Recall : Geostrophic Balance.

Horizontal momentum equation (Boussinesq approx)

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u} + \underline{f}\times\underline{u} = -\frac{1}{\rho}\underline{\nabla}_z p$$

$f\hat{\underline{z}}$

$f = 2\Omega \sin\phi$

reference density (ρ_0)

pressure anomaly $(\tilde{p})$

$$\underline{u} = (u, v) \qquad \underline{\nabla}_z = (\partial_x, \partial_y)$$

$$\underbrace{\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u}}_{U^2/L} + \underbrace{\underline{f}\times\underline{u}}_{fU} = \underbrace{-\frac{1}{\rho}\underline{\nabla}_z p}_{\frac{P}{\rho L}}$$

$T_{adv} = L/U.$

$$\text{Ratio of inertial to Coriolis} = \frac{U^2/L}{fU} = \frac{U}{fL} = Ro.$$

For "rapidly rotating" $Ro \to 0$

$\Rightarrow$ Geostrophic balance: $\underline{f}\times\underline{u} = -\frac{1}{\rho}\underline{\nabla}_z p.$

Pressure scale in geostrophic balance?

$$fU \sim \frac{P}{\rho L} \Rightarrow P \sim fU\rho L.$$

E.g. $f \sim 10^{-4}\,s^{-1}$ (typical midlatitude flow)

$U \sim 10^{-1}\,m\,s^{-1}$ (ocean), $\rho \sim 1000\,kg\,m^{-3}$ (seawater)

$L \sim 10^5\,m \sim 100\,km$ (ocean eddy)

Rossby number in ocean:

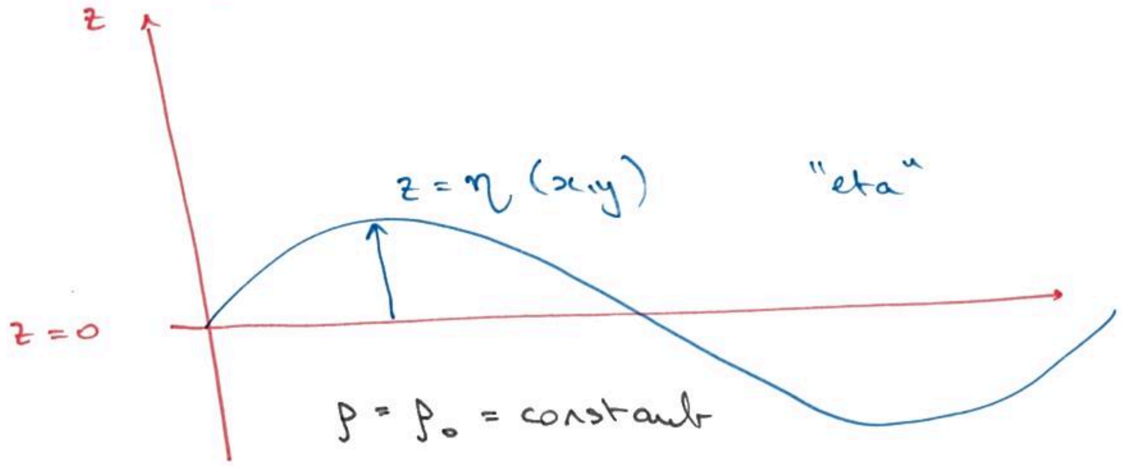
$$R_o \sim \frac{U}{fL} = \frac{10^{-1}}{10^{-4}\,10^{5}} \sim 0.01 \ll 1.$$

Pressure scale:

$$p \sim fU\rho L = 10^{-4} \times 10^{-1} \times 10^{3} \times 10^{5} \sim 10^{3}.$$

$$\sim 1000\ \text{Pa}.$$

Example: geostrophic balance in the ocean



Hydrostatic balance:

$$\frac{dp}{dz} = -\rho_0 g \Rightarrow \int_{p(z)}^{p_0} dp = -\int_{z}^{\eta(x,y)} \rho_0 g\, dz$$

$$p_0 - p(z) = -\rho_0 g(\eta - z)$$

$$\Rightarrow p(z) = \underbrace{p_0}_{\text{air pressure}} - \underbrace{\rho_0 g z}_{\text{water pressure}} + \underbrace{\rho_0 g \eta(x,y)}_{\text{modification due to } \eta(x,y)}$$

