Hankel determinants and Jacobi continued fractions for q-Euler numbers

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(Joint work with Lin Jiu 酒霖)

Introduction

Lin Jiu (Duke Kunshan)

Introduction

Lin Jiu (Duke Kunshan)

Karl Dilcher (Dalhousie)

Introduction

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Orthogonal polynomials and Hankel determinants for certain Bernoulli and Euler polynomials ☆

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ABSTRACT

Using continued fraction expansions of certain polygamma functions as a main tool, we find orthogonal polynomials with respect to the odd-index Bernoulli polynomials $B_{2k+1}(x)$ and the Euler polynomials $E_{2k+\nu}(x)$, for $\nu = 0, 1, 2$. In the process we also determine the corresponding Jacobi continued fractions (or J-fractions) and Hankel determinants. In all these cases the Hankel determinants are polynomials in x which factor completely over the rationals.

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Hankel determinants of sequences related to Bernoulli and Euler polynomials

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HANKEL DETERMINANTS OF SHIFTED SEQUENCES OF BERNOULLI AND EULER NUMBERS

KARL DILCHER AND LIN JIU*

ABSTRACT. Hankel determinants of sequences related to Bernoulli and Euler numbers have been studied before, and numerous identities are known. However, when a sequence is shifted by one unit, the situation often changes significantly. In this paper we use classical orthogonal polynomials and related methods to prove a general result concerning Hankel determinants for shifted sequences. We then apply this result to obtain new Hankel determinant evaluations for a total of 14 sequences related to Bernoulli and Euler numbers, one of which concerns Euler polynomials.

Introduction

- A *Hankel matrix* $(M_{i,j})$ is a square matrix with constant skew diagonals, i.e., $M_{i,j} = M_{i',j'}$ whenever $i + j = i' + j'$:

$$\begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n+1} \\ s_2 & s_3 & s_4 & \cdots & s_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & s_{n+2} & \cdots & s_{2n} \end{pmatrix}$$

- Letting $\{s_n\}_{n\geq 0}$ be a sequence in a field $\mathbb{K}$, one may define its associated Hankel matrices by $(s_{i+j})_{0\leq i,j\leq n}$.
- We are interested in the **determinant** of these matrices.

Example 1:

- *Bernoulli numbers* $B_n = (1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, \ldots)$:

$$\sum_{n\geq 0} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}.$$

Hankel determinants:

$$\det(1) = 1,\ \det\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{6} \end{pmatrix} = -\frac{1}{12},\ \det\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & -\frac{1}{30} \end{pmatrix} = -\frac{1}{540},\ \ldots$$

$$1, -\frac{1}{12}, -\frac{1}{540}, \frac{1}{42000}, \frac{1}{3215625}, -\frac{4}{623959875}, \ldots$$

Al-Salam and Carlitz (1959):

$$\det_{0\leq i,j\leq n}(B_{i+j}) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{(k!)^6}{(2k)!(2k+1)!}.$$

Introduction

Example 2:

- *Euler numbers* $E_n = (1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, -50521, \ldots)$:

$$\sum_{n\geq 0} E_n \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}}.$$

Hankel determinants:

$$1, -1, -4, 144, 82944, -1194393600, \ldots$$

Al-Salam and Carlitz (1959):

$$\det_{0\leq i,j\leq n}(E_{i+j}) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} (k!)^2.$$

The World of q

- *q-integers*: For $m \in \mathbb{Z}$,

$$[m]_q := \frac{1-q^m}{1-q}.$$

- *q-factorials*: For $M \in \mathbb{N}$,

$$[M]_q! := \prod_{m=1}^{M} [m]_q.$$

- *q-Pochhammer symbols*: For $N \in \mathbb{N} \cup \{\infty\}$,

$$(A;q)_N := \prod_{k=0}^{N-1} (1 - Aq^k),$$
$$(A, B, \ldots, C; q)_N := (A;q)_N (B;q)_N \cdots (C;q)_N.$$

The World of q

- *q-Bernoulli numbers* β_n (Carlitz, 1948):

$$\beta_n := \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k+1}{[k+1]_q}.$$

