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A congruence involving the quotients of Euler and its applications. III

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Abstract. In the papers of 2002 and 2007, Cai *et al.* introduced a series of congruences involving binomial coefficients under perfect moduli. This article generalizes these congruences to cubic cases leading to many new statements. For example, the congruence $\prod_{d|n}\binom{kd-1}{\lfloor d/e\rfloor}^{\mu(n/d)}$ modulo n^3 for $e=2$, 3, 4 and 6, and the following congruence

$$\prod_{d|n}\binom{(kd-1)/2}{(d-1)/2}^{\mu(n/d)} \equiv 2^{-(k-1)\phi(n)}\begin{cases}(\text{mod } n^3), & \text{if } 3\nmid n,\\ (\text{mod } n^3/3), & \text{if } 3\mid n.\end{cases}$$

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1. Introduction

In 1895, Morley [12] proved the following elegant and profound congruence involving binomial coefficients: for any prime $p\ge 5$,

$$(-1)^{(p-1)/2}\binom{p-1}{(p-1)/2}\equiv 4^{p-1}\pmod{p^3}. \tag{1.1}$$

Although his proof, which is due to de Moivre's Theorem, is very clever, it fails to work for other binomial coefficients. Nevertheless, there were still a number of generalizations of Morley's congruence subsequently; see, for example, [1]. In 2002, Cai [2] extended Morley's congruence to integer cube moduli through a generalization of Lehmer's congruence. More precisely, he proved that for any positive odd integer n,

$$\prod_{d|n}\binom{d-1}{(d-1)/2}^{\mu(n/d)} \equiv (-1)^{\phi(n)/2}4^{\phi(n)}\begin{cases}(\text{mod } n^3), & \text{if } 3\nmid n,\\ (\text{mod } n^3/3), & \text{if } 3\mid n.\end{cases} \tag{1.2}$$

If n is an odd prime $p\ge 5$, (1.2) reduces to (1.1). In 2007, Cai *et al.* [3] further proposed several new congruences of the same type as (1.2), in which $(d-1)/2$ is replaced by $\lfloor d/3\rfloor$, $\lfloor d/4\rfloor$ and $\lfloor d/6\rfloor$, respectively. Here $\lfloor x\rfloor$ denotes the largest integer not exceeding x.

In this paper, we will further extend the work in [2] and [3].

First, we introduce a generalization of the Euler totient function defined in [14]. For a positive integer k and an arithmetic function f, we define

$$\phi_f^{(k)}(n) := n^k\sum_{d|n} d^{-k}f(d)\mu(d). \tag{1.3}$$

If $f \equiv 1$, $\phi_f^{(k)}(n)$ becomes the Jordan totient function. It is easy to verify that if f is multiplicative, then

$$\phi_f^{(k)}(n) = n^k \prod_{p|n}(1 - f(p)p^{-k}).$$

Our first result is as follows.

Theorem 1.1. *Let n be a positive integer such that $(n, 6) = 1$. For $e = 2, 3, 4$ or 6, we have*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \equiv -J_e(n) n^{\phi(n)-2} \phi_{J_e}^{(2-\phi(n))}(n) \frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1} \pmod{n}, \tag{1.4}$$

where $B_n(x)$ is the Bernoulli polynomial and $J_e(n)$ is the Jacobi symbol $\left(\frac{n}{e}\right)$. Recall that since $(n, 6) = 1$, one has

$$J_e(n) = \left(\frac{n}{e}\right) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{e}, \\ -1, & \text{if } n \equiv -1 \pmod{e}. \end{cases}$$

It is known that for any odd integer v, $B_v(\frac{1}{2}) = 0$ and $B_v(\frac{1}{4}) = -\frac{vE_{v-1}}{4^v}$, where E_m is the m-th Euler number defined by the generating function

$$\frac{1}{\cosh x} = \sum_{m \geq 0} \frac{E_m}{m!} \cdot x^m.$$

The following congruences are corollaries of Theorem 1.1.

