In[1]:= `SetOptions[SelectedNotebook[],`
 `PrintingStyleEnvironment → "Printout", ShowSyntaxStyles → True]`

`(*Importing RISC packages*)`

In[2]:= `<< RISC`qMultiSum``
`<< RISC`qGeneratingFunctions``

> qMultiSum Package version 2.54
> written by Axel Riese
> Copyright Research Institute for Symbolic Computation (RISC),
> Johannes Kepler University, Linz, Austria

> qGeneratingFunctions Package version 1.9.1
> written by Christoph Koutschan
> Copyright Research Institute for Symbolic Computation (RISC),
> Johannes Kepler University, Linz, Austria

`(*Initial values of T_M, using the original definition*)`

In[*]:= $\text{MyQP}[A\_, q\_, n\_] := \prod_{k=0}^{n-1} \left(1 - A\, q^k\right);$

`ClearAll[M, N1, N2, N3, N4, N5, z1, z2];`

$$\text{For}\Big[M = 0,\ M \le 10,\ M\text{++},$$

$$S = \sum_{N3=0}^{\text{Floor}\left[\frac{M}{2}\right]} \sum_{N1=0}^{M-2\,N3} \sum_{N2=0}^{M-2\,N3-N1} \sum_{N4=0}^{M-2\,N3-N1-N2} z1^{N1+N3+N4}\ z2^{N2+N3+(M-N1-N2-2\,N3-N4)}$$

$$q^{N1^2+N2^2+N3^2+N4^2+(M-N1-N2-2\,N3-N4)^2+N1\,N2+N1\,N3+N1\,(M-N1-N2-2\,N3-N4)+N2\,N3+N2\,N4+N3\,N4+N3\,(M-N1-N2-2\,N3-N4)+N4\,(M-N1-N2-2\ldots}$$

$$\Big/\ \Big(\text{MyQP}\big[q^2, q^2, N1\big]\ \text{MyQP}\big[q^2, q^2, N2\big]\ \text{MyQP}\big[q^2, q^2, N3\big]$$

$$\text{MyQP}\big[q^2, q^2, N4\big]\ \text{MyQP}\big[q^2, q^2, (M - N1 - N2 - 2\,N3 - N4)\big]\Big]\Big);$$

 `Print[Factor[S]];`
 `]`

1

$$-\frac{z1 + z2}{-1 + q}$$

$$\frac{q\,(z1 + z2)^2}{(-1 + q)^2\,(1 + q)}$$

$$-\frac{q^2\,(z1 + z2)\,(q\,z1 + z2)\,(z1 + q\,z2)}{(-1 + q)^3\,(1 + q)\,(1 + q + q^2)}$$

$$\frac{q^4\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(z1 + q^2\,z2)}{(-1 + q)^4\,(1 + q)^2\,(1 + q^2)\,(1 + q + q^2)}$$

$$-\frac{q^6\,(z1 + z2)\,(q\,z1 + z2)\,(q^3\,z1 + z2)\,(z1 + q\,z2)\,(z1 + q^3\,z2)}{(-1 + q)^5\,(1 + q)^2\,(1 + q^2)\,(1 + q + q^2)\,(1 + q + q^2 + q^3 + q^4)}$$

$$\frac{q^9\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(q^4\,z1 + z2)\,(z1 + q^2\,z2)\,(z1 + q^4\,z2)}{(-1 + q)^6\,(1 + q)^3\,(1 + q^2)\,(1 - q + q^2)\,(1 + q + q^2)^2\,(1 + q + q^2 + q^3 + q^4)}$$

$$-\Big(\big(q^{12}\,(z1 + z2)\,(q\,z1 + z2)\,(q^3\,z1 + z2)\,(q^5\,z1 + z2)\,(z1 + q\,z2)\,(z1 + q^3\,z2)\,(z1 + q^5\,z2)\big)\,/$$
$$\big((-1 + q)^7\,(1 + q)^3\,(1 + q^2)\,(1 - q + q^2)\,(1 + q + q^2)^2\,(1 + q + q^2 + q^3 + q^4)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)\Big)$$

$$\big(q^{16}\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(q^4\,z1 + z2)\,(q^6\,z1 + z2)\,(z1 + q^2\,z2)\,(z1 + q^4\,z2)\,(z1 + q^6\,z2)\big)\,/$$
$$\big((-1 + q)^8\,(1 + q)^4\,(1 + q^2)^2\,(1 - q + q^2)\,(1 + q + q^2)^2$$
$$(1 + q^4)\,(1 + q + q^2 + q^3 + q^4)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)$$

$$-\Big(\big(q^{20}\,(z1 + z2)\,(q\,z1 + z2)\,(q^3\,z1 + z2)\,(q^5\,z1 + z2)\,(q^7\,z1 + z2)\,(z1 + q\,z2)$$
$$(z1 + q^3\,z2)\,(z1 + q^5\,z2)\,(z1 + q^7\,z2)\big)\,/\,\big((-1 + q)^9\,(1 + q)^4\,(1 + q^2)^2\,(1 - q + q^2)$$
$$(1 + q + q^2)^3\,(1 + q^4)\,(1 + q + q^2 + q^3 + q^4)\,(1 + q^3 + q^6)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)\Big)$$

$$\big(q^{25}\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(q^4\,z1 + z2)$$
$$(q^6\,z1 + z2)\,(q^8\,z1 + z2)\,(z1 + q^2\,z2)\,(z1 + q^4\,z2)\,(z1 + q^6\,z2)\,(z1 + q^8\,z2)\big)\,/$$
$$\big((-1 + q)^{10}\,(1 + q)^5\,(1 + q^2)^2\,(1 - q + q^2)\,(1 + q + q^2)^3\,(1 + q^4)\,(1 - q + q^2 - q^3 + q^4)$$
$$(1 + q + q^2 + q^3 + q^4)^2\,(1 + q^3 + q^6)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)$$

```
(*Initial values of T_M, using the rewritten formula*)
```

```mathematica
In[•]:= MyQP[A_, q_, n_] := Product[(1 - A q^k), {k, 0, n-1}];

ClearAll[M, N1, N2, N3, N4, N5, z1, z2];

SList = {};

For[M = 0, M ≤ 10, M++,
  S = Sum[ z1^(N1+N3) z2^(M-N1-N3) q^(M^2-N1-M N1+N1^2-N2-2 M N2+2 N1 N2+2 N2^2-3 M N3+2 N1 N3+4 N2 N3+3 N3^2)
      MyQP[-z1 z2^-1 q^(1+2 N2+2 N3-M), q^2, M - N1 - N2 - 2 N3] /
      (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3] MyQP[q^2, q^2, M - N1 - N2 - 2 N3]),
    {N3, 0, Floor[M/2]}, {N1, 0, M-2 N3}, {N2, 0, M-2 N3-N1}];
  Print[Factor[S]];
  SList = Append[SList, Factor[S]];
]
```

1

$$-\frac{z1 + z2}{-1 + q}$$

$$\frac{q\,(z1 + z2)^2}{(-1 + q)^2\,(1 + q)}$$

$$-\frac{q^2\,(z1 + z2)\,(q\,z1 + z2)\,(z1 + q\,z2)}{(-1 + q)^3\,(1 + q)\,(1 + q + q^2)}$$

$$\frac{q^4\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(z1 + q^2\,z2)}{(-1 + q)^4\,(1 + q)^2\,(1 + q^2)\,(1 + q + q^2)}$$

$$-\frac{q^6\,(z1 + z2)\,(q\,z1 + z2)\,(q^3\,z1 + z2)\,(z1 + q\,z2)\,(z1 + q^3\,z2)}{(-1 + q)^5\,(1 + q)^2\,(1 + q^2)\,(1 + q + q^2)\,(1 + q + q^2 + q^3 + q^4)}$$

$$\frac{q^9\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(q^4\,z1 + z2)\,(z1 + q^2\,z2)\,(z1 + q^4\,z2)}{(-1 + q)^6\,(1 + q)^3\,(1 + q^2)\,(1 - q + q^2)\,(1 + q + q^2)^2\,(1 + q + q^2 + q^3 + q^4)}$$

$$-\Big(\big(q^{12}\,(z1 + z2)\,(q\,z1 + z2)\,(q^3\,z1 + z2)\,(q^5\,z1 + z2)\,(z1 + q\,z2)\,(z1 + q^3\,z2)\,(z1 + q^5\,z2)\big)\,/$$
$$\big((-1 + q)^7\,(1 + q)^3\,(1 + q^2)\,(1 - q + q^2)\,(1 + q + q^2)^2\,(1 + q + q^2 + q^3 + q^4)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)\Big)$$

$$\big(q^{16}\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(q^4\,z1 + z2)\,(q^6\,z1 + z2)\,(z1 + q^2\,z2)\,(z1 + q^4\,z2)\,(z1 + q^6\,z2)\big)\,/$$
$$\big((-1 + q)^8\,(1 + q)^4\,(1 + q^2)^2\,(1 - q + q^2)\,(1 + q + q^2)^2$$
$$(1 + q^4)\,(1 + q + q^2 + q^3 + q^4)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)$$

$$-\Big(\big(q^{20}\,(z1 + z2)\,(q\,z1 + z2)\,(q^3\,z1 + z2)\,(q^5\,z1 + z2)\,(q^7\,z1 + z2)\,(z1 + q\,z2)$$
$$(z1 + q^3\,z2)\,(z1 + q^5\,z2)\,(z1 + q^7\,z2)\big)\,/\,\big((-1 + q)^9\,(1 + q)^4\,(1 + q^2)^2\,(1 - q + q^2)$$
$$(1 + q + q^2)^3\,(1 + q^4)\,(1 + q + q^2 + q^3 + q^4)\,(1 + q^3 + q^6)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)\Big)$$

$$\big(q^{25}\,(z1 + z2)^2\,(q^2\,z1 + z2)\,(q^4\,z1 + z2)$$
$$(q^6\,z1 + z2)\,(q^8\,z1 + z2)\,(z1 + q^2\,z2)\,(z1 + q^4\,z2)\,(z1 + q^6\,z2)\,(z1 + q^8\,z2)\big)\,/$$
$$\big((-1 + q)^{10}\,(1 + q)^5\,(1 + q^2)^2\,(1 - q + q^2)\,(1 + q + q^2)^3\,(1 + q^4)\,(1 - q + q^2 - q^3 + q^4)$$
$$(1 + q + q^2 + q^3 + q^4)^2\,(1 + q^3 + q^6)\,(1 + q + q^2 + q^3 + q^4 + q^5 + q^6)\big)$$

```
(*Recurrence for T_M, with M odd*)
```

```
In[112]:= ClearAll[M, N1, N2, N3, z];
summand = z^(N1+N3) q^(-N1-M N1+N1^2-N2-2 M N2+2 N1 N2+2 N2^2-3 M N3+2 N1 N3+4 N2 N3+3 N3^2)
    qPochhammer[-z q^(1+2 N2+2 N3-M), q^2, M - N1 - N2 - 2 N3] /
    (qPochhammer[q^2, q^2, N1] qPochhammer[q^2, q^2, N2] qPochhammer[q^2, q^2, N3]
       qPochhammer[q^2, q^2, M - N1 - N2 - 2 N3]) /. {M → 2 M + 1};
stru = qFindStructureSet[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
    {1, 1, 1}, qProtocol → True];
rec = qFindRecurrence[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
    {1, 1, 1}, qProtocol → True, StructSet → stru[[1]]];
sumrec = qSumRecurrence[rec]
```

```
Structure set (600 elts.):
 {{0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
  {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
  {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0}, {2, 2, 0, 1},
  {2, 2, 1, 0}, {2, 2, 2, 0}, {2, 3, 0, 0}, {2, 3, 1, 0}, {2, 4, 0, 0}, {3, 0, 0, 0}, {3, 0, 0, 1},
  {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0}, {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1},
  {3, 1, 0, 0}, {3, 1, 0, 1}, {3, 1, 0, 2}, {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0},
```

