

```
In[ ]:= SetOptions[SelectedNotebook[],
  PrintingStyleEnvironment -> "Printout", ShowSyntaxStyles -> True]
```

(*Importing RISC packages*)

```
In[ ]:= << RISC`qMultiSum`
<< RISC`qGeneratingFunctions`
```

qMultiSum Package version 2.54
 written by Axel Riese
 Copyright Research Institute for Symbolic Computation (RISC),
 Johannes Kepler University, Linz, Austria

qGeneratingFunctions Package version 1.9.1
 written by Christoph Koutschan
 Copyright Research Institute for Symbolic Computation (RISC),
 Johannes Kepler University, Linz, Austria

(*Initial values of \tilde{T}_M , using the original definition*)

```
In[ ]:= MyQP[A_, q_, n_] :=  $\prod_{k=0}^{n-1} (1 - A q^k);$ 
```

```
ClearAll[M, N1, N2, N3, N4, N5];
```

```
For[M = 0, M ≤ 10, M++,
```

$$S = \sum_{N3=0}^{\text{Floor}[\frac{M}{2}]} \sum_{N1=0}^{M-2N3} \sum_{N2=0}^{M-2N3-N1} \sum_{N4=0}^{M-2N3-N1-N2} (-1)^{N1+N2} z1^{N1+N3+N4} z2^{N2+N3+(M-N1-N2-2N3-N4)}$$

$$q^{N1^2+N2^2+N3^2+N4^2+(M-N1-N2-2N3-N4)^2+N1N2+N1N3+N1(M-N1-N2-2N3-N4)+N2N3+N2N4+N3N4+N3(M-N1-N2-2N3-N4)+N4(M-N1-N2-2N3-N4)}$$

$$\bigg/ \left(\text{MyQP}[q^2, q^2, N1] \text{MyQP}[q^2, q^2, N2] \text{MyQP}[q^2, q^2, N3] \right.$$

$$\left. \text{MyQP}[q^2, q^2, N4] \text{MyQP}[q^2, q^2, (M - N1 - N2 - 2N3 - N4)] \right);$$

```
Print[Factor[S]];
]
```

1

$$\begin{aligned}
& \frac{z_1 + z_2}{1 + q} \\
& \frac{q (z_1 - z_2)^2}{(-1 + q) (1 + q)^2} \\
& \frac{q^2 (q z_1 - z_2) (z_1 + z_2) (-z_1 + q z_2)}{(-1 + q) (1 + q)^3 (1 - q + q^2)} \\
& \frac{q^4 (z_1 - z_2)^2 (q^2 z_1 - z_2) (-z_1 + q^2 z_2)}{(-1 + q)^2 (1 + q)^4 (1 + q^2) (1 - q + q^2)} \\
& \frac{q^6 (q z_1 - z_2) (q^3 z_1 - z_2) (z_1 + z_2) (-z_1 + q z_2) (-z_1 + q^3 z_2)}{(-1 + q)^2 (1 + q)^5 (1 + q^2) (1 - q + q^2) (1 - q + q^2 - q^3 + q^4)} \\
& \frac{q^9 (z_1 - z_2)^2 (q^2 z_1 - z_2) (q^4 z_1 - z_2) (-z_1 + q^2 z_2) (-z_1 + q^4 z_2)}{(-1 + q)^3 (1 + q)^6 (1 + q^2) (1 - q + q^2)^2 (1 + q + q^2) (1 - q + q^2 - q^3 + q^4)} \\
& \frac{q^{12} (q z_1 - z_2) (q^3 z_1 - z_2) (q^5 z_1 - z_2) (z_1 + z_2) (-z_1 + q z_2) (-z_1 + q^3 z_2) (-z_1 + q^5 z_2)}{(-1 + q)^3 (1 + q)^7 (1 + q^2) (1 - q + q^2)^2 (1 + q + q^2) (1 - q + q^2 - q^3 + q^4) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6)} \\
& - \left(\frac{q^{16} (z_1 - z_2)^2 (q^2 z_1 - z_2) (q^4 z_1 - z_2) (q^6 z_1 - z_2) (-z_1 + q^2 z_2) (-z_1 + q^4 z_2) (-z_1 + q^6 z_2)}{(1 + q)^8 (1 + q^2)^2 (1 - q + q^2)^2 (1 + q + q^2) (1 + q^4) (1 - q + q^2 - q^3 + q^4) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6)} \right) \\
& - \left(\frac{q^{20} (q z_1 - z_2) (q^3 z_1 - z_2) (q^5 z_1 - z_2) (q^7 z_1 - z_2) (z_1 + z_2) (-z_1 + q z_2) (-z_1 + q^3 z_2) (-z_1 + q^5 z_2) (-z_1 + q^7 z_2)}{(1 + q + q^2) (1 + q^4) (1 - q + q^2 - q^3 + q^4) (1 - q^3 + q^6) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6)} \right) \\
& \frac{q^{25} (z_1 - z_2)^2 (q^2 z_1 - z_2) (q^4 z_1 - z_2) (q^6 z_1 - z_2) (q^8 z_1 - z_2) (-z_1 + q^2 z_2) (-z_1 + q^4 z_2) (-z_1 + q^6 z_2) (-z_1 + q^8 z_2)}{((-1 + q)^5 (1 + q)^{10} (1 + q^2)^2 (1 - q + q^2)^3 (1 + q + q^2) (1 + q^4) (1 - q + q^2 - q^3 + q^4)^2 (1 + q + q^2 + q^3 + q^4) (1 - q^3 + q^6) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6))}
\end{aligned}$$

(*Initial values of \tilde{T}_M , using the rewritten formula*)

$$In[] := \text{MyQP}[A_ , q_ , n_] := \prod_{k=0}^{n-1} (1 - A q^k);$$

ClearAll[M, N1, N2, N3, N4, N5];

SList = {};

For[M = 0, M ≤ 10, M++,

$$\begin{aligned}
S = & \sum_{N3=0}^{\text{Floor}[\frac{M}{2}]} \sum_{N1=0}^{M-2N3} \sum_{N2=0}^{M-2N3-N1} (-1)^{N1+N2} z_1^{N1+N3} z_2^{M-N1-N3} q^{M^2-N1-MN1+N1^2-N2-2MN2+2N1N2+2N2^2-3MN3+2N1N3+4N2N3+3N3^2} \\
& \frac{\text{MyQP}[-z_1 z_2^{-1} q^{1+2N2+2N3-M}, q^2, M-N1-N2-2N3]}{(\text{MyQP}[q^2, q^2, N1] \text{MyQP}[q^2, q^2, N2] \text{MyQP}[q^2, q^2, N3] \text{MyQP}[q^2, q^2, M-N1-N2-2N3])}; \\
& \text{Print[Factor[S]]}; \\
& \text{SList} = \text{Append[SList, Factor[S]]}; \\
&]
\end{aligned}$$

1

$$\begin{aligned}
& \frac{z1 + z2}{1 + q} \\
& \frac{q (z1 - z2)^2}{(-1 + q) (1 + q)^2} \\
& \frac{q^2 (q z1 - z2) (z1 + z2) (-z1 + q z2)}{(-1 + q) (1 + q)^3 (1 - q + q^2)} \\
& \frac{q^4 (z1 - z2)^2 (q^2 z1 - z2) (-z1 + q^2 z2)}{(-1 + q)^2 (1 + q)^4 (1 + q^2) (1 - q + q^2)} \\
& \frac{q^6 (q z1 - z2) (q^3 z1 - z2) (z1 + z2) (-z1 + q z2) (-z1 + q^3 z2)}{(-1 + q)^2 (1 + q)^5 (1 + q^2) (1 - q + q^2) (1 - q + q^2 - q^3 + q^4)} \\
& \frac{q^9 (z1 - z2)^2 (q^2 z1 - z2) (q^4 z1 - z2) (-z1 + q^2 z2) (-z1 + q^4 z2)}{(-1 + q)^3 (1 + q)^6 (1 + q^2) (1 - q + q^2)^2 (1 + q + q^2) (1 - q + q^2 - q^3 + q^4)} \\
& \frac{(q^{12} (q z1 - z2) (q^3 z1 - z2) (q^5 z1 - z2) (z1 + z2) (-z1 + q z2) (-z1 + q^3 z2) (-z1 + q^5 z2))}{((-1 + q)^3 (1 + q)^7 (1 + q^2) (1 - q + q^2)^2 (1 + q + q^2) (1 - q + q^2 - q^3 + q^4) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6))} \\
& - \left(\frac{(q^{16} (z1 - z2)^2 (q^2 z1 - z2) (q^4 z1 - z2) (q^6 z1 - z2) (-z1 + q^2 z2) (-z1 + q^4 z2) (-z1 + q^6 z2))}{(1 + q)^8 (1 + q^2)^2 (1 - q + q^2)^2 (1 + q + q^2) (1 + q^4) (1 - q + q^2 - q^3 + q^4) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6))} \right) \\
& - \left(\frac{(q^{20} (q z1 - z2) (q^3 z1 - z2) (q^5 z1 - z2) (q^7 z1 - z2) (z1 + z2) (-z1 + q z2) (-z1 + q^3 z2) (-z1 + q^5 z2))}{(1 + q + q^2) (1 + q^4) (1 - q + q^2 - q^3 + q^4) (1 - q^3 + q^6) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6))} \right) \\
& \frac{(q^{25} (z1 - z2)^2 (q^2 z1 - z2) (q^4 z1 - z2) (q^6 z1 - z2) (q^8 z1 - z2) (-z1 + q^2 z2) (-z1 + q^4 z2) (-z1 + q^6 z2) (-z1 + q^8 z2))}{((-1 + q)^5 (1 + q)^{10} (1 + q^2)^2 (1 - q + q^2)^3 (1 + q + q^2) (1 + q^4) (1 - q + q^2 - q^3 + q^4)^2 (1 + q + q^2 + q^3 + q^4) (1 - q^3 + q^6) (1 - q + q^2 - q^3 + q^4 - q^5 + q^6))}
\end{aligned}$$

(*Recurrence for \tilde{T}_M , with M odd*)

```

In[125]:= ClearAll[M, N1, N2, N3, N4, N5, z];
summand =
  (-1)^(N1+N2) z^N1+N3 q^(-N1-M N1+N1^2-N2-2 M N2+2 N1 N2+2 N2^2-3 M N3+2 N1 N3+4 N2 N3+3 N3^2) qPochhammer[-z q^(1+2 N2+2 N3-M),
    q^2, M-N1-N2-2 N3] / (qPochhammer[q^2, q^2, N1] qPochhammer[q^2, q^2, N2]
    qPochhammer[q^2, q^2, N3] qPochhammer[q^2, q^2, M-N1-N2-2 N3]) /. {M -> 2 M + 1};
stru = qFindStructureSet[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
  {1, 1, 1}, qProtocol -> True];
rec = qFindRecurrence[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
  {1, 1, 1}, qProtocol -> True, StructSet -> stru[[1]]];
sumrec = qSumRecurrence[rec]

```

Structure set (600 elts.):

```

{{0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
{1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
{2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0}, {2, 2, 0, 1},
{2, 2, 1, 0}, {2, 2, 2, 0}, {2, 3, 0, 0}, {2, 3, 1, 0}, {2, 4, 0, 0}, {3, 0, 0, 0}, {3, 0, 0, 1},
{3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0}, {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1},
{3, 1, 0, 0}, {3, 1, 0, 1}, {3, 1, 0, 2}, {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0},
{3, 1, 2, 1}, {3, 2, 0, 0}, {3, 2, 0, 1}, {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0},

