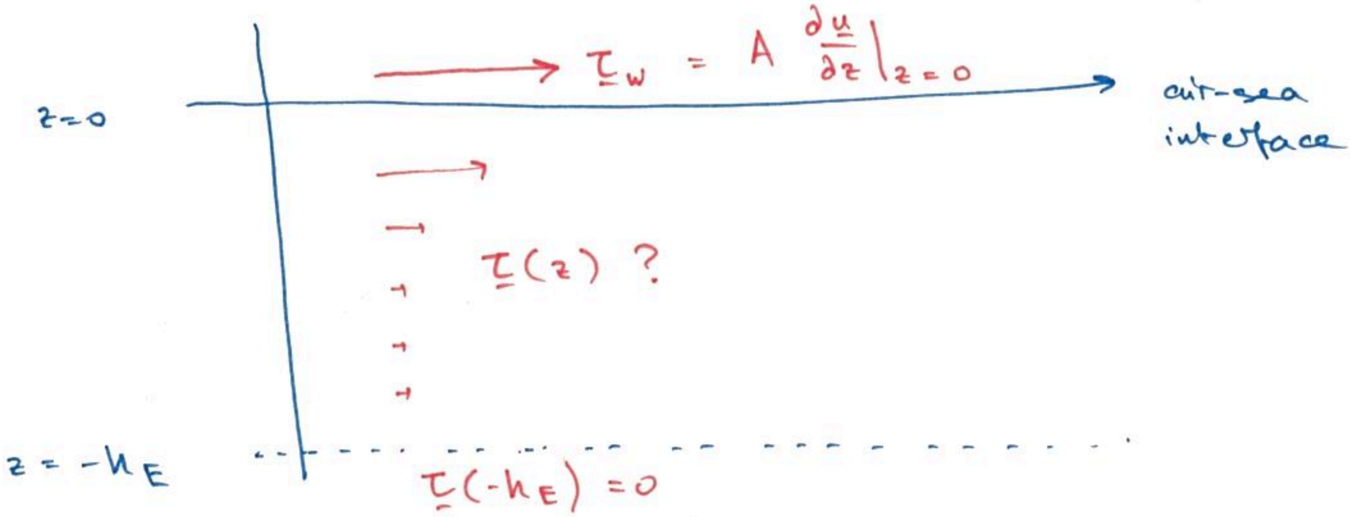
Depth-integrated transport in the Ekman layer

Key idea: friction is a response to the imposed wind stress



$$\underline{f}_{fric} = \frac{\partial \underline{\tau}}{\partial z} = A \frac{\partial^2 u}{\partial z^2} \qquad \left(\text{model } \underline{\tau} = A \frac{\partial u}{\partial z}\right)$$

Integrate over depth:

$$\overline{\underline{f}_{fric}} = \frac{\Delta \underline{\tau}}{h_E} = \frac{\underline{\tau}_w - \underline{0}}{h_E} = \frac{\underline{\tau}_w}{h_E}$$

depth average

Shallow water model of Ekman layer

Horizontal momentum balance in Ekman layer

$$\underline{f} \times \underline{u} = -g \underline{\nabla}_z \eta + \frac{\underline{\tau}_w}{h_E}$$

for simplicity use $\underline{f} = f_0 \hat{\underline{z}}$ (c.f. next lecture)

Decompose the flow:

$$\underline{u} = \underline{u}_g + \underline{u}_E$$

geostrophic component

"Ekman flow" (driven by wind stress)

where we define

$$\underline{f} \times \underline{u}_g = - g \underline{\nabla}_z \eta. \qquad \text{geostrophic}$$

$$\underline{f} \times \underline{u}_E = \frac{\underline{\tau}_w}{h_E} \qquad \text{ageostrophic}$$

