The Shallow Water Model

* primative equations with simplified representation of stratification
* thin layers of constant density with no vertical variation in horizontal velocities:

In each layer : ρ = constant

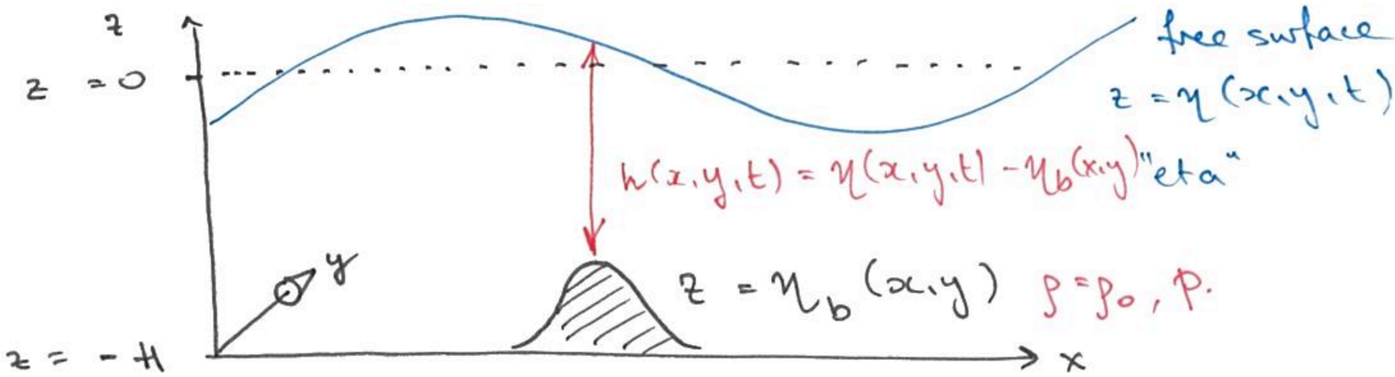
$u = u(x, y, t)$

$v = v(x, y, t)$

$p = p(x, y, z, t)$

* horizontal scales >> depth ("shallow water")

Single-layer SW model



Conservation of Mass

ρ_0 = constant $\Rightarrow$ fluid is incompressible in 3D.

$$\underset{\text{3D velocity}}{\nabla \cdot \underline{v}} = 0 = \underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}} + \frac{\partial w}{\partial z}$$

$$\Rightarrow \boxed{\frac{\partial w}{\partial z} = -\nabla_z \cdot \underline{u}} \qquad \nabla_z = (\partial_x, \partial_y) \quad \underline{u} = (u, v)$$

Integrate from bottom to top:

$$\int_{\eta_b}^{\eta} \frac{\partial w}{\partial z}\, dz = w(\eta) - w(\eta_b)$$

$$= \int_{\eta_b}^{\eta} -\underline{\nabla}_z \cdot \underline{u}\, dz$$

u, v don't depend on z.

$$= -\underline{\nabla}_z \cdot \underline{u} \quad (\eta - \eta_b)$$

$$= -(\underline{\nabla}_z \cdot \underline{u})\, h$$

But position of the fluid at the top and bottom.

$$w(\eta) = \frac{D\eta}{Dt} \qquad w(\eta_b) = \frac{D\eta_b}{Dt} = \underline{u} \cdot \underline{\nabla}_z \eta_b.$$

$$w(\eta) - w(\eta_b) = \frac{D}{Dt}\underbrace{(\eta - \eta_b)}_{h} = -(\underline{\nabla}_z \cdot \underline{u})\underbrace{(\eta - \eta_b)}_{h}$$

$$\Rightarrow \boxed{\frac{Dh}{Dt} = -\underline{\nabla}_z \cdot \underline{u}\, h.}$$

Conservation of mass.

compare with 3D conservation of mass:

$$\frac{D\rho}{Dt} = -\underline{\nabla} \cdot \underline{v} \cdot \rho \qquad \underline{v} \longrightarrow \underline{u} \qquad \rho \longrightarrow h$$

Lagrangian form : $\dfrac{Dh}{Dt} = -h\, \underline{\nabla}_z \cdot \underline{u}$

Eulerian form : $\dfrac{\partial h}{\partial t} + = -\underline{\nabla}_z \cdot (\underline{u} h).$

Pressure from hydrostatic balance

$$\frac{\partial p}{\partial z} = -\rho_0 g \qquad \rho_0 = \text{constant}$$

Integrate:

$$\int_{p(z)}^{p_0} \frac{\partial p}{\partial z}\, dz = -\int_z^{\eta} \rho_0 g\, dz \qquad z < 0$$

$$p_0 - p(z) = -\rho_0 g (\eta - z)$$

$$p(z) = \underbrace{p_0}_{\text{surface pressure}} + \underbrace{\rho_0 g (\eta - z)}_{\text{mass of fluid above depth } z}$$