{3, 1, 2, 1}, {3, 2, 0, 0}, {3, 2, 0, 1}, {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0},
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{3, 4, 0, 1}, {3, 4, 1, 0}, {3, 4, 2, 0}, {3, 5, 0, 0}, {3, 5, 1, 0}, {4, 0, 0, 0}, {4, 0, 0, 1},
{4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4}, {4, 0, 1, 0}, {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3},
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{4, 3, 1, 1}, {4, 3, 1, 2}, {4, 3, 2, 0}, {4, 3, 2, 1}, {4, 4, 0, 0}, {4, 4, 0, 1}, {4, 4, 0, 2},
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{5, 0, 0, 3}, {5, 0, 0, 4}, {5, 0, 1, 0}, {5, 0, 1, 1}, {5, 0, 1, 2}, {5, 0, 1, 3}, {5, 0, 2, 0},
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{5, 1, 2, 2}, {5, 1, 2, 3}, {5, 1, 3, 0}, {5, 2, 0, 0}, {5, 2, 0, 1}, {5, 2, 0, 2}, {5, 2, 0, 3},
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{5, 5, 1, 1}, {5, 5, 1, 2}, {5, 5, 2, 0}, {5, 5, 2, 1}, {5, 6, 0, 1}, {5, 6, 1, 0}, {5, 6, 2, 1},
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{6, 2, 0, 3}, {6, 2, 0, 4}, {6, 2, 0, 5}, {6, 2, 1, 0}, {6, 2, 1, 1}, {6, 2, 1, 2}, {6, 2, 1, 3},
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{7, 2, 0, 3}, {7, 2, 0, 4}, {7, 2, 0, 5}, {7, 2, 0, 6}, {7, 2, 1, 0}, {7, 2, 1, 1}, {7, 2, 1, 2},
{7, 2, 1, 3}, {7, 2, 1, 4}, {7, 2, 1, 5}, {7, 2, 2, 0}, {7, 2, 2, 1}, {7, 2, 2, 2}, {7, 2, 2, 3},
{7, 2, 2, 4}, {7, 2, 3, 1}, {7, 2, 3, 2}, {7, 2, 4, 1}, {7, 2, 4, 2}, {7, 3, 0, 2}, {7, 3, 0, 3},
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{7, 4, 1, 2}, {7, 4, 1, 3}, {7, 4, 1, 4}, {7, 4, 2, 0}, {7, 4, 2, 1}, {7, 4, 2, 2}, {7, 4, 2, 3},
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```
   {8, 1, 3, 2}, {8, 1, 3, 3}, {8, 2, 0, 4}, {8, 2, 0, 5}, {8, 2, 0, 6}, {8, 2, 0, 7}, {8, 2, 1, 2},
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   {8, 3, 1, 6}, {8, 3, 2, 1}, {8, 3, 2, 2}, {8, 3, 2, 3}, {8, 3, 2, 4}, {8, 3, 2, 5}, {8, 3, 3, 1},
   {8, 3, 3, 2}, {8, 3, 4, 2}, {8, 3, 5, 2}, {8, 4, 0, 3}, {8, 4, 0, 4}, {8, 4, 0, 5}, {8, 4, 0, 6},
   {8, 4, 1, 2}, {8, 4, 1, 3}, {8, 4, 1, 4}, {8, 4, 1, 5}, {8, 4, 2, 1}, {8, 4, 2, 2}, {8, 4, 2, 3},
   {8, 4, 2, 4}, {8, 4, 2, 5}, {8, 4, 3, 1}, {8, 4, 3, 2}, {8, 4, 4, 1}, {8, 4, 4, 2}, {8, 4, 5, 2},
   {8, 4, 6, 2}, {8, 5, 0, 4}, {8, 5, 0, 5}, {8, 5, 1, 3}, {8, 5, 1, 4}, {8, 5, 1, 5}, {8, 5, 2, 2},
   {8, 5, 2, 3}, {8, 5, 2, 4}, {8, 5, 3, 1}, {8, 5, 4, 1}, {8, 5, 5, 1}, {8, 6, 0, 4}, {8, 6, 1, 3},
   {8, 6, 2, 2}, {8, 6, 2, 4}, {8, 6, 3, 1}, {8, 6, 4, 0}, {8, 6, 5, 1}, {8, 6, 6, 1}, {9, 1, 3, 4},
   {9, 2, 0, 8}, {9, 2, 1, 6}, {9, 2, 1, 7}, {9, 2, 2, 4}, {9, 2, 2, 5}, {9, 2, 2, 6}, {9, 2, 3, 3},
   {9, 2, 3, 4}, {9, 2, 4, 3}, {9, 2, 4, 4}, {9, 3, 0, 6}, {9, 3, 0, 7}, {9, 3, 1, 4}, {9, 3, 1, 5},
   {9, 3, 1, 6}, {9, 3, 1, 7}, {9, 3, 2, 2}, {9, 3, 2, 3}, {9, 3, 2, 4}, {9, 3, 2, 5}, {9, 3, 2, 6},
   {9, 3, 3, 2}, {9, 3, 3, 3}, {9, 3, 4, 3}, {9, 3, 5, 3}, {9, 4, 0, 4}, {9, 4, 0, 5}, {9, 4, 0, 6},
   {9, 4, 0, 7}, {9, 4, 1, 3}, {9, 4, 1, 4}, {9, 4, 1, 5}, {9, 4, 1, 6}, {9, 4, 2, 2},
   {9, 4, 2, 3}, {9, 4, 2, 4}, {9, 4, 2, 5}, {9, 4, 2, 6}, {9, 4, 3, 2}, {9, 4, 3, 3},
   {9, 4, 4, 2}, {9, 4, 4, 3}, {9, 4, 5, 3}, {9, 5, 0, 5}, {9, 5, 0, 6}, {9, 5, 1, 4},
   {9, 5, 1, 5}, {9, 5, 1, 6}, {9, 5, 2, 3}, {9, 5, 2, 4}, {9, 5, 2, 5}, {9, 5, 3, 2},
   {9, 5, 4, 2}, {9, 5, 5, 2}, {9, 6, 0, 5}, {9, 6, 1, 4}, {9, 6, 2, 3}, {9, 6, 2, 5},
   {9, 6, 3, 2}, {9, 6, 4, 1}, {9, 6, 5, 2}, {9, 6, 6, 2}, {10, 2, 4, 4}, {10, 3, 1, 8},
   {10, 3, 2, 6}, {10, 3, 2, 7}, {10, 3, 3, 4}, {10, 3, 4, 4}, {10, 3, 5, 4}, {10, 4, 0, 8},
   {10, 4, 1, 6}, {10, 4, 1, 7}, {10, 4, 2, 4}, {10, 4, 2, 5}, {10, 4, 2, 6}, {10, 4, 2, 7},
   {10, 4, 3, 3}, {10, 4, 3, 4}, {10, 4, 4, 3}, {10, 4, 4, 4}, {10, 4, 5, 4}, {10, 5, 0, 6},
   {10, 5, 0, 7}, {10, 5, 1, 5}, {10, 5, 1, 6}, {10, 5, 1, 7}, {10, 5, 2, 4}, {10, 5, 2, 5},
   {10, 5, 2, 6}, {10, 5, 3, 3}, {10, 5, 4, 3}, {10, 5, 5, 3}, {10, 6, 0, 6}, {10, 6, 1, 5},
   {10, 6, 2, 4}, {10, 6, 2, 6}, {10, 6, 3, 3}, {10, 6, 4, 2}, {10, 6, 5, 3}, {10, 6, 6, 3},
   {11, 4, 2, 8}, {11, 4, 4, 4}, {11, 4, 4, 5}, {11, 4, 5, 5}, {11, 5, 1, 8}, {11, 5, 2, 6},
   {11, 5, 2, 7}, {11, 5, 3, 4}, {11, 5, 4, 4}, {11, 5, 5, 4}, {11, 6, 1, 6}, {11, 6, 2, 5},
   {11, 6, 2, 7}, {11, 6, 3, 4}, {11, 6, 4, 3}, {11, 6, 5, 4}, {11, 6, 6, 4}, {12, 5, 5, 5},
   {12, 6, 2, 8}, {12, 6, 4, 4}, {12, 6, 5, 5}, {12, 6, 6, 5}, {13, 6, 5, 6}, {13, 6, 6, 6}}
```

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 11346x3116

Simplifying 1 solution(s)

Solution exists

Structure set (143 elts.):
 {{0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
  {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
  {2, 0, 2, 1}, {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0},
  {2, 2, 0, 1}, {2, 2, 1, 0}, {2, 3, 0, 0}, {3, 0, 0, 1}, {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0},
  {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1}, {3, 0, 2, 2}, {3, 1, 0, 0}, {3, 1, 0, 1},
  {3, 1, 0, 2}, {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0}, {3, 1, 2, 1}, {3, 2, 0, 0},
  {3, 2, 0, 1}, {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0}, {3, 3, 0, 0}, {3, 3, 0, 1},
  {3, 3, 1, 0}, {4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4}, {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3},
  {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 0, 2, 3}, {4, 1, 0, 1}, {4, 1, 0, 2}, {4, 1, 0, 3}, {4, 1, 1, 0},
  {4, 1, 1, 1}, {4, 1, 1, 2}, {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1}, {4, 1, 2, 2}, {4, 2, 0, 0},
  {4, 2, 0, 1}, {4, 2, 0, 2}, {4, 2, 0, 3}, {4, 2, 1, 0}, {4, 2, 1, 1}, {4, 2, 1, 2}, {4, 2, 2, 0},
  {4, 2, 2, 1}, {4, 3, 0, 0}, {4, 3, 0, 1}, {4, 3, 0, 2}, {4, 3, 1, 0}, {4, 3, 1, 1}, {4, 3, 2, 0},
  {5, 0, 0, 3}, {5, 0, 0, 4}, {5, 0, 0, 5}, {5, 0, 1, 2}, {5, 0, 1, 3}, {5, 0, 1, 4}, {5, 0, 2, 2},
  {5, 0, 2, 3}, {5, 1, 0, 2}, {5, 1, 0, 3}, {5, 1, 0, 4}, {5, 1, 1, 1}, {5, 1, 1, 2}, {5, 1, 1, 3},
  {5, 1, 1, 4}, {5, 1, 2, 1}, {5, 1, 2, 2}, {5, 1, 2, 3}, {5, 2, 0, 1}, {5, 2, 0, 2}, {5, 2, 0, 3},
  {5, 2, 1, 0}, {5, 2, 1, 1}, {5, 2, 1, 2}, {5, 2, 1, 3}, {5, 2, 2, 0}, {5, 2, 2, 1}, {5, 2, 2, 2},
  {5, 3, 0, 1}, {5, 3, 0, 2}, {5, 3, 1, 0}, {5, 3, 1, 1}, {5, 3, 1, 2}, {5, 3, 2, 0}, {5, 3, 2, 1},
  {6, 0, 0, 5}, {6, 0, 0, 6}, {6, 0, 1, 4}, {6, 0, 1, 5}, {6, 0, 2, 3}, {6, 0, 2, 4},
  {6, 1, 0, 4}, {6, 1, 0, 5}, {6, 1, 1, 3}, {6, 1, 1, 4}, {6, 1, 2, 2}, {6, 1, 2, 3},
  {6, 2, 0, 3}, {6, 2, 0, 4}, {6, 2, 1, 2}, {6, 2, 1, 3}, {6, 2, 2, 1}, {6, 2, 2, 2},
  {6, 3, 0, 2}, {6, 3, 0, 3}, {6, 3, 1, 1}, {6, 3, 1, 2}, {6, 3, 2, 0}, {6, 3, 2, 1}}

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 3104x1095

Simplifying 1 solution(s)

Out[116]= $\{q^4 (q^{6+2M} - z) z^3 (q^{1+2M} + z) (q^{3+2M} + z) (1 + q^{1+2M} z) (1 + q^{3+2M} z) (1 + q^{5+2M} z) \text{SUM}[M] -$
$q^{6M} z^2 (q^{3+2M} + z) (1 + q^{3+2M} z) (1 + q^{5+2M} z)$
$(q^{13+4M} + q^{15+4M} + q^{17+4M} - q^{5+2M} z - q^{7+2M} z + q^{8+2M} z - q^{9+2M} z + q^{10+2M} z - q^{11+2M} z + q^{12+2M} z +$
$q^{14+2M} z + q^{16+2M} z + q^{20+6M} z + 2 q^{21+6M} z + 2 q^{22+6M} z + q^{23+6M} z - z^2 - q^2 z^2 - q^4 z^2 - q^6 z^2 -$
$q^8 z^2 - q^{10} z^2 - q^{12+4M} z^2 - q^{14+4M} z^2 + q^{17+4M} z^2 - q^{7+2M} z^3 - q^{9+2M} z^3 - q^{11+2M} z^3) \text{SUM}[1 + M] +$
$q^{4+12M} z (1 + q^{5+2M} z) (q^{22+6M} + q^{24+6M} + q^{26+6M} - q^{12+4M} z - q^{14+4M} z + q^{15+4M} z - 2 q^{16+4M} z +$
