In[33]:=
```
SetOptions[SelectedNotebook[],
 PrintingStyleEnvironment → "Printout", ShowSyntaxStyles → True]
```

```
(*Importing RISC packages*)
```

In[34]:=
```
<< RISC`qMultiSum`
<< RISC`qGeneratingFunctions`
```

qMultiSum Package version 2.54
written by Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

qGeneratingFunctions Package version 1.9.1
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
(*Initial values of T̃_M, using the original definition*)
```

In[36]:=
$$\text{MyQP}[A\_, q\_, n\_] := \prod_{k=0}^{n-1} \left(1 - A\, q^k\right);$$

```
ClearAll[M, N1, N2, N3, N4, N5];

For[M = 0, M ≤ 10, M++,
 S = Sum[Sum[Sum[Sum[(-1)^(M+N1+N2)
       q^(N1^2+N2^2+N3^2+N4^2+(M-N1-N2-2 N3-N4)^2+N1 N2+N1 N3+N1 (M-N1-N2-2 N3-N4)+N2 N3+N2 N4+N3 N4+N3 (M-N1-N2-2 N3-N4)+N4 (M-N1-N2-2
        / (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3]
           MyQP[q^2, q^2, N4] MyQP[q^2, q^2, (M - N1 - N2 - 2 N3 - N4)]),
      {N4, 0, M - N1 - N2 - 2 N3}],
     {N3, 0, Floor[M/2]}, {N2, 0, M - 2 N3}, {N1, 0, M - 2 N3 - N2}];
 Print[Factor[S]];
]
```

$$1$$

$$\frac{2}{1+q}$$

$$0$$

$$-\frac{2\,(-1+q)\,q^2}{(1+q)^3\,\left(1-q+q^2\right)}$$

$$0$$

$$\frac{2\,(-1+q)^2\,q^6\,\left(1+q+q^2\right)^2}{(1+q)^5\,\left(1+q^2\right)\,\left(1-q+q^2\right)\,\left(1-q+q^2-q^3+q^4\right)}$$

$$0$$

$$-\left(\left(2\,(-1+q)^3\,q^{12}\,\left(1+q+q^2\right)\,\left(1+q+q^2+q^3+q^4\right)^2\right)\Big/\left((1+q)^7\,\left(1+q^2\right)\,\left(1-q+q^2\right)^2\,\left(1-q+q^2-q^3+q^4\right)\,\left(1-q+q^2-q^3+q^4-q^5+q^6\right)\right)\right)$$

$$0$$

$$\left(2\,(-1+q)^4\,q^{20}\,\left(1+q+q^2\right)\,\left(1+q+q^2+q^3+q^4\right)^2\,\left(1+q+q^2+q^3+q^4+q^5+q^6\right)^2\right)\Big/\left((1+q)^9\,\left(1+q^2\right)^2\,\left(1-q+q^2\right)^3\,\left(1+q^4\right)\,\left(1-q+q^2-q^3+q^4\right)\,\left(1-q^3+q^6\right)\,\left(1-q+q^2-q^3+q^4-q^5+q^6\right)\right)$$

$$0$$

```
(*Initial values of T̃_M, using the rewritten formula*)
```

In[39]:=
```
MyQP[A_, q_, n_] := Product[(1 - A q^k), {k, 0, n-1}];

ClearAll[M, N1, N2, N3, N4, N5];

SList = {};

For[M = 0, M ≤ 10, M++,
 S = Sum[Sum[Sum[
     q^(M^2 - N1 - M N1 + N1^2 - N2 - 2 M N2 + 2 N1 N2 + 2 N2^2 - 3 M N3 + 2 N1 N3 + 4 N2 N3 + 3 N3^2)
     MyQP[-q^(1 + 2 N2 + 2 N3 - M), q^2, M - N1 - N2 - 2 N3] /
      (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3] MyQP[q^2, q^2, M - N1 - N2 - 2 N3]),
     {N2, 0, M - 2 N3 - N1}], {N1, 0, M - 2 N3}], {N3, 0, Floor[M/2]}];
 Print[Factor[S]];
 SList = Append[SList, Factor[S]];
]
```

$$1$$

$$\frac{2}{1+q}$$

$$0$$

$$-\frac{2\,(-1+q)\,q^2}{(1+q)^3\,\left(1-q+q^2\right)}$$

$$0$$

$$\frac{2\,(-1+q)^2\,q^6\,\left(1+q+q^2\right)^2}{(1+q)^5\,\left(1+q^2\right)\,\left(1-q+q^2\right)\,\left(1-q+q^2-q^3+q^4\right)}$$

$$0$$

$$-\left(\left(2\,(-1+q)^3\,q^{12}\,\left(1+q+q^2\right)\,\left(1+q+q^2+q^3+q^4\right)^2\right)\Big/ \left((1+q)^7\,\left(1+q^2\right)\,\left(1-q+q^2\right)^2\,\left(1-q+q^2-q^3+q^4\right)\,\left(1-q+q^2-q^3+q^4-q^5+q^6\right)\right)\right)$$

$$0$$

$$\left(2\,(-1+q)^4\,q^{20}\,\left(1+q+q^2\right)\,\left(1+q+q^2+q^3+q^4\right)^2\,\left(1+q+q^2+q^3+q^4+q^5+q^6\right)^2\right)\Big/ \left((1+q)^9\,\left(1+q^2\right)^2\,\left(1-q+q^2\right)^3\,\left(1+q^4\right)\,\left(1-q+q^2-q^3+q^4\right)\,\left(1-q^3+q^6\right)\,\left(1-q+q^2-q^3+q^4-q^5+q^6\right)\right)$$

$$0$$

