## Momentum Equation:

$$Ro\left[\frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \hat{\underline{u}}\cdot\hat{\underline{\nabla}}\hat{\underline{u}}\right] + \hat{\underline{f}}\times\hat{\underline{u}} = -\hat{\underline{\nabla}}\hat{\eta}$$

$$\hat{\underline{u}} = \hat{\underline{u}}_0 + Ro\,\hat{\underline{u}}_1 + Ro^2\hat{\underline{u}}_2 + \cdots$$
$$\hat{\eta} = \hat{\eta}_0 + Ro\,\hat{\eta}_1 + Ro^2\hat{\eta}_2 + \cdots$$
$$\hat{f} = 1 + Ro\,\hat{\beta}\hat{y}$$

Sub in:

$$Ro\left[\frac{\partial \hat{\underline{u}}_0}{\partial \hat{t}} + Ro\frac{\partial \hat{\underline{u}}_1}{\partial \hat{t}} + \cdots + (\hat{\underline{u}}_0 + Ro\hat{\underline{u}}_1 + \cdots)\cdot\hat{\underline{\nabla}}(\hat{\underline{u}}_0 + Ro\hat{\underline{u}}_1 + \cdots\right.$$

$$+ (1 + Ro\,\hat{\beta}\hat{y})\,\hat{\underline{z}}\times(\hat{\underline{u}}_0 + Ro\hat{\underline{u}}_1 + \cdots)$$

$$= -\hat{\underline{\nabla}}(\hat{\eta}_0 + Ro\,\hat{\eta}_1 + \cdots)$$

* Equate all the terms that depend on $Ro^0$ (i.e. no dependence on $Ro$)

$$\boxed{\hat{\underline{z}}\times\hat{\underline{u}}_0 = -\hat{\underline{\nabla}}\eta_0 \qquad (\text{terms at } O(Ro^0)).}$$

Geostrophic balance emerges at $O(Ro^0)$.

* Equate terms that depend on $Ro^1$:

$$\boxed{\frac{\partial \hat{\underline{u}}_0}{\partial \hat{t}} + \hat{\underline{u}}_0\cdot\underline{\nabla}\hat{\underline{u}}_0 + \hat{\underline{z}}\times\hat{\underline{u}}_1 + \hat{\beta}\hat{y}\,\hat{\underline{z}}\times u_0 = -\hat{\underline{\nabla}}\hat{\eta}_1 \qquad \text{terms at } O(Ro^1).}$$

$\hat{\underline{u}}_0, \hat{\eta}_0$ = "geostrophic flow"

$\hat{\underline{u}}_1, \hat{\eta}_1$ etc = "ageostrophic flow"

Eliminate $\hat{\eta}_1$ by taking curl of this equation.

Make use of:

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = \tfrac{1}{2}\underline{\nabla}|\underline{u}|^2 - \underline{u}\times(\underline{\nabla}\times\underline{u})$$

$$= \tfrac{1}{2}\underline{\nabla}|\underline{u}|^2 - \underline{u}\times(\zeta\hat{\underline{z}})$$

$$\underline{\nabla}\times(\underline{u}\cdot\underline{\nabla}\,\underline{u}) = \tfrac{1}{2}\underbrace{\cancel{\underline{\nabla}\times\underline{\nabla}|\underline{u}|^2}}_{0} - \underline{\nabla}\times(\underline{u}\times\zeta\hat{\underline{z}})$$

This gives:

$$\hat{\underline{\nabla}}\times\frac{\partial\hat{\underline{u}}_0}{\partial\hat{t}} = \frac{\partial}{\partial t}\hat{\underline{\nabla}}\times\hat{\underline{u}}_0 = \frac{\partial}{\partial t}(\zeta_0\hat{\underline{z}})$$

$$\hat{\underline{\nabla}}\times(\hat{\underline{u}}_0\cdot\hat{\underline{\nabla}}\,\hat{\underline{u}}_0) = -\hat{\underline{\nabla}}\times(\hat{\underline{u}}_0\times\zeta_0\hat{\underline{z}})$$

$$\hat{\underline{\nabla}}\times(\hat{\underline{z}}\times\hat{\underline{u}}_1) = \cancel{(\hat{\underline{u}}_1\cdot\hat{\underline{\nabla}})\hat{\underline{z}}} + \hat{\underline{z}}(\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_1) - \cancel{\hat{\underline{u}}_1(\hat{\underline{\nabla}}\cdot\hat{\underline{z}})} - \cancel{(\hat{\underline{z}}\cdot\hat{\underline{\nabla}})\hat{\underline{u}}_1} = \hat{\underline{z}}(\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_1)$$

