $\Delta V \to 0$).

Consider



$$\frac{D}{Dt}\Delta V = \frac{D}{Dt}\int_{\Delta V} dV$$

$$= \int_{\Delta S} v_n \, dS$$

$$= \int_{\Delta S} \underline{v}\cdot\hat{\underline{n}}\, dS \overset{\text{divergence}}{\underset{\text{theorem}}{=}} \int_{\Delta V} \underline{\nabla}\cdot\underline{v}\, dV$$

Since ΔV is arbitrary

$$\Rightarrow \quad \frac{D}{Dt}\Delta V = \underline{\nabla}\cdot\underline{v}\,\Delta V \qquad (\text{in limit } \Delta V \to 0).$$

* "incompressible flow" $\underline{\nabla}\cdot\underline{v} = 0 \Rightarrow \Delta V = \text{constant}$

Continuity equation

Conservation of mass : $\frac{D}{Dt}(\underbrace{\rho \Delta V}_{\text{mass = density × volume}}) = 0$

$$\frac{D}{Dt}(\rho\Delta V) = \frac{D\rho}{Dt}\Delta V + \rho\frac{D\Delta V}{Dt} = \frac{D\rho}{Dt}\Delta V + \rho\underline{\nabla}\cdot\underline{v}\,\Delta V$$

$$= \left(\frac{D\rho}{Dt} + \rho\underline{\nabla}\cdot\underline{v}\right)\Delta V = 0$$

$$\Rightarrow \quad \frac{D\rho}{Dt} + \rho\underline{\nabla}\cdot\underline{v} = 0$$

$$\boxed{\frac{D\rho}{Dt} = -\rho \underline{\nabla} \cdot \underline{v}}$$

Continuity equation (Lagrangian form)

$$\overbrace{\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho} = -\rho \underline{\nabla} \cdot \underline{v}$$

$$\frac{\partial \rho}{\partial t} = -\underline{v} \cdot \underline{\nabla} \rho - \rho \underline{\nabla} \cdot \underline{v}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\underbrace{\rho \underline{v}}_{\text{mass flux}})}$$

Continuity equation (Eulerian form)

Conservation of momentum.

- Newton's 2nd law applied to fluid parcels.

$$\underline{p} = m\underline{v} = \text{momentum}$$

$$\frac{d\underline{p}}{dt} = \Sigma \text{ forces.}$$

For a parcel of fluid:

$$\frac{D}{Dt} \Delta \underline{p} = \Sigma \text{ forces.} \qquad \text{where} \quad \Delta \underline{p} = \Delta m . \underline{v}$$

$$= \cancel{\frac{D}{Dt} \Delta m} \cdot \underline{v} + \Delta m \frac{D\underline{v}}{Dt}$$

$$\Rightarrow \boxed{\frac{D\underline{v}}{Dt} = \frac{\Sigma \text{ forces}}{\Delta m.} = \Sigma \underline{f}}$$

Newton's 2nd law (Lagrangian form)

$$\boxed{\frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v} \cdot \underline{\nabla} \underline{v}}_{\text{"flow moves the flow" (nonlinearity)}} = \Sigma \underline{f}}$$

Newton's 2nd law (Eulerian form).

Continuity equation.

$$\frac{\partial \rho}{\partial t} = - \underline{\nabla} \cdot (\rho \underline{v}). \qquad \text{(Eulerian form)}$$

$$\frac{D\rho}{Dt} = - \rho \underline{\nabla} \cdot \underline{v} \qquad \text{(Lagrangian form)}$$

Using $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla}$

Newton's 2nd law

$$\frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v} \cdot \underline{\nabla}\, \underline{v}}_{\text{"flow moves the flow"}} = \Sigma \underline{f} \leftarrow \frac{\text{forces}}{\text{mass}}. \qquad \text{Eulerian form.}$$

$$\frac{D\underline{v}}{Dt} = \Sigma \underline{f}$$

Forces:

Body forces : gravity

$-\underline{\nabla}\Phi$

centrifugal force } psuedo forces.