```
(*Recurrence for T̃_M, with M odd*)
```

In[43]:=
```
ClearAll[M, N1, N2, N3, N4, N5];
summand =
  (-1)^(M+N1+N2) q^(M^2-N1-M N1+N1^2-N2-2 M N2+2 N1 N2+2 N2^2-3 M N3+2 N1 N3+4 N2 N3+3 N3^2) qPochhammer[-q^(1+2 N2+2 N3-M), q^2,
      M - N1 - N2 - 2 N3] / (qPochhammer[q^2, q^2, N1] qPochhammer[q^2, q^2, N2]
       qPochhammer[q^2, q^2, N3] qPochhammer[q^2, q^2, M - N1 - N2 - 2 N3]) /. {M → 2 M + 1};
stru = qFindStructureSet[summand, {M}, {N1, N2, N3}, {1}, {1, 2, 2},
   {1, 1, 1}, qProtocol → True];
rec = qFindRecurrence[summand, {M}, {N1, N2, N3}, {1}, {1, 2, 2},
   {1, 1, 1}, qProtocol → True, StructSet → stru[[1]]];
sumrec = qSumRecurrence[rec]
```

```
Structure set (231 elts.):
 {{0, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 1, 1},
  {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 0, 1}, {1, 1, 1, 0}, {1, 1, 2, 0}, {1, 2, 0, 0}, {1, 2, 1, 0},
  {1, 3, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 1, 2},
  {2, 0, 2, 0}, {2, 0, 2, 1}, {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 0, 2}, {2, 1, 1, 0}, {2, 1, 1, 1},
  {2, 1, 2, 0}, {2, 1, 2, 1}, {2, 2, 0, 0}, {2, 2, 0, 1}, {2, 2, 1, 0}, {2, 2, 1, 1}, {2, 2, 2, 0},
  {2, 3, 0, 0}, {2, 3, 0, 1}, {2, 3, 1, 0}, {2, 3, 2, 0}, {2, 4, 1, 0}, {3, 0, 0, 0}, {3, 0, 0, 1},
  {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0}, {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 1, 3}, {3, 0, 2, 0},
  {3, 0, 2, 1}, {3, 0, 2, 2}, {3, 1, 0, 0}, {3, 1, 0, 1}, {3, 1, 0, 2}, {3, 1, 0, 3}, {3, 1, 1, 0},
  {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0}, {3, 1, 2, 1}, {3, 1, 2, 2}, {3, 2, 0, 0}, {3, 2, 0, 1},
  {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 1, 2}, {3, 2, 2, 0}, {3, 2, 2, 1}, {3, 3, 0, 0},
  {3, 3, 0, 1}, {3, 3, 0, 2}, {3, 3, 1, 0}, {3, 3, 1, 1}, {3, 3, 2, 0}, {3, 3, 2, 1}, {3, 4, 1, 1},
  {3, 4, 2, 0}, {4, 0, 0, 0}, {4, 0, 0, 1}, {4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4}, {4, 0, 1, 0},
  {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3}, {4, 0, 2, 0}, {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 1, 0, 0},
  {4, 1, 0, 1}, {4, 1, 0, 2}, {4, 1, 0, 3}, {4, 1, 0, 4}, {4, 1, 1, 0}, {4, 1, 1, 1}, {4, 1, 1, 2},
  {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1}, {4, 1, 2, 2}, {4, 1, 2, 3}, {4, 1, 3, 0}, {4, 2, 0, 1},
  {4, 2, 0, 2}, {4, 2, 0, 3}, {4, 2, 1, 0}, {4, 2, 1, 1}, {4, 2, 1, 2}, {4, 2, 1, 3}, {4, 2, 2, 0},
  {4, 2, 2, 1}, {4, 2, 2, 2}, {4, 2, 3, 0}, {4, 2, 4, 0}, {4, 3, 0, 1}, {4, 3, 0, 2}, {4, 3, 0, 3},
  {4, 3, 1, 0}, {4, 3, 1, 1}, {4, 3, 1, 2}, {4, 3, 2, 0}, {4, 3, 2, 1}, {4, 3, 2, 2}, {4, 4, 1, 2},
  {4, 4, 2, 1}, {5, 0, 0, 4}, {5, 0, 0, 5}, {5, 0, 1, 2}, {5, 0, 1, 3}, {5, 0, 1, 4}, {5, 0, 2, 1},
  {5, 0, 2, 2}, {5, 0, 2, 3}, {5, 1, 0, 2}, {5, 1, 0, 3}, {5, 1, 0, 4}, {5, 1, 0, 5}, {5, 1, 1, 0},
