GROUP VELOCITY

Wave group / wave packet : superposition of waves of different wavelengths + frequencies that is localized in a small region of space.

For a single wave with wavenumber k.

$$\phi(x,t) = \hat{\phi}_k e^{i\underbrace{(kx - \omega t)}_{\text{phase}}} \qquad \omega = \omega(k)$$

complex-valued amplitude

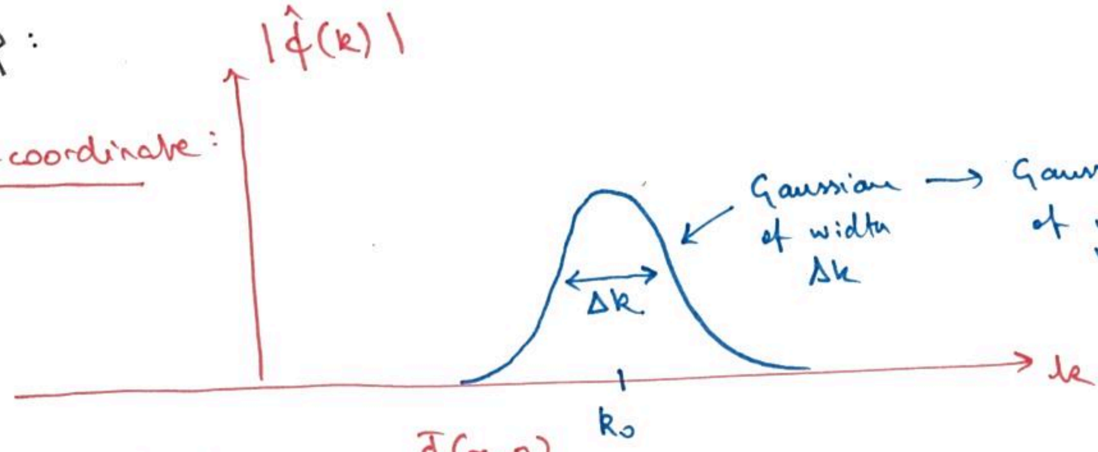
Superposition of infinite number of waves

$$\Phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k) e^{i(kx - \omega(k)t)} \, dk$$

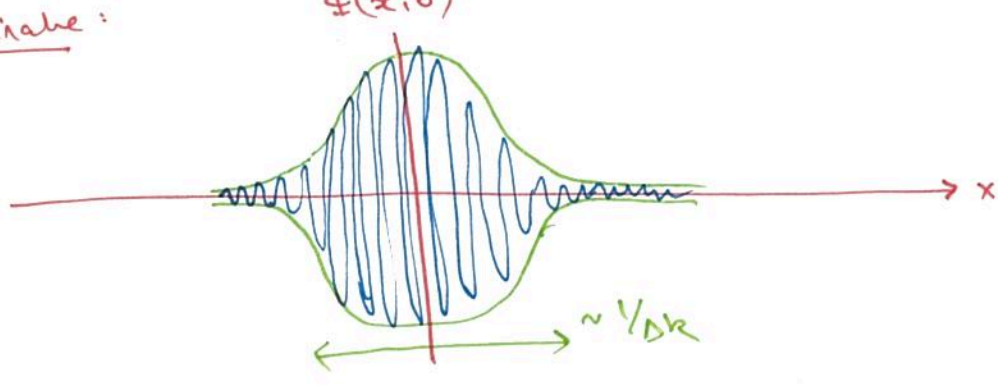
treat k as a continuous variable.

$\hat{\phi}(k)$ is the Fourier transform of $\Phi(x,0)$ and gives the amplitude & phase of each component of the wave group:

in k-coordinate:



in x-coordinate:



Assume that $\hat{\phi}(k)$ is narrowly supported around k_0

$\Rightarrow$ Taylor-series expansion of dispersion relation:

$$\omega(k) \simeq \underbrace{\omega(k_0)}_{\omega_0} + (k-k_0)\underbrace{\frac{\partial \omega}{\partial k}\Big|_{k_0}}_{c_{gr}} + \ldots$$

$$= \omega_0 + c_{gr}(k-k_0) + \cdots$$

Then

$$\Phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k)\, e^{i(kx-\omega t)}\, dk$$

$$\simeq \int_{-\infty}^{\infty} \hat{\phi}(k)\, e^{i(kx-\omega_0 t - c_{gr}(k-k_0)t)}\, dk$$

