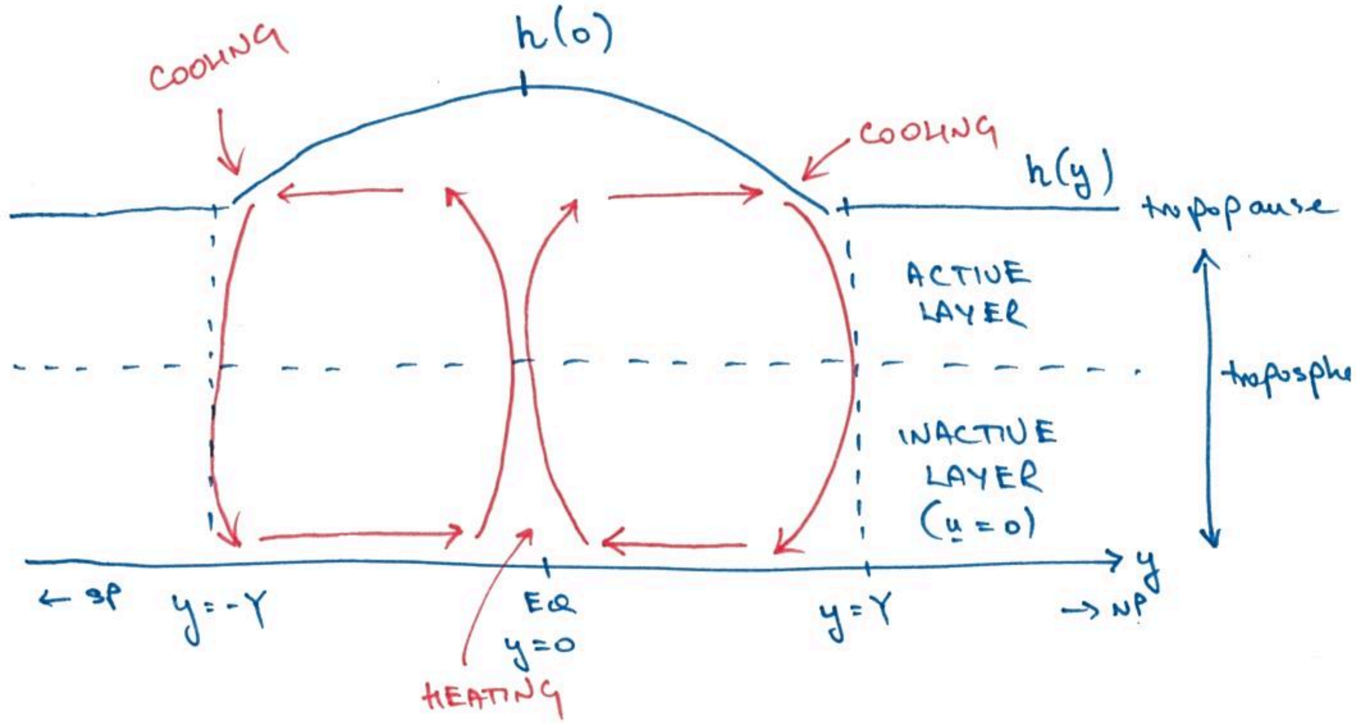
SHALLOW WATER MODEL OF THE HADLEY CELL



Use $1\frac{1}{2}$-layer (reduced gravity) shallow water model to predict:

* height $h(0)$ & shape $h(y)$ of tropopause
* meridional extent of the Hadley cell.

Key features:

* β-plane centred at the equator

$$f(y) = f_0 + \beta y = \beta y \qquad (f_0 = 0).$$

* β-effect (change in f with y) cause acceleration of zonal momentum (u) as air is moved away from the equator.
* zonal momentum balanced by upper sloping surface (geostrophic balance)
* thermal equilibrium gives estimate of Y.