  {5, 1, 1, 1}, {5, 1, 1, 2}, {5, 1, 1, 3}, {5, 1, 1, 4}, {5, 1, 2, 1}, {5, 1, 2, 2}, {5, 1, 2, 3},
  {5, 1, 3, 1}, {5, 2, 0, 2}, {5, 2, 0, 3}, {5, 2, 0, 4}, {5, 2, 1, 1}, {5, 2, 1, 2}, {5, 2, 1, 3},
  {5, 2, 1, 4}, {5, 2, 2, 0}, {5, 2, 2, 1}, {5, 2, 2, 2}, {5, 2, 2, 3}, {5, 2, 3, 1}, {5, 2, 4, 1},
  {5, 3, 0, 2}, {5, 3, 0, 3}, {5, 3, 0, 4}, {5, 3, 1, 1}, {5, 3, 1, 2}, {5, 3, 1, 3}, {5, 3, 2, 0},
  {5, 3, 2, 1}, {5, 3, 2, 2}, {5, 3, 2, 3}, {5, 3, 3, 0}, {5, 4, 1, 3}, {5, 4, 2, 2}, {5, 4, 4, 0},
  {6, 0, 2, 4}, {6, 1, 0, 6}, {6, 1, 1, 4}, {6, 1, 1, 5}, {6, 1, 2, 2}, {6, 1, 2, 3}, {6, 1, 2, 4},
  {6, 1, 3, 2}, {6, 2, 0, 4}, {6, 2, 0, 5}, {6, 2, 1, 2}, {6, 2, 1, 3}, {6, 2, 1, 4}, {6, 2, 1, 5},
  {6, 2, 2, 1}, {6, 2, 2, 2}, {6, 2, 2, 3}, {6, 2, 2, 4}, {6, 2, 3, 2}, {6, 2, 4, 2}, {6, 3, 0, 3},
  {6, 3, 0, 4}, {6, 3, 0, 5}, {6, 3, 1, 2}, {6, 3, 1, 3}, {6, 3, 1, 4}, {6, 3, 2, 1}, {6, 3, 2, 2},
  {6, 3, 2, 3}, {6, 3, 2, 4}, {6, 3, 3, 1}, {6, 4, 1, 4}, {6, 4, 2, 3}, {6, 4, 4, 1}, {7, 2, 1, 6},
  {7, 2, 2, 4}, {7, 2, 2, 5}, {7, 2, 3, 3}, {7, 2, 4, 3}, {7, 3, 0, 6}, {7, 3, 1, 4}, {7, 3, 1, 5},
  {7, 3, 2, 2}, {7, 3, 2, 3}, {7, 3, 2, 4}, {7, 3, 2, 5}, {7, 3, 3, 2}, {7, 4, 1, 5}, {7, 4, 2, 4},
  {7, 4, 4, 2}, {8, 2, 4, 4}, {8, 3, 2, 6}, {8, 4, 1, 6}, {8, 4, 2, 5}, {8, 4, 4, 3}, {9, 4, 4, 4}}

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 4855x1318

Simplifying 9 solution(s)

Solution exists
```

```
Structure set (89 elts.):
 {{0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
  {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
  {2, 0, 2, 1}, {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0},
  {2, 2, 0, 1}, {2, 2, 1, 0}, {3, 0, 0, 1}, {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0}, {3, 0, 1, 1},
  {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1}, {3, 0, 2, 2}, {3, 1, 0, 0}, {3, 1, 0, 1}, {3, 1, 0, 2},
  {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0}, {3, 1, 2, 1}, {3, 2, 0, 0}, {3, 2, 0, 1},
  {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0}, {4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4},
  {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3}, {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 1, 0, 1}, {4, 1, 0, 2},
  {4, 1, 0, 3}, {4, 1, 1, 0}, {4, 1, 1, 1}, {4, 1, 1, 2}, {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1},
  {4, 1, 2, 2}, {4, 2, 0, 1}, {4, 2, 0, 2}, {4, 2, 1, 0}, {4, 2, 1, 1}, {4, 2, 1, 2}, {4, 2, 2, 0},
  {4, 2, 2, 1}, {5, 0, 0, 4}, {5, 0, 0, 5}, {5, 0, 1, 3}, {5, 0, 1, 4}, {5, 0, 2, 2}, {5, 0, 2, 3},
  {5, 1, 0, 3}, {5, 1, 0, 4}, {5, 1, 1, 2}, {5, 1, 1, 3}, {5, 1, 2, 1}, {5, 1, 2, 2},
  {5, 2, 0, 2}, {5, 2, 0, 3}, {5, 2, 1, 1}, {5, 2, 1, 2}, {5, 2, 2, 0}, {5, 2, 2, 1}}