($z<0$ in the ocean)

Geostrophic balance:

$$\underline{f} \times \underline{u} = f\,\hat{\underline{z}} \times \underline{u} = -\frac{1}{\rho_0} \underline{\nabla}_z \left(p_0 - \rho_0 g z + \rho_0 g \eta(x,y) \right)$$

$$= -g \underline{\nabla}_z \eta(x,y)$$

If on an f-plane $(f = f_0)$

$$\Rightarrow \quad \hat{\underline{z}} \times \underline{u} = -\frac{g}{f} \underline{\nabla}_z \eta = -\underline{\nabla}_z \left(\frac{g\eta}{f} \right).$$

Useful identity: if $\underline{u} \perp^{ar} \hat{\underline{z}}$ then

$$\boxed{\hat{\underline{z}} \times (\hat{\underline{z}} \times \underline{u}) = -\underline{u}}$$

"uncross" equations.

Proof: $\underline{u} = u\hat{\underline{x}} + v\hat{\underline{y}}$

$$\hat{\underline{z}} \times \underline{u} = u\,\hat{\underline{z}} \times \hat{\underline{x}} + v\,\hat{\underline{z}} \times \hat{\underline{y}} = u\hat{\underline{y}} - v\hat{\underline{x}}$$

$$\hat{\underline{z}} \times (\hat{\underline{z}} \times \underline{u}) = u\,\hat{\underline{z}} \times \hat{\underline{y}} - v\,\hat{\underline{z}} \times \hat{\underline{x}} = -u\hat{\underline{x}} - v\hat{\underline{y}} = -\underline{u}$$

Exercise: For $\underline{v} = (u, v, w)$ prove that

$$\hat{\underline{z}} \times (\hat{\underline{z}} \times \underline{v}) = w\hat{\underline{z}} - \underline{v} = -u\hat{\underline{x}} - v\hat{\underline{y}}$$ NB

Applying $\hat{\underline{z}} \times$ to each side:

$$\hat{\underline{z}} \times (\hat{\underline{z}} \times \underline{u}) = -\hat{\underline{z}} \times \underline{\nabla}_z \left(\frac{g\eta}{f} \right)$$

$$\boxed{\begin{aligned} \underline{u} &= \hat{\underline{z}} \times \underline{\nabla}_z \psi \\ &= \underline{\nabla}_\perp \psi. \end{aligned}} \qquad \psi = \frac{g\eta}{f}$$

= "geostrophic streamfunction"

$$\underline{\nabla}_\perp = \hat{\underline{z}} \times \underline{\nabla}_z = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right)$$

Implications:

* Flow is incompressible in 2D:

$$\underline{\nabla}_z \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$$

* ψ is a streamfunction

$\Rightarrow$ flow is along lines of constant $\psi = \dfrac{g\eta}{f_0}$

$\Rightarrow$ flow is along lines of constant sea surface height η

* if we know $\eta(x,y)$ we know the flow!

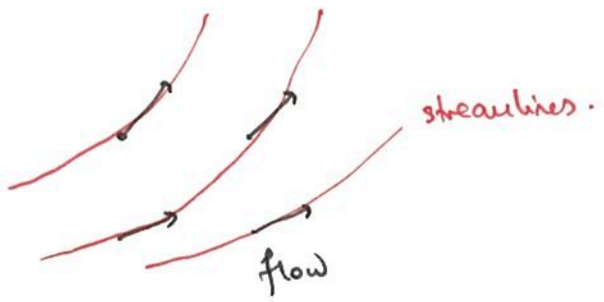
ASIDE: In general can write

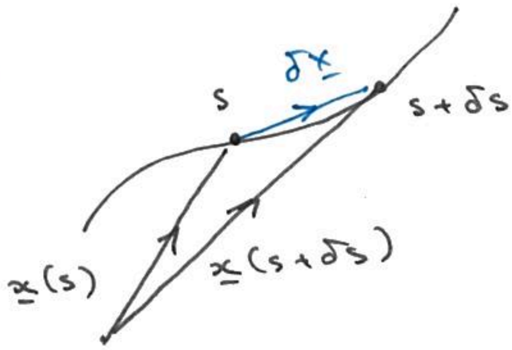
$$\underline{u}_{2D} = \underbrace{\underline{\nabla}\phi}_{\text{potential function}} + \underbrace{\underline{\nabla}_\perp \psi}_{\text{streamfunction.}}$$