$2 q^{17+4M} z - q^{18+4M} z + 3 q^{19+4M} z - q^{20+4M} z + 3 q^{21+4M} z + 2 q^{23+4M} z + q^{25+4M} z + q^{29+8M} z + 2 q^{30+8M} z +$
$3 q^{31+8M} z + 4 q^{32+8M} z + 2 q^{33+8M} z + 2 q^{34+8M} z + q^{36+8M} z - q^{5+2M} z^2 - 2 q^{7+2M} z^2 - 3 q^{9+2M} z^2 +$
$q^{10+2M} z^2 - 4 q^{11+2M} z^2 + q^{12+2M} z^2 - 4 q^{13+2M} z^2 + 2 q^{14+2M} z^2 - 3 q^{15+2M} z^2 + 2 q^{16+2M} z^2 - 2 q^{17+2M} z^2 +$
$2 q^{18+2M} z^2 - q^{19+2M} z^2 + q^{20+2M} z^2 + q^{22+2M} z^2 - q^{19+6M} z^2 - 2 q^{21+6M} z^2 - 2 q^{23+6M} z^2 + 2 q^{24+6M} z^2 +$
$2 q^{25+6M} z^2 + 4 q^{26+6M} z^2 + 4 q^{27+6M} z^2 + 3 q^{28+6M} z^2 + 3 q^{29+6M} z^2 + q^{31+6M} z^2 - q^{32+6M} z^2 + 2 q^{37+10M} z^2 +$
$3 q^{38+10M} z^2 + 5 q^{39+10M} z^2 + 3 q^{40+10M} z^2 + 2 q^{41+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 2 q^6 z^3 - 3 q^8 z^3 -$
$2 q^{10} z^3 - 2 q^{12} z^3 - q^{14} z^3 - q^{16} z^3 - q^{12+4M} z^3 - 3 q^{14+4M} z^3 - 5 q^{16+4M} z^3 - 5 q^{18+4M} z^3 + q^{19+4M} z^3 -$
$3 q^{20+4M} z^3 + q^{21+4M} z^3 - 2 q^{22+4M} z^3 + q^{23+4M} z^3 + q^{24+4M} z^3 + q^{25+4M} z^3 + q^{26+4M} z^3 - q^{28+8M} z^3 -$
$q^{29+8M} z^3 + 3 q^{32+8M} z^3 + 2 q^{33+8M} z^3 + 4 q^{34+8M} z^3 + q^{35+8M} z^3 + 2 q^{36+8M} z^3 - q^{7+2M} z^4 - 2 q^{9+2M} z^4 -$
$3 q^{11+2M} z^4 - 3 q^{13+2M} z^4 - 3 q^{15+2M} z^4 - 2 q^{17+2M} z^4 - q^{19+2M} z^4 - q^{21+6M} z^4 - 2 q^{23+6M} z^4 - q^{24+6M} z^4 -$
$q^{25+6M} z^4 + q^{27+6M} z^4 - q^{28+6M} z^4 - q^{30+6M} z^4 - q^{16+4M} z^5 - q^{18+4M} z^5 - q^{20+4M} z^5) \text{SUM}[2 + M] -$
$q^{16+18M} (q^{28+6M} - q^{16+4M} z - q^{18+4M} z + q^{19+4M} z - q^{20+4M} z + 2 q^{21+4M} z - q^{22+4M} z + 2 q^{23+4M} z +$
$2 q^{25+4M} z + q^{27+4M} z + q^{35+8M} z + 3 q^{36+8M} z + 2 q^{37+8M} z + 3 q^{38+8M} z + 2 q^{40+8M} z + q^{42+8M} z - q^{7+2M} z^2 -$
$2 q^{9+2M} z^2 - 3 q^{11+2M} z^2 + q^{12+2M} z^2 - 4 q^{13+2M} z^2 + q^{14+2M} z^2 - 4 q^{15+2M} z^2 + 2 q^{16+2M} z^2 - 3 q^{17+2M} z^2 +$
$2 q^{18+2M} z^2 - 2 q^{19+2M} z^2 + 2 q^{20+2M} z^2 - q^{21+2M} z^2 + q^{22+2M} z^2 + q^{24+2M} z^2 - q^{23+6M} z^2 - 2 q^{25+6M} z^2 -$

$q^{27+6M} z^2 + 2 q^{28+6M} z^2 + 3 q^{29+6M} z^2 + 3 q^{30+6M} z^2 + 5 q^{31+6M} z^2 + q^{32+6M} z^2 + 4 q^{33+6M} z^2 - 2 q^{34+6M} z^2 +$
$q^{35+6M} z^2 - 3 q^{36+6M} z^2 - q^{38+6M} z^2 + 3 q^{43+10M} z^2 + 3 q^{44+10M} z^2 + 7 q^{45+10M} z^2 + 5 q^{46+10M} z^2 +$
$6 q^{47+10M} z^2 + 3 q^{48+10M} z^2 + 2 q^{49+10M} z^2 + q^{50+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 3 q^6 z^3 - 3 q^8 z^3 -$
$3 q^{10} z^3 - 3 q^{12} z^3 - 2 q^{14} z^3 - q^{16} z^3 - q^{18} z^3 - q^{14+4M} z^3 - 3 q^{16+4M} z^3 - 6 q^{18+4M} z^3 - 6 q^{20+4M} z^3 +$
$q^{21+4M} z^3 - 6 q^{22+4M} z^3 + q^{23+4M} z^3 - 2 q^{24+4M} z^3 + 2 q^{25+4M} z^3 + q^{26+4M} z^3 + q^{27+4M} z^3 + 2 q^{28+4M} z^3 +$
$q^{29+4M} z^3 + 2 q^{30+4M} z^3 + q^{32+4M} z^3 - q^{32+8M} z^3 - q^{33+8M} z^3 - q^{35+8M} z^3 + 5 q^{36+8M} z^3 + q^{37+8M} z^3 +$
$9 q^{38+8M} z^3 + q^{39+8M} z^3 + 6 q^{40+8M} z^3 - 3 q^{41+8M} z^3 - 3 q^{43+8M} z^3 - q^{44+8M} z^3 + q^{51+12M} z^3 + 4 q^{52+12M} z^3 +$
$5 q^{53+12M} z^3 + 5 q^{54+12M} z^3 + 4 q^{55+12M} z^3 + q^{56+12M} z^3 - q^{7+2M} z^4 - 2 q^{9+2M} z^4 - 4 q^{11+2M} z^4 - 5 q^{13+2M} z^4 -$
$6 q^{15+2M} z^4 - 5 q^{17+2M} z^4 - 4 q^{19+2M} z^4 - 2 q^{21+2M} z^4 - q^{23+2M} z^4 - q^{23+6M} z^4 - 3 q^{25+6M} z^4 - q^{26+6M} z^4 -$
$3 q^{27+6M} z^4 - q^{28+6M} z^4 - q^{29+6M} z^4 - 2 q^{30+6M} z^4 + 2 q^{31+6M} z^4 - 2 q^{32+6M} z^4 + 3 q^{33+6M} z^4 - 2 q^{34+6M} z^4 +$
$2 q^{35+6M} z^4 - q^{36+6M} z^4 + q^{37+6M} z^4 - q^{42+10M} z^4 + q^{43+10M} z^4 + 4 q^{45+10M} z^4 + 2 q^{46+10M} z^4 + 3 q^{47+10M} z^4 +$
$q^{48+10M} z^4 - q^{16+4M} z^5 - 2 q^{18+4M} z^5 - 3 q^{20+4M} z^5 - 3 q^{22+4M} z^5 - 2 q^{24+4M} z^5 - q^{26+4M} z^5 - q^{34+8M} z^5 -$
$q^{35+8M} z^5 + q^{36+8M} z^5 - q^{37+8M} z^5 + q^{38+8M} z^5 - 2 q^{39+8M} z^5 - q^{41+8M} z^5 - q^{27+6M} z^6 \big)$ SUM$[3 + M] +$
$q^{36+24M} \big(-q^{20+4M} + q^{23+4M} + q^{25+4M} + q^{40+8M} + q^{42+8M} + q^{44+8M} - q^{9+2M} z - 2 q^{11+2M} z - 2 q^{13+2M} z +$
$q^{14+2M} z - 2 q^{15+2M} z + q^{16+2M} z - 2 q^{17+2M} z + q^{18+2M} z - q^{19+2M} z + q^{20+2M} z + q^{22+2M} z -$
$q^{30+6M} z + 2 q^{31+6M} z + 4 q^{33+6M} z - q^{34+6M} z + 2 q^{35+6M} z - 4 q^{36+6M} z - 3 q^{38+6M} z - q^{40+6M} z +$
$q^{49+10M} z + 2 q^{50+10M} z + 3 q^{51+10M} z + 4 q^{52+10M} z + 2 q^{53+10M} z + 2 q^{54+10M} z + q^{56+10M} z - z^2 -$
$q^2 z^2 - 2 q^4 z^2 - 2 q^6 z^2 - 3 q^8 z^2 - 2 q^{10} z^2 - 2 q^{12} z^2 - q^{14} z^2 - q^{16} z^2 - q^{18+4M} z^2 - 2 q^{20+4M} z^2 -$
$q^{21+4M} z^2 - 2 q^{22+4M} z^2 + q^{24+4M} z^2 + 2 q^{26+4M} z^2 + 4 q^{28+4M} z^2 + 3 q^{30+4M} z^2 + 2 q^{32+4M} z^2 +$
$q^{34+4M} z^2 + q^{38+8M} z^2 - q^{39+8M} z^2 + 3 q^{40+8M} z^2 + 4 q^{42+8M} z^2 - q^{43+8M} z^2 - q^{44+8M} z^2 - 4 q^{45+8M} z^2 -$
$4 q^{46+8M} z^2 - 2 q^{47+8M} z^2 - q^{48+8M} z^2 - q^{50+8M} z^2 + 2 q^{59+12M} z^2 + 3 q^{60+12M} z^2 + 5 q^{61+12M} z^2 +$
$3 q^{62+12M} z^2 + 2 q^{63+12M} z^2 - q^{9+2M} z^3 - 2 q^{11+2M} z^3 - 3 q^{13+2M} z^3 - 3 q^{15+2M} z^3 - 3 q^{17+2M} z^3 -$
$2 q^{19+2M} z^3 - q^{21+2M} z^3 - q^{30+6M} z^3 + q^{31+6M} z^3 - 2 q^{32+6M} z^3 + 3 q^{33+6M} z^3 - 2 q^{34+6M} z^3 + 3 q^{35+6M} z^3 -$
$2 q^{36+6M} z^3 + 2 q^{37+6M} z^3 - q^{38+6M} z^3 + q^{39+6M} z^3 + q^{49+10M} z^3 + q^{50+10M} z^3 + q^{51+10M} z^3 + 2 q^{52+10M} z^3 +$
$q^{54+10M} z^3 - q^{55+10M} z^3 - q^{20+4M} z^4 - q^{22+4M} z^4 - q^{24+4M} z^4 + q^{40+8M} z^4 - q^{41+8M} z^4 - q^{43+8M} z^4 \big)$
SUM$[4 + M] - q^{64+30M} \big(-1 + q^{5+M} \big) \big(1 + q^{5+M} \big) \big(1 + q^{10+2M} \big)$
$\big(q^{11+2M} + q^{13+2M} - q^{16+2M} + q^{34+6M} + q^{36+6M} + q^{38+6M} + z + q^2 z + q^4 z + q^6 z + q^8 z + q^{10} z + q^{22+4M} z +$
$q^{23+4M} z - q^{24+4M} z - 2 q^{26+4M} z - q^{28+4M} z - q^{30+4M} z - q^{32+4M} z + q^{45+8M} z + 2 q^{46+8M} z + 2 q^{47+8M} z +$
$q^{48+8M} z + q^{11+2M} z^2 + q^{13+2M} z^2 + q^{15+2M} z^2 + q^{34+6M} z^2 + q^{36+6M} z^2 - q^{37+6M} z^2 \big)$ SUM$[5 + M] +$
$q^{100+36M} \big(-1 + q^{5+M} \big) \big(1 + q^{5+M} \big) \big(-1 + q^{6+M} \big) \big(1 + q^{6+M} \big) \big(1 + q^{10+2M} \big) \big(1 + q^{12+2M} \big)$
$\big(-1 + q^{13+2M} \big) \big(1 + q^{13+2M} \big)$ SUM$[6 + M] \; == 0 \}$

(*Verifying the recurrence for odd M*)

In[117]:= `ClearAll[M, A, z];`

$\text{MyQP}[A\_, q\_, n\_] := \prod_{k=0}^{n-1} \big(1 - A\, q^k \big);$

$A[k\_, M\_] := \dfrac{z^k\, q^{-3\,(M+k)\,(M+k+1)}\, \text{MyQP}\big[-z\, q^{2M+1}, q^2, k\big]\, \text{MyQP}\big[-z^{-1}\, q^{2M+1}, q^2, k\big]}{q^{-3M\,(M+1)}\, \text{MyQP}\big[q^{2M+2}, q, 2\,k\big]};$

$\big(q^4 \big(q^{6+2M} - z \big) z^3 \big(q^{1+2M} + z \big) \big(q^{3+2M} + z \big) \big(1 + q^{1+2M} z \big) \big(1 + q^{3+2M} z \big) \big(1 + q^{5+2M} z \big) A[0, M] - q^6 z^2$
$\big(q^{3+2M} + z \big) \big(1 + q^{3+2M} z \big) \big(1 + q^{5+2M} z \big) \big(q^{13+4M} + q^{15+4M} + q^{17+4M} - q^{5+2M} z - q^{7+2M} z + q^{8+2M} z - q^{9+2M} z +$
$q^{10+2M} z - q^{11+2M} z + q^{12+2M} z + q^{14+2M} z + q^{16+2M} z + q^{20+6M} z + 2 q^{21+6M} z + 2 q^{22+6M} z + q^{23+6M} z - z^2 -$
$q^2 z^2 - q^4 z^2 - q^6 z^2 - q^8 z^2 - q^{10} z^2 - q^{12+4M} z^2 - q^{14+4M} z^2 + q^{17+4M} z^2 - q^{7+2M} z^3 - q^{9+2M} z^3 - q^{11+2M} z^3 \big)$
$A[1, M] + q^{4+12M} z \big(1 + q^{5+2M} z \big) \big(q^{22+6M} + q^{24+6M} + q^{26+6M} - q^{12+4M} z - q^{14+4M} z + q^{15+4M} z -$