Alternatively, they can be recursively defined by $\beta_0 = 1$ and for $n \geq 1$,

$$\sum_{k=0}^{n}\binom{n}{k}q^{k+1}\beta_k - \beta_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

Nombres de q-Bernoulli–Carlitz et fractions continues
F Chapoton, J Zeng - Journal de théorie des nombres de Bordeaux, 2017 - numdam.org
Carlitz a introduit vers 1950 des q-analogues des nombres de Bernoulli. On obtient une représentation de ces q-analogues (ainsi que de variantes décalées) comme moments de …
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The World of q

- **Hankel determinants for q-Bernoulli numbers:**

$$\det_{0\le i,j\le n}(\beta_{i+j}) = (-1)^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \prod_{k=1}^{n} \frac{([k]_q!)^6}{[2k]_q![2k+1]_q!}.$$

THE STARTING POINT

1. The Carlitz β

Carlitz [1, 2] generated the Bernoulli numbers to the sequence β_n, by the recurrence:

$$\sum_{k=0}^{n}\binom{n}{k}\beta_k q^{k+1} - \beta_m = \begin{cases}1, & m=1;\\ 0, & m>1,\end{cases}$$

with also the value $\beta_0 = 1$.

Definition 1. The q-bracket is defined by

$$[x]_q := \frac{1-q^x}{1-q},$$

for all $x\in\mathbb{R}$ and $q>0$. The q-factorial is then defined by

$$[k]_q! := [k]_q[k-1]_q\cdots[1]_q.$$

Conjecture 2.

$$H_n(\beta_k) = (-1)^{\binom{n+1}{2}} q^{\frac{(n-1)n(n+1)}{6}} \frac{\prod_{k=1}^{n}[k]_q^{6(n+1-k)}}{\prod_{k=1}^{2n+1}[k]_q^{2n+2-k}}.$$

From Lin, Mar 14, 2023

From Shane on Apr 04, 2023.

Throughout, let us use Carlitz's q-Bernoulli numbers β_k given by $\beta_0 = 1$ and for $n\ge 1$,

$$\sum_{k=0}^{n}\binom{n}{k}\beta_k q^{k+1} - \beta_n = \begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n\ge 2.\end{cases}$$

"Need to Prove" 1.

$$\sum_{k\ge 0}\beta_k x^k = \cfrac{\beta_0}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\ddots}}}, \tag{1}$$

where

$$a_n = \frac{1}{(1+q^n)(1+q^{n+1})}\left(-\frac{(1-q^n)(1-q^{n+1})}{1-q}+2q^n\right), \tag{2}$$

$$b_n = -\frac{q^{n-1}(1-q^n)^6}{(1-q)^2(1-q^{2n-1})(1-q^{2n})^2(1-q^{2n+1})}. \tag{3}$$

From Shane, Apr 04, 2023

Déjà vu!

Nombres de q-Bernoulli–Carlitz et fractions continues

F Chapoton, J Zeng

Journal de théorie des nombres de Bordeaux, 2017 • numdam.org

Resume

Carlitz a introduit vers 1950 des q-analogues des nombres de Bernoulli. On obtient une représentation de ces q-analogues (ainsi que de variantes décalées) comme moments de certains polynômes orthogonaux. Ceci donne aussi des factorisations des déterminants de Hankel des nombres de q-Bernoulli, ainsi que des fractions continues pour leurs séries génératrices. Certains de ces résultats sont des q-analogues d'énoncés connus pour les nombres de Bernoulli, mais d'autres sont sans version classique.

Abstract. q-Bernoulli–Carlitz Numbers and continuous fractions.