Start with zonal momentum equation in upper layer:

$$\frac{Du}{Dt} - \underbrace{f(y)v}_{\beta y} = \underbrace{\cancel{-\frac{1}{\rho_0}\frac{\partial p}{\partial x}}}_{\substack{\text{zonal symmetry} \to \text{no variation in } x \\ \Rightarrow \text{no pressure gradient in } x\text{-dir}}} \quad 0$$

Assume steady state : $\frac{\partial}{\partial t} = 0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \beta y v$$

Assume zonal symmetry : $u = u(y)$, $v = v(y)$

$$v\frac{du}{dy} = \beta y v$$

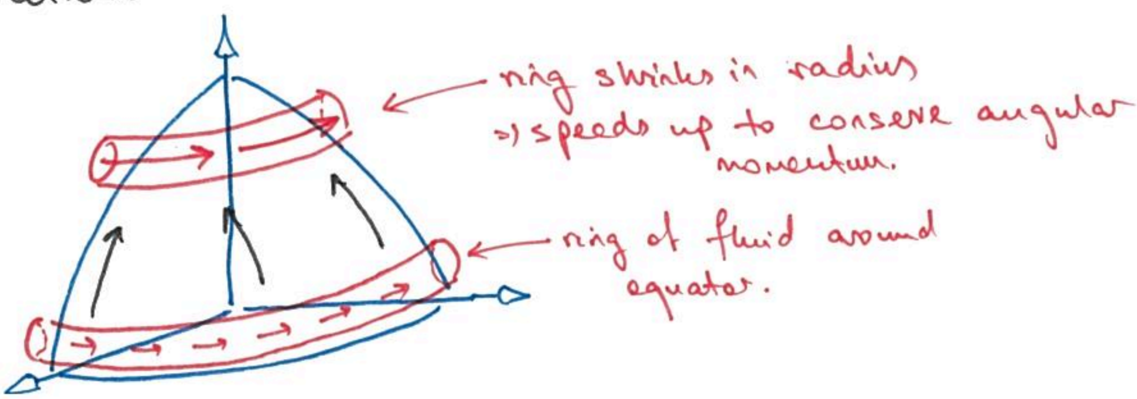
$v \neq 0$ (flow towards pole) so

$$\frac{du}{dy} = \beta y$$

Assume $u(0) = 0$ (no flow at equator)

$$\Rightarrow \boxed{u(y) = \frac{\beta y^2}{2}}$$

Zonal jet in the upper layer increases speed away from equator $\longrightarrow$ consequence of angular momentum conservation!



Geostrophic balance (reduced gravity):

$$fu = -g' \frac{dh}{dy}.$$

g' = reduced gravity

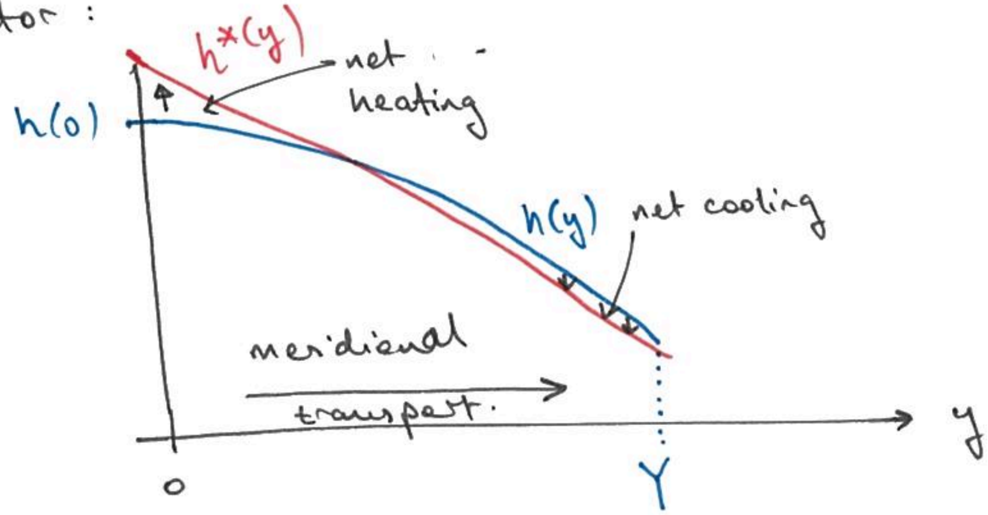
$f = \beta y$.