Helmholtz decomposition.

$$\underline{\nabla} \cdot \underline{u}_{2D} = \nabla^2 \phi \qquad \left(\underline{\nabla} \cdot \underline{\nabla}_\perp \psi = 0\right).$$

Streamline : A line tangent to $\underline{v}$ everywhere at fixed t.





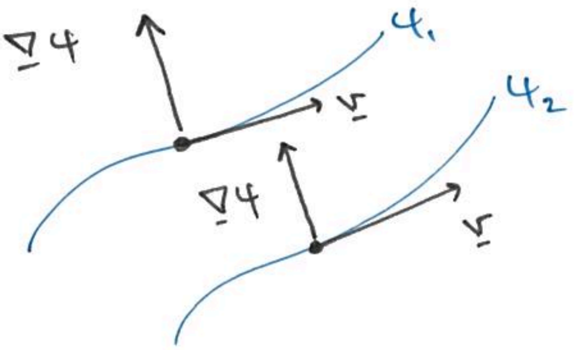
$\delta \underline{x} \,//\, \underline{v}$ as $\delta s \longrightarrow 0$

Def : $\dfrac{d\underline{x}}{ds}$ is tangent to $\underline{v}$: $\dfrac{d\underline{x}}{ds} \times \underline{v} = 0$.

$$\Rightarrow \quad \frac{d\underline{x}}{ds} = \lambda \underline{v} \quad \xrightarrow{\text{rescale } s} \quad \frac{d\underline{x}}{ds} = \underline{v}$$

NB : a streamline is not the same thing as a particle trajectory except when flow is stationary.

Streamfunction: function whose level sets are streamlines [only makes sense for incompressible flow] In 2D:



$$\underline{n} = \underline{\nabla}\psi$$

$$\underline{v} = \hat{\underline{z}} \times \underline{\nabla}\psi = \underline{\nabla}_\perp \psi$$

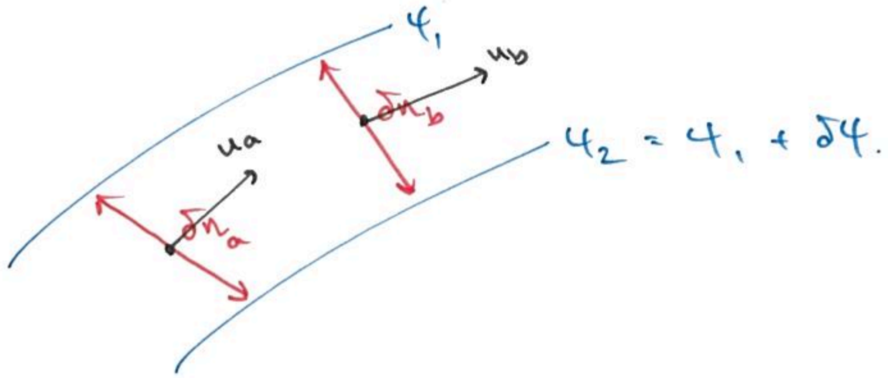
$$= (-\psi_y, \psi_x).$$

By construction: $\underline{\nabla}\psi \cdot \underline{\nabla}_\perp \psi = \underline{n} \cdot \underline{v} = 0$

Note also: $\underline{\nabla} \cdot \underline{\nabla}_\perp \psi = 0$

Velocity + streamfunction :

Distance between streamlines $\longrightarrow$ velocity



$$\underline{u} = \underline{\nabla}_\perp \psi \sim \frac{\delta\psi}{\delta n}$$

$$u_a \sim \frac{\delta\psi_a}{\delta n_a} \qquad u_b \sim \frac{\delta\psi_b}{\delta n_b}. \qquad \text{but } \delta\psi_a = \delta\psi_b = \delta\psi$$