Pressure gradient force:

$$-\frac{1}{\rho_0} \underline{\nabla}_z p = -g \underline{\nabla}_z \eta .$$

Momentum balance:

$$\frac{D\underline{u}}{Dt} + \underline{f} \times \underline{u} = -\frac{1}{\rho_0} \underline{\nabla}_z p = -g \underline{\nabla}_z \eta .$$

$$\uparrow$$

$$\frac{\partial \underline{u}}{\partial t} + u \frac{\partial \underline{u}}{\partial x} + v \frac{\partial \underline{u}}{\partial y} \qquad \left(w \frac{\partial \underline{u}}{\partial z} = 0 \right).$$

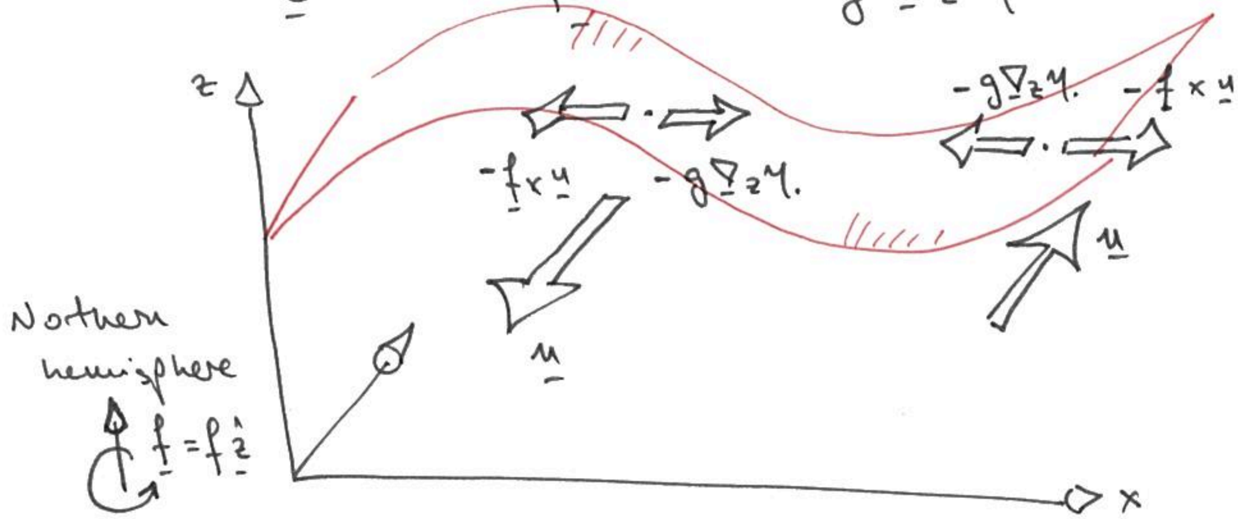
Geostrophic balance in the SW model.

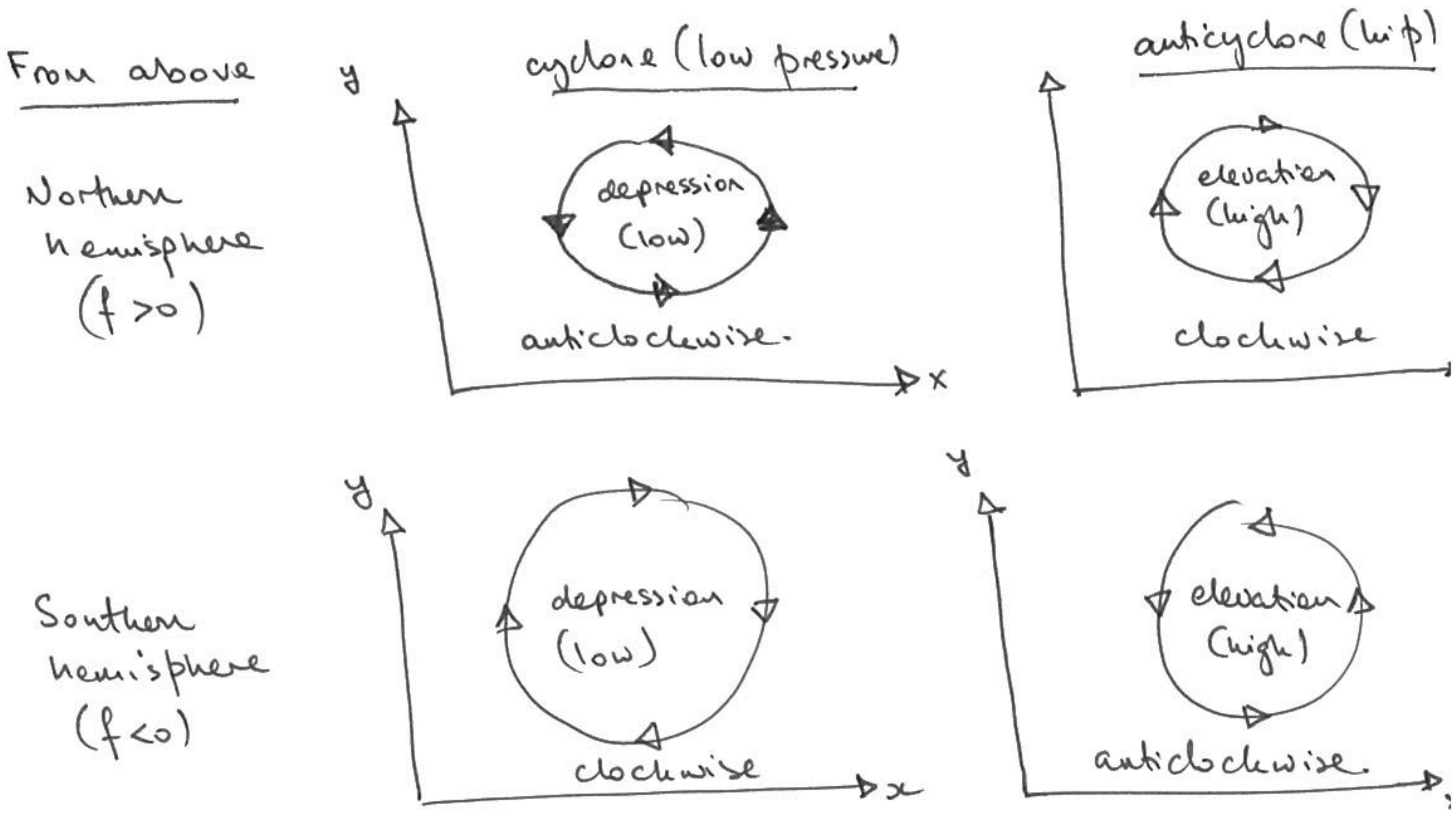
$Ro \ll 1 \Rightarrow$ neglect inertial terms.