$2 q^{16+4M} z + 2 q^{17+4M} z - q^{18+4M} z + 3 q^{19+4M} z - q^{20+4M} z + 3 q^{21+4M} z + 2 q^{23+4M} z + q^{25+4M} z + q^{29+8M} z +$
$2 q^{30+8M} z + 3 q^{31+8M} z + 4 q^{32+8M} z + 2 q^{33+8M} z + 2 q^{34+8M} z + q^{36+8M} z - q^{5+2M} z^2 - 2 q^{7+2M} z^2 -$
$3 q^{9+2M} z^2 + q^{10+2M} z^2 - 4 q^{11+2M} z^2 + q^{12+2M} z^2 - 4 q^{13+2M} z^2 + 2 q^{14+2M} z^2 - 3 q^{15+2M} z^2 + 2 q^{16+2M} z^2 -$
$2 q^{17+2M} z^2 + 2 q^{18+2M} z^2 - q^{19+2M} z^2 + q^{20+2M} z^2 + q^{22+2M} z^2 - q^{19+6M} z^2 - 2 q^{21+6M} z^2 - 2 q^{23+6M} z^2 +$
$2 q^{24+6M} z^2 + 2 q^{25+6M} z^2 + 4 q^{26+6M} z^2 + 4 q^{27+6M} z^2 + 3 q^{28+6M} z^2 + 3 q^{29+6M} z^2 + q^{31+6M} z^2 - q^{32+6M} z^2 +$

$$
\begin{aligned}
&2\,q^{37+10\,M}\,z^2 + 3\,q^{38+10\,M}\,z^2 + 5\,q^{39+10\,M}\,z^2 + 3\,q^{40+10\,M}\,z^2 + 2\,q^{41+10\,M}\,z^2 - z^3 - q^2\,z^3 - 2\,q^4\,z^3 - 2\,q^6\,z^3 - \\
&3\,q^8\,z^3 - 2\,q^{10}\,z^3 - 2\,q^{12}\,z^3 - q^{14}\,z^3 - q^{16}\,z^3 - q^{12+4\,M}\,z^3 - 3\,q^{14+4\,M}\,z^3 - 5\,q^{16+4\,M}\,z^3 - 5\,q^{18+4\,M}\,z^3 + \\
&q^{19+4\,M}\,z^3 - 3\,q^{20+4\,M}\,z^3 + q^{21+4\,M}\,z^3 - 2\,q^{22+4\,M}\,z^3 + q^{23+4\,M}\,z^3 + q^{24+4\,M}\,z^3 + q^{25+4\,M}\,z^3 + q^{26+4\,M}\,z^3 - \\
&q^{28+8\,M}\,z^3 - q^{29+8\,M}\,z^3 + 3\,q^{32+8\,M}\,z^3 + 2\,q^{33+8\,M}\,z^3 + 4\,q^{34+8\,M}\,z^3 + q^{35+8\,M}\,z^3 + 2\,q^{36+8\,M}\,z^3 - q^{7+2\,M}\,z^4 - \\
&2\,q^{9+2\,M}\,z^4 - 3\,q^{11+2\,M}\,z^4 - 3\,q^{13+2\,M}\,z^4 - 3\,q^{15+2\,M}\,z^4 - 2\,q^{17+2\,M}\,z^4 - q^{19+2\,M}\,z^4 - q^{21+6\,M}\,z^4 - 2\,q^{23+6\,M}\,z^4 - \\
&q^{24+6\,M}\,z^4 - q^{25+6\,M}\,z^4 + q^{27+6\,M}\,z^4 - q^{28+6\,M}\,z^4 - q^{30+6\,M}\,z^4 - q^{16+4\,M}\,z^5 - q^{18+4\,M}\,z^5 - q^{20+4\,M}\,z^5\Big)\,A[2,\,M] - \\
&q^{16+18\,M}\,\Big(q^{28+6\,M} - q^{16+4\,M}\,z - q^{18+4\,M}\,z + q^{19+4\,M}\,z - q^{20+4\,M}\,z + 2\,q^{21+4\,M}\,z - q^{22+4\,M}\,z + 2\,q^{23+4\,M}\,z + \\
&2\,q^{25+4\,M}\,z + q^{27+4\,M}\,z + q^{35+8\,M}\,z + 3\,q^{36+8\,M}\,z + 2\,q^{37+8\,M}\,z + 3\,q^{38+8\,M}\,z + 2\,q^{40+8\,M}\,z + q^{42+8\,M}\,z - q^{7+2\,M}\,z^2 - \\
&2\,q^{9+2\,M}\,z^2 - 3\,q^{11+2\,M}\,z^2 + q^{12+2\,M}\,z^2 - 4\,q^{13+2\,M}\,z^2 + q^{14+2\,M}\,z^2 - 4\,q^{15+2\,M}\,z^2 + 2\,q^{16+2\,M}\,z^2 - 3\,q^{17+2\,M}\,z^2 + \\
&2\,q^{18+2\,M}\,z^2 - 2\,q^{19+2\,M}\,z^2 + 2\,q^{20+2\,M}\,z^2 - q^{21+2\,M}\,z^2 + q^{22+2\,M}\,z^2 + q^{24+2\,M}\,z^2 - q^{23+6\,M}\,z^2 - 2\,q^{25+6\,M}\,z^2 - \\
&q^{27+6\,M}\,z^2 + 2\,q^{28+6\,M}\,z^2 + 3\,q^{29+6\,M}\,z^2 + 3\,q^{30+6\,M}\,z^2 + 5\,q^{31+6\,M}\,z^2 + q^{32+6\,M}\,z^2 + 4\,q^{33+6\,M}\,z^2 - 2\,q^{34+6\,M}\,z^2 + \\
&q^{35+6\,M}\,z^2 - 3\,q^{36+6\,M}\,z^2 - q^{38+6\,M}\,z^2 + 3\,q^{43+10\,M}\,z^2 + 3\,q^{44+10\,M}\,z^2 + 7\,q^{45+10\,M}\,z^2 + 5\,q^{46+10\,M}\,z^2 + \\
&6\,q^{47+10\,M}\,z^2 + 3\,q^{48+10\,M}\,z^2 + 2\,q^{49+10\,M}\,z^2 + q^{50+10\,M}\,z^2 - z^3 - q^2\,z^3 - 2\,q^4\,z^3 - 3\,q^6\,z^3 - 3\,q^8\,z^3 - \\
&3\,q^{10}\,z^3 - 3\,q^{12}\,z^3 - 2\,q^{14}\,z^3 - q^{16}\,z^3 - q^{18}\,z^3 - q^{14+4\,M}\,z^3 - 3\,q^{16+4\,M}\,z^3 - 6\,q^{18+4\,M}\,z^3 - 6\,q^{20+4\,M}\,z^3 + \\
&q^{21+4\,M}\,z^3 - 6\,q^{22+4\,M}\,z^3 + q^{23+4\,M}\,z^3 - 2\,q^{24+4\,M}\,z^3 + 2\,q^{25+4\,M}\,z^3 + q^{26+4\,M}\,z^3 + q^{27+4\,M}\,z^3 + 2\,q^{28+4\,M}\,z^3 + \\
&q^{29+4\,M}\,z^3 + 2\,q^{30+4\,M}\,z^3 + q^{32+4\,M}\,z^3 - q^{32+8\,M}\,z^3 - q^{33+8\,M}\,z^3 - q^{35+8\,M}\,z^3 + 5\,q^{36+8\,M}\,z^3 + q^{37+8\,M}\,z^3 + \\
&9\,q^{38+8\,M}\,z^3 + q^{39+8\,M}\,z^3 + 6\,q^{40+8\,M}\,z^3 - 3\,q^{41+8\,M}\,z^3 - 3\,q^{43+8\,M}\,z^3 - q^{44+8\,M}\,z^3 + q^{51+12\,M}\,z^3 + 4\,q^{52+12\,M}\,z^3 + \\
&5\,q^{53+12\,M}\,z^3 + 5\,q^{54+12\,M}\,z^3 + 4\,q^{55+12\,M}\,z^3 + q^{56+12\,M}\,z^3 - q^{7+2\,M}\,z^4 - 2\,q^{9+2\,M}\,z^4 - 4\,q^{11+2\,M}\,z^4 - 5\,q^{13+2\,M}\,z^4 - \\
&6\,q^{15+2\,M}\,z^4 - 5\,q^{17+2\,M}\,z^4 - 4\,q^{19+2\,M}\,z^4 - 2\,q^{21+2\,M}\,z^4 - q^{23+2\,M}\,z^4 - q^{23+6\,M}\,z^4 - 3\,q^{25+6\,M}\,z^4 - q^{26+6\,M}\,z^4 - \\
&3\,q^{27+6\,M}\,z^4 - q^{28+6\,M}\,z^4 - q^{29+6\,M}\,z^4 - 2\,q^{30+6\,M}\,z^4 + 2\,q^{31+6\,M}\,z^4 - 2\,q^{32+6\,M}\,z^4 + 3\,q^{33+6\,M}\,z^4 - 2\,q^{34+6\,M}\,z^4 + \\
&2\,q^{35+6\,M}\,z^4 - q^{36+6\,M}\,z^4 + q^{37+6\,M}\,z^4 - q^{42+10\,M}\,z^4 + q^{43+10\,M}\,z^4 + 4\,q^{45+10\,M}\,z^4 + 2\,q^{46+10\,M}\,z^4 + 3\,q^{47+10\,M}\,z^4 + \\
&q^{48+10\,M}\,z^4 - q^{16+4\,M}\,z^5 - 2\,q^{18+4\,M}\,z^5 - 3\,q^{20+4\,M}\,z^5 - 3\,q^{22+4\,M}\,z^5 - 2\,q^{24+4\,M}\,z^5 - q^{26+4\,M}\,z^5 - q^{34+8\,M}\,z^5 - \\
&q^{35+8\,M}\,z^5 + q^{36+8\,M}\,z^5 - q^{37+8\,M}\,z^5 + q^{38+8\,M}\,z^5 - 2\,q^{39+8\,M}\,z^5 - q^{41+8\,M}\,z^5 - q^{27+6\,M}\,z^6\Big)\,A[3,\,M] + \\
&q^{36+24\,M}\,\Big(-q^{20+4\,M} + q^{23+4\,M} + q^{25+4\,M} + q^{40+8\,M} + q^{42+8\,M} + q^{44+8\,M} - q^{9+2\,M}\,z - 2\,q^{11+2\,M}\,z - 2\,q^{13+2\,M}\,z + \\
&q^{14+2\,M}\,z - 2\,q^{15+2\,M}\,z + q^{16+2\,M}\,z - 2\,q^{17+2\,M}\,z + q^{18+2\,M}\,z - q^{19+2\,M}\,z + q^{20+2\,M}\,z + q^{22+2\,M}\,z - q^{30+6\,M}\,z + \\
&2\,q^{31+6\,M}\,z + 4\,q^{33+6\,M}\,z - q^{34+6\,M}\,z + 2\,q^{35+6\,M}\,z - 4\,q^{36+6\,M}\,z - 3\,q^{38+6\,M}\,z - q^{40+6\,M}\,z + q^{49+10\,M}\,z + \\
&2\,q^{50+10\,M}\,z + 3\,q^{51+10\,M}\,z + 4\,q^{52+10\,M}\,z + 2\,q^{53+10\,M}\,z + 2\,q^{54+10\,M}\,z + q^{56+10\,M}\,z - z^2 - q^2\,z^2 - 2\,q^4\,z^2 - \\
&2\,q^6\,z^2 - 3\,q^8\,z^2 - 2\,q^{10}\,z^2 - 2\,q^{12}\,z^2 - q^{14}\,z^2 - q^{16}\,z^2 - q^{18+4\,M}\,z^2 - 2\,q^{20+4\,M}\,z^2 - q^{21+4\,M}\,z^2 - \\
&2\,q^{22+4\,M}\,z^2 + q^{24+4\,M}\,z^2 + 2\,q^{26+4\,M}\,z^2 + 4\,q^{28+4\,M}\,z^2 + 3\,q^{30+4\,M}\,z^2 + 2\,q^{32+4\,M}\,z^2 + q^{34+4\,M}\,z^2 + q^{38+8\,M}\,z^2 - \\
&q^{39+8\,M}\,z^2 + 3\,q^{40+8\,M}\,z^2 + 4\,q^{42+8\,M}\,z^2 - q^{43+8\,M}\,z^2 - q^{44+8\,M}\,z^2 - 4\,q^{45+8\,M}\,z^2 - 4\,q^{46+8\,M}\,z^2 - 2\,q^{47+8\,M}\,z^2 - \\
&q^{48+8\,M}\,z^2 - q^{50+8\,M}\,z^2 + 2\,q^{59+12\,M}\,z^2 + 3\,q^{60+12\,M}\,z^2 + 5\,q^{61+12\,M}\,z^2 + 3\,q^{62+12\,M}\,z^2 + 2\,q^{63+12\,M}\,z^2 - \\
&q^{9+2\,M}\,z^3 - 2\,q^{11+2\,M}\,z^3 - 3\,q^{13+2\,M}\,z^3 - 3\,q^{15+2\,M}\,z^3 - 3\,q^{17+2\,M}\,z^3 - 2\,q^{19+2\,M}\,z^3 - q^{21+2\,M}\,z^3 - q^{30+6\,M}\,z^3 + \\
&q^{31+6\,M}\,z^3 - 2\,q^{32+6\,M}\,z^3 + 3\,q^{33+6\,M}\,z^3 - 2\,q^{34+6\,M}\,z^3 + 3\,q^{35+6\,M}\,z^3 - 2\,q^{36+6\,M}\,z^3 + 2\,q^{37+6\,M}\,z^3 - q^{38+6\,M}\,z^3 + \\
&q^{39+6\,M}\,z^3 + q^{49+10\,M}\,z^3 + q^{50+10\,M}\,z^3 + q^{51+10\,M}\,z^3 + 2\,q^{52+10\,M}\,z^3 + q^{54+10\,M}\,z^3 - q^{55+10\,M}\,z^3 - q^{20+4\,M}\,z^4 - \\
&q^{22+4\,M}\,z^4 - q^{24+4\,M}\,z^4 + q^{40+8\,M}\,z^4 - q^{41+8\,M}\,z^4 - q^{43+8\,M}\,z^4\Big)\,A[4,\,M] - q^{64+30\,M}\,\big(-1 + q^{5+M}\big)\,\big(1 + q^{5+M}\big) \\
&\big(1 + q^{10+2\,M}\big)\,\big(q^{11+2\,M} + q^{13+2\,M} - q^{16+2\,M} + q^{34+6\,M} + q^{36+6\,M} + q^{38+6\,M} + z + q^2\,z + q^4\,z + q^6\,z + q^8\,z + \\
&q^{10}\,z + q^{22+4\,M}\,z + q^{23+4\,M}\,z - q^{24+4\,M}\,z - 2\,q^{26+4\,M}\,z - q^{28+4\,M}\,z - q^{30+4\,M}\,z - q^{32+4\,M}\,z + q^{45+8\,M}\,z + 2\,q^{46+8\,M}\,z + \\
&2\,q^{47+8\,M}\,z + q^{48+8\,M}\,z + q^{11+2\,M}\,z^2 + q^{13+2\,M}\,z^2 + q^{15+2\,M}\,z^2 + q^{34+6\,M}\,z^2 + q^{36+6\,M}\,z^2 - q^{37+6\,M}\,z^2\big)\,A[5,\,M] + \\
&q^{100+36\,M}\,\big(-1 + q^{5+M}\big)\,\big(1 + q^{5+M}\big)\,\big(-1 + q^{6+M}\big)\,\big(1 + q^{6+M}\big)\,\big(1 + q^{10+2\,M}\big)\,\big(1 + q^{12+2\,M}\big) \\
&\big(-1 + q^{13+2\,M}\big)\,\big(1 + q^{13+2\,M}\big)\,A[6,\,M]\Big)\ \ \text{// Simplify}
\end{aligned}
$$

Out[119]= 0

(*Recurrence for T_M, with M even*)