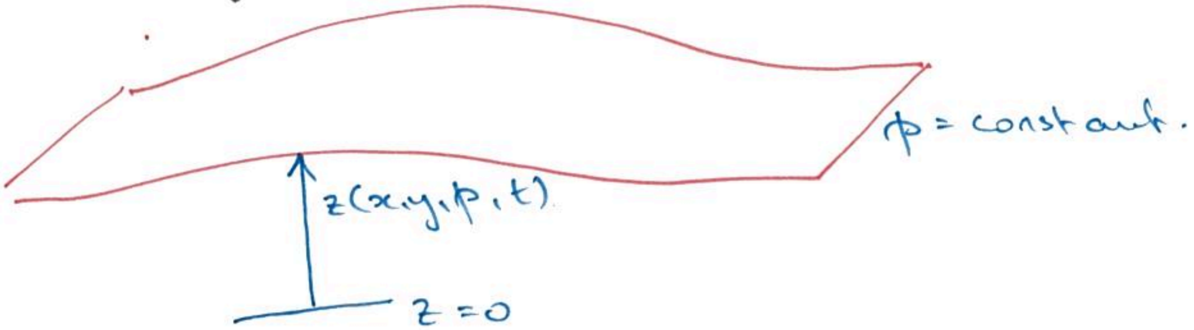
$$\Rightarrow \quad u_a \sim \frac{\delta\psi}{\delta n_a} \qquad u_b \sim \frac{\delta\psi}{\delta n_b}.$$

$\Rightarrow \quad u_a \uparrow \quad$ if $\quad \delta n_a \downarrow.$

Example: Geostrophic balance in the atmosphere

Moral: pressure is a more useful "vertical coordinate" for compressible flows (atmosphere)

$$z = z(x, y, p, t)$$
$$= \text{height of a constant pressure surface}$$



Transforming between "pressure coordinates" and "z-coordinate"

Consider a field $f = f(x, y, z(x, y, p))$

$\underline{\nabla}_z$ = horizontal gradient at fixed height z

$\underline{\nabla}_p$ = horizontal gradient at fixed pressure p.

$$\underline{\nabla}_p f = \underline{\nabla} f \Big|_{p=\text{constant}}$$

$$= \underline{\nabla} f \Big|_{z=\text{constant}} + \frac{\partial f}{\partial z} \underline{\nabla} z \Big|_{p=\text{constant}}$$

E.g. $\displaystyle \frac{\partial f}{\partial x}\Big|_p = \frac{\partial f}{\partial x}\Big|_z + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}\Big|_p .$

Now take $f = p$:

$$\underline{\nabla} p\Big|_{p=\text{constant}} = 0 = \underline{\nabla} p\Big|_z + \frac{\partial p}{\partial z}\,\underline{\nabla} z\Big|_p .$$

$$\boxed{\underline{\nabla}_z p = -\frac{\partial p}{\partial z}\,\underline{\nabla}_p z = \rho g\,\underline{\nabla}_p z}$$

Geostrophic balance in pressure coordinates:

$$\boxed{\underline{f} \times \underline{u} = -\frac{1}{\rho}\,\underline{\nabla}_z p = -g\,\underline{\nabla}_p z}$$

THERMAL WIND BALANCE

Geostrophic balance	+	Hydrostatic balance	=	Thermal wind balance
horizontal p-grad.		vertical p-grad		vertical shear
+		+		+
Coriolis force		gravity		horizontal density gradients

Boussinesq: Geostrophy

$$\underline{f} \times \underline{u} = -\frac{1}{\rho_0}\,\underline{\nabla}_z p .$$

$$\frac{\partial}{\partial z}: \quad \underline{f} \times \frac{\partial \underline{u}}{\partial z} = -\frac{1}{\rho_0}\,\underline{\nabla}_z \boxed{\frac{\partial p}{\partial z}} = -\rho g \quad \text{(HS balance)}$$

$$\boxed{\underline{f} \times \frac{\partial \underline{u}}{\partial z} = \frac{g}{\rho_0}\,\underline{\nabla}_z \rho} \quad \text{Thermal wind balance}$$

