Ocean: treat as incompressible. (density does not change with pressure).

$$\rho_0 = \text{constant.}$$

Hydostatic balance:

$$\frac{dp}{dz} = -\rho_0 g$$

$$\int dp = -\int \rho_0 g \, dz$$

$$p(z) - p(0) = -\rho_0 g z \quad \Rightarrow \quad p(z) = p(0) - \rho_0 g z$$

$$z < 0.$$

| $z$ | $p$ |
|---|---|
| 0 | 1 atm. |
| −10m | 1.99 atm |
| −100m | 10.9 atm. |
| −1000m | 100.2 atm. |

Density does depend on temperature (T) & salinity (S).

$$\rho(T, S).$$

$$S = \frac{\text{mass of "salt"}}{\text{mass of seawater}}$$

$\sigma = \rho - \rho_0$ = density anomaly

$\rho_0 = 1000 \, kg \, m^{-3}$ = reference density.

Equation of state for seawater:

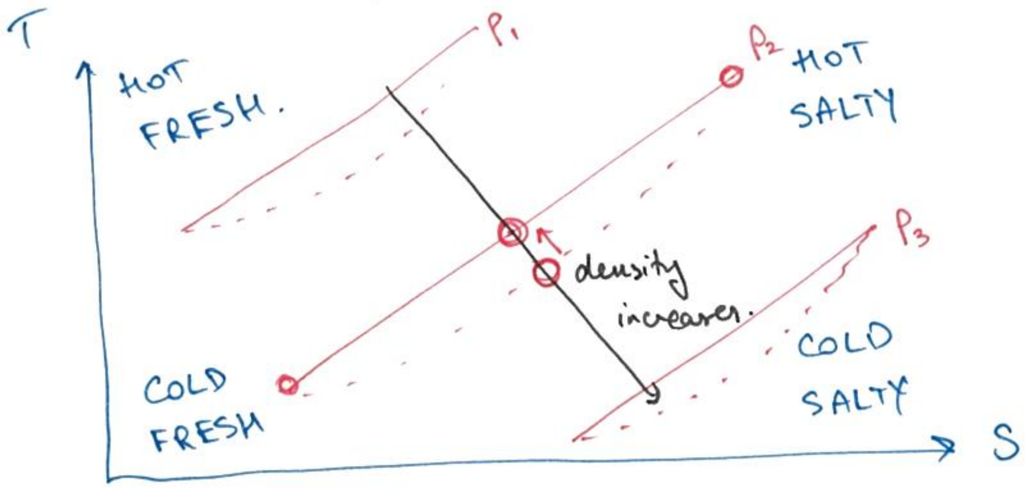
Linearize in $T, S$:

$$\rho(T,S) = \underbrace{\rho(T_0, S_0)}_{\rho_0} + \left.\frac{\partial \rho}{\partial T}\right|_{T_0,S_0} (T-T_0) + \left.\frac{\partial \rho}{\partial S}\right|_{T_0,S_0} (S-S_0) + \cdots$$

$$= \rho_0 \left[ 1 + \underbrace{\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial T}\right|_{T_0,S_0}}_{\text{negative.}} (T-T_0) + \underbrace{\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial S}\right|_{T_0,S_0}}_{\text{positive}} (S-S_0) \right] + \cdots$$

$T\uparrow \ \rho\downarrow$ $\qquad$ $S\uparrow \ \rho\uparrow$

$$= \rho_0 \left( 1 - \alpha_T (T-T_0) + \beta_S (S-S_0) \right)$$




$\rho_1 < \rho_2 < \rho_3$

To close: need. $\dfrac{DT}{Dt} = Q_T$ $\qquad$ $\dfrac{DS}{Dt} = Q_S$

Example : use EOS for dry air and hydrostatic balance to calculate $p(z)$

$$p = \rho R_d T \quad \Rightarrow \quad \rho = \frac{p}{R_d T}$$

Assume an isothermal atmosphere ($T_0$ = constant)

