Topic 4: The quasigeostrophic model

MATH3261/5285: Fluids, Oceans, and Climate

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4.1 Scaling theory

Scaling theory for the shallow water model

$$\frac{\partial}{\partial t} = \frac{U}{L} \frac{\partial}{\partial \hat{t}}$$

$$\nabla = \frac{1}{L} \hat{\nabla}$$

$$\mathbf{u} = U \hat{\mathbf{u}}$$

$$\eta = \mathcal{H} \hat{\eta}$$

$$f = f_0 \hat{f}$$

Dimensional momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{f} \times \mathbf{u} = -g \nabla \eta$$

Non-dimensional momentum equation

$$\mathrm{Ro} \left[\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\mathbf{u}} \right] + \hat{\mathbf{f}} \times \hat{\mathbf{u}} = -\hat{\nabla} \hat{\eta}$$

$$\mathrm{Ro} = \frac{U}{f_0 L} = \text{Rossby number}$$

Dimensional continuity equation

$$\frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h + h \; \nabla \cdot \mathbf{h} = 0$$

Non-dimensional continuity equation

$$\text{Ro F} \; \left[\frac{\partial \hat{\eta}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\eta} \right] + \left(1 + \text{Ro F} \; \hat{\eta} \right) \hat{\nabla} \cdot \hat{\mathbf{u}} = 0$$

$$\text{F} = \frac{L^2}{L_D^2} = \text{Froude number}$$

$$L_D = \frac{\sqrt{gH}}{f_0} = \text{Rossby deformation scale}$$

Quasigeostrophic limit

$$\mathrm{Ro} \ll 1 \quad \text{and} \quad \mathrm{F} \approx O(1) \quad \Longrightarrow \quad L \approx L_D$$

Ocean:

$$L_D \approx 100 \text{ km} \qquad \text{mesoscale (eddy scale)}$$

Atmosphere:

$$L_D \approx 1000 \text{ km} \qquad \text{synoptic scale (weather scale)}$$

4.2 Asymptotic scaling theory

Non-dimensional momentum equation

$$\mathrm{Ro}\left[\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{\mathbf{u}}\right] + \hat{\mathbf{f}} \times \hat{\mathbf{u}} = -\hat{\nabla}\hat{\eta}$$

Non-dimensional continuity equation

$$\mathrm{Ro}\ \mathrm{F}\ \left[\frac{\partial \hat{\eta}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{\eta}\right] + \left(1 + \mathrm{Ro}\ \mathrm{F}\ \hat{\eta}\right)\hat{\nabla} \cdot \hat{\mathbf{u}} = 0$$

Asymptotic scaling theory

Expand fields in powers of Ro and gather terms at each order:

$$\begin{aligned}
\hat{\eta} &= \hat{\eta}_0 + \mathrm{Ro}\ \hat{\eta}_1 + \mathrm{Ro}^2\ \hat{\eta}_2 \\
\hat{\mathbf{u}} &= \hat{\mathbf{u}}_1 + \mathrm{Ro}\ \hat{\mathbf{u}}_1 + \mathrm{Ro}^2\ \hat{\mathbf{u}}_2 + \cdots \\
\hat{f} &= 1 + \mathrm{Ro}\ \hat{\beta}\ \hat{y}
\end{aligned}$$

- geostrophic flow: $\hat{\eta}_0$, $\hat{\mathbf{u}}_0$
- ageostrophic flow: $\hat{\eta}_1$, $\hat{\mathbf{u}}_1$, $\hat{\eta}_2$, $\hat{\mathbf{u}}_2$, $\cdots$

Summary: derivation of the QG system

- non-dimensionalize the momentum and continuity equations and express in terms of Ro and F
- in limit Ro $\ll 1$, $F \approx O(1)$, expand $\hat{\eta}$, $\hat{\mathbf{u}}$, $\hat{f}$ in powers of Ro
- collect terms in each equation at $O(1)$, $O(\mathrm{Ro})$:
  - geostrophic flow $\hat{\eta}_0$, $\hat{\mathbf{u}}_0$, $\hat{\psi}_0$
  - ageostrophic correction $\hat{\eta}_1$, $\hat{\mathbf{u}}_1$
- take curl of $O(\mathrm{Ro})$ momentum equation and combine with $O(\mathrm{Ro})$ continuity equation
  - non-dimensional QG potential vorticity equation
- rescale to get dimensional QG potential vorticity equation

Quasigeostrophic potential vorticity equation

$$\frac{D_0 Q}{Dt} = 0, \qquad Q = \nabla^2\psi + \beta y - \frac{\psi}{L_D^2}$$

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_0\cdot\nabla = \text{advection by geostrophic velocity}$$