Corollary 1.2. *For any positive integer n such that $(n, 6) = 1$, we have*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\frac{n-1}{2}} \frac{1}{r^2} \equiv 0 \pmod{n}. \tag{1.5}$$

Corollary 1.3. *For any positive integer n such that $(n, 6) = 1$, we have*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/4 \rfloor} \frac{1}{r^2} \equiv (-1)^{\frac{n-1}{2}} 4 n^{\phi(n)-2} \phi_{J_4}^{(2-\phi(n))}(n) E_{\phi(n)-2} \pmod{n} \tag{1.6}$$

Corollary 1.4. *For any positive integer n, we have*

(i) *if $(n, 6) = 1$, then*

$$\begin{aligned}\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/3 \rfloor} \frac{1}{r} &\equiv -\frac{3}{2} q_3(n) + \frac{3}{4} n q_3^2(n) \\ &+ \frac{1}{3} J_3(n) n^{\phi(n)-1} \phi_{J_3}^{(2-\phi(n))}(n) \frac{B_{\phi(n)-1}(\frac{1}{3})}{\phi(n)-1} \pmod{n^2};\end{aligned} \tag{1.7}$$

(ii) *if* $(n,6)=1$, *then*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/4\rfloor}\frac{1}{r}\equiv -3q_2(n)+\frac{3}{2}nq_2^2(n)$$

$$+(-1)^{\frac{n+1}{2}}n^{\phi(n)-1}\phi_{J_4}^{(2-\phi(n))}(n)E_{\phi(n)-2}\pmod{n^2};\tag{1.8}$$

(iii) *if* $(n,30)=1$, *then*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/6\rfloor}\frac{1}{r}\equiv -2q_2(n)-\frac{3}{2}q_3(n)+nq_2^2(n)+\frac{3}{4}nq_3^2(n)$$

$$+\frac{1}{6}J_6(n)n^{\phi(n)-1}\phi_{J_6}^{(2-\phi(n))}(n)\frac{B_{\phi(n)-1}(\frac{1}{6})}{\phi(n)-1}\pmod{n^2}.\tag{1.9}$$

Remark. Similar results were also obtained independently in [4] and [6].

Theorem 1.5. *For any positive integer k and positive odd integer n, we have*

$$\prod_{d\mid n}\binom{kd-1}{(d-1)/2}^{\mu(n/d)}\equiv(-1)^{\phi(n)/2}4^{k\phi(n)}\begin{cases}\pmod{n^3}, & \text{if } 3\nmid n,\\ \pmod{n^3/3}, & \text{if } 3\mid n.\end{cases}\tag{1.10}$$

If n is an odd prime $p\ge 5$, (1.10) reduces to the following generalization of Morley's congruence obtained in [8].

Corollary 1.6. *Let $p\ge 5$ be an odd prime. For any positive integer k, we have*

$$(-1)^{(p-1)/2}\binom{kp-1}{(p-1)/2}\equiv 4^{k(p-1)}\pmod{p^3}.\tag{1.11}$$

If n is a product of two distinct odd primes, then we are able to extend [2, Corollary 4] and obtain a congruence resembling the quadratic reciprocity law.

Corollary 1.7. *Let p and q be two distinct odd primes. For any positive integer k, we have*

$$\binom{kpq-1}{(pq-1)/2}\equiv 4^{k(p-1)(q-1)}\binom{kp-1}{(p-1)/2}\binom{kq-1}{(q-1)/2}\pmod{p^3q^3}.\tag{1.12}$$

Next, to state our results more clearly, we define

$$A_e(n):=J_e(n)n^{\phi(n)-2}\phi_{J_e}^{(2-\phi(n))}(n)\frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1}.$$

We have

Theorem 1.8. *For any positive integer k and positive odd integer n, we have*

(i) *if* $(3,n)=1$, *then*

$$\prod_{d\mid n}\binom{kd-1}{\lfloor d/3\rfloor}^{\mu(n/d)}$$

$$\equiv(-1)^{\phi_3(n)}\left\{\frac{1}{2}(27^{k\phi(n)}+1)+k\left(\frac{1}{2}k-\frac{1}{3}\right)n^2A_3(n)\right\}\pmod{n^3};\tag{1.13}$$

(ii) *if* $(3,n)=1$, *then*

$$\prod_{d|n}\binom{kd-1}{\lfloor d/4\rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_4(n)}\left\{8^{k\phi(n)}+(-1)^{\frac{n+1}{2}}k(2k-1)n^{\phi(n)}\phi_{J_4}^{(2-\phi(n))}(n)E_{\phi(n)-2}\right\} \pmod{n^3}; \tag{1.14}$$

(iii) *if* $(15,n)=1$, *then*

$$\prod_{d|n}\binom{kd-1}{\lfloor d/6\rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_6(n)}\left\{\frac{1}{2}(16^{k\phi(n)}+27^{k\phi(n)})+\frac{1}{2}k\left(k-\frac{1}{3}\right)n^2A_6(n)\right\} \pmod{n^3}, \tag{1.15}$$

where

$$\phi_e(n):=\sum_{d|n}\mu\left(\frac{n}{d}\right)\left\lfloor\frac{d}{e}\right\rfloor$$

is the generalized Euler totient function defined in [3].