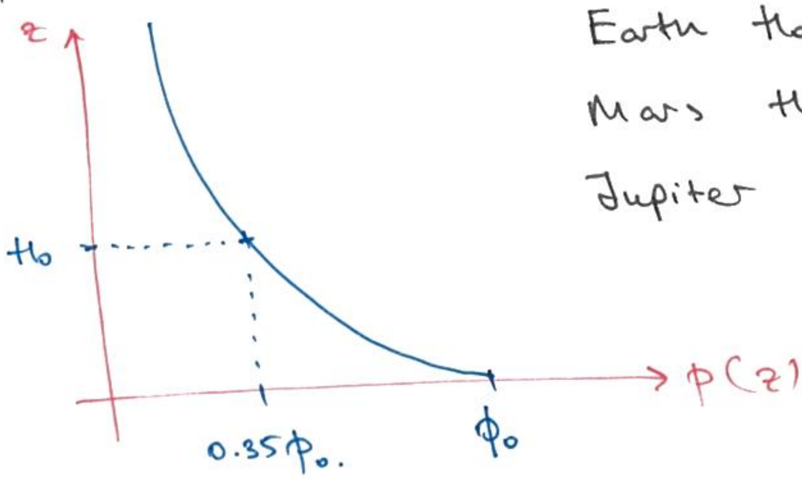
hydrostatic balance: $$\frac{dp}{dz} = -\rho g = -\frac{p g}{R_d T_0}.$$

$$\int \frac{dp}{p} = -\int \frac{g}{R_d T_0} dz \Rightarrow \ln p = -\frac{g z}{R_d T_0} + C.$$

let $C = \ln p_0$. (surface pressure)

$$\ln p - \ln p_0 = -\frac{z}{H_0} \qquad H_0 = \frac{R_d T_0}{g} = \text{"scale height"}$$

$$\Rightarrow p = p_0 e^{-z/H_0}.$$




Earth $H_0 \simeq 7.5$ km.

Mars $H_0 \simeq 11$ km.

Jupiter $H_0 \simeq 27$ km.

# Boussinesq approximation

- exploit smallness of density variations (in the ocean)

$$\rho = \underbrace{\rho_0}_{\text{reference density}} + \underbrace{\tilde{\rho}(x,y,z,t)}_{\text{small perturbation}} \qquad \frac{|\tilde{\rho}|}{\rho_0} \ll 1.$$

- define reference pressure $p_0(z)$ in hydrostatic balance with $\rho_0$

$$p = p_0(z) + \tilde{p}(x,y,z,t) \qquad \frac{|\tilde{p}|}{p_0} \ll 1.$$

where. $$\frac{dp_0}{dz} = -\rho_0 g$$

- sub into momentum equation.

$$\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v} = -\frac{1}{\rho}\underline{\nabla} p - g\hat{\underline{z}}$$

$\times \rho = \rho_0 + \tilde{\rho}$

$$(\rho_0 + \overset{\text{neglect}}{\cancel{\tilde{\rho}}})\left[\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v}\right] = -\underline{\nabla}(p_0 + \overset{\text{keep!}}{\tilde{p}}) - (\rho_0 + \overset{\text{keep!}}{\tilde{\rho}})g\hat{\underline{z}}$$

Boussinesq approx: neglect $\tilde{\rho}$ EXCEPT where multiplied by $g$.

$$\rho_0\left[\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v}\right] \simeq -\frac{dp_0}{dz}\hat{\underline{z}} - \underline{\nabla}\tilde{p} - \rho_0 g\hat{\underline{z}} - \tilde{\rho} g\hat{\underline{z}}$$

cancel (in hydrostatic balance).

$$\Rightarrow \boxed{\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v} = -\frac{1}{\rho_0}\underline{\nabla}\tilde{p} - \underbrace{\frac{\tilde{\rho} g}{\rho_0}}_{\text{reduced gravity}}\hat{\underline{z}}}$$

Boussinesq momentum equation.

## Continuity equation

$$\frac{D\rho}{Dt} = -\rho \underline{\nabla} \cdot \underline{v} \qquad \rho = \rho_0 + \tilde{\rho}$$

$$\Rightarrow \quad \frac{D\tilde{\rho}}{Dt} = \underbrace{-\rho_0 \underline{\nabla} \cdot \underline{v}}_{\text{largest term.}} - \tilde{\rho} \underline{\nabla} \cdot \underline{v}$$

the $\rho_0 \cdot \underline{\nabla} \cdot \underline{v}$ term has no term to balance it so.

$$\Rightarrow \quad \rho_0 \underline{\nabla} \cdot \underline{v} = 0. \qquad !$$

$$\Rightarrow \quad \boxed{\underline{\nabla} \cdot \underline{v} = 0.}$$

In B-approx, continuity equation → incompressibility condition.

