Proof of a conjecture of Lin and Ma on 0012-avoiding inversion sequences

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Abstract. The study of pattern avoidance in inversion sequences recently attracts extensive research interests. In particular, Zhicong Lin and Jun Ma conjectured a formula that counts the number of inversion sequences avoiding the length-four pattern 0012. Our object of the paper is to confirm this conjecture.

Keywords. Inversion sequence, pattern avoidance, generating function, kernel method.

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1. Introduction

An *inversion sequence* of length n is a sequence $e = e_1 e_2 \cdots e_n$ such that $0 \le e_i \le i-1$ for each $1 \le i \le n$. We denote by $\mathbf{I}_n$ the set of inversion sequences of length n. Given any word $w \in \{0, 1, \ldots, n-1\}^n$ of length n, we define its *reduction* by the word obtained via replacing the k-th smallest entries of e with $k-1$. For instance, the reduction of 0023252 is 0012131. We say that an inversion sequence e *contains* a given pattern p if there exists a subsequence of e such that its reduction is the same as p; otherwise, we say that e *avoids* the pattern p. For instance, 0023252 has a subsequence 022 whose reduction is 011 — hence, 0023252 contains the pattern 011. On the other hand, none of the length 3 subsequences of 0023252 have the reduction 110 — hence, 0023252 avoids the pattern 110.

Let p_1, p_2, $\ldots$, p_m be given patterns. We denote by $\mathbf{I}_n(p_1, p_2, \ldots, p_m)$ the set of inversion sequences of length n that avoid all of the patterns p_1, p_2, $\ldots$, p_m. Recently, the study of pattern avoidance in inversion sequences attracts extensive research interests. See [1–8, 10–12, 14–16, 19, 20] for several instances of work on this topic. Among these work, one particular interesting problem is about the enumeration of inversion sequences that avoids fixed patterns. For example, in a pioneering work of Corteel, Martinez, Savage and Weselcouch [7], it was shown that

$$|\mathbf{I}_n(011)| = B_n \quad \text{and} \quad |\mathbf{I}_n(021)| = S_n$$

where B_n is the n-th Bell number (OEIS [18], A000110) and S_n is the n-th large Schröder number (OEIS, A006318).

In a recent paper [19], Yan and Lin proved a conjecture due to Martinez and Savage [16] that claims

$$|\mathbf{I}_n(021, 120)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}. \tag{1.1}$$

This sequence is registered as A279561 in OEIS. Lin and Yan also showed that this sequence as well enumerates $|\mathbf{I}_n(102, 110)|$ and $|\mathbf{I}_n(102, 120)|$. This therefore establishes the Wilf-equivalences

$$\mathbf{I}_n(021,120) \sim \mathbf{I}_n(102,110) \sim \mathbf{I}_n(102,120). \tag{1.2}$$

At the end of [19], a conjecture of Zhicong Lin and Jun Ma discovered in 2019 is recorded.

Conjecture 1.1 (Lin and Ma)**.** For $n \geq 1$,

$$|\mathbf{I}_n(0012)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}. \tag{1.3}$$

In other words, one may extend the balanced Wilf-equivalences (1.2) as the following unbalanced ones:

$$\mathbf{I}_n(0012) \sim \mathbf{I}_n(021,120) \sim \mathbf{I}_n(102,110) \sim \mathbf{I}_n(102,120).$$

It is also worth pointing out that the consideration of length-four pattern avoidance in inversion sequences appears novel in the literature.

Our object of this paper is to confirm the above conjecture of Lin and Ma.

Theorem 1.1. *Conjecture 1.1 is true.*

Let us fix some notation. Given $e = e_1e_2\cdots e_n \in \mathbf{I}_n(0012)$, we define

$$\mathcal{R}(e) := \{m : \exists\, i \neq j \text{ such that } e_i = e_j = m\}.$$

In other words, $\mathcal{R}(e)$ is the set of letters that appear more than once in e. We further define

$$\operatorname{srpt}(e) := \min \mathcal{R}(e),$$

that is, the smallest number in $\mathcal{R}(e)$ — here srpt stands for "smallest repeated." Notice that there is only one sequence $01\cdots(n-1)$ in which none of the letters repeat. For this sequence, we assign that

$$\operatorname{srpt}(01\cdots(n-1)) := n-1.$$

Finally, we define

$$\operatorname{last}(e) := e_n,$$

the last entry of e.

