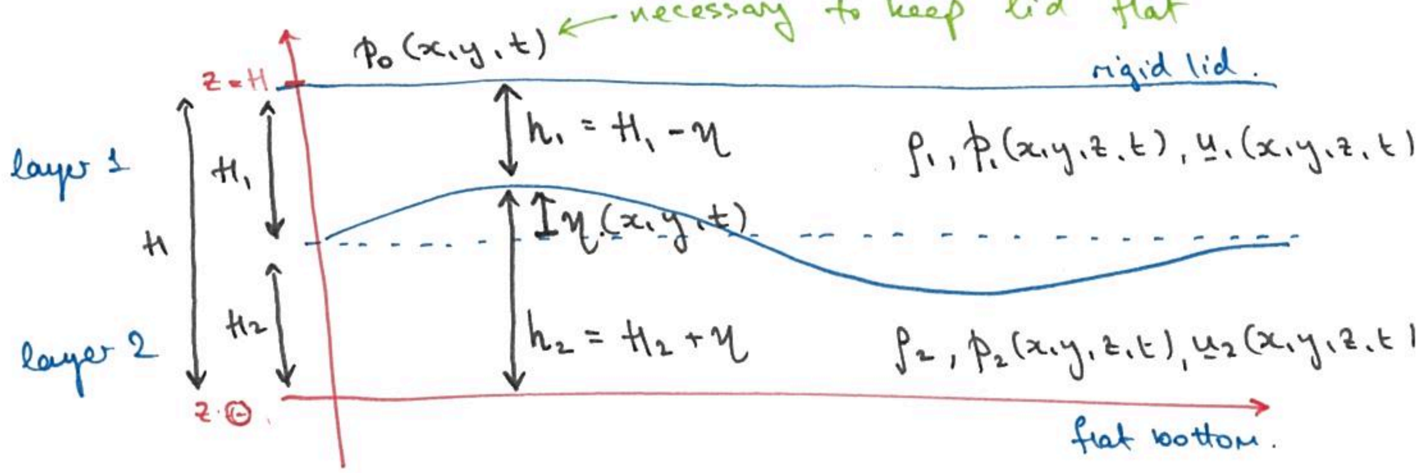
BAROCLINIC INSTABILITY

Two-layer quasigeostrophic model



Pressure in each layer:

$$p_1 = p_0 + \rho_1 g (H - z)$$

$$p_2 = p_0 + \rho_1 g h_1 + \rho_2 g (h_2 - z)$$

$$= p_0 + \rho_1 g (H_1 - \eta) + \rho_2 g (H_2 + \eta - z)$$

Pressure gradients:

$$\underline{\nabla}_z p_1 = \underline{\nabla}_z p_0$$

$$\underline{\nabla}_z p_2 = \underline{\nabla}_z p_0 - \rho_1 g \underline{\nabla} \eta + \rho_2 g \nabla \eta .$$

$$= \underline{\nabla}_z p_0 + \rho_0 g' \nabla \eta .$$

ρ_0 = average density , $g' = \dfrac{\rho_2 - \rho_1}{\rho_0}$ reduced gravity

[Used Boussinesq approximation: $\rho_0 \simeq \rho_1 \simeq \rho_2$]

Geostrophic balance (f-plane approximation).

layer 1: $f_0 \hat{\underline{z}} \times \underline{u}_1 = -\frac{1}{\rho_0} \underline{\nabla}_z p_1$

Boussinesq approximation.

$$\Rightarrow \quad \underline{u}_1 = -\hat{\underline{z}} \times (\hat{\underline{z}} \times \underline{u}_1) = \hat{\underline{z}} \times \underline{\nabla}_z \left(\frac{p_1}{f_0 \rho_0} \right) = \hat{\underline{z}} \times \underline{\nabla} \psi_1$$

$\psi_1 = \frac{p_1}{f_0 \rho_0}$ geostrophic streamfunction in layer 1.

