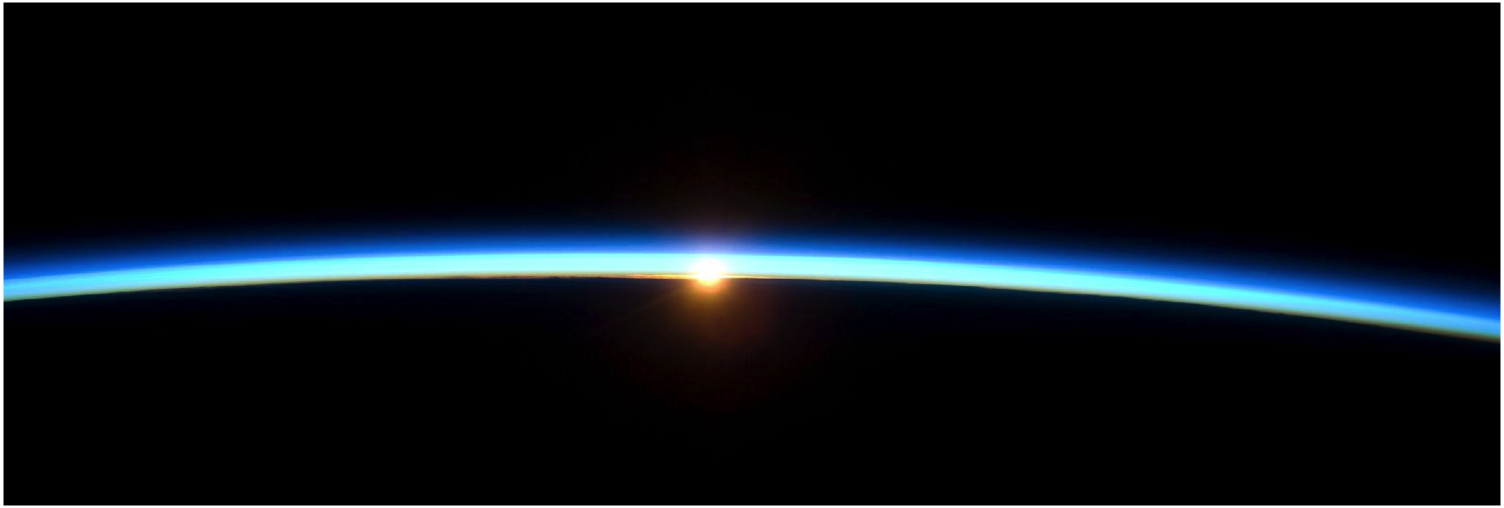

Topic 1: Rotating, Stratified, Thin

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UNSW Sydney, Term 1 2019

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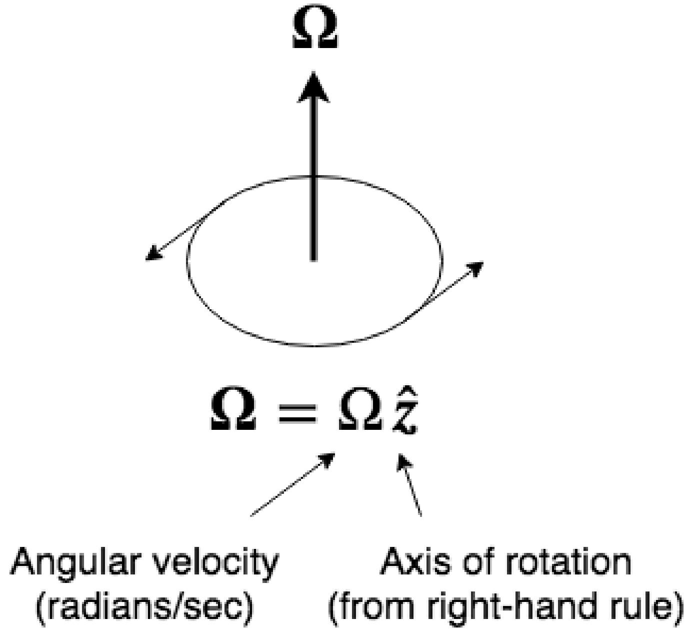
1.1 Calculus on a Carousel

"He that is giddy thinks the world turns around."

William Shakespeare, *Taming of the Shrew Act 5 Scene 2*

How do we represent *rotation* mathematically?