Use identity

$$\underline{\hat{z}} \times (\underline{\hat{z}} \times \underline{u}) = - \underline{u}$$

and $\underline{f} = f_0 \underline{\hat{z}}$

$- \times \frac{\underline{\hat{z}}}{f_0}$

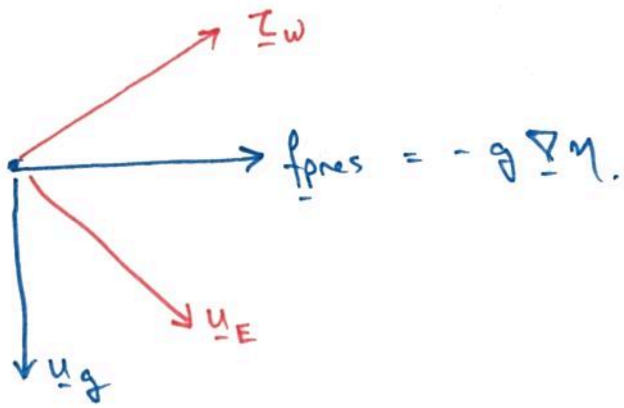
$$- \frac{\underline{\hat{z}}}{f_0} \times (f_0 \underline{\hat{z}} \times \underline{u}_g) = \underline{u}_g = \frac{g}{f_0} \underline{\hat{z}} \times \underline{\nabla} \eta.$$

$$- \frac{\underline{\hat{z}}}{f_0} \times (f_0 \underline{\hat{z}} \times \underline{u}_E) = \underline{u}_E = - \frac{1}{f_0} \underline{\hat{z}} \times \frac{\underline{\tau}_w}{h_E}$$

$$\underline{u}_g = - \frac{1}{f_0} \underline{\hat{z}} \times \underline{f}_{pres} \Rightarrow \underline{u}_g \perp^{ar} \text{ to pressure gradient force}$$

$$\underline{u}_E = - \frac{1}{f_0} \underline{\hat{z}} \times \underline{f}_{fric} \Rightarrow \underline{u}_E \perp^{ar} \text{ to wind stress/ friction force.}$$

$\boxed{f_0 > 0}$



Ekman flow is to $\left\{ \begin{array}{l} \text{RIGHT} \\ \text{LEFT} \end{array} \right\}$ of $\underline{\tau}_w$ in $\left\{ \begin{array}{l} \text{N. HEMI} \\ \text{S. HEMI} \end{array} \right\}$

Ekman pumping (or suction)

Vertical velocity caused by converging / diverging Ekman flow:

$$\underline{\nabla}_z \cdot \underline{u}_E = -\frac{\partial w}{\partial z} \xrightarrow{} -\frac{\Delta w}{\Delta z}$$

$$\simeq -\frac{\cancel{w(0)}^{\,0} - w(-h_E)}{h_E} \quad \text{(in SW model.)}$$

$$= \frac{w_B}{h_E} \leftarrow \text{upward flow thru' the bottom of the Ekman layer.}$$

$$w_B = +ve \quad \text{for divergent flow} \quad (\underline{\nabla}_z \cdot \underline{u}_E > 0)$$

$$= -ve \quad \text{for convergent flow} \quad (\underline{\nabla}_z \cdot \underline{u}_E < 0)$$

Use $\underline{u} = \underline{u}_g + \underline{u}_E$

$$w_B = h_E \underline{\nabla}_z \cdot \Big(\overbrace{-\frac{1}{f_0} \hat{\underline{z}} \times \frac{\underline{\tau}_w}{h_E}} \Big).$$

$$= -\frac{1}{f_0} \underline{\nabla}_z \cdot (\hat{\underline{z}} \times \underline{\tau}_w)$$

Let $\underline{\tau}_w = (\tau_w^x, \tau_w^y, 0)$

$$\underline{\nabla}_z \cdot (\hat{\underline{z}} \times \underline{\tau}_w) = \underline{\nabla}_z \cdot (\tau_w^x \hat{\underline{y}} - \tau_w^y \hat{\underline{x}})$$

$$= -\frac{\partial \tau_w^y}{\partial x} + \frac{\partial \tau_w^x}{\partial y} = -\text{curl}_z \underline{\tau}_w$$

$$\Rightarrow \boxed{w_B = f_0^{-1} \text{curl}_z \underline{\tau}_w}$$

The vertical velocity at base of Ekman layer is proportional to curl of wind stress at upper surface