layer 2: $f_0 \hat{\underline{z}} \times \underline{u}_2 = -\frac{1}{\rho_0} \underline{\nabla}_z p_2$

$$\Rightarrow \quad \underline{u}_2 = -\hat{\underline{z}} \times (\hat{\underline{z}} \times \underline{u}_2) = \hat{\underline{z}} \times \underline{\nabla}_z \left(\frac{p_2}{f_0 \rho_0} \right) = \hat{\underline{z}} \times \underline{\nabla} \psi_2$$

$$\psi_2 = \frac{p_2}{f_0 \rho_0} = \frac{1}{f_0 \rho_0} \Big(p_0 + \rho_1 g (H - \eta) + \rho_2 g (H_2 + \eta - z) \Big)$$

$$= \frac{p_0}{f_0 \rho_0} + \frac{\rho_1 g}{f_0 \rho_0} H + \frac{\rho_2 g}{f_0 \rho_0} H_2 + \frac{g'}{f_0} \eta - \frac{\rho_2 g}{f_0 \rho_0} z$$

$$= \frac{p_0}{f_0 \rho_0} + \frac{g'}{f_0} \eta + \text{constants.}$$

(ignore)

Compare with $\quad \psi_1 = \frac{p_1}{f_0 \rho_0} = \frac{p_0}{f_0 \rho_0} + \frac{\rho_1 g}{f_0 \rho_0} (H - z)$

$$= \frac{p_0}{f_0 \rho_0} + \text{constants}$$

(ignore)

So find $\quad \psi_2 = \psi_1 + \frac{g' \eta}{f_0}$ $\quad$ [ignoring constants]

or $\quad \boxed{\psi_2 - \psi_1 = \frac{g' \eta}{f_0}.}$

PV in each layer

For a single layer we found

$$q = \nabla^2\psi + \beta y - \frac{f_0^2\psi}{gH} \qquad L_d = \frac{\sqrt{gH}}{f_0}$$

To calculate PV in multiple layers use shallow water PV and expand:

$$Q_i = \frac{\zeta_i + f}{h_i} = \frac{\zeta_i + f_0 + \beta y}{H_i(1 + h_i'/H_i)} \qquad i = 1,2$$

$$\simeq \frac{\zeta_i + \beta y + f_0}{H_i}\left(1 - \frac{h_i'}{H_i}\right)$$

$$\simeq \frac{\zeta_i + \beta y + f_0}{H_i} - \frac{f_0}{H_i}\frac{h_i'}{H_i} + \text{higher order terms}$$

define quasigeostrophic PV:

$$\boxed{q_i = \zeta_i + \beta y - \frac{f_0 h_i'}{H_i}}$$

(ignoring factors of f_0, H_i).

$i=1,2$

Then

$$q_1 = \zeta_1 + \beta y - \frac{f_0 h_1'}{H_1} \qquad \zeta_1 = \nabla^2\psi_1$$

$$h_1' = -\eta = -\frac{f_0}{g'}(\psi_2 - \psi_1)$$

$$= \nabla^2\psi_1 + \beta y + \frac{f_0^2}{g'H_1}(\psi_2 - \psi_1)$$

$$q_2 = \zeta_2 + \beta y - \frac{f_0 h_2'}{H_2} \qquad \zeta_2 = \nabla^2\psi_2$$

$$h_2' = \eta = \frac{f_0}{g'}(\psi_2 - \psi_1)$$

$$= \nabla^2\psi_2 + \beta y - \frac{f_0^2}{g'H_2}(\psi_2 - \psi_1)$$

vortex stretching/squeezing

Simplification: $H_1 = H_2 = \frac{H}{2}.$

In that case:

$$\frac{f_0^2}{g' H_1} = \frac{f_0^2}{g' H_2} = \frac{k_d^2}{2}.$$

$$k_d^{-1} = L_d = \frac{\sqrt{g' H}}{2 f_0} = \left\{ \begin{matrix} \text{internal} \\ \text{baroclinic} \end{matrix} \right\} \text{Rossby radius of deformation.}$$

Then,
$$q_1 = \nabla^2 \psi_1 + \beta y + \frac{k_d^2}{2} (\psi_2 - \psi_1)$$
$$q_2 = \nabla^2 \psi_2 + \beta y - \frac{k_d^2}{2} (\psi_2 - \psi_1)$$

