Case 1 : Non-rotating case ($f = 0$) : SW gravity waves

1. Base state

only self-consistent base state is $\bar{h}$ = constant (no rotation ⇒ cannot balance pressure gradient). Can still have a base state zonal flow $\bar{u}$. But for now let's set this to zero.

$$\boxed{\bar{u} = \bar{v} = 0, \quad \bar{h} = H = \text{constant}}$$

2. Small perturbations

$$u = \bar{u} + \tilde{u} = \tilde{u}(x, y, t)$$
$$v = \bar{v} + \tilde{v} = \tilde{v}(x, y, t)$$
$$h = \bar{h} + \tilde{h} = H + \tilde{h}(x, y, t)$$

Sub into equations of motion ($f = 0$) and neglect nonlinear terms:

$$\frac{\partial \tilde{u}}{\partial t} + \cancel{\tilde{u}\frac{\partial \tilde{u}}{\partial x}} + \cancel{\tilde{v}\frac{\partial \tilde{u}}{\partial y}} = -g\frac{\partial \tilde{h}}{\partial x}$$
$$\frac{\partial \tilde{v}}{\partial t} + \cancel{\tilde{u}\frac{\partial \tilde{v}}{\partial x}} + \cancel{\tilde{v}\frac{\partial \tilde{v}}{\partial y}} = -g\frac{\partial \tilde{h}}{\partial y}$$
$$\frac{\partial \tilde{h}}{\partial t} + \cancel{\tilde{u}\frac{\partial \tilde{h}}{\partial x}} + \cancel{\tilde{v}\frac{\partial \tilde{h}}{\partial y}} + H\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) + \cancel{\tilde{h}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)} = 0$$

Neglect nonlinear terms :

$$\boxed{\begin{aligned} \frac{\partial \tilde{u}}{\partial t} + g\frac{\partial \tilde{h}}{\partial x} &= 0 \\ \frac{\partial \tilde{v}}{\partial t} + g\frac{\partial \tilde{h}}{\partial y} &= 0 \\ \frac{\partial \tilde{h}}{\partial t} + H\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) &= 0 \end{aligned}}$$

3. Look for plane-wave solutions

$$\tilde{u} = \hat{u}\, e^{i(kx + ly - \omega t)}$$
$$\tilde{v} = \hat{v}\, e^{i(kx + ly - \omega t)}$$
$$\tilde{h} = \hat{h}\, e^{i(kx + ly - \omega t)}$$

$$\frac{\partial}{\partial t} \longrightarrow -i\omega \qquad \frac{\partial}{\partial x} \longrightarrow +ik \qquad \frac{\partial}{\partial y} \longrightarrow +il.$$

$$\left.\begin{aligned} -i\omega \tilde{u} + igk\tilde{h} &= 0 \\ -i\omega \tilde{v} + igl\tilde{h} &= 0 \\ -i\omega \tilde{h} + iHk\tilde{u} + iHl\tilde{v} &= 0. \end{aligned}\right\} (*)$$

Note : can cancel factors of $e^{i(kx + ly - \omega t)}$

$\Rightarrow \quad \tilde{u} \longrightarrow \hat{u} \qquad \tilde{v} \longrightarrow \hat{v} \qquad \tilde{h} \longrightarrow \hat{h}$

$\Rightarrow$ 3×3 algebraic system for $\hat{u}, \hat{v}, \hat{h}$.

4. Look for non-trivial solution

$(*) \times i$:

$$\omega\hat{u} - gk\hat{h} = 0.$$
$$\omega\hat{v} - gl\hat{h} = 0$$
$$\omega\hat{h} - Hk\hat{u} - Hl\hat{v} = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} \omega & 0 & -gk \\ 0 & \omega & -gl \\ -Hk & -Hl & \omega \end{bmatrix}}_{\mathbb{M}} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For nontrivial solutions must have $\det \mathbb{M} = 0$

$$\omega\left(\omega^2 - gHl^2\right) - gk\left(\omega H k\right) = 0$$

$$\omega^3 - \omega g H \underbrace{\left(k^2 + l^2\right)}_{K^2} = 0$$

$$K = |\underline{k}| = \sqrt{k^2 + l^2}$$

$$\omega\left(\omega^2 - gHK^2\right) = 0$$

Three solutions : i) $\omega = 0$

ii) $\omega = \sqrt{gH}\, K$

iii) $\omega = -\sqrt{gH}\, K.$

Solution i) $\omega = 0$

Sub back into matrix equation & solve for $\hat{u}, \hat{v}, \hat{h}$

$$\begin{bmatrix} 0 & 0 & -gk \\ 0 & 0 & -gl \\ -Hk & -Hl & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -gk\hat{h} &= 0 \\ -gl\hat{h} &= 0 \end{aligned}\right\} \Rightarrow \hat{h} = 0$$

