MEMS 1060: Homework #4

Due on February 28, 2021 at 6:00pm

Professor Sammak Th 6:00PM

Made using LaTeX/Inkscape. Source files available upon request.

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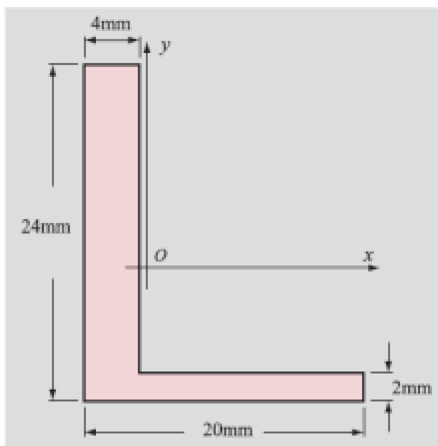
Problem 1

Given the cross-sectional area shown, find the principal moments and axes of inertia, given the following:

- $I_x = 7523[mm^4]$
- $I_y = 3210[mm^4]$
- $I_{xy} = I_{yx} = 2640[mm^4]$
- $I = \begin{bmatrix} I_x & -I_{xy} \\ -I_{yx} & I_y \end{bmatrix}$

Figure 1: Problem 1.



Solution

We must find eigenvalues and eigenvectors of the I matrix. In order to do this, we will solve the characteristic equation for λ (we will call the moment of inertia matrix $[A]$ and the identity matrix $[I]$ for clarity):

$$det[A - \lambda I] = 0$$
$$\begin{vmatrix} I_x - \lambda & -I_{xy} \\ -I_{yx} & I_y - \lambda \end{vmatrix} = 0$$
$$(I_x - \lambda)(I_y - \lambda) - (-I_{xy})(-I_{yx}) = 0$$
$$I_x I_y - \lambda I_y - \lambda I_x + \lambda^2 - I_{xy}^2 = 0$$
$$\lambda^2 - (I_x + I_y)\lambda + (I_x I_y - I_{xy}^2) = 0$$

We now employ the quadratic formula to find our eigenvalues:

$$\lambda_1, \lambda_2 = \frac{(I_x + I_y) \pm \sqrt{(I_x + I_y)^2 - 4(I_x I_y - I_{xy}^2)}}{2}$$
$$\lambda_1 \approx 8775[mm^4]$$
$$\lambda_2 \approx 1958[mm^4]$$

From here, we must find our corresponding normalized eigenvectors for these eigenvalues. We will solve the proper equation to find these eigenvectors.

Problem 1 continued on next page...

$$[A - I\lambda][v] = 0$$

$$\begin{bmatrix} I_x - \lambda_1 & -I_{xy} \\ -I_{yx} & I_y - \lambda_1 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \end{bmatrix} = 0$$

This linear system is to be solved for both λ_1 and λ_2. For both, we will fix v_2 to 1 to find v_1 and then we will normalize. This could be done by hand for a 2x2, but I will reduce the matrix in MATLAB and solve the following equation:

$$(rref([A - I\lambda]) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) * [v] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

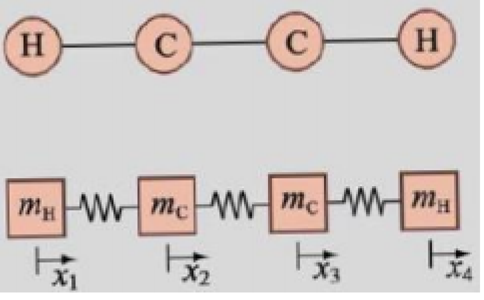
By making this tweak and then normalizing the output, we find our eigenvalues. See the source code.

Using the characteristic polynomial method, we find the principal axes and moments of inertia for the cross section: $\lambda_1 = 8775[mm^4]$**,** $\lambda_2 = 1958[mm^4]$**,** $u_1 = [-0.9035; 0.4286]$**,** $u_2 = [0.4286; 0.9035]$

Problem 2

Given an acetylene molecule idealized as four masses attached by springs and the givens $k_{CH} = 592[kg/s^2]$, $k_{CC} = 1580[kg/s^2]$, $m_H = 1[amu]$, $m_C = 12[amu]$, and $1[amu] = 1.6605 * 10^-27[kg]$, find the frequencies and corresponding eigenvalues.

Figure 2: Problem 2.



Solution

In order to find the frequencies, we must represent the amplitudes of vibration as a matrix and find its eigenvalues. The coefficient matrix $[a]$ is displayed such that it produces the amplitudes of vibration $[A]$:

$$a = \begin{bmatrix} \frac{k_{CH}}{m_H} & -\frac{k_{CH}}{m_H} & 0 & 0 \\ -\frac{k_{CH}}{m_C} & \frac{(k_{CH}+k_{CC})}{m_C} & -\frac{k_{CC}}{m_C} & 0 \\ 0 & -\frac{k_{CC}}{m_C} & \frac{(k_{CH}+k_{CC})}{m_C} & -\frac{k_{CH}}{m_C} \\ 0 & 0 & -\frac{k_{CH}}{m_H} & \frac{k_{CH}}{m_H} \end{bmatrix}$$

$$[a - \omega^2 I][A] = 0$$

Because of the setup of this type of problem, it so happens that the frequencies are simply the roots of the eigenvalues of the $[a]$ matrix. Therefore if we find the eigenvalues and eigenvectors of $[a]$, we have effectively

solved the problem. We will implement QR factorization in order to do find the eigenvalues, and then we will compute the corresponding eigenvectors using row-reduction in MATLAB. For the iteration, I ran 100 iterations and then compared the resulting eigenvalues to those from the stock function. In a more rigorous implementation it would be worth using a residual convergence criteria for the eigenvalues.
Using QR factorization, the following eigenvalues are found. The frequencies are simply the squares of these eigenvalues. Note: one of the eigenvalues coming from the factorization as 16 orders of magnitude less than the others. We will consider this to likely be an eigenvalue of zero clouded by machine error. We will check the eigenvector to be sure.
In order to find the corresponding eigenvectors, one must keep a running product of Q matrices for every iteration.