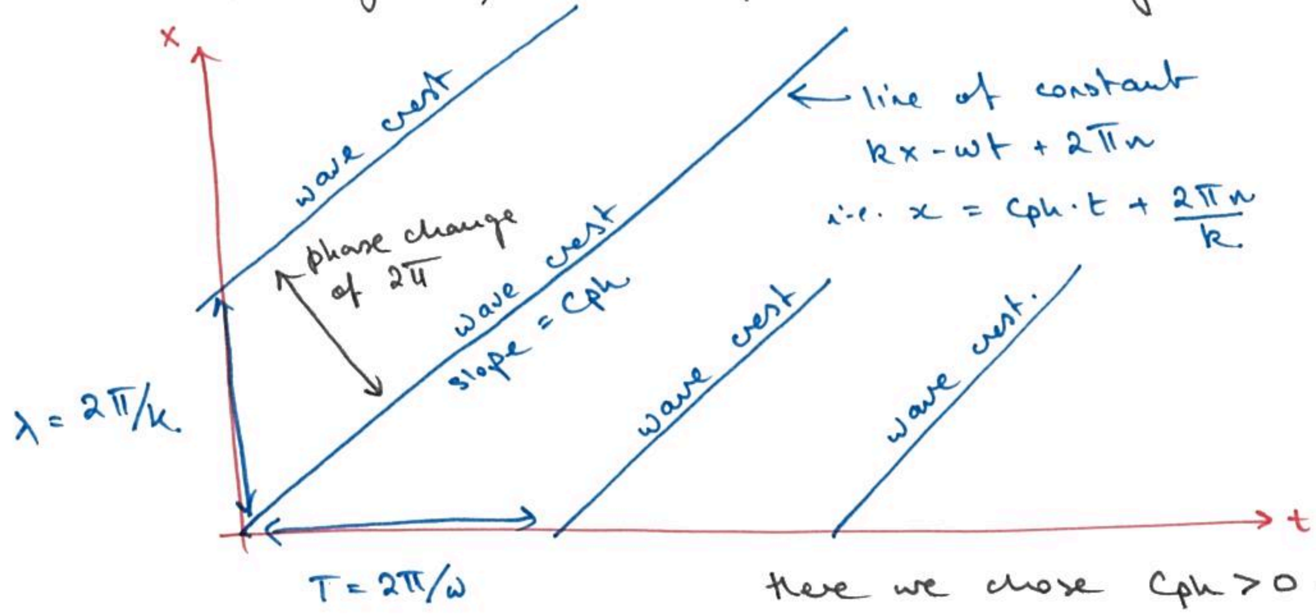
$$= e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \hat{\phi}(k)\, e^{i[(k-k_0)x - c_{gr}(k-k_0)t]}\, d$$

$$= e^{ik_0(x - c_{ph} t)} \int_{-\infty}^{\infty} \hat{\phi}(k)\, e^{i(k-k_0)(x - c_{gr}\cdot t)}\, dk.$$

$c_{ph} = \omega_0 / k_0$

$$= \underbrace{e^{ik_0(x-c_{ph}\cdot t)}}_{\text{wave crests moving with speed } c_{ph}.} \; \underbrace{\psi_{gr}(x - c_{gr}\cdot t)}_{\text{envelope (wave group) moving without changing shape with speed } c_{gr}.}$$

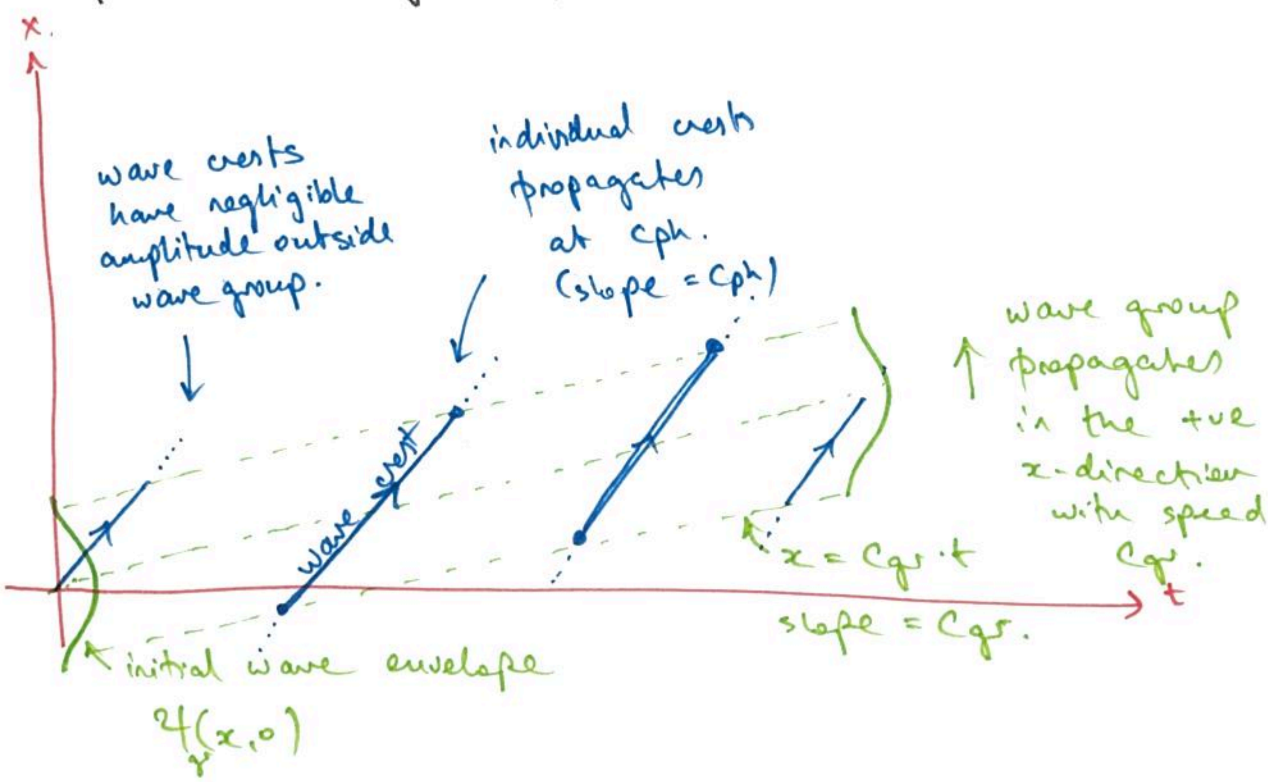
We've seen that monochromatic waves (single k) can be visualized as lines of constant phase ("crests", "troughs") in a space-time diagram:



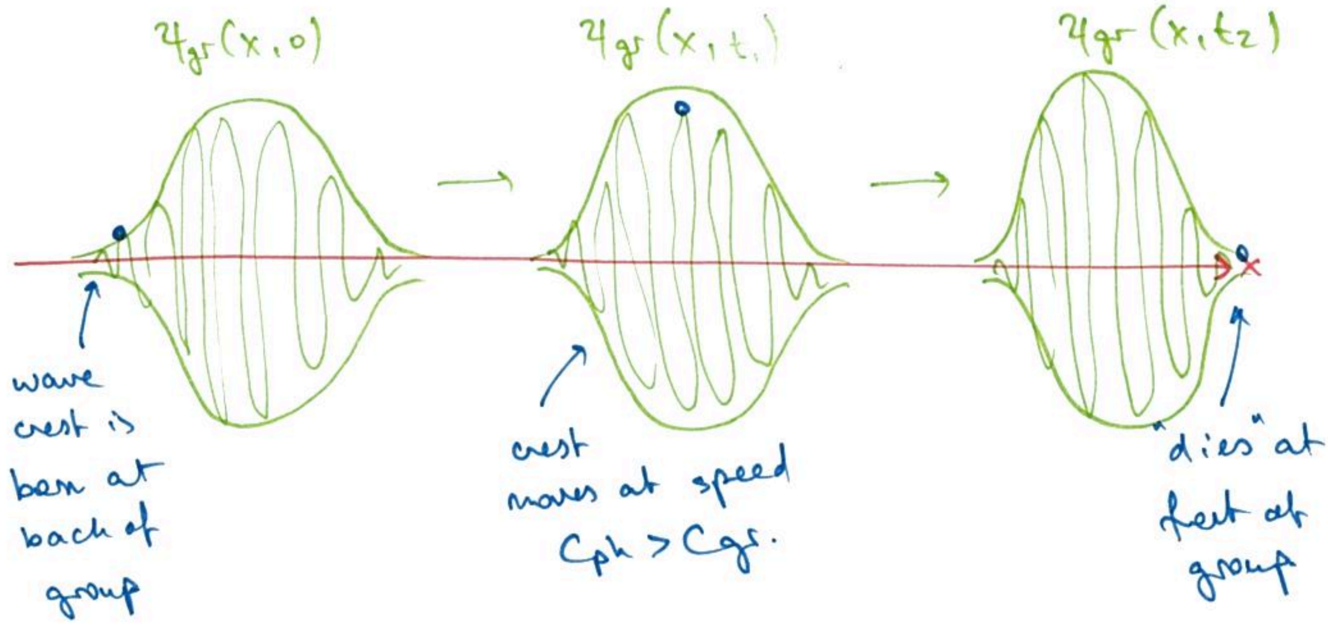
here we chose $c_{ph} > 0$.

For a wave group these crests (and troughs) will be modulated by an envelope function $\psi_{gr}(x - c_{gr} \cdot t)$ that propagates at speed c_{gr}.