(no surface deflection)

$$-Hk\hat{u} - Hl\hat{v} = 0 \rightarrow k\hat{u} + l\hat{v} = 0.$$

Consider the 2D divergence :

$$\underline{\nabla}_2 \cdot \tilde{\underline{u}} = \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = \Big(\underbrace{-ik\hat{u} - il\hat{v}}_{=0.}\Big)\, e^{i(kx + ly - \omega t)}$$

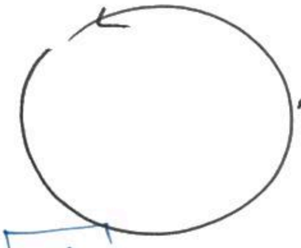
So $\underline{\nabla}_2 \cdot \tilde{\underline{u}} = 0$ for this mode.

"Vortex mode" $(\omega = 0)$

Any solution of the form $\tilde{\underline{u}} = (-\tilde{\psi}_y, \tilde{\psi}_x)$ will satisfy this:

$$\tilde{u} = -\frac{\partial \tilde{\psi}}{\partial y} \qquad \tilde{v} = \frac{\partial \tilde{\psi}}{\partial x}$$

$$\Rightarrow \quad \underline{\nabla}_z \cdot \tilde{\underline{u}} = \frac{\partial}{\partial x}\left(-\frac{\partial \tilde{\psi}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \tilde{\psi}}{\partial x}\right) = 0 .$$



$\tilde{\psi}$ = constant.

Solution ii) $\omega = \sqrt{gH}\, K$

$$\begin{bmatrix} \sqrt{gH}\,K & 0 & -gk \\ 0 & \sqrt{gH}\,K & -gl. \\ -Hk & -Hl & \sqrt{gH}\,K \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \sqrt{gH}\,K\hat{u} - gk\hat{h} &= 0 \\ \sqrt{gH}\,K\hat{v} - gl\hat{h} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \hat{u} &= \frac{k}{K}\sqrt{\frac{g}{H}}\,\hat{h} \\ \hat{v} &= \frac{l}{K}\sqrt{\frac{g}{H}}\,\hat{h} \end{aligned}$$

$$\Big[-Hk\hat{u} - Hl\hat{v} + \sqrt{gH}\,K\hat{h} = 0$$

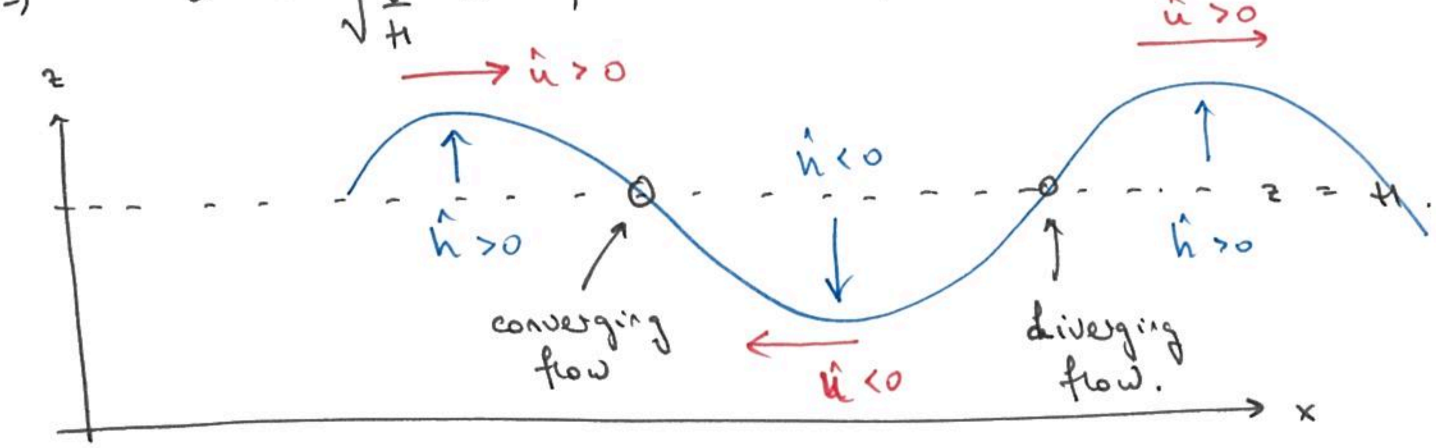
$$\Rightarrow -\sqrt{gH}\,\frac{k^2}{K}\hat{h} - \sqrt{gH}\,\frac{l^2}{K}\hat{h} + \sqrt{gH}\,K\hat{h}$$