4.3 Rossby waves in the QG system

Dispersion relation

$$\omega = kU - k\frac{\beta + k_D^2 U}{k^2 + l^2 + k_D^2}$$

Case 1: $k_D = 0$

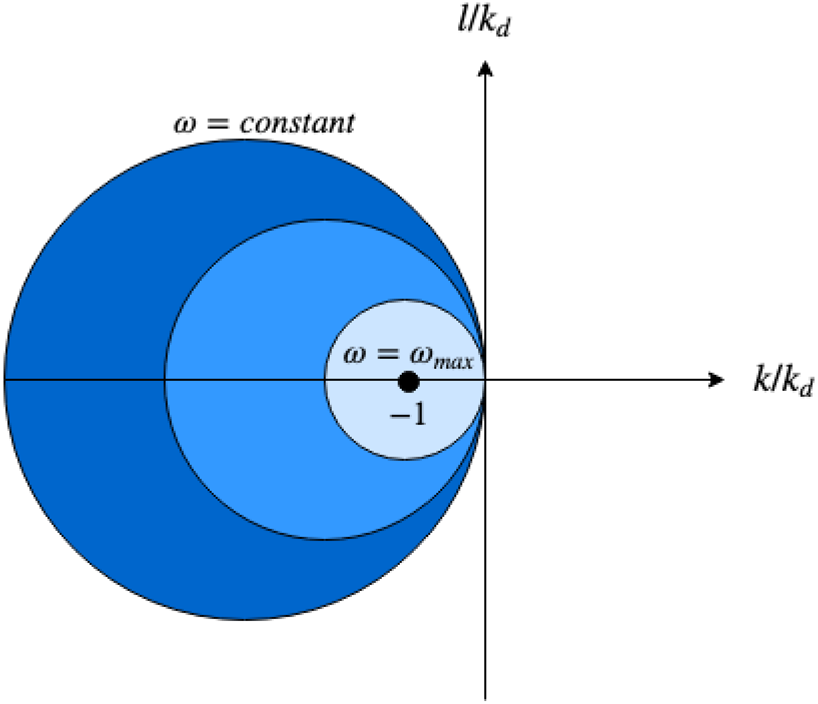
$$\omega = kU - k\frac{\beta}{k^2 + l^2}$$

- barotropic Rossby wave
- effect of U is to Doppler shift the frequency

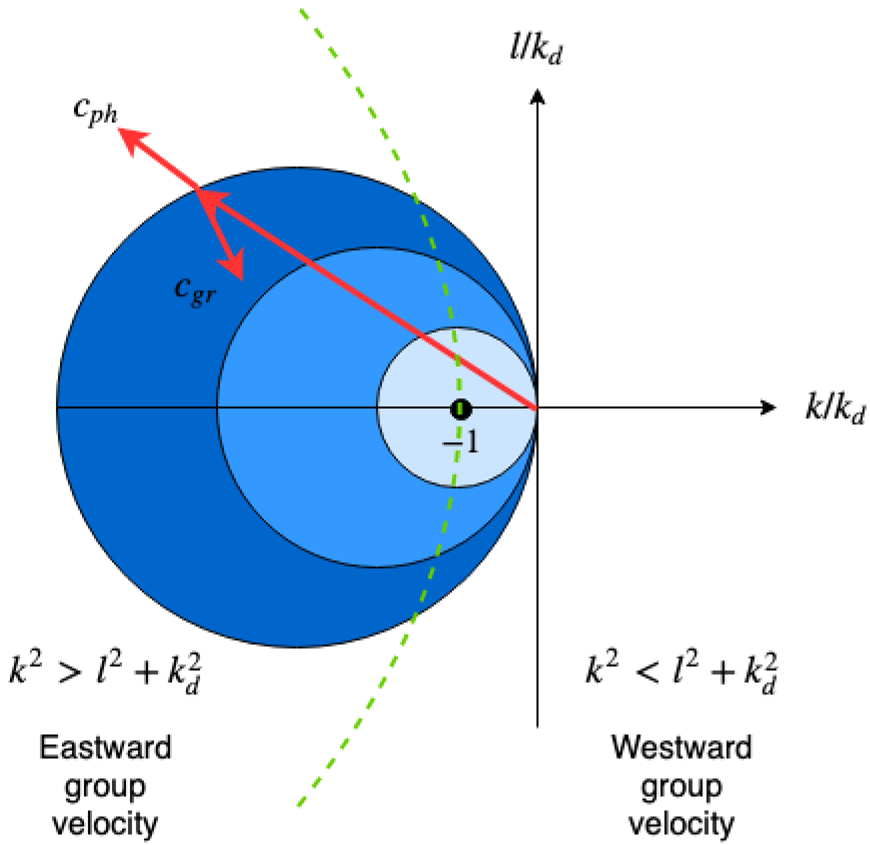
Case 2: $U = 0$

$$\omega = -k\frac{\beta}{k^2 + l^2 + k_D^2}$$

- baroclinic Rossby wave
- get a maximum frequency of $\omega_{max} = \beta/2k_D$.



l/k_d
$\omega = constant$
$\omega = \omega_{max}$
-1
k/k_d


l/k_d
c_{ph}
c_{gr}
k/k_d
-1
$k^2 > l^2 + k_d^2$
Eastward group velocity
$k^2 < l^2 + k_d^2$
Westward group velocity