A different look at Euclidean billiard partitions

Shane Chern

Abstract. Euclidean billiard partitions are introduced by Andrews, Dragović and Radnović in their study of periodic trajectories of ellipsoidal billiards in the Euclidean space. They are integer partitions into distinct parts such that (E1) adjacent parts are never both odd; (E2) the smallest part is even. In this paper, we prove bivariate generating function identities that keep track of both the size and length not only for Euclidean billiard partitions but also for distinct partitions satisfying merely Condition (E1). In analogy, we also investigate distinct partitions such that adjacent parts are never both even.

Keywords. Euclidean billiard partition, generating function, q-difference equation, adjacent parts.

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1. Introduction

Euclidean billiard partitions are introduced by Andrews, Dragović and Radnović [3] in their study of periodic trajectories of ellipsoidal billiards in the Euclidean space.

Definition 1.1. *Euclidean billiard partitions* are integer partitions into distinct parts such that

(E1) adjacent parts are never both odd;
(E2) the smallest part is even.

We denote by $\mathscr{D}^E_{OO\times}$ the set of Euclidean billiard partitions.

Given any integer partition λ, let us adopt the usual notation that $|\lambda|$ and $\sharp(\lambda)$ denote the sum of all parts (namely, the *size*) and the number of parts (namely, the *length*) in λ, respectively. Andrews, Dragović and Radnović first defined a weight ϕ for each Euclidean billiard partition λ as follows:

$$\phi(\lambda) := \begin{cases} \sharp(\lambda) - 2s(\lambda) - 1 & \text{if the largest part in } \lambda \text{ is even,} \\ \sharp(\lambda) - 2s(\lambda) & \text{if the largest part in } \lambda \text{ is odd,} \end{cases}$$

where $s(\lambda)$ is the total number of odd parts in λ. One of the main results in [3] is the following bivariate generating function identity.

Theorem 1.1 (Andrews, Dragović and Radnović)**.** *We have*

$$1 + \sum_{\lambda \in \mathscr{D}^E_{OO\times}} x^{\phi(\lambda)} q^{|\lambda|} = 1 + \sum_{d \geq 1} \sum_{m \geq 0} \frac{s(d, m)}{(q^2; q^2)_d} \tag{1.1}$$

where

$$s(d,m) = \begin{cases} x^{2n-d}q^{d^2+2n^2-2dn+3n} \begin{bmatrix} n-1 \\ 2n-d \end{bmatrix}_{q^2} & \text{if } m = 2n+1, \\[3ex] x^{2n-d-1}q^{d^2+2n^2-2dn+2d-n} \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2} & \text{if } m = 2n, \end{cases}$$

with the q-Pochhammer symbol defined for $n \in \mathbb{N} \cup \{\infty\}$ by

$$(A;q)_n := \prod_{k=0}^{n-1}(1 - Aq^k)$$

and the q-binomial coefficient defined by

$$\begin{bmatrix} A \\ B \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_A}{(q;q)_B(q;q)_{A-B}} & \text{if } 0 \le B \le A, \\[3ex] 0 & \text{otherwise.} \end{cases}$$

We notice that Condition (E2) in Definition 1.1 only constrains the parity of the smallest part in the partition in question. Therefore, Euclidean billiard partitions form a subset of distinct partitions that are only restricted by Condition (E1). Naturally, we may turn our attention to such partitions.

Definition 1.2. We denote by $\mathscr{D}_{OO\times}$ the set of integer partitions into distinct parts such that

(E1) adjacent parts are never both odd.

Instead of focusing on the weight ϕ defined in [3], our first objective is to derive bivariate generating function identities for not only $\mathscr{D}_{OO\times}^E$ but also $\mathscr{D}_{OO\times}$ that keep track of both the size and length of the partitions. Just like Andrews, Dragović and Radnović's Theorem 1.1, the two generating functions are also double series.

Theorem 1.2. *We have*

(a).

$$1 + \sum_{\lambda \in \mathscr{D}_{OO\times}} x^{\sharp(\lambda)}q^{|\lambda|} = \sum_{i \ge 0}\sum_{j=0}^{i+1} \frac{x^{i+j}q^{i^2+j^2+i}}{(q^2;q^2)_{i+j}} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2}; \qquad (1.2)$$

(b).

$$1 + \sum_{\lambda \in \mathscr{D}_{OO\times}^E} x^{\sharp(\lambda)}q^{|\lambda|} = \sum_{i \ge 0}\sum_{j=0}^{i} \frac{x^{i+j}q^{i^2+j^2+i+2j}}{(q^2;q^2)_{i+j}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2}. \qquad (1.3)$$

Recall that to discover the generating function identity in Theorem 1.1, Andrews, Dragović and Radnović utilized the idea of *separable integer partition classes* which was later systematically studied by Andrews in [2]. However, our derivation of Theorem 1.2 relies on a totally different approach, which absorbs ideas from *linked partition ideals* introduced by Andrews in the 1970s. The interested reader may consult [1, Chapter 8] or [4, Section 2] for a detailed account of linked partition ideals.

Next, we see that Condition (E1) in Definition 1.1 constrains the parity of each adjacent parts. Our second objective is to study twin siblings of partitions in $\mathscr{D}_{OO\times}$.

Definition 1.3. We denote by $\mathscr{D}_{EE\times}$ the set of integer partitions into distinct parts such that

(O1) adjacent parts are never both even.

