A central limit theorem for a card shuffling problem

Shane Chern

陈小航

Universität Wien (维也纳大学)

chenxiaohang92@gmail.com

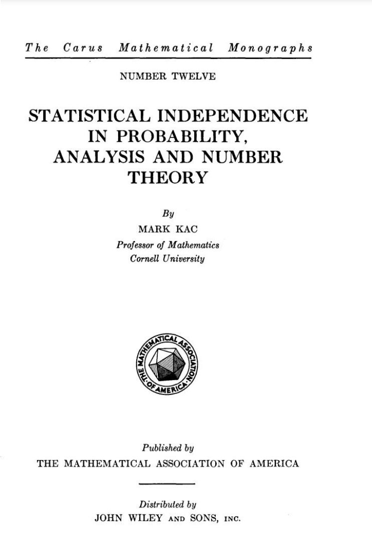
@ Chongqing Normal University

Chongqing

Jul 12, 2024

(Joint work with Lin Jiu and Italo Simonelli)

Mark Kac's *Carus Monographs*


The Carus Mathematical Monographs

NUMBER TWELVE

STATISTICAL INDEPENDENCE IN PROBABILITY, ANALYSIS AND NUMBER THEORY

By
MARK KAC
Professor of Mathematics
Cornell University

Published by
THE MATHEMATICAL ASSOCIATION OF AMERICA

Distributed by
JOHN WILEY AND SONS, INC.

Simple statistical observations of number-theoretic or combinatorial objects "**are often the starting point of rich and fruitful theories.**"

Erdős–Kac Theorem: *Let $\omega(n)$ be the number of distinct prime factors of n. If we randomly choose n from $1 \leq n \leq N$, then the probability distribution of*

$$\frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}$$

tends to the standard normal distribution as N becomes large enough.

Card shuffling

- **(i).** Given a positive integer n, consider a random permutation τ of the set $[n] := \{1, 2, \ldots, n\}$;

  (ii). In τ, we look for sequences of consecutive integers that appear in adjacent positions, and a maximal such a sequence is called a block.

  Initial set: $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$;

  Permuting: $(1, 7, 5, 6, 8, 10, 9, 2, 3, 4)$.

- **(i).** Each block in τ is merged into its first integer.

  (ii). After all the merges the elements of this new set are relabeled from 1 to the current number of elements.

  (iii). Permute this new set.

  Merging: $(1, 7, 5, 8, 10, 9, 2)$;

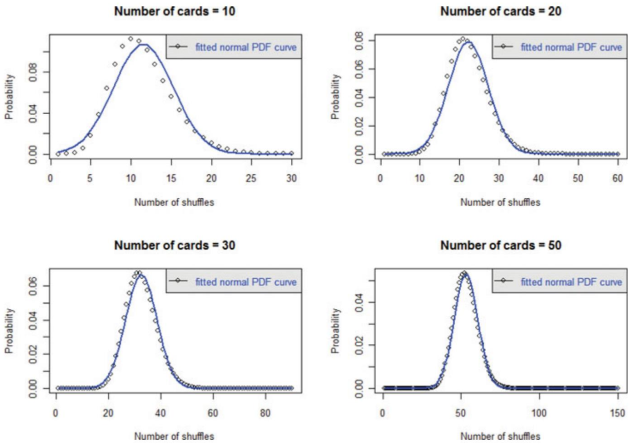
  Relabeling: $(1, 2, 3, 4, 5, 6, 7)$;

  Permuting: $(7, 5, 4, 6, 3, 2, 1)$ — NEW shuffing!

- We continue this merging and permuting until only one integer is left.

Card shuffling

- **The quantity of interest is X_n, the number of permutations needed for the process to end.**
- Rao et al. (2016):



Card shuffling

Theorem (C.–Jiu–Simonelli, 2023)

Let Z denote a standard normal random variable, i.e., $Z \sim \mathcal{N}(0,1)$. Then as $n \to \infty$,

$$\frac{X_n - n}{\sqrt{n}} \xrightarrow{w} Z.$$

Chebyshev's method of moments (1887):

- Z is a standard normal random variable, so its moments are given by

$$\mathbf{E}\big[Z^m\big] = \begin{cases} (m-1)!!, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

- $Z_1, Z_2, \ldots, Z_n, \ldots$ are a series of random variables such that

$$\lim_{n\to\infty} \mathbf{E}\big[Z_n^m\big] = \mathbf{E}\big[Z^m\big].$$

Then Z_n converges to Z in distribution.