$$\underline{f} \times \underline{u} = -g \underline{\nabla}_z \eta.$$

Rewrite this as a force balance.

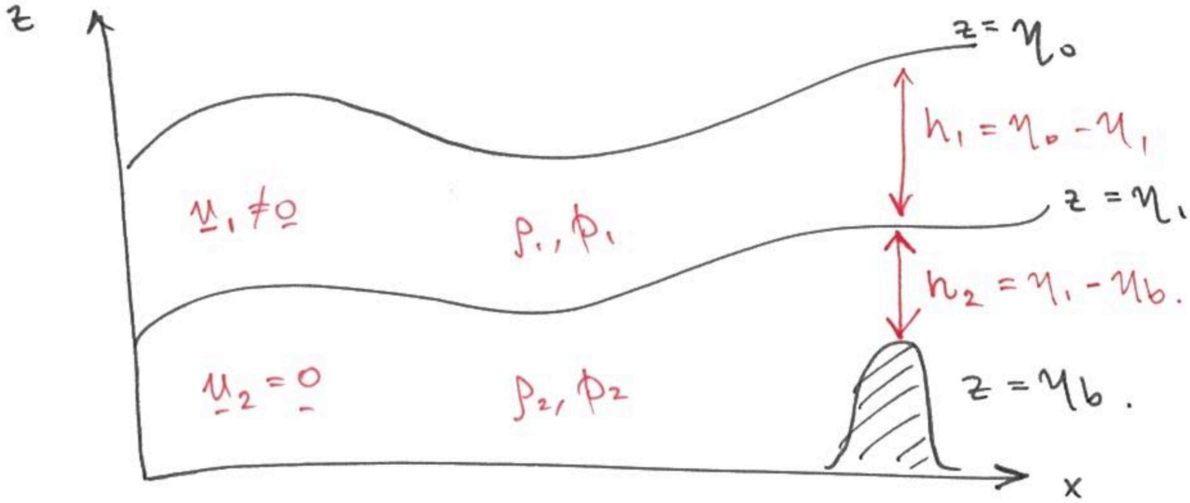
$$\underline{0} = -\underline{f} \times \underline{u} - g \underline{\nabla}_z \eta .$$





$1\frac{1}{2}$ layer shallow water model

* "active layer" ($\underline{u} \neq 0$) on top of "quiescent layer" ($\underline{u} = 0$).