Finally, we obtain a new congruence, which is similar to Theorem 1.5.

Theorem 1.9. *For any positive integer k and positive odd integer n, we have*

$$\prod_{d|n}\binom{(kd-1)/2}{(d-1)/2}^{\mu(n/d)} \equiv 2^{-(k-1)\phi(n)}\begin{cases}\pmod{n^3} & \text{if } 3\nmid n,\\ \pmod{n^3/3} & \text{if } 3\mid n.\end{cases} \tag{1.16}$$

Remark. When k is even, $(kd-1)/2$ is no longer an integer. Hence, we need to generalize the binomial coefficients as follows. Let $x\in\mathbb{C}$. If n is a positive integer, let

$$\binom{x}{n}=\frac{x(x-1)\cdots(x-n+1)}{n(n-1)\cdots 1}.$$

If $n=0$, we put $\binom{x}{0}=1$.

2. Preliminaries

To prove the theorems, we need the following lemmas.

Lemma 2.1 (Cf. [3, Lemma 1])**.** *If $p\geq 5$ is a prime, $k\geq 2$, l and t are positive integers, and s is the smallest positive residue of p^l modulo t, then*

$$\sum_{r=1}^{\lfloor p^l/t\rfloor}(p^l-tr)^{2k}\equiv\frac{t^{2k}}{2k+1}\left\{\frac{2k+1}{t}p^lB_{2k}-B_{2k+1}\left(\frac{s}{t}\right)\right\} \pmod{p^{3l-1}}, \tag{2.1}$$

where B_n is the n-th Bernoulli number.

Lemma 2.2 (Cf. [2, Lemma 1])**.** *Let n be a positive integer. Then,*

$$\sum_{\substack{i=1\\(i,n)=1}}^{n-1}\frac{1}{i^2}\equiv 0\begin{cases}\pmod{n}, & \text{if } 3\nmid n,\ n\neq 2^a,\\ \pmod{n/3}, & \text{if } 3\mid n,\\ \pmod{n/2}, & \text{if } n=2^a.\end{cases} \tag{2.2}$$

Lemma 2.3 (Cf. [13, Corollary 1.3]). *Let $a \in \mathbb{Z}$. Let $k, q, m \in \mathbb{Z}^+$ such that $(m, q) = 1$. Then,*

$$\frac{1}{k}\left(m^k B_k\left(\frac{x+a}{m}\right) - B_k(x)\right) \equiv \sum_{j=0}^{q-1}\left(\left\lfloor\frac{a+jm}{q}\right\rfloor + \frac{1-m}{2}\right)(x+a+jm)^{k-1} \pmod{q}. \tag{2.3}$$

Lemma 2.4 (Cf. [2, Theorem 1]). *For any positive odd integer n, we have*

$$\sum_{\substack{i=1\\(i,n)=1}}^{(n-1)/2} \frac{1}{i} \equiv -2q_2(n) + nq_2^2(n) \pmod{n^2}, \tag{2.4}$$

where $q_r(n)$ with $(n, r) = 1$ is the Euler quotient, that is,

$$q_r(n) = \frac{r^{\phi(n)} - 1}{n}.$$

Remark. According to Cosgrave and Dilcher [5], (2.4) appears to be the first Lehmer type congruence under a composite modulus.

Lemma 2.5 (Cf. [3, Theorem 1]). *For any positive odd integer n, we have*

(i) *if $(3, n) = 1$, then*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/3\rfloor} \frac{1}{n-3r} \equiv \frac{1}{2}q_3(n) - \frac{1}{4}nq_3^2(n) \pmod{n^2}; \tag{2.5}$$

(ii) *if $(3, n) = 1$, then*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/4\rfloor} \frac{1}{n-4r} \equiv \frac{3}{4}q_2(n) - \frac{3}{8}nq_2^2(n) \pmod{n^2}; \tag{2.6}$$

(iii) *if $(15, n) = 1$, then*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/6\rfloor} \frac{1}{n-6r} \equiv \frac{1}{3}q_2(n) + \frac{1}{4}q_3(n) - \frac{1}{6}nq_2^2(n) - \frac{1}{8}nq_3^2(n) \pmod{n^2}. \tag{2.7}$$

Remark. Congruences in Lemmas 2.4 and 2.5 can also be found in [9] and [10].