Card shuffling

Goal: Evaluate

$$\mathbf{E}\left[\left(\frac{X_n - n}{\sqrt{n}}\right)^m\right].$$

- $\mu_n = \mathbf{E}[X_n] \sim n$;
- $\mathbf{Var}[X_n] = \mathbf{E}\big[(X_n - \mu_n)^2\big] \sim n$;

Goal': Evaluate

$$\mathbf{E}\big[(X_n - \mu_n)^m\big].$$

Probability sequence

- Let Y_n be the number of blocks in a random permutation of $[n]$;

Then for $n \geq 2$,

$$X_n = \begin{cases} 1, & \text{with probability } \mathbf{P}(Y_n = 1), \\ 1 + X_k, & \text{with probability } \mathbf{P}(Y_n = k) \text{ for } 2 \leq k \leq n. \end{cases}$$

For $1 \leq k \leq n$,

$$\mathbf{P}(Y_n = k) = \frac{A(n,k)}{n!},$$

where $A(n,k)$ counts the number of permutations of n with k blocks.

Probability sequence

$$A(n,k) = \binom{n-1}{k-1} A(k,k).$$

- $[n]$ is split into k components $B_1, \ldots, B_k$:

$$\underbrace{1\ 2}_{B_1} \mid \underbrace{3\ 4\ 5}_{B_2} \mid \underbrace{6\ 7\ 8}_{B_3} \qquad \Rightarrow \qquad \binom{n-1}{k-1}$$

- The components $B_1, \ldots, B_k$ **cannot** be merged:

$$B_1\ B_3\ B_2 \qquad \Rightarrow \qquad A(k,k)$$

$$(B_3\ B_1\ B_2 \qquad = \qquad \underline{678}\ \underline{12345} \qquad \text{✗})$$

Probability sequence

How to evaluate $A(k,k)$?

- Remove k to get a permutation of size $k-1$;
- In this permutation of size $k-1$, the number of blocks is either $k-1$ or $k-2$.

$$A(k,k) = (k-1)\cdot A(k-1,k-1) + (k-2)\cdot A(k-2,k-2).$$

- 53142 counted by $A(k-1,k-1) = A(5,5)$:

$$☑\ 5\ ☒\ 3\ ☑\ 1\ ☑\ 4\ ☑\ 2\ ☑ \qquad \Rightarrow \qquad (k-1)\cdot A(k-1,k-1)$$

- 52314 counted by $A(k-1,k-2) = A(5,4)$:

$$☒\ 5\ ☒\ 2\ ☑\ 3\ ☒\ 1\ ☒\ 4\ ☒ \qquad \Rightarrow \qquad 1\cdot A(k-1,k-2)$$

$$= 1\cdot \binom{k-2}{k-3} A(k-2,k-2)$$

$$= (k-2)\cdot A(k-2,k-2)$$

Probability sequence

- $A(1,1) = 1$;
- $A(2,2) = 1$ (i.e., the permutation 21);
- For $k \geq 3$,

$$A(k,k) = (k-1)A(k-1,k-1) + (k-2)A(k-2,k-2).$$

The OEIS is supported by the many generous donors to the OEIS Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

1, 1, 3, 11, 53, 309, 2119, 16687, 148329, 1468457 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,3,11,53,309,2119,16687,148329,1468457**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000255 a(n) = n*a(n-1) + (n-1)*a(n-2), a(0) = 1, a(1) = 1. +30
(Formerly M2905 N1166) 101

1, 1, 3, 11, 53, 309, 2119, 16687, 148329, 1468457, 16019531, 190899411, 2467007773, 34361893981, 513137616783, 8178130767479, 138547156531409, 2486151753313617, 47106033220679059, 939765362752547227, 19690321886243846661, 432292066866171724421 (list; graph; refs; listen; history; text; internal format)

Probability sequence

$$\mathbf{P}(Y_n = k) = \frac{A(n,k)}{n!},$$

where the sequence $A(n,k)$ is defined for $1 \leq k \leq n$ by

$$A(n,k) := \binom{n-1}{k-1} A(k-1),$$

with the sequence $A(n)$ being recursively given by

$$A(n) = nA(n-1) + (n-1)A(n-2),$$

in combination with initial values $A(0) = A(1) = 1$.