Continuity equation in upper layer:

$$\frac{Dh_1}{Dt} + h_1 \underline{\nabla}_z \cdot \underline{u} = 0$$

Pressure gradient:

In layer 1:

$$p_1(z) = p_0 + \rho_1 g (\eta_0 - z). \qquad \eta_0 \leq z \leq \eta_1.$$

In layer 2:

$$p_2(z) = p_0 + \rho_1 g (\eta_0 - \eta_1) + \rho_2 g (\eta_1 - z)$$

$$\eta_1 \leq z \leq \eta_b.$$

$$\Rightarrow \underline{f}_{pres,1} = -\frac{1}{\rho_1} \underline{\nabla}_z p_1 = -g \underline{\nabla}_z \eta_0.$$

$$\underline{f}_{pres,2} = -\frac{1}{\rho_2} \underline{\nabla}_z p_2 = -\frac{\rho_1}{\rho_2} g \underline{\nabla}_z (\eta_0 - \eta_1) - g \underline{\nabla}_z \eta_1$$

Momentum equation in each layer:

In layer 1: $\dfrac{D\underline{u}}{Dt} + \underline{f} \times \underline{u} = -g \underline{\nabla}_z \eta_0$

In layer 2: $0 + 0 = -\dfrac{\rho_1}{\rho_2} g \underline{\nabla}_z (\eta_0 - \eta_1) - g \underline{\nabla}_z \eta_1$

$\Rightarrow \quad \underline{\nabla}_z \eta_0 = -\dfrac{\rho_2}{\rho_1} \underline{\nabla}_z \eta_1 + \underline{\nabla}_z \eta_1 = -\dfrac{\rho_2 - \rho_1}{\rho_1} \underline{\nabla}_z \eta_1$

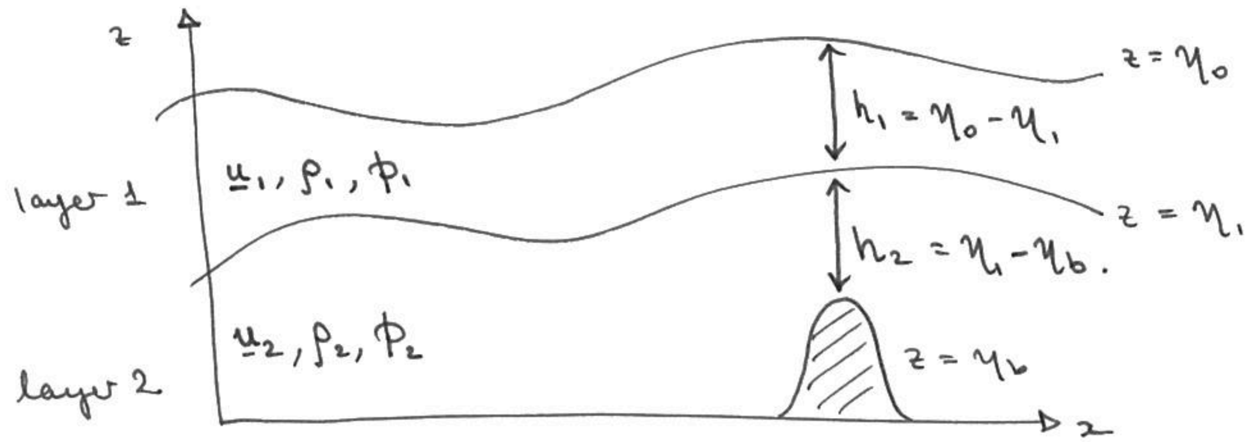
$\Rightarrow \quad \dfrac{D\underline{u}}{Dt} + \underline{f} \times \underline{u} = g \dfrac{\rho_2 - \rho_1}{\rho_1} \underline{\nabla}_z \eta_1 \qquad \rho_2 > \rho_1$

$= g' \underline{\nabla}_z \eta_1 \qquad g' = g \dfrac{\rho_2 - \rho_1}{\rho_1} \ll g$

reduced gravity.

$\Rightarrow \quad |\underline{\nabla}_z \eta_1| \gg |\underline{\nabla}_z \eta_0|.$

TWO LAYER SHALLOW WATER MODEL



Continuity eqns for each layer:

$$\frac{D_1 h_1}{Dt} + h_1 \underline{\nabla}_z \cdot \underline{u}_1 = 0 \qquad \frac{D_1}{Dt} = \frac{\partial}{\partial t} + \underline{u}_1 \cdot \underline{\nabla}_z$$

$$\frac{D_2 h_2}{Dt} + h_2 \underline{\nabla}_z \cdot \underline{u}_2 = 0. \qquad \frac{D_2}{Dt} = \frac{\partial}{\partial t} + \underline{u}_2 \cdot \underline{\nabla}_z$$

Pressure gradient forces:

$$p_1 = p_0 + \rho_1 g(\eta_0 - z) \qquad \eta_0 \geq z \geq \eta_1$$

$$p_2 = p_0 + \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z) \qquad \eta_1 \geq z \geq \eta_b$$

$$\Rightarrow \quad \underline{f}_{pres,1} = -g \underline{\nabla} \eta_0$$

$$\underline{f}_{pres,2} = -g \frac{\rho_1}{\rho_2} \underline{\nabla}_z (\eta_0 - \eta_1) - g \underline{\nabla}_z \eta_1$$

$$= \frac{\rho_1}{\rho_2} \left(-g \underline{\nabla}_z \eta_0 - g' \underline{\nabla}_z \eta_1 \right) \qquad \frac{\rho_1}{\rho_2} \simeq 1$$

reduced gravity

in Boussinesq approx.