```

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 $\{7, 0, 2, 3\}, \{7, 0, 2, 4\}, \{7, 1, 0, 2\}, \{7, 1, 0, 3\}, \{7, 1, 0, 4\}, \{7, 1, 0, 5\}, \{7, 1, 0, 6\},$
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 $\{8, 1, 3, 2\}, \{8, 1, 3, 3\}, \{8, 2, 0, 4\}, \{8, 2, 0, 5\}, \{8, 2, 0, 6\}, \{8, 2, 0, 7\}, \{8, 2, 1, 2\},$

{8, 2, 1, 3}, {8, 2, 1, 4}, {8, 2, 1, 5}, {8, 2, 1, 6}, {8, 2, 2, 1}, {8, 2, 2, 2}, {8, 2, 2, 3},
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 {8, 5, 2, 3}, {8, 5, 2, 4}, {8, 5, 3, 1}, {8, 5, 4, 1}, {8, 5, 5, 1}, {8, 6, 0, 4}, {8, 6, 1, 3},
 {8, 6, 2, 2}, {8, 6, 2, 4}, {8, 6, 3, 1}, {8, 6, 4, 0}, {8, 6, 5, 1}, {8, 6, 6, 1}, {9, 1, 3, 4},
 {9, 2, 0, 8}, {9, 2, 1, 6}, {9, 2, 1, 7}, {9, 2, 2, 4}, {9, 2, 2, 5}, {9, 2, 2, 6}, {9, 2, 3, 3},
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 {9, 3, 1, 6}, {9, 3, 1, 7}, {9, 3, 2, 2}, {9, 3, 2, 3}, {9, 3, 2, 4}, {9, 3, 2, 5}, {9, 3, 2, 6},
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 {9, 4, 0, 7}, {9, 4, 1, 3}, {9, 4, 1, 4}, {9, 4, 1, 5}, {9, 4, 1, 6}, {9, 4, 2, 2},
 {9, 4, 2, 3}, {9, 4, 2, 4}, {9, 4, 2, 5}, {9, 4, 2, 6}, {9, 4, 3, 2}, {9, 4, 3, 3},
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 {9, 5, 1, 5}, {9, 5, 1, 6}, {9, 5, 2, 3}, {9, 5, 2, 4}, {9, 5, 2, 5}, {9, 5, 3, 2},
 {9, 5, 4, 2}, {9, 5, 5, 2}, {9, 6, 0, 5}, {9, 6, 1, 4}, {9, 6, 2, 3}, {9, 6, 2, 5},
 {9, 6, 3, 2}, {9, 6, 4, 1}, {9, 6, 5, 2}, {9, 6, 6, 2}, {10, 2, 4, 4}, {10, 3, 1, 8},
 {10, 3, 2, 6}, {10, 3, 2, 7}, {10, 3, 3, 4}, {10, 3, 4, 4}, {10, 3, 5, 4}, {10, 4, 0, 8},
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 {10, 4, 3, 3}, {10, 4, 3, 4}, {10, 4, 4, 3}, {10, 4, 4, 4}, {10, 4, 5, 4}, {10, 5, 0, 6},
 {10, 5, 0, 7}, {10, 5, 1, 5}, {10, 5, 1, 6}, {10, 5, 1, 7}, {10, 5, 2, 4}, {10, 5, 2, 5},
 {10, 5, 2, 6}, {10, 5, 3, 3}, {10, 5, 4, 3}, {10, 5, 5, 3}, {10, 6, 0, 6}, {10, 6, 1, 5},
 {10, 6, 2, 4}, {10, 6, 2, 6}, {10, 6, 3, 3}, {10, 6, 4, 2}, {10, 6, 5, 3}, {10, 6, 6, 3},
 {11, 4, 2, 8}, {11, 4, 4, 4}, {11, 4, 4, 5}, {11, 4, 5, 5}, {11, 5, 1, 8}, {11, 5, 2, 6},
 {11, 5, 2, 7}, {11, 5, 3, 4}, {11, 5, 4, 4}, {11, 5, 5, 4}, {11, 6, 1, 6}, {11, 6, 2, 5},
 {11, 6, 2, 7}, {11, 6, 3, 4}, {11, 6, 4, 3}, {11, 6, 5, 4}, {11, 6, 6, 4}, {12, 5, 5, 5},
 {12, 6, 2, 8}, {12, 6, 4, 4}, {12, 6, 5, 5}, {12, 6, 6, 5}, {13, 6, 5, 6}, {13, 6, 6, 6}

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 11346x3116

Simplifying 1 solution(s)

Solution exists

Structure set (143 elts.):

```
{ {0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
  {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
  {2, 0, 2, 1}, {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0},
  {2, 2, 0, 1}, {2, 2, 1, 0}, {2, 3, 0, 0}, {3, 0, 0, 1}, {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0},
  {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1}, {3, 0, 2, 2}, {3, 1, 0, 0}, {3, 1, 0, 1},
  {3, 1, 0, 2}, {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0}, {3, 1, 2, 1}, {3, 2, 0, 0},
  {3, 2, 0, 1}, {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0}, {3, 3, 0, 0}, {3, 3, 0, 1},
  {3, 3, 1, 0}, {4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4}, {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3},
  {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 0, 2, 3}, {4, 1, 0, 1}, {4, 1, 0, 2}, {4, 1, 0, 3}, {4, 1, 1, 0},
  {4, 1, 1, 1}, {4, 1, 1, 2}, {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1}, {4, 1, 2, 2}, {4, 2, 0, 0},
  {4, 2, 0, 1}, {4, 2, 0, 2}, {4, 2, 0, 3}, {4, 2, 1, 0}, {4, 2, 1, 1}, {4, 2, 1, 2}, {4, 2, 2, 0},
  {4, 2, 2, 1}, {4, 3, 0, 0}, {4, 3, 0, 1}, {4, 3, 0, 2}, {4, 3, 1, 0}, {4, 3, 1, 1}, {4, 3, 2, 0},
  {5, 0, 0, 3}, {5, 0, 0, 4}, {5, 0, 0, 5}, {5, 0, 1, 2}, {5, 0, 1, 3}, {5, 0, 1, 4}, {5, 0, 2, 2},
  {5, 0, 2, 3}, {5, 1, 0, 2}, {5, 1, 0, 3}, {5, 1, 0, 4}, {5, 1, 1, 1}, {5, 1, 1, 2}, {5, 1, 1, 3},
  {5, 1, 1, 4}, {5, 1, 2, 1}, {5, 1, 2, 2}, {5, 1, 2, 3}, {5, 2, 0, 1}, {5, 2, 0, 2}, {5, 2, 0, 3},
  {5, 2, 1, 0}, {5, 2, 1, 1}, {5, 2, 1, 2}, {5, 2, 1, 3}, {5, 2, 2, 0}, {5, 2, 2, 1}, {5, 2, 2, 2},
  {5, 3, 0, 1}, {5, 3, 0, 2}, {5, 3, 1, 0}, {5, 3, 1, 1}, {5, 3, 1, 2}, {5, 3, 2, 0}, {5, 3, 2, 1},
  {6, 0, 0, 5}, {6, 0, 0, 6}, {6, 0, 1, 4}, {6, 0, 1, 5}, {6, 0, 2, 3}, {6, 0, 2, 4},
  {6, 1, 0, 4}, {6, 1, 0, 5}, {6, 1, 1, 3}, {6, 1, 1, 4}, {6, 1, 2, 2}, {6, 1, 2, 3},
  {6, 2, 0, 3}, {6, 2, 0, 4}, {6, 2, 1, 2}, {6, 2, 1, 3}, {6, 2, 2, 1}, {6, 2, 2, 2},
  {6, 3, 0, 2}, {6, 3, 0, 3}, {6, 3, 1, 1}, {6, 3, 1, 2}, {6, 3, 2, 0}, {6, 3, 2, 1}}
```

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 3104x1095

Simplifying 1 solution(s)

$$\begin{aligned} \text{Out}[129] = & \left\{ -q^4 (q^{1+2M} - z) (q^{3+2M} - z) (q^{6+2M} - z) z^3 (-1 + q^{1+2M} z) (-1 + q^{3+2M} z) (-1 + q^{5+2M} z) \text{SUM}[M] - \right. \\ & q^{6M} (q^{3+2M} - z) z^2 (-1 + q^{3+2M} z) (-1 + q^{5+2M} z) \\ & (q^{13+4M} + q^{15+4M} + q^{17+4M} - q^{5+2M} z - q^{7+2M} z - q^{8+2M} z - q^{9+2M} z - q^{10+2M} z - q^{11+2M} z - q^{12+2M} z - \\ & q^{14+2M} z - q^{16+2M} z - q^{20+6M} z + 2 q^{21+6M} z - 2 q^{22+6M} z + q^{23+6M} z + z^2 + q^2 z^2 + q^4 z^2 + q^6 z^2 + \\ & q^8 z^2 + q^{10} z^2 + q^{12+4M} z^2 + q^{14+4M} z^2 + q^{17+4M} z^2 - q^{7+2M} z^3 - q^{9+2M} z^3 - q^{11+2M} z^3) \text{SUM}[1+M] + \\ & q^{4+12M} z (-1 + q^{5+2M} z) (-q^{22+6M} - q^{24+6M} - q^{26+6M} + q^{12+4M} z + q^{14+4M} z + q^{15+4M} z + 2 q^{16+4M} z + \\ & 2 q^{17+4M} z + q^{18+4M} z + 3 q^{19+4M} z + q^{20+4M} z + 3 q^{21+4M} z + 2 q^{23+4M} z + q^{25+4M} z + q^{29+8M} z - 2 q^{30+8M} z + \\ & 3 q^{31+8M} z - 4 q^{32+8M} z + 2 q^{33+8M} z - 2 q^{34+8M} z - q^{36+8M} z - q^{5+2M} z^2 - 2 q^{7+2M} z^2 - 3 q^{9+2M} z^2 - \\ & q^{10+2M} z^2 - 4 q^{11+2M} z^2 - q^{12+2M} z^2 - 4 q^{13+2M} z^2 - 2 q^{14+2M} z^2 - 3 q^{15+2M} z^2 - 2 q^{16+2M} z^2 - 2 q^{17+2M} z^2 - \\ & 2 q^{18+2M} z^2 - q^{19+2M} z^2 - q^{20+2M} z^2 - q^{22+2M} z^2 - q^{19+6M} z^2 - 2 q^{21+6M} z^2 - 2 q^{23+6M} z^2 - 2 q^{24+6M} z^2 + \\ & 2 q^{25+6M} z^2 - 4 q^{26+6M} z^2 + 4 q^{27+6M} z^2 - 3 q^{28+6M} z^2 + 3 q^{29+6M} z^2 + q^{31+6M} z^2 + q^{32+6M} z^2 + 2 q^{37+10M} z^2 - \\ & 3 q^{38+10M} z^2 + 5 q^{39+10M} z^2 - 3 q^{40+10M} z^2 + 2 q^{41+10M} z^2 + z^3 + q^2 z^3 + 2 q^4 z^3 + 2 q^6 z^3 + 3 q^8 z^3 + \\ & 2 q^{10} z^3 + 2 q^{12} z^3 + q^{14} z^3 + q^{16} z^3 + q^{12+4M} z^3 + 3 q^{14+4M} z^3 + 5 q^{16+4M} z^3 + 5 q^{18+4M} z^3 + q^{19+4M} z^3 + \\ & 3 q^{20+4M} z^3 + q^{21+4M} z^3 + 2 q^{22+4M} z^3 + q^{23+4M} z^3 - q^{24+4M} z^3 + q^{25+4M} z^3 - q^{26+4M} z^3 + q^{28+8M} z^3 - \\ & q^{29+8M} z^3 - 3 q^{32+8M} z^3 + 2 q^{33+8M} z^3 - 4 q^{34+8M} z^3 + q^{35+8M} z^3 - 2 q^{36+8M} z^3 - q^{7+2M} z^4 - 2 q^{9+2M} z^4 - \\ & 3 q^{11+2M} z^4 - 3 q^{13+2M} z^4 - 3 q^{15+2M} z^4 - 2 q^{17+2M} z^4 - q^{19+2M} z^4 - q^{21+6M} z^4 - 2 q^{23+6M} z^4 + q^{24+6M} z^4 - \\ & q^{25+6M} z^4 + q^{27+6M} z^4 + q^{28+6M} z^4 + q^{30+6M} z^4 + q^{16+4M} z^5 + q^{18+4M} z^5 + q^{20+4M} z^5) \text{SUM}[2+M] - \\ & q^{16+18M} (q^{28+6M} - q^{16+4M} z - q^{18+4M} z - q^{19+4M} z - q^{20+4M} z - 2 q^{21+4M} z - q^{22+4M} z - 2 q^{23+4M} z - \\ & 2 q^{25+4M} z - q^{27+4M} z - q^{35+8M} z + 3 q^{36+8M} z - 2 q^{37+8M} z + 3 q^{38+8M} z + 2 q^{40+8M} z + q^{42+8M} z + q^{7+2M} z^2 + \\ & 2 q^{9+2M} z^2 + 3 q^{11+2M} z^2 + q^{12+2M} z^2 + 4 q^{13+2M} z^2 + q^{14+2M} z^2 + 4 q^{15+2M} z^2 + 2 q^{16+2M} z^2 + 3 q^{17+2M} z^2 + \\ & 2 q^{18+2M} z^2 + 2 q^{19+2M} z^2 + 2 q^{20+2M} z^2 + q^{21+2M} z^2 + q^{22+2M} z^2 + q^{24+2M} z^2 + q^{23+6M} z^2 + 2 q^{25+6M} z^2 + \end{aligned}$$