Recurrences

Recurrence for $\mu_n = \mathbf{E}[X_n]$:

$$\begin{aligned}
\mu_n &= \mathbf{P}(Y_n = 1) + \sum_{k=2}^{n} \mathbf{P}(Y_n = k)\mathbf{E}\big[1 + X_k\big] \\
&= \frac{A(n,1)}{n!} + \sum_{k=2}^{n} \frac{A(n,k)}{n!}(1 + \mu_k) \\
&= \sum_{k=1}^{n} \frac{A(n,k)}{n!} + \sum_{k=2}^{n} \frac{A(n,k)}{n!}\mu_k \\
&= 1 + \sum_{k=2}^{n} \frac{A(n,k)}{n!}\mu_k.
\end{aligned}$$

We adopt the convention that $X_1 = 0$, a definite constant, so that the mean value $\mu_1 = \mathbf{E}[X_1] = 0$, as there is no more shuffling needed. Then the above recurrence for μ_n becomes ($\mu_1 = 0$ and $\mu_2 = 2$):

$$\mu_n = 1 + \sum_{k=1}^{n} \frac{A(n,k)}{n!}\mu_k.$$

Recurrences

In general, if we assume that $p(x)$ is an arbitrary polynomial in x, then

$$\mathbf{E}\big[p(X_n)\big] = \sum_{k=1}^{n} \mathbf{P}(Y_n = k)\mathbf{E}\big[p(1+X_k)\big] = \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \mathbf{E}\big[p(1+X_k)\big].$$

Thus,

$$\begin{aligned}\mathbf{E}\left[(X_n-\mu_n)^m\right] &= \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \mathbf{E}\left[\big(1+X_k-\mu_n\big)^m\right] \\ &= \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \mathbf{E}\left[\big((X_k-\mu_k)+(1+\mu_k-\mu_n)\big)^m\right],\end{aligned}$$

so that

$$\left(1-\frac{A(n,n)}{n!}\right)\mathbf{E}\big[(X_n-\mu_n)^m\big] = \sum_{k=1}^{n-1} \frac{A(n,k)}{n!}\mathbf{E}\big[(X_k-\mu_k)^m\big] + \sum_{\ell=1}^{m}\binom{m}{\ell} I_\ell^{(m)},$$

where for each ℓ with $1 \le \ell \le m$,

$$I_\ell^{(m)} := \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \mathbf{E}\big[(X_k-\mu_k)^{m-\ell}\big]\big(1+\mu_k-\mu_n\big)^\ell.$$

Recurrences

Let $\{\lambda_n\}_{n\geq 1}$ be a complex sequence such that $\lambda_n \sim Mn^L$ as $n \to \infty$, wherein L is a fixed nonnegative integer, and M is a fixed complex number, which, in addition, is nonzero when L is nonzero. Write

$$\delta_n := \lambda_n - Mn^L.$$

We define a complex sequence $\{\xi_n\}_{n\geq 1}$ with given initial values $\xi_1, \ldots, \xi_{n_0}$ for a certain $n_0 \geq 2$ by the recurrence

$$\left(1 - \frac{A(n,n)}{n!}\right)\xi_n = \lambda_n + \sum_{k=1}^{n-1} \frac{A(n,k)}{n!}\xi_k,$$

for every $n > n_0$.