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Théorème 4.2. *On a, pour les matrices d'indices* $0 \leq i, j \leq n-1$,

$$\det\left(\beta_{i+j}\right)_{i,j} = (-1)^{\binom{n}{2}} q^{\binom{n}{3}} \prod_{i=1}^{n-1} \frac{[i]!_q^6}{[2i]!_q [2i+1]!_q}, \tag{4.7}$$

The World of q

- *q-Euler numbers* ϵ_n (Carlitz, 1948):

$$\epsilon_n := \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+q}{1+q^{k+1}}.$$

Alternatively, they can be recursively defined by $\epsilon_0 = 1$ and for $n \geq 1$,

$$\sum_{k=0}^{n}\binom{n}{k}q^{k+1}\epsilon_k + \epsilon_n = 0.$$

- At the $q \to 1$ limit, ϵ_n reduces to

$$1, -\tfrac{1}{2}, 0, \tfrac{1}{4}, 0, -\tfrac{1}{2}, 0, \tfrac{17}{8}, 0, -\tfrac{31}{2}, \ldots,$$

which is identical to $E_n(0)$ rather than $2^n E_n(\frac{1}{2})$, the Euler numbers E_n. Here the *Euler polynomials* $E_n(x)$ are given by

$$\sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1}.$$

Theorem (C.–Jiu, 2023)

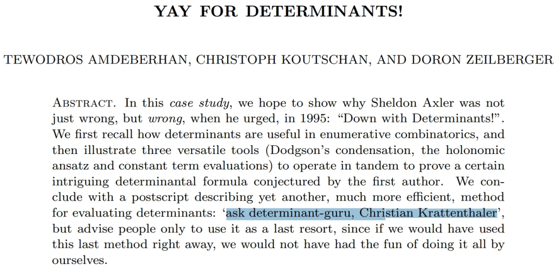
$$\det_{0\le i,j\le n}(\epsilon_{i+j}) = \frac{(-1)^{\binom{n+1}{2}} q^{\frac{1}{4}\binom{2n+2}{3}}}{(1-q)^{n(n+1)}} \prod_{k=1}^{n} \frac{(q^2,q^2;q^2)_k}{(-q,-q^2,-q^2,-q^3;q^2)_k},$$

$$\det_{0\le i,j\le n}(\epsilon_{i+j+1}) = \frac{(-1)^{\binom{n+2}{2}} q^{\frac{1}{4}\binom{2n+4}{3}}}{(1-q)^{n(n+1)}(1+q^2)^{n+1}} \prod_{k=1}^{n} \frac{(q^2,q^4;q^2)_k}{(-q^2,-q^3,-q^3,-q^4;q^2)_k},$$

$$\det_{0\le i,j\le n}(\epsilon_{i+j+2}) = \frac{(-1)^{\binom{n+2}{2}} q^{\frac{1}{4}\binom{2n+4}{3}}(1+q)^n\left(1-(-1)^n q^{(n+2)^2}\right)}{(1-q)^{n(n+1)}(1+q^2)^{2(n+1)}(1+q^3)^{n+1}} \prod_{k=1}^{n} \frac{(q^4,q^4;q^2)_k}{(-q^3,-q^4,-q^4,-q^5;q^2)_k}.$$

Determinant-guru

Christian Krattenthaler



YAY FOR DETERMINANTS!

TEWODROS AMDEBERHAN, CHRISTOPH KOUTSCHAN, AND DORON ZEILBERGER

Abstract. In this *case study*, we hope to show why Sheldon Axler was not just wrong, but *wrong*, when he urged, in 1995: "Down with Determinants!". We first recall how determinants are useful in enumerative combinatorics, and then illustrate three versatile tools (Dodgson's condensation, the holonomic ansatz and constant term evaluations) to operate in tandem to prove a certain intriguing determinantal formula conjectured by the first author. We conclude with a postscript describing yet another, much more efficient, method for evaluating determinants: 'ask determinant-guru, Christian Krattenthaler', but advise people only to use it as a last resort, since if we would have used this last method right away, we would not have had the fun of doing it all by ourselves.

Determinant-guru

- C. Krattenthaler, Advanced determinant calculus, *Sém. Lothar. Combin.* **42** (1999), Art. B42q, 67 pp.
- C. Krattenthaler, Advanced determinant calculus: a complement, *Linear Algebra Appl.* **411** (2005), 68–166.

ADVANCED DETERMINANT CALCULUS

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Dedicated to the pioneer of determinant evaluations (among many other things), George Andrews

Abstract. The purpose of this article is threefold. First, it provides the reader with a few useful and efficient tools which should enable her/him to evaluate nontrivial determinants for the case such a determinant should appear in her/his research. Second, it lists a number of such determinants that have been already evaluated, together with explanations which tell in which contexts they have appeared. Third, it points out references where further such determinant evaluations can be found.