* Note: this doesn't mean $\rho = \rho_0 + \tilde{\rho}$ is constant.

or $\frac{D\tilde{\rho}}{Dt} = 0.$

(Need information about sources of heating/cooling.)

But if fluid is adiabatic (no heating/cooling)

$$\Rightarrow \quad \boxed{\frac{D\tilde{\rho}}{Dt} = 0.}$$

## Boussinesq approximations.

$$\frac{D\underline{v}}{Dt} + 2\underline{\Omega} \times \underline{v} = -\frac{1}{\rho_0} \underline{\nabla} \tilde{p} - \frac{\tilde{\rho}}{\rho_0} g \hat{\underline{z}} \qquad \text{(momentum)}$$

$$\underline{\nabla} \cdot \underline{v} = 0 \qquad \text{(incompressible)}$$

$$\frac{D\tilde{\rho}}{Dt} = 0 \qquad \text{(adiabatic).}$$

Thin layer approximation : $L \gg H$

* $\underline{\nabla} \cdot \underline{v} = 0$ (incompressibility)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{U}{L} \qquad \frac{U}{L}. \qquad \frac{W}{H}$$

$$\Rightarrow \quad \frac{U}{L} \sim \frac{W}{H} \quad \Rightarrow \quad W \sim U \frac{H}{L} \sim \alpha U$$

$\alpha \ll 1.$


$\Rightarrow$ neglect terms involving $w$ (compared with $u, v$)

EXCEPT in terms like $\frac{\partial w}{\partial z}$ , $w \frac{\partial}{\partial z}$

x - momentum:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \underbrace{w \frac{\partial u}{\partial z}}_{\text{keep.}} + \underbrace{\cancel{w f_*}}_{w \ll v} - v f = - \frac{1}{\rho_0} \frac{\partial p}{\partial x}$$

y-momentum:

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \underbrace{w \frac{\partial v}{\partial z}}_{\text{keep.}} + u f = - \frac{1}{\rho_0} \frac{\partial p}{\partial y}$$

z-momentum:

$$\underbrace{\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}}_{\text{neglect } w \ll u.} - u f_* = - \frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}$$

Not important in atm/ocn flows.

Simplified equations for rotating, stratified thin layer flow

$$\frac{Du}{Dt} - vf = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + uf = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{D\rho}{Dt} = 0.$$

Note! * Boussinesq approx

* Tangent plane approx

* Thin layer approx

Compact notation:

$\underline{u} = (u, v)$ $\quad\quad \underline{\nabla}_z = (\partial_x, \partial_y)$

$\underline{f} = f\hat{\underline{z}}$ $\quad\quad f = 2\Omega \sin\phi =$ "Coriolis parameter"

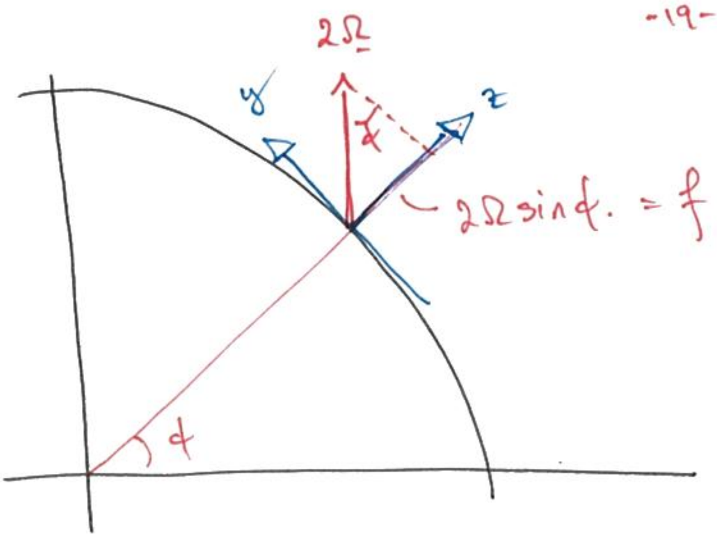
$$\frac{D\underline{u}}{Dt} + \underline{f}\times\underline{u} = -\frac{1}{\rho_0}\underline{\nabla}_z p \quad\quad \text{momentum.}$$

$$\frac{\partial p}{\partial z} = -\rho g \quad\quad \text{hydrostatic balance}$$

$$\frac{D\rho}{Dt} = 0 \quad\quad \text{adiabatic}$$

$$\underline{\nabla}_z \cdot \underline{u} + \frac{\partial w}{\partial z} = 0.$$

