Multi-headed Lattice Green Function (N = 4, M = 3)
Polya Number

```
NN = 4;
MM = 3;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{1}{1-\frac{z}{\binom{N}{M}}\sigma_M(\cos\theta_1,\ldots,\cos\theta_N)}\,d\theta_1\ldots d\theta_N$$

$$R_{M,N}(z) := P_{M,N}\left(2^M\binom{N}{M}z\right) \text{ and } R_{M,N}(z) = \sum_{n\geq 0} r_{M,N}(n)\,z^n$$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define
$\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n\geq 0}\tilde{r}_{M,N}(n)\,z^n = \sum_{n\geq 0} r_{M,N}(2n)\,z^n$

Our goal is to find the associated Polya number of the lattice in question.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan's implementation).

```
(* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals,
      Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Command: SeqLimit

Compute the limit of a convergent sequence (Koutschan's implementation).

```
(* Given the first values {f[0],...,f[m]} of a sequence f[n] and a basis of
 its asymptotic solutions, compute the limit Limit[f[n], n→Infinity]. *)
Clear[SeqLimit];
SeqLimit[data_List, asym_, n_] :=
  Module[{c, d = Length[asym], pos, ansatz, sol},
   pos = Length[data] + Range[-d, -1];
   ansatz = Array[c, d].asym;
   sol = Solve[((ansatz /. n → #) == data[[# + 1]]) & /@ pos, Array[c, d]][[1]];
   Return[N[c[d] /. sol, 200]];
  ];
```

Load RISC packages.

```
In[∘]:= << RISC`HolonomicFunctions`
<< RISC`Asymptotics`
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan


```
--> Type  ?HolonomicFunctions  for help.
```

Asymptotics Package version 0.3
written by Manuel Kauers

Package GeneratingFunctions version 0.9 written by Christian Mallinger

Guess Package version 0.52
written by Manuel Kauers


```
In[∘]:= ClearAll[Seq];
```

Load in advance the REC for $\tilde{r}_{3,4}(n)$ in Theorem 4.3 at the end of this file!

Translate the recurrence in terms of Ore Polynomials.

```
In[∘]:= RECinS = ToOrePolynomial[REC /. {Seq[k_] → S[α]^(k-α)}];
```

Compute the recurrence for the *partial* Green function: $\sum_{0\le n\le n_0} \tilde{r}_{M,N}(n)\left(\frac{1}{2^M\binom{N}{M}}\right)^{2n}$.

```
In[∘]:= RECPartialGreeninS =
  DFiniteTimes[{RECinS}, Annihilator[(1/(2^MM Binomial[NN, MM]))^(2 α), S[α]]][[1]] **
   (S[α] - 1);

In[∘]:= OrePolynomialDegree[RECPartialGreeninS, S[α]]

Out[∘]= 5

In[∘]:= RECPartialGreen = ApplyOreOperator[RECPartialGreeninS, Seq[α]];
```

Compute the initial values of the partial Green function by the values of $\tilde{r}$ and then generate a list.

In[∘]:=
```
RIni = {1, 32, 6048, 2451200, 1391236000, 921422380032, 663895856219904};
PartialGreenIni =
  Table[Sum[RIni[[i]] * (1/(2^MM Binomial[NN, MM]))^(2 (i-1)), {i, 1, m}], {m, 0, Length@RIni}]
```

Out[∘]= $\left\{0, 1, \frac{33}{32}, \frac{33981}{32768}, \frac{4359143}{4194304}, \frac{35753575581}{34359738368}, \frac{1145014245135}{1099511627776}, \frac{4692571691261319}{4503599627370496}\right\}$

In[∘]:=
```
Bound = 1000;

