Multi-headed Lattice Green Function (N = 5, M = 4) Polya Number

```
In[*]:= NN = 5;
MM = 4;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{1}{1-\frac{z}{\binom{N}{M}}\sigma_M(\cos\theta_1,\ldots,\cos\theta_N)}\,d\theta_1\ldots d\theta_N$$

$$R_{M,N}(z) := P_{M,N}\left(2^M\binom{N}{M}z\right) \text{ and } R_{M,N}(z) = \sum_{n\geq 0} r_{M,N}(n)\,z^n$$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define $\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n\geq 0}\tilde{r}_{M,N}(n)\,z^n = \sum_{n\geq 0} r_{M,N}(2n)\,z^n$

Our goal is to find the associated Polya number of the lattice in question.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan's implementation).

```
In[*]:= (* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] ⧴ a, Infinity]];
   Do[
    AppendTo[vals,
      Solve[rec /. n → i /. f[i] → x /. f[a_] ⧴ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Command: SeqLimit

Compute the limit of a convergent sequence (Koutschan's implementation).

```
In[*]:= (* Given the first values {f[0],...,f[m]} of a sequence f[n] and a basis of
 its asymptotic solutions, compute the limit Limit[f[n], n→Infinity]. *)
Clear[SeqLimit];
SeqLimit[data_List, asym_, n_] :=
  Module[{c, d = Length[asym], pos, ansatz, sol},
   pos = Length[data] + Range[-d, -1];
   ansatz = Array[c, d].asym;
   sol = Solve[((ansatz /. n → #) == data[[# + 1]]) & /@ pos, Array[c, d]][[1]];
   Return[N[c[d] /. sol, 200]];
  ];
```

Load RISC packages.

In[∘]:=
```
<< RISC`HolonomicFunctions`
<< RISC`Asymptotics`
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

Asymptotics Package version 0.3
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Package GeneratingFunctions version 0.9 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Guess Package version 0.52
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[∘]:=
```
ClearAll[Seq];
```

Load in advance the REC for $r_{4,5}(n)$ in Theorem 4.7 at the end of this file!

Translate the recurrence in terms of Ore Polynomials.

In[∘]:=
```
RECinS = ToOrePolynomial[REC /. {Seq[k_] → S[α]^(k-α)}];
```

Compute the recurrence for the *partial* Green function: $\sum_{0\le n\le n_0} r_{M,N}(n)\left(\frac{1}{2^M\binom{N}{M}}\right)^n$.

In[∘]:=
```
RECPartialGreeninS =
  DFiniteTimes[{RECinS}, Annihilator[(1/(2^MM Binomial[NN, MM]))^α, S[α]]][[1]] ** (S[α] - 1);
```

In[∘]:=
```
OrePolynomialDegree[RECPartialGreeninS, S[α]]
```

Out[∘]= 7

In[∘]:=
```
RECPartialGreen = ApplyOreOperator[RECPartialGreeninS, Seq[α]];
```

Compute the initial values of the partial Green function by the values of *r* and then generate a list.

In[◦]:=
```
RIni = {1, 0, 80, 0, 58 320, 933 120, 107 360 000, 4 032 000 000,
   305 742 850 000, 16 007 947 200 000, 1 092 754 448 110 080, 66 052 872 139 161 600};
PartialGreenIni = Table[Sum[RIni[[i]] * (1/(2^MM Binomial[NN, MM]))^(i-1), {i, 1, m}],
  {m, 0, Length@RIni}]
```

Out[◦]=
$$\left\{0, 1, 1, \frac{81}{80}, \frac{81}{80}, \frac{519\,129}{512\,000}, \frac{1\,298\,187}{1\,280\,000}, \frac{41\,558\,759}{40\,960\,000}, \frac{20\,783\,317}{20\,480\,000}, \frac{170\,287\,507\,149}{167\,772\,160\,000}, \frac{170\,307\,517\,083}{167\,772\,160\,000}, \frac{4\,258\,114\,784\,281\,293}{4\,194\,304\,000\,000\,000}, \frac{851\,687\,461\,614\,207}{838\,860\,800\,000\,000}\right\}$$

In[◦]:=
```
Bound = 1000;

PartialGreenList = UnrollRecurrence[RECPartialGreen, Seq[α], PartialGreenIni, Bound];
```

Analyze the asymptotic behavior of the sequence of partial Green function values.