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 1992x670

Simplifying 1 solution(s)
```

Out[47]=

$$\Big\{-q^{50+10M}\left(-1+q^{3+M}\right)\left(1+q^{3+M}\right)\left(-1+q^{1+2M}\right)^2\left(-1+q^{3+2M}\right)^2\,\text{SUM}[M]+$$
$$q^{40+8M}\left(-1+q^{3+2M}\right)^2\left(-1-q^2-q^4-q^6-q^8+2q^{5+2M}+2q^{7+2M}+2q^{8+2M}+2q^{9+2M}+q^{10+2M}+q^{12+2M}+\right.$$
$$\left.q^{14+2M}-q^{10+4M}-q^{12+4M}-2q^{13+4M}-2q^{15+4M}-q^{16+4M}+q^{18+6M}-2q^{19+6M}+2q^{20+6M}\right)\,\text{SUM}[1+M]-$$
$$q^{30+6M}\left(-1+q^{5+2M}\right)\left(1+q^2+2q^4+2q^6+2q^8+q^{10}+q^{12}-q^{5+2M}-3q^{7+2M}-q^{8+2M}-4q^{9+2M}-\right.$$
$$2q^{10+2M}-4q^{11+2M}-2q^{12+2M}-2q^{13+2M}-3q^{14+2M}-q^{15+2M}-q^{16+2M}+q^{17+2M}-q^{18+2M}+2q^{12+4M}+$$
$$q^{13+4M}+3q^{14+4M}+2q^{15+4M}+2q^{16+4M}+4q^{17+4M}+q^{18+4M}+q^{19+4M}-q^{20+4M}+q^{21+4M}+q^{22+4M}-q^{23+4M}-$$
$$\left.q^{19+6M}-2q^{22+6M}+3q^{23+6M}-q^{24+6M}-q^{25+6M}+q^{26+6M}+q^{27+6M}-2q^{28+8M}+3q^{29+8M}-4q^{30+8M}+q^{31+8M}\right)$$
$$\text{SUM}[2+M]-q^{20+4M}\left(-1+q^{7+2M}\right)\left(-1-q^2-2q^4-2q^6-2q^8-q^{10}-q^{12}+q^{7+2M}+q^{8+2M}+\right.$$
$$3q^{9+2M}+q^{10+2M}+2q^{11+2M}+3q^{12+2M}+2q^{13+2M}+2q^{14+2M}+2q^{16+2M}-q^{17+2M}+q^{18+2M}-q^{19+2M}-$$
$$\left.q^{15+4M}-2q^{16+4M}-q^{17+4M}-q^{19+4M}+q^{20+4M}+q^{22+4M}+2q^{23+4M}+q^{25+4M}-2q^{25+6M}-q^{26+6M}-\right.$$
$$\left.q^{27+6M}-3q^{28+6M}+q^{29+6M}-2q^{30+6M}-q^{33+8M}+4q^{34+8M}-3q^{35+8M}+2q^{36+8M}\right)\,\text{SUM}[3+M]-$$
$$q^{10+2M}\left(-1+q^{4+M}\right)\left(1+q^{4+M}\right)\left(-1+q^{9+2M}\right)\left(-1-q^2-q^4-q^6-q^8+q^{9+2M}+q^{11+2M}-q^{13+2M}+q^{14+2M}-\right.$$
$$\left.q^{15+2M}-q^{16+2M}-q^{17+2M}+2q^{19+4M}+2q^{20+4M}+q^{22+4M}+q^{23+4M}-q^{25+4M}+2q^{29+6M}-2q^{30+6M}+q^{31+6M}\right)$$
$$\text{SUM}[4+M]+\left(-1+q^{4+M}\right)\left(1+q^{4+M}\right)\left(-1+q^{5+M}\right)\left(1+q^{5+M}\right)\left(1+q^{10+2M}\right)$$
$$\left(-1+q^{11+2M}\right)\left(1+q^{11+2M}\right)\text{SUM}[5+M]==0\Big\}$$

```
(*Verifying the recurrence for odd M*)
```

In[245]:=
```
ClearAll[M, A];

MyQP[A_, q_, n_] := Product[(1 - A q^k), {k, 0, n-1}];

A[k_, M_] := q^((M+k) (M+k+1)) MyQP[q^(2 M+1), q^2, k]^2 / (q^(M (M+1)) MyQP[q^(2 M+2), -q, 2 k]);