3. Proofs of the main results

3.1. Proof of Theorem 1.1. First, we prove that for any prime p and positive integer l,

$$\sum_{\substack{r=1\\p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2} \equiv -J_e(p^l)\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l}. \tag{3.1}$$

Note that for $m \geq 1$, one has

$$B_n(mx) = m^{n-1}\sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right).$$

Hence, we are always able to obtain the values of $B_n(\frac{s}{t})$.

Taking $2k = \phi(p^l) - 2$ and $t = e$ in (2.1) and using the von Staudt–Clauson Theorem, we have

$$\begin{aligned}
\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2} &\equiv \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{e^2}{(p^l-er)^2} \equiv \sum_{r=1}^{\lfloor p^l/e\rfloor} (p^l-er)^{\phi(p^l)-2}\\
&\equiv -\frac{e^{\phi(p^l)-2+2}}{\phi(p^l)-1} B_{\phi(p^l)-1}\left(\frac{s}{e}\right) \equiv -\frac{B_{\phi(p^l)-1}(\frac{s}{e})}{\phi(p^l)-1}\\
&\equiv -J_e(p^l)\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l},
\end{aligned}$$

where s is the smallest positive residue of p^l modulo e. Hence, (3.1) is proven.

Next, we prove that for any positive integer m such that $(m,e)=1$,

$$\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor mp^l/e\rfloor} \frac{1}{r^2} \equiv J_e(m) \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2} \pmod{p^l}. \tag{3.2}$$

If $m \equiv 1 \pmod{e}$, there exists a nonnegative integer k such that $m = ek+1$. It follows from Lemma 2.2 that

$$\begin{aligned}
\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (ek+1)p^l/e\rfloor} \frac{1}{r^2} &= \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor kp^l+p^l/e\rfloor} \frac{1}{r^2} = \sum_{\substack{r=1\\ p\nmid r}}^{kp^l} \frac{1}{r^2} + \sum_{\substack{r=kp^l+1\\ p\nmid r}}^{kp^l+\lfloor p^l/e\rfloor} \frac{1}{r^2}\\
&\equiv \sum_{a=0}^{k-1}\sum_{\substack{b=1\\ p\nmid b}}^{p^l} \frac{1}{(ap^l+b)^2} + \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{(kp^l+r)^2}\\
&\equiv \sum_{a=0}^{k-1}\sum_{\substack{b=1\\ p\nmid b}}^{p^l} \frac{1}{b^2} + \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2} \equiv k\sum_{\substack{b=1\\ p\nmid b}}^{p^l-1} \frac{1}{b^2} + \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2}\\
&\equiv \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2} \pmod{p^l}.
\end{aligned}$$

If $m \equiv -1 \pmod{e}$, then $m = ek-1$ for some positive integer k. Hence,

$$\begin{aligned}
\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (ek-1)p^l/e\rfloor} \frac{1}{r^2} &= \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (k-1)p^l+(e-1)p^l/e\rfloor} \frac{1}{r^2} \equiv \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (e-1)p^l/e\rfloor} \frac{1}{r^2}\\
&\equiv \sum_{\substack{r=1\\ p\nmid r}}^{p^l-\lfloor p^l/e\rfloor-1} \frac{1}{r^2} \equiv \sum_{\substack{r=1\\ p\nmid r}}^{p^l-1} \frac{1}{r^2} - \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2}
\end{aligned}$$

$$\equiv -\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{r^2} \pmod{p^l}.$$

Therefore, (3.2) is proven. Further, if $p^l \| n$, then taking $m = \frac{n}{p^l}$ in (3.2) yields

$$\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor n/e\rfloor} \frac{1}{r^2} \equiv -J_e(n)\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l}. \tag{3.3}$$

Further, we prove that

$$\sum_{\substack{r=1\\ (r,n)=1}}^{\lfloor n/e\rfloor} \frac{1}{r^2} \equiv -J_e(n)n^{\phi(n)-2}\phi_{J_e}^{(2-\phi(n))}(n)\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l}. \tag{3.4}$$