Consider

$$w_g = h_E \, \underline{\nabla}_z \cdot \underline{u}_g$$

$$= h_E \underline{\nabla}_z \cdot \left(\frac{g}{f_0} \, \underline{\hat{z}} \times \underline{\nabla}_z \eta \right)$$

$$= \frac{g h_E}{f_0} \, \underline{\nabla}_z \cdot \left(\underline{\hat{z}} \times \underline{\nabla}_z \eta \right)$$

$$\underline{\nabla}_z \cdot \left(\underline{\hat{z}} \times \underline{\nabla}_z \eta \right) = \underline{\nabla}_z \cdot \left(\frac{\partial \eta}{\partial x} \underline{\hat{y}} - \frac{\partial \eta}{\partial y} \underline{\hat{x}} \right)$$

$$= \frac{\partial^2 \eta}{\partial y \partial x} - \frac{\partial^2 \eta}{\partial x \partial y} = 0$$

$\Rightarrow$ no vertical velocity from geostrophic flow!

Shallow water model of the wind-driven circulation.

Single layer, depth H, $Ro \ll 1$.

$$\underset{\sim}{f} \times \underline{u} = -g \underline{\nabla}_z \eta + \frac{\overbrace{\underline{\tau}_W}^{\text{wind stress}} - \overbrace{\underline{\tau}_B}^{\text{bottom stress}}}{\underbrace{H}_{\text{depth of the ocean}}}$$

Use β-plane approximation : $f = f_0 + \beta y$

$$-fv = -g\frac{\partial \eta}{\partial x} + \frac{\tau_W^x - \tau_B^x}{H} \qquad (1)$$

$$fu = -g\frac{\partial \eta}{\partial y} + \frac{\tau_W^y - \tau_B^y}{H} \qquad (2)$$

$\frac{\partial (1)}{\partial y}$ $$-f\frac{\partial v}{\partial y} - \beta v = -g\frac{\partial^2 \eta}{\partial y \partial x} + H^{-1}\left(\frac{\partial \tau_W^x}{\partial y} - \frac{\partial \tau_B^x}{\partial y}\right)$$

$\frac{\partial (2)}{\partial x}$ $$f\frac{\partial u}{\partial x} = -g\frac{\partial^2 \eta}{\partial x \partial y} + H^{-1}\left(\frac{\partial \tau_W^y}{\partial x} - \frac{\partial \tau_B^y}{\partial x}\right)$$

Subtract $$f\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{=0 \text{ (rigid lid approx)}} + \beta v = H^{-1} \operatorname{curl}_z (\underline{\tau}_W - \underline{\tau}_B).$$

$$\boxed{\beta v = \frac{1}{H} \operatorname{curl}_z (\underline{\tau}_W - \underline{\tau}_B)}$$

Sverdrup balance (Harald Sverdrup 1888-1957)

Neglect bottom friction ($\underline{\tau}_B = 0$)

$$\boxed{\beta V = \frac{1}{H} \operatorname{curl}_z \underline{\tau}_w}$$ Sverdrup balance

Balance between β-effect and wind-stress curl.

In the northern hemisphere:

* -ve wind stress curl. $\Rightarrow$ downward Ekman pumping
* vortex squashing (H decreasing)
* conserve PV by <u>moving</u> water columns <u>south</u> to decrease planetary vorticity.

In <u>both</u> hemispheres:

$$\text{A} \left\{ \begin{array}{c} \text{anticyclonic} \\ \text{cyclonic} \end{array} \right\} \text{wind-stress curl. drives} \left\{ \begin{array}{c} \text{equatorwards} \\ \text{polewards} \end{array} \right\} \text{flow}$$

<u>OR</u>

$$\text{A} \left\{ \begin{array}{c} \text{negative} \\ \text{positive} \end{array} \right\} \text{wind-stress curl. drives} \left\{ \begin{array}{c} \text{southward} \\ \text{northward} \end{array} \right\} \text{flow.}$$

Wind-driven circulation from Sverdrup balance

$$\beta v = H^{-1} \, \mathrm{curl}_z \, \underline{\tau}_w$$

u, v are divergenceless.