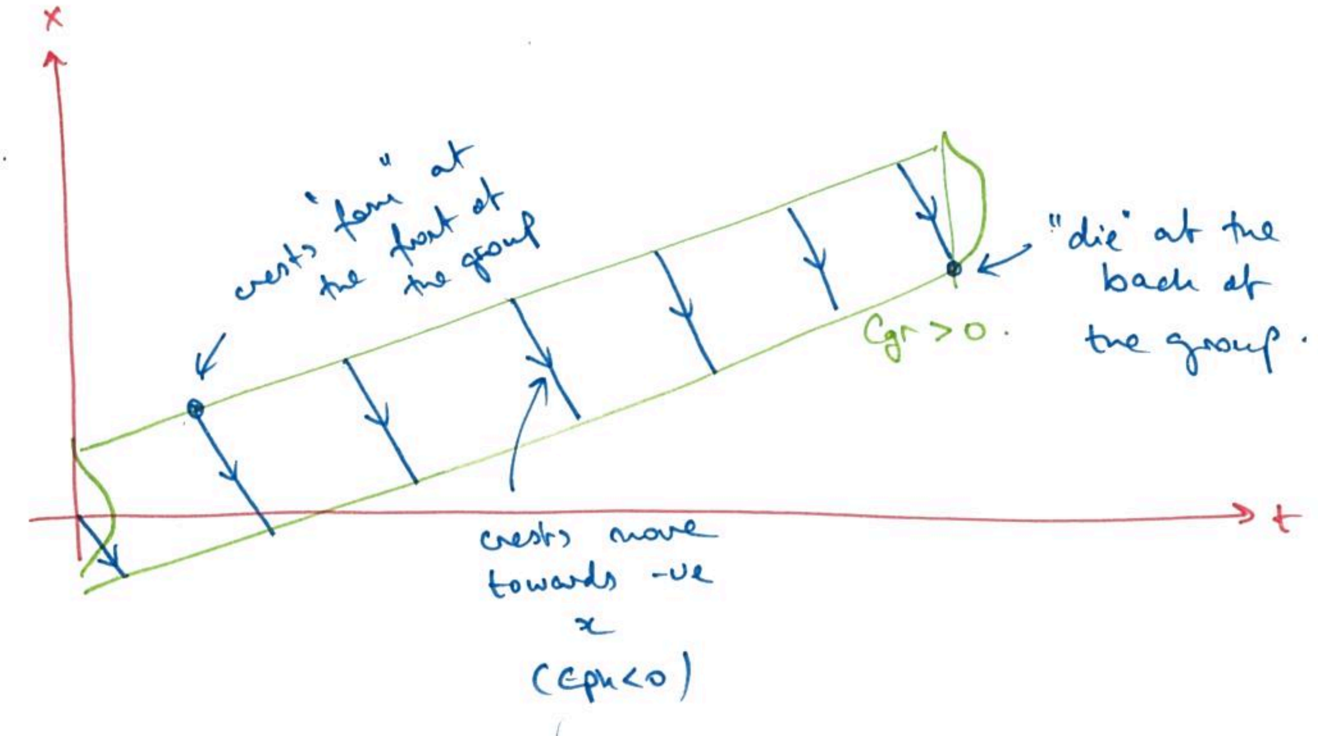
For example, for $c_{gr} > 0$.



Following wave group, we see:



Can also look at phase speed + group velocity with opposite sign (e.g. Rossby waves)



ASYMPTOTIC SCALING THEORY

Recall the "primative equations"

* tangent-plane (f-plane, β-plane)
* Boussinesq fluid
* thin layer approximation.

$\dfrac{D\underline{u}}{Dt} + \underline{f} \times \underline{u}$	$= -\dfrac{1}{\rho_0} \underline{\nabla}_z p$	horizontal momentum.
$\dfrac{\partial p}{\partial z}$	$= -\rho g$	hydrostatic balance
$\dfrac{D\rho}{Dt}$	$= 0$	adiabatic approximation.
$\dfrac{\partial w}{\partial z}$	$= -\underline{\nabla}_z \cdot \underline{u}$	3D incompressibility

To examine specific phenomena we have simplified these equations using ad hoc approximations, e.g.

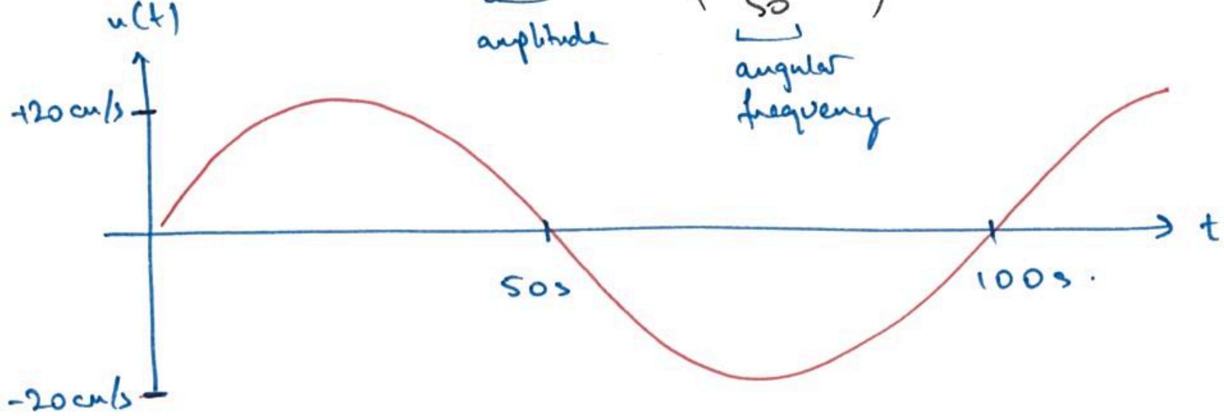
* approximate stratification as layers of constant density
  $\Rightarrow$ shallow water model.
* steady flow with no variation in x-direction
  $\Rightarrow$ Hadley cell.
* neglect nonlinear terms
  $\Rightarrow$ shallow water waves.