2. Combinatorial observations

We collect some combinatorial observations about inversion sequences in $\mathbf{I}_n(0012)$.

Lemma 2.1. *For $n \geq 1$ and $e \in \mathbf{I}_n(0012)$, if $\operatorname{srpt}(e) = k$, then for $1 \leq i \leq k+1$, we have $e_i = i-1$.*

Proof. If $\operatorname{srpt}(e) = n-1$, then $e = 01\cdots(n-1)$ and hence the lemma is true. Let $\operatorname{srpt}(e) \neq n-1$. If in this case the lemma is not true, then since $0 \leq e_i \leq i-1$ for each i, there must exist some $k_1 < k = \operatorname{srpt}(e)$ that appears more than once among $e_1, e_2, \ldots, e_{k+1}$. This violates the assumption that $\operatorname{srpt}(e) = k$. □

Lemma 2.2. *For $n \geq 2$ and $e = e_1e_2\cdots e_n \in \mathbf{I}_n(0012)$, let $\gamma(e) = e_1e_2\cdots e_{n-1}$. We further assume that $e \neq 01\cdots(n-1)$. Then*

(a). *if $\operatorname{last}(e) > \operatorname{srpt}(\gamma(e))$, then*

$$\operatorname{srpt}(e) = \operatorname{srpt}(\gamma(e));$$

(b). *if* $\mathrm{last}(e) \le \mathrm{srpt}(\gamma(e))$, *then*

$$\mathrm{srpt}(e) = \mathrm{last}(e).$$

Proof. A simple observation is that $\gamma(e) \in \mathbf{I}_{n-1}(0012)$. Below let us assume that $\mathrm{last}(e) = \ell$, $\mathrm{srpt}(e) = k$ and $\mathrm{srpt}(\gamma(e)) = k'$.

First, if $\mathcal{R}(\gamma(e)) = \emptyset$, then for each $0 \le i \le n-1$, $e_i = i-1$. Since $e \ne 01\cdots(n-1)$, we have $\mathrm{last}(e) = \ell \le n-2 = \mathrm{srpt}(\gamma(e))$. This fits into Case (b). Further, we find that $\mathcal{R}(e) = \{\ell\}$ and hence $\mathrm{srpt}(e) = \ell$. This implies that $\mathrm{srpt}(e) = \mathrm{last}(e)$.

Now we assume that $\mathcal{R}(\gamma(e)) \ne \emptyset$. Notice that Case (a) is trivial. For Case (b), we first deduce from $\mathcal{R}(\gamma(e)) \ne \emptyset$ that $k' \le n-3$. By Lemma 2.1, we find that for $1 \le i \le k'+1$, $e_i = i-1$. If $\mathrm{last}(e) = \ell \le k'$, then we know that $e_{\ell+1} = \ell = e_n$. Also, we notice that the indices satisfy $\ell+1 \le k'+1 \le n-2 < n$. Hence, $\ell \in \mathcal{R}(e)$. Therefore, $\mathrm{srpt}(e) = \min\{\ell, k'\} = \ell = \mathrm{last}(e)$. $\square$

Corollary 2.3. *For* $e \in \mathbf{I}_n(0012)$,

$$0 \le \mathrm{srpt}(e) \le \mathrm{last}(e) \le n-1.$$

Proof. If $e = 01\cdots(n-1)$, the above inequalities are trivial since $\mathrm{srpt}(e) = \mathrm{last}(e) = n-1$. If $e \ne 01\cdots(n-1)$, the inequalities are direct consequences of Lemma 2.2 and the fact that $\mathrm{srpt}(e) \ge 0$ and $\mathrm{last}(e) \le n-1$. $\square$

Lemma 2.4. *For* $n \ge 2$ *and* $e = e_1e_2\cdots e_n \in \mathbf{I}_n(0012)$, *let* e *be such that* $\mathrm{srpt}(e) = \mathrm{last}(e) = k$ *with* $0 \le k \le n-2$. *Then*