$$\simeq -g \underline{\nabla}_z \eta_0 - g' \underline{\nabla}_z \eta_1$$

Geostrophic balance in each layer ($Ro \ll 1$).

$$\underline{f} \times \underline{u}_1 = -g \underline{\nabla}_z \eta_0 \qquad (1)$$

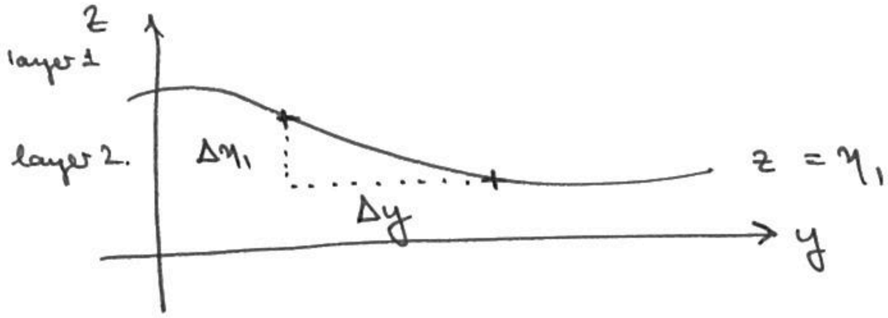
$$\underline{f} \times \underline{u}_2 = -g \underline{\nabla}_z \eta_0 - g' \underline{\nabla}_z \eta_1 \qquad (2)$$

Thermal wind balance in SW system:

(1) - (2)

$$\underline{f} \times (\underline{u}_1 - \underline{u}_2) = g' \underline{\nabla}_z \eta_1 \qquad \underline{f} = f_0 \hat{\underline{z}}$$

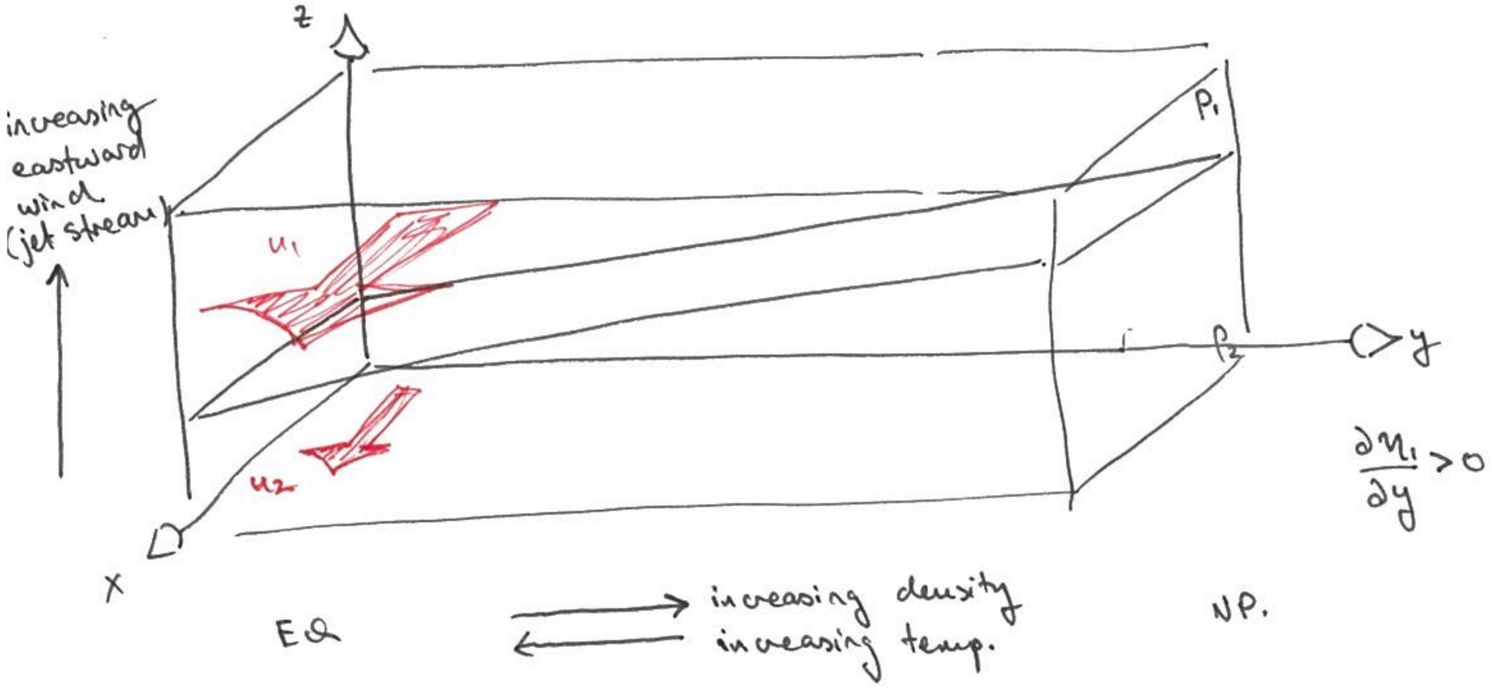
$$u_1 - u_2 = \frac{g'}{f_0} \frac{\partial \eta_1}{\partial y} \qquad v_1 - v_2 = -\frac{g'}{f_0} \frac{\partial \eta_1}{\partial x}$$