Angular velocity $\quad \Omega = \dfrac{2\pi}{T} \quad \left[\text{seconds}^{-1}\right]$

Period $\quad T = \dfrac{2\pi}{\Omega} \quad [\text{seconds}]$

Ω = "omega" (upper case)

Rotation rate of Earth

Sidereal day (wrt fixed stars):

$$T = 23.93 \text{ hours}$$

(Earth's solar day $\approx$ 24 hours combines rotation and motion along orbit.)

$$\Omega = \frac{2\pi}{23.93 \times 60 \times 60} = 7.29 \times 10^{-5} \text{ s}^{-1}$$

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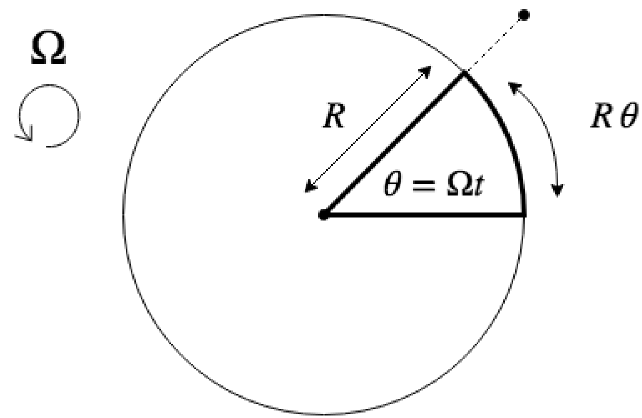
Rotation rate of a carousel

Period of 6 seconds = 10 rpm (revolutions per minute).

$$\Omega = \frac{2\pi}{6} = 1 \text{ s}^{-1} \gg \Omega_{Earth}$$

Tangental velocity

View from above North Pole: Earth rotates counterclockwise
("prograde" = same direction as Sun)



$$\text{Arc length} = R\,\theta = R\,\Omega t$$

$$\text{Tangential velocity} = \frac{d}{dt} R\,\theta = R\,\Omega$$

θ = "theta" (lower case)

Tangential velocity at Equator

At Earth's surface $R_{Eq} = 6378$ km

$$V_T = \frac{\text{Circumference}}{\text{Period}} = \frac{2\pi \times 6378}{23.93} = 1674 \text{ km/hr}$$

In a geostationary orbit $R_{Geo} = 42000$ km

$$V_T = \frac{\text{Circumference}}{\text{Period}} = \frac{2\pi \times 42000}{23.93} = 11000 \text{ km/hr}$$

1.2 Kinematics in a rotating reference frame

Velocity

$$\boldsymbol{v}_0 = \boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}$$

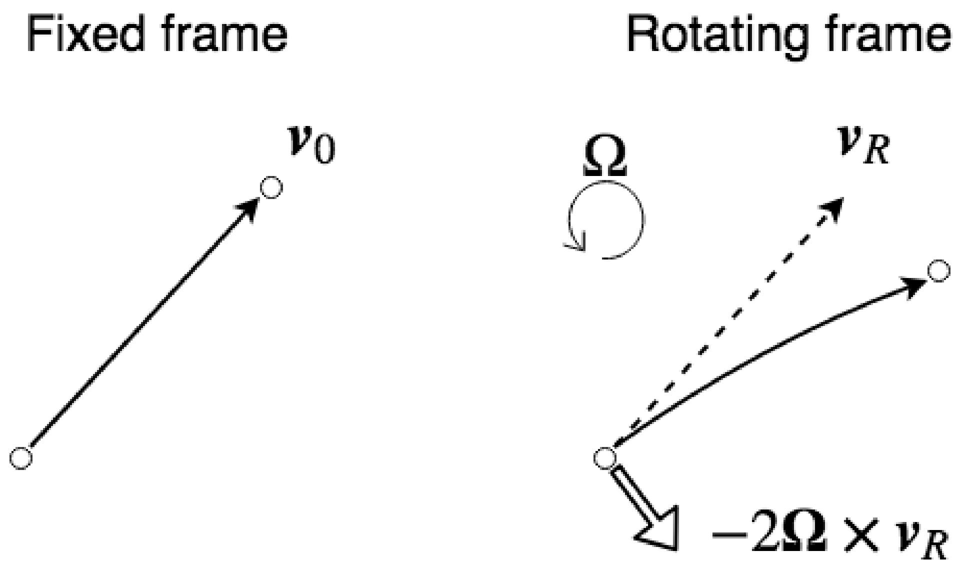
$\boldsymbol{v}_0$	Absolute velocity (measured in fixed frame)
$\boldsymbol{v}_R$	Relative velocity (measured in rotating frame)
$\boldsymbol{\Omega} \times \boldsymbol{r}$	Apparent deflection due to rotation

Acceleration

$$\left.\frac{d\boldsymbol{v}_R}{dt}\right|_R = \left.\frac{d\boldsymbol{v}_0}{dt}\right|_0 - 2\boldsymbol{\Omega}\times\boldsymbol{v}_R - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$$