$$\begin{aligned}
& q^{27+6M} z^2 + 2 q^{28+6M} z^2 - 3 q^{29+6M} z^2 + 3 q^{30+6M} z^2 - 5 q^{31+6M} z^2 + q^{32+6M} z^2 - 4 q^{33+6M} z^2 - 2 q^{34+6M} z^2 - \\
& q^{35+6M} z^2 - 3 q^{36+6M} z^2 - q^{38+6M} z^2 - 3 q^{43+10M} z^2 + 3 q^{44+10M} z^2 - 7 q^{45+10M} z^2 + 5 q^{46+10M} z^2 - \\
& 6 q^{47+10M} z^2 + 3 q^{48+10M} z^2 - 2 q^{49+10M} z^2 + q^{50+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 3 q^6 z^3 - 3 q^8 z^3 - \\
& 3 q^{10} z^3 - 3 q^{12} z^3 - 2 q^{14} z^3 - q^{16} z^3 - q^{18} z^3 - q^{14+4M} z^3 - 3 q^{16+4M} z^3 - 6 q^{18+4M} z^3 - 6 q^{20+4M} z^3 - \\
& q^{21+4M} z^3 - 6 q^{22+4M} z^3 - q^{23+4M} z^3 - 2 q^{24+4M} z^3 - 2 q^{25+4M} z^3 + q^{26+4M} z^3 - q^{27+4M} z^3 + 2 q^{28+4M} z^3 - \\
& q^{29+4M} z^3 + 2 q^{30+4M} z^3 + q^{32+4M} z^3 - q^{32+8M} z^3 + q^{33+8M} z^3 + q^{35+8M} z^3 + 5 q^{36+8M} z^3 - q^{37+8M} z^3 + \\
& 9 q^{38+8M} z^3 - q^{39+8M} z^3 + 6 q^{40+8M} z^3 + 3 q^{41+8M} z^3 + 3 q^{43+8M} z^3 - q^{44+8M} z^3 - q^{51+12M} z^3 + 4 q^{52+12M} z^3 - \\
& 5 q^{53+12M} z^3 + 5 q^{54+12M} z^3 - 4 q^{55+12M} z^3 + q^{56+12M} z^3 + q^{7+2M} z^4 + 2 q^{9+2M} z^4 + 4 q^{11+2M} z^4 + 5 q^{13+2M} z^4 + \\
& 6 q^{15+2M} z^4 + 5 q^{17+2M} z^4 + 4 q^{19+2M} z^4 + 2 q^{21+2M} z^4 + q^{23+2M} z^4 + q^{23+6M} z^4 + 3 q^{25+6M} z^4 - q^{26+6M} z^4 + \\
& 3 q^{27+6M} z^4 - q^{28+6M} z^4 + q^{29+6M} z^4 - 2 q^{30+6M} z^4 - 2 q^{31+6M} z^4 - 2 q^{32+6M} z^4 - 3 q^{33+6M} z^4 - 2 q^{34+6M} z^4 - \\
& 2 q^{35+6M} z^4 - q^{36+6M} z^4 - q^{37+6M} z^4 - q^{42+10M} z^4 - q^{43+10M} z^4 - 4 q^{45+10M} z^4 + 2 q^{46+10M} z^4 - 3 q^{47+10M} z^4 + \\
& q^{48+10M} z^4 - q^{16+4M} z^5 - 2 q^{18+4M} z^5 - 3 q^{20+4M} z^5 - 3 q^{22+4M} z^5 - 2 q^{24+4M} z^5 - q^{26+4M} z^5 - q^{34+8M} z^5 + \\
& q^{35+8M} z^5 + q^{36+8M} z^5 + q^{37+8M} z^5 + q^{38+8M} z^5 + 2 q^{39+8M} z^5 + q^{41+8M} z^5 + q^{27+6M} z^6) \text{ SUM}[3+M] - \\
& q^{36+24M} (q^{20+4M} + q^{23+4M} + q^{25+4M} - q^{40+8M} - q^{42+8M} - q^{44+8M} - q^{9+2M} z - 2 q^{11+2M} z - 2 q^{13+2M} z - \\
& q^{14+2M} z - 2 q^{15+2M} z - q^{16+2M} z - 2 q^{17+2M} z - q^{18+2M} z - q^{19+2M} z - q^{20+2M} z - q^{22+2M} z + \\
& q^{30+6M} z + 2 q^{31+6M} z + 4 q^{33+6M} z + q^{34+6M} z + 2 q^{35+6M} z + 4 q^{36+6M} z + 3 q^{38+6M} z + q^{40+6M} z + \\
& q^{49+10M} z - 2 q^{50+10M} z + 3 q^{51+10M} z - 4 q^{52+10M} z + 2 q^{53+10M} z - 2 q^{54+10M} z - q^{56+10M} z + z^2 + \\
& q^2 z^2 + 2 q^4 z^2 + 2 q^6 z^2 + 3 q^8 z^2 + 2 q^{10} z^2 + 2 q^{12} z^2 + q^{14} z^2 + q^{16} z^2 + q^{18+4M} z^2 + 2 q^{20+4M} z^2 - \\
& q^{21+4M} z^2 + 2 q^{22+4M} z^2 - q^{24+4M} z^2 - 2 q^{26+4M} z^2 - 4 q^{28+4M} z^2 - 3 q^{30+4M} z^2 - 2 q^{32+4M} z^2 - \\
& q^{34+4M} z^2 - q^{38+8M} z^2 - q^{39+8M} z^2 - 3 q^{40+8M} z^2 - 4 q^{42+8M} z^2 - q^{43+8M} z^2 + q^{44+8M} z^2 - 4 q^{45+8M} z^2 + \\
& 4 q^{46+8M} z^2 - 2 q^{47+8M} z^2 + q^{48+8M} z^2 + q^{50+8M} z^2 + 2 q^{59+12M} z^2 - 3 q^{60+12M} z^2 + 5 q^{61+12M} z^2 - \\
& 3 q^{62+12M} z^2 + 2 q^{63+12M} z^2 - q^{9+2M} z^3 - 2 q^{11+2M} z^3 - 3 q^{13+2M} z^3 - 3 q^{15+2M} z^3 - 3 q^{17+2M} z^3 - \\
& 2 q^{19+2M} z^3 - q^{21+2M} z^3 + q^{30+6M} z^3 + q^{31+6M} z^3 + 2 q^{32+6M} z^3 + 3 q^{33+6M} z^3 + 2 q^{34+6M} z^3 + 3 q^{35+6M} z^3 + \\
& 2 q^{36+6M} z^3 + 2 q^{37+6M} z^3 + q^{38+6M} z^3 + q^{39+6M} z^3 + q^{49+10M} z^3 - q^{50+10M} z^3 + q^{51+10M} z^3 - 2 q^{52+10M} z^3 - \\
& q^{54+10M} z^3 - q^{55+10M} z^3 + q^{20+4M} z^4 + q^{22+4M} z^4 + q^{24+4M} z^4 - q^{40+8M} z^4 - q^{41+8M} z^4 - q^{43+8M} z^4) \\
& \text{SUM}[4+M] - q^{64+30M} (-1 + q^{5+M}) (1 + q^{5+M}) (1 + q^{10+2M}) \\
& (-q^{11+2M} - q^{13+2M} - q^{16+2M} + q^{34+6M} + q^{36+6M} + q^{38+6M} + z + q^2 z + q^4 z + q^6 z + q^8 z + q^{10} z + q^{22+4M} z - \\
& q^{23+4M} z - q^{24+4M} z - 2 q^{26+4M} z - q^{28+4M} z - q^{30+4M} z - q^{32+4M} z - q^{45+8M} z + 2 q^{46+8M} z - 2 q^{47+8M} z + \\
& q^{48+8M} z - q^{11+2M} z^2 - q^{13+2M} z^2 - q^{15+2M} z^2 + q^{34+6M} z^2 + q^{36+6M} z^2 + q^{37+6M} z^2) \text{SUM}[5+M] + \\
& q^{100+36M} (-1 + q^{5+M}) (1 + q^{5+M}) (-1 + q^{6+M}) (1 + q^{6+M}) (1 + q^{10+2M}) (1 + q^{12+2M}) \\
& (-1 + q^{13+2M}) (1 + q^{13+2M}) \text{SUM}[6+M] = 0\}
\end{aligned}$$

(*Verifying the recurrence for odd M*)

In[133]:= ClearAll[M, A, z];

MyQP[A_, q_, n_] := $\prod_{k=0}^{n-1} (1 - A q^k)$;

