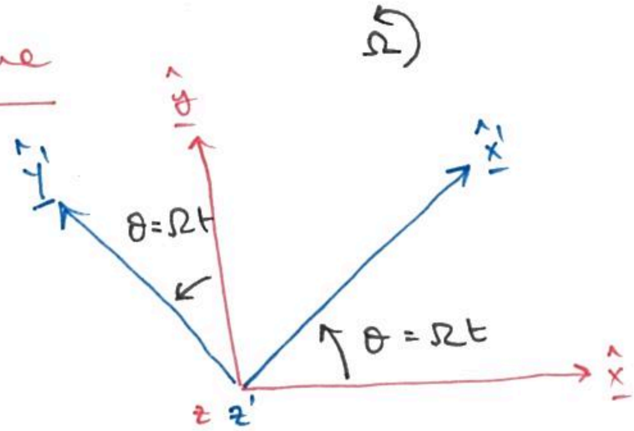
Rotating Reference Frame

fixed axis (x, y, z)

rotating axis (x', y', z')



Express rotating basis vectors in terms of the fixed basis vectors

$$\hat{\underline{x}}' = \cos\Omega t\ \hat{\underline{x}} + \sin\Omega t\ \hat{\underline{y}}$$

$$\hat{\underline{y}}' = -\sin\Omega t\ \hat{\underline{x}} + \cos\Omega t\ \hat{\underline{y}}$$

$$\hat{\underline{z}}' = \hat{\underline{z}}$$

Express rates of change of $\hat{\underline{x}}', \hat{\underline{y}}$ as measured in fixed frame

$$\left.\frac{d}{dt}\hat{\underline{x}}'\right|_0 = -\Omega\sin\Omega t\ \hat{\underline{x}} + \Omega\cos\Omega t\ \hat{\underline{y}}$$

$$\left.\frac{d}{dt}\hat{\underline{y}}'\right|_0 = -\Omega\cos\Omega t\ \hat{\underline{x}} - \Omega\sin\Omega t\ \hat{\underline{y}}$$

$$\left.\frac{d}{dt}\hat{\underline{z}}'\right|_0 = 0$$

"as measured in the fixed frame"

This gives

$$\left.\frac{d}{dt}\hat{\underline{x}}'\right|_0 = \Omega\hat{\underline{y}}' = \underline{\Omega}\times\hat{\underline{x}}'$$

$$\left.\frac{d}{dt}\hat{\underline{y}}'\right|_0 = -\Omega\hat{\underline{x}}' = \underline{\Omega}\times\hat{\underline{y}}'$$

simple circular motion.

Using $\underline{\Omega} = \Omega\hat{\underline{z}} = \Omega\hat{\underline{z}}'$ & $\hat{\underline{z}}'\times\hat{\underline{x}}' = \hat{\underline{y}}'$, $\hat{\underline{z}}'\times\hat{\underline{y}}' = -\hat{\underline{x}}'$

Kinematics in a rotating frame

Let $\underline{a} = \underline{r}(t) = r_1(t)\,\hat{\underline{x}}' + r_2(t)\,\hat{\underline{y}}' + r_3(t)\,\hat{\underline{z}}'$

= particle trajectory in a rotating frame

Def: relative velocity = velocity as measured in rotating frame.

$$\underline{v}_R = \left.\frac{d\underline{r}}{dt}\right|_R = \dot{r}_1\,\hat{\underline{x}}' + \dot{r}_2\,\hat{\underline{y}}' + \dot{r}_3\,\hat{\underline{z}}'$$

Def: absolute velocity = velocity measured in fixed frame.

$$\underline{v}_0 = \left.\frac{d\underline{r}}{dt}\right|_0 = \left.\frac{d\underline{r}}{dt}\right|_R + \underline{\Omega} \times \underline{r}$$

$$\Rightarrow \boxed{\underline{v}_0 = \underline{v}_R + \underbrace{\underline{\Omega} \times \underline{r}}_{\text{apparent extra velocity due to rotation.}}}$$

Acceleration

$$\left.\frac{d\underline{v}_0}{dt}\right|_0 = \left.\frac{d\underline{v}_0}{dt}\right|_R + \underline{\Omega} \times \underline{v}_0$$

$$= \left.\frac{d}{dt}\left(\underline{v}_R + \underline{\Omega} \times \underline{r}\right)\right|_R + \underline{\Omega} \times \left(\underline{v}_R + \underline{\Omega} \times \underline{r}\right)$$

$$= \left.\frac{d\underline{v}_R}{dt}\right|_R + 2\,\underline{\Omega} \times \underline{v}_R + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right).$$

Rearrange:

$$\underbrace{\left.\frac{d\underline{v}_R}{dt}\right|_R}_{\text{acceleration in rotating frame}} = \underbrace{\left.\frac{d\underline{v}_0}{dt}\right|_0}_{\text{force/mass in fixed frame}} \underbrace{- 2\,\underline{\Omega} \times \underline{v}_R}_{\text{Coriolis force}} \underbrace{- \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right)}_{\text{Centrifugal force}}.$$

