Some generating functions and inequalities for the Andrews–Stanley partition functions

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Abstract. Let $\mathcal{O}(\pi)$ denote the number of odd parts in an integer partition π. In 2005, Stanley introduced a new statistic $\mathrm{srank}(\pi) = \mathcal{O}(\pi) - \mathcal{O}(\pi')$, where π' is the conjugate of π. Let $p(r, m; n)$ denote the number of partitions of n with srank congruent to r modulo m. Generating functions, congruences and inequalities for $p(0, 4; n)$ and $p(2, 4; n)$ were then established by a number of mathematicians, including Stanley, Andrews, Swisher, Berkovich and Garvan. Motivated by these work, we deduce some generating functions and inequalities for $p(r, m; n)$ with $m = 16$ and 24. These results are refinements of some inequalities due to Swisher.

Keywords. Andrews–Stanley partition function, rank, crank, partition inequality, asymptotic formula.

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1. Introduction

Let π be an integer partition and π' its conjugate. Stanley [9, 10] introduced a new integral partition statistic

$$\mathrm{srank}(\pi) = \mathcal{O}(\pi) - \mathcal{O}(\pi'),$$

where $\mathcal{O}(\pi)$ denotes the number of odd parts in the partition π. This statistic is called the Stanley rank.

Let $n \geq 1$ and $m \geq 2$ be integers. For any integer r with $0 \leq r \leq m-1$, define

$$p(r, m; n) := \#\{\pi \mid \pi \text{ is a partition of } n \text{ with } \mathrm{srank}(\pi) \equiv r \pmod{m}\}. \tag{1.1}$$

From the fact that

$$n \equiv \mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod{2},$$

it is easy to see that for $n \geq 1$,

$$p(n) = p(0, 4; n) + p(2, 4; n),$$

where $p(n)$ is the number of partitions of n. Moreover, if m is even and r is odd, then

$$p(r, m; n) = 0.$$

Stanley [9, 10] also established the following generating function for $p(0, 4; n) - p(2, 4; n)$:

$$\sum_{n=0}^{\infty} (p(0, 4; n) - p(2, 4; n))q^n = \frac{E(q^2)^4 E(q^8)^2}{E(q)E(q^4)^6}.$$

Here and throughout this paper,

$$E(q) := \prod_{n=1}^{\infty} (1 - q^n).$$

Following the work of Stanley, Andrews [2] then obtained the generating function for $p(0,4;n)$:

$$\sum_{n=0}^{\infty} p(0,4;n)q^n = \frac{E(q^2)^2E(q^{16})^5}{E(q)E(q^4)^5E(q^{32})^2}.$$

Furthermore, he proved that for $n \geq 0$,

$$p(0,4;5n+4) \equiv p(2,4;5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$

which is a refinement of the following famous congruence due to Ramanujan:

$$p(5n+4) \equiv 0 \pmod{5}.$$

At the end of his paper [2], Andrews asked for a partition statistic that would give a combinatorial interpretation of (1.2) since his proof of (1.2) is analytic. Berkovich and Garvan [4] later provided three such statistics and answered Andrews' inquiry.

In 2010, Swisher [13] proved that (1.2) is just one of infinitely many similar congruences satisfied by $p(0,4;n)$. In her PhD thesis [12], Swisher also established the following elegant results:

$$\lim_{n\to+\infty} \frac{p(0,4;n)}{p(n)} = \frac{1}{2} \tag{1.3}$$

and for sufficiently large n,

$$p(0,4;4n+0,1) > p(2,4;4n+0,1), \tag{1.4}$$

$$p(0,4;4n+2,3) < p(2,4;4n+2,3). \tag{1.5}$$

Berkovich and Garvan [3] also gave elementary proofs of (1.3)–(1.5) with the restriction of "n sufficiently large" removed. Further, Berkovich and Garvan presented a handful of new results, including

$$\lim_{n\to+\infty} \frac{p(0,4;2n)-p(2,4;2n)}{p(0,4;2n+1)-p(2,4;2n+1)} = 1+\sqrt{2} \tag{1.6}$$

and for $n \geq 1$,

$$|p(0,4;2n)-p(2,4;2n)| > |p(0,4;2n+1)-p(2,4;2n+1)|.$$

In this paper, we establish the generating functions for $p(r,m;n)$ with $m = 16$ and 24. It should be pointed out that if we define