(-q^(50+10 M) (-1 + q^(3+M)) (1 + q^(3+M)) (-1 + q^(1+2 M))^2 (-1 + q^(3+2 M))^2 A[0, M] +
   q^(40+8 M) (-1 + q^(3+2 M))^2 (-1 - q^2 - q^4 - q^6 - q^8 + 2 q^(5+2 M) + 2 q^(7+2 M) + 2 q^(8+2 M) + 2 q^(9+2 M) + q^(10+2 M) + q^(12+2 M) +
      q^(14+2 M) - q^(10+4 M) - q^(12+4 M) - 2 q^(13+4 M) - 2 q^(15+4 M) - q^(16+4 M) + q^(18+6 M) - 2 q^(19+6 M) + 2 q^(20+6 M)) A[1, M] -
   q^(30+6 M) (-1 + q^(5+2 M)) (1 + q^2 + 2 q^4 + 2 q^6 + 2 q^8 + q^10 + q^12 - q^(5+2 M) - 3 q^(7+2 M) - q^(8+2 M) - 4 q^(9+2 M) -
      2 q^(10+2 M) - 4 q^(11+2 M) - 2 q^(12+2 M) - 2 q^(13+2 M) - 3 q^(14+2 M) - q^(15+2 M) - q^(16+2 M) + q^(17+2 M) - q^(18+2 M) + 2 q^(12+4 M) +
      q^(13+4 M) + 3 q^(14+4 M) + 2 q^(15+4 M) + 2 q^(16+4 M) + 4 q^(17+4 M) + q^(18+4 M) + q^(19+4 M) - q^(20+4 M) + q^(21+4 M) + q^(22+4 M) - q^(23+4 M) -
      q^(19+6 M) - 2 q^(22+6 M) + 3 q^(23+6 M) - q^(24+6 M) - q^(25+6 M) + q^(26+6 M) + q^(27+6 M) - 2 q^(28+8 M) + 3 q^(29+8 M) - 4 q^(30+8 M) + q^(31+8 M)) A[2, M] -
   q^(20+4 M) (-1 + q^(7+2 M)) (-1 - q^2 - 2 q^4 - 2 q^6 - 2 q^8 - q^10 - q^12 + q^(7+2 M) + q^(8+2 M) + 3 q^(9+2 M) +
      q^(10+2 M) + 2 q^(11+2 M) + 3 q^(12+2 M) + 2 q^(13+2 M) + 2 q^(14+2 M) + 2 q^(16+2 M) - q^(17+2 M) + q^(18+2 M) - q^(19+2 M) - q^(15+4 M) -
      2 q^(16+4 M) - q^(17+4 M) - q^(19+4 M) + q^(20+4 M) + q^(22+4 M) + 2 q^(23+4 M) + q^(25+4 M) - 2 q^(25+6 M) - q^(26+6 M) - q^(27+6 M) - 3 q^(28+6 M) +
      q^(29+6 M) - 2 q^(30+6 M) - q^(33+8 M) + 4 q^(34+8 M) - 3 q^(35+8 M) + 2 q^(36+8 M)) A[3, M] - q^(10+2 M) (-1 + q^(4+M)) (1 + q^(4+M))
   (-1 + q^(9+2 M)) (-1 - q^2 - q^4 - q^6 - q^8 + q^(9+2 M) + q^(11+2 M) - q^(13+2 M) + q^(14+2 M) - q^(15+2 M) - q^(16+2 M) - q^(17+2 M) +
      2 q^(19+4 M) + 2 q^(20+4 M) + q^(22+4 M) + q^(23+4 M) - q^(25+4 M) + 2 q^(29+6 M) - 2 q^(30+6 M) + q^(31+6 M)) A[4, M] + (-1 + q^(4+M))
   (1 + q^(4+M)) (-1 + q^(5+M)) (1 + q^(5+M)) (1 + q^(10+2 M)) (-1 + q^(11+2 M)) (1 + q^(11+2 M)) A[5, M]) // Simplify
```

Out[247]=
```
0
```

```
(*Recurrence for T̃_M, with M even*)
```

In[90]:=
```
ClearAll[M, N1, N2, N3, N4, N5];
summand =
  (-1)^(M+N1+N2) q^(M^2-N1-M N1+N1^2-N2-2 M N2+2 N1 N2+2 N2^2-3 M N3+2 N1 N3+4 N2 N3+3 N3^2) qPochhammer[-q^(1+2 N2+2 N3-M), q^2,
      M - N1 - N2 - 2 N3] / (qPochhammer[q^2, q^2, N1] qPochhammer[q^2, q^2, N2]
       qPochhammer[q^2, q^2, N3] qPochhammer[q^2, q^2, M - N1 - N2 - 2 N3]) /. {M → 2 M};
stru = qFindStructureSet[summand, {M}, {N1, N2, N3}, {1}, {1, 2, 2},
   {1, 1, 1}, qProtocol → True];
rec = qFindRecurrence[summand, {M}, {N1, N2, N3}, {1}, {1, 2, 2},
   {1, 1, 1}, qProtocol → True, StructSet → stru[[1]]];
sumrec = qSumRecurrence[rec]
```

