RESEARCH STATEMENT
(FULL VERSION)

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Contents

0. Overview

My current research focuses on number theory, combinatorics, special functions and computer algebra. In particular, I am interested in integer partitions, basic hypergeometric series, permutation patterns, Diophantine equations, circle and sieve methods, modular forms, and the development of algorithmic methods for applications in combinatorics and number theory. Meanwhile, some of my research projects have close connections with representation theory and mathematical physics.

1. Partitions: *Linked partition ideals*

In 2014, Shashank Kanade and Matthew Russell proposed six challenging conjectures on partition identities of Rogers–Ramanujan type. Their conjectures are partition-theoretic, concerning partitions under congruence conditions and partitions under difference conditions. For instance, the Conjecture I_1 can be stated as follows.

Conjecture 1.1 (Kanade–Russell Conjecture I_1)**.** *The number of partitions of a nonnegative integer n into parts congruent to 1, 3, 6 or 8 modulo 9 is the same as the number of partitions of n with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3. Here, we say a partition $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ satisfies the difference at least d at distance k condition if, for all j, $\lambda_j - \lambda_{j+k} \geq d$.*

The analytic counterparts were not discovered until the summer of 2018, partly by Kağan Kurşungöz, and partly by Kanade and Russell themselves, using different combinatorial approaches. As an example, the Conjecture I_1 is equivalent to

$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} \overset{?}{=} \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty}. \tag{1.1}$$

Around the same time, Zhitai Li and I were reading Chapter 8 of George Andrews' monograph "*The Theory of Partitions*", which is on linked partition ideals, a theory for partition identities developed by Andrews in the 1970s. As I was aware of the Kanade–Russell Conjectures several years ago, I had the feeling that the generating functions for the partitions sets under difference conditions in these conjectures (e.g., the sum side in (1.1)) can be uniformly treated by a special type of linked partition ideals, called span one linked partition ideals.

Definition 1.1. Assume that we are given

- a finite set $\Pi = \{\pi_1, \pi_2, \dots, \pi_K\}$ of integer partitions with $\pi_1 = \emptyset$, the empty partition,
- a *map of linking sets*, $\mathcal{L} : \Pi \to P(\Pi)$, the power set of Π, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\emptyset) = \Pi$ and $\pi_1 = \emptyset \in \mathcal{L}(\pi_k)$ for any $1 \leq k \leq K$,
- and a positive integer T, called the *modulus*, which is greater than or equal to the largest part among all partitions in Π.

We say a *span one linked partition ideal* $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, T)$ is the collection of all partitions of the form

$$\begin{aligned}\lambda &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N) \oplus \phi^{(N+1)T}(\pi_1) \oplus \phi^{(N+2)T}(\pi_1) \oplus \cdots \\ &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N),\end{aligned} \tag{1.2}$$

where $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i and λ_N is not the empty partition. We also include in $\mathscr{I}$ the empty partition, which corresponds to $\phi^0(\pi_1) \oplus \phi^T(\pi_1) \oplus \cdots$. Here for any two partitions μ and ν, $\mu \oplus \nu$ gives a partition by collecting all parts in μ and ν, and $\phi^m(\mu)$ gives a partition by adding m to each part of μ.

I shared this idea with Li, and in [14], we obtained generating function identities for all six partition sets under difference conditions in the Kanade–Russell Conjectures, along with several new results.

Briefly speaking, given a partition set under a difference condition, we may construct a system of first-order q-difference equations for several closely-related generating functions, based on the above linked partition ideal decomposition. Then we developed an algorithm, which is a refinement of an approach due to Andrews, to deduce a q-difference equation of higher order for our target generating function. Next, we ran a computer search for a suitable summation expression satisfying this higher-order q-difference equation. Finally, we made use of two *Mathematica* packages `qMultiSum` and `qGeneratingFunctions` developed by the RISC group to confirm that the above summation expression is as desired.

Subsequently, in Summer 2019, I further extended the idea of span one linked partition ideals by viewing the partitions $\pi \in \Pi$ in Definition 1.1 as vertices of a direct graph. Also, there is an edge from π_i to π_j if $\pi_j \in \mathcal{L}(\pi_i)$. Thus, the chain of partitions in (1.2) can be regarded as a path in this direct graph. We may then assign weights to each vertex and finally define an attached generating function for the directed graph. With this broader setting, we may deal with not only span one linked partition ideals, but also other combinatorial objects that can be translated into this language. See [5].

In the same paper [5], I also developed a method to get rid of the computer assistance in [14]. This method is based on the study of a general family of q-multisums:

$$H(\underline{\boldsymbol{\beta}}) := \sum_{n_1,\ldots,n_R\geq 0} \frac{q^{\sum_{r=1}^R \alpha_{r,r}n_r(n_r-1)/2} q^{\sum_{1\leq i<j\leq R}\alpha_{i,j}n_in_j} q^{\sum_{r=1}^R \beta_r n_r} x^{\sum_{r=1}^R \gamma_r n_r}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_R};q^{A_R})_{n_R}}$$

where $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathrm{Mat}_{R\times R}(\mathbb{N})$ is a fixed symmetric matrix, and $\underline{\mathbf{A}} = (A_r) \in \mathbb{N}_{>0}^R$ and $\underline{\boldsymbol{\gamma}} = (\gamma_r) \in \mathbb{N}_{>0}^R$ are fixed vectors. The following identity is a key.

Lemma 1.1. *For $1 \leq r \leq R$, we have*

$$\begin{aligned} H(\beta_1,\ldots,\beta_r,\ldots,\beta_R) &= H(\beta_1,\ldots,\beta_r+A_r,\ldots,\beta_R) \\ &\quad + x^{\gamma_r}q^{\beta_r}H(\beta_1+\alpha_{r,1},\ldots,\beta_r+\alpha_{r,r},\ldots,\beta_R+\alpha_{r,R}). \end{aligned} \tag{1.3}$$

After joining Dalhousie as a postdoc fellow, I started a series of projects (one joint with George Andrews and Zhitai Li) to embed Schur's 1926 partition theorem into the framework of span one linked partition ideals. Schur's theorem, and its variants due to Krishnaswami Alladi and George Andrews, claim that the following partitions of the same size are equinumerous:

- Set $\mathscr{A}$: Partitions into parts congruent to ± 1 modulo 6;
- Set $\mathscr{B}$: Partitions into distinct nonmultiples of 3;
- Set $\mathscr{C}$: Partitions into odd parts with none appearing more than twice;
- Set $\mathscr{D}$: Partitions of the form $\mu_1+\mu_2+\cdots+\mu_s$ where $\mu_i-\mu_{i+1}\geq 3$ with strict inequality if $3 \mid \mu_i$;
- Set $\mathscr{E}$: Partitions (counted with weight $(-1)^\tau$ where τ is the number of parts that appear exactly twice) in which odd parts appear at most once, even parts appear at most twice, and the difference between two parts can never be 1 and can be 2 only if both are odd.