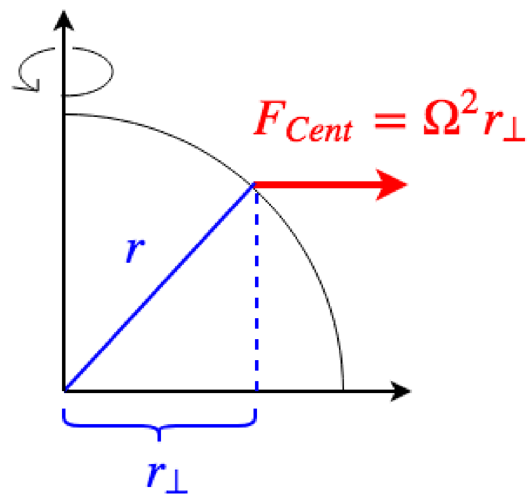
$\left.\frac{d\boldsymbol{v}_R}{dt}\right\|_R$	Acceleration in rotating frame
$\left.\frac{d\boldsymbol{v}_0}{dt}\right\|_0$	Force/mass in fixed frame
$-2\boldsymbol{\Omega}\times\boldsymbol{v}_R$	Coriolis force
$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$	Centrifugal force

Coriolis force



Demonstration: {MIT Department of Physics}

Centrifugal force

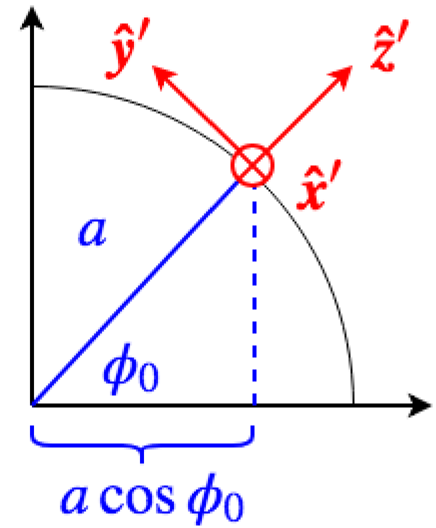


$$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \Omega^2 \boldsymbol{r}_\perp$$

where $\boldsymbol{r}_\perp$ is the component of the vector $\boldsymbol{r}$ perpendicular to the axis of rotation

$$\boldsymbol{r} = \boldsymbol{r}_\perp + \boldsymbol{r}_\parallel$$

Tangent plane approximation



Plane tangent to the Earth's surface at latitude ϕ_0 and longitude λ_0:

$$\begin{aligned} x' &= & a\cos\phi_0\,(\lambda-\lambda_0) & \quad \text{East (zonal)} \\ y' &= & a\,(\phi-\phi_0) & \quad \text{North (meridional)} \\ z' &= & r-a & \quad \text{Up} \end{aligned}$$

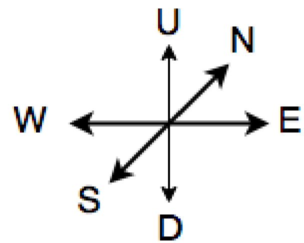
ϕ = "phi" (lower case), λ = "lambda" (lower case)

Coriolis force in tangent plane approximation

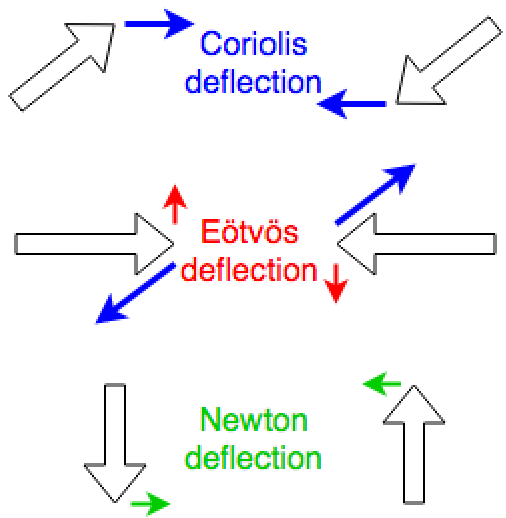
$$-2\boldsymbol{\Omega} \times \boldsymbol{v} = \begin{pmatrix} -wf_* + vf \\ -uf \\ uf_* \end{pmatrix}$$

In the **Northern hemisphere** ($f > 0, f_* > 0$)

Direction of motion	**Coriolis**	**Eötvös**	**Newton**
North ($v > 0$)	East	-	-
South ($v < 0$)	West	-	-
East ($u > 0$)	South	Up	-
West ($u < 0$)	North	Down	-
Up ($w > 0$)	-	-	West
Down ($w < 0$)	-	-	East