(a). *for* $1 \le i \le k+1$,

$$e_i = i-1;$$

(b). *if we denote* $e' = e'_1e'_2\cdots e'_{n-k}$ *by the sequence obtained via* $e'_i = e_{k+i} - k$ *for each* $1 \le i \le n-k$, *then* $e' \in \mathbf{I}_{n-k}(0012)$ *such that*

$$\mathrm{srpt}(e') = \mathrm{last}(e') = 0.$$

Proof. Part (a) simply comes from Lemma 2.1. Also, we know from Part (a) that for $k+1 \le i \le n$, it holds that $e_i \ge k$. On the other hand, $e_i \le i-1$. Hence, e' is still an inversion sequence. Further, it is trivial to see that e' still avoids the pattern 0012. Finally, we have $e'_1 = e_{k+1} - k = k-k = 0$ and $\mathrm{last}(e') = e'_{n-k} = e_n - k = k-k = 0$. Since $n-k \ge 2 > 1$, we have $0 \in \mathcal{R}(e')$ and hence $\mathrm{srpt}(e') = 0$. $\square$

3. Recurrences

Let

$$f_n(k,\ell) := \left\{ \begin{array}{c} \text{the number of sequences } e \in \mathbf{I}_n(0012) \text{ with} \\ \mathrm{srpt}(e) = k \text{ and } \mathrm{last}(e) = \ell \end{array} \right\}.$$

We will establish the following recurrences.

Lemma 3.1. *We have*

(a). *for* $n \ge 1$,

$$f_n(n-1, n-1) = 1;$$

(b). *for* $n \ge 2$,

$$f_n(n-2, n-1) = 0;$$

(c). *for* $n \geq 2$ *and* $0 \leq k \leq n-3$,

$$f_n(k, n-1) = \sum_{k'=k}^{n-2} f_{n-1}(k', n-2);$$

(d). *for* $n \geq 2$ *and* $0 \leq \ell \leq n-2$,

$$f_n(\ell, \ell) = \sum_{\ell'=\ell}^{n-2} \sum_{k'=\ell}^{\ell'} f_{n-1}(k', \ell');$$

(e). *for* $n \geq 2$ *and* $0 \leq k < \ell \leq n-2$,

$$f_n(k, \ell) = \sum_{k'=k}^{\ell} f_{n-1}(k', \ell) + \sum_{\ell'=\ell}^{n-2} f_{n-1}(k, \ell').$$

Proof. Cases (a) and (b) are trivial. In particular, Case (a) enumerates the only inversion sequence $01\cdots(n-1)$ in which none of the letters repeat. Below we always assume that $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n(0012)$. Let $\gamma(e)$ be as in Lemma 2.2.

For Case (c), let e be such that $\mathrm{srpt}(e) = k \leq n-3$ and $\mathrm{last}(e) = n-1$. We first notice that $e_{n-1} = \mathrm{last}(\gamma(e)) \geq \mathrm{srpt}(\gamma(e))$ by Corollary 2.3. Also, it is easy to see that $\mathrm{srpt}(\gamma(e)) = \mathrm{srpt}(e) = k$ since $\mathrm{last}(e) = n-1 > k$. Now we claim that $e_{n-1} = k$. Otherwise, namely, if $e_{n-1} > k$, we may find $i < j < n-1$ such that $e_i = e_j = k$. Hence, $e_i e_j e_{n-1} e_n$ has the reduction 0012, which contradicts the assumption that $e \in \mathbf{I}_n(0012)$. We therefore have a bijection

$$e = e_1 e_2 \cdots e_{n-2}(k)(n-1) \longleftrightarrow e_1 e_2 \cdots e_{n-2}(n-2) = e'.$$

Notice that e' is still an inversion sequence avoiding the pattern 0012. Also, $\mathrm{srpt}(e') \geq k$. Otherwise, there exists some $k' < k$ that appears more than once among $e_1, e_2, \ldots, e_{n-2}$ and therefore $\mathrm{srpt}(e) < k$, which leads to a contradiction. Finally, to prove Case (c), it suffices to show that e' could be any inversion sequence in $\mathbf{I}_{n-1}(0012)$ with $\mathrm{last}(e') = n-2$ (which is of course true) and $\mathrm{srpt}(e') \geq k$. Let e' be such a sequence and assume that $\mathrm{srpt}(e') = k' \geq k$. By Lemma 2.1, we have $e_{k+1} = k$. Pulling back to e, we have $e_{k+1} = e_{n-1} = k$ with the indices $k+1 \leq n-2 < n-1$. Therefore, for this e, we have $k \in \mathcal{R}(e)$ and hence $\mathrm{srpt}(e) = \min\{k', k\} = k$.