Coriolis force (per unit mass) $\quad -2\,\underline{\Omega} \times \underline{v}_R$

* only occurs when $\underline{v}_R \neq 0$
* $\perp^{ar}$ to both $\underline{\Omega}$ and $\underline{v}_R$.
* Work $= \int \underline{F} \cdot d\underline{r} = \int \underline{F} \cdot \underline{v}\, dt \sim \int \underline{\Omega} \times \underline{v} \cdot \underline{v}\, dt = 0$

Centrifugal force (per unit mass) $\quad -\underline{\Omega} \times (\underline{\Omega} \times \underline{r})$

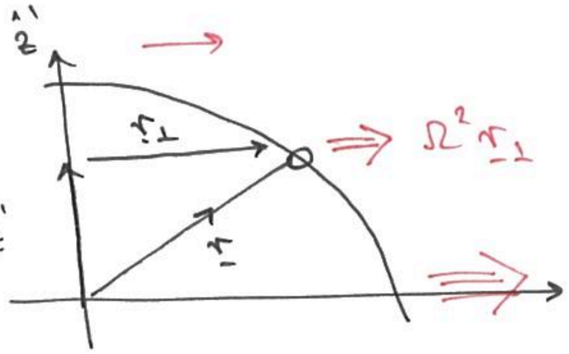
$$= -\Omega^2\, \hat{\underline{z}}' \times (\hat{\underline{z}}' \times \underline{r}).$$

$$= -\Omega^2\, \hat{\underline{z}}' \times \Big(\hat{\underline{z}}' \times \{ \underbrace{r_1 \hat{\underline{x}}' + r_2 \hat{\underline{y}}'}_{\underline{r}_\perp} + r_3 \hat{\underline{z}}' \}\Big)$$

$$= -\Omega^2\, \hat{\underline{z}} \times (\hat{\underline{z}}' \times \underline{r}_\perp)$$

$$= -\Omega^2 (-r_1 \hat{\underline{x}}' - r_2 \hat{\underline{y}}') = \Omega^2 \underline{r}_\perp$$

* Centrifugal force is always OUTWARDS from rotation axis
* increases with distance from axis.



* Think of Centrifugal force as an "anti-gravity".

$$\underline{f}_{grav} = -g\hat{\underline{r}} = -\underline{\nabla}(gr)$$

$$\underline{f}_{cent} = \Omega^2 \underline{r}_\perp = \underline{\nabla}\left(\tfrac{1}{2}\Omega^2 r_\perp^2\right)$$

(Ex: prove these)

$$\underline{f}_{grav} + \underline{f}_{cent} = -\underline{\nabla}\left(gr - \tfrac{1}{2}\Omega^2 r_\perp^2\right)$$

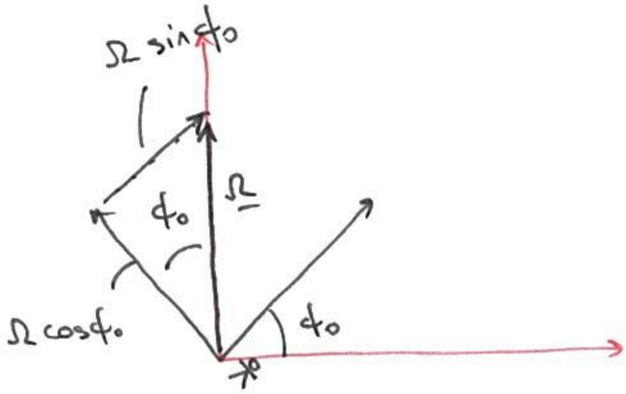
Tangent plane approximation.

Coriolis force /mass

$$\underline{v} = u\,\hat{\underline{x}}' + v\,\hat{\underline{y}}' + w\,\hat{\underline{z}}'$$

$$\underline{\Omega} = \Omega\cos\phi_0 \cdot \hat{\underline{y}}' + \Omega\sin\phi_0\,\hat{\underline{z}}'$$

$$-2\,\underline{\Omega} \times \underline{v}$$



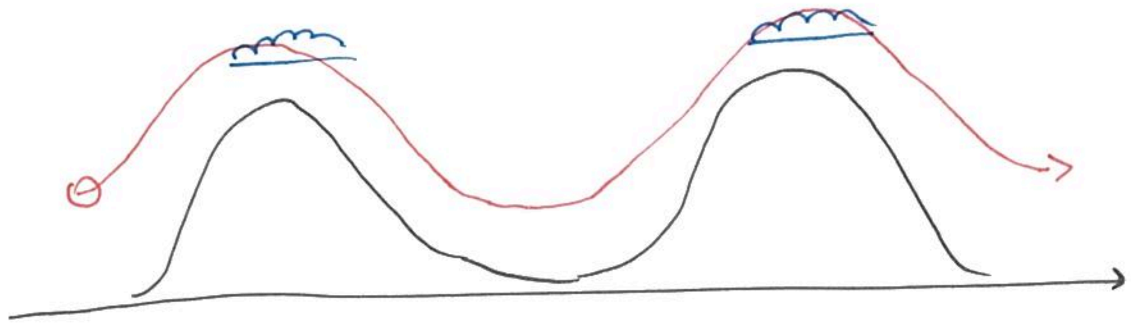
$$= \begin{vmatrix} \hat{\underline{x}}' & \hat{\underline{y}}' & \hat{\underline{z}}' \\ 0 & -2\Omega\cos\phi_0 & -2\Omega\sin\phi_0 \\ u & v & w \end{vmatrix}$$