Further, we denote by $\mathscr{D}_{EE\times}^{O}$ the set of integer partitions in $\mathscr{D}_{EE\times}$ with an additional restriction that

(O2) the smallest part is odd.

Interestingly, bivariate generating functions for $\mathscr{D}_{EE\times}$ and $\mathscr{D}_{EE\times}^{O}$ may be deduced as simple consequences of Theorem 1.2.

Corollary 1.3. *We have*

(a).

$$1 + \sum_{\lambda \in \mathscr{D}_{EE\times}} x^{\sharp(\lambda)} q^{|\lambda|} = (1 + xq) \sum_{i \geq 0} \sum_{j=0}^{i+1} \frac{x^{i+j} q^{i^2 + j^2 + 2i + j}}{(q^2; q^2)_{i+j}} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2}; \qquad (1.4)$$

(b).

$$1 + \sum_{\lambda \in \mathscr{D}_{EE\times}^{O}} x^{\sharp(\lambda)} q^{|\lambda|} = \sum_{i \geq 0} \sum_{j=0}^{i} \frac{x^{i+j} q^{i^2 + j^2 + j}}{(q^2; q^2)_{i+j}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2}. \qquad (1.5)$$

2. Partition set $\mathscr{D}_{OO\times}$ and a q-difference system

2.1. Combinatorial constructions. We start with a finite set of integer partitions

$$\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$$
$$= \{\emptyset_E, \emptyset_O, 1, 2, 1+2\},$$

where $\emptyset_E$ and $\emptyset_O$ are both the empty partition although we deliberately assume that they are different. Let $P(\Pi)$ be the power set of Π. We define the map of *linking sets*, $\mathcal{L} : \Pi \to P(\Pi)$, by

$\pi \in \Pi$	$\mathcal{L}(\pi)$
$\pi_1 = \emptyset_E$	$\{\pi_1, \pi_3, \pi_4, \pi_5\} = \{\emptyset_E, 1, 2, 1+2\}$
$\pi_2 = \emptyset_O$	$\{\pi_2, \pi_4\} = \{\emptyset_O, 2\}$
$\pi_3 = 1$	$\{\pi_2, \pi_4\} = \{\emptyset_O, 2\}$
$\pi_4 = 2$	$\{\pi_1, \pi_3, \pi_4, \pi_5\} = \{\emptyset_E, 1, 2, 1+2\}$
$\pi_5 = 1+2$	$\{\pi_1, \pi_3, \pi_4, \pi_5\} = \{\emptyset_E, 1, 2, 1+2\}$

Now we may construct finite chains $\tilde{\sigma} = \sigma_0 \sigma_1 \sigma_2 \cdots \sigma_N$ by the following rules:

(R1) $\sigma_n \in \Pi$ for $0 \leq n \leq N$;
(R2) $\sigma_N \in \{\pi_3, \pi_4, \pi_5\}$, that is, σ_N is not the empty partition;
(R3) $\sigma_n \in \mathcal{L}(\sigma_{n-1})$ for $1 \leq n \leq N$.

We notice that exactly one of $\emptyset_E$ and $\emptyset_O$ belongs to $\mathcal{L}(\pi)$ for any $\pi \in \Pi$. Therefore, we may extend $\tilde{\sigma}$ to an infinite chain $\sigma = \sigma_0 \sigma_1 \sigma_2 \cdots \sigma_N \sigma_{N+1} \sigma_{N+2} \cdots$ by the rule:

(R4) for $n \geq N + 1$,

$$\sigma_n = \begin{cases} \emptyset_E & \text{if } \sigma_N \in \{\pi_4, \pi_5\}, \\ \emptyset_O & \text{if } \sigma_N \in \{\pi_3\}, \end{cases}$$

that is, σ_n is the empty partition that belongs to $\mathcal{L}(\sigma_{n-1})$.

Let $\mathcal{C}_\Pi$ be the set of such infinite chains σ together with

$$\emptyset_E \emptyset_E \emptyset_E \cdots \quad \text{and} \quad \emptyset_O \emptyset_O \emptyset_O \cdots .$$

Next, for each $\sigma = \sigma_0 \sigma_1 \sigma_2 \cdots \in \mathcal{C}_\Pi$, we attach a counting function κ by

$$\kappa_\sigma(x) = \kappa_\sigma(x; q) := \prod_{n \geq 0} (x q^{2n})^{\sharp(\sigma_n)} q^{|\sigma_n|}. \tag{2.1}$$

Notice that σ eventually ends with either $\emptyset_E \emptyset_E \emptyset_E \cdots$ or $\emptyset_O \emptyset_O \emptyset_O \cdots$ while $\sharp(\emptyset_E) = \sharp(\emptyset_O) = 0$ and $|\emptyset_E| = |\emptyset_O| = 0$. Therefore, the above infinite product is essentially a finite product. A useful property of the counting function κ is that

$$\begin{aligned} \kappa_\sigma(x) &= \prod_{n \geq 0} (x q^{2n})^{\sharp(\sigma_n)} q^{|\sigma_n|} \\ &= x^{\sharp(\sigma_0)} q^{|\sigma_0|} \prod_{n \geq 1} (x q^{2n})^{\sharp(\sigma_n)} q^{|\sigma_n|} \\ &= x^{\sharp(\sigma_0)} q^{|\sigma_0|} \kappa_{\sigma'}(x q^2), \end{aligned} \tag{2.2}$$

where σ' is the subchain $\sigma_1 \sigma_2 \sigma_3 \cdots$ which also belongs to $\mathcal{C}_\Pi$. This relation plays an important role in our derivation of the desired q-difference system.