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Linear Algebra and its Applications 411 (2005) 68–166

LINEAR ALGEBRA AND ITS APPLICATIONS

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Advanced determinant calculus: A complement[☆]

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Abstract

This is a complement to my previous article “Advanced Determinant Calculus” [C. Krattenthaler, Advanced determinant calculus, Séminaire Lotharingien Combin. 42 (1999) (“The Andrews Festschrift”), Article B42q, 67 pp.]. In the present article, I share with the reader my experience of applying the methods described in the previous article in order to solve a particular problem from number theory [G. Almkvist, C. Krattenthaler, J. Petersson, Some new formulas for π, Experiment. Math. 12 (2003) 441–456]. Moreover, I add a list of determinant evaluations which I consider as interesting, which have been found since the appearance of the previous article, or which I failed to mention there, including several conjectures and open problems.

Hankel determinants

Jacobi continued fractions **(a.k.a. *J-fractions*):**

Fact (Heilermann)

Let $\{\mu_k\}_{k\geq 0}$ be a sequence such that its generating function $\sum_{k\geq 0}\mu_k x^k$ has the J-fraction expression

$$\sum_{k\geq 0}\mu_k x^k = \cfrac{\mu_0}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\ddots}}}.$$

Then for $n\geq 0$,

$$\det_{0\leq i,j\leq n}(\mu_{i+j}) = \mu_0^{n+1}b_1^n b_2^{n-1}\cdots b_{n-1}^2 b_n.$$

***Jacobi continued fractions* (a.k.a. *J-fractions*):**

Fact (Heilermann (continued...))

Further, define $\{p_n(z)\}_{n\geq 0}$ a family of polynomials given by a three-term recursive relation for $n \geq 1$,

$$p_{n+1}(z) = (a_n + z)p_n(z) - b_n p_{n-1}(z),$$

with initial conditions $p_0(z) = 1$ and $p_1(z) = a_0 + z$. Then

$$\det_{0\leq i,j\leq n}(\mu_{i+j+1}) = \det_{0\leq i,j\leq n}(\mu_{i+j}) \cdot (-1)^{n+1} p_{n+1}(0).$$

Hankel determinants

Orthogonal polynomials: A family of polynomials $\{p_n(z)\}_{n\geq 0}$ with $p_n(z)$ of degree n is called *orthogonal* if there is a linear functional L on the space of polynomials in z such that $L\big(p_m(z)p_n(z)\big) = \delta_{m,n}\sigma_n$ where $\delta_{m,n}$ is the Kronecker delta and $\{\sigma_n\}_{n\geq 0}$ is a fixed nonzero sequence.

Orthogonal polynomials $\Leftrightarrow$ ***Three-term recurrences***:

Fact (Favard, Stieltjes)

Let $\{p_n(z)\}_{n\geq 0}$ be a family of monic polynomials with $p_n(z)$ of degree n. Then they are orthogonal if and only if there exist sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 1}$ with $b_n \neq 0$ such that $p_0(z) = 1$, $p_1(z) = a_0 + z$, and for $n \geq 1$,

$$p_{n+1}(z) = (a_n + z)p_n(z) - b_n p_{n-1}(z).$$

Hankel determinants

Orthogonal polynomials $\Rightarrow$ ***J-fractions***:

Fact

Let L be an associated linear functional for a family of orthogonal monic polynomials $\{p_n(z)\}_{n\geq 0}$ with $p_n(z)$ of degree n. Then

$$\sum_{k\geq 0} L(z^k)x^k = \cfrac{L(z^0)}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\ddots}}},$$

where $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 1}$ are given by the associated three-term recurrence.

How to evaluate Hankel determinants:

- STEP 1: Guess the J-fraction expression
- STEP 2: Formulate the associated orthogonal polynomials
- STEP 3: Determine a suitable linear functional
- STEP 4: Check orthogonality under this linear functional

Proof

STEP 1: Guess the J-fraction expression.