[From $\underline{\nabla}\times(\underline{a}\times\underline{b}) = (\underline{b}\cdot\underline{\nabla})\underline{a} + \underline{a}(\underline{\nabla}\cdot\underline{b}) - \underline{b}(\underline{\nabla}\cdot\underline{a}) - (\underline{a}\cdot\underline{\nabla})\underline{b}$ ]

$$\hat{\underline{\nabla}}\times(\hat{\beta}\hat{y}\hat{\underline{z}}\times\hat{\underline{u}}_0) = (\hat{\underline{u}}_0\cdot\hat{\underline{\nabla}})\hat{\beta}\hat{y}\hat{\underline{z}} + \hat{\beta}\hat{y}\hat{z}\cancel{(\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_0)} - \cancel{\hat{\underline{u}}_0(\hat{\underline{\nabla}}\cdot\hat{\beta}\hat{y}\hat{\underline{z}})} - \cancel{\hat{\beta}\hat{y}\hat{\underline{z}}\cdot\hat{\underline{\nabla}}\hat{\underline{u}}_0}$$

$$= (\hat{\underline{u}}_0\cdot\hat{\underline{\nabla}})(\hat{\beta}\hat{y}\hat{\underline{z}})$$

$$\hat{\underline{\nabla}}\times(-\hat{\underline{\nabla}}\hat{\eta}_1) = 0$$

Finally find:

$$\frac{\partial}{\partial t}\hat{\zeta}_0\hat{\underline{z}} - \underline{\hat{\underline{\nabla}}\times(\hat{\underline{u}}_0\times\hat{\zeta}_0\hat{\underline{z}})} + \hat{\underline{z}}(\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_1) + (\hat{\underline{u}}_0\cdot\hat{\underline{\nabla}})(\hat{\beta}\hat{y}\hat{\underline{z}}) = 0$$

Ex: show $\hat{\underline{\nabla}}\times(\hat{\underline{u}}_0\times\hat{\zeta}_0\hat{\underline{z}}) = \hat{\underline{u}}_0\cdot\hat{\underline{\nabla}}\hat{\zeta}_0\hat{\underline{z}}$

Which finally gives

$$\frac{\partial}{\partial t}\hat{\zeta}_0\,\hat{\underline{z}} + \hat{\underline{u}}_0\cdot\hat{\underline{\nabla}}\left(\hat{\zeta}_0 + \hat{\beta}\hat{y}\right)\hat{\underline{z}} = -\,\hat{\underline{z}}\,(\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_1)$$

Take the z-component of this:

$$\boxed{\frac{\partial\hat{\zeta}_0}{\partial t} + \hat{\underline{u}}_0\cdot\hat{\underline{\nabla}}(\hat{\zeta}_0 + \hat{\beta}\hat{y}) = -\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_1\,.}$$

Vorticity equation at O(Ro)

## Continuity equation

$$Ro\cdot\underset{\substack{\uparrow\\ O(1).}}{\mathbf{F}}\cdot\left[\frac{\partial\hat{\eta}}{\partial t} + \hat{\underline{u}}\cdot\hat{\underline{\nabla}}\hat{\eta}\right] + \left[1 + Ro\,\mathbf{F}\,\hat{\eta}\right]\hat{\underline{\nabla}}\cdot\hat{\underline{u}} = 0$$

$$\hat{\eta} = \hat{\eta}_0 + Ro\,\hat{\eta}_1 + \cdots$$

$$\hat{\underline{u}} = \hat{\underline{u}}_0 + Ro\,\hat{\underline{u}}_1 + \cdots$$