Let p_1, p_2, …, p_u be distinct prime factors of n. Noting that

$$\phi(n)-1 \geq \phi(p^l)-1 = p^{l-1}(p-1)-1 \geq 4\cdot 5^{l-1}-1 > l,$$

we have

$$\sum_{\substack{r=1\\ (r,n)=1}}^{\lfloor n/e\rfloor} \frac{1}{r^2} = \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor n/e\rfloor} \frac{1}{r^2} - \sum_i \sum_{\substack{r=1\\ p\nmid r\\ p_i|r}}^{\lfloor n/e\rfloor} \frac{1}{r^2} + \sum_{i,j} \sum_{\substack{r=1\\ p\nmid r\\ p_ip_j|r}}^{\lfloor n/e\rfloor} \frac{1}{r^2} + \cdots + (-1)^u \sum_{\substack{r=1\\ p\nmid r\\ p_1p_2\cdots p_u|r}}^{\lfloor n/e\rfloor} \frac{1}{r^2}$$

$$\equiv -\left\{J_e(n) - \sum_i \frac{J_e(n/p_i)}{p_i^2} + \sum_{i,j}\frac{J_e(n/p_ip_j)}{p_i^2p_j^2} + \cdots + (-1)^u \frac{J_e(n/p_1p_2\cdots p_u)}{p_1^2p_2^2\cdots p_u^2}\right\}\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1}$$

$$\equiv -\left\{J_e(n) - \sum_i \frac{J_e(n)}{p_i^2J_e(p_i)} + \sum_{i,j}\frac{J_e(n)}{p_i^2p_j^2J_e(p_ip_j)} + \cdots + (-1)^u \frac{J_e(n)}{p_1^2p_2^2\cdots p_u^2J_e(p_1p_2\cdots p_u)}\right\}\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1}$$

$$\equiv -J_e(n)\prod_{q|\frac{n}{p^l}}\left(1-\frac{1}{q^2J_e(q)}\right)\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1}$$

$$\equiv -J_e(n)\prod_{q|n}(1-q^{\phi(p^l)-2}J_e(q))\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1}$$

$$\equiv -J_e(n)n^{\phi(n)-2}\phi_{J_e}^{(2-\phi(n))}(n)\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l}.$$

Hence, (3.4) is proven.

Finally, taking $k=\phi(p^l)$, $m=e$, $x=0$, $a=1$ and $q=p^l$ in (2.3), we have

$$\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \equiv e\sum_{j=0}^{p^l-1}\left(\left\lfloor\frac{1+je}{p^l}\right\rfloor + \frac{1-e}{2}\right)(1+je)^{\phi(p^l)-2}$$

$$\equiv e \sum_{\substack{j=0 \\ (p,1+je)=1}}^{p^l-1} \left(\left\lfloor \frac{1+je}{p^l} \right\rfloor + \frac{1-e}{2} \right) (1+je)^{-2} \pmod{p^l}.$$

Further, taking $k = \phi(n) - 1$ yields

$$\begin{aligned}\frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1} &\equiv e \sum_{j=0}^{p^l-1} \left(\left\lfloor \frac{1+j}{p^l} \right\rfloor + \frac{1-e}{2} \right) (1+je)^{\phi(n)-2} \\ &\equiv e \sum_{\substack{j=0 \\ (p,1+je)=1}}^{p^l-1} \left(\left\lfloor \frac{1+je}{p^l} \right\rfloor + \frac{1-e}{2} \right) (1+je)^{-2} \pmod{p^l},\end{aligned}$$

which means that for $p^l \| n$,

$$\frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1} \equiv \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l}. \tag{3.5}$$

Hence, we arrive at (1.4).

Remark. Eie and Ong [7] proved that

$$\frac{1}{m}\left\{ B_m\left(\frac{\alpha}{k}\right) - p^{m-1} B_m\left(\frac{\beta}{k}\right) \right\} \equiv \frac{1}{n}\left\{ B_n\left(\frac{\alpha}{k}\right) - p^{n-1} B_n\left(\frac{\beta}{k}\right) \right\} \pmod{p^{l+1}},$$

where $p \geq 5$ is a prime, j is an integer between 0 and p, α and β are nonnegative integers such that $\alpha + jk = p\beta$, and $m \equiv n \pmod{(p-1)p^l}$ are positive integers that are not multiples of $p-1$. Hence, taking $\alpha = \beta = 1$, $k = e$, $m = \phi(p^l) - 1$ in this congruence also yields (3.5).