$$= \begin{bmatrix} -2\Omega w\cos\phi_0 + 2\Omega v\sin\phi_0 \\ 2\Omega u\sin\phi_0 \\ -2\Omega u\cos\phi_0 \end{bmatrix} = \begin{bmatrix} -w f_* + v f \\ u f \\ -u f_* \end{bmatrix}$$

$$f = 2\Omega\sin\phi_0 \qquad f_* = 2\Omega\cos\phi_0 .$$

ADVECTIVE DERIVATIVE

(a.k.a. material derivative, Lagrangian derivative ...).



$C(x, y, z)$ = "cloudiness"

Even though $\left.\frac{\partial C}{\partial t}\right|_{\text{fixed point}} = 0$ everywhere, $\left.\frac{\partial C}{\partial t}\right|_{\text{particle}} \neq 0$

Change in cloudiness:

$$\delta C = \underbrace{\frac{\partial C}{\partial t}}_{=0 \text{ here}} \delta t + \frac{\partial C}{\partial x}\delta x + \frac{\partial C}{\partial y}\delta y + \frac{\partial C}{\partial z}\delta z$$

In the limit $\delta t \to 0$

$$\lim_{\delta t \to 0} \frac{\delta C}{\delta t} = \left.\frac{\partial C}{\partial t}\right|_{\text{particle}} = \underbrace{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}\frac{dx}{dt} + \frac{\partial C}{\partial y}\frac{dy}{dt} + \frac{\partial C}{\partial z}\frac{dz}{dt}}_{\text{evaluated at fixed } \underline{x}}$$

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}u + \frac{\partial C}{\partial y}v + \frac{\partial C}{\partial z}w$$

$$\boxed{\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \underline{v} \cdot \underline{\nabla} C}$$

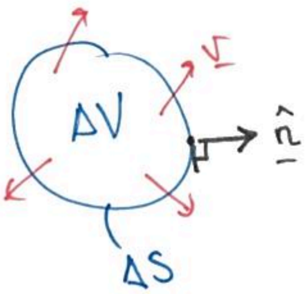
Lagrangian derivative

Eulerian derivative

"Parcels of fluid" have infinitessimal volume but can be squeezed and deformed

ΔV = volume of fluid parcel (limit $\Delta V \to 0$).

Consider



$$\frac{D}{Dt}\Delta V = \frac{D}{Dt}\int_{\Delta V} dV$$

$$= \int_{\Delta S} v_n \, dS$$

$$= \int_{\Delta S} \underline{v}\cdot\hat{\underline{n}}\, dS \overset{\text{divergence}}{\underset{\text{theorem}}{=}} \int_{\Delta V} \underline{\nabla}\cdot\underline{v}\, dV$$

Since ΔV is arbitrary

$$\Rightarrow \quad \frac{D}{Dt}\Delta V = \underline{\nabla}\cdot\underline{v}\;\Delta V \qquad (\text{in limit } \Delta V \to 0).$$

* "incompressible flow" $\underline{\nabla}\cdot\underline{v} = 0 \Rightarrow \Delta V = \text{constant}$

Continuity equation

Conservation of mass : $\frac{D}{Dt}(\underbrace{\rho\Delta V}_{\text{mass = density × volume}}) = 0$

$$\frac{D}{Dt}(\rho\Delta V) = \frac{D\rho}{Dt}\Delta V + \rho\frac{D\Delta V}{Dt} = \frac{D\rho}{Dt}\Delta V + \rho\,\underline{\nabla}\cdot\underline{v}\;\Delta V$$

$$= \left(\frac{D\rho}{Dt} + \rho\,\underline{\nabla}\cdot\underline{v}\right)\Delta V = 0$$

$$\Rightarrow \quad \frac{D\rho}{Dt} + \rho\,\underline{\nabla}\cdot\underline{v} = 0$$

$$\boxed{\frac{D\rho}{Dt} = -\rho \underline{\nabla} \cdot \underline{v}}$$ Continuity equation (Lagrangian form)

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho = -\rho \underline{\nabla} \cdot \underline{v}$$

$$\frac{\partial \rho}{\partial t} = -\underline{v} \cdot \underline{\nabla} \rho - \rho \underline{\nabla} \cdot \underline{v}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\underbrace{\rho \underline{v}}_{\text{mass flux}})}$$ Continuity equation (Eulerian form)

Conservation of momentum.