Sub this in:

$$Ro\,F\left[\frac{\partial\hat{\eta}_0}{\partial\hat{t}} + Ro\frac{\partial\hat{\eta}_1}{\partial\hat{t}} + \cdots + (\hat{\underline{u}}_0 + Ro\,\hat{\underline{u}}_1 + \cdots)\cdot\hat{\underline{\nabla}}(\hat{\eta}_0 + Ro\,\hat{\eta}_1 + \cdots\right.$$

$$+ \left[1 + Ro\,F(\hat{\eta}_0 + Ro\,\hat{\eta}_1 + \cdots)\right]\hat{\underline{\nabla}}\cdot(\hat{\underline{u}}_0 + Ro\,\hat{\underline{u}}_1 + \cdots) =$$

$O(Ro^0)$: $\quad\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_0 = 0$ — incompressibility. at $O(Ro^0)$. (already satisfied by geostrophic balance).

$$O(Ro^1):\ \boxed{F\left[\frac{\partial\hat{\eta}_0}{\partial\hat{t}} + \hat{\underline{u}}_0\cdot\hat{\underline{\nabla}}\hat{\eta}_0\right] + \hat{\underline{\nabla}}\cdot\hat{\underline{u}}_1 + \cancel{F\hat{\eta}_0\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_0} = 0}$$

divergence of $\hat{\underline{u}}_1$

Use continuity equation (to $O(Ro)$) to eliminate the $\hat{\nabla}\cdot\hat{\underline{u}}_1$ in the vorticity equation (to $O(Ro)$):

$$\frac{\partial \hat{\zeta}_0}{\partial \hat{t}} + \hat{\underline{u}}_0\cdot\hat{\nabla}(\hat{\zeta}_0 + \hat{\beta}\hat{y}) = -\hat{\nabla}\cdot\hat{\underline{u}}_1$$

$$F\left[\frac{\partial \hat{\eta}_0}{\partial \hat{t}} + \hat{\underline{u}}_0\cdot\hat{\nabla}\hat{\eta}_0\right] = -\hat{\nabla}\cdot\hat{\underline{u}}_1$$

$$\Rightarrow \frac{\partial \hat{\zeta}_0}{\partial \hat{t}} + \hat{\underline{u}}_0\cdot\hat{\nabla}(\hat{\zeta}_0 + \hat{\beta}\hat{y}) - F\frac{\partial \hat{\eta}_0}{\partial \hat{t}} - F\hat{\underline{u}}_0\cdot\hat{\nabla}\hat{\eta}_0 = 0$$

$$\frac{\partial}{\partial \hat{t}}\left(\hat{\zeta}_0 - F\hat{\eta}_0\right) + \hat{\underline{u}}_0\cdot\hat{\nabla}\left(\hat{\zeta}_0 + \hat{\beta}\hat{y} - F\hat{\eta}_0\right) = 0$$

But $\hat{\beta}\hat{y}$ has <u>no</u> $t$-dependence, so can include it in $\frac{\partial}{\partial \hat{t}}$:

$$\boxed{\frac{D_0}{D\hat{t}}\left(\hat{\zeta}_0 + \hat{\beta}\hat{y} - F\hat{\eta}_0\right) = 0 \qquad \text{where} \quad \frac{D_0}{D\hat{t}} = \frac{\partial}{\partial \hat{t}} + \hat{\underline{u}}_0\cdot\hat{\nabla}}$$

conservation of potential vorticity

advection by geostrophic flow $\hat{\underline{u}}_0$.

Looks like potential vorticity (ignoring $f_0$, $H$)

$$\frac{D_0\hat{Q}}{D\hat{t}} = 0 \qquad \hat{Q} = \hat{\zeta}_0 + \hat{\beta}\hat{y} - F\hat{\eta}_0$$

relative vorticity — planetary vorticity — vortex stretching.