3.2. Proof of Corollary 1.4. It follows from Theorem 1.1 and Lemma 2.5 that

$$\begin{aligned}\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{n-er} &\equiv \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} (n-er)^{\phi(n^2)-1} \\ &\equiv \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \left\{ (-er)^{\phi(n^2)-1} + (\phi(n^2)-1)n(-er)^{\phi(n^2)-2} \right\} \\ &\equiv -\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} (er)^{\phi(n^2)-1} + \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} (n\phi(n)-1)n(er)^{\phi(n^2)-2} \\ &\equiv -\frac{1}{e} \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r} - \frac{n}{e^2} \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \pmod{n^2}.\end{aligned}$$

Therefore,

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r} \equiv -e \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{n-er} - \frac{n}{e} \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \pmod{n^2}. \tag{3.6}$$

Letting $e = 3$, 4 and 6 gives the desired congruences.

3.3. Proofs of Theorems 1.5 and 1.8. For any positive integer e, we have

$$\binom{kn-1}{\lfloor n/e\rfloor} = \prod_{r=1}^{\lfloor n/e\rfloor} \frac{kn-r}{r} = \prod_{d\mid n} \prod_{\substack{r=1\\(r,n)=d}}^{\lfloor n/e\rfloor} \frac{kn-r}{r} = \prod_{d\mid n} T_{n/d} = \prod_{d\mid n} T_d,$$

where

$$T_d = \prod_{\substack{r=1\\(r,d)=1}}^{\lfloor d/e\rfloor} \frac{kd-r}{r}. \tag{3.7}$$

It follows from the multiplicative version of the Möbius inversion formula that

$$T_n = \prod_{d\mid n} \binom{kd-1}{\lfloor d/e\rfloor}^{\mu(n/d)}. \tag{3.8}$$

We have

$$\begin{aligned} T_n &= \prod_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e\rfloor} \frac{kn-r}{r} = (-1)^{\phi_e(n)} \prod_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e\rfloor} \left(1-\frac{kn}{r}\right) \\ &\equiv (-1)^{\phi_e(n)} \left\{ 1 - kn \sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e\rfloor} \frac{1}{r} \right. \\ &\qquad \left. + \frac{k^2n^2}{2} \left(\left(\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e\rfloor} \frac{1}{r} \right)^2 - \sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e\rfloor} \frac{1}{r^2} \right) \right\} \pmod{n^3}. \end{aligned}$$

If $e=2$, then $\lfloor n/2\rfloor = \frac{n-1}{2}$. Note that

$$\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2} \frac{1}{r^2} = \frac{1}{2} \sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2} \left\{ \frac{1}{r^2} + \frac{1}{(n-r)^2} \right\} = \frac{1}{2} \sum_{\substack{r=1\\(r,n)=1}}^{n-1} \frac{1}{r^2} \pmod{n}.$$

Further, it follows from Lemmas 2.2 and 2.4 that if $3\nmid n$, then

$$\begin{aligned} T_n &\equiv (-1)^{\phi(n)/2} \left\{ 1 + 2knq_2(n) + (2k^2-k)(nq_2(n))^2 \right\} \\ &\equiv (-1)^{\phi(n)/2} (1+nq_2(n))^{2k} \\ &\equiv (-1)^{\phi(n)/2} 4^{k\phi(n)} \pmod{n^3}. \end{aligned}$$

If $3\mid n$, we replace the modulus by $n^3/3$. In light of (3.8), we have

$$\prod_{d\mid n} \binom{kd-1}{(d-1)/2}^{\mu(n/d)} \equiv (-1)^{\phi(n)/2} 4^{k\phi(n)} \begin{cases} \pmod{n^3}, & \text{if } 3\nmid n, \\ \pmod{n^3/3}, & \text{if } 3\mid n. \end{cases}$$

This completes the proof of Theorem 1.5.