For Case (d), let e be such that $\mathrm{srpt}(e) = \mathrm{last}(e) = \ell$ with $0 \leq \ell \leq n-2$. We first find that $\mathrm{srpt}(\gamma(e)) \geq \mathrm{srpt}(e) = \ell$. On the other hand, let $e' = e'_1 e'_2 \cdots e'_{n-1} \in \mathbf{I}_{n-1}(0012)$ be such that $\mathrm{srpt}(e') \geq \ell$. By Lemma 2.1, $e'_{\ell+1} = \ell$. Hence, by appending ℓ to the end of e', we obtain a sequence with both srpt and last equal to ℓ. We therefore arrive at a bijection between e and e',

$$e = e_1 e_2 \cdots e_{n-1}(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-1} = e',$$

and the desired relation follows.

For Case (e), let e be such that $\mathrm{srpt}(e) = k$ and $\mathrm{last}(e) = \ell$ with $0 \leq k < \ell \leq n-2$. Notice that $e_{n-1} \geq k$. Otherwise, we assume that $e_{n-1} = k' < k$. Then by Lemma 2.1, $e_{k'+1} = k' = e_{n-1}$. However, $k'+1 < k+1 < n-1$ and hence $k' \in \mathcal{R}(e)$. But this violates the fact that $k = \min \mathcal{R}(e)$. Now we have two cases.

- $e_{n-1} < e_n$. We claim that $e_{n-1} = k$. Otherwise, we may find $i < j < n-1$ such that $e_i = e_j = k$. Hence, $e_i e_j e_{n-1} e_n$ has the reduction 0012, which violates the assumption that $e \in \mathbf{I}_n(0012)$. Now we have a bijection between e and $e' \in \mathbf{I}_{n-1}(0012)$ such that $\mathrm{srpt}(e') \geq k$ and $\mathrm{last}(e') = \ell$ by

$$e = e_1 e_2 \cdots e_{n-2}(k)(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-2}(\ell) = e'.$$

The argument is similar to that for Case (c). This bijection leads to the first term in the right-hand side of the recurrence relation in Case (e).

- $e_{n-1} \geq e_n$. We have a bijection between e and $e' \in \mathbf{I}_{n-1}(0012)$ such that $\mathrm{srpt}(e') = k$ and $\mathrm{last}(e') \geq \ell$ by

$$e = e_1 e_2 \cdots e_{n-1}(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-1} = e'.$$

The argument is similar to that for Case (d). This bijection leads to the second term in the right-hand side of the recurrence relation in Case (e).

The proof of the lemma is therefore complete. □

We may therefore determine the support of $f_n(k, \ell)$.

Corollary 3.2. *For $n \geq 1$, $f_n(k, \ell)$ is supported on*

$$\{(k, \ell) \in \mathbb{N}^2 : 0 \leq k \leq \ell \leq n-1\} \backslash \{(n-2, n-1)\}.$$

Proof. By Corollary 2.3, $f_n(k, \ell) = 0$ if

$$(k, \ell) \notin \{(k, \ell) \in \mathbb{N}^2 : 0 \leq k \leq \ell \leq n-1\}.$$

Also, $f_n(n-2, n-1) = 0$ by Lemma 3.1(b). Finally, for the remaining (k, ℓ), we have $f_n(k, \ell) \neq 0$ with the help of the recurrences in Lemma 3.1. □

Finally, we have another recurrence.