- Newton's 2nd law applied to fluid parcels.

$$\underline{p} = m\underline{v} = \text{momentum}$$

$$\frac{d\underline{p}}{dt} = \Sigma \text{ forces.}$$

For a parcel of fluid:

$$\frac{D}{Dt} \Delta \underline{p} = \Sigma \text{ forces.} \qquad \text{where} \quad \Delta \underline{p} = \Delta m . \underline{v}$$

$$= \cancel{\frac{D}{Dt} \Delta m} \cdot \underline{v} + \Delta m \frac{D\underline{v}}{Dt}$$

$$\Rightarrow \boxed{\frac{D\underline{v}}{Dt} = \frac{\Sigma \text{ forces}}{\Delta m} = \Sigma \underline{f}}$$ Newton's 2nd law (Lagrangian form)

$$\boxed{\frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v} \cdot \underline{\nabla} \underline{v}}_{\text{"flow moves the flow" (nonlinearity)}} = \Sigma \underline{f}}$$ Newton's 2nd law (Eulerian form).

Continuity equation.

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla}\cdot(\rho \underline{v}). \qquad \text{(Eulerian form)}$$

$$\frac{D\rho}{Dt} = -\rho\,\underline{\nabla}\cdot\underline{v} \qquad \text{(Lagrangian form)}$$

Using $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}$

Newton's 2nd law

$$\frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v}\cdot\underline{\nabla}\,\underline{v}}_{\text{"flow moves the flow"}} = \Sigma\,\underline{f} \leftarrow \frac{\text{forces}}{\text{mass}}. \qquad \text{Eulerian form.}$$

$$\frac{D\underline{v}}{Dt} = \Sigma\,\underline{f}$$

Forces:

Body forces : gravity

$-\underline{\nabla}\Phi$

centrifugal force } psuedo forces.

coriolis force

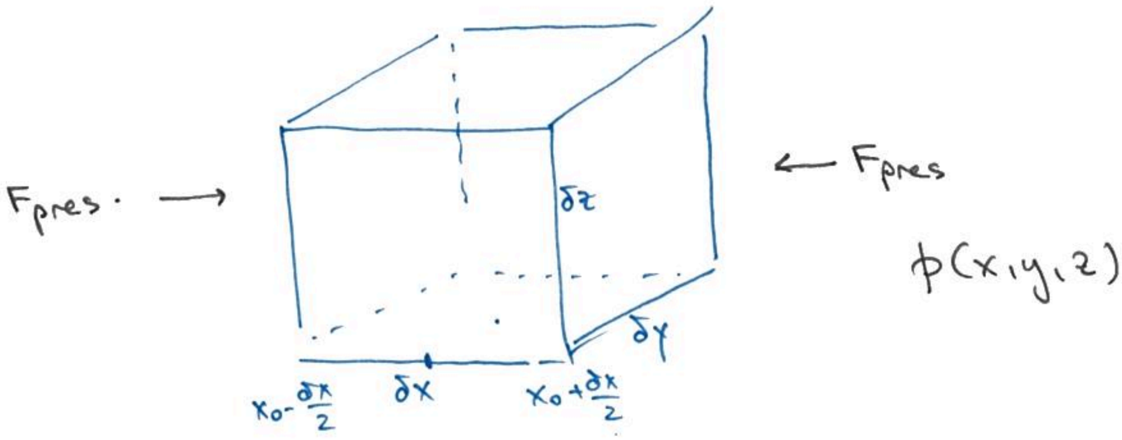
$-2\underline{\Omega}\times\underline{v}$

Contact forces: pressure gradient force.

Viscous stresses. } not usually important

friction forces. } in GFD.

Pressure gradient force.



Force = pressure × area.

$$= p\left(x_0 - \frac{\delta x}{2}, y_0, z_0\right)\delta y \delta z - p\left(x_0 + \frac{\delta x}{2}, y_0, z_0\right)\delta y \delta z$$

$$\underset{\text{expand}}{\overset{\text{Taylor}}{=}} \left[\cancel{p(x_0, y_0, z_0)} - \frac{\delta x}{2}\frac{\partial p}{\partial x}(x_0, y_0, z_0)\right]\delta y \delta z + \cdots$$

$$- \left[\cancel{p(x_0, y_0, z_0)} + \frac{\delta x}{2}\frac{\partial p}{\partial x}(x_0, y_0, z_0)\right]\delta y \delta z + \cdots$$

$$\simeq -\delta x \frac{\partial p}{\partial x}(x_0, y_0, z_0)\delta y \delta z.$$