Equations of motion

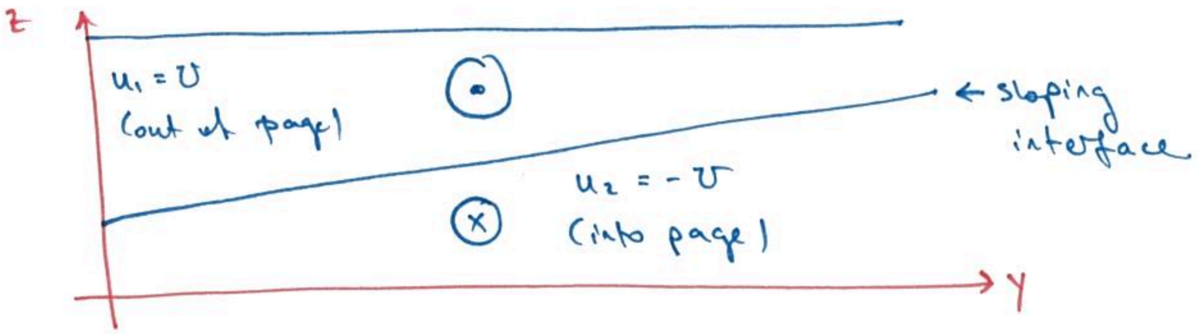
$$\frac{D q_1}{D t} = \frac{\partial q_1}{\partial t} + \underline{u}_1 \cdot \underline{\nabla} q_1 = 0 \qquad \underline{u}_1 = \left(-\frac{\partial \psi_1}{\partial y}, \frac{\partial \psi_1}{\partial x} \right)$$

$$\frac{D q_2}{D t} = \frac{\partial q_2}{\partial t} + \underline{u}_2 \cdot \underline{\nabla} q_2 = 0 \qquad \underline{u}_2 = \left(-\frac{\partial \psi_2}{\partial y}, \frac{\partial \psi_2}{\partial x} \right)$$

BASE STATE

Thermal wind balance

$$\bar{\psi}_1 = -U y \quad \Rightarrow \quad \bar{u}_1 = -\bar{\psi}_{1,y} = U$$
$$\bar{\psi}_2 = U y \quad \Rightarrow \quad \bar{u}_2 = -\bar{\psi}_{2,y} = -U$$



Add small perturbations:

$$\psi_1 = \bar{\psi}_1 + \tilde{\psi}_1 = -Uy + \tilde{\psi}_1$$

$$\psi_2 = \bar{\psi}_2 + \tilde{\psi}_2 = Uy + \tilde{\psi}_2$$

base state has zero relative vorticity

$$q_1 = \underbrace{\cancel{\nabla^2\bar{\psi}_1} + \beta y + \frac{k_d^2}{2}(\bar{\psi}_2 - \bar{\psi}_1)}_{\bar{q}_1} + \underbrace{\nabla^2\tilde{\psi}_1 + \frac{k_d^2}{2}(\tilde{\psi}_2 - \tilde{\psi}_1)}_{\tilde{q}_1}$$

$$= (\beta + k_d^2 U)\,y + \nabla^2\tilde{\psi}_1 + \frac{k_d^2}{2}(\tilde{\psi}_2 - \tilde{\psi}_1)$$

$$q_2 = (\beta - k_d^2 U)\,y + \nabla^2\tilde{\psi}_2 - \frac{k_d^2}{2}(\tilde{\psi}_2 - \tilde{\psi}_1)$$

Linearized equations of motion:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{q}_1 + \frac{\partial\tilde{\psi}_1}{\partial x}(\beta + k_d^2 U) = 0$$

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\tilde{q}_2 + \frac{\partial\tilde{\psi}_2}{\partial x}(\beta - k_d^2 U) = 0$$

Plane wave solutions:

$$\tilde{\psi}_1 = \hat{\psi}_1 e^{i(kx + ly - \omega t)} \qquad \omega = \omega_r + i\omega_i$$

$$\tilde{\psi}_2 = \hat{\psi}_2 e^{i(kx + ly - \omega t)} \qquad \omega_i \neq 0 \Rightarrow \text{instability.}$$