Primative equations




$$\underline{f} = f(\phi)\,\underline{\hat{z}}$$

Two simplifications

1. f-plane approximation.

treat $f = 2\Omega\sin\phi$ as a constant $(\phi = \phi_0)$

$$f = f_0 = 2\Omega\sin\phi_0.$$

2. β-plane approximation.

allow linear changes of $f$ with latitude $\phi = \phi_0 + \Delta\phi$

$$f = 2\Omega\sin(\phi_0 + \Delta\phi)$$

$$\simeq 2\Omega\sin\phi_0 + \Delta\phi\left.\frac{\partial f}{\partial\phi}\right|_{\phi_0} \qquad \text{(Taylor approx)}$$

$$= \underbrace{2\Omega\sin\phi_0}_{f_0} + \underbrace{\frac{2\Omega\cos\phi_0}{a}}_{\beta}\,.\,\underbrace{a\Delta\phi}_{y}.$$

$$f = f_0 + \beta y \qquad y = a\Delta\phi = \text{distance north.}$$

$$\beta = \frac{1}{a}\left.\frac{\partial f}{\partial\phi}\right|_{\phi_0} = \frac{2\Omega\cos\phi_0}{a}$$

## Geostrophic balance

Rapidly rotating flows: $Ro \ll 1$

$$Ro = \frac{\text{period of rotation}}{\text{time to move } L} \sim \frac{2\pi/\Omega}{L/U} \sim \frac{U}{L\Omega}$$

Compare sizes of terms (forces) in the horizontal momentum equation.

$$Ro \sim \frac{\text{inertial terms } (Du/Dt)}{\text{Coriolis term } (-vf)} \sim \frac{U/T}{U\Omega} \sim \frac{1}{\Omega T}$$

So if we <u>choose</u> relevant timescale $T \sim L/U$ (advective timescale)

$$\Rightarrow \quad R \sim \frac{U}{L\Omega} \quad \text{as before}$$

Rapidly rotating flows are equivalent to neglecting inertial terms:

$$\underbrace{\cancel{\frac{D\underline{u}}{Dt}}}_{Ro \ll 1.} + \underbrace{\underline{f} \times \underline{u}}_{\text{Coriolis}} = \underbrace{-\frac{1}{\rho_0} \underline{\nabla}_z p.}_{\text{horizontal pressure gradient}}$$

$\Rightarrow$ "geostrophic balance"

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad \Rightarrow \quad v = \frac{1}{\rho_0 f}\frac{\partial p}{\partial x}$$

$$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \qquad\quad u = -\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}$$

In f-plane approximation $f = f_0$

$$\Rightarrow \quad v = \frac{\partial}{\partial x}\left(\frac{\Phi}{\rho_0 f_0}\right) = \frac{\partial \psi}{\partial x}$$

$$u = -\frac{\partial}{\partial y}\left(\frac{\Phi}{\rho_0 f_0}\right) = -\frac{\partial \psi}{\partial y}$$

$$\psi = \frac{\Phi}{\rho_0 f_0}.$$

"geostrophic streamfunction"

* Flow is incompressible in 2D.

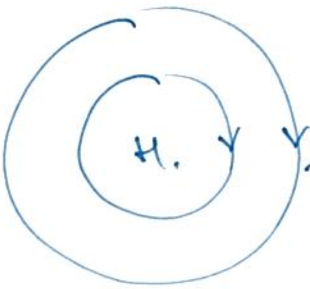
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0.$$

* $\psi$ is a streamfunction

$\Rightarrow$ flow is along contours of $\psi$.

$\Rightarrow$ flow is along contours of $\Phi$.

NH




constant $\Phi, \psi$.

"anticyclone" $\rightarrow$ clockwise in NH

$\rightarrow$ anticlockwise in SH.