A[k_, M_] := $\frac{z^k q^{-3(M+k)} \text{MyQP}[z q^{2M+1}, q^2, k] \text{MyQP}[z^{-1} q^{2M+1}, q^2, k]}{q^{-3M(M+1)} \text{MyQP}[q^{2M+2}, -q, 2k]}$;

$$\begin{aligned}
& (-q^4 (q^{1+2M} - z) (q^{3+2M} - z) (q^{6+2M} - z) z^3 (-1 + q^{1+2M} z) (-1 + q^{3+2M} z) (-1 + q^{5+2M} z) A[0, M] - \\
& q^{6M} (q^{3+2M} - z) z^2 (-1 + q^{3+2M} z) (-1 + q^{5+2M} z) (q^{13+4M} + q^{15+4M} + q^{17+4M} - q^{5+2M} z - q^{7+2M} z - q^{8+2M} z - \\
& q^{9+2M} z - q^{10+2M} z - q^{11+2M} z - q^{12+2M} z - q^{14+2M} z - q^{16+2M} z - q^{20+6M} z + 2 q^{21+6M} z - 2 q^{22+6M} z + \\
& q^{23+6M} z + z^2 + q^2 z^2 + q^4 z^2 + q^6 z^2 + q^8 z^2 + q^{10} z^2 + q^{12+4M} z^2 + q^{14+4M} z^2 + q^{17+4M} z^2 - q^{7+2M} z^3 - \\
& q^{9+2M} z^3 - q^{11+2M} z^3) A[1, M] + q^{4+12M} z (-1 + q^{5+2M} z) (-q^{22+6M} - q^{24+6M} - q^{26+6M} + q^{12+4M} z + \\
& q^{14+4M} z + q^{15+4M} z + 2 q^{16+4M} z + 2 q^{17+4M} z + q^{18+4M} z + 3 q^{19+4M} z + q^{20+4M} z + 3 q^{21+4M} z + 2 q^{23+4M} z + \\
& q^{25+4M} z + q^{29+8M} z - 2 q^{30+8M} z + 3 q^{31+8M} z - 4 q^{32+8M} z + 2 q^{33+8M} z - 2 q^{34+8M} z - q^{36+8M} z - q^{5+2M} z^2 - \\
& 2 q^{7+2M} z^2 - 3 q^{9+2M} z^2 - q^{10+2M} z^2 - 4 q^{11+2M} z^2 - q^{12+2M} z^2 - 4 q^{13+2M} z^2 - 2 q^{14+2M} z^2 - 3 q^{15+2M} z^2 - \\
& 2 q^{16+2M} z^2 - 2 q^{17+2M} z^2 - 2 q^{18+2M} z^2 - q^{19+2M} z^2 - q^{20+2M} z^2 - q^{22+2M} z^2 - q^{19+6M} z^2 - 2 q^{21+6M} z^2 - \\
& 2 q^{23+6M} z^2 - 2 q^{24+6M} z^2 + 2 q^{25+6M} z^2 - 4 q^{26+6M} z^2 + 4 q^{27+6M} z^2 - 3 q^{28+6M} z^2 + 3 q^{29+6M} z^2 + q^{31+6M} z^2 +
\end{aligned}$$

$$\begin{aligned}
& q^{32+6M} z^2 + 2 q^{37+10M} z^2 - 3 q^{38+10M} z^2 + 5 q^{39+10M} z^2 - 3 q^{40+10M} z^2 + 2 q^{41+10M} z^2 + z^3 + q^2 z^3 + \\
& 2 q^4 z^3 + 2 q^6 z^3 + 3 q^8 z^3 + 2 q^{10} z^3 + 2 q^{12} z^3 + q^{14} z^3 + q^{16} z^3 + q^{12+4M} z^3 + 3 q^{14+4M} z^3 + 5 q^{16+4M} z^3 + \\
& 5 q^{18+4M} z^3 + q^{19+4M} z^3 + 3 q^{20+4M} z^3 + q^{21+4M} z^3 + 2 q^{22+4M} z^3 + q^{23+4M} z^3 - q^{24+4M} z^3 + q^{25+4M} z^3 - \\
& q^{26+4M} z^3 + q^{28+8M} z^3 - q^{29+8M} z^3 - 3 q^{32+8M} z^3 + 2 q^{33+8M} z^3 - 4 q^{34+8M} z^3 + q^{35+8M} z^3 - 2 q^{36+8M} z^3 - \\
& q^{7+2M} z^4 - 2 q^{9+2M} z^4 - 3 q^{11+2M} z^4 - 3 q^{13+2M} z^4 - 3 q^{15+2M} z^4 - 2 q^{17+2M} z^4 - q^{19+2M} z^4 - q^{21+6M} z^4 - \\
& 2 q^{23+6M} z^4 + q^{24+6M} z^4 - q^{25+6M} z^4 + q^{27+6M} z^4 + q^{28+6M} z^4 + q^{30+6M} z^4 + q^{16+4M} z^5 + q^{18+4M} z^5 + q^{20+4M} z^5) \\
& A[2, M] - q^{16+18M} (q^{28+6M} - q^{16+4M} z - q^{18+4M} z - q^{19+4M} z - q^{20+4M} z - q^{21+4M} z - q^{22+4M} z - \\
& 2 q^{23+4M} z - 2 q^{25+4M} z - q^{27+4M} z - q^{35+8M} z + 3 q^{36+8M} z - 2 q^{37+8M} z + 3 q^{38+8M} z + 2 q^{40+8M} z + \\
& q^{42+8M} z + q^{7+2M} z^2 + 2 q^{9+2M} z^2 + 3 q^{11+2M} z^2 + q^{12+2M} z^2 + 4 q^{13+2M} z^2 + q^{14+2M} z^2 + 4 q^{15+2M} z^2 + \\
& 2 q^{16+2M} z^2 + 3 q^{17+2M} z^2 + 2 q^{18+2M} z^2 + 2 q^{19+2M} z^2 + 2 q^{20+2M} z^2 + q^{21+2M} z^2 + q^{22+2M} z^2 + q^{24+2M} z^2 + \\
& q^{23+6M} z^2 + 2 q^{25+6M} z^2 + q^{27+6M} z^2 + 2 q^{28+6M} z^2 - 3 q^{29+6M} z^2 + 3 q^{30+6M} z^2 - 5 q^{31+6M} z^2 + q^{32+6M} z^2 - \\
& 4 q^{33+6M} z^2 - 2 q^{34+6M} z^2 - q^{35+6M} z^2 - 3 q^{36+6M} z^2 - q^{38+6M} z^2 - 3 q^{43+10M} z^2 + 3 q^{44+10M} z^2 - 7 q^{45+10M} z^2 + \\
& 5 q^{46+10M} z^2 - 6 q^{47+10M} z^2 + 3 q^{48+10M} z^2 - 2 q^{49+10M} z^2 + q^{50+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 3 q^6 z^3 - \\
& 3 q^8 z^3 - 3 q^{10} z^3 - 3 q^{12} z^3 - 2 q^{14} z^3 - q^{16} z^3 - q^{18} z^3 - q^{14+4M} z^3 - 3 q^{16+4M} z^3 - 6 q^{18+4M} z^3 - \\
& 6 q^{20+4M} z^3 - q^{21+4M} z^3 - 6 q^{22+4M} z^3 - q^{23+4M} z^3 - 2 q^{24+4M} z^3 - 2 q^{25+4M} z^3 + q^{26+4M} z^3 - q^{27+4M} z^3 + \\
& 2 q^{28+4M} z^3 - q^{29+4M} z^3 + 2 q^{30+4M} z^3 + q^{32+4M} z^3 - q^{32+8M} z^3 + q^{33+8M} z^3 + q^{35+8M} z^3 + 5 q^{36+8M} z^3 - \\
& q^{37+8M} z^3 + 9 q^{38+8M} z^3 - q^{39+8M} z^3 + 6 q^{40+8M} z^3 + 3 q^{41+8M} z^3 + 3 q^{43+8M} z^3 - q^{44+8M} z^3 - q^{51+12M} z^3 + \\
& 4 q^{52+12M} z^3 - 5 q^{53+12M} z^3 + 5 q^{54+12M} z^3 - 4 q^{55+12M} z^3 + q^{56+12M} z^3 + q^{7+2M} z^4 + 2 q^{9+2M} z^4 + 4 q^{11+2M} z^4 + \\
& 5 q^{13+2M} z^4 + 6 q^{15+2M} z^4 + 5 q^{17+2M} z^4 + 4 q^{19+2M} z^4 + 2 q^{21+2M} z^4 + q^{23+2M} z^4 + q^{23+6M} z^4 + 3 q^{25+6M} z^4 - \\
& q^{26+6M} z^4 + 3 q^{27+6M} z^4 - q^{28+6M} z^4 + q^{29+6M} z^4 - 2 q^{30+6M} z^4 - 2 q^{31+6M} z^4 - 2 q^{32+6M} z^4 - 3 q^{33+6M} z^4 - \\
& 2 q^{34+6M} z^4 - 2 q^{35+6M} z^4 - q^{36+6M} z^4 - q^{37+6M} z^4 - q^{42+10M} z^4 - q^{43+10M} z^4 - 4 q^{45+10M} z^4 + 2 q^{46+10M} z^4 - \\
& 3 q^{47+10M} z^4 + q^{48+10M} z^4 - q^{16+4M} z^5 - 2 q^{18+4M} z^5 - 3 q^{20+4M} z^5 - 3 q^{22+4M} z^5 - 2 q^{24+4M} z^5 - q^{26+4M} z^5 - \\
& q^{34+8M} z^5 + q^{35+8M} z^5 + q^{36+8M} z^5 + q^{37+8M} z^5 + q^{38+8M} z^5 + 2 q^{39+8M} z^5 + q^{41+8M} z^5 + q^{27+6M} z^6) A[3, M] - \\
& q^{36+24M} (q^{20+4M} + q^{23+4M} + q^{25+4M} - q^{40+8M} - q^{42+8M} - q^{44+8M} - q^{9+2M} z - 2 q^{11+2M} z - 2 q^{13+2M} z - \\
& q^{14+2M} z - 2 q^{15+2M} z - q^{16+2M} z - 2 q^{17+2M} z - q^{18+2M} z - q^{19+2M} z - q^{20+2M} z - q^{22+2M} z + q^{30+6M} z + \\
& 2 q^{31+6M} z + 4 q^{33+6M} z + q^{34+6M} z + 2 q^{35+6M} z + 4 q^{36+6M} z + 3 q^{38+6M} z + q^{40+6M} z + q^{49+10M} z - \\
& 2 q^{50+10M} z + 3 q^{51+10M} z - 4 q^{52+10M} z + 2 q^{53+10M} z - 2 q^{54+10M} z - q^{56+10M} z + z^2 + q^2 z^2 + 2 q^4 z^2 + \\
& 2 q^6 z^2 + 3 q^8 z^2 + 2 q^{10} z^2 + 2 q^{12} z^2 + q^{14} z^2 + q^{16} z^2 + q^{18+4M} z^2 + 2 q^{20+4M} z^2 - q^{21+4M} z^2 + 2 q^{22+4M} z^2 - \\
& q^{24+4M} z^2 - 2 q^{26+4M} z^2 - 4 q^{28+4M} z^2 - 3 q^{30+4M} z^2 - 2 q^{32+4M} z^2 - q^{34+4M} z^2 - q^{38+8M} z^2 - q^{39+8M} z^2 - \\
& 3 q^{40+8M} z^2 - 4 q^{42+8M} z^2 - q^{43+8M} z^2 + q^{44+8M} z^2 - 4 q^{45+8M} z^2 + 4 q^{46+8M} z^2 - 2 q^{47+8M} z^2 + q^{48+8M} z^2 + \\
& q^{50+8M} z^2 + 2 q^{59+12M} z^2 - 3 q^{60+12M} z^2 + 5 q^{61+12M} z^2 - 3 q^{62+12M} z^2 + 2 q^{63+12M} z^2 - q^{9+2M} z^3 - \\
& 2 q^{11+2M} z^3 - 3 q^{13+2M} z^3 - 3 q^{15+2M} z^3 - 3 q^{17+2M} z^3 - 2 q^{19+2M} z^3 - q^{21+2M} z^3 + q^{30+6M} z^3 + q^{31+6M} z^3 + \\
& 2 q^{32+6M} z^3 + 3 q^{33+6M} z^3 + 2 q^{34+6M} z^3 + 3 q^{35+6M} z^3 + 2 q^{36+6M} z^3 + 2 q^{37+6M} z^3 + q^{38+6M} z^3 + q^{39+6M} z^3 + \\
& q^{49+10M} z^3 - q^{50+10M} z^3 + q^{51+10M} z^3 - 2 q^{52+10M} z^3 - q^{54+10M} z^3 - q^{55+10M} z^3 + q^{20+4M} z^4 + q^{22+4M} z^4 + \\
& q^{24+4M} z^4 - q^{40+8M} z^4 - q^{41+8M} z^4 - q^{43+8M} z^4) A[4, M] - q^{64+30M} (-1 + q^{5+M}) (1 + q^{5+M}) (1 + q^{10+2M}) \\
& (-q^{11+2M} - q^{13+2M} - q^{16+2M} + q^{34+6M} + q^{36+6M} + q^{38+6M} + z + q^2 z + q^4 z + q^6 z + q^8 z + q^{10} z + \\
& q^{22+4M} z - q^{23+4M} z - q^{24+4M} z - 2 q^{26+4M} z - q^{28+4M} z - q^{30+4M} z - q^{32+4M} z - q^{45+8M} z + 2 q^{46+8M} z - \\
& 2 q^{47+8M} z + q^{48+8M} z - q^{11+2M} z^2 - q^{13+2M} z^2 - q^{15+2M} z^2 + q^{34+6M} z^2 + q^{36+6M} z^2 + q^{37+6M} z^2) A[5, M] + \\
& q^{100+36M} (-1 + q^{5+M}) (1 + q^{5+M}) (-1 + q^{6+M}) (1 + q^{6+M}) (1 + q^{10+2M}) (1 + q^{12+2M}) \\
& (-1 + q^{13+2M}) (1 + q^{13+2M}) A[6, M]) // Simplify
\end{aligned}$$