Can express this in terms of geostrophic streamfunction $\hat{\psi}_0$

$$\hat{\zeta}_0 = \nabla^2\hat{\psi}_0, \quad \hat{\underline{u}}_0 = \left(-\frac{\partial\hat{\psi}_0}{\partial y}, \frac{\partial\hat{\psi}_0}{\partial x}\right), \quad \hat{\eta}_0 = \hat{\psi}_0$$

Put back dimensional scales :

$$u_0 = U\hat{u}_0 \Rightarrow \hat{u}_0 = \frac{1}{U}u_0$$

$$\psi_0 = UL\hat{\psi}_0 \Rightarrow \hat{\psi}_0 = \frac{1}{UL}\psi_0 \quad \left(\text{c.f. } u_0 = -\frac{\partial \psi_0}{\partial y}\right)$$

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \underline{u}_0\cdot\underline{\nabla} = \frac{1}{T}\frac{D_0}{D\hat{t}} \Rightarrow \frac{D_0}{D\hat{t}} = T\frac{D_0}{Dt}$$

$$\zeta_0 = \nabla^2\psi_0 = \frac{1}{T}\hat{\zeta}_0 \Rightarrow \hat{\zeta}_0 = \hat{\nabla}^2\hat{\psi} = T\zeta_0$$

$$\beta y = \frac{1}{T}\hat{\beta}\hat{y} \Rightarrow \hat{\beta}\hat{y} = T\beta y$$

$$F\hat{\psi}_0 = \left(\frac{L}{L_D}\right)^2\frac{1}{UL}\psi_0 = \frac{L}{U}\frac{1}{L_D^2}\psi_0 = \frac{T}{L_D^2}\psi_0$$

Sub into non-dim PV equation:

$$\frac{D_0}{D\hat{t}}\left(\hat{\zeta}_0 + \hat{\beta}\hat{y} - F\hat{\psi}_0\right) = 0$$

$$T\frac{D_0}{Dt}\left(T\zeta_0 + T\beta y - \frac{T}{L_D^2}\psi_0\right) = 0$$

$$\Rightarrow \boxed{\frac{D_0 Q}{Dt} = 0 \qquad Q = \zeta_0 + \beta y - \frac{\psi_0}{L_D^2}}$$

Dimensional QGPV conservation.

$Q$ : quasigeostrophic potential vorticity

Can also express geostrophic streamfunction in terms of $\eta$:

$$\hat{\underline{z}}\times\hat{\underline{u}}_0 = -\hat{\underline{\nabla}}\eta_0$$

$$\Rightarrow \hat{\underline{u}}_0 = -\hat{\underline{z}}\times(\hat{\underline{z}}\times\hat{\underline{u}}_0) = \hat{\underline{z}}\times\hat{\underline{\nabla}}\hat{\eta}_0 \Rightarrow \boxed{\psi_0 = \frac{g\eta_0}{f_0}}$$

## Summary: derivation of QG system:

* non-dimensionalized the momentum & conservation equations $\longrightarrow$ two non-dimensional numbers.

$$Ro = \frac{U}{f_0 L} \qquad \text{Rossby number}$$

$$F = \left(\frac{L}{L_D}\right)^2 \qquad \text{Froude number}$$

* In the limit of $Ro \ll 1$ but $F \sim O(1)$ did an asymptotic expansion of $\hat{\underline{u}}, \hat{\eta}, \hat{f}$

* Collected terms in each equation at $O(Ro^0)$:

  $\Rightarrow$ geostrophic balance
  $\Rightarrow$ "geostrophic flow": $\hat{\underline{u}}_0, \hat{\eta}_0, \hat{\psi}_0$

* Collected terms in each equation at $O(Ro^1)$:

  $\Rightarrow$ correction to geostrophic balance
  $\Rightarrow$ "ageostrophic flow": $\hat{\underline{u}}_1, \hat{\eta}_1$

* Took the curl of the $O(Ro^1)$ momentum equation and combined it with $O(Ro^1)$ continuity equation

  ↳ non-dimensional QG potential vorticity equation.

