Linked partition ideals and a Schur-type identity of Andrews

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Abstract. In his contribution to the *Proceedings of the 1998 AMS–IMS–SIAM Joint Summer Research Conference on q-Series, Combinatorics, and Computer Algebra*, Andrews considered a variant of Schur's partition theorem, concerning partitions in which odd parts appear at most once, even parts appear at most twice, and the difference between two parts can never be 1 and can be 2 only if both are odd. Fitting into the framework of linked partition ideals, we obtain a non-standard trivariate generating function for such partitions that counts both the number of parts and the number of different parts that appear twice; the latter statistic plays an important role in Andrews' identity. In particular, we are led to an application of using q-Borel operators in solving certain q-difference equations. Finally, we show that the regular trivariate generating function for such partitions has an interesting connection with the continuous q-Hermite polynomials.

Keywords. Linked partition ideals, Schur's partition theorem, generating function, q-Borel operator, continuous q-Hermite polynomials.

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1. Introduction

In 1926, Schur [13] proved the following result:

Theorem S. *Let $A(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 6. Let $B(n)$ denote the number of partitions of n into distinct nonmultiples of 3. Let $D(n)$ denote the number of partitions of n of the form $\mu_1+\mu_2+\cdots+\mu_s$ where $\mu_i - \mu_{i+1} \geq 3$ with strict inequality if $3 \mid \mu_i$. Then*

$$A(n) = B(n) = D(n).$$

This partition theorem has many variants, one of which is due to Andrews [5]:

Theorem A. *We consider partitions in which odd parts appear at most once, even parts appear at most twice, and the difference between two parts can never be 1 and can be 2 only if both are odd. Let $E(n)$ denote the weighted count of these partitions with weight $(-1)^\tau$ for each partition that has exactly τ parts that appear twice. Then*

$$A(n) = B(n) = D(n) = E(n).$$

A combinatorial proof of the fact that $D(n) = E(n)$ was later provided by Yee [14].

In recent years, there are a substantial amount of papers studying generating functions for certain partition sets that can be represented as an *Andrews–Gordon type series* of the form

$$\sum_{n_1,\ldots,n_r\geq 0} \frac{(-1)^{L_1(n_1,\ldots,n_r)} q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}}, \tag{1.1}$$

in which L_1 and L_2 are linear forms and Q is a quadratic form in $n_1, \ldots, n_r$, and the q-Pochhammer symbol is defined for $n \in \mathbb{N} \cup \{\infty\}$,

$$(A;q)_n := \prod_{k=0}^{n-1} (1 - Aq^k).$$

In particular, in the previous papers of this series [7–10], such representations are associated with the framework of linked partition ideals, with, especially, [7] and [9] dealing with identities born out of Schur's Theorem S.

For any partition λ, we denote by $|\lambda|$ the sum of all parts in λ, and by $\sharp(\lambda)$ the number of parts in λ. We also denote by $\tau(\lambda)$ the number of different parts in λ that appear twice.

Let $\mathscr{A}$ denote the set of partitions counted by $E(n)$ for all nonnegative n.

Although it looks like the trivariate generating function for partitions λ in $\mathscr{A}$ that counts both statistics $\sharp(\lambda)$ and $\tau(\lambda)$ does not have a simple representation as an Andrews–Gordon type series, our object is the following non-standard generating function identity.

Theorem 1.1. *We have*

$$\sum_{\lambda \in \mathscr{A}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda| - \sharp(\lambda)(\sharp(\lambda)-1)} = \frac{(-xq^2;q^2)_\infty}{\prod\limits_{n \geq 0} (1 - xq^{2n+1} - x^2 y q^{4n+2})}. \tag{1.2}$$

Setting $y = -1$ yields a new proof of Theorem A.

Corollary 1.2. *We have*

$$\sum_{\lambda \in \mathscr{A}} (-1)^{\tau(\lambda)} x^{\sharp(\lambda)} q^{|\lambda|} = \sum_{n_1, n_2 \geq 0} \frac{(-1)^{n_2} q^{3\binom{n_1}{2} + 18\binom{n_2}{2} + 6n_1 n_2 + n_1 + 9n_2} x^{n_1 + 3n_2}}{(q;q)_{n_1} (q^6;q^6)_{n_2}}. \tag{1.3}$$