Out[135]= 0

(*Recurrence for \tilde{T}_M , with M even*)


```

In[136]:= ClearAll[M, N1, N2, N3, N4, N5, z];
summand =
  (-1)^(N1+N2) z^(N1+N3) q^(-N1-M N1+N1^2-N2-2 M N2+2 N1 N2+2 N2^2-3 M N3+2 N1 N3+4 N2 N3+3 N3^2) qPochhammer[-z q^(1+2 N2+2 N3-M),
    q^2, M-N1-N2-2 N3] / (qPochhammer[q^2, q^2, N1] qPochhammer[q^2, q^2, N2]
    qPochhammer[q^2, q^2, N3] qPochhammer[q^2, q^2, M-N1-N2-2 N3]) /. {M -> 2 M};
stru = qFindStructureSet[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
  {1, 1, 1}, qProtocol -> True];
rec = qFindRecurrence[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
  {1, 1, 1}, qProtocol -> True, StructSet -> stru[[1]]];
sumrec = qSumRecurrence[rec]

Structure set (600 elts.):
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{1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
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 $\{7, 5, 2, 3\}, \{7, 5, 3, 0\}, \{7, 5, 4, 0\}, \{7, 5, 5, 0\}, \{7, 6, 0, 3\}, \{7, 6, 1, 2\}, \{7, 6, 2, 1\},$
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 $\{8, 1, 3, 2\}, \{8, 1, 3, 3\}, \{8, 2, 0, 4\}, \{8, 2, 0, 5\}, \{8, 2, 0, 6\}, \{8, 2, 0, 7\}, \{8, 2, 1, 2\},$
 $\{8, 2, 1, 3\}, \{8, 2, 1, 4\}, \{8, 2, 1, 5\}, \{8, 2, 1, 6\}, \{8, 2, 2, 1\}, \{8, 2, 2, 2\}, \{8, 2, 2, 3\},$
 $\{8, 2, 2, 4\}, \{8, 2, 2, 5\}, \{8, 2, 3, 2\}, \{8, 2, 3, 3\}, \{8, 2, 4, 2\}, \{8, 2, 4, 3\}, \{8, 3, 0, 3\},$
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 $\{8, 4, 1, 2\}, \{8, 4, 1, 3\}, \{8, 4, 1, 4\}, \{8, 4, 1, 5\}, \{8, 4, 2, 1\}, \{8, 4, 2, 2\}, \{8, 4, 2, 3\},$
 $\{8, 4, 2, 4\}, \{8, 4, 2, 5\}, \{8, 4, 3, 1\}, \{8, 4, 3, 2\}, \{8, 4, 4, 1\}, \{8, 4, 4, 2\}, \{8, 4, 5, 2\},$
 $\{8, 4, 6, 2\}, \{8, 5, 0, 4\}, \{8, 5, 0, 5\}, \{8, 5, 1, 3\}, \{8, 5, 1, 4\}, \{8, 5, 1, 5\}, \{8, 5, 2, 2\},$
 $\{8, 5, 2, 3\}, \{8, 5, 2, 4\}, \{8, 5, 3, 1\}, \{8, 5, 4, 1\}, \{8, 5, 5, 1\}, \{8, 6, 0, 4\}, \{8, 6, 1, 3\},$
 $\{8, 6, 2, 2\}, \{8, 6, 2, 4\}, \{8, 6, 3, 1\}, \{8, 6, 4, 0\}, \{8, 6, 5, 1\}, \{8, 6, 6, 1\}, \{9, 1, 3, 4\},$
 $\{9, 2, 0, 8\}, \{9, 2, 1, 6\}, \{9, 2, 1, 7\}, \{9, 2, 2, 4\}, \{9, 2, 2, 5\}, \{9, 2, 2, 6\}, \{9, 2, 3, 3\},$
 $\{9, 2, 3, 4\}, \{9, 2, 4, 3\}, \{9, 2, 4, 4\}, \{9, 3, 0, 6\}, \{9, 3, 0, 7\}, \{9, 3, 1, 4\}, \{9, 3, 1, 5\},$
 $\{9, 3, 1, 6\}, \{9, 3, 1, 7\}, \{9, 3, 2, 2\}, \{9, 3, 2, 3\}, \{9, 3, 2, 4\}, \{9, 3, 2, 5\}, \{9, 3, 2, 6\},$
 $\{9, 3, 3, 2\}, \{9, 3, 3, 3\}, \{9, 3, 4, 3\}, \{9, 3, 5, 3\}, \{9, 4, 0, 4\}, \{9, 4, 0, 5\}, \{9, 4, 0, 6\},$
 $\{9, 4, 0, 7\}, \{9, 4, 1, 3\}, \{9, 4, 1, 4\}, \{9, 4, 1, 5\}, \{9, 4, 1, 6\}, \{9, 4, 2, 2\},$
 $\{9, 4, 2, 3\}, \{9, 4, 2, 4\}, \{9, 4, 2, 5\}, \{9, 4, 2, 6\}, \{9, 4, 3, 2\}, \{9, 4, 3, 3\},$
 $\{9, 4, 4, 2\}, \{9, 4, 4, 3\}, \{9, 4, 5, 3\}, \{9, 5, 0, 5\}, \{9, 5, 0, 6\}, \{9, 5, 1, 4\},$
 $\{9, 5, 1, 5\}, \{9, 5, 1, 6\}, \{9, 5, 2, 3\}, \{9, 5, 2, 4\}, \{9, 5, 2, 5\}, \{9, 5, 3, 2\},$
 $\{9, 5, 4, 2\}, \{9, 5, 5, 2\}, \{9, 6, 0, 5\}, \{9, 6, 1, 4\}, \{9, 6, 2, 3\}, \{9, 6, 2, 5\},$
 $\{9, 6, 3, 2\}, \{9, 6, 4, 1\}, \{9, 6, 5, 2\}, \{9, 6, 6, 2\}, \{10, 2, 4, 4\}, \{10, 3, 1, 8\},$
 $\{10, 3, 2, 6\}, \{10, 3, 2, 7\}, \{10, 3, 3, 4\}, \{10, 3, 4, 4\}, \{10, 3, 5, 4\}, \{10, 4, 0, 8\},$
 $\{10, 4, 1, 6\}, \{10, 4, 1, 7\}, \{10, 4, 2, 4\}, \{10, 4, 2, 5\}, \{10, 4, 2, 6\}, \{10, 4, 2, 7\},$
 $\{10, 4, 3, 3\}, \{10, 4, 3, 4\}, \{10, 4, 4, 3\}, \{10, 4, 4, 4\}, \{10, 4, 5, 4\}, \{10, 5, 0, 6\},$
 $\{10, 5, 0, 7\}, \{10, 5, 1, 5\}, \{10, 5, 1, 6\}, \{10, 5, 1, 7\}, \{10, 5, 2, 4\}, \{10, 5, 2, 5\},$
 $\{10, 5, 2, 6\}, \{10, 5, 3, 3\}, \{10, 5, 4, 3\}, \{10, 5, 5, 3\}, \{10, 6, 0, 6\}, \{10, 6, 1, 5\},$
 $\{10, 6, 2, 4\}, \{10, 6, 2, 6\}, \{10, 6, 3, 3\}, \{10, 6, 4, 2\}, \{10, 6, 5, 3\}, \{10, 6, 6, 3\},$
 $\{11, 4, 2, 8\}, \{11, 4, 4, 4\}, \{11, 4, 4, 5\}, \{11, 4, 5, 5\}, \{11, 5, 1, 8\}, \{11, 5, 2, 6\},$
 $\{11, 5, 2, 7\}, \{11, 5, 3, 4\}, \{11, 5, 4, 4\}, \{11, 5, 5, 4\}, \{11, 6, 1, 6\}, \{11, 6, 2, 5\},$
 $\{11, 6, 2, 7\}, \{11, 6, 3, 4\}, \{11, 6, 4, 3\}, \{11, 6, 5, 4\}, \{11, 6, 6, 4\}, \{12, 5, 5, 5\},$
 $\{12, 6, 2, 8\}, \{12, 6, 4, 4\}, \{12, 6, 5, 5\}, \{12, 6, 6, 5\}, \{13, 6, 5, 6\}, \{13, 6, 6, 6\}$

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 11346x3116

Simplifying 1 solution(s)

Solution exists

Structure set (143 elts.):