$$p(k;n) := \#\{\pi \mid \pi \text{ is a partition of } n \text{ with } \mathrm{srank}(\pi) = k\}, \tag{1.7}$$

then in view of (1.1) and (1.7),

$$p(r,m;n) = \sum_{k\equiv r \pmod{m}} p(k;n). \tag{1.8}$$

It follows from [4, (2.8) and (2.9)] that

$$\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} p(k;n)z^kq^n = \frac{E(q^2)^2}{E(q)E(q^4)^2(z^2q^2;q^4)_\infty(q^2/z^2;q^4)_\infty}, \tag{1.9}$$

where the q-Pochhammer symbol is defined as usual by

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

From (1.9), we observe that

$$p(k;n) = p(-k;n)$$

and

$$p(r,m;n) = p(m-r,m;n).$$

Therefore, we merely list the generating functions for $p(r,m;n)$ with $m \in \{16, 24\}$ and $0 \le r \le \frac{m}{2}$.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty} p(0,16;n)q^n = \frac{S_1(q)}{8} + \frac{S_2(q)}{2} + \frac{S_3(q)}{4} + \frac{S_4(q)}{8}, \tag{1.10}$$

$$\sum_{n=0}^{\infty} p(2,16;n)q^n = \frac{S_1(q)}{8} - \frac{S_4(q)}{8} + \frac{S_5(q)}{2}, \tag{1.11}$$

$$\sum_{n=0}^{\infty} p(4,16;n)q^n = \frac{S_1(q)}{8} - \frac{S_3(q)}{4} + \frac{S_4(q)}{8}, \tag{1.12}$$

$$\sum_{n=0}^{\infty} p(6,16;n)q^n = \frac{S_1(q)}{8} - \frac{S_4(q)}{8} - \frac{S_5(q)}{2}, \tag{1.13}$$

$$\sum_{n=0}^{\infty} p(8,16;n)q^n = \frac{S_1(q)}{8} - \frac{S_2(q)}{2} + \frac{S_3(q)}{4} + \frac{S_4(q)}{8}, \tag{1.14}$$

where

$$S_1(q) = \frac{1}{E(q)}, \quad S_2(q) = \frac{E(q^2)^2E(q^8)E(q^{32})^3}{E(q)E(q^4)^3E(q^{16})^2E(q^{64})}, \quad S_3(q) = \frac{E(q^2)^2E(q^{16})}{E(q)E(q^4)E(q^8)^2},$$

$$S_4(q) = \frac{E(q^2)^4E(q^8)^2}{E(q)E(q^4)^6}, \quad S_5(q) = q^2\frac{E(q^2)^2E(q^8)E(q^{64})}{E(q)E(q^4)^3E(q^{16})}.$$

Theorem 1.2. *We have*

$$\sum_{n=0}^{\infty} p(0,24;n)q^n = \frac{F_1(q)}{12} + \frac{F_2(q)}{3} + \frac{F_3(q)}{6} + \frac{F_4(q)}{6} + \frac{F_5(q)}{6} + \frac{F_6(q)}{12}, \tag{1.15}$$

$$\sum_{n=0}^{\infty} p(2,24;n)q^n = \frac{F_1(q)}{12} + \frac{F_4(q)}{12} - \frac{F_5(q)}{12} - \frac{F_6(q)}{12} + \frac{F_7(q)}{2}, \tag{1.16}$$

$$\sum_{n=0}^{\infty} p(4,24;n)q^n = \frac{F_1(q)}{12} + \frac{F_2(q)}{6} - \frac{F_3(q)}{6} - \frac{F_4(q)}{12} - \frac{F_5(q)}{12} + \frac{F_6(q)}{12}, \tag{1.17}$$

$$\sum_{n=0}^{\infty} p(6,24;n)q^n = \frac{F_1(q)}{12} - \frac{F_4(q)}{6} + \frac{F_5(q)}{6} - \frac{F_6(q)}{12}, \tag{1.18}$$

$$\sum_{n=0}^{\infty} p(8,24;n)q^n = \frac{F_1(q)}{12} - \frac{F_2(q)}{6} + \frac{F_3(q)}{6} - \frac{F_4(q)}{12} - \frac{F_5(q)}{12} + \frac{F_6(q)}{12}, \tag{1.19}$$