```
Structure set (231 elts.):
 {{0, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 1, 1},
  {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 0, 1}, {1, 1, 1, 0}, {1, 1, 2, 0}, {1, 2, 0, 0}, {1, 2, 1, 0},
  {1, 3, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 1, 2},
  {2, 0, 2, 0}, {2, 0, 2, 1}, {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 0, 2}, {2, 1, 1, 0}, {2, 1, 1, 1},
  {2, 1, 2, 0}, {2, 1, 2, 1}, {2, 2, 0, 0}, {2, 2, 0, 1}, {2, 2, 1, 0}, {2, 2, 1, 1}, {2, 2, 2, 0},
  {2, 3, 0, 0}, {2, 3, 0, 1}, {2, 3, 1, 0}, {2, 3, 2, 0}, {2, 4, 1, 0}, {3, 0, 0, 0}, {3, 0, 0, 1},
  {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0}, {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 1, 3}, {3, 0, 2, 0},
  {3, 0, 2, 1}, {3, 0, 2, 2}, {3, 1, 0, 0}, {3, 1, 0, 1}, {3, 1, 0, 2}, {3, 1, 0, 3}, {3, 1, 1, 0},
  {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0}, {3, 1, 2, 1}, {3, 1, 2, 2}, {3, 2, 0, 0}, {3, 2, 0, 1},
  {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 1, 2}, {3, 2, 2, 0}, {3, 2, 2, 1}, {3, 3, 0, 0},
  {3, 3, 0, 1}, {3, 3, 0, 2}, {3, 3, 1, 0}, {3, 3, 1, 1}, {3, 3, 2, 0}, {3, 3, 2, 1}, {3, 4, 1, 1},
  {3, 4, 2, 0}, {4, 0, 0, 0}, {4, 0, 0, 1}, {4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4}, {4, 0, 1, 0},
  {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3}, {4, 0, 2, 0}, {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 1, 0, 0},
  {4, 1, 0, 1}, {4, 1, 0, 2}, {4, 1, 0, 3}, {4, 1, 0, 4}, {4, 1, 1, 0}, {4, 1, 1, 1}, {4, 1, 1, 2},
  {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1}, {4, 1, 2, 2}, {4, 1, 2, 3}, {4, 1, 3, 0}, {4, 2, 0, 1},
  {4, 2, 0, 2}, {4, 2, 0, 3}, {4, 2, 1, 0}, {4, 2, 1, 1}, {4, 2, 1, 2}, {4, 2, 1, 3}, {4, 2, 2, 0},
  {4, 2, 2, 1}, {4, 2, 2, 2}, {4, 2, 3, 0}, {4, 2, 4, 0}, {4, 3, 0, 1}, {4, 3, 0, 2}, {4, 3, 0, 3},
  {4, 3, 1, 0}, {4, 3, 1, 1}, {4, 3, 1, 2}, {4, 3, 2, 0}, {4, 3, 2, 1}, {4, 3, 2, 2}, {4, 4, 1, 2},
  {4, 4, 2, 1}, {5, 0, 0, 4}, {5, 0, 0, 5}, {5, 0, 1, 2}, {5, 0, 1, 3}, {5, 0, 1, 4}, {5, 0, 2, 1},
  {5, 0, 2, 2}, {5, 0, 2, 3}, {5, 1, 0, 2}, {5, 1, 0, 3}, {5, 1, 0, 4}, {5, 1, 0, 5}, {5, 1, 1, 0},
  {5, 1, 1, 1}, {5, 1, 1, 2}, {5, 1, 1, 3}, {5, 1, 1, 4}, {5, 1, 2, 1}, {5, 1, 2, 2}, {5, 1, 2, 3},
  {5, 1, 3, 1}, {5, 2, 0, 2}, {5, 2, 0, 3}, {5, 2, 0, 4}, {5, 2, 1, 1}, {5, 2, 1, 2}, {5, 2, 1, 3},
  {5, 2, 1, 4}, {5, 2, 2, 0}, {5, 2, 2, 1}, {5, 2, 2, 2}, {5, 2, 2, 3}, {5, 2, 3, 1}, {5, 2, 4, 1},
  {5, 3, 0, 2}, {5, 3, 0, 3}, {5, 3, 0, 4}, {5, 3, 1, 1}, {5, 3, 1, 2}, {5, 3, 1, 3}, {5, 3, 2, 0},
  {5, 3, 2, 1}, {5, 3, 2, 2}, {5, 3, 2, 3}, {5, 3, 3, 0}, {5, 4, 1, 3}, {5, 4, 2, 2}, {5, 4, 4, 0},
  {6, 0, 2, 4}, {6, 1, 0, 6}, {6, 1, 1, 4}, {6, 1, 1, 5}, {6, 1, 2, 2}, {6, 1, 2, 3}, {6, 1, 2, 4},
  {6, 1, 3, 2}, {6, 2, 0, 4}, {6, 2, 0, 5}, {6, 2, 1, 2}, {6, 2, 1, 3}, {6, 2, 1, 4}, {6, 2, 1, 5},
  {6, 2, 2, 1}, {6, 2, 2, 2}, {6, 2, 2, 3}, {6, 2, 2, 4}, {6, 2, 3, 2}, {6, 2, 4, 2}, {6, 3, 0, 3},
  {6, 3, 0, 4}, {6, 3, 0, 5}, {6, 3, 1, 2}, {6, 3, 1, 3}, {6, 3, 1, 4}, {6, 3, 2, 1}, {6, 3, 2, 2},
  {6, 3, 2, 3}, {6, 3, 2, 4}, {6, 3, 3, 1}, {6, 4, 1, 4}, {6, 4, 2, 3}, {6, 4, 4, 1}, {7, 2, 1, 6},
  {7, 2, 2, 4}, {7, 2, 2, 5}, {7, 2, 3, 3}, {7, 2, 4, 3}, {7, 3, 0, 6}, {7, 3, 1, 4}, {7, 3, 1, 5},
  {7, 3, 2, 2}, {7, 3, 2, 3}, {7, 3, 2, 4}, {7, 3, 2, 5}, {7, 3, 3, 2}, {7, 4, 1, 5}, {7, 4, 2, 4},
  {7, 4, 4, 2}, {8, 2, 4, 4}, {8, 3, 2, 6}, {8, 4, 1, 6}, {8, 4, 2, 5}, {8, 4, 4, 3}, {9, 4, 4, 4}}