```
In[120]:= ClearAll[M, N1, N2, N3, z];
        summand = z^(N1+N3) q^(-N1-M N1+N1^2-N2-2 M N2+2 N1 N2+2 N2^2-3 M N3+2 N1 N3+4 N2 N3+3 N3^2)
            qPochhammer[-z q^(1+2 N2+2 N3-M), q^2, M - N1 - N2 - 2 N3] / (qPochhammer[q^2, q^2, N1] qPochhammer[q^2,
                q^2, N2] qPochhammer[q^2, q^2, N3] qPochhammer[q^2, q^2, M - N1 - N2 - 2 N3]) /. {M → 2 M};
        stru = qFindStructureSet[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
            {1, 1, 1}, qProtocol → True];
        rec = qFindRecurrence[summand, {M}, {N1, N2, N3}, {2}, {2, 2, 2},
            {1, 1, 1}, qProtocol → True, StructSet → stru[[1]]];
        sumrec = qSumRecurrence[rec]
```

Structure set (600 elts.):
 {{0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
   {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
   {2, 1, 0, 0}, {2, 1, 0, 1}, {2, 1, 1, 0}, {2, 1, 1, 1}, {2, 1, 2, 0}, {2, 2, 0, 0}, {2, 2, 0, 1},
   {2, 2, 1, 0}, {2, 2, 2, 0}, {2, 3, 0, 0}, {2, 3, 1, 0}, {2, 4, 0, 0}, {3, 0, 0, 0}, {3, 0, 0, 1},
   {3, 0, 0, 2}, {3, 0, 0, 3}, {3, 0, 1, 0}, {3, 0, 1, 1}, {3, 0, 1, 2}, {3, 0, 2, 0}, {3, 0, 2, 1},
   {3, 1, 0, 0}, {3, 1, 0, 1}, {3, 1, 0, 2}, {3, 1, 1, 0}, {3, 1, 1, 1}, {3, 1, 1, 2}, {3, 1, 2, 0},
   {3, 1, 2, 1}, {3, 2, 0, 0}, {3, 2, 0, 1}, {3, 2, 0, 2}, {3, 2, 1, 0}, {3, 2, 1, 1}, {3, 2, 2, 0},
   {3, 2, 2, 1}, {3, 3, 0, 0}, {3, 3, 0, 1}, {3, 3, 1, 0}, {3, 3, 1, 1}, {3, 3, 2, 0}, {3, 4, 0, 0},
   {3, 4, 0, 1}, {3, 4, 1, 0}, {3, 4, 2, 0}, {3, 5, 0, 0}, {3, 5, 1, 0}, {4, 0, 0, 0}, {4, 0, 0, 1},
   {4, 0, 0, 2}, {4, 0, 0, 3}, {4, 0, 0, 4}, {4, 0, 1, 0}, {4, 0, 1, 1}, {4, 0, 1, 2}, {4, 0, 1, 3},
   {4, 0, 2, 0}, {4, 0, 2, 1}, {4, 0, 2, 2}, {4, 1, 0, 0}, {4, 1, 0, 1}, {4, 1, 0, 2}, {4, 1, 0, 3},
   {4, 1, 1, 0}, {4, 1, 1, 1}, {4, 1, 1, 2}, {4, 1, 1, 3}, {4, 1, 2, 0}, {4, 1, 2, 1}, {4, 1, 2, 2},
   {4, 2, 0, 0}, {4, 2, 0, 1}, {4, 2, 0, 2}, {4, 2, 0, 3}, {4, 2, 1, 0}, {4, 2, 1, 1}, {4, 2, 1, 2},
   {4, 2, 2, 0}, {4, 2, 2, 1}, {4, 2, 2, 2}, {4, 3, 0, 0}, {4, 3, 0, 1}, {4, 3, 0, 2}, {4, 3, 1, 0},
   {4, 3, 1, 1}, {4, 3, 1, 2}, {4, 3, 2, 0}, {4, 3, 2, 1}, {4, 4, 0, 0}, {4, 4, 0, 1}, {4, 4, 0, 2},
   {4, 4, 1, 0}, {4, 4, 1, 1}, {4, 4, 2, 0}, {4, 4, 2, 1}, {4, 5, 0, 0}, {4, 5, 0, 1}, {4, 5, 1, 0},
   {4, 5, 1, 1}, {4, 5, 2, 0}, {4, 6, 0, 0}, {4, 6, 2, 0}, {5, 0, 0, 0}, {5, 0, 0, 1}, {5, 0, 0, 2},
   {5, 0, 0, 3}, {5, 0, 0, 4}, {5, 0, 1, 0}, {5, 0, 1, 1}, {5, 0, 1, 2}, {5, 0, 1, 3}, {5, 0, 2, 0},
   {5, 0, 2, 1}, {5, 0, 2, 2}, {5, 1, 0, 0}, {5, 1, 0, 1}, {5, 1, 0, 2}, {5, 1, 0, 3}, {5, 1, 0, 4},
   {5, 1, 1, 0}, {5, 1, 1, 1}, {5, 1, 1, 2}, {5, 1, 1, 3}, {5, 1, 1, 4}, {5, 1, 2, 0}, {5, 1, 2, 1},
   {5, 1, 2, 2}, {5, 1, 2, 3}, {5, 1, 3, 0}, {5, 2, 0, 0}, {5, 2, 0, 1}, {5, 2, 0, 2}, {5, 2, 0, 3},
   {5, 2, 0, 4}, {5, 2, 1, 0}, {5, 2, 1, 1}, {5, 2, 1, 2}, {5, 2, 1, 3}, {5, 2, 2, 0}, {5, 2, 2, 1},
   {5, 2, 2, 2}, {5, 2, 2, 3}, {5, 2, 3, 0}, {5, 2, 4, 0}, {5, 3, 0, 0}, {5, 3, 0, 1}, {5, 3, 0, 2},
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   {5, 3, 2, 2}, {5, 4, 0, 0}, {5, 4, 0, 1}, {5, 4, 0, 2}, {5, 4, 0, 3}, {5, 4, 1, 0}, {5, 4, 1, 1},
   {5, 4, 1, 2}, {5, 4, 2, 0}, {5, 4, 2, 1}, {5, 4, 2, 2}, {5, 5, 0, 1}, {5, 5, 0, 2}, {5, 5, 1, 0},
   {5, 5, 1, 1}, {5, 5, 1, 2}, {5, 5, 2, 0}, {5, 5, 2, 1}, {5, 6, 0, 1}, {5, 6, 1, 0}, {5, 6, 2, 1},
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   {6, 0, 1, 1}, {6, 0, 1, 2}, {6, 0, 1, 3}, {6, 0, 1, 4}, {6, 0, 2, 0}, {6, 0, 2, 1}, {6, 0, 2, 2},
   {6, 0, 2, 3}, {6, 1, 0, 0}, {6, 1, 0, 1}, {6, 1, 0, 2}, {6, 1, 0, 3}, {6, 1, 0, 4}, {6, 1, 0, 5},
   {6, 1, 1, 0}, {6, 1, 1, 1}, {6, 1, 1, 2}, {6, 1, 1, 3}, {6, 1, 1, 4}, {6, 1, 2, 0}, {6, 1, 2, 1},
   {6, 1, 2, 2}, {6, 1, 2, 3}, {6, 1, 3, 0}, {6, 1, 3, 1}, {6, 2, 0, 0}, {6, 2, 0, 1}, {6, 2, 0, 2},
   {6, 2, 0, 3}, {6, 2, 0, 4}, {6, 2, 0, 5}, {6, 2, 1, 0}, {6, 2, 1, 1}, {6, 2, 1, 2}, {6, 2, 1, 3},
   {6, 2, 1, 4}, {6, 2, 2, 0}, {6, 2, 2, 1}, {6, 2, 2, 2}, {6, 2, 2, 3}, {6, 2, 2, 4}, {6, 2, 3, 0},
   {6, 2, 3, 1}, {6, 2, 4, 0}, {6, 2, 4, 1}, {6, 3, 0, 1}, {6, 3, 0, 2}, {6, 3, 0, 3}, {6, 3, 0, 4},
   {6, 3, 1, 0}, {6, 3, 1, 1}, {6, 3, 1, 2}, {6, 3, 1, 3}, {6, 3, 1, 4}, {6, 3, 2, 0}, {6, 3, 2, 1},
   {6, 3, 2, 2}, {6, 3, 2, 3}, {6, 3, 3, 0}, {6, 3, 4, 0}, {6, 3, 5, 0}, {6, 4, 0, 1}, {6, 4, 0, 2},
   {6, 4, 0, 3}, {6, 4, 0, 4}, {6, 4, 1, 0}, {6, 4, 1, 1}, {6, 4, 1, 2}, {6, 4, 1, 3}, {6, 4, 2, 0},
   {6, 4, 2, 1}, {6, 4, 2, 2}, {6, 4, 2, 3}, {6, 4, 3, 0}, {6, 4, 4, 0}, {6, 4, 5, 0}, {6, 4, 6, 0},
   {6, 5, 0, 2}, {6, 5, 0, 3}, {6, 5, 1, 1}, {6, 5, 1, 2}, {6, 5, 1, 3}, {6, 5, 2, 0}, {6, 5, 2, 1},