In particular, for $n \geq 0$,

$$D(n) = E(n).$$

Finally, we recall that the *continuous q-Hermite polynomials* $H_n(x;q)$ are given by

$$H_n(x;q) := e^{\mathrm{i}n\theta} {}_2\phi_0 \left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix} ; q, q^n e^{-2\mathrm{i}\theta} \right) \qquad (\text{with } x = \cos\theta),$$

where the basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s \left(\begin{matrix} a_1, a_2 \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right) := \sum_{n \geq 0} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s-r+1} z^n.$$

The continuous q-Hermite polynomials are a family of q-orthogonal polynomials in the basic Askey scheme. See [11, Section 3.26] for details. In particular, they satisfy a second-order recurrence for $n \geq 1$,

$$H_{n+1}(x;q) = 2xH_n(x;q) - (1 - q^n) H_{n-1}(x;q) \tag{1.4}$$

with $H_0(x;q) = 1$ and $H_1(x;q) = 2x$.

We show that the generating function for the partition set $\mathscr{A}$ is related to the continuous q-Hermite polynomials.

Corollary 1.3. *We have*

$$\sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)}y^{\tau(\lambda)}q^{|\lambda|} = \sum_{M,N\geq 0} \frac{q^{2\binom{M}{2}+4\binom{N}{2}+2MN+M+2N}x^{M+N}t_M(y)}{(q^2;q^2)_M(q^2;q^2)_N}, \tag{1.5}$$

where

$$t_M(y) = (-\mathrm{i})^M y^{M/2} H_M(\tfrac{\mathrm{i}}{2}y^{-1/2};q^2). \tag{1.6}$$

Remark 1.1. By (1.4), we know that as a polynomial in x, $H_n(x;q)$ has degree n. Further, when n is even, then terms in $H_n(x;q)$ with an odd exponent of x vanish; when n is odd, then terms in $H_n(x;q)$ with an even exponent of x vanish. Therefore, $t_M(y)$ is a polynomial in y of degree $\lfloor M/2\rfloor$.

2. Linked partition ideals and a matrix equation

The general theory of linked partition ideals was proposed by Andrews [1–3] in the 1970s; see [4, Chapter 8] for an introduction. In recent years, a special type of linked partition ideals, called *span one linked partition ideals*, was revisited by Chern and Li [10] and Chern [8] to associate this theory with Andrews–Gordon type series.

Definition 2.1. Assume that we are given

- a finite set $\Pi = \{\pi_1,\pi_2,\ldots,\pi_K\}$ of integer partitions with $\pi_1 = \emptyset$, the empty partition,
- a *map of linking sets*, $\mathcal{L} : \Pi \to P(\Pi)$, the power set of Π, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\emptyset) = \Pi$ and $\pi_1 = \emptyset \in \mathcal{L}(\pi_k)$ for any $1 \leq k \leq K$,
- and a positive integer T, called the *modulus*, which is greater than or equal to the largest part among all partitions in Π.

We say a *span one linked partition ideal* $\mathscr{I} = \mathscr{I}(\langle\Pi,\mathcal{L}\rangle,T)$ is the collection of all partitions of the form

$$\begin{aligned}\lambda &= \phi^0(\lambda_0)\oplus\phi^T(\lambda_1)\oplus\cdots\oplus\phi^{NT}(\lambda_N)\oplus\phi^{(N+1)T}(\pi_1)\oplus\phi^{(N+2)T}(\pi_1)\oplus\cdots\\ &= \phi^0(\lambda_0)\oplus\phi^T(\lambda_1)\oplus\cdots\oplus\phi^{NT}(\lambda_N),\end{aligned} \tag{2.1}$$

where $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i and λ_N is not the empty partition. We also include in $\mathscr{I}$ the empty partition, which corresponds to $\phi^0(\pi_1)\oplus\phi^T(\pi_1)\oplus\cdots$. Here for any two partitions μ and ν, $\mu\oplus\nu$ gives a partition by collecting all parts in μ and ν, and $\phi^m(\mu)$ gives a partition by adding m to each part of μ.