* Re-dimensionalize the QG potential vorticity equation

$$\frac{D_0}{Dt} Q = 0 \qquad Q = \nabla^2 \psi + \beta y - \frac{\psi}{L_D^2}$$

↑ advection by $\underline{u}_0$

A non-rigorous derivation: (c.f. Assignment 3)

SW potential vorticity:

$$Q_{sw} = \frac{f + \zeta}{h}$$

Let $h = H + \eta$

$$Q_{sw} = \frac{f + \zeta}{H + \eta}$$

$$= \frac{f + \zeta}{H}\left(1 + \frac{\eta}{H}\right)^{-1}$$

$$\simeq \frac{f + \zeta}{H}\left(1 - \frac{\eta}{H}\right) + \cdots$$

$$= \left(f_0 + \beta y + \zeta - \frac{f_0 \eta}{H}\right)\frac{1}{H} + \cdots$$

This is the same as QG potential vorticity except for constants like $f_0$, $H$.

## Rossby waves in the QG system

(Include effects of stratification through vortex-stretching term)

* Single-layer QG shallow water model : PV conservation

$$\frac{D}{Dt} Q = 0 \qquad Q = \zeta + \beta y - L_D^{-2} \psi$$

advection by the geostrophic flow.

vortex stretching

Introduce $k_D = L_D^{-1}$ = deformation wavenumber.

* Base state : uniform zonal flow

$$\bar{u} = U \,,\quad \bar{v} = 0 \,,\quad \bar{\psi} = -Uy$$

$\Rightarrow$ sloping upper surface : $\bar{\psi} = \dfrac{g\bar{\eta}}{f_0} \Rightarrow \bar{\eta}(y) = -\dfrac{fU}{g} \cdot y$.

* PV of the base state is

$$\bar{Q} = \cancel{\nabla^2 \bar{\psi}} + \beta y - k_D^2 \bar{\psi}$$

$$= \beta y + k_D^2 U y$$

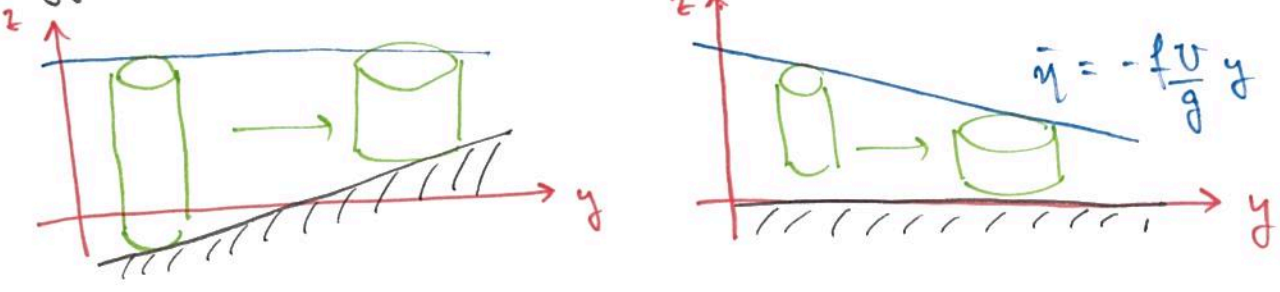
gradient in PV in y-direction.

planetary vorticity

vortex stretching / squeezing due to sloping upper surface

$\Rightarrow$ can get a PV gradient even when $\beta = 0$ !

* Analogy with topographic $\beta$-effect :



Add small perturbations:

$$\psi = -Uy + \tilde{\psi} \quad \Rightarrow \quad u = U - \frac{\partial \tilde{\psi}}{\partial y}$$

$$v = \frac{\partial \tilde{\psi}}{\partial x}$$

$$Q = \bar{Q}(y) + \tilde{Q}(x,y,t)$$

$$= (\beta + k_D^2 U)y + \nabla^2 \tilde{\psi} - k_D^2 \tilde{\psi}$$

Then

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \underline{u} \cdot \underline{\nabla} Q$$

$$= \frac{\partial}{\partial t}\left(\nabla^2 \tilde{\psi} - k_D^2 \tilde{\psi}\right) + \left(U - \frac{\partial \tilde{\psi}}{\partial y}\right)\frac{\partial}{\partial x}\left(\cancel{\bar{Q}(y)} + \nabla^2 \tilde{\psi} - k_D^2 \tilde{\psi}\right)$$

$$+ \frac{\partial \tilde{\psi}}{\partial x}\frac{\partial}{\partial y}\left(\bar{Q}(y) + \nabla^2 \tilde{\psi} - k_D^2 \tilde{\psi}\right) = 0$$