In the **Northern** hemisphere



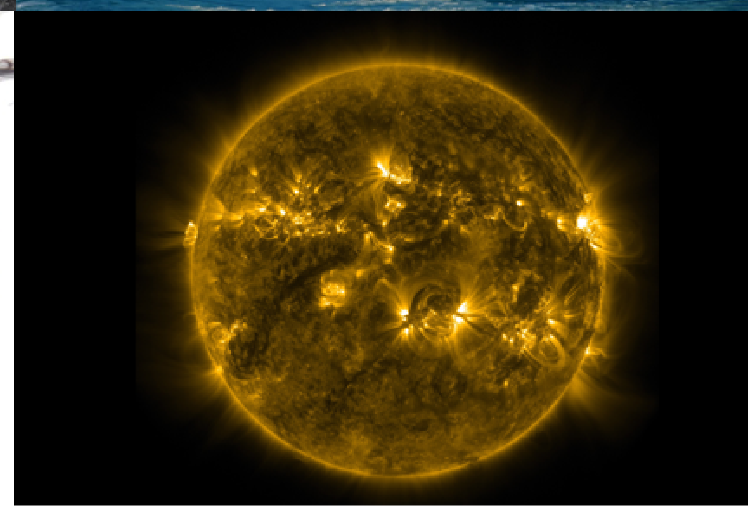
1.3 Go with the flow

"How can we know the dancer from the dance?"

W.B. Yeats, *Among School Children*

What is a fluid?



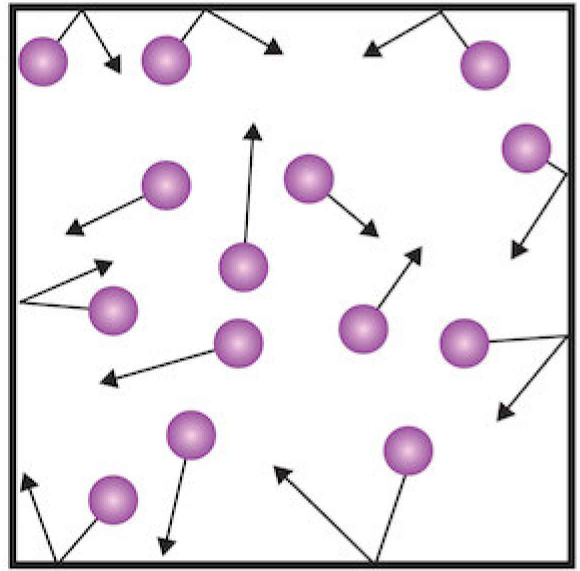



A **fluid** is a *continuous medium* that *deforms under stress*

- **continuous medium**: consists of infinitessimal parcels of fluid "material"
- **deforms under stress**: changes shape due to contact forces with neighboring fluid parcels

Classical states of matter: solid, liquid, gas, plasma

- Liquids, gases, and plasmas are fluids. 99% of the visibile universe is a fluid (plasma)
- **Plastic solids** behave like solids on short times but like fluids on long times



In fluid dynamics, the average effect of *microscopic* interactions among individual atoms is represented by equations describing the evolution of *macroscopic* properties:

Microscopic	**Macroscopic**
conservation of mass	continuity equation
conservation of momentum	Newton's second law
conservation of energy	energy equation
thermodynamic properties of the fluid	equation of state
mechanical properties of the fluid	constitutive equation

Eulerian and Lagrangian representations

Equivalent representations of a flow:

Eulerian representation: field evaluated at fixed points in space - "fixed frame"

Scalar fields:

- Temperature $T(x, y, z, t) = T(\boldsymbol{x}, t)$
- Pressure $p(\boldsymbol{x}, t)$
- Density $\rho(\boldsymbol{x}, t)$

Vector fields:

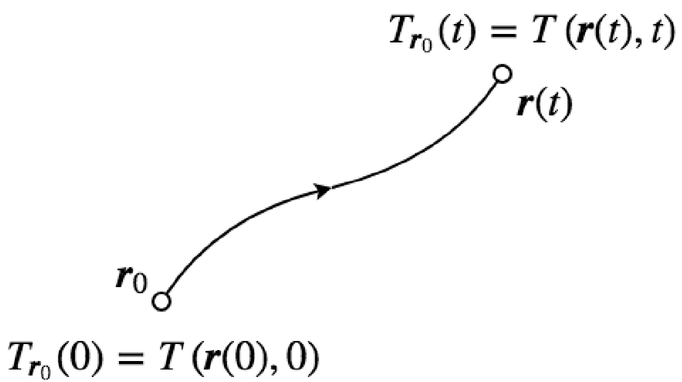
- Velocity $\boldsymbol{v}(\boldsymbol{x}, t) = (u, v, w)$
- Vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{v}$

ρ = "rho" (lower case), ω = "omega" (lower case)

Lagrangian representation: field evaluated at points moving with the flow - "flow following"

$$T(\boldsymbol{r}(t), t), \quad \text{where} \quad \frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}\,(\boldsymbol{r}(t), t)$$

Can be labelled by initial position $\boldsymbol{r}_0 = \boldsymbol{r}(0)$, i.e. $T_{\boldsymbol{r}_0}(t)$.



1.4 Heat, Air, Water, Salt

"This isn't your typical cookbook."

Samin Nosrat, *Salt, Fat, Acid, Heat: Mastering the Elements of Good Cooking*

The story so far...