2.2. A q-difference system. For notational convenience, given any infinite chain $\sigma \in \mathcal{C}_\Pi$, we always assume that it is of the form $\sigma_0 \sigma_1 \sigma_2 \cdots$. We define five functions: for $1 \leq i \leq 5$,

$$H_i(x) = H_i(x; q) := \sum_{\substack{\sigma \in \mathcal{C}_\Pi \\ \sigma_0 = \pi_i}} \kappa_\sigma(x). \tag{2.3}$$

By Rule (R3), if $\sigma_0 = \pi_i$, then $\sigma_1 \in \mathcal{L}(\pi_i)$. In light of (2.2), we have

$$\begin{aligned} H_i(x) &= \sum_{\substack{\sigma \in \mathcal{C}_\Pi \\ \sigma_0 = \pi_i}} \kappa_\sigma(x) \\ &= x^{\sharp(\pi_i)} q^{|\pi_i|} \sum_{j : \pi_j \in \mathcal{L}(\pi_i)} \sum_{\substack{\sigma' \in \mathcal{C}_\Pi \\ \sigma_0' = \pi_j}} \kappa_{\sigma'}(x q^2) \\ &= x^{\sharp(\pi_i)} q^{|\pi_i|} \sum_{j : \pi_j \in \mathcal{L}(\pi_i)} H_j(x q^2). \end{aligned} \tag{2.4}$$

This yields the following q-difference system:

$$\begin{cases} H_1(x) = H_1(x q^2) + H_3(x q^2) + H_4(x q^2) + H_5(x q^2), & \text{(2.5a)} \\ H_2(x) = H_2(x q^2) + H_4(x q^2), & \text{(2.5b)} \\ H_3(x) = x q \big(H_2(x q^2) + H_4(x q^2) \big), & \text{(2.5c)} \\ H_4(x) = x q^2 \big(H_1(x q^2) + H_3(x q^2) + H_4(x q^2) + H_5(x q^2) \big), & \text{(2.5d)} \\ H_5(x) = x^2 q^3 \big(H_1(x q^2) + H_3(x q^2) + H_4(x q^2) + H_5(x q^2) \big). & \text{(2.5e)} \end{cases}$$

Notice that by (2.5a), (2.5d) and (2.5e),

$$\begin{cases} H_4(x) = xq^2 H_1(x), \\ H_5(x) = x^2 q^3 H_1(x), \end{cases}$$

and by (2.5b) and (2.5c),

$$H_3(x) = xq H_2(x).$$

Therefore, the above q-difference system can be simplified as follows.

Lemma 2.1. *We have*

$$\begin{cases} H_1(x) = (1 + xq^4 + x^2 q^7) H_1(xq^2) + xq^3 H_2(xq^2), & (2.6a) \\ H_2(x) = xq^4 H_1(xq^2) + H_2(xq^2). & (2.6b) \end{cases}$$

2.3. Partition sets $\mathscr{D}_{OO\times}$ and $\mathscr{D}_{OO\times}^E$. Now we will turn our attention to the connections between the two partition sets $\mathscr{D}_{OO\times}$ and $\mathscr{D}_{OO\times}^E$ and the set of infinite chains $\mathcal{C}_\Pi$. What we are going to show are the following two relations.

Lemma 2.2. *We have*

(a).

$$1 + \sum_{\lambda \in \mathscr{D}_{OO\times}} x^{\sharp(\lambda)} q^{|\lambda|} = H_1(xq^{-2}); \tag{2.7}$$

(b).

$$1 + \sum_{\lambda \in \mathscr{D}_{OO\times}^E} x^{\sharp(\lambda)} q^{|\lambda|} = H_2(xq^{-2}). \tag{2.8}$$

Proof. We start our proof by introducing some notation. Given an integer partition λ, let $\lambda_{(k)}$ be the collection of parts p in λ such that $2k + 1 \le p \le 2k + 2$ for each $k \ge 0$. Notice that there exists some K such that $\lambda_{(k)}$ is the empty partition for all $k \ge K$. Then λ can be uniquely decomposed as

$$\lambda_{(0)} \oplus \lambda_{(1)} \oplus \lambda_{(2)} \oplus \cdots.$$

We also define operators ϕ^ℓ with $\ell \ge 0$ for partitions by adding ℓ to each part of the partition. In particular, $\phi^\ell(\emptyset) = \emptyset$ for all $\ell \ge 0$. Then for each $\lambda_{(k)}$, we may find a unique $\lambda_{(k)}^*$ with largest part at most 2 such that

$$\lambda_{(k)} = \phi^{2k}(\lambda_{(k)}^*).$$

For instance, we decompose the partition $1 + 2 + 3 + 8 + 9 + 10$ as

$$\begin{aligned} 1 + 2 &+ 3 + 8 + 9 + 10 \\ &= (1 + 2) \oplus (3) \oplus \emptyset \oplus (8) \oplus (9 + 10) \oplus \emptyset \oplus \cdots \\ &= \phi^0(1 + 2) \oplus \phi^2(1) \oplus \phi^4(\emptyset) \oplus \phi^6(2) \oplus \phi^8(1 + 2) \oplus \phi^{10}(\emptyset) \oplus \cdots \end{aligned}$$