Neglect nonlinear terms:

$$\frac{\partial}{\partial t}\left(\nabla^2 \tilde{\psi} - k_D^2 \tilde{\psi}\right) + U\frac{\partial}{\partial x}\left(\nabla^2 \tilde{\psi} - k_D^2 \tilde{\psi}\right)$$

$$+ \frac{\partial \tilde{\psi}}{\partial x}\left(\beta + k_D^2 U\right) = 0$$

$$\boxed{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2 \tilde{\psi} - k_D^2 \tilde{\psi}\right) + \left(\beta + k_D^2 U\right)\frac{\partial \tilde{\psi}}{\partial x} = 0}$$

linearized QG PV equation.

↳ PDE with constant coefficients.

Plane wave solutions: $\tilde{\psi} = \Psi e^{i(kx + ly - \omega t)}$

$$\left(-i\omega + ikU\right)\left(-(k^2+l^2) - k_D^2\right)\cancel{\Psi}$$

$$+ ik\left(\beta + k_D^2 U\right)\cancel{\Psi} = 0$$

non-trivial solns $\Psi \neq 0$

$$\Rightarrow \boxed{\omega = kU - k\,\frac{\beta + k_D^2 U}{k^2 + l^2 + k_D^2}}$$

Dispersion reln.

Two key differences from SW:

* $k_D = L_D^{-1} \Rightarrow$ effect of stratification.
* $U \Rightarrow$ effect of mean flow

Case 1 : $k_D = 0$ $\quad (L_D \to \infty)$

$$\boxed{\omega = kU - \frac{\beta k}{k^2 + l^2}}$$

$\omega_R$

$k_D = 0$

"barotropic Rossby wave"

(as before, but with $U$)

Effect of $U$:

- changes the frequency by $kU$
- changed the $c_{ph}$ by $U$.

} Doppler shift.

## Case 2: $U = 0$

$$\boxed{\omega = -\frac{\beta k}{k^2 + l^2 + k_D^2}}$$

new term.

$k_D \neq 0$:

"baroclinic Rossby wave"

Rewrite as

$$\omega(k^2 + l^2 + k_D^2) = -\beta k$$

Treat $\omega$ as a constant (find contours of constant $\omega$)

Completing the square (check!):

$$\left(k + \frac{\beta}{2\omega}\right)^2 + l^2 = \left(\frac{\beta}{2\omega}\right)^2 - k_D^2$$

$$\left(\frac{k}{k_D} + \frac{\beta}{2\omega k_D}\right)^2 + \left(\frac{l}{k_D}\right)^2 = \left(\frac{\beta}{2\omega k_D}\right)^2 - 1.$$

$\Rightarrow$ Circle centred at $\left(\frac{k}{k_D}, \frac{l}{k_D}\right) = \left(-\frac{\beta}{2\omega k_D}, 0\right)$