How do we rigorously determine the importance of each term in the primitive equations.

SCALING ANALYSIS.

Example: eastward current speed at a current meter.

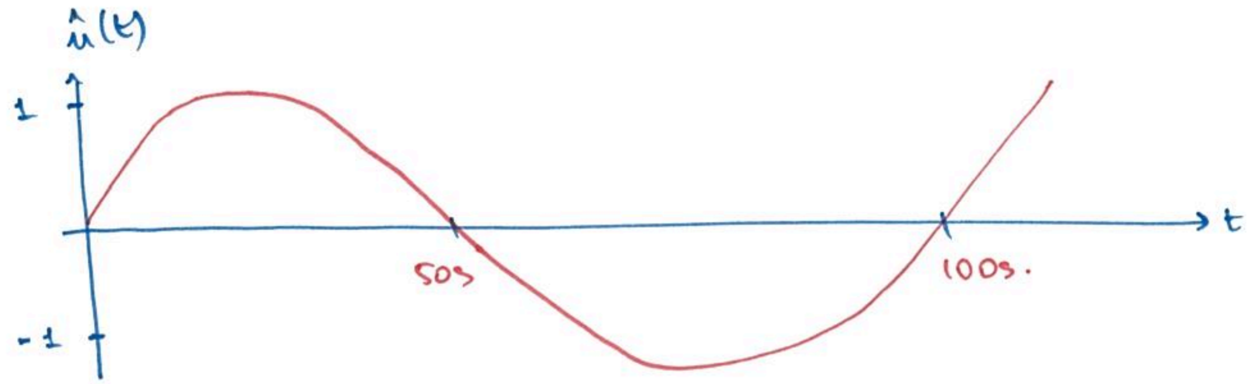
$$u(t) \approx \underbrace{20}_{\text{amplitude}} \sin\left(\underbrace{\frac{\pi}{50}}_{\substack{\text{angular}\\\text{frequency}}} \cdot t\right) \quad \text{cm/s}.$$



$$\omega = \frac{\pi}{50} \quad s^{-1} \qquad T = \frac{2\pi}{\omega} = 100 \text{ s}.$$

Rewrite as

$$u(t) = \underset{\substack{\text{dimensional}\\\text{scale}}}{U}\ \underset{\substack{\text{non-dimensional}\\O(1)\ \text{function.}}}{\hat{u}(t)} \qquad U = 20 \text{ cm/s}.$$

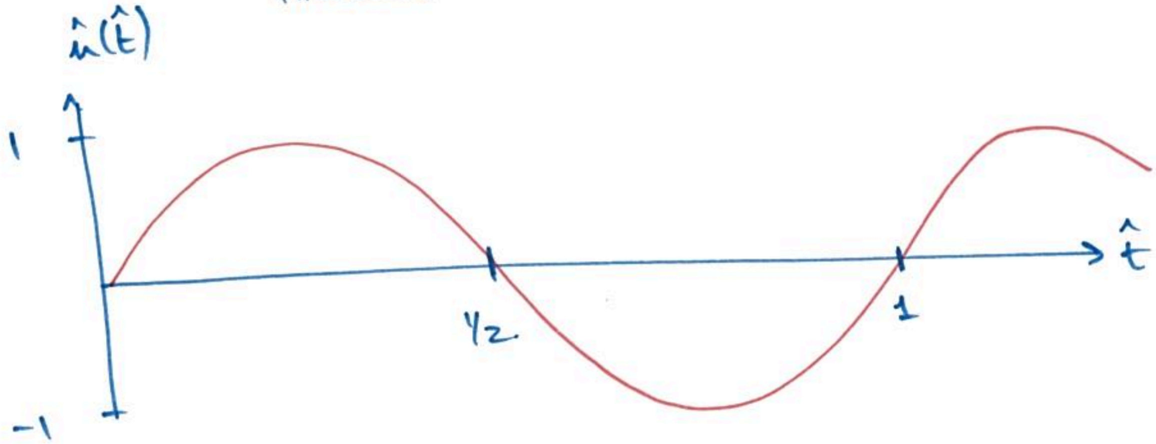


Can also write time as

$$t = T \hat{t} \qquad T = 100 \text{ s.}$$

dimensional timescale

dimensionless $O(1)$ variable



What about terms like $\frac{\partial u}{\partial t}$?