Lemma 3.3. *We have, for $n \geq 2$ and $0 \leq k \leq n-2$,*

$$f_n(k, k) = f_{n-k}(0, 0).$$

Proof. This is an immediate consequence of Lemma 2.4. □

In the sequel, we require three auxiliary functions with q *within a sufficiently small neighborhood of* 0:

$$\mathcal{L}(x; q) = \sum_{n \geq 1} L_n(x) q^n := \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} f_n(k, n-1) x^k \right) q^n,$$

$$\mathcal{D}(x; q) = \sum_{n \geq 1} D_n(x) q^n := \sum_{n \geq 1} \left(\sum_{\ell=0}^{n-2} f_n(\ell, \ell) x^\ell \right) q^n,$$

$$\mathcal{F}(x, y; q) = \sum_{n \geq 1} F_n(x, y) q^n := \sum_{n \geq 1} \left(\sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_n(k, \ell) x^k y^\ell \right) q^n.$$

Notice that $L_1(x) = 1$, $D_1(x) = 0$ and $F_1(x, y) = 1$. Also, since $f_n(n-1, n-1) = 1$, we have

$$\sum_{\ell=0}^{n-1} f_n(\ell, \ell) x^\ell = D_n(x) + x^{n-1}.$$

4. 001-Avoidance and a result of Corteel et al.

The following result on 001-avoidance was shown by Corteel et al. [7].

Theorem 4.1 (Corteel et al.)**.** *For $n \geq 1$,*

$$|\mathbf{I}_n(001)| = 2^{n-1}. \tag{4.1}$$

One readily observes that, for $n \geq 2$, there is a natural bijection between 001-avoiding inversion sequences of length $n-1$ and 0012-avoiding inversion sequences of length n in which the last entry equals $n-1$. Such a bijection could be simply constructed by appending $n-1$ to the end of the 001-avoiding inversion sequences. Therefore, we have an enumeration result as follows.

Corollary 4.2. *For $n \geq 1$,*

$$|\{e \in \mathbf{I}_n(0012) : \mathrm{last}(e) = n-1\}| = \begin{cases} 1 & \text{if } n = 1, \\ 2^{n-2} & \text{if } n \geq 2. \end{cases} \tag{4.2}$$

Notice that Corollary 4.2 is equivalent to

$$\begin{aligned}
\mathcal{L}(1;q) &= \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} f_n(k,n-1)\right) q^n \\
&= q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + \cdots \\
&= \frac{q(1-q)}{1-2q}.
\end{aligned}$$

Now we prove a bivariate strengthening of the above that will be utilized in our proof of Theorem 1.1.

Theorem 4.3. *We have*

$$\mathcal{L}(x;q) = \frac{q(1-q)^2}{(1-2q)(1-xq)}. \tag{4.3}$$

Proof. For $n \geq 2$, it follows from (a), (b) and (c) of Lemma 3.1 that

$$\begin{aligned}
\sum_{k=0}^{n-1} f_n(k,n-1)x^k &= x^{n-1} + \sum_{k=0}^{n-3}\sum_{k'=k}^{n-2} f_{n-1}(k',n-2)x^k \\
&= x^{n-1} + \sum_{k'=0}^{n-3} f_{n-1}(k',n-2)\sum_{k=0}^{k'} x^k + f_{n-1}(n-2,n-2)\sum_{k=0}^{n-3} x^k \\
&= x^{n-1} + \sum_{k'=0}^{n-3} f_{n-1}(k',n-2)\frac{1-x^{k'+1}}{1-x} + \frac{1-x^{n-2}}{1-x}.
\end{aligned}$$

Therefore,

$$L_n(x) = x^{n-1} + \frac{1}{1-x}\big(L_{n-1}(1) - xL_{n-1}(x)\big) - \frac{1-x^{n-1}}{1-x} + \frac{1-x^{n-2}}{1-x}.$$

Multiplying the above by q^n and summing over $n \geq 2$, we have

$$\mathcal{L}(x;q) - q = \frac{q}{1-x}\mathcal{L}(1;q) - \frac{xq}{1-x}\mathcal{L}(x;q) - \frac{q^2(1-x)}{1-xq},$$

or

$$(1-xq)(1-x+xq)\mathcal{L}(x;q)=q(1-xq)\mathcal{L}(1;q)+q(1-q)(1-x). \tag{4.4}$$