Can show that equations of motion become

$$① \quad (c - U)\left(K^2\hat{\psi}_1 + \frac{k_d^2}{2}(\hat{\psi}_1 - \hat{\psi}_2)\right) + (\beta + k_d^2 U)\hat{\psi}_1 = 0$$

$$② \quad (c + U)\left(K^2\hat{\psi}_2 - \frac{k_d^2}{2}(\hat{\psi}_1 - \hat{\psi}_2)\right) + (\beta - k_d^2 U)\hat{\psi}_2 = 0$$

where

$$c = \frac{\omega}{k} = \text{phase speed.}$$

$$K = \sqrt{k^2 + l^2} = \text{wavenumber magnitude.}$$

① + ② ⇒

$$(cK^2+\beta)(\hat\psi_1+\hat\psi_2) - UK^2(\hat\psi_1-\hat\psi_2) = 0$$

① − ② ⇒

$$[c(K^2+K_d^2)+\beta](\hat\psi_1-\hat\psi_2) - U(K^2-K_d^2)(\hat\psi_1+\hat\psi_2) = 0$$

Introduce

$$\hat\psi_s = \tfrac{1}{2}(\hat\psi_1+\hat\psi_2)$$ barotropic mode (symmetric)

$$\hat\psi_A = \tfrac{1}{2}(\hat\psi_1-\hat\psi_2)$$ baroclinic mode (antisymmetric)

$$(cK^2+\beta)\hat\psi_s - UK^2\hat\psi_A = 0$$

$$(c(K^2+K_d^2)+\beta)\hat\psi_A - U(K^2-K_d^2)\hat\psi_s = 0$$

$$\boxed{\begin{vmatrix} cK^2+\beta & -UK^2 \\ -U(K^2-K_d^2) & c(K^2+K^2_d)+\beta \end{vmatrix} \begin{vmatrix} \hat\psi_s \\ \hat\psi_A \end{vmatrix} = 0.}$$

For non-trivial solutions, determinant = 0.

Case 1 : $U = 0$

$$\begin{vmatrix} cK^2+\beta & 0 \\ 0 & c(K^2+K_d^2)+\beta \end{vmatrix}$$

$$= (cK^2+\beta)(c(K^2+K_d^2)+\beta) = 0$$

$$c = -\frac{\beta}{K^2}$$ phase speed of barotropic Rossby wave

$$c = -\frac{\beta}{K^2+K_d^2}$$ phase speed of baroclinic Rossby wave

<u>NO INSTABILITY !</u>

Case 2 : $U \neq 0$ (nonzero shear, tilting interface)

determinant $= (cK^2+\beta)\left[c(K^2+K_d^2)+\beta\right] - U^2K^2(K^2-K_d^2)$

$= 0$

Quadratic in c :

$$c^2K^2(K^2+K_d^2) + \beta(2K^2+K_d^2)c + \beta^2 - U^2K^2(K^2-K_d^2) = 0$$

$$c = -\beta\frac{2K^2+K_d^2}{2K^2(K^2+K_d^2)} \pm \frac{\sqrt{D}}{2K^2(K^2+K_d^2)}$$

Discriminant

$D = \beta^2K_d^4 + 4U^2K^4(K^4-K_d^4)$. $<0 \Rightarrow$ instability

For $D<0$ get complex $c = c_r + ic_i$

$\Rightarrow \psi \sim e^{\omega_i t} = e^{c_i k t}$

When is $D<0$?

$$\beta^2K_d^4 < 4U^2K^4(K_d^4-K^4).$$

$$\Rightarrow \quad 1 < \frac{4U^2K_d^4}{\beta^2}\left\{\frac{K^4}{K_d^4}\right\}\left(1-\left\{\frac{K^4}{K_d^4}\right\}\right).$$

let $\hat{U} = \dfrac{UK_d^2}{\beta}$ non-dimensional shear velocity

$\hat{K} = \dfrac{K}{K_d}$ non-dimensional wavenumber

$$\boxed{1 < 4\hat{U}^2\hat{K}^4(1-\hat{K}^4)}.$$ condition for instability

Rewrite as

$$(\hat{K}^4)^2 - \hat{K}^4 + \frac{1}{4\hat{U}^2} < 0 .$$

quadratic inequality in $\hat{K}^4$.

Roots : $\hat{K}_c^4 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{1}{\hat{U}^2}}$

For <u>real</u> roots : $\hat{U} > 1$.

$$\Rightarrow \boxed{U > \frac{\beta}{Kd^2}}$$

threshold shear velocity for instability.