$$\frac{\partial \eta_1}{\partial y} \simeq \frac{\Delta \eta_1}{\Delta y}. \Rightarrow$$ horizontal gradients in pressure (height)

$\Updownarrow$

sloping interfaces.



Rearrange :

$$\underbrace{\frac{\partial \underline{u}}{\partial t}}_{①} + \underbrace{\underline{\omega} \times \underline{u}}_{②} + \underbrace{\underline{f} \times \underline{u}}_{③} = \underbrace{-\underline{\nabla}\left(g\eta + \tfrac{1}{2}\,\underline{u}\cdot\underline{u}\right)}_{④}.$$

Take curl term-by-term :

① $\underline{\nabla} \times \dfrac{\partial \underline{u}}{\partial t} = \dfrac{\partial}{\partial t}\underline{\nabla}\times\underline{u} = \dfrac{\partial \underline{\omega}}{\partial t} = \dfrac{\partial \zeta}{\partial t}\,\hat{\underline{z}}$

② $\underline{\nabla} \times (\underline{\omega}\times\underline{u}) \overset{\#3}{=} \underline{\omega}\,(\underline{\nabla}\cdot\underline{u}) + (\underline{u}\cdot\underline{\nabla})\,\underline{\omega}$

$$- \underline{u}\,(\cancelto{0}{\underline{\nabla}\cdot\underline{\omega}}) - \underbrace{(\underline{\omega}\cdot\underline{\nabla})\,\underline{u}}$$

#1. $\underline{\nabla}\cdot\underline{\omega} = \underline{\nabla}\cdot(\underline{\nabla}\times\underline{u})$
$= 0.$

$\zeta \dfrac{\partial \underline{u}}{\partial z} = 0$

$\underline{u}$ has no z-dependence.

③ $\underline{\nabla} \times (\underline{f}\times\underline{u}) \overset{\#3}{=} \underline{f}\,(\underline{\nabla}\cdot\underline{u}) + (\underline{u}\cdot\underline{\nabla})\,\underline{f}$

$$- \underline{u}\,\underbrace{(\cancelto{0}{\underline{\nabla}\cdot\underline{f}})} - \underbrace{(\cancelto{0}{\underline{f}\cdot\underline{\nabla}})}\,\underline{u}$$

$\underline{\nabla}\cdot\underline{f} = \dfrac{\partial}{\partial z} f(y) = 0 \qquad f\dfrac{\partial}{\partial z}\underline{u} = 0$

④ $\underline{\nabla} \times \underline{\nabla}\left(g\eta + \tfrac{1}{2}\,\underline{u}\cdot\underline{u}\right) = 0$ (Identity #4 $\underline{\nabla}\times\underline{\nabla}\phi = 0$)

Thus

$$\underbrace{\frac{\partial}{\partial t}\underline{\omega}}_{①} + \underbrace{\underline{\omega}\,(\underline{\nabla}\cdot\underline{u}) + \underline{u}\cdot\underline{\nabla}\,\underline{\omega}}_{②} + \underbrace{\underline{f}\,(\underline{\nabla}\cdot\underline{u}) + (\underline{u}\cdot\underline{\nabla})\,\underline{f}}_{③} = \underbrace{\underline{0}}_{④}$$

$$\frac{\partial}{\partial t}\zeta\,\hat{\underline{z}} + \zeta\,(\underline{\nabla}\cdot\underline{u})\,\hat{\underline{z}} + (\underline{u}\cdot\underline{\nabla}\,\zeta)\,\hat{\underline{z}} + f(\underline{\nabla}\cdot\underline{u})\,\hat{\underline{z}} + (\underline{u}\cdot\underline{\nabla}\,f)\,\hat{\underline{z}} = \underline{0}$$

Take z-component :

$$\frac{\partial}{\partial t}(\zeta + f) + (\underline{u}\cdot\underline{\nabla})(\zeta + f) + (\zeta + f)(\underline{\nabla}\cdot\underline{u}) = 0$$