A

B

Out[69]=
$$\left\{\frac{q}{1+q^2},\ \frac{-1-q+2q^3+q^4+q^5}{(1+q^2)(1+q^4)},\ \frac{-1-q-q^2-q^3+2q^5+q^6+q^7+q^8+q^9}{(1+q^2)(1+q^4)(1-q^2+q^4)},\right.$$
$$\frac{-1-q-q^2-q^3-q^4-q^5+2q^7+q^8+q^9+q^{10}+q^{11}+q^{12}+q^{13}}{(1+q^2)(1-q^2+q^4)(1+q^8)},$$
$$\left.\frac{-1-q-q^2-q^3-q^4-q^5-q^6-q^7+2q^9+q^{10}+q^{11}+q^{12}+q^{13}+q^{14}+q^{15}+q^{16}+q^{17}}{(1+q^2)(1+q^8)(1-q^2+q^4-q^6+q^8)}\right\}$$

Out[70]=
$$\left\{-\frac{q}{(1+q^2)^2(1-q+q^2)},\ -\frac{q^3(1+q^2)^2}{(1-q+q^2)(1+q^4)^2(1-q+q^2-q^3+q^4)},\right.$$
$$-\frac{q^5(1-q+q^2)^2(1+q+q^2)^2}{(1+q^2)^2(1-q^2+q^4)^2(1-q+q^2-q^3+q^4)(1-q+q^2-q^3+q^4-q^5+q^6)},$$
$$-\frac{q^7(1+q^2)^2(1+q^4)^2}{(1-q+q^2)(1-q^3+q^6)(1-q+q^2-q^3+q^4-q^5+q^6)(1+q^8)^2},$$
$$-\Big(\big(q^9(1-q+q^2-q^3+q^4)^2(1+q+q^2+q^3+q^4)^2\big)\Big/\Big(\big(1+q^2\big)^2(1-q+q^2)(1-q^3+q^6)$$
$$\left.(1-q^2+q^4-q^6+q^8)^2(1-q+q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9+q^{10})\Big)\Big)\right\}$$

Proof

STEP 1: Guess the J-fraction expression.

Guess(!)

Let $\ell \in \{0, 1\}$. *Then*

$$\sum_{k\geq 0} \epsilon_{k+\ell} x^k \overset{?}{=} \cfrac{\epsilon_\ell}{1 + a_{\ell,0}x - \cfrac{b_{\ell,1}x^2}{1 + a_{\ell,1}x - \cfrac{b_{\ell,2}x^2}{1 + a_{\ell,2}x - \ddots}}},$$

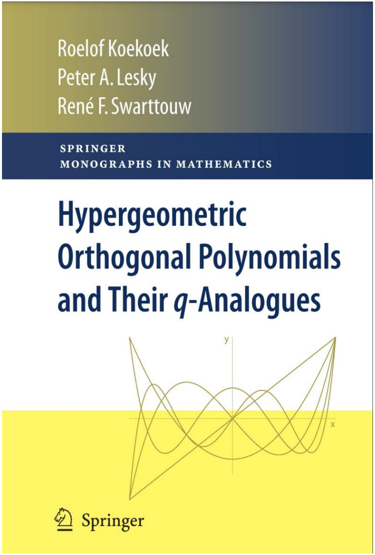
where

$$a_{\ell,n} = \frac{q^{2n+\ell}(1+q)(1+q^\ell)}{(1-q)(1+q^{2n+\ell})(1+q^{2n+\ell+2})} - \frac{1}{1-q},$$

$$b_{\ell,n} = -\frac{q^{2n+2\ell-1}(1-q^{2n})(1-q^{2n+2\ell})}{(1-q)^2(1+q^{2n+\ell-1})(1+q^{2n+\ell})^2(1+q^{2n+\ell+1})}.$$

Proof

STEP 2: Formulate the associated orthogonal polynomials.



R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer-Verlag, Berlin, 2010.

Proof

STEP 2: Formulate the associated orthogonal polynomials.