Use $u(y) = \beta y^2/2$:

$$\frac{dh}{dy} = -\frac{f(y)u(y)}{g'} = -\frac{\beta^2 y^3}{2g'}$$

Integrate: $$\boxed{h(y) = h(0) - \frac{\beta^2 y^4}{8g'}}$$

Thus, height of tropopause will <u>decrease</u> away from equator:



Simple thermodynamic model:

$$\frac{Dh}{Dt} = -\frac{1}{\tau}(h - h^*).$$

if $h < h^* \Rightarrow h \uparrow$
if $h > h^* \Rightarrow h \downarrow$

where $h^* = h^*(y)$ is a prescribed "forcing" of the tropopause height due to heating + cooling

τ = timescale on which h "relaxes" to $h^*(y)$.

For steady $\left(\frac{\partial}{\partial t}=0\right)$ and zonally symmetric $\left(\frac{\partial}{\partial x}=0\right)$ flow

$$\frac{Dh}{Dt} = \cancel{\frac{\partial h}{\partial t}} + u\cancel{\frac{\partial h}{\partial x}} + \underbrace{v\frac{\partial h}{\partial y}}_{\text{meridional advection}} = \underbrace{-\frac{1}{\tau}\left(h - h^*(y)\right)}_{\text{heating / cooling}}$$

Choose a reasonable model for $h^*(y)$

$$h^*(y) = h_0(1 - \alpha|y|)$$

To be in thermal equilibrium we must have heating and cooling cancel when integrated over the cell:

$$\int_0^Y (h - h^*)\,dy = 0.$$

$y \geq 0$:

$$\int_0^Y \left[h(0) - \frac{\beta^2 y^4}{8g'} - h_0(1-\alpha y)\right] dy$$

$$= (h(0) - h_0)Y - \frac{\beta^2 Y^5}{40g'} + \frac{h_0 \alpha Y^2}{2} = 0$$

$$\Rightarrow \boxed{\frac{\beta^2 Y^4}{40g'} - \frac{h_0 \alpha Y}{2} = h(0) - h_0}$$

Constraint #1 on $Y, h(0)$

Next: outside the Hadley cell there is no motion ($u, v = 0$) so must have:

$$h(y) = h^*(y) \qquad y \geq Y.$$

This gives: $h(Y) = h^*(Y)$ at $y = Y$

$$\boxed{h(0) - \frac{\beta^2 Y^4}{8g'} = h_0(1 - \alpha Y)}$$

Constraint #2 on $Y, h(0)$

Exercise: put #1 & #2 together to solve for Y, $h(0)$

$$Y = \left(\frac{5 h_0 \alpha g'}{\beta^2} \right)^{1/3}$$

$$h(0) = h_0 \left(1 - \frac{3\alpha}{8} Y \right)$$

SHALLOW WATER WAVES: LINEAR WAVE THEORY

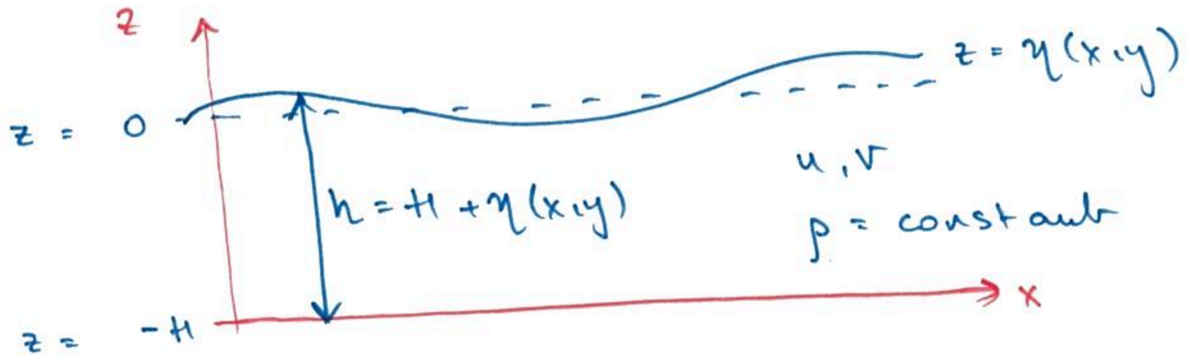
Single Shallow water layer : $\eta_b = -H = \text{constant}.$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

$$h = \eta - \eta_b = H + \eta \quad , \quad f = f_0 + \beta y.$$



Linear wave theory:

studying small perturbations to a background base state

1. Determine the base state (unchanging in time)

$$u = \bar{u}(x,y) \quad , \quad v = \bar{v}(x,y) \quad , \quad h = \bar{h}(x,y)$$