PartialGreenList = UnrollRecurrence[RECPartialGreen, Seq[α], PartialGreenIni, Bound];
```

Analyze the asymptotic behavior of the sequence of partial Green function values.

In[∘]:=
```
Asymptotics[RECPartialGreen, Seq[α]]
```

Out[∘]= $\left\{\frac{64^{-\alpha}}{\alpha^2}, \frac{4^{-\alpha}}{\alpha^2}, \frac{1}{\alpha^2}, \frac{1}{\alpha}, 1\right\}$

Compute the limit of partial Green function sequence and the associated Polya number.

In[∘]:=
```
lim1 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 30], α]
```

Out[∘]=
```
1.0452879180865911417843270133824978673752777397256790765800746729908972504362795260587280053325719131916418189882225607533866047201082307920379467818546491857995110796787292822423937716338115597824133
```

In[∘]:=
```
lim2 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 32], α]
```

Out[∘]=
```
1.0452879180865911417843270133824978673752777397256790765800746729908978874655508327043747385428792555875743875662020296967016288176778352654553918272341499670218733469078853054713472431146366985461419
```

In[∘]:=
```
lim1 - lim2
```

Out[∘]= $-6.370292713066456467332103073423959325685779794689433150240975696044734174450453795007812223622672291560232289534714808251387637287 \times 10^{-70}$

In[∘]:=
```
1 - 1/lim2
```

Out[∘]=
```
0.04332578355013523911985002959583436982621694420764581838291342263347221652221583425454986885705193317718084864475491816063041260745264175452555003996602296568042967176528834796305034315419700621172431
```

Load the REC for $\tilde{r}_{3,4}(n)$ in Theorem 4.3.

In[ ]:= REC = $(221\,086\,792\,032\,258\,663\,383\,040 + 3\,002\,581\,182\,281\,579\,476\,549\,632\,\alpha + 18\,896\,284\,453\,973\,181\,469\,818\,880\,\alpha^2 + 73\,337\,056\,136\,834\,742\,984\,114\,176\,\alpha^3 + 197\,017\,275\,538\,043\,925\,583\,364\,096\,\alpha^4 + 389\,745\,626\,428\,476\,129\,286\,291\,456\,\alpha^5 + 589\,529\,476\,016\,351\,811\,509\,157\,888\,\alpha^6 + 698\,690\,177\,713\,813\,455\,561\,031\,680\,\alpha^7 + 659\,396\,154\,092\,196\,671\,988\,432\,896\,\alpha^8 + 500\,766\,687\,956\,261\,350\,615\,810\,048\,\alpha^9 + 307\,887\,490\,552\,535\,839\,569\,608\,704\,\alpha^{10} + 153\,616\,793\,330\,862\,792\,246\,296\,576\,\alpha^{11} + 62\,125\,104\,506\,185\,984\,379\,977\,728\,\alpha^{12} + 20\,265\,270\,278\,609\,884\,774\,662\,144\,\alpha^{13} + 5\,282\,843\,409\,745\,454\,510\,899\,200\,\alpha^{14} + 1\,084\,193\,901\,809\,507\,676\,192\,768\,\alpha^{15} + 171\,154\,981\,038\,855\,165\,050\,880\,\alpha^{16} + 20\,040\,031\,539\,432\,857\,272\,320\,\alpha^{17} + 1\,638\,003\,152\,561\,664\,688\,128\,\alpha^{18} + 83\,373\,097\,696\,100\,352\,000\,\alpha^{19} + 1\,988\,330\,027\,074\,191\,360\,\alpha^{20})\,\mathrm{Seq}[\alpha] + (-123\,596\,648\,884\,357\,621\,088\,256 - 1\,387\,410\,081\,329\,207\,115\,251\,712\,\alpha - 7\,308\,010\,505\,383\,031\,273\,947\,136\,\alpha^2 - 24\,020\,604\,752\,075\,269\,740\,691\,456\,\alpha^3 - 55\,262\,591\,055\,735\,725\,773\,815\,808\,\alpha^4 - 94\,607\,549\,345\,038\,165\,436\,006\,400\,\alpha^5 - 125\,070\,786\,847\,359\,746\,869\,821\,440\,\alpha^6 - 130\,760\,992\,638\,503\,780\,446\,109\,696\,\alpha^7 - 109\,819\,712\,522\,499\,293\,630\,693\,376\,\alpha^8 - 74\,830\,049\,897\,678\,615\,099\,736\,064\,\alpha^9 - 41\,599\,115\,200\,046\,517\,939\,601\,408\,\alpha^{10} - 18\,902\,277\,196\,351\,684\,209\,803\,264\,\alpha^{11} - 