coriolis force

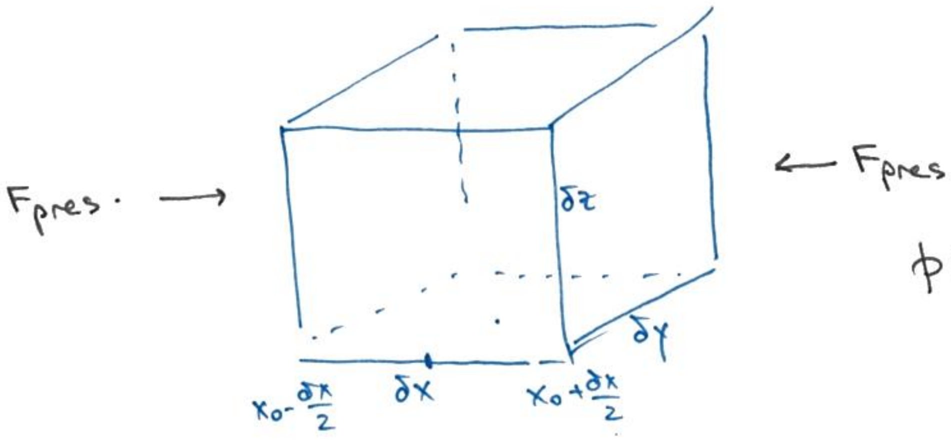
$-2\underline{\Omega} \times \underline{v}$

Contact forces: pressure gradient force.

Viscous stresses. } not usually important

friction forces. } in GFD.

Pressure gradient force.



$p(x, y, z)$

Force = pressure × area.

$$= p\left(x_0 - \frac{\delta x}{2}, y_0, z_0\right)\delta y \delta z - p\left(x_0 + \frac{\delta x}{2}, y_0, z_0\right)\delta y \delta z$$

Taylor expand

$$\simeq \left[\cancel{p(x_0, y_0, z_0)} - \frac{\delta x}{2}\frac{\partial p}{\partial x}(x_0, y_0, z_0)\right]\delta y \delta z + \cdots$$

$$- \left[\cancel{p(x_0, y_0, z_0)} + \frac{\delta x}{2}\frac{\partial p}{\partial x}(x_0, y_0, z_0)\right]\delta y \delta z + \cdots$$

$$\simeq -\delta x \frac{\partial p}{\partial x}(x_0, y_0, z_0)\,\delta y \delta z.$$

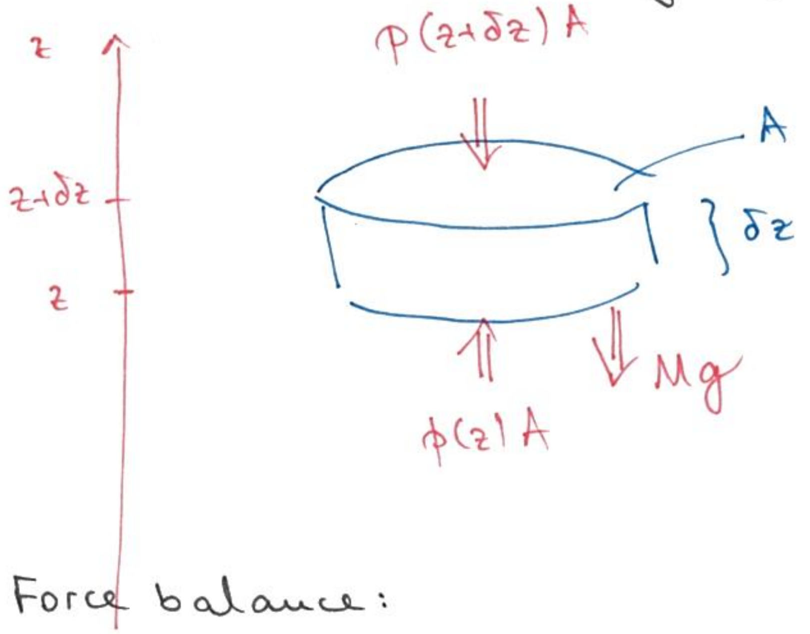
$$f_{pres} = \frac{\text{force}}{\text{mass}} = \frac{-\frac{\partial p}{\partial x}\delta x \delta y \delta z}{\rho\, \delta x \delta y \delta z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Pressure gradient force:

$$\underline{f}_{pres} = -\frac{1}{\rho}\underline{\nabla} p.$$

STATICS: forces in balance

Hydrostatic balance: no flow, pressure gradient force is balanced by gravity



Force balance:

$$p(z)A - p(z+\delta z)A - \rho(z)Ag\delta z = 0.$$

$$\frac{p(z+\delta z) - p(z)}{\delta z} = -\rho(z)g.$$

$\delta z \to 0$: $$\boxed{\frac{dp}{dz} = -\rho g}$$ "hydrostatic balance".

Geostrophic balance:

Horizontal pressure gradient = Coriolis force

Cyclostrophic balance:

Horizontal pressure gradient = Centrifugal force of rotating fluid.