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 4855x1318

Simplifying 9 solution(s)

Solution exists
```

```
Structure set (89 elts.):
 {{0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
  {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
  {2, 0, 2, 1}, {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0},
  {2, 2, 0, 1}, {2, 2, 1, 0}, {3, 0, 0, 1}, {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0}, {3, 0, 1, 1},
  {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1}, {3, 0, 2, 2}, {3, 1, 0, 0}, {3, 1, 0, 1}, {3, 1, 0, 2},
  {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0}, {3, 1, 2, 1}, {3, 2, 0, 0}, {3, 2, 0, 1},
  {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0}, {4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4},
  {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3}, {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 1, 0, 1}, {4, 1, 0, 2},
  {4, 1, 0, 3}, {4, 1, 1, 0}, {4, 1, 1, 1}, {4, 1, 1, 2}, {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1},
  {4, 1, 2, 2}, {4, 2, 0, 1}, {4, 2, 0, 2}, {4, 2, 1, 0}, {4, 2, 1, 1}, {4, 2, 1, 2}, {4, 2, 2, 0},
  {4, 2, 2, 1}, {5, 0, 0, 4}, {5, 0, 0, 5}, {5, 0, 1, 3}, {5, 0, 1, 4}, {5, 0, 2, 2}, {5, 0, 2, 3},
  {5, 1, 0, 3}, {5, 1, 0, 4}, {5, 1, 1, 2}, {5, 1, 1, 3}, {5, 1, 2, 1}, {5, 1, 2, 2},
  {5, 2, 0, 2}, {5, 2, 0, 3}, {5, 2, 1, 1}, {5, 2, 1, 2}, {5, 2, 2, 0}, {5, 2, 2, 1}}

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 1992x670

Simplifying 1 solution(s)
```

Out[94]=

$$\Big\{-q^{45+10M}\left(-1+q^{M}\right)^{2}\left(1+q^{M}\right)^{2}\left(-1+q^{1+M}\right)^{2}\left(1+q^{1+M}\right)^{2}\left(-1+q^{5+2M}\right)\text{SUM}[M]+$$
$$q^{36+8M}\left(-1+q^{1+M}\right)^{2}\left(1+q^{1+M}\right)^{2}$$
$$\left(-1-q^{2}-q^{4}-q^{6}-q^{8}+2q^{4+2M}+2q^{6+2M}+2q^{7+2M}+2q^{8+2M}+q^{9+2M}+q^{11+2M}+q^{13+2M}-q^{8+4M}-q^{10+4M}-\right.$$
$$\left.2q^{11+4M}-2q^{13+4M}-q^{14+4M}+q^{15+6M}-2q^{16+6M}+2q^{17+6M}\right)\text{SUM}[1+M]-q^{27+6M}\left(-1+q^{2+M}\right)\left(1+q^{2+M}\right)$$
$$\left(1+q^{2}+2q^{4}+2q^{6}+2q^{8}+q^{10}+q^{12}-q^{4+2M}-3q^{6+2M}-q^{7+2M}-4q^{8+2M}-2q^{9+2M}-4q^{10+2M}-\right.$$
$$2q^{11+2M}-2q^{12+2M}-3q^{13+2M}-q^{14+2M}-q^{15+2M}+q^{16+2M}-q^{17+2M}+2q^{10+4M}+q^{11+4M}+3q^{12+4M}+$$
$$\left.2q^{13+4M}+2q^{14+4M}+4q^{15+4M}+q^{16+4M}+q^{17+4M}-q^{18+4M}+q^{19+4M}+q^{20+4M}-q^{21+4M}-q^{16+6M}-2q^{19+6M}+\right.$$
$$\left.3q^{20+6M}-q^{21+6M}-q^{22+6M}+q^{23+6M}+q^{24+6M}-2q^{24+8M}+3q^{25+8M}-4q^{26+8M}+q^{27+8M}\right)\text{SUM}[2+M]-$$
$$q^{18+4M}\left(-1+q^{3+M}\right)\left(1+q^{3+M}\right)\left(-1-q^{2}-2q^{4}-2q^{6}-2q^{8}-q^{10}-q^{12}+q^{6+2M}+q^{7+2M}+3q^{8+2M}+\right.$$
$$q^{9+2M}+2q^{10+2M}+3q^{11+2M}+2q^{12+2M}+2q^{13+2M}+2q^{15+2M}-q^{16+2M}+q^{17+2M}-q^{18+2M}-q^{13+4M}-$$
$$\left.2q^{14+4M}-q^{15+4M}-q^{17+4M}+q^{18+4M}+q^{20+4M}+2q^{21+4M}+q^{23+4M}-2q^{22+6M}-q^{23+6M}-q^{24+6M}-3q^{25+6M}+\right.$$
$$\left.q^{26+6M}-2q^{27+6M}-q^{29+8M}+4q^{30+8M}-3q^{31+8M}+2q^{32+8M}\right)\text{SUM}[3+M]-q^{9+2M}\left(-1+q^{4+M}\right)\left(1+q^{4+M}\right)$$
$$\left(-1+q^{7+2M}\right)\left(-1-q^{2}-q^{4}-q^{6}-q^{8}+q^{8+2M}+q^{10+2M}-q^{12+2M}+q^{13+2M}-q^{14+2M}-q^{15+2M}-\right.$$
$$\left.q^{16+2M}+2q^{17+4M}+2q^{18+4M}+q^{20+4M}+q^{21+4M}-q^{23+4M}+2q^{26+6M}-2q^{27+6M}+q^{28+6M}\right)\text{SUM}[4+M]+$$
$$\left(-1+q^{5+M}\right)\left(1+q^{5+M}\right)\left(-1+q^{7+2M}\right)\left(-1+q^{9+2M}\right)\left(1+q^{9+2M}\right)\left(1+q^{10+2M}\right)\text{SUM}[5+M]==0\Big\}$$