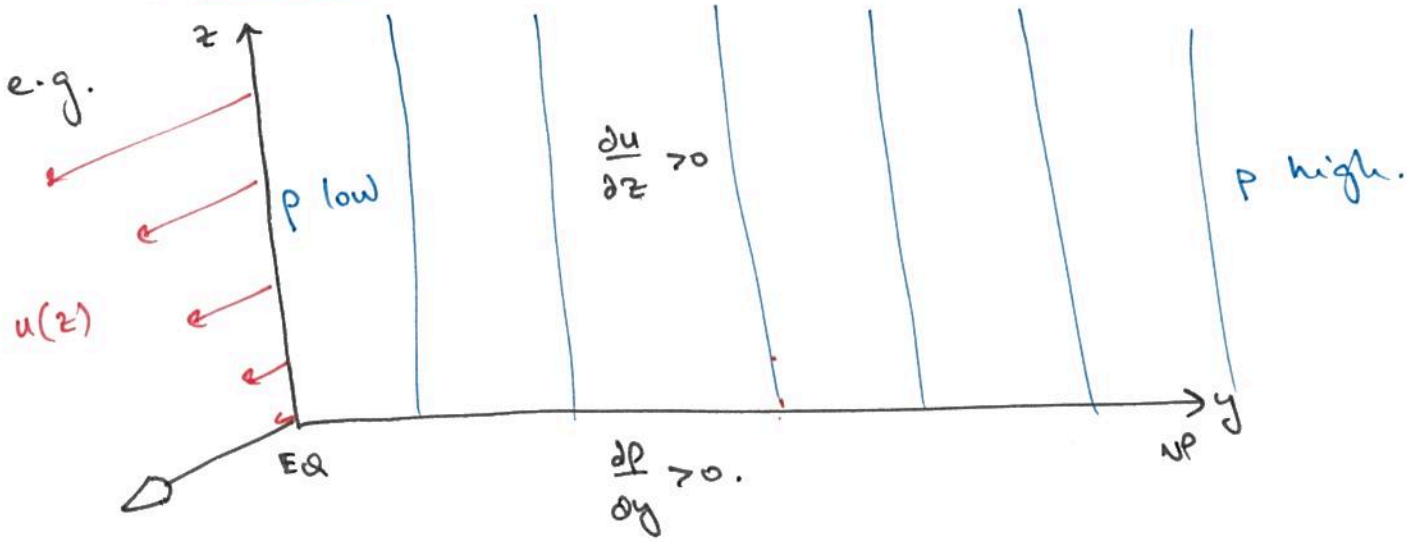
$$\hat{\underline{z}} \times \frac{\partial \underline{u}}{\partial z} = \frac{g}{f\rho_0} \underline{\nabla}_z \rho \qquad \underline{f} = f\hat{\underline{z}}$$

$$\hat{\underline{z}} \times \left(\hat{\underline{z}} \times \frac{\partial \underline{u}}{\partial z}\right) = -\frac{\partial \underline{u}}{\partial z}$$

$$\Rightarrow \quad \boxed{\frac{\partial \underline{u}}{\partial z} = -\frac{g}{f\rho_0} \hat{\underline{z}} \times \underline{\nabla}_z \rho .}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{g}{f\rho_0} \frac{\partial \rho}{\partial y} \qquad \frac{\partial v}{\partial z} = -\frac{g}{f\rho_0} \frac{\partial \rho}{\partial x}}$$

e.g.



Taylor - Proudman Theorem and Thermal Wind

3D Navier Stokes equation (no approximation):

$$\frac{D\underline{v}}{Dt} + 2\underline{\Omega} \times \underline{v} = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla}\Phi$$

full 3D velocity (u, v, w)

$\underline{\Omega} = \Omega \hat{\underline{z}}$

Gravitational potential.