Lemma 2.1. *$\mathscr{A}$ is the span one linked partition ideal $\mathscr{I}(\langle\Pi,\mathcal{L}\rangle,2)$, where $\Pi = \{\pi_1=\emptyset,\pi_2=(1),\pi_3=(2),\pi_4=(2+2)\}$ and*

$$\begin{cases}\mathcal{L}(\pi_1)=\mathcal{L}(\pi_2)=\{\pi_1,\pi_2,\pi_3,\pi_4\},\\ \mathcal{L}(\pi_3)=\mathcal{L}(\pi_4)=\{\pi_1\}.\end{cases}$$

Proof. We decompose each partition in $\mathscr{A}$ into blocks $B_0,B_1,\ldots$ such that all parts between $2i+1$ and $2i+2$ fall into block B_i. By the definition of $\mathscr{A}$, we find that if we apply the operator ϕ^{-2i} to the block B_i, then it is among Π. If $\phi^{-2i}(B_i)$ is π_1 or π_2, then $\phi^{-2(i+1)}(B_{i+1})$ can be any among Π. If $\phi^{-2i}(B_i)$ is π_3 or π_4, then this partition has a part of size $2i+2$ and therefore the next different part is at least $2i+5$ since its difference with $2i+2$ cannot be 1 or 2. Thus, in this case, the block B_{i+1} is empty, that is $\phi^{-2(i+1)}(B_{i+1}) = \pi_1$. Conversely, it is straightforward to verify that all partitions in $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 2)$ satisfy the difference conditions defined for $\mathscr{A}$. $\square$

From now on, we always decompose any partition $\lambda \in \mathscr{A} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 2)$ as in (2.1). Further, for each $1 \le k \le 4$, we define

$$G_k(x) := \sum_{\substack{\lambda \in \mathscr{A} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|},$$

the generating function for partitions whose first decomposed block is π_k.

By the definition of span one linked partition ideals, we have

$$G_k(x) = x^{\sharp(\pi_k)} y^{\tau(\pi_k)} q^{|\pi_k|} \sum_{j : \pi_j \in \mathcal{L}(\pi_k)} G_j(xq^2). \tag{2.2}$$

Therefore,

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & xq & & \\ & & xq^2 & \\ & & & x^2yq^4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1(xq^2) \\ G_2(xq^2) \\ G_3(xq^2) \\ G_4(xq^2) \end{pmatrix}. \tag{2.3}$$

We then define

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix}. \tag{2.4}$$

Substituting (2.3) into (2.4) yields the following matrix equation:

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq & & \\ & & xq^2 & \\ & & & x^2yq^4 \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \\ F_4(xq^2) \end{pmatrix}. \tag{2.5}$$

3. q-Borel operators

In this section, following [12,15], we introduce a family of operators $\mathcal{B}_k$ for integers k, which can be treated as q-analogs of the Borel transformation.

Definition 3.1. Let $\mathbb{K}$ be a field. Let $F(x) = \sum_{n\ge 0} f(n)x^n \in \mathbb{K}(q)[[x]]$. We define the operator $\mathcal{B}_k$ for $k \in \mathbb{Z}$ by

$$\mathcal{B}_k\big(F(x)\big) := \sum_{n\ge 0} f(n) q^{-k\binom{n}{2}} x^n. \tag{3.1}$$

The following property of $\mathcal{B}_k$ will play an important role.

Lemma 3.1. *Let $F(x) \in \mathbb{K}(q)[[x]]$. For any integers k and N, and nonnegative integer M, we have*

$$\mathcal{B}_k\big(x^M F(xq^N)\big) = x^M q^{-k\binom{M}{2}} \mathcal{B}_k\big(F(xq^{N-kM})\big). \tag{3.2}$$

Proof. Let us write $F(x)=\sum_{n\ge 0} f(n)x^n$. Then

$$\begin{aligned}\mathcal{B}_k\big(x^M F(xq^N)\big) &= \sum_{n\ge 0} f(n) q^{-k\binom{M+n}{2}+Nn} x^{M+n}\\ &= x^M q^{-k\binom{M}{2}} \sum_{n\ge 0} f(n) q^{-k\left(Mn+\binom{n}{2}\right)+Nn} x^n\\ &= x^M q^{-k\binom{M}{2}} \sum_{n\ge 0} f(n) q^{-k\binom{n}{2}} (xq^{N-kM})^n\\ &= x^M q^{-k\binom{M}{2}} \mathcal{B}_k\big(F(xq^{N-kM})\big),\end{aligned}$$

which is our desired result. □

4. Non-standard generating function

In this section, we solve the matrix equation (2.5) and give a proof of Theorem 1.1. First, we observe from (2.4) that

$$\begin{aligned}\sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|} &= G_1(x)+G_2(x)+G_3(x)+G_4(x)\\ &= F_1(x). \end{aligned}\tag{4.1}$$