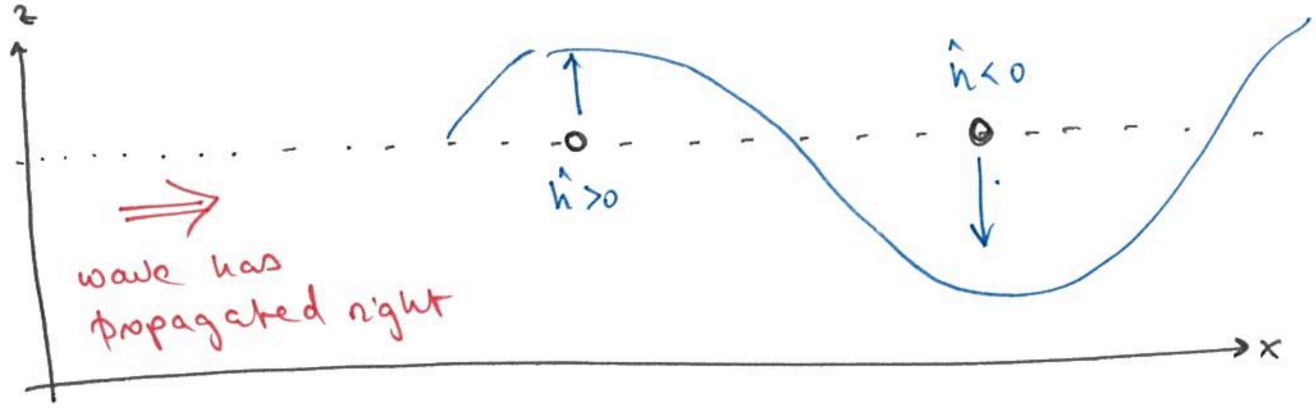
$$= -\sqrt{gH}\,\frac{K^2}{K}\hat{h} + \sqrt{gH}\,K\hat{h} = 0 \quad \text{as expected} \Big]$$

Consider a wave propagating in the positive x-direction

$\underline{k} = (|\underline{k}|, 0) = (K, 0)$ $\quad [l = 0,\ K = \sqrt{k^2}]$

$\Rightarrow \quad \hat{u} = \sqrt{\frac{g}{H}}\,\hat{h}, \quad \hat{v} = 0, \quad \hat{h} = \hat{h}$





$\Rightarrow$ SW gravity propagating in the positive x-direction.

Ex: Confirm that solution (iii) $\omega = -\sqrt{gH}\,K$ gives a SW gravity wave propagating in the negative x-direction.

Shallow water gravity waves have a dispersion relation

$$\boxed{\omega = \pm\sqrt{gH}\,K.}$$

SW gravity wave dispersion relation.

Phase speed and Group velocity

Phase speed : $c_{ph} = \dfrac{\omega}{K}$ = speed of wave crests.
(scalar)

To get a velocity, multiply by $\hat{\underline{k}} = \dfrac{\underline{k}}{K} = \left(\dfrac{k}{K}, \dfrac{l}{K}\right)$

Phase velocity: $\underline{c}_{ph} = \dfrac{\omega}{K}\hat{\underline{k}} = \dfrac{\omega \underline{k}}{K^2} = \left(\dfrac{\omega k}{K^2}, \dfrac{\omega l}{K^2}\right)$