Let $\lambda \in \mathscr{D}_{OO\times}$. Since it is a partition into distinct parts, then each $\lambda_{(k)}^*$ is among $\{\emptyset, 1, 2, 1 + 2\}$; otherwise, some parts repeat. Let $\lambda_{(M)}^*$ and $\lambda_{(N)}^*$ be the first and last non-empty partition among $\{\lambda_{(k)}^*\}_{k \ge 0}$. We construct an infinite chain $\sigma(\lambda) = \sigma_0 \sigma_1 \sigma_2 \cdots$ of partitions as follows:

(C1) for $0 \le n < M$, $\sigma_n = \emptyset_E = \pi_1$;

(C2) for $n > N$,

$$\sigma_n = \begin{cases} \emptyset_E = \pi_1 & \text{if } \lambda^*_{(N)} \in \{2, 1+2\} = \{\pi_4, \pi_5\}, \\ \emptyset_O = \pi_2 & \text{if } \lambda^*_{(N)} \in \{1\} = \{\pi_3\}; \end{cases}$$

(C3) for $M \le n \le N$, we iteratively assign

$$\sigma_n = \begin{cases} \lambda^*_{(n)} & \text{if } \lambda^*_{(n)} \neq \emptyset, \\ \emptyset_E = \pi_1 & \text{if } \lambda^*_{(n)} = \emptyset \text{ and } \lambda^*_{(n-1)} \in \{\emptyset_E, 2, 1+2\} = \{\pi_1, \pi_4, \pi_5\}, \\ \emptyset_O = \pi_2 & \text{if } \lambda^*_{(n)} = \emptyset \text{ and } \lambda^*_{(n-1)} \in \{\emptyset_O, 1\} = \{\pi_2, \pi_3\}. \end{cases}$$

For instance, the partition $1 + 2 + 3 + 8 + 9 + 10$ induces the chain

$$\pi_5 \pi_3 \pi_2 \pi_4 \pi_5 \pi_1 \pi_1 \pi_1 \cdots.$$

We shall show that $\sigma(\lambda) \in \mathcal{C}_\Pi$ and in particular, $\sigma_0 \in \{\pi_1, \pi_3, \pi_4, \pi_5\}$.

First, if $\lambda^*_{(0)} = \emptyset$, then $\sigma_0 = \emptyset_E = \pi_1$; if $\lambda^*_{(0)} \neq \emptyset$, then $\sigma_0 \in \{\pi_3, \pi_4, \pi_5\}$. Therefore, the latter claim holds true. Also, by the construction, all elements in $\sigma(\lambda)$ belong to Π and thus Rule (R1) is satisfied. Further, (C2) ensures Rules (R2) and (R4). Finally, (C1) ensures that for $1 \le n < M$, Rule (R3) holds true since $\emptyset_E \in \mathcal{L}(\emptyset_E)$. Also, since $\lambda^*_{(M)}$ is not the empty partition, it is among $\{\pi_3, \pi_4, \pi_5\}$. If $M > 0$, we have $\sigma_{M-1} = \emptyset_E$ by (C1) while $\pi_3, \pi_4, \pi_5 \in \mathcal{L}(\emptyset_E) = \mathcal{L}(\sigma_{M-1})$. Hence, (R3) is true for $n = M$ if $M > 0$. It therefore remains to check Rule (R3) for $M + 1 \le n \le N$. Let us assume that $\lambda^*_{(m)}$ is not the empty partition for some $M \le m \le N$ and that $\lambda^*_{(n)}$ with $n > m$ is the next non-empty partition. We have two cases.

- $\lambda^*_{(m)} \in \{1\} = \{\pi_3\}$. Then $\lambda^*_{(n)} \in \{2\} = \{\pi_4\}$; otherwise, there exist adjacent parts of size $2m + 1$ and $2n + 1$ in λ so Condition (E1) is violated. Hence, by (C3), $\sigma_m \in \{\pi_3\}$ and $\sigma_n \in \{\pi_4\}$. Further, if $n > m + 1$, then the empty partitions $\lambda^*_{(k)} = \emptyset$ where $m < k < n$ are mapped to $\sigma_k = \emptyset_O = \pi_2$ by (C3). Therefore, we always have $\sigma_{n'} \in \mathcal{L}(\sigma_{n'-1})$ for $m + 1 \le n' \le n$ by recalling that $\mathcal{L}(\pi_2) = \mathcal{L}(\pi_3) = \{\pi_2, \pi_4\}$.
- $\lambda^*_{(m)} \in \{2, 1+2\} = \{\pi_4, \pi_5\}$. Then $\lambda^*_{(n)} \in \{1, 2, 1+2\} = \{\pi_3, \pi_4, \pi_5\}$ since $\lambda^*_{(m)}$ induces a part of size $2m + 2$ in λ, which is even. By (C3), $\sigma_m \in \{\pi_4, \pi_5\}$ and $\sigma_n \in \{\pi_3, \pi_4, \pi_5\}$. Further, if $n > m + 1$, then the empty partitions $\lambda^*_{(k)} = \emptyset$ where $m < k < n$ are mapped to $\sigma_k = \emptyset_E = \pi_1$ by (C3). Therefore, we also have $\sigma_{n'} \in \mathcal{L}(\sigma_{n'-1})$ for $m + 1 \le n' \le n$ by recalling that $\mathcal{L}(\pi_1) = \mathcal{L}(\pi_4) = \mathcal{L}(\pi_5) = \{\pi_1, \pi_3, \pi_4, \pi_5\}$.

The above argument is enough to ensure (R3) for $M + 1 \le n \le N$. Thus, $\sigma(\lambda) \in \mathcal{C}_\Pi$.