```
(*No need to verify the recurrence for even M, which is trivial.*)

(*Verifying the Rogers-Ramanujan type identity*)
```

In[239]:=
```
MyQP[A_, q_, n_] := Product[(1 - A q^k), {k, 0, n-1}];
f[k_] := QPochhammer[q^k, q^k];

SumMax = 6;
PowerMax = 30;

Normal[Series[Sum[(-1)^(N4+N5+N6)
    q^(N1^2+N2^2+N3^2+N4^2+N5^2+N6^2+N1 N2+N1 N3+N1 N5+N1 N6+N2 N3+N2 N4+N2 N6+N3 N4+N3 N5+2 N3 N6+N4 N5+N4 N6+N5 N6-(N1+N2+N6))
    / (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3] MyQP[q^2, q^2, N4]
       MyQP[q^2, q^2, N5] MyQP[q^2, q^2, N6]),
  {N1, 0, SumMax}, {N2, 0, SumMax}, {N3, 0, SumMax}, {N4, 0, SumMax}, {N5, 0, SumMax}, {N6, 0, SumMax}],
 {q, 0, PowerMax}]]
Normal[Series[2 f[1]^2 f[4]^2 / f[2]^4, {q, 0, PowerMax}]]
```

Out[243]=
$$2 - 4\,q + 6\,q^2 - 12\,q^3 + 18\,q^4 - 28\,q^5 + 44\,q^6 - 64\,q^7 + 92\,q^8 - 132\,q^9 + 186\,q^{10} - 256\,q^{11} + 352\,q^{12} - 476\,q^{13} + 638\,q^{14} - 852\,q^{15} + 1124\,q^{16} - 1472\,q^{17} + 1920\,q^{18} - 2484\,q^{19} + 3196\,q^{20} - 4096\,q^{21} + 5216\,q^{22} - 6612\,q^{23} + 8350\,q^{24} - 10\,496\,q^{25} + 13\,140\,q^{26} - 16\,396\,q^{27} + 20\,380\,q^{28} - 25\,244\,q^{29} + 31\,178\,q^{30}$$

Out[244]=
$$2 - 4\,q + 6\,q^2 - 12\,q^3 + 18\,q^4 - 28\,q^5 + 44\,q^6 - 64\,q^7 + 92\,q^8 - 132\,q^9 + 186\,q^{10} - 256\,q^{11} + 352\,q^{12} - 476\,q^{13} + 638\,q^{14} - 852\,q^{15} + 1124\,q^{16} - 1472\,q^{17} + 1920\,q^{18} - 2484\,q^{19} + 3196\,q^{20} - 4096\,q^{21} + 5216\,q^{22} - 6612\,q^{23} + 8350\,q^{24} - 10\,496\,q^{25} + 13\,140\,q^{26} - 16\,396\,q^{27} + 20\,380\,q^{28} - 25\,244\,q^{29} + 31\,178\,q^{30}$$