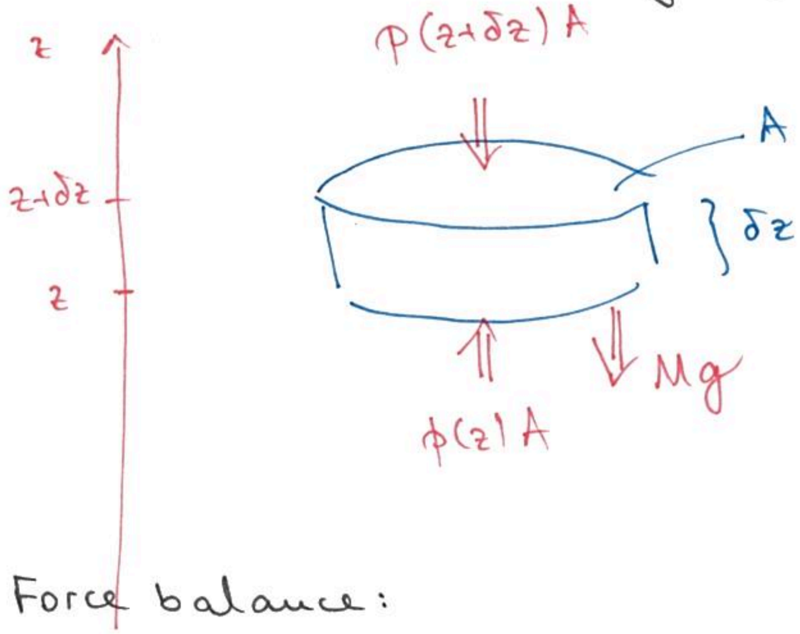
$$f_{pres} = \frac{\text{force}}{\text{mass}} = \frac{-\frac{\partial p}{\partial x}\delta x \delta y \delta z}{\rho \delta x \delta y \delta z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Pressure gradient force:

$$\underline{f}_{pres} = -\frac{1}{\rho}\underline{\nabla} p.$$

STATICS: forces in balance

Hydrostatic balance: no flow, pressure gradient force is balanced by gravity



Force balance:

$$p(z)A - p(z+\delta z)A - \rho(z)Ag\,\delta z = 0.$$

$$\frac{p(z+\delta z) - p(z)}{\delta z} = -\rho(z)\,g.$$

$\delta z \to 0$:

$$\boxed{\frac{dp}{dz} = -\rho g}$$ "hydrostatic balance".

Geostrophic balance:

Horizontal pressure gradient = Coriolis force

Cyclostrophic balance:

Horizontal pressure gradient = Centrifugal force of rotating fluid.

Ocean: treat as incompressible. (density does not change with pressure).

$$\rho_0 = \text{constant}.$$

Hydostatic balance:

$$\frac{dp}{dz} = -\rho_0 g$$

$$\int dp = -\int \rho_0 g \, dz$$

$$p(z) - p(0) = -\rho_0 g z \quad \Rightarrow \quad p(z) = p(0) - \rho_0 g z$$

$$z < 0.$$

z	p
0	1 atm.
−10m	1.99 atm
−100m	10.9 atm.
−1000m	100.2 atm.

Density <u>does</u> depend on temperature (T) & salinity (S).

$$\rho(T, S).$$

$$S = \frac{\text{mass of "salt"}}{\text{mass of seawater}}$$

$$\sigma = \rho - \rho_0 = \text{density anomaly}$$

$$\rho_0 = 1000 \, kg \, m^{-3} = \text{reference density}.$$

Equation of state for seawater:

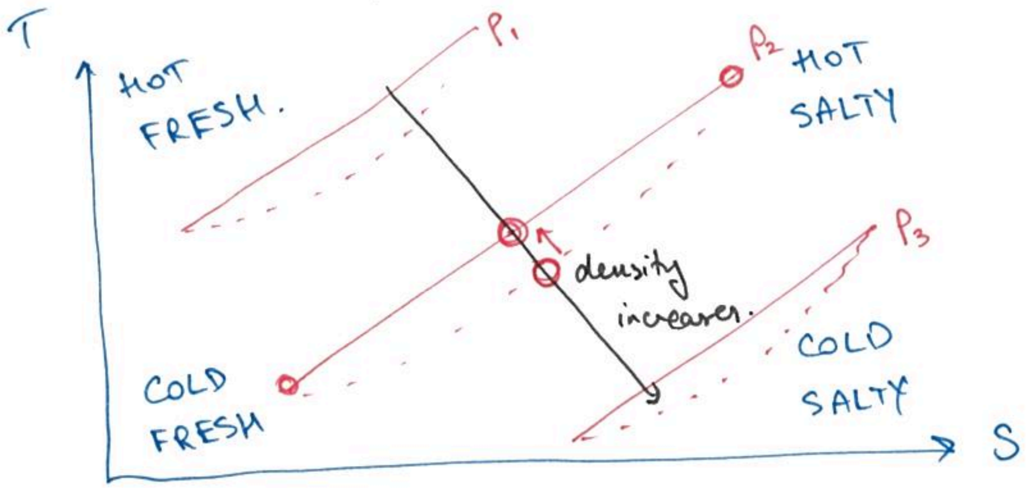
Linearize in T, S:

$$\rho(T,S) = \underbrace{\rho(T_0, S_0)}_{\rho_0} + \left.\frac{\partial \rho}{\partial T}\right|_{T_0,S_0} (T-T_0) + \left.\frac{\partial \rho}{\partial S}\right|_{T_0,S_0} (S-S_0) + \cdots$$

$$= \rho_0 \left[1 + \underbrace{\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial T}\right|_{T_0,S_0}}_{\text{negative.}} (T-T_0) + \underbrace{\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial S}\right|_{T_0,S_0}}_{\text{positive}} (S-S_0) \right] + \cdots$$