Conversely, if $\sigma \in \mathcal{C}_\Pi$ with $\sigma_0 \in \{\pi_1, \pi_3, \pi_4, \pi_5\}$, we may induce a partition as follows:

$$\lambda : \phi^0(\sigma_0) \oplus \phi^2(\sigma_1) \oplus \cdots \oplus \phi^{2k}(\sigma_k) \oplus \cdots. \tag{2.9}$$

Apparently, this is a distinct partition. Hence, to show $\lambda \in \mathscr{D}_{OO\times}$, it suffices to check Condition (E1). Let σ_m be a non-empty partition and σ_n with $n > m$ be the next non-empty partition. We again have two cases.

- $\sigma_m \in \{1\} = \{\pi_3\}$. Then the empty partitions σ_k where $m < k < n$ must be $\emptyset_O = \pi_2$ as $\mathcal{L}(\pi_2) = \mathcal{L}(\pi_3) = \{\pi_2, \pi_4\}$. Thus, $\sigma_n \in \{\pi_4\}$. Now the adjacent parts induces from σ_m and σ_n are $2m + 1$ and $2n + 2$ and therefore (E1) is satisfied.

- $\sigma_m \in \{2, 1+2\} = \{\pi_4, \pi_5\}$. Then the empty partitions σ_k where $m < k < n$ must be $\emptyset_E = \pi_1$ as $\mathcal{L}(\pi_1) = \mathcal{L}(\pi_4) = \mathcal{L}(\pi_5) = \{\pi_1, \pi_3, \pi_4, \pi_5\}$. Thus, $\sigma_n \in \{\pi_3, \pi_4, \pi_5\}$. Now the adjacent parts induces from σ_m and σ_n are $2m+2$ and $2n+1$ or $2n+2$ and therefore (E1) is also satisfied.

The above reveals a bijective map from $\sigma \in \mathcal{C}_\Pi$ with $\sigma_0 \in \{\pi_1, \pi_3, \pi_4, \pi_5\}$ to partitions in $\mathscr{D}_{OO\times}$. This map is given by (2.9). Thus,

$$
\begin{aligned}
1 + \sum_{\lambda \in \mathscr{D}_{OO\times}} x^{\sharp(\lambda)} q^{|\lambda|} &= \sum_{\substack{\sigma \in \mathcal{C}_\Pi \\ \sigma_0 \in \{\pi_1, \pi_3, \pi_4, \pi_5\}}} \prod_{k \geq 0} x^{\sharp(\phi^{2k}(\sigma_k))} q^{|\phi^{2k}(\sigma_k)|} \\
&= \sum_{\substack{\sigma \in \mathcal{C}_\Pi \\ \sigma_0 \in \{\pi_1, \pi_3, \pi_4, \pi_5\}}} \prod_{k \geq 0} (xq^{2k})^{\sharp(\sigma_k)} q^{|\sigma_k|} \\
&= \sum_{\substack{\sigma \in \mathcal{C}_\Pi \\ \sigma_0 \in \{\pi_1, \pi_3, \pi_4, \pi_5\}}} \kappa_\sigma(x) && \text{(by (2.1))} \\
&= H_1(x) + H_3(x) + H_4(x) + H_5(x) && \text{(by (2.3))} \\
&= H_1(xq^{-2}). && \text{(by (2.5a))}
\end{aligned}
$$

This completes Part (a) of the lemma.

For Part (b) of the lemma, we carry out a similar argument. First, let $\lambda \in \mathscr{D}_{OO\times}^E$ and let $\lambda_{(M)}^*$ and $\lambda_{(N)}^*$ be the first and last non-empty partition among $\{\lambda_{(k)}^*\}_{k \geq 0}$. We construct a slightly different infinite chain $\sigma(\lambda) = \sigma_0 \sigma_1 \sigma_2 \cdots$ of partitions in Π by

(C1') for $0 \leq n < M$, $\sigma_n = \emptyset_O = \pi_2$;
(C2') for $n > N$,

$$
\sigma_n = \begin{cases} \emptyset_E = \pi_1 & \text{if } \lambda_{(N)}^* \in \{2, 1+2\} = \{\pi_4, \pi_5\}, \\ \emptyset_O = \pi_2 & \text{if } \lambda_{(N)}^* \in \{1\} = \{\pi_3\}; \end{cases}
$$

(C3') for $M \leq n \leq N$, we iteratively assign

$$
\sigma_n = \begin{cases} \lambda_{(n)}^* & \text{if } \lambda_{(n)}^* \neq \emptyset, \\ \emptyset_E = \pi_1 & \text{if } \lambda_{(n-1)}^* \in \{\emptyset_E, 2, 1+2\} = \{\pi_1, \pi_4, \pi_5\}, \\ \emptyset_O = \pi_2 & \text{if } \lambda_{(n-1)}^* \in \{\emptyset_O, 1\} = \{\pi_2, \pi_3\}. \end{cases}
$$

Here the only difference is that we use $\emptyset_O$ in (C1') instead of $\emptyset_E$ in (C1). Similarly, we have $\sigma(\lambda) \in \mathcal{C}_\Pi$ and in particular, $\sigma_0 \in \{\pi_2, \pi_4\}$. Conversely, if $\sigma \in \mathcal{C}_\Pi$ with $\sigma_0 \in \{\pi_2, \pi_4\}$, we may still induce a partition by

$$
\lambda : \phi^0(\sigma_0) \oplus \phi^2(\sigma_1) \oplus \cdots \oplus \phi^{2k}(\sigma_k) \oplus \cdots. \tag{2.10}
$$