```
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```

Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 11 346x3116

Simplifying 1 solution(s)

Solution exists

Structure set (143 elts.):
  {{0, 0, 0, 0}, {1, 0, 0, 0}, {1, 0, 0, 1}, {1, 0, 1, 0}, {1, 0, 2, 0}, {1, 1, 0, 0}, {1, 1, 1, 0},
   {1, 2, 0, 0}, {2, 0, 0, 0}, {2, 0, 0, 1}, {2, 0, 0, 2}, {2, 0, 1, 0}, {2, 0, 1, 1}, {2, 0, 2, 0},
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Setting up equations

Eliminating trivial equations

Starting equation solver

Dimension of matrix: 3104x1095

Simplifying 1 solution(s)

Out[124]= $\Big\{ q^7 \left(q^{5+2\,M} - z \right) z^3 \left(q^{2\,M} + z \right) \left(q^{2+2\,M} + z \right) \left(1 + q^{2\,M}\,z \right) \left(1 + q^{2+2\,M}\,z \right) \left(1 + q^{4+2\,M}\,z \right) \mathrm{SUM}[M] -$
$\quad q^{6\,M}\,z^2 \left(q^{2+2\,M} + z \right) \left(1 + q^{2+2\,M}\,z \right) \left(1 + q^{4+2\,M}\,z \right)$
$\quad\quad \big(q^{11+4\,M} + q^{13+4\,M} + q^{15+4\,M} - q^{4+2\,M}\,z - q^{6+2\,M}\,z + q^{7+2\,M}\,z - q^{8+2\,M}\,z + q^{9+2\,M}\,z - q^{10+2\,M}\,z + q^{11+2\,M}\,z +$
$\quad\quad\quad q^{13+2\,M}\,z + q^{15+2\,M}\,z + q^{17+6\,M}\,z + 2\,q^{18+6\,M}\,z + 2\,q^{19+6\,M}\,z + q^{20+6\,M}\,z - z^2 - q^2\,z^2 - q^4\,z^2 - q^6\,z^2 -$
$\quad\quad\quad q^8\,z^2 - q^{10}\,z^2 - q^{10+4\,M}\,z^2 - q^{12+4\,M}\,z^2 + q^{15+4\,M}\,z^2 - q^{6+2\,M}\,z^3 - q^{8+2\,M}\,z^3 - q^{10+2\,M}\,z^3 \big) \,\mathrm{SUM}[1+M] +$
$\quad q^{1+12\,M}\,z \left(1 + q^{4+2\,M}\,z \right) \big(q^{19+6\,M} + q^{21+6\,M} + q^{23+6\,M} - q^{10+4\,M}\,z - q^{12+4\,M}\,z + q^{13+4\,M}\,z - 2\,q^{14+4\,M}\,z +$
$\quad\quad\quad 2\,q^{15+4\,M}\,z - q^{16+4\,M}\,z + 3\,q^{17+4\,M}\,z - q^{18+4\,M}\,z + 3\,q^{19+4\,M}\,z + 2\,q^{21+4\,M}\,z + q^{23+4\,M}\,z + q^{25+8\,M}\,z + 2\,q^{26+8\,M}\,z +$
$\quad\quad\quad 3\,q^{27+8\,M}\,z + 4\,q^{28+8\,M}\,z + 2\,q^{29+8\,M}\,z + 2\,q^{30+8\,M}\,z + q^{32+8\,M}\,z - q^{4+2\,M}\,z^2 - 2\,q^{6+2\,M}\,z^2 - 3\,q^{8+2\,M}\,z^2 +$
$\quad\quad\quad q^{9+2\,M}\,z^2 - 4\,q^{10+2\,M}\,z^2 + q^{11+2\,M}\,z^2 - 4\,q^{12+2\,M}\,z^2 + 2\,q^{13+2\,M}\,z^2 - 3\,q^{14+2\,M}\,z^2 + 2\,q^{15+2\,M}\,z^2 - 2\,q^{16+2\,M}\,z^2 +$
$\quad\quad\quad 2\,q^{17+2\,M}\,z^2 - q^{18+2\,M}\,z^2 + q^{19+2\,M}\,z^2 + q^{21+2\,M}\,z^2 - q^{16+6\,M}\,z^2 - 2\,q^{18+6\,M}\,z^2 - 2\,q^{20+6\,M}\,z^2 + 2\,q^{21+6\,M}\,z^2 +$
$\quad\quad\quad 2\,q^{22+6\,M}\,z^2 + 4\,q^{23+6\,M}\,z^2 + 4\,q^{24+6\,M}\,z^2 + 3\,q^{25+6\,M}\,z^2 + 3\,q^{26+6\,M}\,z^2 + q^{28+6\,M}\,z^2 - q^{29+6\,M}\,z^2 + 2\,q^{32+10\,M}\,z^2 +$
$\quad\quad\quad 3\,q^{33+10\,M}\,z^2 + 5\,q^{34+10\,M}\,z^2 + 3\,q^{35+10\,M}\,z^2 + 2\,q^{36+10\,M}\,z^2 - z^3 - q^2\,z^3 - 2\,q^4\,z^3 - 2\,q^6\,z^3 - 3\,q^8\,z^3 -$
$\quad\quad\quad 2\,q^{10}\,z^3 - 2\,q^{12}\,z^3 - q^{14}\,z^3 - q^{16}\,z^3 - q^{10+4\,M}\,z^3 - 3\,q^{12+4\,M}\,z^3 - 5\,q^{14+4\,M}\,z^3 - 5\,q^{16+4\,M}\,z^3 + q^{17+4\,M}\,z^3 -$
$\quad\quad\quad 3\,q^{18+4\,M}\,z^3 + q^{19+4\,M}\,z^3 - 2\,q^{20+4\,M}\,z^3 + q^{21+4\,M}\,z^3 + q^{22+4\,M}\,z^3 + q^{23+4\,M}\,z^3 + q^{24+4\,M}\,z^3 - q^{24+8\,M}\,z^3 -$
$\quad\quad\quad q^{25+8\,M}\,z^3 + 3\,q^{28+8\,M}\,z^3 + 2\,q^{29+8\,M}\,z^3 + 4\,q^{30+8\,M}\,z^3 + q^{31+8\,M}\,z^3 + 2\,q^{32+8\,M}\,z^3 - q^{6+2\,M}\,z^4 - 2\,q^{8+2\,M}\,z^4 -$
$\quad\quad\quad 3\,q^{10+2\,M}\,z^4 - 3\,q^{12+2\,M}\,z^4 - 3\,q^{14+2\,M}\,z^4 - 2\,q^{16+2\,M}\,z^4 - q^{18+2\,M}\,z^4 - q^{18+6\,M}\,z^4 - 2\,q^{20+6\,M}\,z^4 - q^{21+6\,M}\,z^4 -$

$q^{22+6M} z^4 + q^{24+6M} z^4 - q^{25+6M} z^4 - q^{27+6M} z^4 - q^{14+4M} z^5 - q^{16+4M} z^5 - q^{18+4M} z^5)$ SUM$[2+M]$ -

$q^{10+18M} (q^{25+6M} - q^{14+4M} z - q^{16+4M} z + q^{17+4M} z - q^{18+4M} z + 2 q^{19+4M} z - q^{20+4M} z + 2 q^{21+4M} z +$

$\qquad 2 q^{23+4M} z + q^{25+4M} z + q^{31+8M} z + 3 q^{32+8M} z + 2 q^{33+8M} z + 3 q^{34+8M} z + 2 q^{36+8M} z + q^{38+8M} z - q^{6+2M} z^2 -$

$\qquad 2 q^{8+2M} z^2 - 3 q^{10+2M} z^2 + q^{11+2M} z^2 - 4 q^{12+2M} z^2 + q^{13+2M} z^2 - 4 q^{14+2M} z^2 + 2 q^{15+2M} z^2 - 3 q^{16+2M} z^2 +$

$\qquad 2 q^{17+2M} z^2 - 2 q^{18+2M} z^2 + 2 q^{19+2M} z^2 - q^{20+2M} z^2 + q^{21+2M} z^2 + q^{23+2M} z^2 - q^{20+6M} z^2 - 2 q^{22+6M} z^2 -$

$\qquad q^{24+6M} z^2 + 2 q^{25+6M} z^2 + 3 q^{26+6M} z^2 + 3 q^{27+6M} z^2 + 5 q^{28+6M} z^2 + q^{29+6M} z^2 + 4 q^{30+6M} z^2 - 2 q^{31+6M} z^2 +$

$\qquad q^{32+6M} z^2 - 3 q^{33+6M} z^2 - q^{35+6M} z^2 + 3 q^{38+10M} z^2 + 3 q^{39+10M} z^2 + 7 q^{40+10M} z^2 + 5 q^{41+10M} z^2 +$

$\qquad 6 q^{42+10M} z^2 + 3 q^{43+10M} z^2 + 2 q^{44+10M} z^2 + q^{45+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 3 q^6 z^3 - 3 q^8 z^3 -$

$\qquad 3 q^{10} z^3 - 3 q^{12} z^3 - 2 q^{14} z^3 - q^{16} z^3 - q^{18} z^3 - q^{12+4M} z^3 - 3 q^{14+4M} z^3 - 6 q^{16+4M} z^3 - 6 q^{18+4M} z^3 +$

$\qquad q^{19+4M} z^3 - 6 q^{20+4M} z^3 + q^{21+4M} z^3 - 2 q^{22+4M} z^3 + 2 q^{23+4M} z^3 + q^{24+4M} z^3 + q^{25+4M} z^3 + 2 q^{26+4M} z^3 +$

$\qquad q^{27+4M} z^3 + 2 q^{28+4M} z^3 + q^{30+4M} z^3 - q^{28+8M} z^3 - q^{29+8M} z^3 - q^{31+8M} z^3 + 5 q^{32+8M} z^3 + q^{33+8M} z^3 +$

$\qquad 9 q^{34+8M} z^3 + q^{35+8M} z^3 + 6 q^{36+8M} z^3 - 3 q^{37+8M} z^3 - 3 q^{39+8M} z^3 - q^{40+8M} z^3 + q^{45+12M} z^3 + 4 q^{46+12M} z^3 +$

$\qquad 5 q^{47+12M} z^3 + 5 q^{48+12M} z^3 + 4 q^{49+12M} z^3 + q^{50+12M} z^3 - q^{6+2M} z^4 - 2 q^{8+2M} z^4 - 4 q^{10+2M} z^4 - 5 q^{12+2M} z^4 -$

$\qquad 6 q^{14+2M} z^4 - 5 q^{16+2M} z^4 - 4 q^{18+2M} z^4 - 2 q^{20+2M} z^4 - q^{22+2M} z^4 - q^{20+6M} z^4 - 3 q^{22+6M} z^4 - q^{23+6M} z^4 -$