$$\sum_{n=0}^{\infty} p(10,24;n)q^n = \frac{F_1(q)}{12} + \frac{F_4(q)}{12} - \frac{F_5(q)}{12} - \frac{F_6(q)}{12} - \frac{F_7(q)}{2}, \tag{1.20}$$

$$\sum_{n=0}^{\infty} p(12,24;n)q^n = \frac{F_1(q)}{12} - \frac{F_2(q)}{3} - \frac{F_3(q)}{6} + \frac{F_4(q)}{6} + \frac{F_5(q)}{6} + \frac{F_6(q)}{12}, \tag{1.21}$$

where

$$F_1(q) = \frac{1}{E(q)}, \quad F_2(q) = \frac{E(q^2)^2E(q^8)E(q^{12})E(q^{16})}{E(q)E(q^4)^4E(q^{24})}, \quad F_3(q) = \frac{E(q^2)^2E(q^{16})}{E(q)E(q^4)E(q^8)^2},$$

$$F_4(q) = \frac{E(q^2)E(q^6)E(q^{24})}{E(q)E(q^8)E(q^{12})^2}, \quad F_5(q) = \frac{E(q^2)^3E(q^{12})}{E(q)E(q^4)^3E(q^6)}, \quad F_6(q) = \frac{E(q^2)^4E(q^8)^2}{E(q)E(q^4)^6},$$

$$F_7(q) = q^2\frac{E(q^2)^2E(q^8)^2E(q^{12})E(q^{48})^2}{E(q)E(q^4)^4E(q^{16})E(q^{24})^2}.$$

Remark 1.1. Noticing that

$$p(r,m;n) = p(r,2m;n) + p(m+r,2m;n),$$

one may therefore obtain the generating functions for $p(r,m;n)$ with $m \in \{6,8,12\}$ with the assistance of Theorems 1.1 and 1.2.

In light of Theorems 1.1 and 1.2, we prove the following results which are refinements of (1.3)–(1.5).

Theorem 1.3. *Let $m \in \{4,6\}$ and i be an integer with $0 \le i \le m-1$. Then*

$$\lim_{n\to+\infty} \frac{p(2i,4m;n)}{p(n)} = \frac{1}{2m} \tag{1.22}$$

and

$$\lim_{n\to+\infty} \frac{p(4i,4m;2n) - p(4i+2,4m;2n)}{p(4i,4m;2n+1) - p(4i+2,4m;2n+1)} = 1+\sqrt{2}. \tag{1.23}$$

Also, for sufficiently large n,

$$p(4i,4m;n) > p(4i+2,4m;n), \text{ if } n \equiv 0,1 \pmod 4, \tag{1.24}$$

$$p(4i,4m;n) < p(4i+2,4m;n), \text{ if } n \equiv 2,3 \pmod 4. \tag{1.25}$$

2. Proof of Theorem 1.1

In this section, we always set $\zeta = e^{\pi \mathrm{i}/8}$. In order to prove Theorem 1.1, we first establish a lemma.

Lemma 2.1. *We have*

$$\prod_{k=0}^{\infty} \frac{1}{(1-\sqrt{2}q^{4k+2}+q^{8k+4})} = \frac{E(q^8)E(q^{32})^3}{E(q^4)E(q^{16})^2E(q^{64})} + \sqrt{2}q^2\frac{E(q^8)E(q^{64})}{E(q^4)E(q^{16})} \tag{2.1}$$

and

$$\prod_{k=0}^{\infty} \frac{1}{(1+\sqrt{2}q^{4k+2}+q^{8k+4})} = \frac{E(q^8)E(q^{32})^3}{E(q^4)E(q^{16})^2E(q^{64})} - \sqrt{2}q^2\frac{E(q^8)E(q^{64})}{E(q^4)E(q^{16})}. \tag{2.2}$$

Proof. Noticing that $\zeta^2=\frac{\sqrt{2}}{2}(1+\mathrm{i})$, we have

$$\prod_{k=0}^{\infty}\frac{1}{(1-\sqrt{2}q^{4k+2}+q^{8k+4})}=\prod_{k=0}^{\infty}\frac{(1+\sqrt{2}q^{4k+2}+q^{8k+4})}{(1+q^{16k+8})}$$
$$=\frac{E(q^8)E(q^{32})}{E(q^4)E(q^{16})^2}f(\zeta^2q^2,q^2/\zeta^2), \tag{2.3}$$

where

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

It follows from Entry 30 (ii) and (iii) on page 46 of Berndt's book [5] that

$$f(a,b)=f(a^3b,ab^3)+af(b/a,a^5b^3). \tag{2.4}$$

Taking $a=\zeta^2q^2$ and $b=q^2/\zeta^2$ in (2.4) yields

$$f(\zeta^2q^2,q^2/\zeta^2)=f(\zeta^4q^8,q^8/\zeta^4)+\zeta^2q^2f(\zeta^{-4},\zeta^4q^{16}). \tag{2.5}$$