Applying the kernel method (see [9, Exercise 4, §2.2.1, p. 243] or [17]) yields

$$\begin{cases} 1-x+xq=0, \\ q(1-xq)\mathcal{L}(1;q)+q(1-q)(1-x)=0. \end{cases}$$

Solving the first equation of the system for x gives

$$x=\frac{1}{1-q}.$$

Substituting the above into the second equation of the system, we have

$$\mathcal{L}(1;q)=\frac{q(1-q)}{1-2q}.$$

Substituting the above back to (4.4), we arrive at (4.3). □

5. Proof of Theorem 1.1

We first establish two relations concerning $\mathcal{D}(x;q)$.

Lemma 5.1. *We have*

$$\begin{aligned} \mathcal{D}(x;q) &= \frac{1}{1-xq}\mathcal{D}(0;q) && (5.1)\\ &= \frac{q}{1-xq}\mathcal{F}(1,1;q). && (5.2) \end{aligned}$$

Proof. We know from Lemma 3.3 that

$$\begin{aligned} \sum_{n\geq 2}\sum_{k=0}^{n-2} f_n(k,k)x^k q^n &= \sum_{n\geq 2}\sum_{k=0}^{n-2} f_{n-k}(0,0)x^k q^n \\ {\scriptstyle (\text{with } n'=n-k)} &= \sum_{n'\geq 2}\sum_{n\geq n'} f_{n'}(0,0)x^{n-n'}q^n \\ &= \sum_{n'\geq 2} f_{n'}(0,0)x^{-n'}\sum_{n\geq n'}(xq)^n \\ &= \frac{1}{1-xq}\sum_{n'\geq 2} f_{n'}(0,0)q^{n'}. \end{aligned}$$

Noticing that $D_1(x)=0$, we have

$$\mathcal{D}(x;q)=\frac{1}{1-xq}\mathcal{D}(0;q),$$

which is the first part of the lemma. For the second part, we deduce from Lemma 3.1(d) that

$$\begin{aligned} \mathcal{D}(0;q) &= \sum_{n\geq 2} f_n(0,0)q^n \\ &= \sum_{n\geq 2}\sum_{\ell'=0}^{n-2}\sum_{k'=0}^{\ell'} f_{n-1}(k',\ell')q^n \\ &= q\mathcal{F}(1,1;q). \end{aligned}$$

Therefore, (5.2) follows. □

Next, we show a relation between $\mathcal{F}(x,1;q)$ and $\mathcal{F}(1,1;q)$.

Lemma 5.2. *We have*

$$\mathcal{F}(x,1;q) = \frac{1-q}{1-xq}\mathcal{F}(1,1;q). \tag{5.3}$$

Proof. For $n \geq 2$, it follows from Lemma 3.1(d) that

$$\begin{aligned} D_n(x) &= \sum_{\ell=0}^{n-2} f_n(\ell,\ell)x^\ell \\ &= \sum_{\ell=0}^{n-2}\sum_{\ell'=\ell}^{n-2}\sum_{k'=\ell}^{\ell'} f_{n-1}(k',\ell')x^\ell \\ &= \sum_{\ell'=0}^{n-2}\sum_{k'=0}^{\ell'} f_{n-1}(k',\ell')\sum_{k=0}^{k'} x^\ell \\ &= \sum_{\ell'=0}^{n-2}\sum_{k'=0}^{\ell'} f_{n-1}(k',\ell')\frac{1-x^{k'+1}}{1-x} \\ &= \frac{1}{1-x}\big(F_{n-1}(1,1) - xF_{n-1}(x,1)\big). \end{aligned}$$

Therefore,

$$\mathcal{D}(x;q) = \frac{q}{1-x}\big(\mathcal{F}(1,1;q) - x\mathcal{F}(x,1;q)\big).$$

Substituting (5.2) into the above yields

$$\frac{q}{1-xq}\mathcal{F}(1,1;q) = \frac{q}{1-x}\big(\mathcal{F}(1,1;q) - x\mathcal{F}(x,1;q)\big),$$

from which (5.3) follows. □

We then construct a functional equation for $\mathcal{F}(x,y;q)$.