Assume:

* $Ro \longrightarrow 0$ (neglect inertial term)
* $\underline{\nabla} \cdot \underline{v} = 0$ (3D incompressible)

$$2\underline{\Omega} \times \underline{v} = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla}\Phi$$

horizontal components $\longrightarrow$ Geostrophic balance

vertical component $\longrightarrow$ Hydrostatic balance

Curl both sides:

$$\underline{\nabla} \times (2\underline{\Omega} \times \underline{v}) = -\underline{\nabla}\times\left(\frac{1}{\rho}\underline{\nabla} p\right) - \cancelto{0}{\underline{\nabla} \times \underline{\nabla}\Phi}$$

$$\overset{\text{product}}{\underset{\text{rule}}{=}} -\underline{\nabla}\left(\frac{1}{\rho}\right) \times \underline{\nabla} p - \frac{1}{\rho}\cancelto{0}{\underline{\nabla} \times \underline{\nabla} p}$$

$$= \frac{1}{\rho^2}\underline{\nabla}\rho \times \underline{\nabla} p.$$

Vector identity:

$$\underline{\nabla} \times (\underline{A} \times \underline{B}) = \underline{A}(\underline{\nabla}\cdot\underline{B}) - \underline{B}(\underline{\nabla}\cdot\underline{A}) + \underline{B}\cdot\underline{\nabla}\,\underline{A} - \underline{A}\cdot\underline{\nabla}\,\underline{B}$$

$$\underline{\nabla} \times (2\underline{\Omega} \times \underline{v}) = \underbrace{\cancel{2\underline{\Omega}(\underline{\nabla}\cdot\underline{v})}}_{\text{incompressible.}} - \underbrace{\cancel{\underline{v}(\underline{\nabla}\cdot 2\underline{\Omega})}}_{\Omega = \text{const}} + \underbrace{\cancel{\underline{v}\cdot\underline{\nabla}\,2\underline{\Omega}}}_{\Omega = \text{const.}} - 2\underline{\Omega}\cdot\underline{\nabla}\,\underline{v}$$

$$= -2\underline{\Omega}\cdot\underline{\nabla}\,\underline{v}$$

So:
$$\boxed{-2\underline{\Omega}\cdot\underline{\nabla}\,\underline{v} = \frac{1}{\rho^2}\underline{\nabla}\rho \times \underline{\nabla}p.} \qquad (*)$$

Taylor - Proudman theorem.

J. Proudman (1916) & G. I. Taylor (1921) + experimental verification.

<u>Definition</u>: A fluid is called a <u>barotropic fluid</u> if its density is a function of pressure only:

$$\rho = \rho(p).$$

<u>Theorem</u>: For a rapidly rotating, incompressible barotropic fluid, the velocity $\underline{v} = (u, v, w)$ has no variation along the axis of rotation.

<u>Proof</u>: From $(*)$ a rapidly rotating incompressible fluid satisfies

$$-2\underline{\Omega}\cdot\underline{\nabla}\,\underline{v} = \frac{1}{\rho^2}\underline{\nabla}\rho \times \underline{\nabla}p.$$

For a barotropic fluid $\rho = \rho(p)$

$$\underline{\nabla}\rho = \frac{d\rho}{dp}\underline{\nabla}p. \quad \Rightarrow \quad \underline{\nabla}\rho \times \underline{\nabla}p = \frac{d\rho}{dp}\underline{\nabla}p \times \underline{\nabla}p = \underline{0}$$

From (*) then

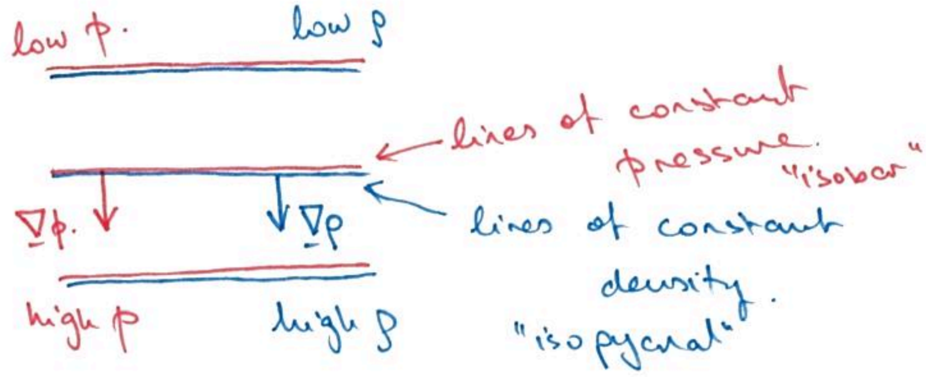
$$-2\underline{\Omega}\cdot\underline{\nabla}\,\underline{v} = \underline{0}. \qquad \underline{\Omega} = \Omega\,\hat{\underline{z}}$$