Similarly, this partition is in $\lambda \in \mathscr{D}_{OO\times}$. To show it is indeed in $\lambda \in \mathscr{D}_{OO\times}^E$, we only need to check the parity of the smallest part, that is, Condition (E2). Let σ_m be the first non-empty partition in $\{\sigma_k\}_{k \geq 0}$. If $m = 0$, then it is $\pi_4 = 2$ by our assumption $\sigma_0 \in \{\pi_2, \pi_4\}$. If $m > 0$, then it is the successor of $\pi_2 = \emptyset_O$ and therefore is in $\mathcal{L}(\pi_2) = \{\pi_2, \pi_4\}$. So it is also $\pi_4 = 2$. Since σ_m is always π_4, then the smallest part in the induced partition is $2m+2$, which is even. So $\lambda \in \mathscr{D}_{OO\times}^E$.

From the above bijection, we conclude that

$$1 + \sum_{\lambda \in \mathscr{D}_{OO\times}^{E}} x^{\sharp(\lambda)} q^{|\lambda|} = \sum_{\substack{\sigma \in \mathcal{C}_{\Pi} \\ \sigma_0 \in \{\pi_2, \pi_4\}}} \prod_{k \geq 0} x^{\sharp(\phi^{2k}(\sigma_k))} q^{|\phi^{2k}(\sigma_k)|}$$

$$= \sum_{\substack{\sigma \in \mathcal{C}_{\Pi} \\ \sigma_0 \in \{\pi_2, \pi_4\}}} \prod_{k \geq 0} (xq^{2k})^{\sharp(\sigma_k)} q^{|\sigma_k|}$$

$$= \sum_{\substack{\sigma \in \mathcal{C}_{\Pi} \\ \sigma_0 \in \{\pi_2, \pi_4\}}} \kappa_\sigma(x) \qquad \text{(by (2.1))}$$

$$= H_2(x) + H_4(x) \qquad \text{(by (2.3))}$$

$$= H_2(xq^{-2}). \qquad \text{(by (2.5b))}$$

Therefore, Part (b) of the lemma is established. $\qquad \square$

3. Proof of Theorem 1.2

3.1. Proof of Theorem 1.2(a). Let us write

$$F(x) = 1 + \sum_{\lambda \in \mathscr{D}_{OO\times}} x^{\sharp(\lambda)} q^{|\lambda|}.$$

Then by (2.7),

$$F(x) = H_1(xq^{-2}). \tag{3.1}$$

We first deduce from (2.6a) that

$$H_2(xq^2) = x^{-1}q^{-3}\big(H_1(x) - (1 + xq^4 + x^2q^7)H_1(xq^2)\big).$$

Substituting the above into (2.6b) yields

$$q^2 H_1(xq^{-2}) - (1 + q^2 + xq^4 + x^2q^5)H_1(x) + (1 + xq^4)H_1(xq^2) = 0.$$

Recalling (3.1), we have a q-difference equation for $F(x)$:

$$q^2 F(x) - (1 + q^2 + xq^4 + x^2q^5)F(xq^2) + (1 + xq^4)F(xq^4) = 0. \tag{3.2}$$

Next, we write

$$F(x) = \sum_{n \geq 0} f_n x^n.$$

Apparently,

$$f_0 = 1, \tag{3.3}$$

$$f_1 = q + q^2 + q^3 + \cdots = \frac{q}{1 - q}. \tag{3.4}$$

We may translate the q-difference equation (3.2) into a recurrence of f_n: for $n \geq 2$,

$$(1 - q^{2n})(1 - q^{2n-2})f_n = q^{2n}(1 - q^{2n-2})f_{n-1} + q^{2n-1}f_{n-2}. \tag{3.5}$$

Now let us define, for $n \geq 0$,

$$g_n = f_n(q^2; q^2)_n. \tag{3.6}$$

Then by (3.3) and (3.4),

$$g_0 = 1,$$

$$g_1 = q + q^2.$$

Further, for $n \geq 2$, (3.5) becomes

$$g_n = q^{2n} g_{n-1} + q^{2n-1} g_{n-2}. \tag{3.7}$$

We write

$$G(x) = \sum_{n \geq 0} g_n x^n.$$

Therefore,

$$G(x) - 1 - x(q + q^2) = xq^2 \left(G(xq^2) - 1 \right) + x^2 q^3 G(xq^2),$$

that is,

$$G(x) - xq^2(1 + xq)G(xq^2) = 1 + xq. \tag{3.8}$$

Now we give an explicit expression of $G(x)$.

Lemma 3.1. *We have*

$$G(x) = \sum_{i \geq 0} x^i q^{i(i+1)} (-xq; q^2)_{i+1}. \tag{3.9}$$

Proof. We prove a truncated result: for $N \geq 1$,

$$G(x) - x^N q^{N(N+1)} (-xq; q^2)_N G(xq^{2N}) = \sum_{i=0}^{N-1} x^i q^{i(i+1)} (-xq; q^2)_{i+1}. \tag{3.10}$$

We will see our lemma follows if we let $N \to \infty$.