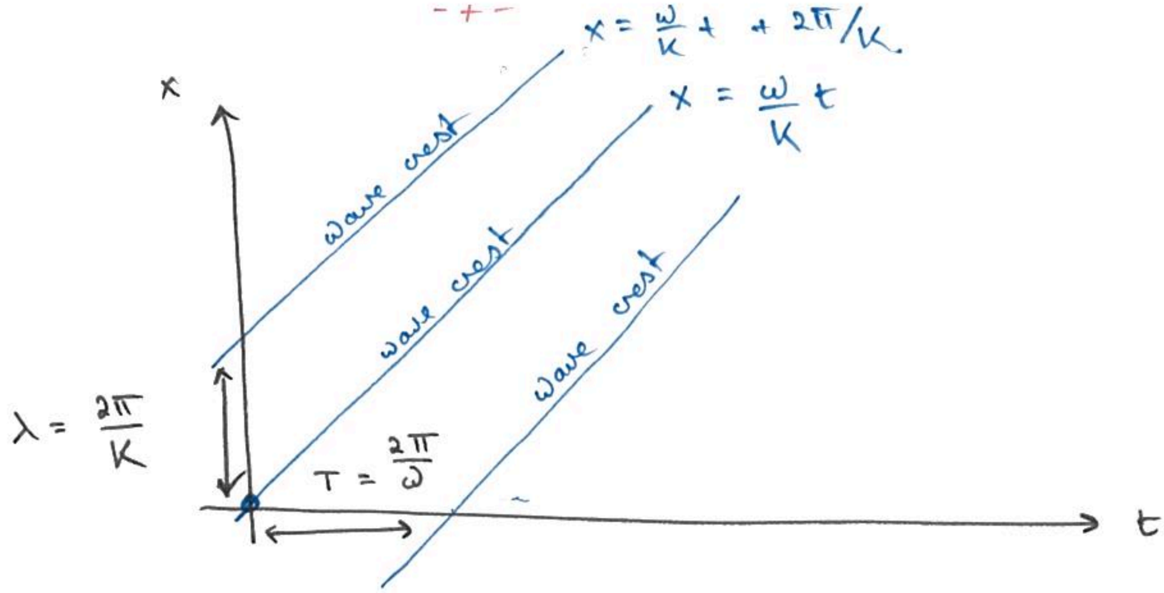
NOTE: phase velocity $\neq \left(\dfrac{\omega}{k}, \dfrac{\omega}{l}\right)$

Recall, lines of constant phase (wave crest, for example)

$$\phi = kx + ly - \omega t = \text{constant}.$$

For simplicity, rotate coordinate system so that $l = 0$,
$k = K. > 0.$

$$\phi = Kx - \omega t = \text{constant}.$$



$$\phi = kx - \omega t + n2\pi$$

Slope of lines of constant phase in space-time plot give the speed of propagation of the wave.

For SW gravity waves: $(\omega \neq 0)$

$$\omega = \pm\sqrt{gH}\,K$$

$$c_{ph} = \frac{\omega}{K} = \pm\sqrt{gH}$$

$\Rightarrow$ wave crests propagating in +ve and -ve x-direction with uniform speed $\sqrt{gH}$ that does not depend on the wavelength (i.e. on k).

Group Velocity: (vector, not scalar)

$$\underline{c}_{gr} = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}\right) \qquad \omega = \pm\sqrt{gH}\,(k^2+l^2)^{1/2}$$

$$= \left(\pm\sqrt{gH}\,\frac{k}{K}, \pm\sqrt{gH}\,\frac{l}{K}\right) = \pm\sqrt{gH}\,\hat{\underline{k}}$$

Group speed $= |\underline{c}_{gr}| = \sqrt{gH}$

$\Rightarrow$ wave group moves with uniform speed = phase speed

$\Rightarrow$ waves remain "in formation"

$\Rightarrow$ SW gravity waves are non-dispersive.

In general.

* For a linear dispersion relation

$$\omega \propto K$$

$\Rightarrow$ $c_{ph} = |\underline{c}_{gr}| = \dfrac{\omega}{K}$ = constant

$\Rightarrow$ non-dispersive waves

* For a nonlinear dispersion relation

$$\omega \not\propto K$$

$\Rightarrow$ $c_{ph} \neq |c_{gr}|$

$\Rightarrow$ dispersive waves.