For any partition λ, we denote by $|\lambda|$ the sum of all parts in λ, by $\sharp(\lambda)$ the number of parts in λ, and by $\sharp_{a,M}(\lambda)$ the number of parts in λ that are congruent to a modulo M. Different methods were applied to treat the q-difference equations, as discussed beforehand, satisfied by our desired generating functions.

Computer algebra: Andrews, Li and I [2] still made use of the *Mathematica* packages `qMultiSum` and `qGeneratingFunctions`, and proved, for example, that

Theorem 1.2.

$$\sum_{\lambda\in\mathscr{D}} x^{\sharp(\lambda)}y^{\sharp_{0,2}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3}x^{n_1+n_2+2n_3}y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}} \times q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

This formula corresponds to Andrews' identity of Gleißberg type that links the partition sets $\mathscr{C}$ and $\mathscr{D}$.

***q*-Series:** I [9] extended Lemma 1.1 by allowing more free parameters $x_1, x_2, \ldots$, and proved, for example, that

Theorem 1.3.

$$\sum_{\lambda\in\mathscr{D}} x^{\sharp(\lambda)}y^{\sharp_{0,3}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^3;q^3)_{n_1}(q^3;q^3)_{n_2}(q^3;q^3)_{n_3}} \times q^{3\binom{n_1}{2}+3\binom{n_2}{2}+6\binom{n_3}{2}+3n_1n_2+3n_2n_3+3n_3n_1+n_1+2n_2+3n_3}.$$

A set of general results include some earlier generating function identities due to Krishnaswami Alladi and Basil Gordon.

Operator theory: I [10] borrowed the q-Borel operator, which was first developed in analysis, and proved that

Theorem 1.4.

$$\sum_{\lambda\in\mathscr{E}} x^{\sharp(\lambda)}y^{\tau(\lambda)}q^{|\lambda|-\sharp(\lambda)(\sharp(\lambda)-1)} = \frac{(-xq^2;q^2)_\infty}{\prod_{n\geq 0}(1-xq^{2n+1}-x^2yq^{4n+2})}.$$

This formula also implies that the standard generating function for $\mathscr{E}$ is related to the continuous q-Hermite polynomials.

Future work. At the *27th International Conference on Applications of Computer Algebra (ACA 2022)*, I presented a talk on "*Linked Partition Ideals and Computer Algebra*." At the same conference, Ali Uncu (U Bath & Austrian Academy of Sciences) also presented his results on cylindric partitions. It appears very feasible to consider cylindric partitions from the perspective of linked partition ideals. Uncu and I also had a lot of discussions on the applications of computer algebra in solving related problems. The largest obstacle we are facing is the high complexity of searching for a suitable sum-like expression for our desired generating function. In practice, we need to compute the first several coefficients of the formal power series solution to a higher-order q-difference equation, and find patterns from them, thereby relying highly on human intuitions. We hope to develop an algorithmic way to reduce the searching complexity.

2. Basic Hypergeometric Functions: *Andrews–Uncu Conjecture and Rogers–Ramanujan type identities*

In 2021, George Andrews and Ali Uncu investigated sequences in overpartitions, and as a consequence, they derived an asymmetric Rogers–Ramanujan type identity:

$$\sum_{m,n\geq 0} \frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+m+6n}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q;q^3)_\infty}. \tag{2.1}$$

Meanwhile, a similar identity was conjectured:

$$\sum_{m,n\geq 0} \frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+2m+7n}}{(q;q)_m (q^3;q^3)_n} = \frac{1}{(q^2,q^3;q^6)_\infty}. \tag{2.2}$$

This conjectural identity is much harder for the reason that the functional equation technique for (2.1), which is also commonly used in the proof of many Rogers–Ramanujan type identities, does not work well in this case.

I first learned this conjecture from George Andrews in our weekly online meeting at the end of 2021, and eventually in April 2022, I confirmed this conjecture in [11].

Theorem 2.1. *The Andrews–Uncu Conjecture* (2.2) *is true.*

The basic idea relies on rewriting the sum side in (2.2) as a contour integral:

$$\sum_{m,n\geq 0} \frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+2m+7n}}{(q;q)_m (q^3;q^3)_n} = (q;q)_\infty \oint \frac{(q^2z, qz, 1/z; q)_\infty}{(q^{4/3}z, \omega q^{4/3}z, \omega^2 q^{4/3}z; q)_\infty} \frac{\mathrm{d}\, z}{2\pi \mathrm{i} z},$$

where $\omega = \mathrm{e}^{2\pi i/3}$ and the integral is over a positively oriented contour separating 0 from all poles of the integrand. Further, this contour integral can be transformed in terms of basic hypergeometric series. This technique is recorded in Section 4.10 of George Gasper and Mizan Rahman's book "*Basic Hypergeometric Series*", and recently, Hjalmar Rosengren had already used this idea in his proofs of some Mod 12 Kanade–Russell Conjectures.

However, even with this powerful method, we still require a mysterious identity, which is *completely weird*, as pointed out by George Andrews in a personal communication.

Theorem 2.2.

$$\sum_{n\geq 0} \frac{(-1)^n q^{3\binom{n}{2}+4n}(q;q^3)_n}{(q^9;q^9)_n} = \frac{(q^4;q^6)_\infty (q^{12};q^{18})_\infty}{(q^5;q^6)_\infty (q^9;q^{18})_\infty}. \tag{2.3}$$

The difficulty of (2.3) mainly appears in two aspects: (i). its finitization remains unknown after a thorough computer search; (ii). it cannot fit into the framework of Bailey pairs. Now, one of the last attacks is based on finding the so-called a-generalization of (2.3), i.e., attaching one more free parameter a.