$$u(t) = U \hat{u}(t) \qquad t = T\hat{t}$$

$$\underbrace{\frac{\partial u}{\partial t}}_{[\text{cm/s}]} = \frac{\partial u}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial t} = \underbrace{\frac{U}{T}}_{[\text{cm/s}]} \underbrace{\frac{\partial \hat{u}}{\partial \hat{t}}}_{\text{unitless and } O(1)}$$

Thus can formally write

$$\frac{\partial}{\partial t} \longrightarrow \frac{1}{T} \frac{\partial}{\partial \hat{t}}$$

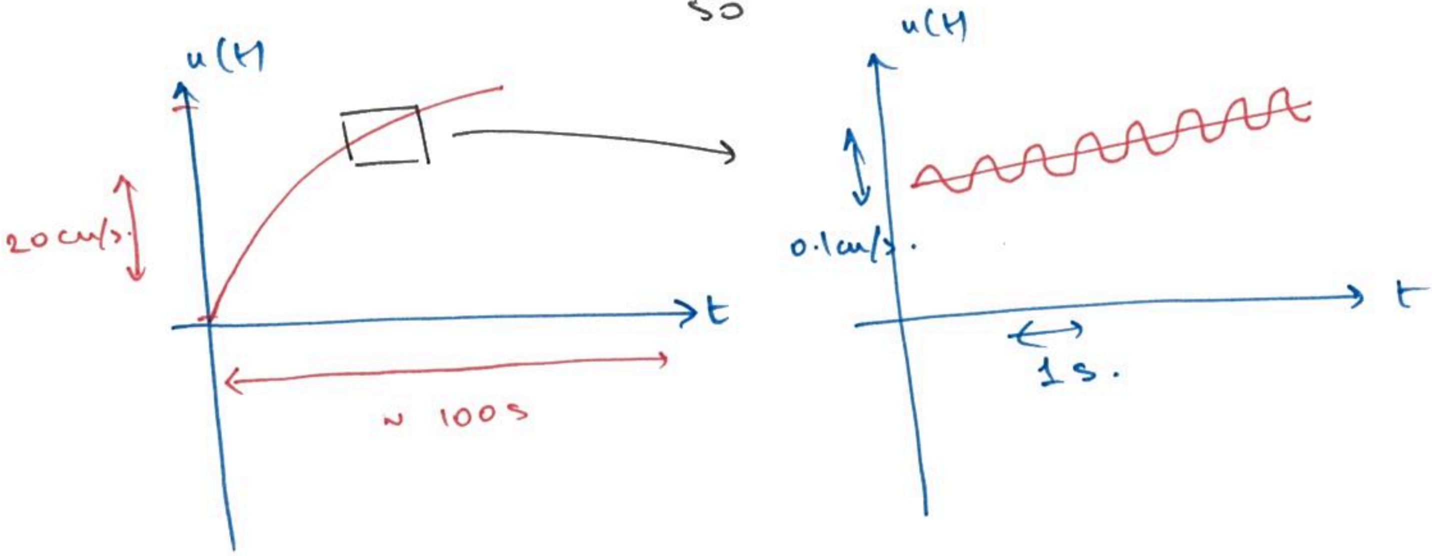
$$\frac{\partial}{\partial x} \longrightarrow \frac{1}{L} \frac{\partial}{\partial \hat{x}}$$

"hatted" variables are always $O(1)$ and dimensionless.

NB: scaling is a <u>choice</u> that determines the dynamics we're interested in.

→ "zooming in"

$$u(t) = 20 \sin \frac{\pi t}{50} + 0.1 \sin \pi t \cdot \quad \text{cm/s}.$$



SCALING THEORY IN THE SHALLOW WATER MODEL

Momentum equation:

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla}\, \underline{u} + \underline{f} \times \underline{u} = -g \underline{\nabla}_z \eta .$$

Write:

$$\underline{u} = U \hat{\underline{u}}$$
$$t = T \hat{t}$$
$$\underline{\nabla} = L^{-1} \hat{\underline{\nabla}}$$
$$\partial/\partial t = T^{-1} \partial/\partial \hat{t}$$
$$f = f_0 \hat{f}$$
$$\eta = \mathcal{H} \hat{\eta}$$

$$\frac{U}{T} \frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \frac{U^2}{L} \hat{\underline{u}} \cdot \hat{\underline{\nabla}} \hat{\underline{u}} + f_0 U \hat{\underline{f}} \times \hat{\underline{u}} = -\frac{g\mathcal{H}}{L} \hat{\underline{\nabla}}_z \hat{\eta}$$