Case 2: rotating on an f-plane ($f = f_0$)

1. Base state:

$$\boxed{\bar{u} = \bar{v} = 0 \qquad \bar{h} = H.}$$

2. Small perturbations:

$$\boxed{\begin{aligned} &\frac{\partial \tilde{u}}{\partial t} - f_0 \tilde{v} + g \frac{\partial \tilde{h}}{\partial x} = 0 \\ &\frac{\partial \tilde{v}}{\partial t} + f_0 \tilde{u} + g \frac{\partial \tilde{h}}{\partial y} = 0 \\ &\frac{\partial \tilde{h}}{\partial t} + H \left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} \right) = 0. \end{aligned}}$$

3. Plane wave solutions

$$\frac{\partial}{\partial t} \to -i\omega \quad , \quad \frac{\partial}{\partial x} \to ik \quad , \quad \frac{\partial}{\partial y} \to il$$

$$\tilde{u} \to \hat{u} \quad , \quad \tilde{v} \to \hat{v} \quad , \quad \tilde{h} \to \hat{h}$$

$$\Rightarrow \quad \begin{aligned} &-i\omega \hat{u} - f_0 \hat{v} + igk\hat{h} = 0 \\ &-i\omega \hat{v} + f_0 \hat{u} + igl\hat{h} = 0 \\ &-i\omega \hat{h} + iHk\hat{u} + iHl\hat{v} = 0. \end{aligned}$$

$(\times i)$

$$\begin{bmatrix} \omega & -if_0 & -gk \\ if_0 & \omega & -gl \\ -Hk & -Hl & \omega \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

4. Nontrivial solutions

$$\det \mathbb{M} = \omega(\omega^2 - gHl^2) + if_0(if_0\omega - \cancel{gHkl}) - gk(-\cancel{if_0Hl} + \omega Hk) = 0$$

$$\omega(\omega^2 - \underbrace{gHK^2}_{\text{SW gravity waves}} - \underbrace{f_0^2}_{\text{new term}}) = 0$$

Solutions:
i) $\omega = 0$ (Vortex mode)

ii) $\omega = \sqrt{gHK^2 + f_0^2}$

iii) $\omega = -\sqrt{gHK^2 + f_0^2}$

$\Big\}$ Poincaré waves.

* Consider first the limit of large K (short wavelength)

specifically, look $K \gg f_0/\sqrt{gH}$

Then $\omega = \pm\sqrt{gHK^2}\sqrt{1 + \frac{f_0^2}{gHK^2}}$ $\ll 1$.

$\simeq \pm\sqrt{gH}\,K$ recover SW gravity waves (non-rotating) NON-DISPERSIVE

* Consider the limit of small K (long wavelength)

specifically, look $K \ll f_0/\sqrt{gH}$

Then $\omega = \pm|f_0|\sqrt{1 + \frac{gHK^2}{f_0^2}}$ $\ll 1$

$\simeq \pm|f_0|$ recover inertial oscillations.

$c_{ph} = \frac{\omega}{K} = \pm\frac{|f_0|}{K} \neq$ constant DISPERSIVE.

Case 3: β-plane $(f = f_0 + \beta y)$ - Rossby waves

* To simplify algebra, neglect effects of surface gravity waves by assuming

$$\boxed{h = H = \text{constant}}$$

This is also known as the "rigid lid approximation".

1. Choose base state.

$$\bar{u} = \bar{v} = 0 \quad , \quad \bar{h} = H.$$

2. Small perturbations.

$$u = \tilde{u} \quad , \quad v = \tilde{v} \quad , \quad h = \bar{h} = H \quad (\tilde{h} = 0)$$

① $$\frac{\partial \tilde{u}}{\partial t} - (f_0 + \beta y)\tilde{v} = 0$$

② $$\frac{\partial \tilde{v}}{\partial t} + (f_0 + \beta y)\tilde{u} = 0$$

③ $$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0$$ $\Rightarrow$ flow is incompressible in 2D.

Non-constant coefficients: $\beta y \tilde{u}$, $\beta y \tilde{v}$. !