$T\uparrow \ \rho\downarrow$ $\qquad$ $S\uparrow \ \rho\uparrow$

$$= \rho_0 \left(1 - \alpha_T (T-T_0) + \beta_S (S-S_0) \right)$$



$\rho_1 < \rho_2 < \rho_3$

To close: need. $\quad \frac{DT}{Dt} = Q_T \qquad \frac{DS}{Dt} = Q_S$

Example : use EOS for dry air and hydrostatic balance to calculate $p(z)$

$$p = \rho R_d T \quad \Rightarrow \quad \rho = \frac{p}{R_d T}$$

Assume an isothermal atmosphere (T_o = constant)

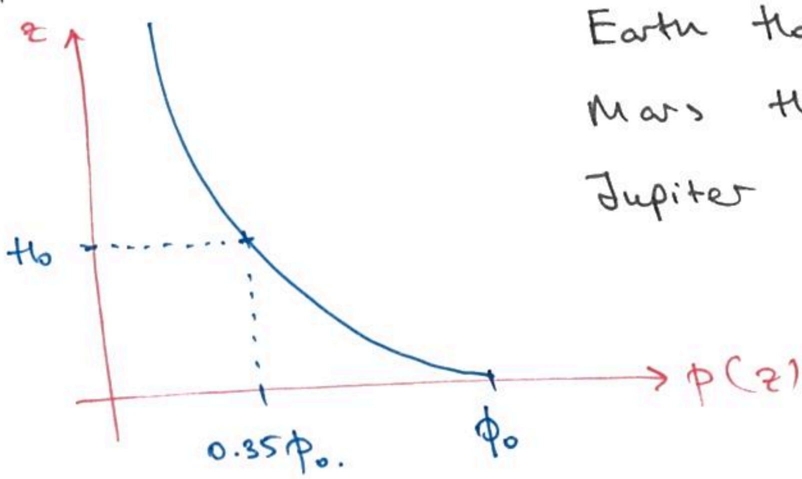
hydrostatic balance: $\frac{dp}{dz} = -\rho g = -\frac{p g}{R_d T_o}$.

$$\int \frac{dp}{p} = -\int \frac{g}{R_d T_o} dz \Rightarrow \ln p = -\frac{g z}{R_d T_o} + C.$$

let $C = \ln p_o$. (surface pressure)

$$\ln p - \ln p_o = -\frac{z}{H_o} \qquad H_o = \frac{R_d T_o}{g} = \text{"scale height"}$$

$$\Rightarrow p = p_o e^{-z/H_o}.$$



Earth $H_o \simeq 7.5$ km.

Mars $H_o \simeq 11$ km.

Jupiter $H_o \simeq 27$ km.

Boussinesq approximation

- exploit smallness of density variations (in the ocean)

$$\rho = \underbrace{\rho_0}_{\text{reference density}} + \underbrace{\tilde{\rho}(x,y,z,t)}_{\text{small perturbation}} \qquad \frac{|\tilde{\rho}|}{\rho_0} \ll 1.$$

- define reference pressure $p_0(z)$ in hydrostatic balance with ρ_0

$$p = p_0(z) + \tilde{p}(x,y,z,t) \qquad \frac{|\tilde{p}|}{p_0} \ll 1.$$

where. $$\frac{dp_0}{dz} = -\rho_0 g$$

- sub into momentum equation.

$$\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v} = -\frac{1}{\rho}\underline{\nabla} p - g\hat{\underline{z}}$$

$\times \rho = \rho_0 + \tilde{\rho}$

neglect, keep!, keep!

$$(\rho_0 + \cancel{\tilde{\rho}})\left[\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v}\right] = -\underline{\nabla}(p_0 + \tilde{p}) - (\rho_0 + \tilde{\rho})g\hat{\underline{z}}$$

Boussinesq approx: neglect $\tilde{\rho}$ EXCEPT where multiplied by g.

$$\rho_0\left[\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v}\right] \simeq -\frac{dp_0}{dz}\hat{\underline{z}} - \underline{\nabla}\tilde{p} - \rho_0 g\hat{\underline{z}} - \tilde{\rho} g\hat{\underline{z}}$$

cancel (in hydrostatic balance).

$$\Rightarrow \boxed{\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v} = -\frac{1}{\rho_0}\underline{\nabla}\tilde{p} - \underbrace{\frac{\tilde{\rho} g}{\rho_0}}_{\text{reduced gravity}}\hat{\underline{z}}}$$

Boussinesq momentum equation.