Recurrences

Theorem

As $n \to \infty$,

$$\xi_n \sim \frac{M}{L+1}\, n^{L+1},$$

where the asymptotic relation depends only on L and M. More precisely, letting

$$\eta_n := \xi_n - \frac{M}{L+1}\, n^{L+1},$$

there exists a positive constant C, depending only on L and M, such that for all $n \geq 1$,

$$\left|\eta_n\right| < C \sum_{j=1}^{n} \left(\left|\delta_j\right| + j^{L-1}\right).$$

Recurrences

Basic idea:

- Define the partial sums $S_n(t)$ with $1 \leq t \leq n$:

$$S_n(t) := \sum_{k=1}^{t} \frac{A(n,k)}{n!}.$$

- Rewrite the recurrence $\left(1 - \frac{A(n,n)}{n!}\right) \xi_n = \lambda_n + \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \xi_k$ as

$$S_n(n-1) \cdot \xi_n = \lambda_n + \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 := \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \frac{M}{L+1}\, k^{L+1},$$

$$\Sigma_2 := \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \eta_k.$$

Recurrences

Abel summation formula: Let $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ be sequences of complex numbers. Then for any $N \geq 1$,

$$\sum_{n=1}^{N} u_n v_n = U(N) v_{N+1} + \sum_{n=1}^{N} U(n)\big(v_n - v_{n+1}\big),$$

where $U(n) := \sum_{k=1}^{n} u_k$.

Recurrences

- Apply the Abel summation formula to Σ_1:

$$\begin{aligned}\Sigma_1 &= \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \frac{M}{L+1} k^{L+1} \\ &= S_n(n-1) \cdot \frac{M}{L+1} n^{L+1} + \sum_{k=1}^{n-1} S_n(k) \cdot \frac{M}{L+1} \big(k^{L+1} - (k+1)^{L+1}\big) \\ &= S_n(n-1) \cdot \frac{M}{L+1} n^{L+1} - \sum_{k=1}^{n-1} S_n(k) \cdot \big(Mk^L + O(k^{L-1})\big) \\ &= S_n(n-1) \cdot \frac{M}{L+1} n^{L+1} - Mn^L + O(n^{L-1}).\end{aligned}$$

- Substitute back to the recurrence for ξ_n:

$$S_n(n-1) \cdot \eta_n = \delta_n + \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \eta_k + O(n^{L-1}).$$

Recurrences

Further applying the Abel summation formula to Σ_2 gives an elaboration on the error term η_n when δ_n is efficiently bounded:

Theorem

We further have

1. *In the case where $L = 0$, if we further require that $\delta_n = O(n^{-1})$ as $n \to \infty$, then*
$$\left|\eta_n - \eta_{n-1}\right| = O(n^{-1}).$$
2. *In the case where $L \geq 1$, if we further require that $\delta_n = O(n^{L-1} \log n)$ as $n \to \infty$, then*
$$\left|\eta_n - \eta_{n-1}\right| = O(n^{L-1} \log n).$$

The above asymptotic relations depend only on L and M.

Recurrences

- Recall that for $\mu_n = \mathbf{E}[X_n]$,

$$\left(1 - \frac{A(n,n)}{n!}\right)\mu_n = 1 + \sum_{k=1}^{n-1} \frac{A(n,k)}{n!}\mu_k.$$

By choosing $\lambda_n = 1$ in the generic recurrence, it is immediate that

$$\mu_n \sim n.$$

- Let $\mathcal{H}_n := \sum_{k=1}^{n} \frac{1}{k}$ denote the n-th harmonic number.

$$S_n(n-1) \cdot \mu_n = 1 + \Sigma_1' + \Sigma_2' + \Sigma_3',$$

where

$$\Sigma_1' := \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} k, \qquad \Sigma_2' := \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \mathcal{H}_{k-1}, \qquad \Sigma_3' := \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \varepsilon_k.$$

Recurrences

Theorem

We have

$$\mu_n = n + \mathcal{H}_{n-1} + \varepsilon_n,$$

where the limit $\lim_{n\to\infty} \varepsilon_n$ *exists. In particular, for* $n \geq 2$*,*

$$0 < \varepsilon_n - \varepsilon_{n+1} < \frac{1}{n^2}.$$

Why do we care about an explicit formula of μ_n?

- To understand the behavior of μ_n in a more accurate way.

Open Problem. Find an explicit formula for $\lim_{n\to\infty} \varepsilon_n$.

Expectation:

$$\lim_{n\to\infty} \varepsilon_n = \square \cdot \zeta(2) + \square \cdot \zeta(3) + \cdots .$$

Recurrences

Why do we care about an explicit formula of μ_n?