Theorem 2.3.

$$\sum_{n\geq 0} \frac{(a;q)_n (a^{-1}q^2;q^2)_n}{(a^2q;q^2)_n (q^3;q^3)_n} (-1)^n a^n q^{\binom{n}{2}+n} = \frac{(aq;q^2)_\infty (a^3q^3;q^6)_\infty}{(a^2q;q^2)_\infty (q^3;q^6)_\infty}. \tag{2.4}$$

As one may find, the unexpectedness rises as two extra q-shifted factorials are attached to the summand in (2.4) in comparison to the original (2.3), and in fact the construction of the two extra terms took me around four months. Finally, the proof of (2.4) relies on a trick from complex analysis, which I first saw in Mourad Ismail's proof of Ramanujan's ${}_1\psi_1$ sum published in 1977.

Future work. After posting the proof of the Andrews–Uncu Conjecture on arXiv, Wadim Zudilin (Radboud U Nijmegen) and Ali Uncu (U Bath & Austrian Academy of Sciences) had a lot of discussions with me on related problems. Our first ambition is to use the same technique to work on the Mod 9 Kanade–Russell Conjectures. Yet another important problem is from the limiting case of (2.3) with $q \to 1^-$:

$$ {}_2F_1\left(\begin{matrix} 2A, -A+1 \\ 2A+\frac{1}{2} \end{matrix}; -\frac{1}{3}\right) = \frac{\sqrt{\pi}}{3^A}\frac{\Gamma(2A+\frac{1}{2})}{\Gamma(A+\frac{1}{2})^2}. $$

It is known that Akihito Ebisu had developed an algorithm to produce strange Γ-evaluations of hypergeometric series. Therefore, finding a q-analog of Ebisu's algorithm might lead to more identities of Rogers–Ramanujan type.

3. *q*-Series: *General coefficient-vanishing results associated with theta series*

Given a Laurent series $G(q) = \sum_n g_n q^n \in \mathbb{C}[[q, q^{-1}]]$, a particularly interesting problem is to determine if there exists an arithmetic progression $Mn + w$ such that the coefficients of $G(q)$ indexed by this arithmetic progression vanish. In other words, $g_{Mn+w} = 0$ always holds true. For $G(q)$ a quotient of several Ramanujan's theta functions,

$$ f(a, b) := \sum_{n\in\mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_\infty, $$

the study of its vanishing coefficients indexed by an arithmetic progression has a long history, with one of the very first results obtained by Bruce Richmond and George Szekeres in their work on the Ramanujan–Göllnitz–Gordon continued fraction. More recently, Michael Hirschhorn considered the theta product

$$ \sum_{n\geq 0} \gamma(n) q^n := (-q, -q^4; q^5)_\infty (q, q^9; q^{10})_\infty^3, $$

and proved, for example, that $\gamma(5n + 2) = 0$.

In Spring 2022, based on his empirical discoveries, Dazhao Tang and I [16] started a project on a systematic investigation of this coefficient-vanishing phenomenon. Our ambitious target is to prove general families of such results, rather than repeating the endless case studies in the literature. One example of our results is as follows, which includes Hirschhorn's result as the $(\ell, m, \mu, k) = (0, 1, 1, 1)$ case.

Theorem 3.1. *Let*

$$ \sum_n \gamma_{i,j,r,\ell,m}(n) q^n := (-q^i, -q^{r-i}; q^r)_\infty^\ell (q^j, q^{2r-j}; q^{2r})_\infty^m. \tag{3.1} $$

Then for integers $\ell \geq 0$, $m \geq 0$, $\mu \geq 1$ and $k \geq 1$ with $\gcd(k, 4\ell + 2m + 3) = 1$,

$$ \gamma_{k,k,(4\ell+2m+3)\mu,2\ell+1,2m+1}\big((4\ell + 2m + 3)n + (\ell + m + 1)k\big) = 0. \tag{3.2} $$

Our starting point is James Mc Laughlin's recent generalization of a formula of Heinrich Schröter discovered in the 19th century, which tells us that for any theta power or any product of two theta powers, we may expand it as a linear combination, $\sum \mathscr{A}\mathscr{F}$, where $\mathscr{A}$ is a theta series, usually times a power of (-1) and a power of q, and $\mathscr{F}$ is a series in q^M for a certain M. Recall that the huffing operator for $G(q) = \sum_n g_n q^n \in \mathbb{C}[[q, q^{-1}]]$ and M a positive integer is defined by

$$ H_M\big(G(q)\big) := \sum_n g_{Mn} q^{Mn}. $$

Therefore, under the action of H_M, only $\mathscr{A}$ is effective in each summand of the above expansion $\sum \mathscr{A}\mathscr{F}$. Also, a surprising fact of this linear expansion is its hidden symmetry. To be precise, if we write the theta power

$$f\big((-1)^\kappa q^{k+A'},(-1)^\kappa q^{-k+(A-A')}\big)^m=\sum \mathscr{A}\mathscr{F}$$

as above, then for each summand $\mathscr{A}\mathscr{F}$ with only one exception, there exists a companion summand $\mathscr{A}'\mathscr{F}'$ such that $\mathscr{F}=\mathscr{F}'$. In other words, we may pair the summands and write the theta power as

$$f\big((-1)^\kappa q^{k+A'},(-1)^\kappa q^{-k+(A-A')}\big)^m=\mathscr{A}_0\mathscr{F}_0+\sum\big(\mathscr{A}_I+\mathscr{A}_{II}\big)\mathscr{F}.$$

With the above preparation in mind, the theta products in our general results, including that in (3.1), can be reformulated using our pairing process,

$$\Big(\mathscr{A}_0\mathscr{F}_0+\sum\big(\mathscr{A}_I+\mathscr{A}_{II}\big)\mathscr{F}\Big)\cdot\Big(\mathscr{B}_0\mathscr{G}_0+\sum\big(\mathscr{B}_I+\mathscr{B}_{II}\big)\mathscr{G}\Big),$$

where $\mathscr{F}$ and $\mathscr{G}$ (including $\mathscr{F}_0$ and $\mathscr{G}_0$) are series in q^M for a certain M.