Continuity equation

$$\frac{D\rho}{Dt} = -\rho \underline{\nabla} \cdot \underline{v} \qquad \rho = \rho_0 + \tilde{\rho}$$

$$\Rightarrow \quad \frac{D\tilde{\rho}}{Dt} = \underbrace{-\rho_0 \underline{\nabla} \cdot \underline{v}}_{\text{largest term.}} - \tilde{\rho}\, \underline{\nabla} \cdot \underline{v}$$

the $\rho_0 \cdot \underline{\nabla} \cdot \underline{v}$ term has no term to balance it so.

$$\Rightarrow \quad \rho_0 \underline{\nabla} \cdot \underline{v} = 0. \qquad !$$

$$\Rightarrow \quad \boxed{\underline{\nabla} \cdot \underline{v} = 0.}$$

In B-approx, continuity equation → incompressibility condition.

* Note: this doesn't mean $\rho = \rho_0 + \tilde{\rho}$ is constant.

or $\frac{D\tilde{\rho}}{Dt} = 0.$

(Need information about sources of heating/cooling.)

But if fluid is adiabatic (no heating/cooling)

$$\Rightarrow \quad \boxed{\frac{D\tilde{\rho}}{Dt} = 0.}$$

Boussinesq approximations.

$$\frac{D\underline{v}}{Dt} + 2\underline{\Omega} \times \underline{v} = -\frac{1}{\rho_0}\underline{\nabla}\tilde{p} - \frac{\tilde{\rho}}{\rho_0} g \hat{\underline{z}} \qquad \text{(momentum)}$$

$$\underline{\nabla} \cdot \underline{v} = 0 \qquad \text{(incompressible)}$$

$$\frac{D\tilde{\rho}}{Dt} = 0 \qquad \text{(adiabatic).}$$

Thin layer approximation : $L \gg H$

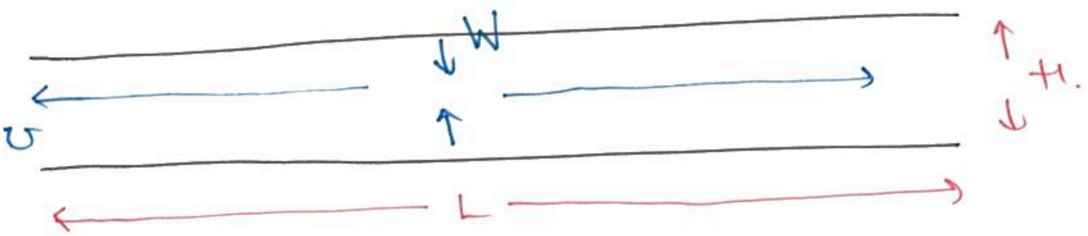
* $\underline{\nabla} \cdot \underline{v} = 0$ (incompressibility)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{U}{L} \quad \frac{U}{L}. \quad \frac{W}{H}$$

$$\Rightarrow \quad \frac{U}{L} \sim \frac{W}{H} \quad \Rightarrow W \sim U\frac{H}{L} \sim \alpha U$$

$\alpha \ll 1.$



$\Rightarrow$ neglect terms involving w (compared with u, v)

EXCEPT in terms like $\frac{\partial w}{\partial z}$, $w\frac{\partial}{\partial z}$

x - momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \underbrace{w\frac{\partial u}{\partial z}}_{\text{keep.}} + \underbrace{\cancel{w f_*}}_{w \ll v} - vf = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

y-momentum:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \underbrace{w\frac{\partial v}{\partial z}}_{\text{keep.}} + uf = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

z-momentum:

$$\underbrace{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{neglect } w \ll u.} - uf_* = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0} g$$

Not important in atm/ocn flows.

Simplified equations for rotating, stratified thin layer flow

$$\frac{Du}{Dt} - vf = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + uf = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{D\rho}{Dt} = 0.$$

Note! * Boussinesq approx

* Tangent plane approx

* Thin layer approx

Compact notation:

$\underline{u} = (u, v)$ $\qquad \underline{\nabla}_z = (\partial_x, \partial_y)$

$\underline{f} = f\hat{\underline{z}}$ $\qquad f = 2\Omega\sin\phi =$ "Coriolis parameter"

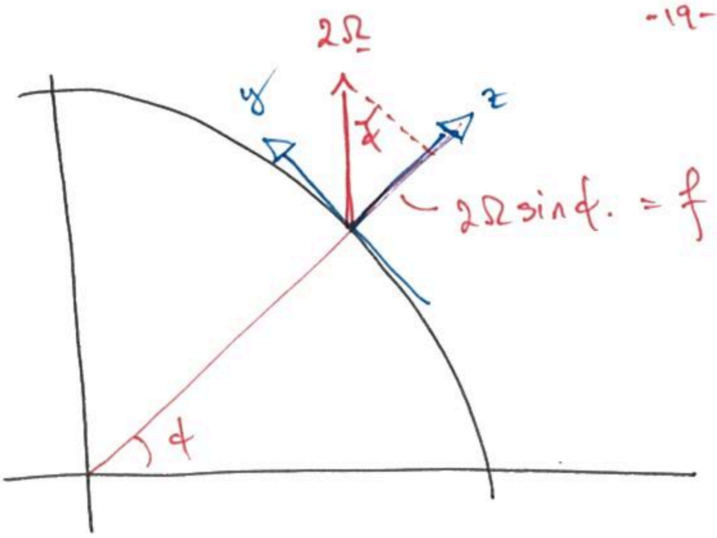
$$\frac{D\underline{u}}{Dt} + \underline{f}\times\underline{u} = -\frac{1}{\rho_0}\underline{\nabla}_z p \qquad \text{momentum.}$$