```
{ {0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
  {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
  {2, 0, 2, 1}, {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0},
  {2, 2, 0, 1}, {2, 2, 1, 0}, {2, 3, 0, 0}, {3, 0, 0, 1}, {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0},
  {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1}, {3, 0, 2, 2}, {3, 1, 0, 0}, {3, 1, 0, 1},
  {3, 1, 0, 2}, {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0}, {3, 1, 2, 1}, {3, 2, 0, 0},
  {3, 2, 0, 1}, {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0}, {3, 3, 0, 0}, {3, 3, 0, 1},
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  {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 0, 2, 3}, {4, 1, 0, 1}, {4, 1, 0, 2}, {4, 1, 0, 3}, {4, 1, 1, 0},
  {4, 1, 1, 1}, {4, 1, 1, 2}, {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1}, {4, 1, 2, 2}, {4, 2, 0, 0},
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  {5, 0, 2, 3}, {5, 1, 0, 2}, {5, 1, 0, 3}, {5, 1, 0, 4}, {5, 1, 1, 1}, {5, 1, 1, 2}, {5, 1, 1, 3},
  {5, 1, 1, 4}, {5, 1, 2, 1}, {5, 1, 2, 2}, {5, 1, 2, 3}, {5, 2, 0, 1}, {5, 2, 0, 2}, {5, 2, 0, 3},
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  {6, 1, 0, 4}, {6, 1, 0, 5}, {6, 1, 1, 3}, {6, 1, 1, 4}, {6, 1, 2, 2}, {6, 1, 2, 3},
  {6, 2, 0, 3}, {6, 2, 0, 4}, {6, 2, 1, 2}, {6, 2, 1, 3}, {6, 2, 2, 1}, {6, 2, 2, 2},
  {6, 3, 0, 2}, {6, 3, 0, 3}, {6, 3, 1, 1}, {6, 3, 1, 2}, {6, 3, 2, 0}, {6, 3, 2, 1}}
```

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 3104x1095

Simplifying 1 solution(s)

$$\text{Out}[140]= \left\{ q^7 (q^{2+2M} - z) (q^{5+2M} - z) z^3 (-q^{2M} + z) (-1 + q^{2M} z) (-1 + q^{2+2M} z) (-1 + q^{4+2M} z) \text{SUM}[M] - \right. \\ q^{6M} (q^{2+2M} - z) z^2 (-1 + q^{2+2M} z) (-1 + q^{4+2M} z) \\ (q^{11+4M} + q^{13+4M} + q^{15+4M} - q^{4+2M} z - q^{6+2M} z - q^{7+2M} z - q^{8+2M} z - q^{9+2M} z - q^{10+2M} z - q^{11+2M} z - \\ q^{13+2M} z - q^{15+2M} z - q^{17+6M} z + 2 q^{18+6M} z - 2 q^{19+6M} z + q^{20+6M} z + z^2 + q^2 z^2 + q^4 z^2 + q^6 z^2 + \\ q^8 z^2 + q^{10} z^2 + q^{10+4M} z^2 + q^{12+4M} z^2 + q^{15+4M} z^2 - q^{6+2M} z^3 - q^{8+2M} z^3 - q^{10+2M} z^3) \text{SUM}[1+M] + \\ q^{1+12M} z (-1 + q^{4+2M} z) (-q^{19+6M} - q^{21+6M} - q^{23+6M} + q^{10+4M} z + q^{12+4M} z + q^{13+4M} z + 2 q^{14+4M} z + \\ 2 q^{15+4M} z + q^{16+4M} z + 3 q^{17+4M} z + q^{18+4M} z + 3 q^{19+4M} z + 2 q^{21+4M} z + q^{23+4M} z + q^{25+8M} z - 2 q^{26+8M} z + \\ 3 q^{27+8M} z - 4 q^{28+8M} z + 2 q^{29+8M} z - 2 q^{30+8M} z - q^{32+8M} z - q^{4+2M} z^2 - 2 q^{6+2M} z^2 - 3 q^{8+2M} z^2 - \\ q^{9+2M} z^2 - 4 q^{10+2M} z^2 - q^{11+2M} z^2 - 4 q^{12+2M} z^2 - 2 q^{13+2M} z^2 - 3 q^{14+2M} z^2 - 2 q^{15+2M} z^2 - 2 q^{16+2M} z^2 - \\ 2 q^{17+2M} z^2 - q^{18+2M} z^2 - q^{19+2M} z^2 - q^{21+2M} z^2 - q^{16+6M} z^2 - 2 q^{18+6M} z^2 - 2 q^{20+6M} z^2 - 2 q^{21+6M} z^2 + \\ 2 q^{22+6M} z^2 - 4 q^{23+6M} z^2 + 4 q^{24+6M} z^2 - 3 q^{25+6M} z^2 + 3 q^{26+6M} z^2 + q^{28+6M} z^2 + q^{29+6M} z^2 + 2 q^{32+10M} z^2 - \\ 3 q^{33+10M} z^2 + 5 q^{34+10M} z^2 - 3 q^{35+10M} z^2 + 2 q^{36+10M} z^2 + z^3 + q^2 z^3 + 2 q^4 z^3 + 2 q^6 z^3 + 3 q^8 z^3 + \\ 2 q^{10} z^3 + 2 q^{12} z^3 + q^{14} z^3 + q^{16} z^3 + q^{10+4M} z^3 + 3 q^{12+4M} z^3 + 5 q^{14+4M} z^3 + 5 q^{16+4M} z^3 + q^{17+4M} z^3 + \\ 3 q^{18+4M} z^3 + q^{19+4M} z^3 + 2 q^{20+4M} z^3 + q^{21+4M} z^3 - q^{22+4M} z^3 + q^{23+4M} z^3 - q^{24+4M} z^3 + q^{24+8M} z^3 -$$

$$\begin{aligned}
& q^{25+8M} z^3 - 3 q^{28+8M} z^3 + 2 q^{29+8M} z^3 - 4 q^{30+8M} z^3 + q^{31+8M} z^3 - 2 q^{32+8M} z^3 - q^{6+2M} z^4 - 2 q^{8+2M} z^4 - \\
& 3 q^{10+2M} z^4 - 3 q^{12+2M} z^4 - 3 q^{14+2M} z^4 - 2 q^{16+2M} z^4 - q^{18+2M} z^4 - q^{18+6M} z^4 - 2 q^{20+6M} z^4 + q^{21+6M} z^4 - \\
& q^{22+6M} z^4 + q^{24+6M} z^4 + q^{25+6M} z^4 + q^{27+6M} z^4 + q^{14+4M} z^5 + q^{16+4M} z^5 + q^{18+4M} z^5) \text{SUM}[2+M] - \\
& q^{10+18M} (q^{25+6M} - q^{14+4M} z - q^{16+4M} z - q^{17+4M} z - q^{18+4M} z - 2 q^{19+4M} z - q^{20+4M} z - 2 q^{21+4M} z - \\
& 2 q^{23+4M} z - q^{25+4M} z - q^{31+8M} z + 3 q^{32+8M} z - 2 q^{33+8M} z + 3 q^{34+8M} z + 2 q^{36+8M} z + q^{38+8M} z + q^{6+2M} z^2 + \\
& 2 q^{8+2M} z^2 + 3 q^{10+2M} z^2 + q^{11+2M} z^2 + 4 q^{12+2M} z^2 + q^{13+2M} z^2 + 4 q^{14+2M} z^2 + 2 q^{15+2M} z^2 + 3 q^{16+2M} z^2 + \\
& 2 q^{17+2M} z^2 + 2 q^{18+2M} z^2 + 2 q^{19+2M} z^2 + q^{20+2M} z^2 + q^{21+2M} z^2 + q^{23+2M} z^2 + q^{20+6M} z^2 + 2 q^{22+6M} z^2 + \\
& q^{24+6M} z^2 + 2 q^{25+6M} z^2 - 3 q^{26+6M} z^2 + 3 q^{27+6M} z^2 - 5 q^{28+6M} z^2 + q^{29+6M} z^2 - 4 q^{30+6M} z^2 - 2 q^{31+6M} z^2 - \\
& q^{32+6M} z^2 - 3 q^{33+6M} z^2 - q^{35+6M} z^2 - 3 q^{38+10M} z^2 + 3 q^{39+10M} z^2 - 7 q^{40+10M} z^2 + 5 q^{41+10M} z^2 - \\
& 6 q^{42+10M} z^2 + 3 q^{43+10M} z^2 - 2 q^{44+10M} z^2 + q^{45+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 3 q^6 z^3 - 3 q^8 z^3 - \\
& 3 q^{10} z^3 - 3 q^{12} z^3 - 2 q^{14} z^3 - q^{16} z^3 - q^{18} z^3 - q^{12+4M} z^3 - 3 q^{14+4M} z^3 - 6 q^{16+4M} z^3 - 6 q^{18+4M} z^3 - \\
& q^{19+4M} z^3 - 6 q^{20+4M} z^3 - q^{21+4M} z^3 - 2 q^{22+4M} z^3 - 2 q^{23+4M} z^3 + q^{24+4M} z^3 - q^{25+4M} z^3 + 2 q^{26+4M} z^3 - \\
& q^{27+4M} z^3 + 2 q^{28+4M} z^3 + q^{30+4M} z^3 - q^{28+8M} z^3 + q^{29+8M} z^3 + q^{31+8M} z^3 + 5 q^{32+8M} z^3 - q^{33+8M} z^3 + \\
& 9 q^{34+8M} z^3 - q^{35+8M} z^3 + 6 q^{36+8M} z^3 + 3 q^{37+8M} z^3 + 3 q^{39+8M} z^3 - q^{40+8M} z^3 - q^{45+12M} z^3 + 4 q^{46+12M} z^3 - \\
& 5 q^{47+12M} z^3 + 5 q^{48+12M} z^3 - 4 q^{49+12M} z^3 + q^{50+12M} z^3 + q^{6+2M} z^4 + 2 q^{8+2M} z^4 + 4 q^{10+2M} z^4 + 5 q^{12+2M} z^4 + \\
& 6 q^{14+2M} z^4 + 5 q^{16+2M} z^4 + 4 q^{18+2M} z^4 + 2 q^{20+2M} z^4 + q^{22+2M} z^4 + q^{20+6M} z^4 + 3 q^{22+6M} z^4 - q^{23+6M} z^4 + \\
& 3 q^{24+6M} z^4 - q^{25+6M} z^4 + q^{26+6M} z^4 - 2 q^{27+6M} z^4 - 2 q^{28+6M} z^4 - 2 q^{29+6M} z^4 - 3 q^{30+6M} z^4 - 2 q^{31+6M} z^4 - \\
& 2 q^{32+6M} z^4 - q^{33+6M} z^4 - q^{34+6M} z^4 - q^{37+10M} z^4 - q^{38+10M} z^4 - 4 q^{40+10M} z^4 + 2 q^{41+10M} z^4 - 3 q^{42+10M} z^4 + \\
& q^{43+10M} z^4 - q^{14+4M} z^5 - 2 q^{16+4M} z^5 - 3 q^{18+4M} z^5 - 3 q^{20+4M} z^5 - 2 q^{22+4M} z^5 - q^{24+4M} z^5 - q^{30+8M} z^5 + \\
& q^{31+8M} z^5 + q^{32+8M} z^5 + q^{33+8M} z^5 + q^{34+8M} z^5 + 2 q^{35+8M} z^5 + q^{37+8M} z^5 + q^{24+6M} z^6) \text{SUM}[3+M] - \\
& q^{27+24M} (q^{18+4M} + q^{21+4M} + q^{23+4M} - q^{36+8M} - q^{38+8M} - q^{40+8M} - q^{8+2M} z - 2 q^{10+2M} z - 2 q^{12+2M} z - \\
& q^{13+2M} z - 2 q^{14+2M} z - q^{15+2M} z - 2 q^{16+2M} z - q^{17+2M} z - q^{18+2M} z - q^{19+2M} z - q^{21+2M} z + q^{27+6M} z + \\
& 2 q^{28+6M} z + 4 q^{30+6M} z + q^{31+6M} z + 2 q^{32+6M} z + 4 q^{33+6M} z + 3 q^{35+6M} z + q^{37+6M} z + q^{44+10M} z - \\
& 2 q^{45+10M} z + 3 q^{46+10M} z - 4 q^{47+10M} z + 2 q^{48+10M} z - 2 q^{49+10M} z - q^{51+10M} z + z^2 + q^2 z^2 + 2 q^4 z^2 + \\
& 2 q^6 z^2 + 3 q^8 z^2 + 2 q^{10} z^2 + 2 q^{12} z^2 + q^{14} z^2 + q^{16} z^2 + q^{16+4M} z^2 + 2 q^{18+4M} z^2 - q^{19+4M} z^2 + \\
& 2 q^{20+4M} z^2 - q^{22+4M} z^2 - 2 q^{24+4M} z^2 - 4 q^{26+4M} z^2 - 3 q^{28+4M} z^2 - 2 q^{30+4M} z^2 - q^{32+4M} z^2 - \\
& q^{34+8M} z^2 - q^{35+8M} z^2 - 3 q^{36+8M} z^2 - 4 q^{38+8M} z^2 - q^{39+8M} z^2 + q^{40+8M} z^2 - 4 q^{41+8M} z^2 + 4 q^{42+8M} z^2 - \\
& 2 q^{43+8M} z^2 + q^{44+8M} z^2 + q^{46+8M} z^2 + 2 q^{53+12M} z^2 - 3 q^{54+12M} z^2 + 5 q^{55+12M} z^2 - 3 q^{56+12M} z^2 + \\
& 2 q^{57+12M} z^2 - q^{8+2M} z^3 - 2 q^{10+2M} z^3 - 3 q^{12+2M} z^3 - 3 q^{14+2M} z^3 - 3 q^{16+2M} z^3 - 2 q^{18+2M} z^3 - \\
& q^{20+2M} z^3 + q^{27+6M} z^3 + q^{28+6M} z^3 + 2 q^{29+6M} z^3 + 3 q^{30+6M} z^3 + 2 q^{31+6M} z^3 + 3 q^{32+6M} z^3 + 2 q^{33+6M} z^3 + \\
& 2 q^{34+6M} z^3 + q^{35+6M} z^3 + q^{36+6M} z^3 + q^{44+10M} z^3 - q^{45+10M} z^3 + q^{46+10M} z^3 - 2 q^{47+10M} z^3 - q^{49+10M} z^3 - \\
& q^{50+10M} z^3 + q^{18+4M} z^4 + q^{20+4M} z^4 + q^{22+4M} z^4 - q^{36+8M} z^4 - q^{37+8M} z^4 - q^{39+8M} z^4) \text{SUM}[4+M] - \\