Our next observation comes from the known coefficient-vanishing results in the literature. Briefly speaking, if a coefficient-vanishing phenomenon appears, we will encounter a series of cancelations according to the pairing process:

$$\begin{aligned}
&H_M\big(\mathscr{A}_0\mathscr{B}_0\big)=0,\\
&H_M\big(\mathscr{A}_0\mathscr{B}_I\big)=\pm H_M\big(\mathscr{A}_0\mathscr{B}_{II}\big), &&H_M\big(\mathscr{B}_0\mathscr{A}_I\big)=\pm H_M\big(\mathscr{B}_0\mathscr{A}_{II}\big),\\
&H_M\big(\mathscr{A}_I\mathscr{B}_I\big)=\pm H_M\big(\mathscr{A}_{II}\mathscr{B}_{II}\big), &&H_M\big(\mathscr{A}_I\mathscr{B}_{II}\big)=\pm H_M\big(\mathscr{A}_{II}\mathscr{B}_I\big).
\end{aligned}$$

Finally, the verification of the above relations is closely related to the behaviors of the following theta products,

$$\begin{aligned}
\mathscr{H}(q)&:=q^w f\big((-1)^\kappa q^{u+A'M},(-1)^\kappa q^{-u+(A-A')M}\big)f\big((-1)^\lambda q^{v+B'M},(-1)^\lambda q^{-v+(B-B')M}\big),\\
\hat{\mathscr{H}}(q)&:=q^{\hat{w}} f\big((-1)^\kappa q^{u+A'M},(-1)^\kappa q^{-u+(A-A')M}\big)f\big((-1)^\lambda q^{v+(B-B')M},(-1)^\lambda q^{-v+B'M}\big),\\
\check{\mathscr{H}}(q)&:=q^{\check{w}} f\big((-1)^\kappa q^{u+(A-A')M},(-1)^\kappa q^{-u+A'M}\big)f\big((-1)^\lambda q^{v+(B-B')M},(-1)^\lambda q^{-v+B'M}\big),
\end{aligned}$$

under the action of H_M. So we provide some effective criteria to check if $H_M\big(\mathscr{H}(q)\big)=0$ and if $H_M\big(\mathscr{H}(q)\big)$ equals $H_M\big(\hat{\mathscr{H}}(q)\big)$ or $H_M\big(\check{\mathscr{H}}(q)\big)$, and hence make our approach complete.

Future work. One interesting problem that Dazhao Tang (Chongqing Normal U) and I are continuing working on comes from the above criteria for checking $H_M\big(\mathscr{H}(q)\big)=0$ and $H_M\big(\mathscr{H}(q)\big)=H_M\big(\hat{\mathscr{H}}(q)\big)$ or $H_M\big(\check{\mathscr{H}}(q)\big)$. These criteria rely on the validity of a set of divisibility conditions. However, there are occasional cases that even if one or more of the divisibility conditions fail to hold, we still encounter vanishing coefficients in a general theta product or quotient. One such example is that for $M=4\ell+6m+8$, $\sigma=-(2\ell+2m+3)k$, $\kappa\in\{1\}$, $\lambda\in\{0,1\}$ and any k such that $\gcd(k,M)=1$,

$$H_M\left(q^\sigma\cdot f\big((-1)^\kappa q^{2k},(-1)^\kappa q^{\mu M-2k}\big)^{2\ell+1}\left(\frac{f\big(-q^{2k},-q^{\mu M-2k}\big)}{f\big((-1)^\lambda q^k,(-1)^\lambda q^{\mu M-k}\big)}\right)^{4m+4}\right)\stackrel{?}{=}0,$$

which indicates the existence of possible alternatives for these criteria.

4. Analytic Number Theory: *Seo–Yee Conjecture and nonmodular infinite products*

In their work on the index of seaweed algebras and integer partitions, Seunghyun Seo and Ae Ja Yee proved that an earlier conjecture of Vincent Coll, Andrew Mayers and Nick Mayers is equivalent to the following nonnegativity conjecture.

Conjecture 4.1 (Seo–Yee Conjecture)**.** *The series expansion of*

$$\frac{1}{(q,-q^3;q^4)_\infty} \tag{4.1}$$

has nonnegative coefficients.

As a q-hypergeometric proof of this conjecture is notoriously difficult to find, one may give hope to the approach of deriving an asymptotic formula for the coefficients. However, unlike products of Dedekind eta functions or Jacobi theta functions, which exhibit modular properties (see my papers [4, 6] for general results on the asymptotics of the two cases), a Rademacher-type proof fails for (4.1) in no longer modular. In the literature, one of the few works about the asymptotics for nonmodular infinite products is due to Emil Grosswald. In his paper, the infinite product $1/(q^a;q^M)_\infty$ with a prime modulus M is considered. However, a closer examination of Grosswald's paper reveals several mistakes. Also, a natural question is about the case where the modulus is composite.

I first learned about the Seo–Yee Conjecture in September of 2019, from Ae Ja Yee's talk at the Penn State Combinatorics/Partitions Seminar. I then spent that fall semester on the investigation of the asymptotic behavior of the logarithm of a general nonmodular infinite product:

$$\Phi_{a,M}(q) := \log\left(\frac{1}{(q^a;q^M)_\infty}\right), \tag{4.2}$$

when the complex variable q with $|q|<1$ approaches the unit circle. Here, M is a positive integer and a is any of $1,2,\ldots,M$.

In [12], the following result is proved.

Theorem 4.1. *Let X be a sufficiently large positive number. Let $q=e^{-\tau+2\pi ih/k}$ where $1\le h\le k\le\lfloor\sqrt{2\pi X}\rfloor =: N$ with $(h,k)=1$ (throughout, (m,n) denotes the greatest common divisor of integers m and n) and $\tau = X^{-1}+2\pi iY$ with $|Y|\le 1/(kN)$. Let M be a positive integer and a be any of $1,2,\ldots,M$. If we denote by b the unique integer between 1 and (k,M) such that $b\equiv -ha \pmod{(k,M)}$ and write*

$$b^* = \begin{cases} (k,M)-b & \text{if } b\neq (k,M),\\ (k,M) & \text{if } b=(k,M),\end{cases}$$

then

$$\log\left(\frac{1}{(q^a;q^M)_\infty}\right) = \frac{1}{\tau}\frac{(k,M)^2}{k^2M}\left(\pi^2\left(\frac{b^2}{(k,M)^2}-\frac{b}{(k,M)}+\frac{1}{6}\right)\right.$$
$$\left. +\, 2\pi i\left(-\zeta'\left(-1,\frac{b}{(k,M)}\right)+\zeta'\left(-1,\frac{b^*}{(k,M)}\right)\right)\right)+E$$

where

$$|\Re(E)| \ll_{a,M} X^{1/2}\log X.$$

Equipped with Theorem 4.1 and the circle method, I arrived at an asymptotic formula for the coefficients in the series expansion of the infinite product in the Seo–Yee Conjecture, which leads to its proof except for an examination up to an explicit upper bound.