$\Rightarrow$ introduce streamfunction

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \quad \beta \frac{\partial \psi}{\partial x} = H^{-1} \, \mathrm{curl}_z \, \underline{\tau}_w .$$

Remove dimensions using

$$x = L\hat{x}, \quad y = L\hat{y}, \quad \underline{\tau}_w = \tau_0 \hat{\underline{\tau}}_w, \quad \psi = \Psi_0 \hat{\psi}$$

$$\frac{\beta \Psi_0}{L} \frac{\partial \hat{\psi}}{\partial \hat{x}} = \frac{\tau_0}{HL} \, \widehat{\mathrm{curl}}_z \, \hat{\underline{\tau}}_w$$

Both sides balance

$$\frac{\beta \Psi_0}{L} = \frac{\tau_0}{HL}$$

And non-dimensional Sverdrup balance

$$\boxed{\frac{\partial \hat{\psi}}{\partial \hat{x}} = \widehat{\mathrm{curl}}_z \, \hat{\underline{\tau}}_w}$$

Sverdrup balance (Harald Sverdrup 1888-1957)

Neglect bottom friction $\underline{\tau}_B = 0$

$$\boxed{\beta v = \frac{1}{H} \operatorname{curl}_z \underline{\tau}_W}$$ Sverdrup balance

Balance between β-effect and wind-stress curl.

<u>In NH</u>

* -ve wind-stress curl $\Rightarrow$ down Ekman pumping
* vortex squashing (H decreases)
* conserve PV by moving <u>south</u> to decrease planetary vorticity $f = f_0 + \beta y$.

In <u>both</u> hemispheres

a $\left\{\begin{matrix}\text{negative}\\ \text{positive}\end{matrix}\right\}$ wind-stress curl drives $\left\{\begin{matrix}\text{equatorward}\\ \text{poleward}\end{matrix}\right\}$ flow.

Case 1 : no flow thru' the western boundary.

$$\hat{u} = -\hat{\psi}_y = 0 \quad \text{at} \quad \hat{x} = 0$$

$$\Rightarrow \quad \hat{\psi}(0,\hat{y}) = \text{constant} = 0$$

compare with $\hat{\psi}(\hat{x},\hat{y}) - \hat{\psi}(0,\hat{y}) = \pi\,(c(\hat{y}) - \hat{x})\sin\pi\hat{y}$

$$\Rightarrow \quad 0 = \pi\,(c(\hat{y}) - 0)\sin\pi\hat{y}$$

$$\Rightarrow \quad c(\hat{y}) = 0.$$

$$\Rightarrow \quad \boxed{\hat{\psi}(\hat{x},\hat{y}) = -\pi\,\hat{x}\sin\pi\hat{y}}$$

no flow thru' WESTERN BDY

Case 2 : no flow thru' the eastern boundary.