- We can effectively bound $\mu_n - \mu_k$ in terms of $n - k$:

$$n - k \leq \mu_n - \mu_k \leq (n-k)\left(1 + \tfrac{1}{k}\right).$$

- We can prove that

$$\sum_{k=1}^{n} \frac{A(n,k)}{n!} \left(\mu_n - \mu_k\right)^{\ell} = B_\ell + O(n^{-1}),$$

  where B_ℓ is the ℓ-th Bell number defined by the exponential generating function

$$\sum_{\ell=0}^{\infty} B_\ell \frac{x^\ell}{\ell!} := e^{e^x - 1}.$$

Central moments

Theorem

For every $m \geq 2$, we have, as $n \to +\infty$,

$$\mathbf{E}\big[(X_n-\mu_n)^m\big] = \begin{cases} (2M-1)!!\cdot n^M + O(n^{M-1}\log n), & \text{if } m=2M,\\ \frac{2}{3}M(2M+1)!!\cdot n^M + O(n^{M-1}\log n), & \text{if } m=2M+1. \end{cases}$$

In particular, if we define

$$\varepsilon_n^{(m)} := \mathbf{E}\big[(X_n-\mu_n)^m\big] - \begin{cases} (2M-1)!!\cdot n^M, & \text{if } m=2M,\\ \frac{2}{3}M(2M+1)!!\cdot n^M, & \text{if } m=2M+1, \end{cases}$$

then

$$\big|\varepsilon_n^{(m)} - \varepsilon_{n-1}^{(m)}\big| = \begin{cases} O(n^{-1}), & \text{if } m=2 \text{ or } 3,\\ O(n^{\lfloor \frac{m}{2}\rfloor-2}\log n), & \text{if } m\geq 4. \end{cases}$$

Central moments

Recall that

$$\left(1-\frac{A(n,n)}{n!}\right)\mathbf{E}\big[(X_n-\mu_n)^m\big]=\sum_{k=1}^{n-1}\frac{A(n,k)}{n!}\,\mathbf{E}\big[(X_k-\mu_k)^m\big]+\sum_{\ell=1}^{m}\binom{m}{\ell}I_\ell^{(m)},$$

where for each ℓ with $1\le\ell\le m$,

$$I_\ell^{(m)}:=\sum_{k=1}^{n}\frac{A(n,k)}{n!}\cdot\mathbf{E}\big[(X_k-\mu_k)^{m-\ell}\big]\big(1+\mu_k-\mu_n\big)^{\ell}.$$

Basic idea:

- Induction on m;
- Split each $\mathbf{E}\big[(X_k-\mu_k)^{m-\ell}\big]$ as a main term $?\cdot k^{?}$ (by an inductive argument) and an error term $\varepsilon_k^{(m-\ell)}$.

Central moments

Variance $\mathbf{Var}[X_n] = \mathbf{E}\big[(X_n - \mu_n)^2\big]$**:**

$$\left(1 - \frac{A(n,n)}{n!}\right)\mathbf{Var}[X_n] = \sum_{k=1}^{n-1} \frac{A(n,k)}{n!}\,\mathbf{Var}[X_k] + I_1^{(2)} + I_2^{(2)},$$

where

$$I_1^{(2)} = \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \mathbf{E}\big[X_k - \mu_k\big]\big(1 + \mu_k - \mu_n\big) = 0,$$

$$I_2^{(2)} = \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \big(1 + \mu_k - \mu_n\big)^2 = 1 + O(n^{-1}).$$

Thus,

$$\left(1 - \frac{A(n,n)}{n!}\right)\mathbf{Var}[X_n] = \sum_{k=1}^{n-1} \frac{A(n,k)}{n!}\,\mathbf{Var}[X_k] + 1 + O(n^{-1}),$$

so that

$$\mathbf{Var}[X_n] \sim n.$$

Central moments

Third central moment $\mathbf{Var}[X_n] = \mathbf{E}\big[(X_n - \mu_n)^3\big]$:

$$\left(1 - \frac{A(n,n)}{n!}\right)\mathbf{E}\big[(X_n-\mu_n)^3\big] = \sum_{k=1}^{n-1}\frac{A(n,k)}{n!}\,\mathbf{E}\big[(X_k-\mu_k)^3\big] + 3I_1^{(3)} + 3I_2^{(3)} + I_3^{(3)},$$