Theorem 4.2. *Let*

$$G(q) := \sum_{n\geq 0} g(n)q^n = \frac{1}{(q,-q^3;q^4)_\infty}.$$

We have, as $n\to\infty$,

$$g(n) \sim \frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + (-1)^n \frac{\pi^{3/4}\Gamma(3/4)}{2^{11/4}3^{5/8}n^{5/8}} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right)$$

where $I_s(x)$ is the modified Bessel function of the first kind. Further, when $n\geq 2.4\times 10^{14}$, we have $g(n)>0$.

The basic idea of approximating $\Phi_{a,M}(q)$ is based on the standard Mellin transform, which allows us to rewrite $\Phi_{a,M}(q)$ as an integral over vertical lines. However, an important trick I used is the introduction of an auxiliary function

$$\Psi_{a,M}(q^*) := \log\left(\prod_{\substack{m\geq 1\\ m\equiv -ha \bmod M^*}} \frac{1}{1-e^{2\pi i\alpha a/M}(q^*)^m}\right),$$

where $M^* := \gcd(k,M)$ and $q^* := \exp\left(\frac{2\pi i\beta h'}{k} - \frac{2\pi}{Kz}\right)$ with some additionally defined parameters. It is this function that makes an explicit bound of the error terms possible.

Future work. Seunghyun Seo and Ae Ja Yee also have a general conjecture.

Conjecture 4.2. *For $m\geq 4$, every coefficient of $1/(q,-q^{m-1};q^m)_\infty$ is nonnegative.*

In Spring 2022, William Craig (U Virginia) communicated with me the idea that the major arcs for the coefficients in the general infinite product of Seo–Yee may be approached by the Euler–Maclaurin estimation, and I pointed out that the minor arcs may be effectively bounded by my Theorem 4.1. Also, for the initial coefficients, Craig and I are working with Hannah Burson (U Minnesota) and Dennis Eichhorn (UC Irvine) from a combinatorial perspective through a deep investigation of copartitions, which was introduced recently by the latter two. Actually, we have convinced ourselves that even the general conjecture of Seo–Yee is just the tip of the iceberg.

5. Representation Theory: *Ariki–Koike algebras and Andrews–Gordon type identities*

The Ariki–Koike algebras, denoted by $\mathcal{H}_{\mathbb{C},q;Q_1,\ldots,Q_m}\big(G(m,1,n)\big)$, can be viewed as the Iwahori–Hecke algebras associated to the complex reflection groups $G(m,1,n)\cong S_n\ltimes(\mathbb{Z}/m\mathbb{Z})^n$, where q and Q_i ($i=1,\ldots,m$) are parameters. They were introduced by Susumu Ariki and Kazuhiko Koike, and independently by Michel Broué and Gunter Malle, where Broué and Malle defined cyclotomic Hecke algebras for all complex reflection groups. It is known that the simple modules of the Ariki–Koike algebras (when the parameters are roots of unity) are labeled by the so-called Kleshchev multipartitions. In particular, for $Q_1=\cdots=Q_a=-1$, $Q_{a+1}=\cdots=Q_m=1$, and $q=-1$, Andrew Mathas constructed a set $\Lambda^{a,m}(n)$ of multipartitions to parametrize the simple modules of $\mathcal{H}_{\mathbb{C},q;Q_1,\ldots,Q_m}\big(G(m,1,n)\big)$.

Definition 5.1. We denote by $\Lambda^{a,m}$ the set of m-multipartitions (i.e. m-tuples of partitions) $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ such that

(1) each $\lambda^{(i)}$ is 2-restricted;
(2) $\ell(\lambda^{(i)}) + t_i \le \lambda_1^{(i+1)} + t_{i+1}$ for $1 \le i \le m-1$ where $t_1 = \cdots = t_a = 0$ and $t_{a+1} = \cdots = t_m = 1$.

Also, $\Lambda^{a,m}(n)$ denotes the subset of m-multipartitions of size n in $\Lambda^{a,m}$.

In a joint paper with Zhitai Li, Dennis Stanton, Ting Xue and Ae Ja Yee [15], we prove that the generating function for $\Lambda^{a,m}$ is essentially equivalent to the following new q-theoretic identity.

Theorem 5.1. *For $m \ge 1$ and $0 \le a \le m$,*

$$\frac{(-q;q)_\infty (q^{a+1}, q^{m+1-a}, q^{m+2}; q^{m+2})_\infty}{(q;q)_\infty}$$
$$= \sum_{N_1,\dots,N_m \ge 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q;q)_{N_1}} \begin{bmatrix} N_1 + \delta_{a,1} \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 + \delta_{a,2} \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} + \delta_{a,m-1} \\ N_m \end{bmatrix}, \tag{5.1}$$

where $\begin{bmatrix} M \\ N \end{bmatrix}$ is the q-binomial coefficient and δ is the Kronecker delta function.

Recall also the following identity of George Andrews

$$\frac{(q^{a+1}, q^{2k+2-a}, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty} = \sum_{N_1,\dots,N_{2k+1} \ge 0} \frac{q^{2\sum_{i=1}^k \binom{N_i+1}{2} - \sum_{i=1}^a N_i}}{(q;q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{k-1} \\ N_k \end{bmatrix},$$

which is an analytic counterpart of Basil Gordon's partition-theoretic extension of the Rogers–Ramanujan identities. It can be seen that our (5.1) is an overpartition analog of the above relation. What makes (5.1) more extraordinary is the fact that its sum side also exhibits symmetry for a; this is different from other overpartition analogs given by, for instance, George Andrews, and Sun Kim and Ae Ja Yee.

We also investigate multipartitions λ in $\Lambda^{a,2}$ from the perspective of a partition statistic called 2-residue, denoted as $\omega(\lambda)$. This statistic is in essence the BG-rank, which was introduced by Alexander Berkovich and Frank Garvan in their study of the Andrews–Stanley srank.