$$\hat{u} = -\hat{\psi}_y = 0 \quad \text{at} \quad \hat{x} = 1$$

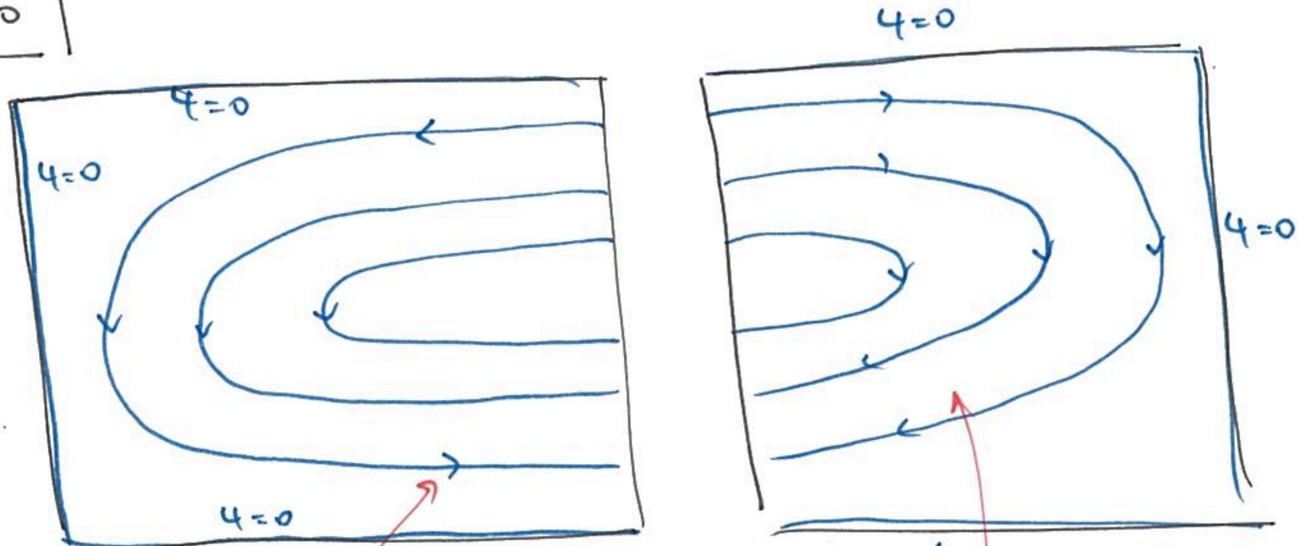
$$\Rightarrow \quad \hat{\psi}(1,\hat{y}) = \text{constant} = 0.$$

$$\Rightarrow \quad c(\hat{y}) = 1$$

$$\boxed{\hat{\psi}(\hat{x},\hat{y}) = \pi\,(1-\hat{x})\sin\pi\hat{y}}$$

no flow thru' EASTERN BDY

$\boxed{f > 0}$



Case 1 : no flow to west — valid but unstable

Case 2 : no flow to east. — physical

Western boundary currents & the Stommel Problem

Stommel: boundary layer to close the circulation.
Do this by adding bottom friction in a thin boundary layer near one of boundaries.

Parameterize bottom stress:

$$\underline{\tau}_B = r\,\underline{u}$$

$$\Rightarrow \quad \mathrm{curl}_z\,\underline{\tau}_B = r \cdot \mathrm{curl}_z\,\underline{u} = r\zeta = r\nabla^2\psi$$

Then $\quad \beta v = H^{-1}\left(\mathrm{curl}_z\,\underline{\tau}_w - \mathrm{curl}_z\,\underline{\tau}_B\right)$

$$\beta\frac{\partial\psi}{\partial x} = H^{-1}\,\mathrm{curl}_z\,\underline{\tau}_w - H^{-1} r\nabla^2\psi$$

Non-dimensionalize:

$$\epsilon\hat{\nabla}^2\hat{\psi} + \frac{\partial\hat{\psi}}{\partial\hat{x}} = \widehat{\mathrm{curl}}_z\,\hat{\underline{\tau}}_w \qquad \epsilon = \frac{r}{\beta H L}$$

increases the PDE to second order

Asymptotic matching

Write full solution as

$$\hat{\psi} = \hat{\psi}_I + \hat{\psi}_B$$

interior solution (Sverdrup) boundary layer solution.

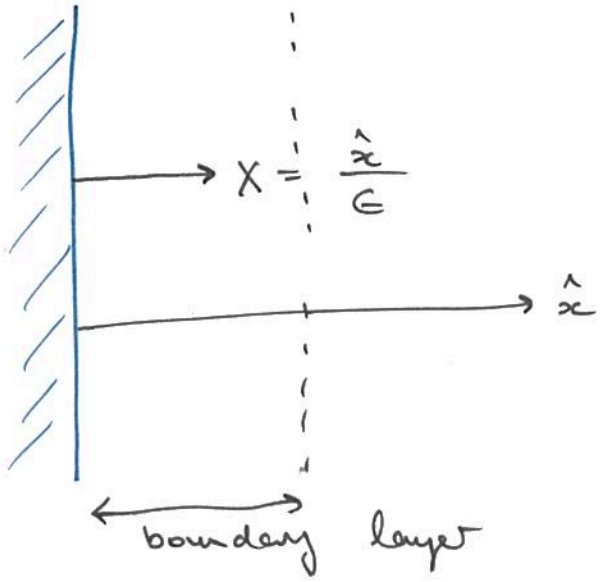
For $\quad \hat{\underline{\tau}}_w = (-\cos\pi\hat{y}\,,0) \qquad c = 0$ no flow to WB