If $e=3$, it follows from Theorem 1.1 and Corollary 1.4 that for $(n,6)=1$, we have

$$T_n \equiv (-1)^{\phi_3(n)} \left\{ 1 - kn\left(-\frac{3}{2}q_3(n) + \frac{3}{4}nq_3^2(n) \right) - \frac{1}{3}kn^2 A_3(n) \right.$$

$$
\begin{aligned}
&\qquad + \frac{1}{2}k^2n^2\left(\frac{9}{4}q_3^2(n)+A_3(n)\right)\Bigg\}\\
&\equiv (-1)^{\phi_3(n)}\left\{1+\frac{3}{2}knq_3(n)-\frac{3}{4}kn^2q_3^2(n)+\frac{9}{8}k^2n^2q_3^2(n)+k\left(\frac{1}{2}-\frac{1}{3}\right)n^2A_3(n)\right\}\\
&\equiv (-1)^{\phi_3(n)}\left\{\frac{1}{2}((1+nq_3(n))^{3k}+1)+k\left(\frac{1}{2}-\frac{1}{3}\right)n^2A_3(n)\right\}\\
&\equiv (-1)^{\phi_3(n)}\left\{\frac{1}{2}(27^{k\phi(n)}+1)+k\left(\frac{1}{2}k-\frac{1}{3}\right)n^2A_3(n)\right\} \pmod{n^3}.
\end{aligned}
$$

For the cases in which $e=4$ and 6, one may apply similar arguments and derive (1.14) and (1.15). Hence, Theorem 1.8 is proven.

3.4. Proof of Corollary 1.7. If $3\nmid pq$, a direct verification gives the congruence. Now we assume that $p=3$ and $q\geq 5$. Let $n=3q$. Recall that in the proof of Theorem 1.5, we have shown that

$$\binom{3kq-1}{(3q-1)/2}\equiv\left\{4^{2k(q-1)}+\frac{3^2k^2q^2}{4}\sum_{\substack{r=1\\(r,3q)=1}}^{3q-1}\frac{1}{r^2}\right\}\binom{3k-1}{1}\binom{kq-1}{(q-1)/2} \pmod{3^3q^3}.$$

From Lemma 2.2, we have

$$\sum_{\substack{r=1\\(r,3q)=1}}^{3q-1}\frac{1}{r^2}\equiv 0 \pmod{q}.$$

It also follows from the Fermat's Little Theorem that

$$\sum_{\substack{r=1\\(r,3q)=1}}^{3q-1}\frac{1}{r^2}\equiv\sum_{\substack{r=1\\(r,3q)=1}}^{3q-1}1=2q-2 \pmod{3}.$$

By the Chinese Remainder Theorem, we know that for integers n and x, and any coprime integers a and b, $n\equiv x \pmod{a}$ and $n\equiv x \pmod{b}$ if and only if $n\equiv x \pmod{ab}$. Hence, if $q\equiv 1 \pmod{6}$, then

$$\sum_{\substack{r=1\\(r,3q)=1}}^{3q-1}\frac{1}{r^2}\equiv 0 \pmod{3q}.$$

We therefore arrive at (1.12). If $q\equiv 5 \pmod{6}$, then

$$\sum_{\substack{r=1\\(r,3q)=1}}^{3q-1}\frac{1}{r^2}\equiv 4q \pmod{3q}.$$

Hence, to prove Corollary 1.7, it suffices to show that

$$3^2k^2q^3\binom{3k-1}{1}\binom{kq-1}{(q-1)/2}\equiv 0 \pmod{3^3q^3}. \tag{3.9}$$

For any prime p, let $\mathrm{ord}_p(n) := \max\{i \in \mathbb{N} : p^i \mid n\}$. The Legendre Theorem tells us that

$$\mathrm{ord}_p(n!) = \sum_{i\geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor .$$

We therefore have

$$\begin{aligned}
\mathrm{ord}_3\left(\binom{kq-1}{(q-1)/2}\right) &= \sum_{i\geq 1}\left(\left\lfloor \frac{kq-1}{3^i}\right\rfloor - \left\lfloor \frac{(q-1)/2}{3^i}\right\rfloor - \left\lfloor \frac{((2k-1)q-1)/2}{3^i}\right\rfloor\right) \\
&\geq \left\lfloor \frac{kq-1}{3}\right\rfloor - \left\lfloor \frac{q-1}{6}\right\rfloor - \left\lfloor \frac{(2k-1)q-1}{6}\right\rfloor \\
&= 1 + \left\lfloor \frac{5k-1}{3}\right\rfloor - \left\lfloor \frac{5k}{3}\right\rfloor .
\end{aligned}$$

If $3 \mid k$, it is obvious that (3.9) holds. If $3 \nmid k$, then

$$\mathrm{ord}_3\left(\binom{kq-1}{(q-1)/2}\right) \geq 1,$$

which implies that $3 \mid \binom{kq-1}{(q-1)/2}$. Hence, (3.9) is proven.