Theorem 5.2.

$$\sum_{\lambda \in \Lambda^{1,2}} x^{\omega(\lambda)} q^{|\lambda|} = \frac{(-q^2;q^2)_\infty (-xq, -x^{-1}q, q^2; q^2)_\infty}{(q^2;q^2)_\infty}$$

and

$$\sum_{\lambda \in \Lambda^{2,2}} x^{\omega(\lambda)} q^{|\lambda|} = \frac{(-q;q^2)_\infty (-x, -x^{-1}q^2, q^2; q^2)_\infty}{2(q^2;q^2)_\infty} + \frac{(q;q^2)_\infty (x, x^{-1}q^2, q^2; q^2)_\infty}{2(q^2;q^2)_\infty}.$$

In particular, these results imply that the number of simple modules in the block of $\mathcal{H}_{-1;-1,1}(W_n)$ (resp. $\mathcal{H}_{-1;-1,-1}(W_n)$), labeled by ω, equals the number of finite dimensional simple modules of $\mathcal{H}^{rat}_{\frac{1}{2},1}(W_{n-\omega^2})$ (resp. $\mathcal{H}^{rat}_{\frac{1}{2},\frac{1}{2}}(W_{n-\omega^2+\omega})$). Here, W_n is $G(2,1,n)$ for the Weyl group of type B_n (or C_n), and $\mathcal{H}^{rat}_{c_1,c_2}(W_n)$ denotes the rational Cherednik algebra of W_n, with parameter c_1 (resp. c_2) assigned to the reflections associated to hyperplanes $z_i = z_j$

(resp. $z_i = 0$). This provides evidence that maximal support and minimal support simple modules in dual blocks of category $\mathcal{O}$'s of cyclotomic rational double affine Hecke algebras correspond to each other under the so-called level-rank duality.

6. Mathematical Physics: *Theory of parafermions*

For $\mathfrak{g}$ a finite-dimensional Lie algebra, we denote by $\widehat{\mathfrak{g}}_k$ the (untwisted) affine Kac–Moody algebra at level k associated to $\mathfrak{g}$. Let Λ be an integrable dominant weight of $\widehat{\mathfrak{g}}_k$, and λ be a maximal weight in the representation with highest weight Λ. Then the so-called string functions c^Λ_λ are defined by

$$c^\Lambda_\lambda(\tau) = q^{s_\Lambda(\lambda)} \sum_{n\geq 0} \mathrm{mult}_\Lambda(\lambda - n\delta) e^{2\pi i\tau}$$

with δ the basic imaginary root and

$$s_\Lambda(\lambda) = \frac{|\Lambda+\rho|^2}{2(k+h^\vee)} - \frac{|\rho|^2}{2h^\vee} - \frac{|\lambda|^2}{2k},$$

where ρ is half of the sum of all positive roots of $\hat{\mathfrak{g}}$ and $h^\vee$ is the dual coexter number of $\hat{\mathfrak{g}}$. According to Victor Kač and Dale Peterson, these string functions are modular.

It is known that given $\widehat{\mathfrak{g}}_k$, one can construct a parafermionic conformal field theory. Algebraically, this theory can be thought of the GKO coset construction of the (generalized) WZW model based on $\widehat{\mathfrak{g}}_k$ constrained by $\widehat{\mathfrak{u}(1)}^n$, where $n = \mathrm{rank}(\mathfrak{g})$. We denote such a construction as $\mathrm{PF}_k(\mathfrak{g})$. On the other hand, the modules of $\mathrm{PF}_k(\mathfrak{g})$ are again characterized by Λ and λ, denoted by L^Λ_λ. The characters (or partition functions) of the modules are given by

$$b^\Lambda_\lambda(\tau) := \mathrm{Tr}\; e^{(L_0 - c/24)2\pi i\tau}$$

restricting to the module L^Λ_λ, where L_0 is the operator that diagonalizes the fock space of $\mathrm{PF}_k(\mathfrak{g})$. It is known that the parafermionic characters b^Λ_λ of $\mathrm{PF}_k(\mathfrak{g})$ are simply related to the string functions c^Λ_λ by

$$b^\Lambda_\lambda(\tau) = \eta(\tau)^n c^\Lambda_\lambda(\tau),$$

where $\eta(\tau)$ is the Dedekind eta function.

Recall that the representation theory of $\mathrm{PF}_2(\widehat{\mathfrak{sl}_n})$ involves intertwiners between various representations, so it is natural to think of fractional exclusion statistics where the statistical interactions of intertwiners are encoded into a matrix $\mathbf{G}$. Further, generalized commutation relations allow us to analyze the exclusion statistics of intertwiners, thereby connecting parafermionic characters b^Λ_λ with the so-called universal chiral partition function (UCPF), which is in the form of

$$\mathrm{UCPF}\,(\mathbf{G};\mathbf{a}) = q^\delta \sum_{N_i \geq 0} \frac{q^{\frac{1}{2}\mathbf{N}^\mathsf{T}\cdot\mathbf{G}\cdot\mathbf{N} - \mathbf{a}\cdot\mathbf{N}}}{\prod_i (q;q)_{N_i}}$$

where q^δ is a prefactor, $\mathbf{N} = (N_1, N_2, \ldots, N_m)^\mathsf{T}$, $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ and m is the number of quasi-particles.

In a joint paper with Peter Bouwknegt and Bolin Han [3], we focus on the cases of $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$. For instance, in the $\mathfrak{sl}_4$ case, we choose

$$\mathbf{G}_4 := \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}$$

and thus an untwisted representation is given by

$$\mathrm{UCPF}\left(\mathbf{G}_4;\mathbf{0}\right) := q^{-\frac{1}{12}} \sum_{N_i\geq 0} \frac{q^{\frac{1}{2}\mathbf{N}^\mathsf{T}\cdot\mathbf{G}_4\cdot\mathbf{N}}}{\prod_i (q;q)_{N_i}}$$

with $\mathbf{N}^\mathsf{T} = (N_1,N_2,N_3,N_4,N_5,N_6)$.

Theorem 6.1.

$$\mathrm{UCPF}\left(\mathbf{G}_4;\mathbf{0}\right) = b^{2000}_{2000} + b^{2000}_{0020} + 6b^{2000}_{0101},$$

where b^{2000}_{2000}, b^{2000}_{0020} and b^{2000}_{0101} are explicitly expressed string functions.

Such results also lead us to interesting connections with lattice characters for $\mathbb{Z}_2$-orbifolds of scaled $\mathfrak{sl}_n$-root lattices. For the root lattice of $\mathfrak{sl}_{n+1}$, denoted by A_n, there is a standard process to construct a bosonic conformal field theory from A_n. Also, for $A_n/\sqrt{2}$, a fermionic conformal field theory obtained by scaling the root lattice A_n by a factor of $1/\sqrt{2}$, its modules are in correspondence with $\left(A_n/\sqrt{2}\right)^*/\left(A_n/\sqrt{2}\right)$, where L^* denotes the dual lattice of L. The expressions of characters of these modules are also straightforward:

$$\chi_\gamma(\tau) = \eta(\tau)^{-n} \sum_{v\in A_n/\sqrt{2}+\gamma} q^{\frac{1}{2}|\gamma|^2},$$

where $\gamma \in \left(A_n/\sqrt{2}\right)^*/\left(A_n/\sqrt{2}\right)$.