In[◦]:=
```
Asymptotics[RECPartialGreen, Seq[α]]
```

Out[◦]=
$$\left\{\frac{\left(-\frac{27}{5}\right)^{\alpha}}{\alpha^{5/2}}, \frac{\left(-\frac{3}{5}\right)^{\alpha}}{\alpha^{5/2}}, \frac{\left(-\frac{1}{15}\right)^{\alpha}}{\alpha^{5/2}}, \frac{5^{-\alpha}}{\alpha^{9/4}}, \frac{5^{-\alpha}}{\alpha^{7/4}}, \frac{1}{\alpha^{3/2}}, 1\right\}$$

Compute the limit of partial Green function sequence and the associated Polya number.

In[◦]:=
```
lim1 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 30], α]
```

Out[◦]=
```
1.015856017164931405950838954209415274441607110200291467804237447012782152267217772631
 279430408132198538241157035115208702470469371602890624494501584873337340520733333747
 13792055872396589021016254245244
```

In[◦]:=
```
lim2 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 32], α]
```

Out[◦]=
```
1.015856017164931405950838954209415274441607110200291467804237447012782152267217772241
 064138795694872298063065697955262404387120553855542042808653749305881193334920569014
 849489342962207704996011643707501
```

In[◦]:=
```
lim1 - lim2
```

Out[◦]=
$3.92152916124373262401780913371599462980833488177473485816858478355674561471858127647\
3228843121576175818521415089874494 \times 10^{-82}$

In[◦]:=
```
1 - 1/lim2
```

Out[◦]=
```
0.015608528075841548586769127274108770624182765130661967601633528522347978982895167971
 060341336707951989636807987031514019070077255777471785315870137498737007454138633757
 24666493590721135797984637717835
```

Load the REC for $r_{4,5}(n)$ in Theorem 4.7.

In[◦]:=
```
REC =
  (2 364 822 061 925 891 270 067 722 649 600 000 + 24 311 763 241 480 737 290 507 853 496 320 000 α +
      118 884 714 388 336 585 062 289 753 767 936 000 α^2 +
      368 251 136 151 853 255 846 369 719 798 988 800 α^3 +
      811 793 640 582 985 414 140 746 797 028 474 880 α^4 +
      1 356 499 120 040 750 577 583 138 444 526 223 360 α^5 +
      1 786 835 040 377 781 128 110 811 754 937 712 640 α^6 +
      1 904 958 007 246 824 509 445 186 467 125 002 240 α^7 +
      1 674 545 402 297 600 373 785 511 713 251 000 320 α^8 +
      1 230 194 808 706 317 371 163 067 050 208 788 480 α^9 +
      762 791 807 513 049 677 466 384 009 532 538 880 α^10 +
```

$402\,079\,430\,499\,218\,110\,643\,393\,128\,200\,929\,280\,\alpha^{11} +$
$181\,085\,303\,893\,806\,582\,831\,390\,648\,576\,245\,760\,\alpha^{12} +$
$69\,909\,566\,044\,762\,687\,837\,271\,137\,604\,075\,520\,\alpha^{13} +$
$23\,174\,037\,389\,797\,607\,720\,091\,614\,796\,840\,960\,\alpha^{14} +$
$6\,597\,237\,647\,955\,223\,324\,018\,009\,760\,071\,680\,\alpha^{15} + 1\,610\,851\,715\,462\,724\,269\,782\,004\,410\,613\,760$
$\alpha^{16} + 336\,382\,193\,033\,012\,242\,367\,855\,858\,810\,880\,\alpha^{17} +$
$59\,795\,770\,083\,083\,316\,221\,336\,805\,703\,680\,\alpha^{18} + 8\,987\,061\,025\,545\,721\,077\,834\,511\,810\,560\,\alpha^{19} +$
$1\,131\,237\,375\,988\,193\,565\,613\,353\,861\,120\,\alpha^{20} + 117\,704\,523\,870\,056\,936\,584\,154\,972\,160\,\alpha^{21} +$
$9\,941\,030\,662\,497\,120\,749\,554\,237\,440\,\alpha^{22} + 664\,040\,244\,922\,741\,425\,721\,835\,520\,\alpha^{23} +$