Eulerian representation:

$$\frac{d\rho}{dt} + \boldsymbol{v} \cdot \nabla\rho = -\rho\nabla \cdot \boldsymbol{v} \qquad \text{continuity equation}$$

$$\frac{d\boldsymbol{v}}{dt} + \boldsymbol{v} \cdot \nabla\boldsymbol{v} = \sum \boldsymbol{f} \qquad \text{momentum equation}$$

Lagrangian representation:

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \boldsymbol{v} \qquad \text{continuity equation}$$

$$\frac{D\boldsymbol{v}}{Dt} = \sum \boldsymbol{f} \qquad \text{momentum equation}$$

Forces:

$$\sum \boldsymbol{f} = -\frac{1}{\rho}\nabla p - \nabla\Phi - 2\boldsymbol{\Omega} \times \boldsymbol{v}$$

4 equations:

$$\left(\frac{d\rho}{dt}, \frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}\right)$$

5 unknowns:

$$(\rho, u, v, w, p)$$

Need an additional **equation of state** to close the system

$$p = p\left(\rho, T, \cdots\right)$$

In some problems also need an **energy equation**, e.g.

$$\frac{DT}{Dt} = Q$$

Relates state variables (e.g. p, V, T, $\cdots$) and properties of the fluid (e.g. N, ρ, $\cdots$).

$$
\begin{aligned}
p &= \text{pressure} && \left[\mathrm{N\ m^{-2}}\right] = [\mathrm{Pa}] \\
V &= \text{volume} && \left[\mathrm{m^{3}}\right] \\
T &= \text{temperature} && [\mathrm{K}] = [^{\circ}\mathrm{C} + 273.15] \\
N &= \text{number of moles} && [\mathrm{mol}] = \left[6.022 \times 10^{23}\right] \\
\rho &= \text{mass density} && \left[\mathrm{kg\ m^{-3}}\right]
\end{aligned}
$$

Typical values

Pressure at sea level

$$\begin{aligned} 1 \text{ atm} &= 101325 \text{ Pa} && \text{[Pascals]} \\ &= 1013.25 \text{ hPa, mbar} && \text{[hectoPascals, millibar]} \\ &= 1.01325 \text{ bar} && \text{[bar]} \end{aligned}$$

Temperature at sea level

$$20\ ^\circ\text{C} = 293.15 \text{ K}$$

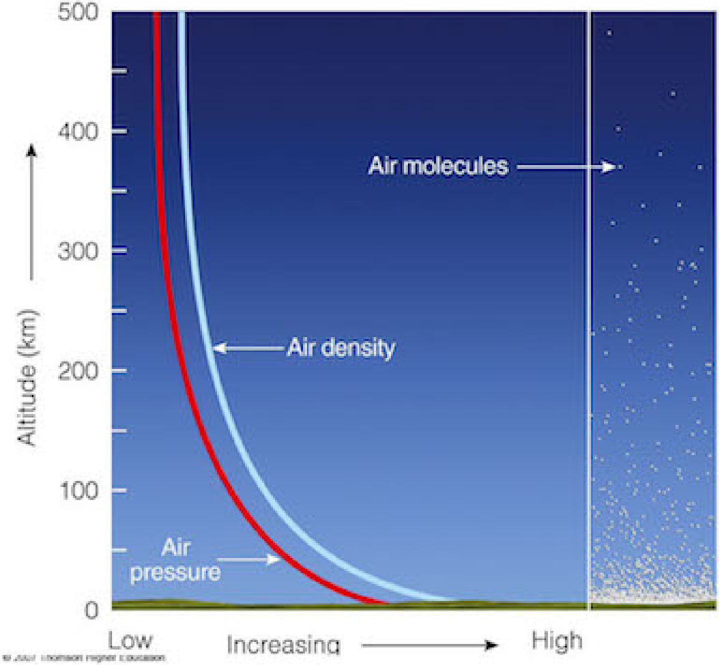
Density of dry air and seawater

$$\rho_{\text{air}} \approx 1.23 \text{ kg m}^{-3}$$
$$\rho_{\text{sea}} \approx 1020 \text{ kg m}^{-2}$$