$\Rightarrow$ Can't use plane wave assumption

Look at vorticity instead!

$$\tilde{\zeta} = \tilde{v}_x - \tilde{u}_y$$

For SW system we saw that

$$\frac{D}{Dt}\left(\frac{f + \zeta}{h}\right) = 0 \qquad \text{SW PV conservation.}$$

Explicity: $(h = H = \text{constant})$

$$\frac{\partial}{\partial t}\left(\frac{f+\tilde{\zeta}}{H}\right) + \tilde{\underline{u}}\cdot\underline{\nabla}\left(\frac{f+\tilde{\zeta}}{H}\right) = 0$$

Neglect nonlinear terms:

$$\frac{\partial}{\partial t}\frac{\tilde{\zeta}}{H} + \tilde{\underline{u}}\cdot\underline{\nabla}\frac{f(y)}{H} = 0$$

$\times H$

$$\boxed{\frac{\partial}{\partial t}\tilde{\zeta} + \tilde{v}\beta = 0}$$

linearized potential vorticity equation for SW system with a rigid lid.

This has constant coefficients $\Rightarrow$ plane wave approx. okay.

Have not made use of (3) yet (2D incompressibility)

$$\frac{\partial\tilde{u}}{\partial x} + \frac{\partial\tilde{v}}{\partial y} = 0$$

$\Rightarrow$ can introduce a streamfunction $\tilde{\psi}$ s.t.

$$\tilde{u} = -\frac{\partial\tilde{\psi}}{\partial y}, \quad \tilde{v} = \frac{\partial\tilde{\psi}}{\partial x}.$$

(automatically satisfies incompressibility)

$\Rightarrow$ Vorticity : $\tilde{\zeta} = \tilde{v}_x - \tilde{u}_y$

$$= \tilde{\psi}_{xx} + \tilde{\psi}_{yy} = \nabla^2\tilde{\psi}$$

Thus have single equation for one unknown!

$$\boxed{\frac{\partial}{\partial t}\nabla^2\tilde{\psi} + \beta\frac{\partial\tilde{\psi}}{\partial x} = 0}$$

3. Plane wave solution:

$$\tilde{\psi} = \hat{\psi}\, e^{i(kx + ly - \omega t)}$$

$$\partial_t \rightarrow -i\omega \,, \quad \partial_x \longrightarrow ik \,, \quad \partial_y \rightarrow il \,, \quad \nabla^2 \longrightarrow -k^2 - l^2$$

$$\Rightarrow \quad -i\omega \underbrace{\left(-k^2 - l^2\right)}_{K^2 = k^2 + l^2} \hat{\psi} + i\beta k \hat{\psi} = 0 .$$

$$\omega K^2 \hat{\psi} = -\beta k \hat{\psi}$$

4. Non-trivial solutions

Solution if $\hat{\psi} \neq 0$

$$\Rightarrow \quad \omega K^2 = -\beta k$$

$$\Rightarrow \quad \boxed{\omega = -\frac{\beta k}{K^2}}$$

NB: only one solution !

dispersion relation for Rossby waves.

Phase speed & Group velocity.

$$c_{ph} = \frac{\omega}{K} = -\frac{\beta k}{K^3}.$$

in the direction $\hat{\underline{k}} = \left(\frac{k}{K}, \frac{l}{K}\right)$.

$$\Rightarrow \quad \underline{c}_{ph} = c_{ph} \cdot \hat{\underline{k}} = \left(c_{ph,x}, c_{ph,y}\right) = \Bigg(\underbrace{-\frac{\beta k^2}{K^4}}_{\text{always } -ve}, \underbrace{-\beta\frac{kl}{K^4}}_{\pm ve}\Bigg)$$

$\Rightarrow c_{ph,x} < 0 \quad \Rightarrow$ Rossby wave crests always propagate to the <u>WEST</u>.

$$\underline{c}_{gr} = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l} \right) = \left(\underbrace{\beta \frac{k^2 - l^2}{K^4}}_{\pm ve}, \frac{2\beta k l}{K^4} \right).$$

Group can propagate EASTWARD or WESTWARD.

$k^2 > l^2 \Rightarrow$ eastward propagation

$k^2 < l^2 \Rightarrow$ westward propagation.

Mechanism of Rossby waves:

Conservation of Potential Vorticity provides "elasticity" that restores columns of fluid that have been perturbed.

Since we're on a β-plane, there's a background gradient of PV : in order to conserve total PV, fluid columns moving NORTH must DECREASE ζ to compensate. Likewise, fluid columns moving SOUTH must INCREASE ζ.

$$q = (\zeta + f_0 + \beta y)/H.$$