Lemma 5.3. *We have*

$$\begin{aligned} &\left(1+\frac{xq}{1-x}+\frac{yq}{1-y}\right)\mathcal{F}(x,y;q) \\ &\quad= \frac{q}{1-x}\mathcal{F}(1,y;q) + \frac{q(1-q)}{(1-y)(1-xyq)}\mathcal{F}(1,1;q) + \frac{q(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)}. \end{aligned} \tag{5.4}$$

Proof. We first observe that

$$\begin{aligned} \sum_{\ell=0}^{n-2} f_n(\ell,\ell)x^\ell y^\ell + \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_n(k,\ell)x^k y^\ell &= F_n(x,y) - \sum_{k=0}^{n-1} f_n(k,n-1)x^k y^{n-1} \\ &= F_n(x,y) - y^{n-1}L_n(x). \end{aligned} \tag{5.5}$$

Notice also that

$$\sum_{\ell=0}^{n-2} f_n(\ell,\ell)x^\ell y^\ell = D_n(xy). \tag{5.6}$$

Now, by Lemma 3.1(e), we may separate

$$\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_n(k,\ell)x^k y^\ell = \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell} f_{n-1}(k',\ell)x^k y^\ell + \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell')x^k y^\ell.$$

We further notice that the first term on the right-hand side can be separated as

$$\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell} f_{n-1}(k',\ell)x^k y^\ell = \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell-1} f_{n-1}(k',\ell)x^k y^\ell + \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_{n-1}(\ell,\ell)x^k y^\ell.$$

We have

$$\begin{aligned}
&\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell-1} f_{n-1}(k',\ell)x^k y^\ell \\
&\quad= \sum_{\ell=1}^{n-2}\sum_{k'=0}^{\ell-1} f_{n-1}(k',\ell)y^\ell \sum_{k=0}^{k'} x^k \\
&\quad= \sum_{\ell=1}^{n-2}\sum_{k'=0}^{\ell-1} f_{n-1}(k',\ell)y^\ell \frac{1-x^{k'+1}}{1-x} \\
&\quad= \sum_{\ell=0}^{n-2}\sum_{k'=0}^{\ell} f_{n-1}(k',\ell)y^\ell \frac{1-x^{k'+1}}{1-x} - \sum_{\ell=0}^{n-2} f_{n-1}(\ell,\ell)y^\ell \frac{1-x^{\ell+1}}{1-x} \\
&\quad= \frac{1}{1-x}\big(F_{n-1}(1,y) - xF_{n-1}(x,y)\big) \\
&\qquad - \frac{1}{1-x}\big(D_{n-1}(y) + y^{n-2} - xD_{n-1}(xy) - x^{n-1}y^{n-2}\big).
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_{n-1}(\ell,\ell)x^k y^\ell &= \sum_{\ell=1}^{n-2} f_{n-1}(\ell,\ell)y^\ell \frac{1-x^\ell}{1-x} \\
&= \sum_{\ell=0}^{n-2} f_{n-1}(\ell,\ell)y^\ell \frac{1-x^\ell}{1-x} \\
&= \frac{1}{1-x}\big(D_{n-1}(y) + y^{n-2} - D_{n-1}(xy) - x^{n-2}y^{n-2}\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell')x^k y^\ell &= \sum_{\ell'=1}^{n-2}\sum_{k=0}^{\ell'-1} f_{n-1}(k,\ell')x^k \sum_{\ell=k+1}^{\ell'} y^\ell \\
&= \sum_{\ell'=1}^{n-2}\sum_{k=0}^{\ell'-1} f_{n-1}(k,\ell')x^k \frac{y^{k+1}-y^{\ell'+1}}{1-y} \\
&= \sum_{\ell'=0}^{n-2}\sum_{k=0}^{\ell'} f_{n-1}(k,\ell')x^k \frac{y^{k+1}-y^{\ell'+1}}{1-y}
\end{aligned}$$

$$= \frac{y}{1-y}\big(F_{n-1}(xy,1) - F_{n-1}(x,y)\big).$$

Therefore,

$$\begin{aligned}
&\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_n(k,\ell)x^k y^\ell \\
&\quad = \frac{1}{1-x}\big(F_{n-1}(1,y) - xF_{n-1}(x,y)\big) + \frac{y}{1-y}\big(F_{n-1}(xy,1) - F_{n-1}(x,y)\big) \\
&\qquad - D_{n-1}(xy) - x^{n-2}y^{n-2}.
\end{aligned} \tag{5.7}$$