☞ *Big q-Jacobi polynomials*:

$$\mathcal{J}_{\ell,n}(z) := {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, z \\ q^{\ell+1}, 0 \end{matrix}; q, q\right).$$

Theorem

Define

$$\mathcal{P}_{\ell,n}(z) := \frac{(-1)^n (q^{\ell+1};q)_n}{q^n (1-q)^n (-q^{n+\ell+1};q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, q(1-(1-q)z) \\ q^{\ell+1}, 0 \end{matrix}; q, q\right).$$

We have $\mathcal{P}_{\ell,0}(z) = 1$, $\mathcal{P}_{\ell,1}(z) = a_{\ell,0} + z$, *and for* $n \geq 1$,

$$\mathcal{P}_{\ell,n+1}(z) = (a_{\ell,n} + z)\mathcal{P}_{\ell,n}(z) - b_{\ell,n}\mathcal{P}_{\ell,n-1}(z).$$

Proof

STEP 3: Determine a suitable linear functional.

Want to study the linear function Φ with

$$\Phi(z^n) = \epsilon_n.$$

☞ Choose a "nice" basis for the vector space of polynomials over $\mathbb{Q}(q)$:

$$\left\{ \begin{bmatrix} n, z \\ n \end{bmatrix}_q \right\}_{n \geq 0},$$

where (cf. Chapoton–Essouabri, 2015)

$$\begin{bmatrix} m, z \\ n \end{bmatrix}_q := \frac{1}{[n]_q!} \prod_{k=m-n+1}^{m} \left([k]_q + q^k z \right).$$

Proof

STEP 3: Determine a suitable linear functional.

Let Φ be the linear functional on $\mathbb{Q}(q)[z]$ given by

$$\Phi\left(\begin{bmatrix} n, z \\ n \end{bmatrix}_q\right) := \frac{1}{(-q^2;q)_n} \qquad (n \geq 0).$$

Lemma

For $0 \leq m \leq n$,

$$\Phi\left(\begin{bmatrix} m, z \\ n \end{bmatrix}_q\right) = \frac{(-1)^{n-m} q^{n-m}}{(-q^2;q)_n}.$$

Lemma

For $n \geq 0$,

$$\Phi\left(\begin{bmatrix} n+1, z \\ n \end{bmatrix}_q\right) = \frac{1+q}{q} - \frac{1}{q(-q^2;q)_n}.$$

Proof

STEP 3: Determine a suitable linear functional.

Theorem

For any $P(z) \in \mathbb{Q}(q)[z]$,

$$q\Phi\big(P(1+qz)\big) + \Phi\big(P(z)\big) = (1+q)P(0).$$

Theorem

For $n \geq 0$,

$$\Phi(z^n) = \epsilon_n.$$

Proof. We first notice that $\Phi(z^0) = \Phi(1) = 1 = \epsilon_0$. Now for $n \geq 1$, we apply the above theorem with $P(z) = z^n$, and derive that

$$q\Phi\big((1+qz)^n\big) + \Phi(z^n) = 0,$$

namely,

$$\sum_{k=0}^{n} \binom{n}{k} q^{k+1} \Phi(z^k) + \Phi(z^n) = 0.$$

This recursive relation for $\Phi(z^n)$ is identical to that for ϵ_n!!!

STEP 4: Check orthogonality under this linear functional.

Let $\ell \in \{0, 1\}$. We define two linear functionals Φ_ℓ on $\mathbb{Q}(q)[z]$ by

$$\Phi_\ell(z^n) := \Phi(z^{n+\ell}) \qquad (n \geq 0).$$

Theorem

Let $\ell \in \{0, 1\}$. The family of monic polynomials $\{\mathcal{P}_{\ell,n}(z)\}_{n\geq 0}$ is orthogonal under the linear functional Φ_ℓ.