where

$$I_1^{(3)} = \sum_{k=1}^{n}\frac{A(n,k)}{n!}\cdot\mathbf{E}\big[(X_k-\mu_k)^2\big]\big(1+\mu_k-\mu_n\big),$$

$$I_2^{(3)} = \sum_{k=1}^{n}\frac{A(n,k)}{n!}\cdot\mathbf{E}\big[X_k-\mu_k\big]\big(1+\mu_k-\mu_n\big)^2 = 0,$$

$$I_3^{(3)} = \sum_{k=1}^{n}\frac{A(n,k)}{n!}\cdot\big(1+\mu_k-\mu_n\big)^3 = -1 + O(n^{-1}).$$

Central moments

Recalling that $\mathbf{E}\big[(X_k-\mu_k)^2\big] = k + O(\log k)$, we split $I_1^{(3)}$ as

$$I_1^{(3)} = I_{1,1}^{(3)} + I_{1,2}^{(3)},$$

where

$$I_{1,1}^{(3)} := \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot k\big(1+\mu_k-\mu_n\big),$$
$$I_{1,2}^{(3)} := \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \varepsilon_k^{(2)}\big(1+\mu_k-\mu_n\big).$$

Then,

$$I_{1,1}^{(3)} = 1 + O(n^{-1}).$$

In general,

$$\sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot k^L\big(1+\mu_k-\mu_n\big) = Ln^{L-1} + O(n^{L-2}).$$

Central moments

Since $\varepsilon_k^{(2)} = O(\log k)$, if we trivially bound $I_{1,2}^{(3)}$, we only have

$$I_{1,2}^{(3)} \ll \log n \sum_{k=1}^{n} \frac{A(n,k)}{n!} |1 + \mu_k - \mu_n| \ll \log n.$$

Hence, we shall invoke the bound $\left|\varepsilon_n^{(2)} - \varepsilon_{n-1}^{(2)}\right| = O(n^{-1})$ and apply the Abel summation formula (in a **very technical** way) to get

$$I_{1,2}^{(3)} = O(n^{-1}).$$

Thus,

$$\left(1 - \frac{A(n,n)}{n!}\right) \mathbf{E}\big[(X_n - \mu_n)^3\big] = \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \mathbf{E}\big[(X_k - \mu_k)^3\big] + 2 + O(n^{-1}),$$

so that

$$\mathbf{E}\big[(X_n - \mu_n)^3\big] \sim 2n.$$

Central moments

Even-order central moment $\mathbf{Var}[X_n] = \mathbf{E}\big[(X_n - \mu_n)^{2M}\big]$:

$$\left(1 - \frac{A(n,n)}{n!}\right) \mathbf{E}\big[(X_n - \mu_n)^{2M}\big] = \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \mathbf{E}\big[(X_k - \mu_k)^{2M}\big] + \sum_{\ell=1}^{2M} \binom{2M}{\ell} I_\ell^{(2M)},$$

where

$$I_1^{(2M)} = \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \mathbf{E}\big[(X_k - \mu_k)^{2M-1}\big] (1 + \mu_k - \mu_n) = O(n^{M-2} \log n),$$

$$I_2^{(2M)} = \sum_{k=1}^{n} \frac{A(n,k)}{n!} \cdot \mathbf{E}\big[(X_k - \mu_k)^{2M-2}\big] (1 + \mu_k - \mu_n)^2 = (2M-3)!! \cdot n^{M-1} + O(n^{M-2} \log n),$$

$$\vdots$$

Thus,

$$\left(1 - \frac{A(n,n)}{n!}\right) \mathbf{E}\big[(X_n - \mu_n)^{2M}\big] = \sum_{k=1}^{n-1} \frac{A(n,k)}{n!} \mathbf{E}\big[(X_k - \mu_k)^{2M}\big] + \binom{2M}{2} (2M-3)!! \cdot n^{M-1} + O(n^{M-2} \log n),$$