By the fact that $\zeta^4=\mathrm{i}$, we have

$$f(\zeta^4q^8,q^8/\zeta^4)=(-\mathrm{i}\,q^8;q^{16})_\infty(\mathrm{i}\,q^8;q^{16})_\infty E(q^{16})=\frac{E(q^{32})^2}{E(q^{64})} \tag{2.6}$$

and

$$f(\zeta^{-4},\zeta^4q^{16})=(\mathrm{i};q^{16})_\infty(-\mathrm{i}\,q^{16};q^{16})_\infty E(q^{16})=(1-\mathrm{i})\frac{E(q^{16})E(q^{64})}{E(q^{32})}. \tag{2.7}$$

Based on (2.5)–(2.7) and the fact that $\zeta^2=\frac{\sqrt{2}}{2}(1+\mathrm{i})$, we arrive at

$$f(\zeta^2q^2,q^2/\zeta^2)=\frac{E(q^{32})^2}{E(q^{64})}+\sqrt{2}q^2\frac{E(q^{16})E(q^{64})}{E(q^{32})}. \tag{2.8}$$

Thanks to (2.3) and (2.8), we obtain (2.1). Also, replacing q by $\mathrm{i}\,q$ in (2.1) leads to (2.2). □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Employing (1.8), (1.9) and the fact that

$$\sum_{j=0}^{15}\zeta^{kj}=\begin{cases}16, & \text{if } k\equiv 0 \pmod{16},\\[2ex] 0, & \text{if } k\not\equiv 0 \pmod{16},\end{cases} \tag{2.9}$$

we have

$$\sum_{n=0}^{\infty}p(a,16;n)q^n=\frac{1}{16}\sum_{j=0}^{15}\zeta^{-aj}\sum_{n=0}^{\infty}\sum_{r=-\infty}^{\infty}p(r;n)\zeta^{jr}q^n$$
$$=\frac{1}{16}\frac{E(q^2)^2}{E(q)E(q^4)^2}\sum_{j=0}^{15}\zeta^{-aj}G(\zeta^j,q), \tag{2.10}$$

where

$$G(z,q)=\frac{1}{(z^2q^2;q^4)_\infty(q^2/z^2;q^4)_\infty}. \tag{2.11}$$

It is easy to check that for $k,j\geq 0$,

$$(1-\zeta^{2j}q^{4k+2})(1-q^{4k+2}/\zeta^{2j})=1-(\zeta^{2j}+\zeta^{-2j})q^{4k+2}+q^{8k+4}. \tag{2.12}$$

In light of (2.11) and (2.12),

$$G(\zeta^j,q)=\begin{cases}\dfrac{E(q^4)^2}{E(q^2)^2}, & \text{if } j\in\{0,8\},\\ \displaystyle\prod_{k=0}^{\infty}\frac{1}{(1-\sqrt{2}q^{4k+2}+q^{8k+4})}, & \text{if } j\in\{1,7,9,15\},\\ \dfrac{E(q^4)E(q^{16})}{E(q^8)^2}, & \text{if } j\in\{2,6,10,14\},\\ \displaystyle\prod_{k=0}^{\infty}\frac{1}{(1+\sqrt{2}q^{4k+2}+q^{8k+4})}, & \text{if } j\in\{3,5,11,13\},\\ \dfrac{E(q^2)^2E(q^8)^2}{E(q^4)^4}, & \text{if } j\in\{4,12\}.\end{cases} \tag{2.13}$$