☞ Show that for $\ell \in \{0, 1\}$, the identity $\Phi_\ell\big(\mathcal{P}_{\ell,n}(z)\big) = 0$ holds whenever $n \geq 1$:

$$\mathcal{P}_{0,n}(z) = \frac{(-1)^n (q;q)_n}{q^n (1-q)^n (-q^{n+1};q)_n} \sum_{k=0}^{n} \frac{q^k (q^{-n}, -q^{n+1}; q)_k}{(q;q)_k} \begin{bmatrix} k, z \\ k \end{bmatrix}_q,$$

$$z \cdot \mathcal{P}_{1,n}(z) = \frac{(-1)^n (q^2;q)_n}{q^n (1-q)^n (-q^{n+2};q)_n} \sum_{k=0}^{n} \frac{q^k (q^{-n}, -q^{n+2}; q)_k}{(q;q)_k} \begin{bmatrix} k, z \\ k+1 \end{bmatrix}_q.$$

Question A: Families of q-Hankel determinants?

Note that

$$\mathcal{P}_{\ell,n}(z) = \frac{(-1)^n}{q^n(1-q)^n}\widetilde{\mathcal{J}}_{\ell,n}\big((q^2-q)z+q\big),$$

where $\widetilde{\mathcal{J}}_{\ell,n}(z)$ are normalizations of the big q-Jacobi $\mathcal{J}_{\ell,n}(z)$ as monic polynomials:

$$\widetilde{\mathcal{J}}_{\ell,n}(z) := \frac{(q^{\ell+1};q)_n}{(-q^{n+\ell+1};q)_n}\mathcal{J}_{\ell,n}(z),$$

Theorem (C.–Jiu, 2023)

For each nonnegative integer ℓ, define a sequence $\{\xi_{\ell,n}\}_{n\geq 0}$ by

$$\xi_{\ell,n} := \frac{q^{(\ell+1)n}(-q;q)_n}{(-q^{\ell+2};q)_n}.$$

Then

$$\det_{0\leq i,j\leq n}(\xi_{\ell,i+j}) = (-1)^{\binom{n+1}{2}} q^{2\binom{n+2}{3}+(2\ell+1)\binom{n+1}{2}} \prod_{k=1}^{n} \frac{(q^2,q^{2\ell+2};q^2)_k}{(-q^{\ell+1},-q^{\ell+2},-q^{\ell+2},-q^{\ell+3};q^2)_k}.$$

Question B: Bernoulli/Euler polynomials?

- *Bernoulli polynomials* $B_n(x)$ and *Euler polynomials* $E_n(x)$:

$$\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1} \qquad \text{and} \qquad \sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1}.$$

In partucular,

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \qquad \text{and} \qquad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} x^{n-k}.$$

- **Invariance of Hankel determinants under the binomial transform:** For every $n \geq 0$, define

$$s_n(x) := \sum_{k=0}^{n} \binom{n}{k} s_k x^{n-k}.$$

Then

$$\det_{0\leq i,j\leq n} \left(s_{i+j}\right) = \det_{0\leq i,j\leq n} \big(s_{i+j}(x)\big).$$

Question B: Bernoulli/Euler polynomials?

- Trivial determinant evaluations(!):

$$\det_{0\le i,j\le n}\left(B_{i+j}(x)\right) = (-1)^{\binom{n+1}{2}}\prod_{k=1}^{n}\frac{(k!)^6}{(2k)!(2k+1)!},$$

$$\det_{0\le i,j\le n}\left(E_{i+j}(x)\right) = \left(-\tfrac{1}{4}\right)^{\binom{n+1}{2}}\prod_{k=1}^{n}(k!)^2,$$

- **Dilcher–Jiu (2021):**

$$\det_{0\le i,j\le n}\left(B_{2(i+j)}\left(\tfrac{x+1}{2}\right)\right) = ???,$$

$$\det_{0\le i,j\le n}\left(E_{2(i+j)}\left(\tfrac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\prod_{k=1}^{n}\left(\frac{k^2(x^2-(2k-1)^2)}{4}\right)^{n-k+1},$$

$$\det_{0\le i,j\le n}\left(B_{2(i+j)+1}\left(\tfrac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{k=1}^{n}\left(\frac{k^4(x^2-k^2)}{4(2k-1)(2k+1)}\right)^{n-k+1},$$

$$\det_{0\le i,j\le n}\left(E_{2(i+j)+1}\left(\tfrac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{k=1}^{n}\left(\frac{k^2(x^2-4k^2)}{4}\right)^{n-k+1}.$$

Question B: Bernoulli/Euler polynomials?