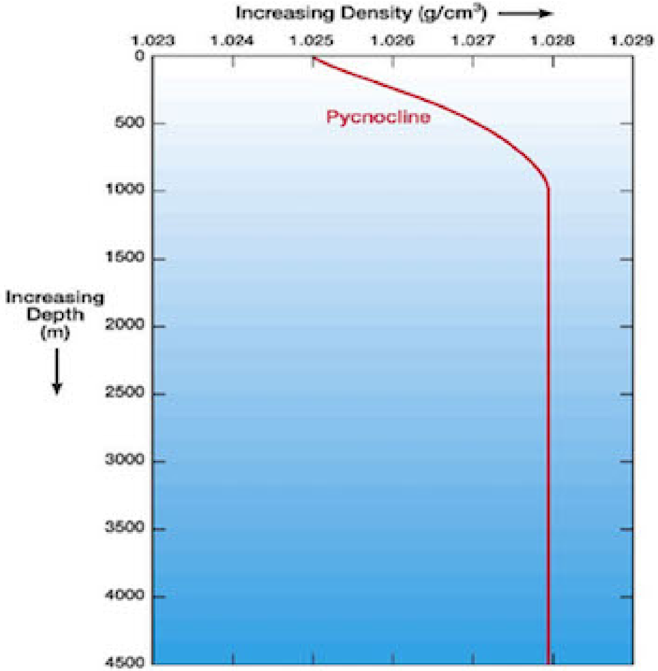
Mass of 1 mole of dry air and seawater

$$m_{\text{air}} \approx 28.97 \text{ g mol}^{-1}$$
$$m_{\text{sea}} \approx 18.64 \text{ g mol}^{-1}$$

Atmosphere



Ocean



1.5 Rotating, Stratified, and Thin

"If you had a globe covered with a coat of varnish, the thickness of that varnish would be about the same as the thickness of Earth's atmosphere compared to the Earth itself."

Carl Sagan

Rotating, stratified, thin

The large-scale circulation of the atmosphere and ocean is...

- Rapidly rotating

$$\text{Rossby number} \ll 1$$

- Stratified

$$\text{Richardson number} > 1$$

- Thin

$$\text{aspect ratio} \ll 1$$

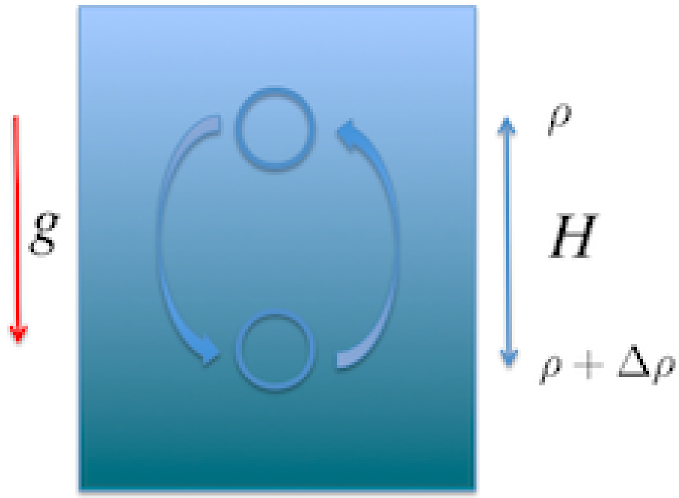
When is rotation important?

- Rossby number

$$\mathrm{Ro} = \frac{\text{Time for one rotation}}{\text{Time to move distance } L} = \frac{2\pi/\Omega}{L/U}.$$

- For $\mathrm{Ro} \ll 1$, rotation is important: small U or large L:
  - $L = 1\,\mathrm{m} : U \le 0.012\,\mathrm{mm/s}$
  - $L = 1\,\mathrm{km} : U \le 1.2\,\mathrm{cm/s}$
  - $L = 1000\,\mathrm{km} : U \le 12\,\mathrm{m/s}$

When is stratification important?