To show (3.10), we induct on N. First, the base case $N = 1$ holds true by (3.8). Now we assume that (3.10) is valid for some $N \geq 1$. Replacing x by xq^{2N} in (3.8), we have

$$G(xq^{2N}) - xq^{2N+2}(1 + xq^{2N+1})G(xq^{2N+2}) = 1 + xq^{2N+1}.$$

Multiplying by $x^N q^{N(N+1)} (-xq; q^2)_N$ on both sides of the above yields

$$x^N q^{N(N+1)} (-xq; q^2)_N G(xq^{2N}) - x^{N+1} q^{(N+1)(N+2)} (-xq; q^2)_{N+1} G(xq^{2N+2})$$
$$= x^N q^{N(N+1)} (-xq; q^2)_{N+1}.$$

Combining the above with (3.10), we have

$$G(x) - x^{N+1} q^{(N+1)(N+2)} (-xq; q^2)_{N+1} G(xq^{2N+2}) = \sum_{i=0}^{N} x^i q^{i(i+1)} (-xq; q^2)_{i+1}.$$

This is exactly the $N + 1$ case of (3.10) and therefore the desired result holds true. $\qquad \square$

Finally, to deduce an explicit expression for $F(x)$, we need to reformulate $G(x)$. We require the q-binomial theorem [1, Eq. (3.3.6)]: for $n \geq 0$,

$$(z; q)_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}} z^j. \tag{3.11}$$

By (3.9),

$$G(x) = \sum_{i \geq 0} x^i q^{i(i+1)} (-xq; q^2)_{i+1}$$

$$= \sum_{i \geq 0} x^i q^{i(i+1)} \sum_{j=0}^{i+1} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2} q^{2\binom{j}{2}} (xq)^j \qquad \text{(by (3.11))}$$

$$= \sum_{i \geq 0} \sum_{j=0}^{i+1} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2} q^{i^2+j^2+i} x^{i+j}.$$

Recall that

$$F(x) = \sum_{n \geq 0} f_n x^n$$

$$= \sum_{n \geq 0} \frac{g_n}{(q^2; q^2)_n} x^n. \qquad \text{(by (3.6))}$$

Therefore,

$$F(x) = \sum_{i \geq 0} \sum_{j=0}^{i+1} \frac{x^{i+j} q^{i^2+j^2+i}}{(q^2; q^2)_{i+j}} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2}.$$

This completes Part (a) of Theorem 1.2.

3.2. Proof of Theorem 1.2(b). The proof of Theorem 1.2(b) is similar to that for Part (a) so we only give a sketch. We as well write

$$S(x) = 1 + \sum_{\lambda \in \mathscr{D}_{OO\times}^E} x^{\sharp(\lambda)} q^{|\lambda|}.$$

Then by (2.8),

$$S(x) = H_2(xq^{-2}). \qquad (3.12)$$

We then deduce from (2.6b) that

$$H_1(xq^2) = x^{-1} q^{-4} \big(H_2(x) - H_2(xq^2) \big).$$

Substituting the above into (2.6a) and recalling (3.12), we have

$$q^2 S(x) - (1 + q^2 + xq^4 + x^2 q^7) S(xq^2) + (1 + xq^4) S(xq^4) = 0. \qquad (3.13)$$

Next, we write

$$S(x) = \sum_{n \geq 0} s_n x^n.$$

Then,

$$s_0 = 1,$$

$$s_1 = q^2 + q^4 + q^6 + \cdots = \frac{q^2}{1 - q^2},$$

and for $n \geq 2$,

$$(1 - q^{2n})(1 - q^{2n-2}) s_n = q^{2n}(1 - q^{2n-2}) s_{n-1} + q^{2n+1} s_{n-2}. \qquad (3.14)$$

Let

$$t_n = s_n (q^2; q^2)_n$$

and

$$T(x) = \sum_{n \geq 0} t_n x^n.$$

Then,

$$T(x) - xq^2(1 + xq^3)T(xq^2) = 1. \tag{3.15}$$

We find that

$$T(x) = \sum_{i \geq 0} x^i q^{i(i+1)}(-xq^3; q^2)_i, \tag{3.16}$$

whose truncated version is that for $N \geq 1$,

$$T(x) - x^N q^{N(N+1)}(-xq^3; q^2)_N T(xq^{2N}) = \sum_{i=0}^{N-1} x^i q^{i(i+1)}(-xq^3; q^2)_i, \tag{3.17}$$

which can be easily shown by induction on N. Finally, by (3.11),

$$T(x) = \sum_{i \geq 0} \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2} q^{i^2 + j^2 + i + 2j} x^{i+j},$$

and therefore,

$$S(x) = \sum_{i \geq 0} \sum_{j=0}^{i} \frac{x^{i+j} q^{i^2 + j^2 + i + 2j}}{(q^2; q^2)_{i+j}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2}.$$

4. Partition set $\mathscr{D}_{EE\times}$ and Proof of Corollary 1.3

We prove Corollary 1.3 by some simple combinatorial arguments. First, if two adjacent parts satisfy Condition (E1), that is, they are not simultaneously odd, then by adding 1 to or deleting 1 from both parts, the resulting parts are not simultaneously even and therefore Condition (O1) is satisfied. Conversely, if two adjacent parts satisfy Condition (O1), then by adding 1 to or deleting 1 from both parts, the resulting parts satisfy Condition (E1).