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[Submitted on 30 Jan 2024]

Leading coefficient in the Hankel determinants related to binomial and q-binomial transforms

Shane Chern, Lin Jiu, Shuhan Li, Liuquan Wang

It is a standard result that the Hankel determinants for a sequence stay invariant after performing the binomial transform on this sequence. In this work, we extend the scenario to q-binomial transforms and study the behavior of the leading coefficient in such Hankel determinants. We also investigate the leading coefficient in the Hankel determinants for even-indexed Bernoulli polynomials with recourse to a curious binomial transform. In particular, the degrees of these Hankel determinants share the same nature as those in one of the q-binomial cases.

Subjects: **Number Theory (math.NT)**; Combinatorics (math.CO)

MSC classes: Primary 11C20, Secondary 11B68, 33D45

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Question B: Bernoulli/Euler polynomials?

Theorem (C.–Jiu–Li–Wang, 2024)

For every $n \geq 0$, $\det_{0\leq i,j\leq n}\left(B_{2(i+j)}\left(\frac{x+1}{2}\right)\right)$ is a polynomial in x of degree $n(n+1)$ with leading coefficient

$$\left[x^{n(n+1)}\right]\det_{0\leq i,j\leq n}\left(B_{2(i+j)}\left(\tfrac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\prod_{j=1}^{n}\frac{(j!)^6}{(2j)!(2j+1)!}.$$

- It is notable that the Hankel determinant $\det_{0\leq i,j\leq n}\left(B_{2(i+j)}\left(\frac{x+1}{2}\right)\right)$ is a linear combination of the terms

$$B_{2j_0}\left(\tfrac{1+x}{2}\right) B_{2(1+j_1)}\left(\tfrac{1+x}{2}\right)\cdots B_{2(n+j_n)}\left(\tfrac{1+x}{2}\right)$$

with $j_0, j_1, \ldots, j_n$ a permutation of $0, 1, \ldots, n$. Here, each term is of degree $\sum_{i=0}^{n} 2(i+j_i) = 2n(n+1)$. However, our result states that the degree of the Hankel determinant is $n(n+1)$, which is only **half** of the above terms, thereby indicating abundant cancelations of higher powers of x in this determinant expansion.

Question C: q-Binomial transform?

- Consider the q-binomial transform

$$\widetilde{\alpha}_n(x) := \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha_k x^{n-k}.$$

- We *no longer* have invariance of Hankel determinants under this q-binomial transform.
- There is a "**half** the degree" phenomenon.

Theorem (C.–Jiu–Li–Wang, 2024)

For every $n \geq 0$, $\det\limits_{0\leq i,j\leq n}\big(\widetilde{\alpha}_{i+j}(x)\big)$ is a polynomial in x of degree $\frac{n(n+1)}{2}$ with leading coefficient

$$\big[x^{\frac{n(n+1)}{2}}\big]\det_{0\leq i,j\leq n}\big(\widetilde{\alpha}_{i+j}(x)\big) = \alpha_0\alpha_1\cdots\alpha_n(-1)^{\binom{n+1}{2}} q^{2\binom{n+1}{3}} \prod_{j=1}^{n}(1-q^j)^{n+1-j}.$$

Question C: q-Binomial transform?

- Special choices of α?

Theorem (C.–Jiu–Li–Wang, 2024)

Choose the sequence $\alpha_k^{u,v} := q^{-\binom{k}{2}}(u;q)_k\, v^k$ *with* u *and* v *indeterminates so as to let*

$$\widetilde{\alpha}_k^{u,v}(x) := \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q (u;q)_\ell\, v^\ell x^{k-\ell} = \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q (u;q)_{k-\ell}\, v^{k-\ell} x^\ell.$$

Then

$$\det_{0\le i,j\le n}\left(\widetilde{\alpha}_{i+j}^{u,v}(x)\right) = v^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \prod_{j=1}^{n} (uvq^{j-1} - x)^{n+1-j} (u,q;q)_{n+1-j}.$$

Thank You!