It follows from (5.5), (5.6) and (5.7) that

$$\begin{aligned}
&F_n(x,y) - y^{n-1}L_n(x) \\
&\quad = D_n(xy) + \frac{1}{1-x}\big(F_{n-1}(1,y) - xF_{n-1}(x,y)\big) \\
&\qquad + \frac{y}{1-y}\big(F_{n-1}(xy,1) - F_{n-1}(x,y)\big) - D_{n-1}(xy) - x^{n-2}y^{n-2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{F}(x,y;q) - y^{-1}\mathcal{L}(x;yq) \\
&\quad = \mathcal{D}(xy;q) + \frac{q}{1-x}\big(\mathcal{F}(1,y;q) - x\mathcal{F}(x,y;q)\big) \\
&\qquad + \frac{yq}{1-y}\big(\mathcal{F}(xy,1;q) - \mathcal{F}(x,y;q)\big) - q\mathcal{D}(xy;q) - \frac{q^2}{1-xyq}.
\end{aligned}$$

Applying (4.3), (5.2) and (5.3) gives the desired result. □

With the assistance of the kernel method, we may deduce a functional equation satisfied by $\mathcal{F}(1,y;q)$.

Lemma 5.4. *We have*

$$\mathcal{F}(1,y;q) = \frac{q}{1-y+y^2q}\mathcal{F}(1,1;q) + \frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)(1-y+y^2q)}. \tag{5.8}$$

Proof. We multiply both sides of (5.4) by $(1-x)(1-y)$. Then

$$\begin{aligned}
&\big((1-y+yq) - x(1-y-q+2yq)\big)\mathcal{F}(x,y;q) \\
&\quad = q(1-y)\mathcal{F}(1,y;q) + \frac{q(1-q)(1-x)}{1-xyq}\mathcal{F}(1,1;q) \\
&\qquad + \frac{q(1-x)(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)}.
\end{aligned}$$

We treat the kernel polynomial as a function in x and solve

$$(1-y+yq) - x(1-y-q+2yq) = 0$$

so that

$$x = \frac{1-y+yq}{1-y-q+2yq}.$$

Substituting the above into

$$0 = q(1-y)\mathcal{F}(1,y;q) + \frac{q(1-q)(1-x)}{1-xyq}\mathcal{F}(1,1;q)$$

$$+\frac{q(1-x)(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)},$$

we arrive at (5.8) after simplification. □

Finally, we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. It is known that (cf. [18, A279561])

$$1+\sum_{n\geq 1}\left(1+\sum_{i=1}^{n-1}\binom{2i}{i-1}\right)q^n=\frac{1-4q+(1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)}. \tag{5.9}$$

We then rewrite (5.8) as

$$(1-y+y^2q)\mathcal{F}(1,y;q)=q\mathcal{F}(1,1;q)+\frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)}.$$

We treat the kernel polynomial as a function in y and solve

$$1-y+y^2q=0.$$

Then

$$y_{1,2}=\frac{1\mp\sqrt{1-4q}}{2q}.$$

We choose the solution

$$y_1=\frac{1-\sqrt{1-4q}}{2q}$$

since $y_1=1+q+O(q^2)$ as $q\to 0$. Substituting $y=y_1$ into

$$0=q\mathcal{F}(1,1;q)+\frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)},$$

we find that

$$\begin{aligned}\mathcal{F}(1,1;q)&=\frac{-(1-2q)(1-4q)+(1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)}\\&=\frac{1-4q+(1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)}-1.\end{aligned} \tag{5.10}$$

This implies that for $n\geq 1$,

$$1+\sum_{i=1}^{n-1}\binom{2i}{i-1}=\sum_{\ell=0}^{n-1}\sum_{k=0}^{\ell}f_n(k,\ell)=|\mathbf{I}_n(0012)|.$$

Therefore, Conjecture 1.1 is true. □

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