Therefore, there is a bijection from partitions in $\mathscr{D}_{OO\times}$ to partitions in $\mathscr{D}_{EE\times}$ with the smallest part at least 2, by adding 1 to each part of the partition. Thus,

$$1 + \sum_{\substack{\lambda \in \mathscr{D}_{EE\times} \\ \varsigma(\lambda) \geq 2}} x^{\sharp(\lambda)} q^{|\lambda|} = \left[1 + \sum_{\lambda \in \mathscr{D}_{OO\times}} x^{\sharp(\lambda)} q^{|\lambda|} \right]_{x \mapsto xq}$$

$$= \sum_{i \geq 0} \sum_{j=0}^{i+1} \frac{(xq)^{i+j} q^{i^2 + j^2 + i}}{(q^2; q^2)_{i+j}} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2}$$

$$= \sum_{i \geq 0} \sum_{j=0}^{i+1} \frac{x^{i+j} q^{i^2 + j^2 + 2i + j}}{(q^2; q^2)_{i+j}} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2}, \tag{4.1}$$

where $\varsigma(\lambda)$ denotes the smallest part of λ. Further, for any $\lambda \in \mathscr{D}_{EE\times}$ with $\varsigma(\lambda) = 1$, if we delete the part of size 1, then the resulting partition is still in $\mathscr{D}_{EE\times}$ and its smallest part is at least 2. Hence,

$$1 + \sum_{\lambda \in \mathscr{D}_{EE\times}} = (1 + xq) \left(1 + \sum_{\substack{\lambda \in \mathscr{D}_{EE\times} \\ \varsigma(\lambda) \geq 2}} x^{\sharp(\lambda)} q^{|\lambda|} \right)$$

$$= (1 + xq) \sum_{i \geq 0} \sum_{j=0}^{i+1} \frac{x^{i+j} q^{i^2+j^2+2i+j}}{(q^2;q^2)_{i+j}} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{q^2}, \tag{4.2}$$

and therefore Corollary 1.3(a) holds true.

For Corollary 1.3(b), we simply notice that given a partition in $\mathscr{D}_{OO\times}^{E}$, its smallest part is even and therefore is at least 2. Then by deleting 1 from each part in this partition, the smallest part of the resulting partition is an odd number that is at least 1. Therefore, we have a bijection from partitions in $\mathscr{D}_{OO\times}^{E}$ to partitions in $\mathscr{D}_{EE\times}^{O}$ by deleting 1 from each part of the partition. Thus,

$$1 + \sum_{\lambda \in \mathscr{D}_{EE\times}^{O}} x^{\sharp(\lambda)} q^{|\lambda|} = \left[1 + \sum_{\lambda \in \mathscr{D}_{OO\times}^{E}} x^{\sharp(\lambda)} q^{|\lambda|} \right]_{x \mapsto xq^{-1}}$$

$$= \sum_{i \geq 0} \sum_{j=0}^{i} \frac{(xq^{-1})^{i+j} q^{i^2+j^2+i+2j}}{(q^2;q^2)_{i+j}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2}$$

$$= \sum_{i \geq 0} \sum_{j=0}^{i} \frac{x^{i+j} q^{i^2+j^2+j}}{(q^2;q^2)_{i+j}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2}. \tag{4.3}$$

5. Final remarks

We remark that setting $x = 1$ in both (1.1) and (1.3) yields the following identity.

Corollary 5.1. *We have*

$$\sum_{i \geq 0} \sum_{j=0}^{i} \frac{q^{i^2+j^2+i+2j}}{(q^2;q^2)_{i+j}} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2} = 1 + \sum_{d \geq 1} \sum_{m \geq 0} \frac{s(d,m)}{(q^2;q^2)_d} \tag{5.1}$$

where

$$s(d,m) = \begin{cases} q^{d^2+2n^2-2dn+3n} \begin{bmatrix} n-1 \\ 2n-d \end{bmatrix}_{q^2} & \text{if } m = 2n+1, \\[20pt] q^{d^2+2n^2-2dn+2d-n} \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2} & \text{if } m = 2n, \end{cases}$$

We also provide an alternative analytic proof of this fact.

Analytic Proof of Corollary 5.1. We have

$$1 + \sum_{d \geq 1} \sum_{m \geq 0} \frac{s(d,m)}{(q^2;q^2)_d}$$

$$= 1 + \sum_{d \geq 1} \sum_{n \geq 0} \frac{q^{d^2+2n^2-2dn+2d-n}}{(q^2;q^2)_d} \left(q^{2(2n-d)} \begin{bmatrix} n-1 \\ 2n-d \end{bmatrix}_{q^2} + \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2} \right)$$

$$= 1 + \sum_{d \geq 1} \sum_{n \geq 0} \frac{q^{d^2+2n^2-2dn+2d-n}}{(q^2;q^2)_d} \begin{bmatrix} n \\ 2n-d \end{bmatrix}_{q^2} \qquad \text{(by [1, Eq. (3.3.4)])}$$

$$= \sum_{n \geq 0} \sum_{d=n}^{2n} \frac{q^{d^2+2n^2-2dn+2d-n}}{(q^2;q^2)_d} \begin{bmatrix} n \\ 2n-d \end{bmatrix}_{q^2}$$

$$= \sum_{n \geq 0} \sum_{d=0}^{n} \frac{q^{n^2+d^2+n+2d}}{(q^2;q^2)_{n+d}} \begin{bmatrix} n \\ n-d \end{bmatrix}_{q^2}. \qquad \text{(by } d \mapsto n+d\text{)}$$

This is exactly the left-hand side of (5.1) since $\begin{bmatrix} n \\ n-d \end{bmatrix}_{q^2} = \begin{bmatrix} n \\ d \end{bmatrix}_{q^2}.$ □

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DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA

E-mail address: shanechern@psu.edu; chenxiaohang92@gmail.com