with radius $\sqrt{\left(\frac{\beta}{2\omega k_D}\right)^2 - 1}$

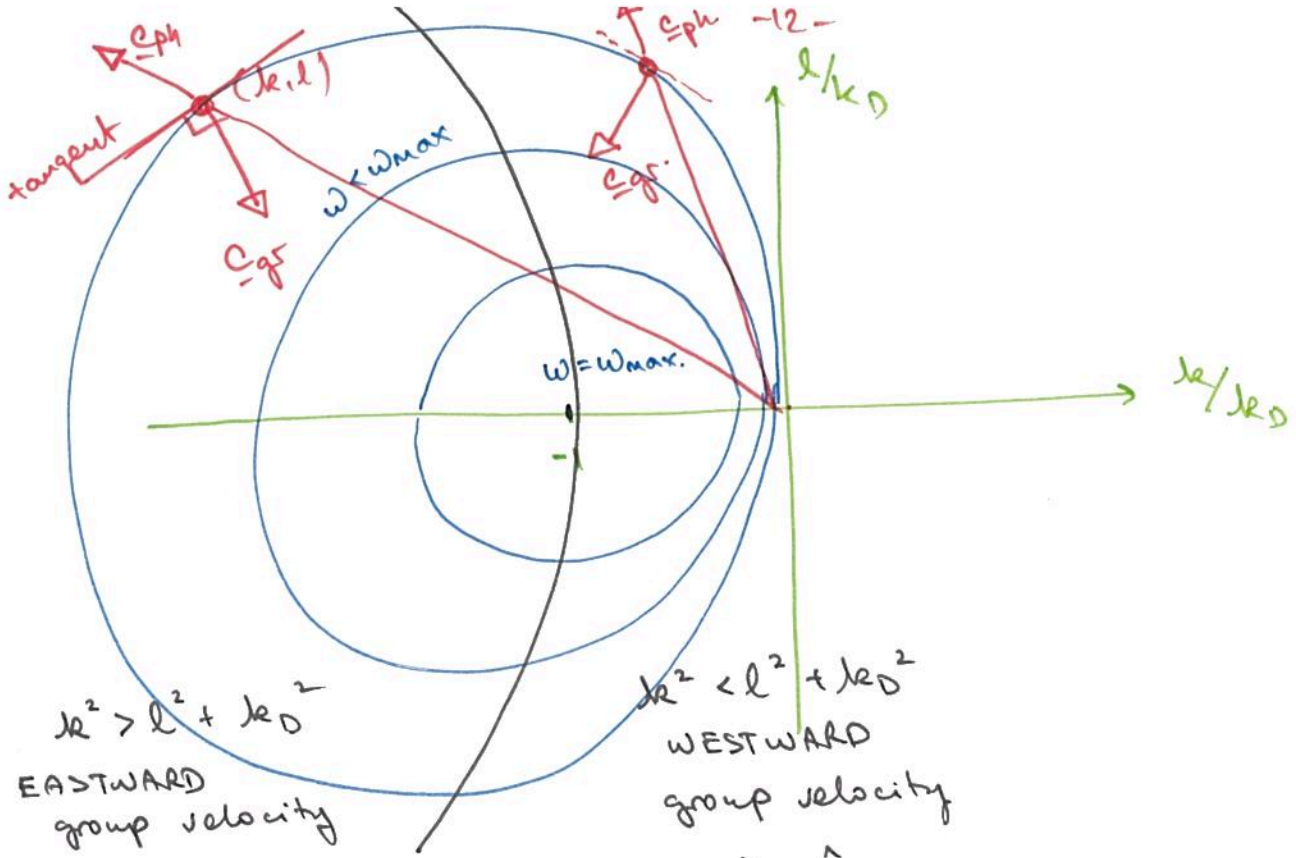
This gives a condition on $\omega$:

$$\left(\frac{\beta}{2\omega k_D}\right)^2 - 1 > 0$$

$$\Rightarrow \quad \omega < \frac{\beta}{2k_D} = \omega_{max}$$ ← maximum frequency

Then

$$\left(\frac{k}{k_D} + \frac{\omega_{max}}{\omega}\right)^2 + \left(\frac{l}{k_D}\right)^2 = \left(\frac{\omega_{max}}{\omega}\right)^2 - 1.$$




Phase velocity: $\underline{c}_{ph} = \dfrac{\omega}{K}\hat{\underline{k}}$

Group velocity: $\underline{c}_{gr} = \left(\dfrac{\partial \omega}{\partial k}, \dfrac{\partial \omega}{\partial l}\right) = \underline{\nabla}_K \omega$

Recall: $c_{gr,x} = \dfrac{\partial \omega}{\partial k} = \beta \dfrac{k^2 - l^2 - k_D^2}{(k^2 + l^2 + k_D^2)^2}.$

Case 3: $k_D \neq 0, U \neq 0.$

$$\omega = kU - k\frac{\beta + k_D^2 U}{k^2 + l^2 + k_D^2}$$

also modifies background PV gradient.

Doppler shift

Exercise: find flow speed $U$ that gives stationary waves. $(l = 0)$.

$$c_{ph} = \frac{\omega}{k} = U - \frac{\beta + k_D^2 U}{k^2 + k_D^2} = 0$$

$\Rightarrow \boxed{U = \dfrac{\beta}{K^2}}$ stationary Rossby waves.