First choice: timescale T

$\hookrightarrow$ Choose: $T = \dfrac{L}{U}$

$$\frac{U^2}{L}\left[\frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \hat{\underline{u}} \cdot \hat{\underline{\nabla}}\, \hat{\underline{u}}\right] + f_0 U\, \hat{\underline{f}} \times \hat{\underline{u}} = -\frac{g\mathcal{H}}{L}\, \hat{\underline{\nabla}}_z \hat{\eta}$$

$\times\ ^1/_{f_0 U}$

$$\boxed{\frac{U}{f_0 L}}\left[\frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \hat{\underline{u}} \cdot \nabla \hat{\underline{u}}\right] + \hat{\underline{f}} \times \hat{\underline{u}} = \boxed{-\frac{g\mathcal{H}}{f_0 U L}}\, \hat{\underline{\nabla}}_z \hat{\eta}$$

$\hookrightarrow$ Rossby number $Ro = \dfrac{U}{f_0 L}$

depends on our choice of $\mathcal{H}$

Second choice: vertical perturbation $\mathcal{H}$

To preserve geostrophic balance must have

$$\frac{g\mathcal{H}}{f_0 U L} = 1. \quad \Rightarrow \quad \mathcal{H} = \frac{f_0 U L}{g}.$$

Non-dimensional horizontal momentum equation:

$$\boxed{Ro \left[\frac{\partial \hat{\underline{u}}}{\partial t} + \hat{\underline{u}} \cdot \hat{\underline{\nabla}}\, \hat{\underline{u}}\right] + \hat{\underline{f}} \times \hat{\underline{u}} = -\hat{\underline{\nabla}}_z \hat{\eta}}$$

* $Ro \longrightarrow 0 \quad \Rightarrow$ geostrophic balance

$$\hat{\underline{f}} \times \hat{\underline{u}} = -\hat{\underline{\nabla}}_z \hat{\eta}.$$

* Ro small but not zero $\Rightarrow$ go beyond geostrophy

$$\hat{\underline{u}} = \hat{\underline{u}}_0 + Ro\, \hat{\underline{u}}_1 + Ro^2\, \hat{\underline{u}}_2 + \cdots$$

$$\hat{\eta} = \hat{\eta}_0 + Ro\, \hat{\eta}_1 + Ro^2\, \hat{\eta}_1 + \cdots$$

Continuity equation.

$$\frac{\partial h}{\partial t} + \underline{u}\cdot\underline{\nabla} h + h\,\underline{\nabla}\cdot\underline{u} = 0.$$

Write $h = H + \eta$ where H = constant (assume flat bottom).

$$\Rightarrow \quad \frac{\partial \eta}{\partial t} + \underline{u}\cdot\underline{\nabla}\eta + (H+\eta)\,\underline{\nabla}\cdot\underline{u} = 0.$$

Now: $\eta = \mathcal{H}\hat{\eta}$, $\underline{u} = U\underline{\hat{u}}$, $\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial \hat{t}}$, $\underline{\nabla} = \frac{1}{L}\underline{\hat{\nabla}}$

where again we choose $T = \frac{L}{U}$, $\mathcal{H} = \frac{f_0 U L}{g}$.

$$\cancel{\frac{\mathcal{H}U}{L}}\frac{\partial\hat{\eta}}{\partial\hat{t}} + \cancel{\frac{\mathcal{H}U}{L}}\,\underline{\hat{u}}\cdot\underline{\hat{\nabla}}\hat{\eta} + (H + \mathcal{H}\hat{\eta})\cancel{\frac{U}{L}}\,\underline{\hat{\nabla}}\cdot\underline{\hat{u}} = 0$$

$$\times 1/H \Rightarrow \quad \boxed{\frac{\mathcal{H}}{H}\left(\frac{\partial\hat{\eta}}{\partial\hat{t}} + \underline{\hat{u}}\cdot\underline{\hat{\nabla}}\hat{\eta}\right) + \left(1 + \frac{\mathcal{H}}{H}\hat{\eta}\right)\underline{\hat{\nabla}}\cdot\underline{\hat{u}} = 0}$$

$$\text{Consider} \quad \frac{\mathcal{H}}{H} = \frac{f_0 U L}{g H} = \underbrace{\frac{U}{f_0 L}}_{Ro}\cdot\underbrace{\frac{f_0^2 L^2}{gH}}_{F}$$

where $Ro = \frac{U}{f_0 L}$.

$F = \frac{L^2}{L_D^2}$, $L_D = \frac{\sqrt{gH}}{f_0}$. Rossby deformation scale

L_d = Rossby deformation scale

$$= \frac{\text{"stratification"}}{\text{"rotation"}}$$

$L \gg L_d$: rotation very important (can neglect surface waves)
($F \gg 1$)

$L \ll L_d$: rotation not important compared to stratification / gravity
($F \ll 1$)

$L \sim L_d$: both stratification & rotation are important
($F \sim 1$)

Ocean : $L_d \sim 100$ km "mesoscale" (scale of ocean eddies)

so $L \gg 100$ km $\sim$ "basin scale".