$\Rightarrow \quad \hat{\psi}_I = \pi(c - x)\sin\pi y \qquad c = 1$ no flow to EB

Look at case of $c = 1$ (try $c = 0$ yourself!)

Introduce a "stretched" variable to deal with rapid variation of $\hat{\psi}_B$ near the boundary

$$X = \frac{\hat{x}}{\epsilon}$$



$$\hat{\psi}_B = \hat{\psi}_B(X, y)$$

$$\frac{\partial \hat{\psi}_B}{\partial \hat{x}} = \frac{\partial X}{\partial \hat{x}} \frac{\partial \hat{\psi}_B}{\partial X} = \frac{1}{\epsilon} \frac{\partial \hat{\psi}_B}{\partial X}$$

$$\frac{\partial^2 \hat{\psi}_B}{\partial \hat{x}^2} = \frac{1}{\epsilon^2} \frac{\partial \hat{\psi}_B}{\partial X}$$

Within the boundary layer : $\hat{\psi} = \hat{\psi}_I + \hat{\psi}_B$

$$\epsilon \hat{\nabla}^2 \hat{\psi} + \frac{\partial \hat{\psi}}{\partial \hat{x}} = \widehat{\mathrm{curl}}_z \, \underline{\hat{\tau}}_w$$

$$\epsilon \hat{\nabla}^2 (\hat{\psi}_I + \hat{\psi}_B) + \cancel{\frac{\partial \hat{\psi}_I}{\partial \hat{x}}} + \frac{\partial \hat{\psi}_B}{\partial \hat{x}} = \cancel{\mathrm{curl}_z \hat{\tau}_w}$$

$$\epsilon \hat{\nabla}^2 \hat{\psi}_I + \epsilon \left[\frac{1}{\epsilon^2} \frac{\partial^2 \hat{\psi}_B}{\partial X^2} + \frac{\partial^2 \hat{\psi}_B}{\partial \hat{y}^2} \right] + \frac{1}{\epsilon} \frac{\partial \hat{\psi}_B}{\partial X} = 0$$

$$\epsilon \hat{\nabla}^2 \hat{\psi}_I + \underbrace{\frac{1}{\epsilon} \frac{\partial^2 \hat{\psi}_B}{\partial X^2} + \epsilon \frac{\partial^2 \hat{\psi}_B}{\partial \hat{y}^2} + \frac{1}{\epsilon} \frac{\partial \hat{\psi}_B}{\partial X}}_{\text{dominant terms.}} = 0$$

To leading order :

$$\frac{\partial^2 \hat{\psi}_B}{\partial X^2} + \frac{\partial \hat{\psi}_B}{\partial X} = 0 .$$

$$\hat{\psi}_B(X, \hat{y}) = A(\hat{y}) + B(\hat{y}) e^{-X}$$

BC 1 : $\hat{\psi}_B \longrightarrow 0$ as $X \longrightarrow \infty$

$$\Rightarrow \quad A(\hat{y}) = 0$$

BC 2 : $\hat{\psi} = \hat{\psi}_I + \hat{\psi}_B = 0$ at $\hat{x} = 0$.

$$\Rightarrow \quad \pi \sin \pi \hat{y} + B(\hat{y}) = 0$$

$$\Rightarrow \quad B(\hat{y}) = -\pi \sin \pi \hat{y}$$

Full solution is

$$\hat{\psi}(\hat{x},\hat{y}) = \pi\left(1 - \hat{x} - e^{-X}\right) \sin \pi \hat{y}$$
$$= \pi\left(1 - \hat{x} - e^{-\hat{x}/\epsilon}\right) \sin \pi \hat{y}$$