& q^{52+30M} (-1 + q^{9+2M}) (1 + q^{9+2M}) (-q^{10+2M} - q^{12+2M} - q^{15+2M} + q^{31+6M} + q^{33+6M} + q^{35+6M} + \\
& z + q^2 z + q^4 z + q^6 z + q^8 z + q^{10} z + q^{20+4M} z - q^{21+4M} z - q^{22+4M} z - 2 q^{24+4M} z - \\
& q^{26+4M} z - q^{28+4M} z - q^{30+4M} z - q^{41+8M} z + 2 q^{42+8M} z - 2 q^{43+8M} z + q^{44+8M} z - \\
& q^{10+2M} z^2 - q^{12+2M} z^2 - q^{14+2M} z^2 + q^{31+6M} z^2 + q^{33+6M} z^2 + q^{34+6M} z^2) \text{SUM}[5+M] + \\
& q^{85+36M} (-1 + q^{6+M}) (1 + q^{6+M}) (-1 + q^{9+2M}) (1 + q^{9+2M}) (-1 + q^{11+2M}) \\
& (1 + q^{11+2M}) (1 + q^{12+2M}) \text{SUM}[6+M] = 0\}
\end{aligned}$$

(*Verifying the recurrence for even M*)

In[141]:= ClearAll[M, A, z];

MyQP[A_, q_, n_] := $\prod_{k=0}^{n-1} (1 - A q^k)$;

A[k_, M_] := $\frac{z^k q^{-3(M+k)^2} \text{MyQP}[z q^{2M}, q^2, k] \text{MyQP}[z^{-1} q^{2M}, q^2, k]}{q^{-3M^2} \text{MyQP}[-q^{2M+1}, -q, 2k]}$;

(q^7 (q^{2+2M} - z) (q^{5+2M} - z) z^3 (-q^{2M} + z) (-1 + q^{2M} z) (-1 + q^{2+2M} z) (-1 + q^{4+2M} z) A[0, M] - q^{6M} (q^{2+2M} - z) z^2 (-1 + q^{2+2M} z) (-1 + q^{4+2M} z) (q^{11+4M} + q^{13+4M} + q^{15+4M} - q^{4+2M} z - q^{6+2M} z - q^{7+2M} z - q^{8+2M} z - q^{9+2M} z - q^{10+2M} z - q^{11+2M} z - q^{13+2M} z - q^{15+2M} z - q^{17+6M} z + 2 q^{18+6M} z - 2 q^{19+6M} z + q^{20+6M} z + z^2 +

$$\begin{aligned}
& q^2 z^2 + q^4 z^2 + q^6 z^2 + q^8 z^2 + q^{10} z^2 + q^{10+4M} z^2 + q^{12+4M} z^2 + q^{15+4M} z^2 - q^{6+2M} z^3 - q^{8+2M} z^3 - q^{10+2M} z^3) \\
& A[1, M] + q^{1+12M} z (-1 + q^{4+2M} z) (-q^{19+6M} - q^{21+6M} - q^{23+6M} + q^{10+4M} z + q^{12+4M} z + q^{13+4M} z + \\
& 2 q^{14+4M} z + 2 q^{15+4M} z + q^{16+4M} z + 3 q^{17+4M} z + q^{18+4M} z + 3 q^{19+4M} z + 2 q^{21+4M} z + q^{23+4M} z + q^{25+8M} z - \\
& 2 q^{26+8M} z + 3 q^{27+8M} z - 4 q^{28+8M} z + 2 q^{29+8M} z - 2 q^{30+8M} z - q^{32+8M} z - q^{4+2M} z^2 - 2 q^{6+2M} z^2 - \\
& 3 q^{8+2M} z^2 - q^{9+2M} z^2 - 4 q^{10+2M} z^2 - q^{11+2M} z^2 - 4 q^{12+2M} z^2 - 2 q^{13+2M} z^2 - 3 q^{14+2M} z^2 - 2 q^{15+2M} z^2 - \\
& 2 q^{16+2M} z^2 - 2 q^{17+2M} z^2 - q^{18+2M} z^2 - q^{19+2M} z^2 - q^{21+2M} z^2 - q^{16+6M} z^2 - 2 q^{18+6M} z^2 - 2 q^{20+6M} z^2 - \\
& 2 q^{21+6M} z^2 + 2 q^{22+6M} z^2 - 4 q^{23+6M} z^2 + 4 q^{24+6M} z^2 - 3 q^{25+6M} z^2 + 3 q^{26+6M} z^2 + q^{28+6M} z^2 + q^{29+6M} z^2 + \\
& 2 q^{32+10M} z^2 - 3 q^{33+10M} z^2 + 5 q^{34+10M} z^2 - 3 q^{35+10M} z^2 + 2 q^{36+10M} z^2 + z^3 + q^2 z^3 + 2 q^4 z^3 + 2 q^6 z^3 + \\
& 3 q^8 z^3 + 2 q^{10} z^3 + 2 q^{12} z^3 + q^{14} z^3 + q^{16} z^3 + q^{10+4M} z^3 + 3 q^{12+4M} z^3 + 5 q^{14+4M} z^3 + 5 q^{16+4M} z^3 + \\
& q^{17+4M} z^3 + 3 q^{18+4M} z^3 + q^{19+4M} z^3 + 2 q^{20+4M} z^3 + q^{21+4M} z^3 - q^{22+4M} z^3 + q^{23+4M} z^3 - q^{24+4M} z^3 + \\
& q^{24+8M} z^3 - q^{25+8M} z^3 - 3 q^{28+8M} z^3 + 2 q^{29+8M} z^3 - 4 q^{30+8M} z^3 + q^{31+8M} z^3 - 2 q^{32+8M} z^3 - q^{6+2M} z^4 - \\
& 2 q^{8+2M} z^4 - 3 q^{10+2M} z^4 - 3 q^{12+2M} z^4 - 3 q^{14+2M} z^4 - 2 q^{16+2M} z^4 - q^{18+2M} z^4 - q^{18+6M} z^4 - 2 q^{20+6M} z^4 + \\
& q^{21+6M} z^4 - q^{22+6M} z^4 + q^{24+6M} z^4 + q^{25+6M} z^4 + q^{27+6M} z^4 + q^{14+4M} z^5 + q^{16+4M} z^5 + q^{18+4M} z^5) A[2, M] - \\
& q^{10+18M} (q^{25+6M} - q^{14+4M} z - q^{16+4M} z - q^{17+4M} z - q^{18+4M} z - 2 q^{19+4M} z - q^{20+4M} z - 2 q^{21+4M} z - \\
& 2 q^{23+4M} z - q^{25+4M} z - q^{31+8M} z + 3 q^{32+8M} z - 2 q^{33+8M} z + 3 q^{34+8M} z + 2 q^{36+8M} z + q^{38+8M} z + q^{6+2M} z^2 + \\
& 2 q^{8+2M} z^2 + 3 q^{10+2M} z^2 + q^{11+2M} z^2 + 4 q^{12+2M} z^2 + q^{13+2M} z^2 + 4 q^{14+2M} z^2 + 2 q^{15+2M} z^2 + 3 q^{16+2M} z^2 + \\
& 2 q^{17+2M} z^2 + 2 q^{18+2M} z^2 + 2 q^{19+2M} z^2 + q^{20+2M} z^2 + q^{21+2M} z^2 + q^{23+2M} z^2 + q^{20+6M} z^2 + 2 q^{22+6M} z^2 + \\
& q^{24+6M} z^2 + 2 q^{25+6M} z^2 - 3 q^{26+6M} z^2 + 3 q^{27+6M} z^2 - 5 q^{28+6M} z^2 + q^{29+6M} z^2 - 4 q^{30+6M} z^2 - 2 q^{31+6M} z^2 - \\
& q^{32+6M} z^2 - 3 q^{33+6M} z^2 - q^{35+6M} z^2 - 3 q^{38+10M} z^2 + 3 q^{39+10M} z^2 - 7 q^{40+10M} z^2 + 5 q^{41+10M} z^2 - \\
& 6 q^{42+10M} z^2 + 3 q^{43+10M} z^2 - 2 q^{44+10M} z^2 + q^{45+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 3 q^6 z^3 - 3 q^8 z^3 - \\
& 3 q^{10} z^3 - 3 q^{12} z^3 - 2 q^{14} z^3 - q^{16} z^3 - q^{18} z^3 - q^{12+4M} z^3 - 3 q^{14+4M} z^3 - 6 q^{16+4M} z^3 - 6 q^{18+4M} z^3 - \\
& q^{19+4M} z^3 - 6 q^{20+4M} z^3 - q^{21+4M} z^3 - 2 q^{22+4M} z^3 - 2 q^{23+4M} z^3 + q^{24+4M} z^3 - q^{25+4M} z^3 + 2 q^{26+4M} z^3 - \\
& q^{27+4M} z^3 + 2 q^{28+4M} z^3 + q^{30+4M} z^3 - q^{28+8M} z^3 + q^{29+8M} z^3 + q^{31+8M} z^3 + 5 q^{32+8M} z^3 - q^{33+8M} z^3 + \\
& 9 q^{34+8M} z^3 - q^{35+8M} z^3 + 6 q^{36+8M} z^3 + 3 q^{37+8M} z^3 + 3 q^{39+8M} z^3 - q^{40+8M} z^3 - q^{45+12M} z^3 + 4 q^{46+12M} z^3 - \\
& 5 q^{47+12M} z^3 + 5 q^{48+12M} z^3 - 4 q^{49+12M} z^3 + q^{50+12M} z^3 + q^{6+2M} z^4 + 2 q^{8+2M} z^4 + 4 q^{10+2M} z^4 + 5 q^{12+2M} z^4 + \\
& 6 q^{14+2M} z^4 + 5 q^{16+2M} z^4 + 4 q^{18+2M} z^4 + 2 q^{20+2M} z^4 + q^{22+2M} z^4 + q^{20+6M} z^4 + 3 q^{22+6M} z^4 - q^{23+6M} z^4 + \\
& 3 q^{24+6M} z^4 - q^{25+6M} z^4 + q^{26+6M} z^4 - 2 q^{27+6M} z^4 - 2 q^{28+6M} z^4 - 2 q^{29+6M} z^4 - 3 q^{30+6M} z^4 - 2 q^{31+6M} z^4 - \\
& 2 q^{32+6M} z^4 - q^{33+6M} z^4 - q^{34+6M} z^4 - q^{37+10M} z^4 - q^{38+10M} z^4 - 4 q^{40+10M} z^4 + 2 q^{41+10M} z^4 - 3 q^{42+10M} z^4 + \\
& q^{43+10M} z^4 - q^{14+4M} z^5 - 2 q^{16+4M} z^5 - 3 q^{18+4M} z^5 - 3 q^{20+4M} z^5 - 2 q^{22+4M} z^5 - q^{24+4M} z^5 - q^{30+8M} z^5 + \\
& q^{31+8M} z^5 + q^{32+8M} z^5 + q^{33+8M} z^5 + q^{34+8M} z^5 + 2 q^{35+8M} z^5 + q^{37+8M} z^5 + q^{24+6M} z^6) A[3, M] - \\
& q^{27+24M} (q^{18+4M} + q^{21+4M} + q^{23+4M} - q^{36+8M} - q^{38+8M} - q^{40+8M} - q^{8+2M} z - 2 q^{10+2M} z - 2 q^{12+2M} z - \\
& q^{13+2M} z - 2 q^{14+2M} z - q^{15+2M} z - 2 q^{16+2M} z - q^{17+2M} z - q^{18+2M} z - q^{19+2M} z - q^{21+2M} z + q^{27+6M} z + \\
& 2 q^{28+6M} z + 4 q^{30+6M} z + q^{31+6M} z + 2 q^{32+6M} z + 4 q^{33+6M} z + 3 q^{35+6M} z + q^{37+6M} z + q^{44+10M} z - \\
& 2 q^{45+10M} z + 3 q^{46+10M} z - 4 q^{47+10M} z + 2 q^{48+10M} z - 2 q^{49+10M} z - q^{51+10M} z + z^2 + q^2 z^2 + 2 q^4 z^2 + \\
& 2 q^6 z^2 + 3 q^8 z^2 + 2 q^{10} z^2 + 2 q^{12} z^2 + q^{14} z^2 + q^{16} z^2 + q^{16+4M} z^2 + 2 q^{18+4M} z^2 - q^{19+4M} z^2 + \\
& 2 q^{20+4M} z^2 - q^{22+4M} z^2 - 2 q^{24+4M} z^2 - 4 q^{26+4M} z^2 - 3 q^{28+4M} z^2 - 2 q^{30+4M} z^2 - q^{32+4M} z^2 - q^{34+8M} z^2 - \\
& q^{35+8M} z^2 - 3 q^{36+8M} z^2 - 4 q^{38+8M} z^2 - q^{39+8M} z^2 + q^{40+8M} z^2 - 4 q^{41+8M} z^2 + 4 q^{42+8M} z^2 - 2 q^{43+8M} z^2 + \\
& q^{44+8M} z^2 + q^{46+8M} z^2 + 2 q^{53+12M} z^2 - 3 q^{54+12M} z^2 + 5 q^{55+12M} z^2 - 3 q^{56+12M} z^2 + 2 q^{57+12M} z^2 - \\
& q^{8+2M} z^3 - 2 q^{10+2M} z^3 - 3 q^{12+2M} z^3 - 3 q^{14+2M} z^3 - 3 q^{16+2M} z^3 - 2 q^{18+2M} z^3 - q^{20+2M} z^3 + q^{27+6M} z^3 + \\
& q^{28+6M} z^3 + 2 q^{29+6M} z^3 + 3 q^{30+6M} z^3 + 2 q^{31+6M} z^3 + 3 q^{32+6M} z^3 + 2 q^{33+6M} z^3 + 2 q^{34+6M} z^3 + q^{35+6M} z^3 + \\
& q^{36+6M} z^3 + q^{44+10M} z^3 - q^{45+10M} z^3 + q^{46+10M} z^3 - 2 q^{47+10M} z^3 - q^{49+10M} z^3 - q^{50+10M} z^3 + q^{18+4M} z^4 + \\
& q^{20+4M} z^4 + q^{22+4M} z^4 - q^{36+8M} z^4 - q^{37+8M} z^4 - q^{39+8M} z^4) A[4, M] - q^{52+30M} (-1 + q^{9+2M}) (1 + q^{9+2M}) \\
& (-q^{10+2M} - q^{12+2M} - q^{15+2M} + q^{31+6M} + q^{33+6M} + q^{35+6M} + z + q^2 z + q^4 z + q^6 z + q^8 z + q^{10} z + \\
& q^{20+4M} z - q^{21+4M} z - q^{22+4M} z - 2 q^{24+4M} z - q^{26+4M} z - q^{28+4M} z - q^{30+4M} z - q^{41+8M} z + 2 q^{42+8M} z - \\
& 2 q^{43+8M} z + q^{44+8M} z - q^{10+2M} z^2 - q^{12+2M} z^2 - q^{14+2M} z^2 + q^{31+6M} z^2 + q^{33+6M} z^2 + q^{34+6M} z^2) A[5, M] + \\
& q^{85+36M} (-1 + q^{6+M}) (1 + q^{6+M}) (-1 + q^{9+2M}) (1 + q^{9+2M}) (-1 + q^{11+2M}) \\
& (1 + q^{11+2M}) (1 + q^{12+2M}) A[6, M]) // Simplify
\end{aligned}$$

Out[143]= 0

(*Verifying the Rogers-Ramanujan type identity*)