$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{hydrostatic balance}$$

$$\frac{D\rho}{Dt} = 0 \qquad \text{adiabatic}$$

$$\underline{\nabla}_z\cdot\underline{u} + \frac{\partial w}{\partial z} = 0.$$

Primitive equations



$$\underline{f} = f(\phi)\,\hat{\underline{z}}$$

Two simplifications

1. f-plane approximation.

treat $f = 2\Omega\sin\phi$ as a constant $(\phi = \phi_0)$

$$f = f_0 = 2\Omega\sin\phi_0.$$

2. β-plane approximation.

allow linear changes of f with latitude $\phi = \phi_0 + \Delta\phi$

$$f = 2\Omega\sin(\phi_0 + \Delta\phi)$$

$$\simeq 2\Omega\sin\phi_0 + \Delta\phi\left.\frac{\partial f}{\partial\phi}\right|_{\phi_0} \qquad \text{(Taylor approx)}$$

$$= 2\Omega\sin\phi_0 + \underbrace{\frac{2\Omega\cos\phi_0}{a}}_{\beta}.\underbrace{a\Delta\phi}_{y}.$$

$$= f_0 + \beta y$$

$$f = f_0 + \beta y \qquad y = a\Delta\phi = \text{distance north.}$$

$$\beta = \frac{1}{a}\left.\frac{\partial f}{\partial\phi}\right|_{\phi_0} = \frac{2\Omega\cos\phi_0}{a}$$

Geostrophic balance

Rapidly rotating flows: $Ro \ll 1$

$$Ro = \frac{\text{period of rotation}}{\text{time to move } L} \sim \frac{2\pi/\Omega}{L/U} \sim \frac{U}{L\Omega}$$

Compare sizes of terms (forces) in the horizontal momentum equation.

$$Ro \sim \frac{\text{inertial terms } (Du/Dt)}{\text{Coriolis term } (-vf)} \sim \frac{U/T}{U\Omega} \sim \frac{1}{\Omega T}$$

So if we $\underline{\text{choose}}$ relevant timescale $T \sim L/U$ (advective timescale)

$$\Rightarrow \quad R \sim \frac{U}{L\Omega} \quad \text{as before}$$

Rapidly rotating flows are equivalent to neglecting inertial terms:

$$\underbrace{\cancel{\frac{D\underline{u}}{Dt}}}_{Ro \ll 1} + \underbrace{\underline{f} \times \underline{u}}_{\text{Coriolis}} = \underbrace{- \frac{1}{\rho_0} \underline{\nabla}_z p}_{\text{horizontal pressure gradient}}.$$

$\Rightarrow$ "geostrophic balance"

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad \Rightarrow \quad v = \frac{1}{\rho_0 f}\frac{\partial p}{\partial x}$$

$$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \qquad\quad u = -\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}$$

In f-plane approximation $f = f_0$

$$\Rightarrow \quad v = \frac{\partial}{\partial x}\left(\frac{\Phi}{\rho_0 f_0}\right) = \frac{\partial \psi}{\partial x}$$

$$u = -\frac{\partial}{\partial y}\left(\frac{\Phi}{\rho_0 f_0}\right) = -\frac{\partial \psi}{\partial y}$$

$\psi = \dfrac{\Phi}{\rho_0 f_0}$. "geostrophic streamfunction"

* Flow is incompressible in 2D.

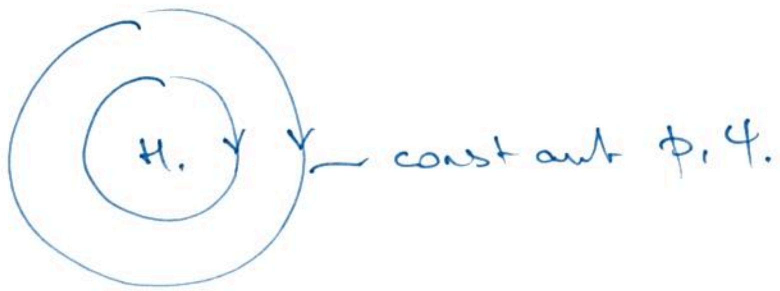
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0.$$

* ψ is a streamfunction

$\Rightarrow$ flow is along contours of ψ.

$\Rightarrow$ flow is along contours of Φ.

NH



"anticyclone" $\rightarrow$ clockwise in NH

$\rightarrow$ anticlockwise in SH.