$33\,746\,986\,442\,943\,554\,031\,452\,160\,\alpha^{24} + 1\,225\,566\,587\,608\,656\,091\,545\,600\,\alpha^{25} +$
$28\,320\,365\,528\,012\,449\,382\,400\,\alpha^{26} + 312\,808\,771\,118\,086\,225\,920\,\alpha^{27}\Big)\ \mathrm{Seq}[\alpha] +$
$\Big(880\,540\,948\,213\,763\,261\,498\,004\,602\,880\,000 + 8\,086\,612\,414\,279\,581\,582\,690\,097\,299\,456\,000\,\alpha +$
$35\,535\,843\,625\,080\,580\,938\,628\,852\,403\,404\,800\,\alpha^{2} +$
$99\,482\,199\,073\,846\,865\,130\,149\,987\,053\,731\,840\,\alpha^{3} +$
$199\,278\,215\,238\,194\,877\,084\,174\,219\,759\,058\,944\,\alpha^{4} +$
$304\,147\,288\,569\,704\,121\,767\,283\,668\,058\,636\,288\,\alpha^{5} +$
$367\,726\,422\,460\,034\,552\,713\,877\,456\,306\,307\,072\,\alpha^{6} +$
$361\,508\,986\,147\,801\,089\,153\,130\,211\,095\,805\,952\,\alpha^{7} +$
$294\,331\,319\,744\,750\,632\,422\,172\,167\,712\,997\,376\,\alpha^{8} +$
$201\,108\,607\,972\,501\,732\,293\,906\,606\,562\,934\,784\,\alpha^{9} +$
$116\,437\,788\,942\,848\,727\,536\,075\,769\,222\,856\,704\,\alpha^{10} +$
$57\,524\,299\,296\,878\,619\,402\,424\,939\,339\,382\,784\,\alpha^{11} +$
$24\,367\,165\,878\,769\,872\,656\,509\,536\,747\,061\,248\,\alpha^{12} +$
$8\,877\,402\,295\,660\,764\,714\,512\,245\,808\,234\,496\,\alpha^{13} +$
$2\,785\,748\,984\,068\,408\,698\,625\,918\,477\,467\,648\,\alpha^{14} +$
$752\,972\,653\,647\,501\,430\,958\,086\,738\,673\,664\,\alpha^{15} + 175\,049\,743\,314\,674\,169\,771\,167\,299\,534\,848$
$\alpha^{16} + 34\,895\,534\,864\,837\,208\,484\,258\,292\,957\,184\,\alpha^{17} +$
$5\,936\,277\,532\,573\,962\,980\,718\,997\,929\,984\,\alpha^{18} + 855\,818\,515\,821\,739\,179\,539\,429\,326\,848\,\alpha^{19} +$
$103\,560\,073\,600\,267\,246\,364\,541\,321\,216\,\alpha^{20} + 10\,380\,185\,487\,431\,012\,018\,005\,475\,328\,\alpha^{21} +$
$846\,180\,664\,706\,397\,472\,693\,420\,032\,\alpha^{22} + 54\,656\,640\,176\,185\,180\,963\,209\,216\,\alpha^{23} +$
$2\,690\,612\,916\,385\,314\,156\,576\,768\,\alpha^{24} + 94\,804\,345\,329\,795\,433\,758\,720\,\alpha^{25} +$
$2\,128\,785\,749\,082\,227\,343\,360\,\alpha^{26} + 22\,881\,382\,331\,785\,936\,896\,\alpha^{27}\Big)\ \mathrm{Seq}[1+\alpha] +$
$\Big(-664\,078\,540\,666\,702\,251\,488\,371\,015\,680\,000 - 5\,805\,956\,958\,011\,506\,960\,041\,778\,348\,032\,000\,\alpha -$
$24\,298\,272\,789\,380\,152\,495\,188\,221\,126\,246\,400\,\alpha^{2} -$
$64\,810\,405\,629\,301\,547\,428\,216\,819\,254\,558\,720\,\alpha^{3} -$
$123\,755\,374\,367\,469\,269\,296\,809\,845\,353\,611\,264\,\alpha^{4} -$
$180\,149\,375\,502\,996\,189\,202\,275\,648\,542\,982\,144\,\alpha^{5} -$
$207\,865\,771\,244\,125\,682\,287\,781\,841\,861\,722\,112\,\alpha^{6} -$
$195\,153\,222\,041\,523\,657\,876\,484\,723\,267\,989\,504\,\alpha^{7} -$
$151\,846\,270\,858\,495\,120\,363\,896\,477\,860\,167\,680\,\alpha^{8} -$
$99\,230\,231\,828\,276\,421\,932\,960\,434\,682\,314\,752\,\alpha^{9} -$
$54\,993\,115\,047\,787\,497\,911\,079\,580\,675\,899\,392\,\alpha^{10} -$