$$\Rightarrow \quad \frac{\partial \underline{v}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0.$$

Therefore u, v, w have no variation in z-direction.

Barotropic fluid.

$\rho = \rho(p)$



· surfaces of constant ρ = surfaces of constant p

$$\Rightarrow \quad \underline{\nabla}\rho \;\; // \;\; \underline{\nabla}p.$$

Corollary :

If there exists a solid boundary within the fluid then

* $w = 0$ everywhere (flow is effectively 2D)
* flow is confined into cylinders aligned along axis of rotation ("Taylor columns")

Proof : since $\dfrac{\partial w}{\partial z} = 0$, then if $w = 0$ on a boundary then $w = 0$ everywhere.

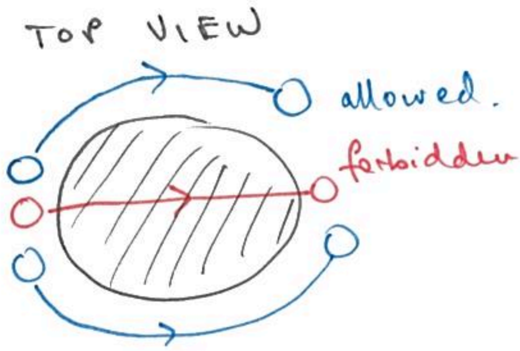
Additionally, since $\dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0$, u and v are constant in the z-direction. $\Rightarrow$ effectively 2D "barotropic flow"

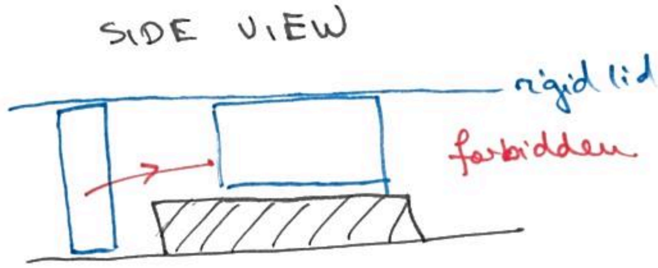
Physically: rotation makes the fluid "rigid" → it resists deformation in the z-direction and instead form columns that cannot be tilted, squashed, stretched

u, v are constant in z

$w = 0$

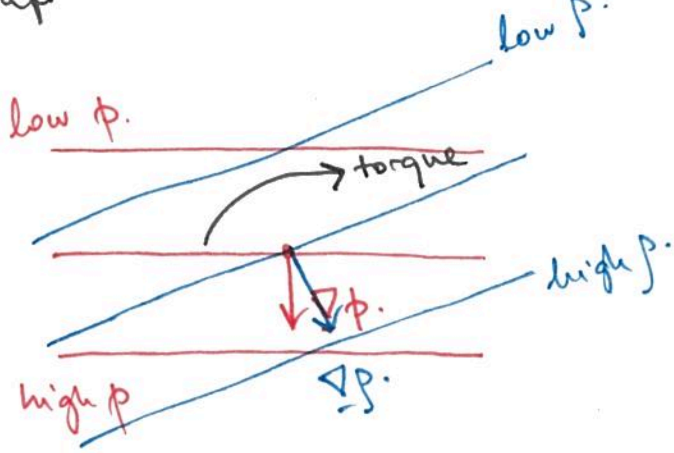
Example: flow around a mountain / sea mount