3.5. Proof of Theorem 1.9. The proof of Theorem 1.9 is more complicated. Note that

$$\begin{aligned}
2^{(n-1)/2}\binom{(kn-1)/2}{(n-1)/2} &= \prod_{r=1}^{(n-1)/2} \frac{kn-(2r-1)}{r} \\
&= \prod_{r=1}^{(n-1)/2}\left(\frac{kn-r}{r}\cdot\frac{kn-(n-r)}{r}\cdot\frac{r}{kn-2r}\right) \\
&= \prod_{d\mid n}\prod_{\substack{r=1\\(r,n)=d}}^{(n-1)/2}\left(\frac{kn-r}{r}\cdot\frac{kn-(n-r)}{r}\cdot\frac{r}{kn-2r}\right) \\
&= \prod_{d\mid n} S_{n/d} = \prod_{d\mid n} S_d,
\end{aligned}$$

where

$$S_d = \prod_{\substack{r=1\\(r,d)=1}}^{(d-1)/2}\left(\frac{kd-r}{r}\cdot\frac{kd-(d-r)}{r}\cdot\frac{r}{kd-2r}\right).$$

Again, it follows from the multiplicative version of the Möbius inversion formula that

$$S_n = \prod_{d\mid n}\left(2^{(d-1)/2}\binom{(kd-1)/2}{(d-1)/2}\right)^{\mu(n/d)} = 2^{\phi(n)/2}\prod_{d\mid n}\binom{(kd-1)/2}{(d-1)/2}^{\mu(n/d)}. \quad (3.10)$$

We assume that $3 \nmid n$. In the proof of Theorem 1.5, we have shown that

$$\prod_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{kn-r}{r} \equiv (-1)^{\phi(n)/2}4^{k\phi(n)} \pmod{n^3}. \quad (3.11)$$

Further,

$$\begin{aligned}
\prod_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{kn-(n-r)}{r} &= \prod_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\left(1+\frac{(k-1)n}{r}\right)\\
&\equiv 1+(k-1)n\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{1}{r}+\frac{(k-1)^2n^2}{2}\left(\left(\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{1}{r}\right)^2-\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{1}{r^2}\right)\\
&\equiv 1-2(k-1)nq_2(n)+(2(k-1)^2+(k-1))(nq_2(n))^2\\
&\equiv (1+nq_2(n))^{-2(k-1)}\\
&\equiv 4^{-(k-1)\phi(n)} \pmod{n^3}. \qquad (3.12)
\end{aligned}$$

Also,

$$\begin{aligned}
\prod_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{kn-2r}{r} &= (-1)^{\phi(n)/2}\prod_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\left(2-\frac{kn}{r}\right)\\
&\equiv (-1)^{\phi(n)/2}2^{\phi(n)/2}\left\{1-\frac{kn}{2}\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{1}{r}\right.\\
&\qquad\left.+\frac{1}{4}\frac{k^2n^2}{2}\left(\left(\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{1}{r}\right)^2-\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\frac{1}{r^2}\right)\right\}\\
&\equiv (-1)^{\phi(n)/2}2^{\phi(n)/2}\left\{1+knq_2(n)+\frac{k^2-k}{2}(nq_2(n))^2\right\}\\
&\equiv (-1)^{\phi(n)/2}2^{\phi(n)/2}(1+nq_2(n))^k\\
&\equiv (-1)^{\phi(n)/2}2^{\phi(n)/2}2^{k\phi(n)} \pmod{n^3}. \qquad (3.13)
\end{aligned}$$

It follows from (3.11), (3.12) and (3.13) that

$$S_n=\prod_{\substack{r=1\\(r,n)=1}}^{(n-1)/2}\left(\frac{kn-r}{r}\cdot\frac{kn-(n-r)}{r}\cdot\frac{r}{kn-2r}\right)\equiv 2^{-(k-3/2)\phi(n)} \pmod{n^3}.$$

The case in which $3 \mid n$ is similar. Combining the above congruence with (3.10), we therefore have

$$\prod_{d\mid n}\binom{(kd-1)/2}{(d-1)/2}^{\mu(n/d)}\equiv 2^{-(k-1)\phi(n)}\begin{cases}\pmod{n^3}, & \text{if } 3\nmid n,\\ \pmod{n^3/3}, & \text{if } 3\mid n.\end{cases}$$

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