Let $\{\alpha_1,\alpha_2\}$ be simple roots of $\mathfrak{sl}_3$ and they form a basis of $A_2/\sqrt{2}$. Then the dual lattice $\left(A_2/\sqrt{2}\right)^*$ has a basis $\{\frac{\sqrt{2}}{3}(\alpha_1-\alpha_2), \sqrt{2}\alpha_1\}$, so there are three cosets in $\left(A_2/\sqrt{2}\right)^*/\left(A_2/\sqrt{2}\right)$, represented by $\gamma_0 := 0$, $\gamma_1 := \frac{2}{3\sqrt{2}}(\alpha_1-\alpha_2)$ and $\gamma_2 := \frac{1}{3\sqrt{2}}(\alpha_1-\alpha_2)$.

Theorem 6.2.

$$\chi_{\gamma_0} = \mathrm{UCPF}\left(\mathbf{G}_4;\mathbf{0}\right).$$

The proofs of these results are based on some new multiple Rogers–Ramanujan type identities. One example is as follows.

Theorem 6.3.

$$\sum_{N_1,N_2,N_3,N_4,N_5,N_6\geq 0} \frac{q^{\mathbf{N}^\mathsf{T}\cdot\mathbf{G}_4\cdot\mathbf{N}}}{(q^2;q^2)_{N_1}(q^2;q^2)_{N_2}(q^2;q^2)_{N_3}(q^2;q^2)_{N_4}(q^2;q^2)_{N_5}(q^2;q^2)_{N_6}}$$
$$= \frac{(q^2;q^2)^3_\infty (q^6;q^6)^5_\infty}{(q;q)^2_\infty (q^3;q^3)^2_\infty (q^4;q^4)^2_\infty (q^{12};q^{12})^2_\infty} + \frac{4q(q^4;q^4)^2_\infty (q^{12};q^{12})^2_\infty}{(q^2;q^2)^3_\infty (q^6;q^6)_\infty}.$$

In fact, such identities are special cases of several general q-hypergeometric transformations. Here, I give one instance.

Theorem 6.4. *For $\delta_1, \delta_2 \in \{0,1\}$,*

$$\sum_{N_1,N_2,N_3,N_4,N_5,N_6\geq 0} \frac{z_1^{N_1+N_3-N_4-N_6} z_2^{N_2+N_3-N_5-N_6} q^{\mathbf{N}^\intercal\cdot\mathbf{G}_4\cdot\mathbf{N}-2\delta_1(N_4+N_6)-2\delta_2(N_5+N_6)}}{(q^2;q^2)_{N_1}(q^2;q^2)_{N_2}(q^2;q^2)_{N_3}(q^2;q^2)_{N_4}(q^2;q^2)_{N_5}(q^2;q^2)_{N_6}}$$
$$= \frac{1}{(q^2;q^2)_\infty^2} \sum_{M_1,M_2=-\infty}^{\infty} z_1^{M_1} z_2^{M_2} q^{M_1^2+M_2^2-M_1M_2}(1+\delta_1 q^{2M_1})(1+\delta_2 q^{2M_2}).$$

Future work. In general, Doron Gepner conjectured an identity, which is of Rogers–Ramanujan type, for the characters of the parafermionic field theories for all Lie algebras at any level. For the sum side of his identity, Gepner already constructed conjectural explicit expressions. However, for the other side, although known to be modular by the Kač–Peterson theory, it is, in most cases, not a single quotient of Dedekind eta functions (or Jacobi theta functions). But when we are facing linear combinations of such quotients, it is usually very hard to formulate an explicit expression. Bolin Han (Australian National U) communicated with me the idea of directly analyzing the sum side through q-hypergeometric techniques and we are continuing along this line.

7. Enumerative Combinatorics: *Patterns in permutations and inversion sequences*

Let $\mathfrak{S}_n$ be the set of permutations on $\{1,2,\ldots,n\} =: [n]$. If entries in a permutation are treated in a cyclic way, we are naturally led to an equivalence relation $\sim_{\text{cyc}}$ on $\mathfrak{S}_n$ (e.g., $\pi_1\cdots\pi_n \sim_{\text{cyc}} \pi_{k+1}\cdots\pi_n\pi_1\cdots\pi_k$). From this perspective, we may define cycles on $[n]$ as equivalence classes in the quotient set $\mathfrak{C}_n := \mathfrak{S}_n/\sim_{\text{cyc}}$.

We say a consecutive pair (π_i,π_{i+1}) (with $1\leq i\leq n$) is a *drop* in a cycle $[\pi]\in\mathfrak{C}_n$ if $\pi_i > \pi_{i+1}$. Also, a drop (π_i,π_{i+1}) is called *odd-odd* (resp. *even-odd*) if π_i is odd (resp. π_i is even) and π_{i+1} is odd. Now, for each cycle $[\pi]$, we denote by $\operatorname{drop}_{oo}([\pi])$ the number of odd-odd drops and by $\operatorname{drop}_{eo}([\pi])$ the number of even-odd drops.

In 2019, Alexander Lazar and Michelle Wachs made an intriguing conjecture.

Conjecture 7.1. *For all $n\geq 1$, the number of cycles on $[2n]$ with only even-odd drops equals the n-th unsigned Genocchi number.*

After seeing proofs of this conjecture due to Zhicong Lin and Sherry Yan, and independently to Qiongqiong Pan and Jiang Zeng, my interest was drawn to an analogous problem on the enumeration of cycles with only odd-odd drops. Since each $[\pi]\in\mathfrak{C}_n$ must have a drop of the form $(a,1)$ for some a, my attention was then restricted to a subset of $\mathfrak{C}_n$:

$$\mathfrak{C}_n^o := \{\pi\in\mathfrak{C}_n : \pi_{i+1} \text{ is odd for all drops } (\pi_i,\pi_{i+1}) \text{ in } [\pi]\}.$$

In [7], I proved two generating function identities.