$\qquad 3 q^{24+6M} z^4 - q^{25+6M} z^4 - q^{26+6M} z^4 - 2 q^{27+6M} z^4 + 2 q^{28+6M} z^4 - 2 q^{29+6M} z^4 + 3 q^{30+6M} z^4 - 2 q^{31+6M} z^4 +$

$\qquad 2 q^{32+6M} z^4 - q^{33+6M} z^4 + q^{34+6M} z^4 - q^{37+10M} z^4 + q^{38+10M} z^4 + 4 q^{40+10M} z^4 + 2 q^{41+10M} z^4 + 3 q^{42+10M} z^4 +$

$\qquad q^{43+10M} z^4 - q^{14+4M} z^5 - 2 q^{16+4M} z^5 - 3 q^{18+4M} z^5 - 3 q^{20+4M} z^5 - 2 q^{22+4M} z^5 - q^{24+4M} z^5 - q^{30+8M} z^5 -$

$\qquad q^{31+8M} z^5 + q^{32+8M} z^5 - q^{33+8M} z^5 + q^{34+8M} z^5 - 2 q^{35+8M} z^5 - q^{37+8M} z^5 - q^{24+6M} z^6)$ SUM$[3+M]$ +

$q^{27+24M} (-q^{18+4M} + q^{21+4M} + q^{23+4M} + q^{36+8M} + q^{38+8M} + q^{40+8M} - q^{8+2M} - 2 q^{10+2M} z - 2 q^{12+2M} z +$

$\qquad q^{13+2M} z - 2 q^{14+2M} z + q^{15+2M} z - 2 q^{16+2M} z + q^{17+2M} z - q^{18+2M} z + q^{19+2M} z + q^{21+2M} z - q^{27+6M} z +$

$\qquad 2 q^{28+6M} z + 4 q^{30+6M} z - q^{31+6M} z + 2 q^{32+6M} z - 4 q^{33+6M} z - 3 q^{35+6M} z - q^{37+6M} z + q^{44+10M} z +$

$\qquad 2 q^{45+10M} z + 3 q^{46+10M} z + 4 q^{47+10M} z + 2 q^{48+10M} z + 2 q^{49+10M} z + q^{51+10M} z - z^2 - q^2 z^2 - 2 q^4 z^2 -$

$\qquad 2 q^6 z^2 - 3 q^8 z^2 - 2 q^{10} z^2 - 2 q^{12} z^2 - q^{14} z^2 - q^{16} z^2 - q^{16+4M} z^2 - 2 q^{18+4M} z^2 - q^{19+4M} z^2 -$

$\qquad 2 q^{20+4M} z^2 + q^{22+4M} z^2 + 2 q^{24+4M} z^2 + 4 q^{26+4M} z^2 + 3 q^{28+4M} z^2 + 2 q^{30+4M} z^2 + q^{32+4M} z^2 +$

$\qquad q^{34+8M} z^2 - q^{35+8M} z^2 + 3 q^{36+8M} z^2 + 4 q^{38+8M} z^2 - q^{39+8M} z^2 - q^{40+8M} z^2 - 4 q^{41+8M} z^2 - 4 q^{42+8M} z^2 -$

$\qquad 2 q^{43+8M} z^2 - q^{44+8M} z^2 - q^{46+8M} z^2 + 2 q^{53+12M} z^2 + 3 q^{54+12M} z^2 + 5 q^{55+12M} z^2 + 3 q^{56+12M} z^2 +$

$\qquad 2 q^{57+12M} z^2 - q^{8+2M} z^3 - 2 q^{10+2M} z^3 - 3 q^{12+2M} z^3 - 3 q^{14+2M} z^3 - 3 q^{16+2M} z^3 - 2 q^{18+2M} z^3 -$

$\qquad q^{20+2M} z^3 - q^{27+6M} z^3 + q^{28+6M} z^3 - 2 q^{29+6M} z^3 + 3 q^{30+6M} z^3 - 2 q^{31+6M} z^3 + 3 q^{32+6M} z^3 - 2 q^{33+6M} z^3 +$

$\qquad 2 q^{34+6M} z^3 - q^{35+6M} z^3 + q^{36+6M} z^3 + q^{44+10M} z^3 + q^{45+10M} z^3 + q^{46+10M} z^3 + 2 q^{47+10M} z^3 + q^{49+10M} z^3 -$

$\qquad q^{50+10M} z^3 - q^{18+4M} z^4 - q^{20+4M} z^4 - q^{22+4M} z^4 + q^{36+8M} z^4 - q^{37+8M} z^4 - q^{39+8M} z^4)$ SUM$[4+M]$ -

$q^{52+30M} (-1 + q^{9+2M}) (1 + q^{9+2M}) (q^{10+2M} + q^{12+2M} - q^{15+2M} + q^{31+6M} + q^{33+6M} + q^{35+6M} +$

$\qquad z + q^2 z + q^4 z + q^6 z + q^8 z + q^{10} z + q^{20+4M} z + q^{21+4M} z - q^{22+4M} z - 2 q^{24+4M} z -$

$\qquad q^{26+4M} z - q^{28+4M} z - q^{30+4M} z + q^{41+8M} z + 2 q^{42+8M} z + 2 q^{43+8M} z + q^{44+8M} z +$

$\qquad q^{10+2M} z^2 + q^{12+2M} z^2 + q^{14+2M} z^2 + q^{31+6M} z^2 + q^{33+6M} z^2 - q^{34+6M} z^2)$ SUM$[5+M]$ +

$q^{85+36M} (-1 + q^{6+M}) (1 + q^{6+M}) (-1 + q^{9+2M}) (1 + q^{9+2M}) (-1 + q^{11+2M})$

$\qquad (1 + q^{11+2M}) (1 + q^{12+2M})$ SUM$[6+M]$ $== 0\}$

(*Verifying the recurrence for even M*)

In[130]:= `ClearAll[M, A, z];`

$\text{MyQP}[A\_, q\_, n\_] := \prod_{k=0}^{n-1} (1 - A\,q^k);$

$A[k\_, M\_] := \dfrac{z^k\, q^{-3\,(M+k)^2}\, \text{MyQP}[-z\,q^{2M}, q^2, k]\, \text{MyQP}[-z^{-1}\,q^{2M}, q^2, k]}{q^{-3\,M^2}\, \text{MyQP}[q^{2M+1}, q, 2\,k]};$

$(q^7\,(q^{5+2M} - z)\,z^3\,(q^{2M} + z)\,(q^{2+2M} + z)\,(1 + q^{2M}\,z)\,(1 + q^{2+2M}\,z)\,(1 + q^{4+2M}\,z)\,A[0, M] - q^{6M}\,z^2\,(q^{2+2M} + z)$

$\qquad (1 + q^{2+2M}\,z)\,(1 + q^{4+2M}\,z)\,(q^{11+4M} + q^{13+4M} + q^{15+4M} - q^{4+2M}\,z - q^{6+2M}\,z + q^{7+2M}\,z - q^{8+2M}\,z + q^{9+2M}\,z -$

$\qquad q^{10+2M}\,z + q^{11+2M}\,z + q^{13+2M}\,z + q^{15+2M}\,z + q^{17+6M}\,z + 2\,q^{18+6M}\,z + 2\,q^{19+6M}\,z + q^{20+6M}\,z - z^2 - q^2\,z^2 -$

$\qquad q^4\,z^2 - q^6\,z^2 - q^8\,z^2 - q^{10}\,z^2 - q^{10+4M}\,z^2 - q^{12+4M}\,z^2 + q^{15+4M}\,z^2 - q^{6+2M}\,z^3 - q^{8+2M}\,z^3 - q^{10+2M}\,z^3)$

$\qquad A[1, M] + q^{1+12M}\,z\,(1 + q^{4+2M}\,z)\,(q^{19+6M} + q^{21+6M} + q^{23+6M} - q^{10+4M}\,z - q^{12+4M}\,z + q^{13+4M}\,z -$