$\dfrac{\partial f}{\partial t} = 0$

Vorticity & Potential Vorticity in 1-layer SW model

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u} + \underline{f}\times\underline{u} = -g\,\underline{\nabla}_z \eta.$$

$$\frac{\partial h}{\partial t} + \underline{u}\cdot\underline{\nabla} h + h\,\underline{\nabla}\cdot\underline{u} = 0$$

$$\underline{u} = (u, v, 0) \qquad \underline{\nabla}_z = (\partial_x, \partial_z, 0)$$

$$\eta = \eta(x,y,t) \qquad u = u(x,y,t) \qquad v = v(x,y,t)$$

$$\underline{f} = f\,\hat{\underline{z}} = (f_0 + \beta y)\,\hat{\underline{z}}$$

SW vorticity.

$$\underline{\omega} = \underline{\nabla}\times\underline{u} = \begin{pmatrix} \cancel{w_y} - \cancel{v_z} \\ \cancel{u_z} - \cancel{w_x} \\ v_x - u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_x - u_y \end{pmatrix} = \zeta\,\hat{\underline{z}}$$

"omega" 3D 2D $(u,v,0)$ "zeta"

$\zeta = v_x - u_y = \hat{\underline{z}}\cdot\underline{\omega}$ = vertical component of vorticity.

Some vector calculus identities:

1. $\underline{\nabla}\times(\underline{\nabla}\phi) = 0$
2. $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{A}) = 0$
3. $\underline{A}\times(\underline{\nabla}\times\underline{A}) = \frac{1}{2}\underline{\nabla}(\underline{A}\cdot\underline{A}) - \underline{A}\cdot\underline{\nabla}\,\underline{A}$
4. $\underline{\nabla}\times(\underline{A}\times\underline{B}) = \underline{A}(\underline{\nabla}\cdot\underline{B}) + (\underline{B}\cdot\underline{\nabla})\underline{A} - \underline{B}(\underline{\nabla}\cdot\underline{A}) - (\underline{A}\cdot\underline{\nabla})\underline{B}$

Example : $\underline{u}\cdot\underline{\nabla}\,\underline{u} = u\frac{\partial \underline{u}}{\partial x} + v\frac{\partial \underline{u}}{\partial y} + \cancelto{0}{w\frac{\partial \underline{u}}{\partial z}}$

#3 with $\underline{A} = \underline{u}$:

$$\underline{u}\cdot\underline{\nabla}\,\underline{u} = \frac{1}{2}\underline{\nabla}\,\underline{u}\cdot\underline{u} - \underline{u}\times(\underline{\nabla}\times\underline{u})$$

$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} - \underline{u}\times\underline{\omega} + \frac{1}{2}\underline{\nabla}(\underline{u}\cdot\underline{u}) + \underline{f}\times\underline{u} = -g\underline{\nabla}_z\eta.$$

Rearrange:

$$\underbrace{\frac{\partial \underline{u}}{\partial t}}_{①} + \underbrace{\underline{\omega}\times\underline{u}}_{②} + \underbrace{\underline{f}\times\underline{u}}_{③} = -\underbrace{\underline{\nabla}_z\left(g\eta + \frac{1}{2}\underline{u}\cdot\underline{u}\right)}_{④}$$

Take curl term-by-term:

① $$\underline{\nabla}\times\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}\underline{\nabla}\times\underline{u} = \frac{\partial \underline{\omega}}{\partial t}$$

② $$\underline{\nabla}\times(\underline{\omega}\times\underline{u}) \overset{\#4}{=} \underline{\omega}(\underline{\nabla}\cdot\underline{u}) + (\underline{u}\cdot\underline{\nabla})\underline{\omega} - \underline{u}\cancelto{0}{(\underline{\nabla}\cdot\underline{\omega})} - \cancelto{0}{(\underline{\omega}\cdot\underline{\nabla})\underline{u}}$$

$$\underline{\nabla}\cdot\underline{\omega} = \underline{\nabla}\cdot(\underline{\nabla}\times\underline{u}) = 0 \qquad \underline{\omega}\cdot\underline{\nabla}\,\underline{u} = \zeta\frac{\partial \underline{u}}{\partial z} = 0$$

③ $$\underline{\nabla}\times(\underline{f}\times\underline{u}) \overset{\#4}{=} \underline{f}(\underline{\nabla}\cdot\underline{u}) + (\underline{u}\cdot\underline{\nabla})\underline{f} - \underline{u}\cancelto{0}{(\underline{\nabla}\cdot\underline{f})} - \cancelto{0}{(\underline{f}\cdot\underline{\nabla})\underline{u}}$$

$$\underline{\nabla}\cdot\underline{f} = \frac{\partial f}{\partial z} = 0 \qquad \underline{f}\cdot\underline{\nabla}\,\underline{u} = f\frac{\partial \underline{u}}{\partial z} = 0.$$