```
In[ ]:= MyQP[A_, q_, n_] :=  $\prod_{k=0}^{n-1} (1 - A q^k);$ 
```

```
f[k_] := QPochhammer[q^k, q^k];
```

```
ClearAll[M, N1, N2, N3, N4, N5, N6, z1, z2];
```

```
SumMax = 5;
```

```
PowerMax = 20;
```

```
S1 = Normal[Series[ $\sum_{N1=0}^{SumMax} \sum_{N2=0}^{SumMax} \sum_{N3=0}^{SumMax} \sum_{N4=0}^{SumMax} \sum_{N5=0}^{SumMax} \sum_{N6=0}^{SumMax} (-1)^{N1+N2+N6} z1^{N1+N3+N4+N6} z2^{N2+N3+N5+N6} q^{N1^2+N2^2+N3^2+N4^2+N5^2+N6^2+N1 N2+N1 N3+N1 N5+N1 N6+N2 N3+N2 N4+N2 N6+N3 N4+N3 N5+2 N3 N6+N4 N5+N4 N6+N5 N6-(N1+N2+N6)}$   
/ (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3]  
MyQP[q^2, q^2, N4] MyQP[q^2, q^2, N5] MyQP[q^2, q^2, N6]), {q, 0, PowerMax}]]];
```

```
S2 = Normal[Series[ $\frac{QPochhammer[z1 z2, -q]}{QPochhammer[-q, q^2]} \sum_{M=0}^{SumMax} \frac{q^{M^2} MyQP[z1^2, q^2, M] MyQP[z2^2, q^2, M]}{MyQP[q^2, q^2, M] MyQP[z1 z2, -q, 2 M]}$   
 $\frac{QPochhammer[-z1 z2, -q]}{QPochhammer[-q, q^2]} \sum_{M=0}^{SumMax} \frac{(z1 + z2) q^{M^2+2 M} MyQP[z1^2, q^2, M] MyQP[z2^2, q^2, M]}{MyQP[q^2, q^2, M] MyQP[-z1 z2, -q, 2 M + 1]}$ , {q,  
0, PowerMax}]]];
```

```
S1 -
```

```
S2 //
```

```
Simplify
```

```
Out[ ]:= 0
```

```
In[ ]:= MyQP[A_, q_, n_] :=  $\prod_{k=0}^{n-1} (1 - A q^k);$ 
```

```
f[k_] := QPochhammer[q^k, q^k];
```

```
ClearAll[M, N1, N2, N3, N4, N5, N6, z1, z2];
```

```
SumMax = 5;
```

```
PowerMax = 20;
```

```
z2 = 1;
```

```
S1 = Normal[Series[ $\sum_{N1=0}^{SumMax} \sum_{N2=0}^{SumMax} \sum_{N3=0}^{SumMax} \sum_{N4=0}^{SumMax} \sum_{N5=0}^{SumMax} \sum_{N6=0}^{SumMax} (-1)^{N1+N2+N6} z1^{N1+N3+N4+N6} z2^{N2+N3+N5+N6} q^{N1^2+N2^2+N3^2+N4^2+N5^2+N6^2+N1 N2+N1 N3+N1 N5+N1 N6+N2 N3+N2 N4+N2 N6+N3 N4+N3 N5+2 N3 N6+N4 N5+N4 N6+N5 N6-(N1+N2+N6)}$   
/ (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3]  
MyQP[q^2, q^2, N4] MyQP[q^2, q^2, N5] MyQP[q^2, q^2, N6]), {q, 0, PowerMax}]]];
```

```
S2 = Normal[Series[ $\frac{QPochhammer[z1, -q]}{QPochhammer[-q, q^2]} - \frac{QPochhammer[-z1, -q]}{QPochhammer[-q, q^2]}$ , {q, 0, PowerMax}]]];
```

```
S1 - S2 // Simplify
```

```
Out[ ]:= 0
```

```

In[ ]:= MyQP[A_, q_, n_] :=  $\prod_{k=0}^{n-1} (1 - A q^k)$ ;

f[k_] := QPochhammer[q^k, q^k];

ClearAll[M, N1, N2, N3, N4, N5, N6, z1, z2];

SumMax = 5;
PowerMax = 20;

z2 = -1;

S1 = Normal[Series[ $\sum_{N1=0}^{SumMax} \sum_{N2=0}^{SumMax} \sum_{N3=0}^{SumMax} \sum_{N4=0}^{SumMax} \sum_{N5=0}^{SumMax} \sum_{N6=0}^{SumMax} (-1)^{N1+N2+N6} z1^{N1+N3+N4+N6} z2^{N2+N3+N5+N6}$ 
 $q^{N1^2+N2^2+N3^2+N4^2+N5^2+N6^2+N1 N2+N1 N3+N1 N5+N1 N6+N2 N3+N2 N4+N2 N6+N3 N4+N3 N5+2 N3 N6+N4 N5+N4 N6+N5 N6-(N1+N2+N6)}$ 
/ (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3]
MyQP[q^2, q^2, N4] MyQP[q^2, q^2, N5] MyQP[q^2, q^2, N6]), {q, 0, PowerMax}]]];

S2 = Normal[Series[ $\frac{QPochhammer[-z1, -q]}{QPochhammer[-q, q^2]} + \frac{QPochhammer[z1, -q]}{QPochhammer[-q, q^2]}$ , {q, 0, PowerMax}]]];

S1 - S2 // Simplify

Out[ ]:= 0

```