Atmosphere : $L_d \sim 1000$ km "synoptic scale" (scale of weather)

so $L \gg 1000$ km $\sim$ "planetary scale"

For now, let F take any value :

$$\frac{\mathcal{H}}{H} = Ro\, F \cdot$$

$$\boxed{Ro\, F \left(\frac{\partial \hat{\eta}}{\partial \hat{t}} + \hat{\underline{u}} \cdot \hat{\underline{\nabla}} \hat{\eta} \right) + \left(1 + Ro\, F\, \hat{\eta} \right) \hat{\underline{\nabla}} \cdot \hat{\underline{u}} = 0}$$

non-dimensional continuity equation.

Two limits :

* $Ro \ll 1$, $F \gg 1$ $\Rightarrow$ planetary geostrophic equations
* $Ro \ll 1$, $F \sim 1$ $\Rightarrow$ quasigeostrophic equations.

Planetary geostrophic equations:

Assume that $Ro \cdot F \sim O(1)$

non-dimensional momentum equation:

$$\cancel{Ro\left(\frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \hat{\underline{u}} \cdot \hat{\underline{\nabla}} \underline{u}\right)} + \hat{\underline{f}} \times \hat{\underline{u}} = -\hat{\underline{\nabla}} \eta .$$

$$\Rightarrow \boxed{\hat{\underline{f}} \times \hat{\underline{u}} = -\hat{\underline{\nabla}} \hat{\eta}}$$ geostrophic balance

non-dimensional continuity equation:

$$\boxed{Ro \cdot F\left(\frac{\partial \hat{\eta}}{\partial \hat{t}} + \hat{\underline{u}} \cdot \hat{\underline{\nabla}} \hat{\eta}\right) + \left(1 + Ro \cdot F\, \hat{\eta}\right) \hat{\underline{\nabla}} \cdot \hat{\underline{u}} = 0}$$

Re-express in terms of dimensional quantities:

$$\boxed{\begin{aligned} \underline{f} \times \underline{u} &= -g \underline{\nabla} \eta . \\ \frac{\partial h}{\partial t} + \underline{u} \cdot \underline{\nabla} h + h \cancel{\underline{\nabla} \cdot \underline{u}} &= 0 \end{aligned}}$$

$\rightarrow$ $\underline{u} = (-\psi_y, \psi_x)$

$\underline{\nabla} \cdot \underline{u} = 0$

Dimensional planetary geostrophic equations.

Quasigeostrophic (QG) model: $Ro \ll 1$, $F \sim O(1)$
($L \sim L_D$).

$\Rightarrow$ asymptotic expansion of the equations of motion in the limit of small Ro.

* Write fields as an expansion in powers of $Ro \ll 1$:

$$\hat{\eta} = \underbrace{\hat{\eta}_0}_{O(1)} + \underbrace{Ro \cdot \hat{\eta}_1}_{O(Ro)} + \underbrace{Ro^2 \cdot \hat{\eta}_2}_{O(Ro^2)} + \cdots$$

where $\hat{\eta}_0, \hat{\eta}_1, \hat{\eta}_2, \cdots$ are all $O(1)$.

$$\hat{\underline{u}} = \hat{\underline{u}}_0 + Ro \cdot \hat{\underline{u}}_1 + Ro^2 \, \hat{\underline{u}}_2 + \cdots$$

where $\hat{\underline{u}}_0, \hat{\underline{u}}_1, \hat{\underline{u}}_2, \cdots$ are all $O(1)$

$\hat{\underline{u}}_0, \hat{\eta}_0$ = "geostrophic flow" ($Ro = 0$)
$\hat{\underline{u}}_1, \hat{\eta}_1, \cdots$ = "ageostrophic flow" ($Ro \neq 0$)

$$\hat{f} = \frac{f_0 + \beta y}{f_0} = 1 + \frac{\beta y}{f_0} = 1 + \frac{\beta L}{f_0} \hat{y}$$

$$= 1 + \boxed{\frac{U}{f_0 L}} \boxed{\frac{\beta L^2}{U}} \hat{y}$$

$\downarrow$ Ro $\qquad \downarrow$ $\hat{\beta} = \frac{\beta L^2}{U}$ $\qquad [\beta] = \frac{1}{TL} = \frac{U}{L^2}$

$$\Rightarrow \boxed{\hat{f} = 1 + Ro \, \hat{\beta} \, \hat{y}} \qquad \text{where } \hat{\beta} = O(1)$$

NEXT WEEK: sub. into non-dimensional momentum + continuity equation.
→ match coefficients of each power of Ro.
→ hierarchy of equations for $\hat{\underline{u}}_0, \hat{\eta}_0, \hat{\underline{u}}_1, \hat{\eta}_1 \cdots$