7\,008\,965\,526\,989\,775\,347\,122\,176\,\alpha^{12} - 2\,109\,519\,207\,312\,665\,281\,560\,576\,\alpha^{13} - 510\,375\,764\,108\,304\,797\,663\,232\,\alpha^{14} - 97\,744\,104\,267\,386\,959\,429\,632\,\alpha^{15} - 14\,472\,279\,363\,085\,494\,386\,688\,\alpha^{16} - 1\,596\,811\,738\,769\,963\,089\,920\,\alpha^{17} - 123\,530\,156\,260\,699\,668\,480\,\alpha^{18} - 5\,975\,058\,303\,292\,538\,880\,\alpha^{19} - 135\,920\,997\,944\,524\,800\,\alpha^{20})\,\mathrm{Seq}[1+\alpha] + (2\,413\,729\,498\,666\,800\,513\,024 + 25\,435\,086\,835\,865\,925\,058\,560\,\alpha + 125\,542\,481\,225\,411\,227\,975\,680\,\alpha^2 + 386\,097\,946\,352\,750\,392\,590\,336\,\alpha^3 + 830\,183\,396\,028\,360\,968\,208\,384\,\alpha^4 + 1\,327\,255\,653\,860\,270\,011\,465\,728\,\alpha^5 + 1\,637\,850\,112\,836\,596\,110\,688\,256\,\alpha^6 + 1\,598\,197\,760\,043\,557\,807\,628\,288\,\alpha^7 + 1\,252\,980\,911\,862\,994\,173\,739\,008\,\alpha^8 + 797\,358\,770\,338\,813\,407\,952\,896\,\alpha^9 + 414\,276\,959\,391\,975\,941\,603\,328\,\alpha^{10} + 176\,103\,421\,096\,866\,815\,410\,176\,\alpha^{11} + 61\,159\,515\,859\,482\,838\,548\,480\,\alpha^{12} + 17\,263\,930\,413\,062\,410\,149\,888\,\alpha^{13} + 3\,923\,295\,133\,237\,310\,914\,560\,\alpha^{14} + 706\,924\,713\,366\,338\,125\,824\,\alpha^{15} + 98\,652\,029\,401\,005\,981\,696\,\alpha^{16} + 10\,278\,087\,291\,823\,325\,184\,\alpha^{17} + 752\,234\,327\,699\,226\,624\,\alpha^{18} + 34\,490\,272\,274\,841\,600\,\alpha^{19} + 745\,214\,176\,788\,480\,\alpha^{20})\,\mathrm{Seq}[2+\alpha] + (-9\,569\,617\,440\,812\,835\,840 - 97\,443\,791\,378\,162\,009\,856\,\alpha - 463\,583\,339\,186\,644\,316\,800\,\alpha^2 - 1\,370\,837\,922\,368\,778\,354\,176\,\alpha^3 - 2\,827\,452\,328\,200\,593\,850\,560\,\alpha^4 - 4\,326\,575\,055\,112\,730\,856\,640\,\alpha^5 - 5\,099\,519\,612\,920\,329\,528\,000\,\alpha^6 - 4\,743\,666\,552\,937\,883\,189\,952\,\alpha^7 - 3\,539\,068\,890\,050\,114\,722\,112\,\alpha^8 - 2\,139\,750\,587\,880\,300\,657\,856\,\alpha^9 - 1\,054\,730\,779\,373\,468\,537\,920\,\alpha^{10} - 424\,824\,967\,934\,147\,228\,480\,\alpha^{11} - 139\,643\,546\,214\,642\,867\,648\,\alpha^{12} - 37\,274\,084\,807\,088\,072\,384\,\alpha^{13} - 8\,003\,802\,897\,605\,020\,608\,\alpha^{14} - 1\,361\,866\,764\,260\,304\,576\,\alpha^{15} - 179\,386\,646\,751\,384\,192\,\alpha^{16} - 17\,635\,678\,788\,631\,680\,\alpha^{17} - 1\,217\,772\,669\,657\,600\,\alpha^{18} - 52\,679\,537\,809\,920\,\alpha^{19} - 1\,074\,030\,451\,200\,\alpha^{20})\,\mathrm{Seq}[3+\alpha] + (9\,051\,531\,325\,562\,880 + 90\,332\,029\,095\,081\,984\,\alpha + 420\,333\,410\,362\,428\,416\,\alpha^2 + 1\,213\,206\,945\,955\,473\,664\,\alpha^3 + 2\,437\,377\,188\,874\,087\,136\,\alpha^4 + 3\,625\,291\,113\,645\,770\,712\,\alpha^5 + 4\,144\,688\,219\,837\,114\,384\,\alpha^6 + 3\,731\,957\,019\,300\,871\,994\,\alpha^7 + 2\,689\,507\,840\,271\,682\,912\,\alpha^8 + 1\,567\,534\,832\,320\,365\,967\,\alpha^9 + 743\,334\,125\,295\,350\,476\,\alpha^{10} + 287\,455\,002\,784\,035\,524\,\alpha^{11} + 90\,539\,774\,552\,500\,272\,\alpha^{12} + 23\,112\,095\,925\,472\,389\,\alpha^{13} + 4\,737\,102\,973\,509\,780\,\alpha^{14} + 767\,930\,664\,461\,310\,\alpha^{15} + 96\,195\,146\,877\,576\,\alpha^{16} + 8\,977\,485\,504\,456\,\alpha^{17} + 587\,451\,930\,408\,\alpha^{18} + 24\,041\,253\,600\,\alpha^{19} + 462\,944\,160\,\alpha^{20})\,\mathrm{Seq}[4+\alpha];$