MEMS 1060: Homework #7

Due on March 18, 2021 at 6:00pm

Professor Sammak Th 6:00PM

Made using LaTeX/Inkscape. Source files available upon request.

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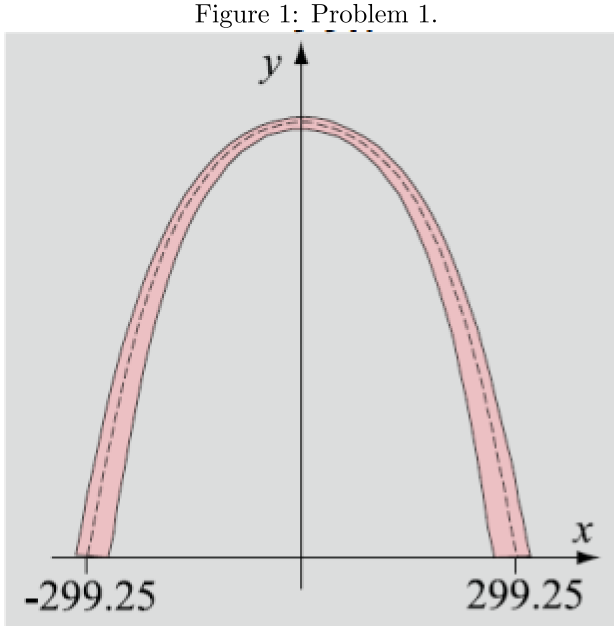
Problem 1

Given f(x) as the y-per-x for the St. Louis Gateway Arch, use the following numerical integration techniques to compute the arc length L:

$$f(x) = 693.9 - 68.8 * cosh(\frac{x}{99.7})$$
$$-299.25 \leq x \leq 299.25$$
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

1. Rectangular Method (n=8)
2. 3/8 Simpson's Method (n=9)
3. Gauss Quadrature (n=2)

Figure 1: Problem 1.



Solution

For all methods, a derivative of $f(x)$ is required. This is found analytically, along with the integrand:

$$f'(x) = (\frac{68.8}{99.7} sinh(\frac{x}{99.7})$$
$$l(x) = \sqrt{1 + [f'(x)]^2}$$
$$L = \int_a^b l(x)\, dx$$

Starting with the rectangular method. A left sum strategy is used in this case. The integral is approximated as a sum of rectangle areas:

Problem 1 continued on next page...

Table 1: Gateway Arch Length – Numerical Integration

Method	n	Arc Length
Left Rect.	8	1543.8
3/8 Simp.	9	1483.1
Gauss	2	1280.2

$$I(l) \approx \sum_{i=1}^{n-1} l(x_i) * h$$
$$h = \frac{b-a}{n}$$

For Simpson's 3/8, we use cubic areas instead of rectangular areas. The final implementation (taken from the book) looks like:

$$I(l) \approx \frac{3h}{8}[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{i=4,7,10}^{N-2} f(x_i) + f(b)]$$

For Gauss Quadrature, vectors for weights and x-values are specified using n. The transformation method is applied to adjust the problem to a $[-1, 1]$ domain. Once that is done, the vectors are used with the function to approximate the integral:

$$lg(x) = l(\frac{(b-a)x + a + b}{2}) * \frac{b-a}{2}$$
$$\int_a^b f(x)dx = \int_{-1}^1 lg(t)dt$$
$$= \sum_{i=1}^{N} C_i lg(x_i)$$

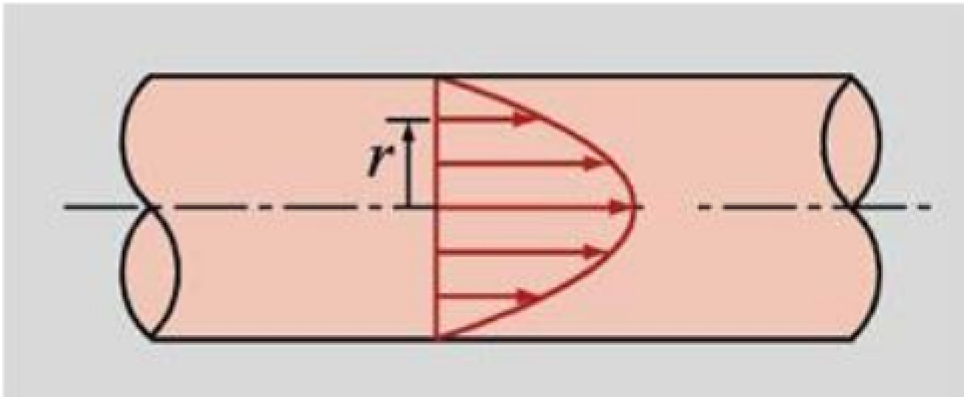
For the $n = 2$ case, $C = [1, 1]$ and $x = [\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$. All three solutions are implemented in MATLAB, and the following solutions are presented. See the source code for implementation details:
The answer reached by running the integral in a stock implementation is ≈ 1480. This shows that the Simpson 3/8 method yielded the closest approximation compared to stock.