Solutions to time-independent equations of motion ($\frac{\partial}{\partial t} = 0$)

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - f\bar{v} = -g\frac{\partial \bar{h}}{\partial x}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + f\bar{u} = -g\frac{\partial \bar{h}}{\partial y}$$

$$\bar u \frac{\partial \bar h}{\partial x} + \bar v \frac{\partial \bar h}{\partial y} + \bar h \left(\frac{\partial \bar u}{\partial x} + \frac{\partial \bar v}{\partial y} \right) = 0$$

Note: can have a flow, just has to be steady, e.g.

$$\bar u = \text{constant}, \quad \bar v = 0, \quad \bar h = ?$$

$$\Rightarrow \quad \frac{\partial \bar h}{\partial x} = 0. \quad \Rightarrow \quad \bar h \text{ does not depend on } x.$$

$$f \bar u = -g \frac{\partial \bar h}{\partial y}. \quad \Rightarrow \quad \bar h(y) = -\frac{f \bar u}{g} y + \text{const.}$$

zonal (eastward) flow balanced by meridional (northward sloping upper surface

2. Add <u>small perturbations</u> to base state and <u>neglect nonlinear terms.</u>

$$u(x,y,t) = \bar u(x,y) + \tilde u(x,y,t) = \bar u + \tilde u(x,y,t)$$
$$v(x,y,t) = \bar v(x,y) + \tilde v(x,y,t) = \tilde v(x,y,t)$$
$$h(x,y,t) = \bar h(x,y) + \tilde h(x,y,t) = \bar h(y) + \tilde h(x,y,t$$

where $\tilde u, \tilde v, \tilde h$ are small perturbations.

$$\frac{\partial \tilde u}{\partial t} + (\bar u + \underline{\underline{\tilde u}}) \frac{\partial \tilde u}{\partial x} + \underline{\underline{\tilde v}} \frac{\partial \tilde u}{\partial y} - f \tilde v = -g \frac{\partial \tilde h}{\partial x}$$

$$\frac{\partial \tilde v}{\partial t} + (\bar u + \underline{\underline{\tilde u}}) \frac{\partial \tilde v}{\partial x} + \underline{\underline{\tilde v}} \frac{\partial \tilde v}{\partial y} + f(\tilde u + \bar u) = -g \frac{\partial \bar h}{\partial y} - g \frac{\partial \tilde h}{\partial y}$$

cancel!

$$\frac{\partial \tilde h}{\partial t} + (\bar u + \underline{\underline{\tilde u}}) \frac{\partial \tilde h}{\partial x} + \tilde v \frac{\partial}{\partial y} (\bar h + \underline{\underline{\tilde h}}) + (\bar h + \underline{\underline{\tilde h}}) \left(\frac{\partial \tilde u}{\partial x} + \frac{\partial \tilde v}{\partial y} \right) = 0$$

Neglect all nonlinear terms in $(\tilde{u}, \tilde{v}, \tilde{h})$ e.g $\tilde{u}\frac{\partial \tilde{u}}{\partial x}$, $\tilde{h}\frac{\partial \tilde{u}}{\partial x}$

This gives us the <u>linearized</u> equations of motion:

$$\frac{\partial \tilde{u}}{\partial t} + \bar{u}\frac{\partial \tilde{u}}{\partial x} - f\tilde{v} + g\frac{\partial \tilde{h}}{\partial x} = 0$$

$$\frac{\partial \tilde{v}}{\partial t} + \bar{u}\frac{\partial \tilde{v}}{\partial x} + f\tilde{u} + g\frac{\partial \tilde{h}}{\partial y} = 0$$

$$\frac{\partial \tilde{h}}{\partial t} + \bar{u}\frac{\partial \tilde{h}}{\partial x} + \tilde{v}\underbrace{\frac{\partial \bar{h}}{\partial y}}_{= -\frac{f\bar{u}}{g}} + \bar{h}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) = 0$$

3. Look for linear plane wave solutions

Assume solutions have form

$$\tilde{u} = \hat{u}\, e^{i(kx + ly - \omega t)}$$
$$\tilde{v} = \hat{v}\, e^{i(kx + ly - \omega t)}$$
$$\tilde{h} = \hat{h}\, e^{i(kx + ly - \omega t)}$$

understood that only the real part is physical.