Theorem 7.1.

$$\sum_{n\geq 1}\sum_{[\pi]\in\mathfrak{C}_n^o} x^{\operatorname{drop}_{oo}([\pi])} t^n = \sum_{m\geq 1} \frac{m!(m-1)!t^{2m}}{\prod_{k=1}^m \big(1+k^2(1-x)t^2\big)} + \sum_{m\geq 1} \frac{\big((m-1)!\big)^2 t^{2m-1}}{\prod_{k=1}^m \big(1+k^2(1-x)t^2\big)}$$

and

$$\sum_{n\geq 1}\sum_{[\pi]\in\mathfrak{C}_n^o} y^{\operatorname{drop}_{eo}([\pi])} t^n$$

$$= (y-1)t^2 + \sum_{m\geq 1} \frac{m!(m-1)!t^{2m}}{\prod_{k=1}^{m}\big(1+k(k+1)(1-y)t^2\big)} + \sum_{m\geq 1} \frac{\big((m-1)!\big)^2 t^{2m-1}}{\prod_{k=1}^{m}\big(1+k(k-1)(1-y)t^2\big)}.$$

Now, the first generating function in the above yields an alternative proof of the Lazar–Wachs Conjecture, while the latter implies the following corollary.

Corollary 7.2. *For all $n \geq 2$, the number of cycles on $[2n-1]$ with only odd-odd drops equals the $(n-2)$-th unsigned Genocchi median.*

Also, it is known that permutations have a natural coding through the so-called *inversion sequences*, which are sequences $e = e_1e_2\cdots e_n$ such that $0 \leq e_i \leq i-1$ for all $i \in [n]$. We usually denote by $\mathbf{I}_n$ the set of inversion sequences of length n.

In 2016, Megan Martinez and Carla Savage considered inversion sequences under pattern restrictions according to a given triple of binary relations (ρ_1, ρ_2, ρ_3) with $\rho_1, \rho_2, \rho_3 \in \{<, >, \leq, \geq, =, \neq, -\}$.

Definition 7.1. We denote by $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ where $\rho_1, \rho_2, \rho_3 \in \{<, >, \leq, \geq, =, \neq, -\}$ the set of inversion sequences $e = e_1e_2\cdots e_n \in \mathbf{I}_n$ such that there are no indices $1 \leq i < j < k \leq n$ with

$$e_i\,\rho_1\,e_j, \quad e_j\,\rho_2\,e_k \quad \text{and} \quad e_i\,\rho_3\,e_k.$$

Subsequently, a handful of Wilf equivalences among the 343 possible sets of inversion sequences avoiding patterns of relation triples were established. A further direction for the study of pattern avoidance is to take account of various statistics and investigate their distribution over pattern avoiding sequences. Along this line, Zhicong Lin conjectured a curious identity concerning the ascent statistic over $\mathbf{I}_n(\geq, \neq, >)$ and $\mathbf{I}_n(>, \neq, \geq)$.

Conjecture 7.2. *For $n \geq 1$,*

$$\sum_{e\in\mathbf{I}_n(\geq,\neq,>)} z^{\mathrm{asc}(e)} = \sum_{e\in\mathbf{I}_n(>,\neq,\geq)} z^{n-1-\mathrm{asc}(e)}.$$

In [1], George Andrews and I confirmed Lin's Conjecture algebraically through the kernel method.

Theorem 7.3. *Lin's Conjecture is true.*

The main trick in our proof is based on separating $\mathbf{I}_n(\geq, \neq, >)$ into four disjoint subclasses. Then with the help of a generating tree, we are able to construct a system for the generating functions for the four cases. Finally, the kernel method allows us to deduce their "explicit" expressions. With a similar process applied to $\mathbf{I}_n(>, \neq, \geq)$, Lin's Conjecture is proved by comparing the two sets of generating functions.

Future work. We say that a sequence e avoids a given pattern $P = p_1p_2\cdots p_k$ if none of the subsequences of e are order isomorphic to P. The enumerations for inversion sequences avoiding any length-3 pattern were well categorized in the past. For length-4 patterns, the first nontrivial result,

$$|\mathbf{I}_n(0012)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1},$$

was conjectured by Zhicong Lin and Jun Ma in 2019, and was confirmed by myself [8] in 2020. Subsequently, Letong Hong and Rupert Li made further investigations on length-4 patterns.

In particular, they conjectured that $|\mathbf{I}_n(0021)|$ corresponds to the OEIS sequence A218225. This was confirmed by Shishuo Fu (Chongqing U), Zhicong Lin (Shandong U) and me [13] in Summer 2022. It was also conjectured by Alexander Burstein that the same sequence also counts pattern restricted permutations $\mathfrak{S}_n(3124, 42153, 24153)$, $\mathfrak{S}_n(2134, 42153, 24153)$, and $\mathfrak{S}_n(2143, 42135, 24135)$, the first of which was also proved by us in [13]. For the remaining two permutation pattern sets, we expect that more complicated decompositions are necessary, and we are still working on them.

8. Problems for students

One important proportion of research concerns the guidance of young students. Although I have no formal mentoring experience, George Andrews, my PhD advisor, invited me to give advice to Zhitai Li, my academic brother on his research over the past years. With this experience, and other chances from working with my collaborators, I started compiling a collection of problems at the research level that are suitable for new graduate students or undergraduates. Here, I will present three of them.

Student Problem 1. *Analyze other partition sets using linked partition ideals.*

As I have explained earlier in this research statement, the framework of linked partition ideals serves as an important tool in the study of partition generating functions. Aside from the instances related to the Kanade–Russell conjectures and Schur's partition theorem, one may find more applications to other partition sets. Working on these parallel problems will help undergraduates get a foot in the colorful world of partitions. On some occasions, one also needs to make slight modifications to this framework, like what I did for the Euclidean billiard partitions. This would then be a good topic for a Master's thesis.

Student Problem 2. *Case studies of George Andrews' "Separable Integer Partition Classes."*

This newly introduced concept, Separable Integer Partition Classes (SIPs), attributes to George Andrews. It can be viewed as a younger sibling of linked partition ideals, for it is also designed for the study of partition identities. Since the theory of SIPs is still in its infancy, careful case studies will undoubtedly lead to a decent development for SIPs. There are many sources for such case studies. For example, all of the Rogers–Ramanujan type identities in Lucy Slater's famous list were polynomialized by Andrew Sills in his PhD thesis. It would be an interesting project for students to transplant Andrews' methodology for these polynomial identities.

Student Problem 3. *Search for more instances of the coefficient-vanishing phenomena for quotients of theta series.*

The general coefficient-vanishing results proved by Dazhao Tang and I are still the tip of the iceberg. We have numerical evidence that there are more uncovered cases to be unearthed. One important step along this road is to search for, with computer assistance, more instances of this nature. I believe such new examples will reveal general relations, and proofs of these examples will become a blueprint of a broad theory. So if students go into depth, their work on this topic will be a promising component of their thesis.

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