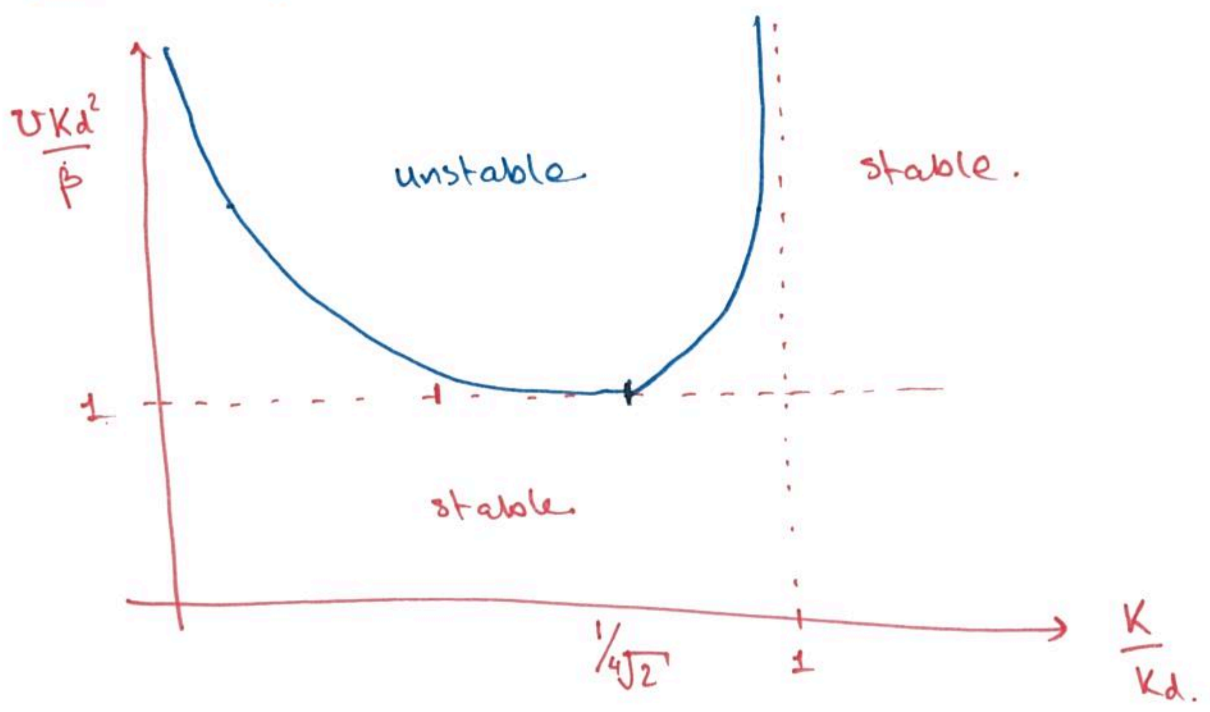
Value of $\hat{K}_c$ for $U = \beta / Kd^2$ $(\hat{U} = 1)$

$$\Rightarrow \hat{K}_c^4 = \frac{1}{2} \Rightarrow \hat{K}_c = \frac{1}{\sqrt[4]{2}}$$

When $\hat{U} \longrightarrow \infty$

$$\hat{K}_c \longrightarrow 0 , 1.$$

<u>Regime diagram</u> :



Why $\pi/2$?

$$\hat{\psi}_A = \frac{cK^2 + \beta}{K^2 U} \hat{\psi}_S$$

For simplicity set $\beta = 0$.

$$\Rightarrow \quad \hat{\psi}_A = \frac{c}{U} \hat{\psi}_S$$

For an unstable wave with $\beta = 0$

$$\Rightarrow \quad c = i c_i$$

$$\hat{\psi}_A = \frac{i c_i}{U} \hat{\psi}_S = e^{i\pi/2} \frac{c_i}{U} \hat{\psi}_S .$$

$\Rightarrow$ $\pi/2$ phase shift between $\hat{\psi}_A$ e $\hat{\psi}_S$.