$\hat{u}, \hat{v}, \hat{h}$ = wave amplitudes (can be complex)

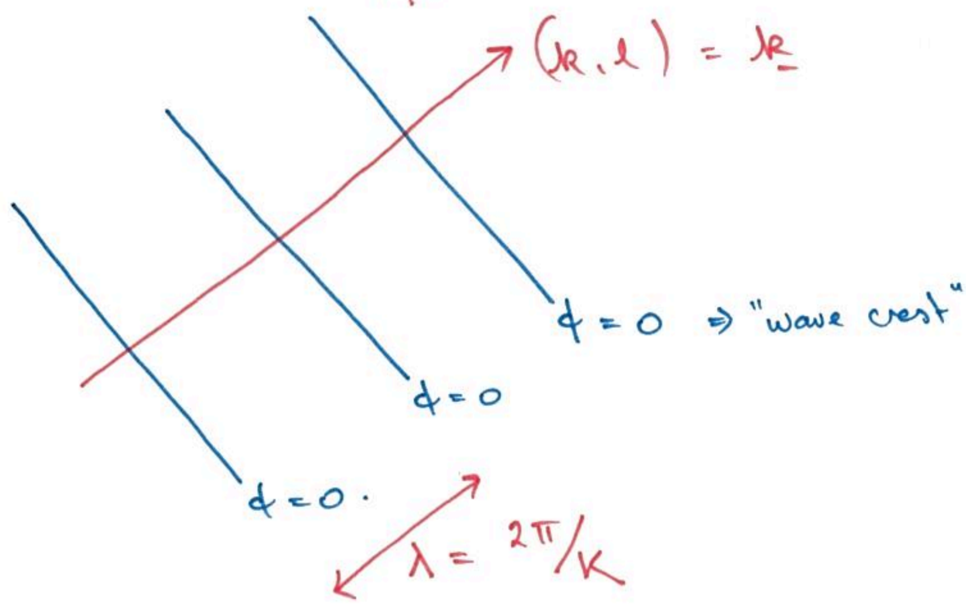
$kx + ly - \omega t = \phi$ = phase. (real, $0 \le \phi \le 2\pi$).

ω = (angular) frequency of wave $= \frac{2\pi}{T}$ # wave crests passing per second

$(k, l) = \underline{k}$ = 2D wave vector

$K = |\underline{k}| = \sqrt{k^2 + l^2}$ = wavenumber $= \frac{2\pi}{\lambda}$ # wave crests per meter.

$\hat{\underline{k}} = \frac{\underline{k}}{K} = \left(\frac{k}{K}, \frac{l}{K}\right)$ = orientation of wave



$$\phi = kx + ly - \omega t = 0$$

at $t = 0$: $kx + ly = 0.$

at $t = \Delta t$: $kx + ly = \omega \Delta t$

4. Look for non-trivial solutions to linearized eqns.

Plane wave assumption transforms differential equations into algebraic equation.

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial t}\left(\hat{u}\, e^{i(kx+ly-\omega t)}\right) = -i\omega \hat{u}\, e^{i(kx+ly-\omega t)}$$
$$= -i\omega \tilde{u}$$

$$\frac{\partial \tilde{v}}{\partial x} = \frac{\partial}{\partial x}\left(\hat{v}\, e^{i(kx+ly-\omega t)}\right) = ik\tilde{v}$$

$$\frac{\partial \tilde{h}}{\partial y} = \frac{\partial}{\partial y}\left(\hat{h}\, e^{i(kx+ly-\omega t)}\right) = il \cdot \tilde{h}$$

Thus, linear 3x3 PDE system ⇒ linear 3x3 algebraic. system

$$\boxed{\mathbb{M}} \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbb{M}$ = linear operator (3×3) depending on k, l, ω.

For non-trivial solutions :

$$\boxed{\det \mathbb{M} = 0}$$

function of k, l, ω

$$\boxed{\mathcal{F}(k, l, \omega) = 0}$$ for non-trivial solutions.

Dispersion relation relating wave properties.

Solutions to $\mathcal{F}(k, l, \omega) = 0$:

$$\boxed{\omega = \omega(k, l)}$$

For each solution find $\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{h} \end{pmatrix}$ with a fixed phase

relationship between $\tilde{u}, \tilde{v}, \tilde{h}$ $\Rightarrow$ wave mode.