$$[A - I\lambda] * [v] = 0$$

Found eigenvalues:

$$\begin{aligned} \lambda_1 &= 4.053 * 10^{29} \\ \lambda_2 &= 3.862 * 10^{29} \\ \lambda_3 &= 1.395 * 10^{29} \\ \lambda_4 &= 0 \end{aligned}$$

Frequencies:

$$\begin{aligned} \omega_1 &= 6.366 * 10^{14} [s^{-1}] \\ \omega_2 &= 6.215 * 10^{14} [s^{-1}] \\ \omega_3 &= 3.735 * 10^{14} [s^{-1}] \\ \omega_4 &= 0 [s^{-1}] \end{aligned}$$

Eigenvectors (normalized):

$$v_1 = \begin{bmatrix} 0.701 \\ -0.096 \\ 0.096 \\ -0.701 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0.705 \\ -0.059 \\ -0.059 \\ 0.705 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0.604 \\ 0.368 \\ -0.368 \\ -0.604 \end{bmatrix}$$

$$v_4 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Using the QR factorization method, the frequencies and corresponding eigenvectors are found.

```
%% MEMS 1060 Homework 4
% Author: Shane Riley
% Date: 2/28/2021
format long
%% Problem 1
% Given the area, find the principal moments and axes of inertia
disp(" ");
disp("Problem 1");

I_x = 7523;     % [mm^4]
I_y = 3210;     % [mm^4]
I_xy = 2640;    % [mm^4]

A = [...
    I_x     -I_xy
    -I_xy     I_y
    ];

% Solve using the characteristic equation. Done on paper. Roots/eigenvalues
% checked using stock function

coeff = [...
    1
    - (I_x + I_y)
    I_x * I_y - I_xy^2
    ];
e = roots(coeff);

% Find eigenvectors from eigenvalues

char = @(lambda) A - lambda .* eye(2);

% Find v such that Av = 0

% Find matrix
r_char = @(x) rref(char(x));  % row reduce
r_char_e1 = r_char(e(1));
r_char_e2 = r_char(e(2));

% Find v
v_e1 = [-r_char_e1(1,:); 0,1]\[0;1]; % fix v_2 to 1
v_e2 = [-r_char_e2(1,:); 0,1]\[0;1]; % fix v_2 to 1

% Normalize v
normalize = @(x) x/norm(x);

v_e1_norm = normalize(v_e1);
v_e2_norm = normalize(v_e2);

```

```matlab
disp("Eigenvalues: ")
fprintf('%1.4f\n', e(1));
fprintf('%1.4f\n', e(2));

disp("Eigenvectors: ")
disp(v_e1_norm);
disp(v_e2_norm);

%% Problem 2
% Write a user-defined function to find eigenvalues. Use it to solve the
% for the vibration frequencies and corresponding eigenvectors of an
% acetylene molecule
disp(" ");
disp("Problem 2");

amu = 1.6605 * 10^-27; % [kg/amu]
k_ch = 592;           % [kg/s^2]
k_cc = 1580;          % [kg/s^2]
m_h = 1*amu;          % [amu]
m_c = 12*amu;         % [amu]


% Matrix for amplitude vibrations
coeff = [...
    (k_ch/m_h)   -(k_ch/m_h)   0   0
    -(k_ch/m_c)   (k_ch+k_cc)/m_c   -(k_cc/m_c)   0
    0   -(k_cc/m_c)   (k_ch+k_cc)/m_c   -(k_ch/m_c)
    0   0   -(k_ch/m_h)   (k_ch/m_h)
    ];

% Get eigenvalues
e = AllEig(coeff);

% Handle machine error
eround = round(e,5,'significant');
eround(4) = 0;

omega = eround.^(0.5);

disp("Eigenvalues: ")
for i=1:length(eround)
    disp(eround(i));
end

disp("Omega: ")
for i=1:length(eround)
    disp(omega(i));
end

```

```
%% Supporting functions

function e = AllEig(A)
% ALLEIG returns a vector of eigenvalues for a given matrix A (uses QR
% factorization)
% A - nxn matrix
% e - vector of eigenvalues

NUM_ITER = 1000;  % TODO: add a convergence criteria

[n, ~] = size(A);
I = eye(n);

pQ = I;
Anew = A;
for iteration=1:NUM_ITER

    % Seed QR
    Q = I;
    R = Anew;
    for i=1:n-1

        % c vector
        c = R(:,i);
        c(1:i-1)=0;

        % e vector
        e1(1:n,1) = 0;
        if c(i) > 0
            e1(i)=1;
        else
            e1(i)=-1;
        end
        v = c + norm(c)*e1;

        % householder
        H = I-(2/(v'*v))*(v*v');

        % Iterate values
        Q = Q*H;
        R = H*R;
    end

    % New A
    Anew = R*Q;
    pQ = pQ*Q;
end

```

```
% Get eigenvalues using Q and R
e=diag(Anew);
disp(pQ);
end
```