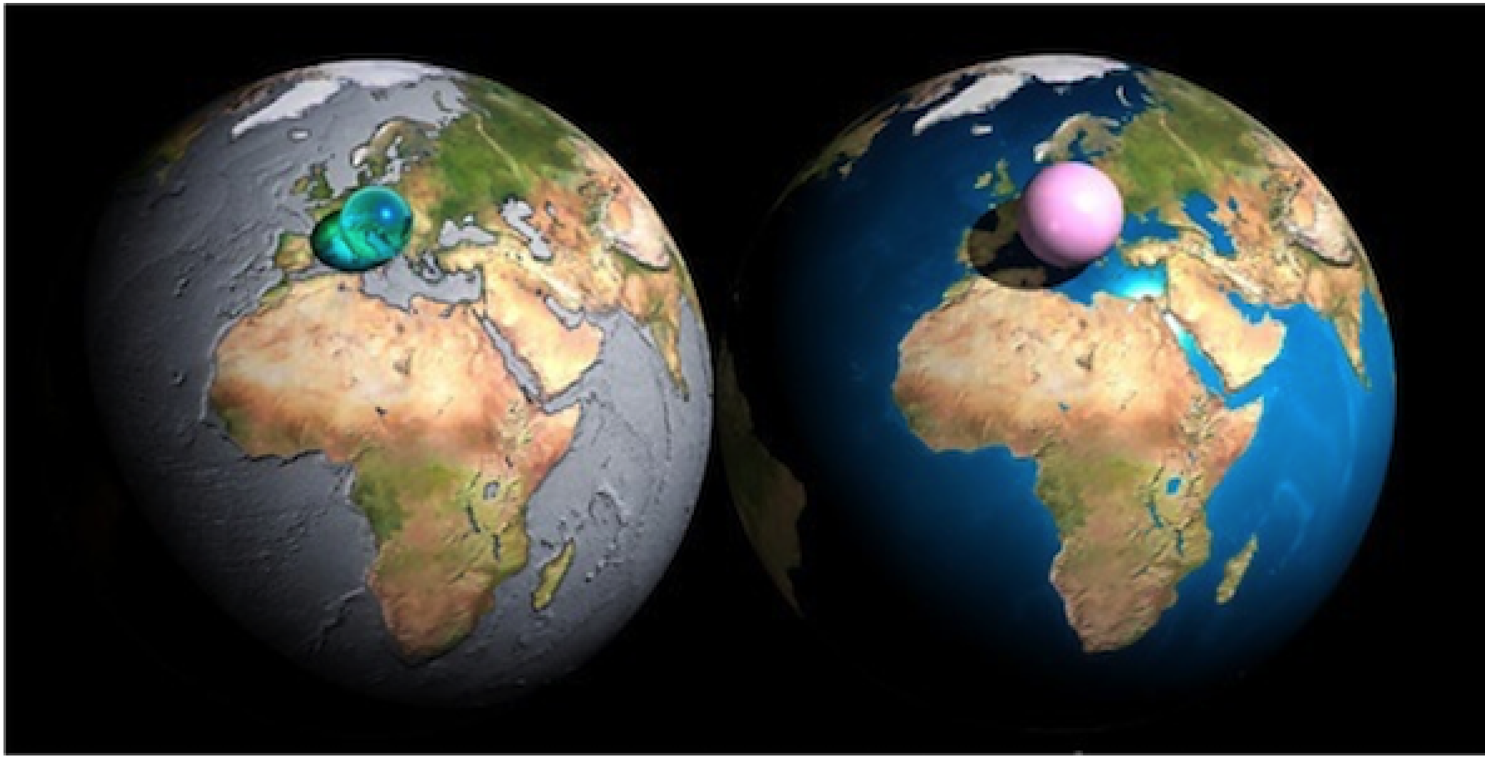


- Richardson number

$$Ri = \frac{\text{Change in potential energy}}{\text{Kinetic energy of motion}} = \frac{\Delta\rho g H}{\frac{1}{2}\rho U^2}.$$

- For $Ri > 1$, insufficient KE to perturb stratification
- For $Ri < 1$, stratification is negligible

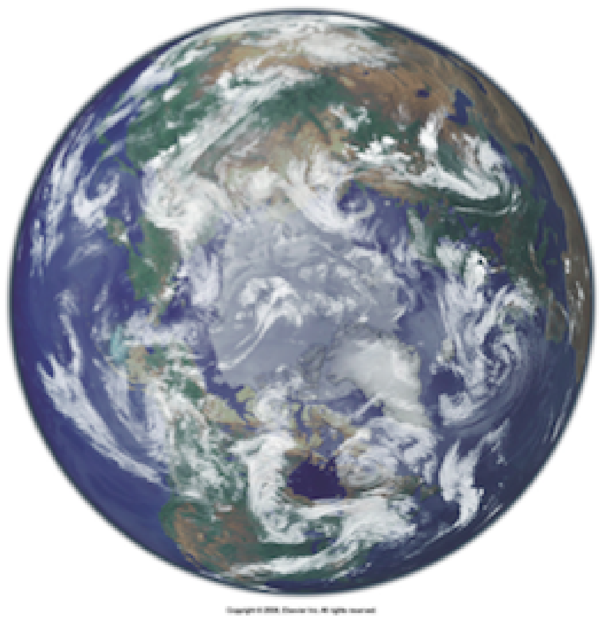
Thin

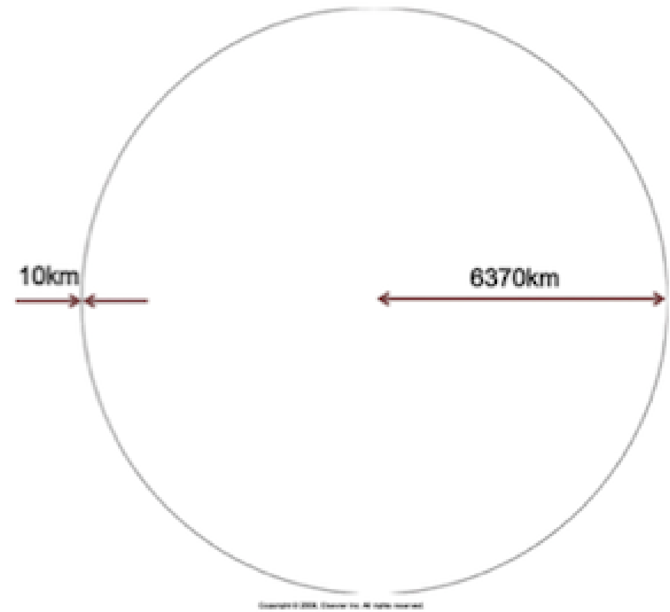


Geophysical fluids are very thin!