④ $$\underline{\nabla}\times(\underline{\nabla}(\cdots)) \overset{\#1}{=} 0.$$

Thus: $$\overbrace{\frac{\partial \underline{\omega}}{\partial t}}^{①} + \overbrace{\underline{u}\cdot\underline{\nabla}\,\underline{\omega} + \underline{\omega}(\underline{\nabla}\cdot\underline{u})}^{②} + \overbrace{\underline{u}\cdot\underline{\nabla}\,\underline{f} + \underline{f}(\underline{\nabla}\cdot\underline{u})}^{③} = \overbrace{0}^{④}$$

$$\frac{\partial \zeta}{\partial t}\hat{\underline{z}} + (\underline{u}\cdot\underline{\nabla})\zeta\hat{\underline{z}} + \zeta\hat{\underline{z}}(\underline{\nabla}\cdot\underline{u}) + (\underline{u}\cdot\underline{\nabla})f\hat{\underline{z}} + f\hat{\underline{z}}(\underline{\nabla}\cdot\underline{u}) = 0$$

Take z-component

$$\frac{\partial \zeta}{\partial t} + (\underline{u}\cdot\underline{\nabla})(\zeta + f) + (\zeta + f)\underline{\nabla}\cdot\underline{u} = 0$$

Write as : $q = \zeta + f$ = absolute vorticity

ζ = relative vorticity

f = planetary vorticity

The SW system can be rewritten as

$$\frac{\partial q}{\partial t} + (\underline{u} \cdot \underline{\nabla}) q + q(\underline{\nabla} \cdot \underline{u}) = 0. \qquad \left(\frac{\partial q}{\partial t} = \frac{\partial \zeta}{\partial t} \right.$$

$$\frac{\partial h}{\partial t} + (\underline{u} \cdot \underline{\nabla}) h + h(\underline{\nabla} \cdot \underline{u}) = 0. \qquad \left. \frac{\partial f}{\partial t} = 0 \right)$$

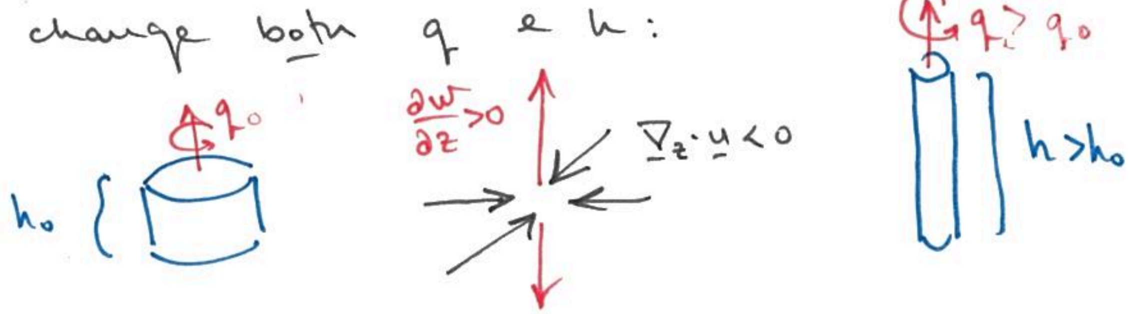
OR

$$\frac{Dq}{Dt} = -q(\underline{\nabla} \cdot \underline{u}) \quad , \quad \frac{Dh}{Dt} = -h(\underline{\nabla} \cdot \underline{u})$$

* For horizontally non-divergent flow $(\underline{\nabla} \cdot \underline{u} = 0)$ both height and vorticity are conserved.

$$\frac{Dq}{Dt} = 0 \qquad \frac{Dh}{Dt} = 0.$$

* Horizontally diverging / converging flows will change both q & h:



* Can change $q = \zeta + f$ by
  - spinning the flow (change ζ)
  - moving the column north / south (change f)

Potential Vorticity in SW system

Combine effects of q & h into a single quantity:

$$Q = \frac{q}{h} = \frac{\zeta + f}{h} = \frac{\text{relative + planetary vorticity}}{\text{height}}$$

$$= \text{"potential vorticity"} \qquad \cancel{[s^{-1}]} \quad !$$

$$\frac{DQ}{Dt} = \frac{D}{Dt}\left(\frac{q}{h}\right) = \frac{1}{h}\frac{Dq}{Dt} + q\frac{D}{Dt}\left(\frac{1}{h}\right)$$

$$= \frac{1}{h}\frac{Dq}{Dt} - \frac{q}{h^2}\frac{Dh}{Dt}$$

$$= \frac{1}{h}\left(-q\,\underline{\nabla}_z\cdot\underline{u}\right) - \frac{q}{h^2}\left(-h\,\underline{\nabla}_z\cdot\underline{u}\right)$$

$$= -\frac{q}{h}\underline{\nabla}_z\cdot\underline{u} + \frac{q}{h}\underline{\nabla}_z\cdot\underline{u} = 0.$$

$$\boxed{\frac{DQ}{Dt} = 0 \qquad Q = \frac{\zeta + f}{h}}$$

Conservation of potential vorticity (PV)

Q is conserved following the flow.