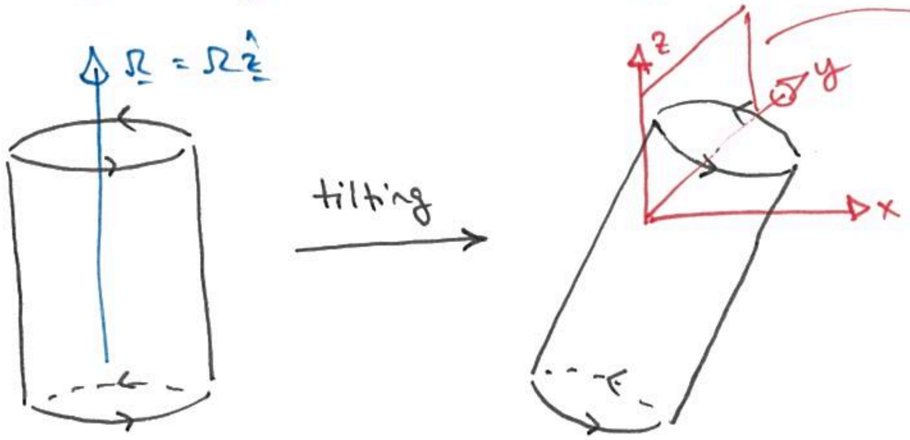
BAROCLINIC FLUID.

When $\rho \neq \rho(p)$ then surfaces of constant ρ & p. do not overlap

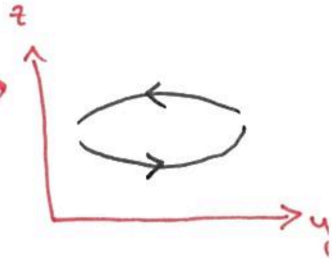


Thus $\underline{\nabla}\rho \times \underline{\nabla} p \neq 0$ (pointed into page)

$$\underbrace{-2\underline{\Omega}\cdot\underline{\nabla}\,\underline{v}}_{\text{vorticity production by tilting}} = \underbrace{\frac{1}{\rho^2}\underline{\nabla}\rho\times\underline{\nabla}p}_{\text{"baroclinic torque"}}.$$



taylor column.



tilting a rotating column "creates" vorticity in the direction of tilt.

Baroclinic torque will balance vorticity production by tilting Taylor columns.

($\Rightarrow$ Thermal wind balance).

A tilting Taylor column will no longer have $\frac{\partial \underline{v}}{\partial z} = 0$.

In fact:

$$-2\underline{\Omega}\cdot\underline{\nabla}\,\underline{v} = -2\Omega\frac{\partial \underline{v}}{\partial z} = \frac{1}{\rho^2}\underline{\nabla}\rho\times\underline{\nabla}p.$$

In atmosphere + ocean: f-plane, thin layer, Boussinesq.

$$-f_0\frac{\partial \underline{u}}{\partial z} = \frac{1}{\rho_0^2}\left(\underline{\nabla}_z\rho + \hat{\underline{z}}\frac{\partial\rho}{\partial z}\right)\times\left(\underline{\nabla}_z p + \hat{\underline{z}}\frac{\partial p}{\partial z}\right).$$

$$= \frac{1}{\rho_0^2}\Bigg(\underline{\nabla}_z\tilde{\rho}\times\hat{\underline{z}}\frac{\partial p_0}{\partial z} + \frac{\partial\tilde{\rho}}{\partial z}\hat{\underline{z}}\times\cancel{\underline{\nabla}_z p_0} + \cancel{\underline{\nabla}_z\rho_0}\times\hat{\underline{z}}\frac{\partial\tilde{p}}{\partial z} + \cancel{\frac{\partial\rho_0}{\partial z}}\hat{\underline{z}}\times\underline{\nabla}\tilde{p}\Bigg) + \text{h.o.t}$$

$$= \frac{1}{\rho_0^2}\underline{\nabla}_z\tilde{\rho}\times\hat{\underline{z}}(-\rho_0 g) = \frac{g}{\rho_0}\hat{\underline{z}}\times\underline{\nabla}_z\tilde{\rho}$$

so $$\frac{\partial \underline{u}}{\partial z} = -\frac{g}{f_0 \rho_0}\, \hat{\underline{z}} \times \underline{\nabla}_z \tilde{\rho} .$$