so that

$$\mathbf{E}\big[(X_n - \mu_n)^{2M}\big] \sim (2M-1)!! \cdot n^M.$$

Central moments

Odd-order central moment $\mathbf{Var}[X_n] = \mathbf{E}\big[(X_n-\mu_n)^{2M+1}\big]$:

$$\left(1-\frac{A(n,n)}{n!}\right)\mathbf{E}\big[(X_n-\mu_n)^{2M+1}\big] = \sum_{k=1}^{n-1}\frac{A(n,k)}{n!}\,\mathbf{E}\big[(X_k-\mu_k)^{2M+1}\big] + \sum_{\ell=1}^{2M+1}\binom{2M+1}{\ell} I_\ell^{(2M+1)},$$

where

$$I_1^{(2M+1)} = \sum_{k=1}^{n}\frac{A(n,k)}{n!}\cdot\mathbf{E}\big[(X_k-\mu_k)^{2M}\big]\big(1+\mu_k-\mu_n\big)$$

$$= M(2M-1)!!\cdot n^{M-1} + O(n^{M-2}\log n), \qquad \text{[DIFFICULT! Abel summation used.]}$$

$$I_2^{(2M+1)} = \sum_{k=1}^{n}\frac{A(n,k)}{n!}\cdot\mathbf{E}\big[(X_k-\mu_k)^{2M-1}\big]\big(1+\mu_k-\mu_n\big)^2$$

$$= \tfrac{2}{3}(M-1)(2M-1)!!\cdot n^{M-1} + O(n^{M-2}\log n),$$

$$I_3^{(2M+1)} = \sum_{k=1}^{n}\frac{A(n,k)}{n!}\cdot\mathbf{E}\big[(X_k-\mu_k)^{2M-2}\big]\big(1+\mu_k-\mu_n\big)^3,$$

$$= -(2M-3)!!\cdot n^{M-1} + O(n^{M-2}\log n),$$

$$\vdots$$

Central moments

Thus,

$$
\begin{aligned}
&\left(1-\frac{A(n,n)}{n!}\right)\mathbf{E}\big[(X_n-\mu_n)^{2M+1}\big]\\
&=\sum_{k=1}^{n-1}\frac{A(n,k)}{n!}\,\mathbf{E}\big[(X_k-\mu_k)^{2M+1}\big]\\
&\quad+\left(\tbinom{2M+1}{1}M(2M-1)!!+\tbinom{2M+1}{2}\tfrac{2}{3}(M-1)(2M-1)!!-\tbinom{2M+1}{3}(2M-3)!!\right)\cdot n^{M-1}\\
&\quad+O(n^{M-2}\log n),
\end{aligned}
$$

so that

$$\mathbf{E}\big[(X_n-\mu_n)^{2M+1}\big]\sim\tfrac{2}{3}M(2M+1)!!\cdot n^M.$$

Central moments

Let

$$Z_n := \frac{X_n - \mu_n}{\sqrt{\mathbf{Var}[X_n]}}.$$

Then $\mathbf{E}[Z_n] = 0$, $\mathbf{Var}[Z_n] = 1$, and more importantly, as $n \to \infty$,

$$\mathbf{E}\big[Z_n^m\big] = \frac{\mathbf{E}\big[(X_n - \mu_n)^m\big]}{\mathbf{Var}[X_n]^{m/2}} \to \begin{cases} (m-1)!!, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd,} \end{cases}$$

matching with the moments of the standard normal distribution Z. So Chebyshev's method of moments asserts the weak convergence of $Z_n \Rightarrow Z$.

Further thoughts

(Ambitious) Goal: Find a criterion for the merging rules that result in a certain central limit theorem.

In principle, we need to look at recurrences of the form

$$\mathbf{E}\big[p(X_n)\big] = \sum_{k=1}^{n} \alpha(n,k) \cdot \mathbf{E}\big[p(1+X_k)\big],$$

which holds for any polynomial $p(x)$. Here each $\alpha(n,k)$ is a probability evaluation so that $\sum_{k=1}^{n} \alpha(n,k) = 1$. We wish to find a way to characterize these $\alpha(n,k)$ so as to ensure that the moments $\mathbf{E}[Z_n^m]$, with Z_n the normalization of X_n, match with those of the standard normal distribution.

Thank You!