- 80% of mass of the atmosphere within 10 km altitude
- mean depth of the ocean is 3.7 km

When is thinness important?





$$\text{Aspect ratio} = \frac{\text{vertical scale}}{\text{horizontal scale}} = \frac{H}{L}$$

When aspect ration $\ll 1$, horizontal motions dominates vertical motion.

When are rotation, stratification *and* thinness important?

- Rossby number $\ll 1$, Richardson number ≈ 1

$$\mathrm{Ro} \ll 1 \quad \Longrightarrow \quad L \gg \frac{U}{\Omega}$$

$$\mathrm{Ri} \approx 1 \quad \Longrightarrow \quad U \approx \sqrt{\frac{\Delta\rho}{\rho} gH}$$

- Defines a characteristic lengthscale: Rossby radius of deformation

$$L \gg L_D \approx \frac{1}{\Omega}\sqrt{\frac{\Delta\rho}{\rho} gH}$$

- Small aspect ratio $H \approx \alpha L_D$

$$\begin{aligned} \text{atmosphere:} \quad & L_D \approx 500 \text{ km} & H \approx 5 \text{ km} \\ \text{ocean:} \quad & L_D \approx 50 \text{ km} & H \approx 500 \text{ m} \end{aligned}$$

1.6 Geophysical fluid dynamics

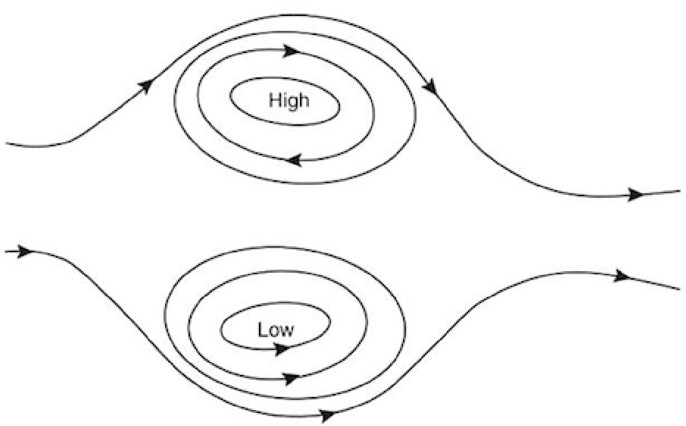
> "Science, my lad, is made up of mistakes, but they are mistakes which it is useful to make, because they lead little by little to the truth."
>
> Jules Verne, *A Journey to the Centre of the Earth*

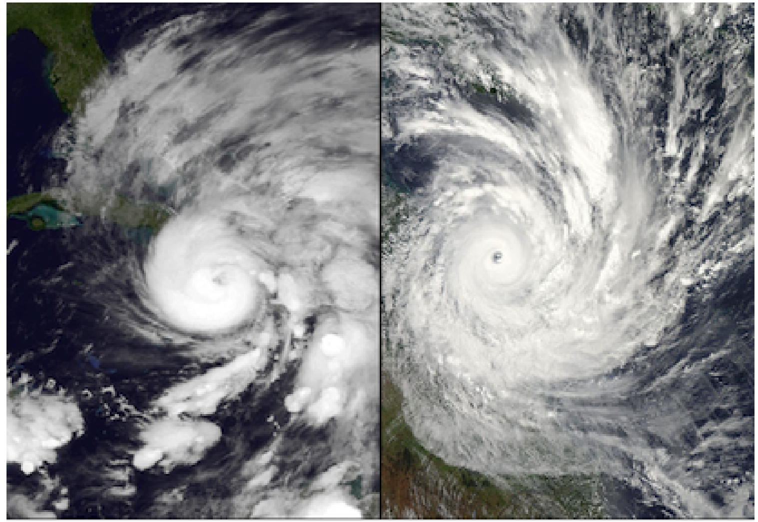
Geostrophic balance

For flows that are dominated by rotation and stratification and have a small aspect ratio, the dominant force balance is

$$\text{Coriolis force} \approx \text{Horizontal pressure gradient}$$

This is called *geostrophic balance*. It implies that flow follows lines of constant pressure (isobars).





Cyclone Yasi, Southern hemisphere (Feb 2011) Hurricane Sandy, Northern hemisphere (Oct 2012)

In the Northern hemisphere the flow is

- clockwise around high pressure
- anticlockwise around low pressure

In the Southern hemisphere the flow is

- anticlockwise around high pressure
- clockwise around low pressure

Low pressure systems are called *cyclones* and high pressure systems are called *anticyclones*.