See results and commentary.

Problem 2

Given v-per-r point data (evenly-spaced), find the volumetric flow rate of a fluid in a pipe of radius $R = 2[m]$.
Note: the problem says v-per-y but since the v vector is not symmetric, we will assume velocity per radius was intended.
Solution

Figure 2: Problem 2.



With 9 points, we will create 8 subdivisions and perform a simple right-handed rectangular summation. Since v per r is always decreasing, we know for certain that this method will yield a conservative underestimate of the volumetric flow:

$$Q = \int_0^R 2\pi r v(r) dr$$
$$\approx \sum_{i=2}^{n} 2\pi r h * v(r_i)$$
$$h = \frac{b-a}{n}$$

The solution is implemented in MATLAB to generate the flow rate.
Using a right-handed rectangular approximation with $n = 8$, we find that $Q = 245.9[m^3/s]$. Since v should always decrease with increasing radius, we know our solution to be an underestimate of the true solution.

```
%% MEMS 1060 Homework 7
% Author: Shane Riley
% Date: 3/15/2021
format long
%% Problem 1
% Given y per x for the Gateway Arch, find the the length of the arch using
% Rectangle (8 subs), Simpson's 3/8 (9 subs), Gauss quadrature (n=2)
disp(" ");
disp("Problem 1");

f = @(x) 693.9 - 68.8 * cosh(x/99.7);
xspan = [-299.25 299.25];

% To find arc length we need f'(x). Doing so analytically:
df = @(x) (68.8/99.7) .* sinh(x/99.7);

arcint = @(x) sqrt(1 + (df(x)).^2);

% Rectangular method (left sum)
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```
n = 8;
h = range(xspan)/n;
xval = @(i) (i-1)*h + xspan(1);
rectsum = sum(arcint(xval(1:n)) * h);

% Simpson 3/8 with 9 subs
n = 9;
h = range(xspan)/n;
m = n/3;
xval = @(i) (i-1).*h + xspan(1);
simpsum = (3*h/8)* sum([...
    arcint(xval(1))
    3*sum(arcint(xval(2:3:(n))))
    3*sum(arcint(xval(3:3:(n))))
    2*sum(arcint(xval(4:3:(n-1))))
    arcint(xval(n+1))
]);

% Gauss Quadrature with n=2
% Weights and points from book
weights = [1 1];
points = [-0.57735027, 0.57735027];
garcint = @(x) arcint(((xspan(2) - xspan(1)).*x + sum(xspan))/2).*(xspan(2)-
    xspan(1))./2;
gausssum = sum(weights .* garcint(points));

% Displays
disp("Gateway Arch Length");
disp(['Left Rectangular sum (n=8): ', sprintf('%5.5f', rectsum)]);
disp(['3/8 Simpson Sum (n=9): ', sprintf('%5.5f', simpsum)]);
disp(['Gauss Quadrature (n=2): ',  sprintf('%5.5f', gausssum)]);


%% Problem 2
% Given v(y) data points, find the volumetric flow rate through the pipe
disp(" ");
disp("Problem 2");

r = [...
    0
    0.25
    0.50
    0.75
    1.0
    1.25
    1.5
    1.75
    2.00];
v = [...
```

```
      38
      37.6
      36.2
      33.6
      29.7
      24.5
      17.8
      9.6
      0];

R = max(r);

% Q = int(0,R, 2 pi r v(r))

flowint = @(i) 2.*pi.*r(i).*v(i);

% Right reimann sum (n=8):
n = 8;
h = 0.25;
flow = sum(flowint(2:length(r)).*h);
disp(['Flow rate [m^3/s]: ', sprintf('%5.5f', flow)]);
```