$26\,028\,017\,908\,489\,825\,928\,212\,462\,245\,453\,824\,\alpha^{11} -$
$10\,572\,113\,416\,646\,586\,933\,511\,582\,698\,766\,336\,\alpha^{12} -$
$3\,696\,722\,231\,163\,815\,760\,173\,082\,026\,344\,448\,\alpha^{13} - 1\,114\,468\,173\,061\,041\,282\,670\,805\,399\,093\,248$
$\alpha^{14} - 289\,688\,969\,845\,746\,113\,335\,461\,572\,931\,584\,\alpha^{15} -$
$64\,831\,091\,647\,102\,802\,811\,533\,842\,055\,168\,\alpha^{16} - 12\,454\,053\,009\,083\,005\,771\,527\,566\,163\,968\,\alpha^{17} -$
$2\,043\,760\,292\,966\,696\,499\,523\,264\,184\,320\,\alpha^{18} - 284\,532\,912\,366\,921\,324\,027\,166\,588\,928\,\alpha^{19} -$
$33\,284\,416\,956\,384\,385\,896\,458\,223\,616\,\alpha^{20} - 3\,228\,606\,478\,351\,534\,833\,828\,626\,432\,\alpha^{21} -$
$254\,974\,947\,491\,313\,890\,128\,560\,128\,\alpha^{22} - 15\,972\,126\,457\,377\,261\,067\,698\,176\,\alpha^{23} -$
$763\,333\,007\,662\,980\,725\,211\,136\,\alpha^{24} - 26\,138\,887\,552\,462\,651\,129\,856\,\alpha^{25} -$
$570\,997\,443\,951\,748\,710\,400\,\alpha^{26} - 5\,976\,795\,675\,008\,958\,464\,\alpha^{27}\Big)\ \mathrm{Seq}[2+\alpha] +$
$\Big(36\,337\,840\,931\,616\,555\,318\,702\,833\,664\,000 + 310\,343\,693\,247\,202\,072\,877\,171\,431\,833\,600\,\alpha +$
$1\,268\,062\,726\,217\,635\,641\,408\,454\,051\,430\,400\,\alpha^{2} + 3\,300\,521\,955\,790\,071\,740\,463\,976\,232\,263\,680$
$\alpha^{3} + 6\,146\,984\,578\,367\,464\,065\,862\,054\,879\,242\,240\,\alpha^{4} +$

$8723512529514925026222139080468480\,\alpha^{5}+9808817646565897068529809213239808\,\alpha^{6}+8970447157798999809214350039412224\,\alpha^{7}+$
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$2327102570668214059453238664192\,\alpha^{16}+434708874971795823099840116736\,\alpha^{17}+$
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$\Big(1737772868400007324872130560000+14528609204414291845066255564800\,\alpha+$
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$392186219850629120\,\alpha^{26}+3900964176134144\,\alpha^{27}\Big)\,\mathrm{Seq}[4+\alpha]+$

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$8261380192874790912\,\alpha^{24}-262301388296421376\,\alpha^{25}-$
$5312632953241600\,\alpha^{26}-51561082388480\,\alpha^{27}\Big)\,\mathrm{Seq}[5+\alpha]+$

$\Big(-154404486709237819219968000-1265327918255018927110348800\,\alpha-$
$4953641658930095511385751040\,\alpha^{2}-12335446851783544166937390720\,\alpha^{3}-$
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$998451712547824111144656513\,\alpha^{12}-320337381856256276567115789\,\alpha^{13}-$
$88485146094830787771471525\,\alpha^{14}-21045641782461353200898049\,\alpha^{15}-$
$4304140182149530399276227\,\alpha^{16}-754678659252915954749073\,\alpha^{17}-$
$112910766050133819763020\,\alpha^{18}-14316213223182938203068\,\alpha^{19}-$
$1523679350645560062336\,\alpha^{20}-134345128624663841280\,\alpha^{21}-$
$9635762018738626560\,\alpha^{22}-547760583383666688\,\alpha^{23}-23739371943886848\,\alpha^{24}-$
$736693272182784\,\alpha^{25}-14575541944320\,\alpha^{26}-138110042112\,\alpha^{27}\Big)\,\mathrm{Seq}[6+\alpha];$