Theorem 4.1. *Let*

$$P(x) := \sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|}.$$

Then

$$P(x) = (1+xq)P(xq^2) + (xq^2+x^2yq^4)P(xq^4). \tag{4.2}$$

Proof. By (2.5), we have

$$F_2(x) = F_1(x) \tag{4.3}$$

and

$$F_3(x) = F_4(x) = F_1(xq^2). \tag{4.4}$$

Also,

$$F_1(x) = F_1(xq^2) + xqF_2(xq^2) + xq^2F_3(xq^2) + x^2yq^4F_4(xq^2).$$

Inserting (4.3) and (4.4) into the above and recalling that $P(x)=F_1(x)$, we arrive at the desired result. □

It looks not easy to solve the q-difference equation in (4.2) directly. Now, we show how to take advantage of the q-Borel operators to prove Theorem 1.1.

Proof of Theorem 1.1. We apply $\mathcal{B}_2$ to both sides of (4.2). Then

$$\begin{aligned}\mathcal{B}_2\big(P(x)\big) &= \mathcal{B}_2\big(P(xq^2)\big) + xq\mathcal{B}_2\big(P(x)\big)\\ &\quad + xq^2\mathcal{B}_2\big(P(xq^2)\big) + x^2yq^2\mathcal{B}_2\big(P(x)\big).\end{aligned}$$

For convenience, we define

$$Q(x) := \mathcal{B}_2\big(P(x)\big).$$

Then,

$$(1 - xq - x^2yq^2)Q(x) = (1 + xq^2)Q(xq^2).$$

Recalling that $Q(0) = P(0) = 1$, we have

$$Q(x) = \prod_{n\geq 0} \frac{1 + xq^{2n+2}}{1 - xq^{2n+1} - x^2yq^{4n+2}}.$$

Finally, we notice that

$$\begin{aligned} Q(x) &= \mathcal{B}_2\big(P(x)\big) \\ &= \mathcal{B}_2\left(\sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)}y^{\tau(\lambda)}q^{|\lambda|}\right) \\ &= \sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)}y^{\tau(\lambda)}q^{|\lambda|-2\binom{\sharp(\lambda)}{2}}. \end{aligned}$$

We are therefore led to (1.2). □

5. Theorem A

The object of this section is an alternative proof of Theorem A. Our starting point is (1.3).

Proof of (1.3). Setting $y = -1$ in (1.2) gives

$$\begin{aligned} \sum_{\lambda\in\mathscr{A}} (-1)^{\tau(\lambda)} x^{\sharp(\lambda)} q^{|\lambda|-\sharp(\lambda)(\sharp(\lambda)-1)} &= \frac{(-xq^2;q^2)_\infty}{\prod_{n\geq 0}(1 - xq^{2n+1} + x^2q^{4n+2})} \\ &= (-xq^2;q^2)_\infty \frac{(-xq;q^2)_\infty}{(-x^3q^3;q^6)_\infty} \\ &= \frac{(-xq;q)_\infty}{(-x^3q^3;q^6)_\infty}. \end{aligned}$$

Recall Euler's first and second summations [4, Corollary 2.2, p. 19]:

$$\frac{1}{(t;q)_\infty} = \sum_{m\geq 0} \frac{t^m}{(q;q)_m} \tag{5.1}$$

and

$$(t;q)_\infty = \sum_{m\geq 0} \frac{(-t)^m q^{m(m-1)/2}}{(q;q)_m}. \tag{5.2}$$

We have

$$\sum_{\lambda\in\mathscr{A}} (-1)^{\tau(\lambda)} x^{\sharp(\lambda)} q^{|\lambda|-\sharp(\lambda)(\sharp(\lambda)-1)} = \sum_{n_1\geq 0} \frac{x^{n_1} q^{\binom{n_1}{2}+n_1}}{(q;q)_{n_1}} \sum_{n_2\geq 0} \frac{(-1)^{n_2} x^{3n_2} q^{3n_2}}{(q^6;q^6)_{n_2}}.$$

Therefore,

$$\sum_{\lambda\in\mathscr{A}} (-1)^{\tau(\lambda)} x^{\sharp(\lambda)} q^{|\lambda|} = \sum_{n_1,n_2\geq 0} \frac{(-1)^{n_2} q^{\binom{n_1}{2}+n_1+3n_2} x^{n_1+3n_2} q^{2\binom{n_1+3n_2}{2}}}{(q;q)_{n_1}(q^6;q^6)_{n_2}}$$

$$= \sum_{n_1,n_2\ge 0} \frac{(-1)^{n_2} q^{3\binom{n_1}{2}+18\binom{n_2}{2}+6n_1n_2+n_1+9n_2} x^{n_1+3n_2}}{(q;q)_{n_1}(q^6;q^6)_{n_2}}.$$

This is our desired result. $\square$

Now, we are ready to show that $D(n) = E(n)$.