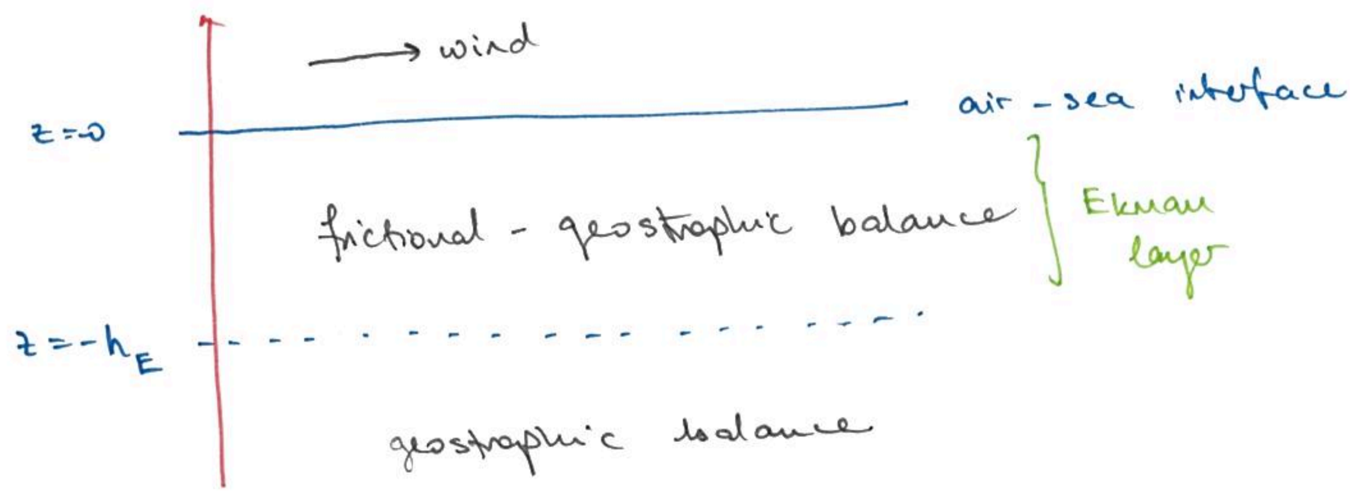
$\Rightarrow$ $\pi/2$ phase shift between $\hat{\psi}_1$ e $\hat{\psi}_2$.

The Ekman Layer.

* frictional boundary layer that connects geostrophic flow in the atmosphere / ocean. to friction - dominated flow near boundaries.

- Atmospheric Ekman layer near ground.
- Weak bottom Ekman layer at ocean floor
- Surface Ekman layer at air-sea interface.

THIS is the Ekman layer we will focus on.



In the Ekman layer

$$\underline{f} \times \underline{u} = -\frac{1}{\rho_0} \underline{\nabla}_z p + \underbrace{A \frac{\partial^2 \underline{u}}{\partial z^2}}_{\text{model for friction.}}$$

A = "eddy viscosity" = models the effect of turbulent motions to remove/diffuse momentum

Also have hydrostatic balance ($\rho = \rho_0$ = constant)

$$\frac{\partial p}{\partial z} = 0 \qquad (p \text{ is the pressure perturbation})$$

$$\underline{\nabla} \cdot \underline{v} = 0 \qquad (\text{3D incompressibility})$$

If friction is not present / important (far from surface) then have geostrophic balance.

$$f_0 U \sim \frac{P}{\rho_0 L}$$

$\Rightarrow$ gives us a scaling for pressure $P \sim f_0 U \rho_0 L$.

This gives us (with friction)

from geostrophic scaling for pressure

$$f_0 U \, \hat{\underline{f}} \times \hat{\underline{u}} = f_0 U \left(-\hat{\nabla}_z \hat{p}\right) + \frac{AU}{H^2} \frac{\partial^2 \hat{\underline{u}}}{\partial \hat{z}^2}$$

gives non-dimensional frictional - geostrophic balance:

$$\hat{\underline{f}} \times \hat{\underline{u}} = -\hat{\underline{\nabla}}_z \hat{p} + Ek \frac{\partial^2 \hat{\underline{u}}}{\partial \hat{z}^2}$$

where $Ek = \frac{A}{f_0 H^2}$ = Ekman number (dimensionless)

This determines the importance of friction:

$Ek \ll 1$: friction negligible

$Ek \gg 1$: friction dominates.

Define thickness of the Ekman layer (h_E) as the depth where $Ek \sim O(1)$:

$$Ek = 1 = \frac{A}{f_0 h_E^2} \Rightarrow \boxed{h_E = \sqrt{A/f_0}}$$

h_E = Ekman layer thickness $= \begin{cases} 1 \text{ km} & \text{in atm.} \\ 50 \text{ m} & \text{in ocn.} \end{cases}$