$$
\begin{aligned}
&2 q^{14+4M} z + 2 q^{15+4M} z - q^{16+4M} z + 3 q^{17+4M} z - q^{18+4M} z + 3 q^{19+4M} z + 2 q^{21+4M} z + q^{23+4M} z + q^{25+8M} z + \\
&2 q^{26+8M} z + 3 q^{27+8M} z + 4 q^{28+8M} z + 2 q^{29+8M} z + 2 q^{30+8M} z + q^{32+8M} z - q^{4+2M} z^2 - 2 q^{6+2M} z^2 - \\
&3 q^{8+2M} z^2 + q^{9+2M} z^2 - 4 q^{10+2M} z^2 + q^{11+2M} z^2 - 4 q^{12+2M} z^2 + 2 q^{13+2M} z^2 - 3 q^{14+2M} z^2 + 2 q^{15+2M} z^2 - \\
&2 q^{16+2M} z^2 + 2 q^{17+2M} z^2 - q^{18+2M} z^2 + q^{19+2M} z^2 + q^{21+2M} z^2 - q^{16+6M} z^2 - 2 q^{18+6M} z^2 - 2 q^{20+6M} z^2 + \\
&2 q^{21+6M} z^2 + 2 q^{22+6M} z^2 + 4 q^{23+6M} z^2 + 4 q^{24+6M} z^2 + 3 q^{25+6M} z^2 + 3 q^{26+6M} z^2 + q^{28+6M} z^2 - q^{29+6M} z^2 + \\
&2 q^{32+10M} z^2 + 3 q^{33+10M} z^2 + 5 q^{34+10M} z^2 + 3 q^{35+10M} z^2 + 2 q^{36+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 2 q^6 z^3 - \\
&3 q^8 z^3 - 2 q^{10} z^3 - 2 q^{12} z^3 - q^{14} z^3 - q^{16} z^3 - q^{10+4M} z^3 - 3 q^{12+4M} z^3 - 5 q^{14+4M} z^3 - 5 q^{16+4M} z^3 + \\
&q^{17+4M} z^3 - 3 q^{18+4M} z^3 + q^{19+4M} z^3 - 2 q^{20+4M} z^3 + q^{21+4M} z^3 + q^{22+4M} z^3 + q^{23+4M} z^3 + q^{24+4M} z^3 - \\
&q^{24+8M} z^3 - q^{25+8M} z^3 + 3 q^{28+8M} z^3 + 2 q^{29+8M} z^3 + 4 q^{30+8M} z^3 + q^{31+8M} z^3 + 2 q^{32+8M} z^3 - q^{6+2M} z^4 - \\
&2 q^{8+2M} z^4 - 3 q^{10+2M} z^4 - 3 q^{12+2M} z^4 - 3 q^{14+2M} z^4 - 2 q^{16+2M} z^4 - q^{18+2M} z^4 - q^{18+6M} z^4 - 2 q^{20+6M} z^4 - \\
&q^{21+6M} z^4 - q^{22+6M} z^4 + q^{24+6M} z^4 - q^{25+6M} z^4 - q^{27+6M} z^4 - q^{14+4M} z^5 - q^{16+4M} z^5 - q^{18+4M} z^5) \, A[2, M] - \\
&q^{10+18M} \, (q^{25+6M} - q^{14+4M} z - q^{16+4M} z + q^{17+4M} z - q^{18+4M} z + 2 q^{19+4M} z - q^{20+4M} z + 2 q^{21+4M} z + \\
&2 q^{23+4M} z + q^{25+4M} z + q^{31+8M} z + 3 q^{32+8M} z + 2 q^{33+8M} z + 3 q^{34+8M} z + 2 q^{36+8M} z + q^{38+8M} z - q^{6+2M} z^2 - \\
&2 q^{8+2M} z^2 - 3 q^{10+2M} z^2 + q^{11+2M} z^2 - 4 q^{12+2M} z^2 + q^{13+2M} z^2 - 4 q^{14+2M} z^2 + 2 q^{15+2M} z^2 - 3 q^{16+2M} z^2 + \\
&2 q^{17+2M} z^2 - 2 q^{18+2M} z^2 + 2 q^{19+2M} z^2 - q^{20+2M} z^2 + q^{21+2M} z^2 + q^{23+2M} z^2 - q^{20+6M} z^2 - 2 q^{22+6M} z^2 - \\
&q^{24+6M} z^2 + 2 q^{25+6M} z^2 + 3 q^{26+6M} z^2 + 3 q^{27+6M} z^2 + 5 q^{28+6M} z^2 + q^{29+6M} z^2 + 4 q^{30+6M} z^2 - 2 q^{31+6M} z^2 + \\
&q^{32+6M} z^2 - 3 q^{33+6M} z^2 - q^{35+6M} z^2 + 3 q^{38+10M} z^2 + 3 q^{39+10M} z^2 + 7 q^{40+10M} z^2 + 5 q^{41+10M} z^2 + \\
&6 q^{42+10M} z^2 + 3 q^{43+10M} z^2 + 2 q^{44+10M} z^2 + q^{45+10M} z^2 - z^3 - q^2 z^3 - 2 q^4 z^3 - 3 q^6 z^3 - 3 q^8 z^3 - \\
&3 q^{10} z^3 - 3 q^{12} z^3 - 2 q^{14} z^3 - q^{16} z^3 - q^{18} z^3 - q^{12+4M} z^3 - 3 q^{14+4M} z^3 - 6 q^{16+4M} z^3 - 6 q^{18+4M} z^3 + \\
&q^{19+4M} z^3 - 6 q^{20+4M} z^3 + q^{21+4M} z^3 - 2 q^{22+4M} z^3 + 2 q^{23+4M} z^3 + q^{24+4M} z^3 + q^{25+4M} z^3 + 2 q^{26+4M} z^3 + \\
&q^{27+4M} z^3 + 2 q^{28+4M} z^3 + q^{30+4M} z^3 - q^{28+8M} z^3 - q^{29+8M} z^3 - q^{31+8M} z^3 + 5 q^{32+8M} z^3 + q^{33+8M} z^3 + \\
&9 q^{34+8M} z^3 + q^{35+8M} z^3 + 6 q^{36+8M} z^3 - 3 q^{37+8M} z^3 - 3 q^{39+8M} z^3 - q^{40+8M} z^3 + q^{45+12M} z^3 + 4 q^{46+12M} z^3 + \\
&5 q^{47+12M} z^3 + 5 q^{48+12M} z^3 + 4 q^{49+12M} z^3 + q^{50+12M} z^3 - q^{6+2M} z^4 - 2 q^{8+2M} z^4 - 4 q^{10+2M} z^4 - 5 q^{12+2M} z^4 - \\
&6 q^{14+2M} z^4 - 5 q^{16+2M} z^4 - 4 q^{18+2M} z^4 - 2 q^{20+2M} z^4 - q^{22+2M} z^4 - q^{20+6M} z^4 - 3 q^{22+6M} z^4 - q^{23+6M} z^4 - \\
&3 q^{24+6M} z^4 - q^{25+6M} z^4 - q^{26+6M} z^4 - 2 q^{27+6M} z^4 + 2 q^{28+6M} z^4 - 2 q^{29+6M} z^4 + 3 q^{30+6M} z^4 - 2 q^{31+6M} z^4 + \\
&2 q^{32+6M} z^4 - q^{33+6M} z^4 + q^{34+6M} z^4 - q^{37+10M} z^4 + q^{38+10M} z^4 + 4 q^{40+10M} z^4 + 2 q^{41+10M} z^4 + 3 q^{42+10M} z^4 + \\
&q^{43+10M} z^4 - q^{14+4M} z^5 - 2 q^{16+4M} z^5 - 3 q^{18+4M} z^5 - 3 q^{20+4M} z^5 - 2 q^{22+4M} z^5 - q^{24+4M} z^5 - q^{30+8M} z^5 - \\
&q^{31+8M} z^5 + q^{32+8M} z^5 - q^{33+8M} z^5 + q^{34+8M} z^5 - 2 q^{35+8M} z^5 - q^{37+8M} z^5 - q^{24+6M} z^6) \, A[3, M] + \\
&q^{27+24M} \, (-q^{18+4M} + q^{21+4M} + q^{23+4M} + q^{36+8M} + q^{38+8M} + q^{40+8M} - q^{8+2M} z - 2 q^{10+2M} z - 2 q^{12+2M} z + \\
&q^{13+2M} z - 2 q^{14+2M} z + q^{15+2M} z - 2 q^{16+2M} z + q^{17+2M} z - q^{18+2M} z + q^{19+2M} z + q^{21+2M} z - q^{27+6M} z - \\
&2 q^{28+6M} z + 4 q^{30+6M} z - q^{31+6M} z + 2 q^{32+6M} z - 4 q^{33+6M} z - 3 q^{35+6M} z - q^{37+6M} z + q^{44+10M} z + \\
&2 q^{45+10M} z + 3 q^{46+10M} z + 4 q^{47+10M} z + 2 q^{48+10M} z + 2 q^{49+10M} z + q^{51+10M} z - z^2 - q^2 z^2 - 2 q^4 z^2 - \\
&2 q^6 z^2 - 3 q^8 z^2 - 2 q^{10} z^2 - 2 q^{12} z^2 - q^{14} z^2 - q^{16} z^2 - q^{16+4M} z^2 - 2 q^{18+4M} z^2 - q^{19+4M} z^2 - \\
&2 q^{20+4M} z^2 + q^{22+4M} z^2 + 2 q^{24+4M} z^2 + 4 q^{26+4M} z^2 + 3 q^{28+4M} z^2 + 2 q^{30+4M} z^2 + q^{32+4M} z^2 + q^{34+8M} z^2 - \\
&q^{35+8M} z^2 + 3 q^{36+8M} z^2 + 4 q^{38+8M} z^2 - q^{39+8M} z^2 - q^{40+8M} z^2 - 4 q^{41+8M} z^2 - 4 q^{42+8M} z^2 - 2 q^{43+8M} z^2 - \\
&q^{44+8M} z^2 - q^{46+8M} z^2 + 2 q^{53+12M} z^2 + 3 q^{54+12M} z^2 + 5 q^{55+12M} z^2 + 3 q^{56+12M} z^2 + 2 q^{57+12M} z^2 - \\
&q^{8+2M} z^3 - 2 q^{10+2M} z^3 - 3 q^{12+2M} z^3 - 3 q^{14+2M} z^3 - 3 q^{16+2M} z^3 - 2 q^{18+2M} z^3 - q^{20+2M} z^3 - q^{27+6M} z^3 + \\
&q^{28+6M} z^3 - 2 q^{29+6M} z^3 + 3 q^{30+6M} z^3 - 2 q^{31+6M} z^3 + 3 q^{32+6M} z^3 - 2 q^{33+6M} z^3 + 2 q^{34+6M} z^3 - q^{35+6M} z^3 + \\
&q^{36+6M} z^3 + q^{44+10M} z^3 + q^{45+10M} z^3 + q^{46+10M} z^3 + 2 q^{47+10M} z^3 + q^{49+10M} z^3 - q^{50+10M} z^3 - q^{18+4M} z^4 - \\
&q^{20+4M} z^4 - q^{22+4M} z^4 + q^{36+8M} z^4 - q^{37+8M} z^4 - q^{39+8M} z^4) \, A[4, M] - q^{52+30M} \, (-1 + q^{9+2M}) \, (1 + q^{9+2M}) \\
&(q^{10+2M} + q^{12+2M} - q^{15+2M} + q^{31+6M} + q^{33+6M} + q^{35+6M} + z + q^2 z + q^4 z + q^6 z + q^8 z + q^{10} z + q^{20+4M} z + q^{21+4M} \\
&z - q^{22+4M} z - 2 q^{24+4M} z - q^{26+4M} z - q^{28+4M} z - q^{30+4M} z + q^{41+8M} z + 2 q^{42+8M} z + 2 q^{43+8M} z + q^{44+8M} z + \\
&q^{10+2M} z^2 + q^{12+2M} z^2 + q^{14+2M} z^2 + q^{31+6M} z^2 + q^{33+6M} z^2 - q^{34+6M} z^2) \, A[5, M] + q^{85+36M} \, (-1 + q^{6+M}) \\
&(1 + q^{6+M}) \, (-1 + q^{9+2M}) \, (1 + q^{9+2M}) \, (-1 + q^{11+2M}) \, (1 + q^{11+2M}) \, (1 + q^{12+2M}) \, A[6, M]) \ // \ \text{Simplify}
\end{aligned}
$$

Out[132]= 0

(*Verifying the Rogers-Ramanujan type identities*)

```
In[•]:= MyQP[A_, q_, n_] := Product[k=0, n-1] (1 - A q^k);

     f[k_] := QPochhammer[q^k, q^k];

     ClearAll[M, N1, N2, N3, N4, N5, N6];

     SumMax = 5;
     PowerMax = 20;

     S1 = Normal[Series[Sum[N1=0, SumMax] Sum[N2=0, SumMax] Sum[N3=0, SumMax] Sum[N4=0, SumMax] Sum[N5=0, SumMax] Sum[N6=0, SumMax] z1^(N1+N3+N4+N6) z2^(N2+N3+N5+N6)
          q^(N1^2+N2^2+N3^2+N4^2+N5^2+N6^2+N1 N2+N1 N3+N1 N5+N1 N6+N2 N3+N2 N4+N2 N6+N3 N4+N3 N5+2 N3 N6+N4 N5+N4 N6+N5 N6 - (N1+N2+N6))
             / (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3]
                MyQP[q^2, q^2, N4] MyQP[q^2, q^2, N5] MyQP[q^2, q^2, N6]), {q, 0, PowerMax}]];

     S2 = Normal[Series[(QPochhammer[-z1 z2, q])/(QPochhammer[q, q^2]) Sum[M=0, SumMax] ((-1)^M q^(M^2) MyQP[z1^2, q^2, M] MyQP[z2^2, q^2, M])/(MyQP[q^2, q^2, M] MyQP[-z1 z2, q, 2 M]) +
          (QPochhammer[-z1 z2, q])/(QPochhammer[q, q^2]) Sum[M=0, SumMax] ((-1)^M (z1 + z2) q^(M^2+2 M) MyQP[z1^2, q^2, M] MyQP[z2^2, q^2, M])/(MyQP[q^2, q^2, M] MyQP[-z1 z2, q, 2 M + 1]), {q,
          0, PowerMax}]];
     S1 -
       S2 //
      Simplify

Out[•]= 0


In[•]:= MyQP[A_, q_, n_] := Product[k=0, n-1] (1 - A q^k);

     f[k_] := QPochhammer[q^k, q^k];

     ClearAll[M, N1, N2, N3, N4, N5, N6, z1, z2];

     SumMax = 5;
     PowerMax = 20;

     z2 = 1;

     S1 = Normal[Series[Sum[N1=0, SumMax] Sum[N2=0, SumMax] Sum[N3=0, SumMax] Sum[N4=0, SumMax] Sum[N5=0, SumMax] Sum[N6=0, SumMax] z1^(N1+N3+N4+N6) z2^(N2+N3+N5+N6)
          q^(N1^2+N2^2+N3^2+N4^2+N5^2+N6^2+N1 N2+N1 N3+N1 N5+N1 N6+N2 N3+N2 N4+N2 N6+N3 N4+N3 N5+2 N3 N6+N4 N5+N4 N6+N5 N6 - (N1+N2+N6))
             / (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3]
                MyQP[q^2, q^2, N4] MyQP[q^2, q^2, N5] MyQP[q^2, q^2, N6]), {q, 0, PowerMax}]];
     S2 = Normal[Series[(2 QPochhammer[-z1, q])/(QPochhammer[q, q^2]), {q, 0, PowerMax}]];

     S1 - S2 // Simplify

Out[•]= 0
```

```mathematica
In[ ]:= MyQP[A_, q_, n_] := Product[(1 - A q^k), {k, 0, n-1}];

f[k_] := QPochhammer[q^k, q^k];

ClearAll[M, N1, N2, N3, N4, N5, N6, z1, z2];

SumMax = 5;
PowerMax = 20;

z2 = -1;

S1 = Normal[Series[
    Sum[Sum[Sum[Sum[Sum[Sum[
    z1^(N1+N3+N4+N6) z2^(N2+N3+N5+N6)
    q^(N1^2+N2^2+N3^2+N4^2+N5^2+N6^2+N1 N2+N1 N3+N1 N5+N1 N6+N2 N3+N2 N4+N2 N6+N3 N4+N3 N5+2 N3 N6+N4 N5+N4 N6+N5 N6-(N1+N2+N6))
    / (MyQP[q^2, q^2, N1] MyQP[q^2, q^2, N2] MyQP[q^2, q^2, N3]
       MyQP[q^2, q^2, N4] MyQP[q^2, q^2, N5] MyQP[q^2, q^2, N6]),
    {N6, 0, SumMax}], {N5, 0, SumMax}], {N4, 0, SumMax}], {N3, 0, SumMax}], {N2, 0, SumMax}], {N1, 0, SumMax}],
    {q, 0, PowerMax}]];
S2 = Normal[Series[0, {q, 0, PowerMax}]];
S1 - S2 // Simplify

Out[ ]= 0
```