Let $\mathscr{S}$ denote the set of partitions counted by $D(n)$ in Theorem S for all nonnegative n. Andrews, Bringmann and Mahlburg [6] proved that the generating function for the partition set $\mathscr{S}$ can be represented as a double series:

$$\sum_{\lambda\in\mathscr{S}} x^{\sharp(\lambda)} q^{|\lambda|} = \sum_{n_1,n_2\ge 0} \frac{(-1)^{n_2} q^{3\binom{n_1}{2}+18\binom{n_2}{2}+6n_1n_2+n_1+9n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^6;q^6)_{n_2}}. \tag{5.3}$$

We therefore have

$$\sum_{\lambda\in\mathscr{A}} (-1)^{\tau(\lambda)} q^{|\lambda|} = \sum_{\lambda\in\mathscr{S}} q^{|\lambda|} = \sum_{n_1,n_2\ge 0} \frac{(-1)^{n_2} q^{3\binom{n_1}{2}+18\binom{n_2}{2}+6n_1n_2+n_1+9n_2}}{(q;q)_{n_1}(q^6;q^6)_{n_2}}.$$

This implies that $D(n) = E(n)$.

6. The continuous q-Hermite polynomials

Here, we prove the generating function identity in Corollary 1.3. Let

$$S(x) = \sum_{M\ge 0} s_M(y) x^M := \frac{1}{\prod_{n\ge 0}(1 - xq^{2n+1} - x^2yq^{4n+2})}.$$

We may compute that $s_0(y) = 1$ and $s_1(y) = q/(1-q^2)$. Also, we have

$$(1 - xq - x^2yq^2)S(x) = S(xq^2).$$

Therefore, for $M \ge 1$,

$$(1 - q^{2M+2}) s_{M+1}(y) = q s_M(y) + yq^2 s_{M-1}(y). \tag{6.1}$$

Now, we define, for $M \ge 0$,

$$t_M(y) := (q^2;q^2)_M q^{-M} s_M(y).$$

Then $t_0(y) = t_1(y) = 1$. Also, (6.1) becomes

$$t_{M+1}(y) = t_M(y) + y(1 - q^{2M}) t_{M-1}(y) \tag{6.2}$$

for $M \ge 1$. To build the connection between $t_M(y)$ and the continuous q-Hermite polynomials, we define, for $M \ge 0$,

$$r_M(y) = (2y)^M t_M(-\tfrac{1}{4y^2}).$$

Then $r_0(y) = 1$ and $r_1(y) = 2y$. Further, (6.2) becomes

$$r_{M+1}(y) = 2y r_M(y) - (1 - q^{2M}) r_{M-1}(y) \tag{6.3}$$

for $M \ge 1$. Comparing with (1.4), we have

$$r_M(y) = H_M(y; q^2)$$

for $M \ge 0$. Thus, (1.6) is established.

Finally,

$$\begin{aligned} S(x) &= \sum_{M\geq 0} s_M(y)x^M \\ &= \sum_{M\geq 0} \frac{x^M q^M t_M(y)}{(q^2;q^2)_M}. \end{aligned}$$

Thus, by Euler's second summation (5.2),

$$\begin{aligned} \sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|-\sharp(\lambda)(\sharp(\lambda)-1)} &= \frac{(-xq^2;q^2)_\infty}{\prod_{n\geq 0}(1-xq^{2n+1}-x^2yq^{4n+2})} \\ &= \sum_{M\geq 0} \frac{x^M q^M t_M(y)}{(q^2;q^2)_M} \sum_{N\geq 0} \frac{x^N q^{2\binom{N}{2}+2N}}{(q^2;q^2)_N}. \end{aligned}$$

We conclude that

$$\sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)} y^{\tau(\lambda)} q^{|\lambda|} = \sum_{M,N\geq 0} \frac{t_M(y) q^{2\binom{N}{2}+M+2N} x^{M+N} q^{2\binom{M+N}{2}}}{(q^2;q^2)_M (q^2;q^2)_N},$$

which yields (1.5).

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