Using (2.1), (2.2), (2.10) and (2.13), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}p(a,16;n)q^n=\frac{1}{16}\frac{E(q^2)^2}{E(q)E(q^4)^2}\Bigg\{&(1+(-1)^a)\frac{E(q^4)^2}{E(q^2)^2}\\ &+\left(\zeta^{-a}+\zeta^{-7a}+\zeta^{-9a}+\zeta^{-15a}\right)\left(\frac{E(q^8)E(q^{32})^3}{E(q^4)E(q^{16})^2E(q^{64})}+\sqrt{2}q^2\frac{E(q^8)E(q^{64})}{E(q^4)E(q^{16})}\right)\\ &+\left(\zeta^{-2a}+\zeta^{-6a}+\zeta^{-10a}+\zeta^{-14a}\right)\frac{E(q^4)E(q^{16})}{E(q^8)^2}\\ &+\left(\zeta^{-3a}+\zeta^{-5a}+\zeta^{-11a}+\zeta^{-13a}\right)\left(\frac{E(q^8)E(q^{32})^3}{E(q^4)E(q^{16})^2E(q^{64})}-\sqrt{2}q^2\frac{E(q^8)E(q^{64})}{E(q^4)E(q^{16})}\right)\\ &+(\mathrm{i}^a+(-\mathrm{i})^a)\frac{E(q^2)^2E(q^8)^2}{E(q^4)^4}\Bigg\}.\end{aligned} \tag{2.14}$$

Theorem 1.1 follows from (2.14) and the fact that $\zeta=\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}\mathrm{i}$. □

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Throughout our proof, we always write $\omega=e^{\pi\mathrm{i}/12}$. We first show the following lemma.

Lemma 3.1. *We have*

$$\prod_{k=0}^{\infty}\frac{1}{(1-\sqrt{3}q^{4k+2}+q^{8k+4})}=\frac{E(q^8)E(q^{12})E(q^{16})}{E(q^4)^2E(q^{24})}+\sqrt{3}q^2\frac{E(q^8)^2E(q^{12})E(q^{48})^2}{E(q^4)^2E(q^{16})E(q^{24})^2} \tag{3.1}$$

and

$$\prod_{k=0}^{\infty}\frac{1}{(1+\sqrt{3}q^{4k+2}+q^{8k+4})}=\frac{E(q^8)E(q^{12})E(q^{16})}{E(q^4)^2E(q^{24})}-\sqrt{3}q^2\frac{E(q^8)^2E(q^{12})E(q^{48})^2}{E(q^4)^2E(q^{16})E(q^{24})^2}. \tag{3.2}$$

Proof. Notice that $\omega^2=\frac{\sqrt{3}}{2}+\frac{1}{2}\,\mathrm{i}$. Therefore,

$$\prod_{k=0}^{\infty}\frac{1}{(1-\sqrt{3}q^{4k+2}+q^{8k+4})}=\frac{E(q^4)}{f(-\omega^2q^2,-q^2/\omega^2)} \tag{3.3}$$

where $f(a,b)$ is as defined in (2.4). It follows from the quintuple product identity [5, (38.2), p. 80] that

$$\frac{E(q^4)}{f(Bq^2,q^2/B)}=\frac{1}{f(-B^2,-q^4/B^2)}\left(f(B^3q^2,q^{10}/B^3)-B^2f(q^2/B^3,B^3q^{10})\right). \tag{3.4}$$

Setting $B=-\omega^2$ in (3.4), we deduce that

$$\frac{E(q^4)}{f(-\omega^2q^2,-q^2/\omega^2)}=\frac{1}{f(-\omega^4,-q^4/\omega^4)}\left(f(-\mathrm{i}\,q^2,-q^{10}/\mathrm{i})-\omega^4f(-q^2/\mathrm{i},-\mathrm{i}\,q^{10})\right). \tag{3.5}$$

By the fact that $\omega^4=\frac{1}{2}+\frac{\sqrt{3}}{2}\,\mathrm{i}$,

$$\begin{aligned}
f(-\omega^4,-q^4/\omega^4)&=E(q^4)(\omega^4;q^4)_\infty(q^4/\omega^4;q^4)_\infty\\
&=(1-\omega^4)E(q^4)\prod_{k=1}^{\infty}\left(1-\left(\omega^4+\frac{1}{\omega^4}\right)q^{4k}+q^{8k}\right)\\
&=(1-\omega^4)E(q^4)\prod_{k=1}^{\infty}\frac{1+q^{12k}}{1+q^{4k}}\\
&=(1-\omega^4)\frac{E(q^4)^2E(q^{24})}{E(q^8)E(q^{12})}.
\end{aligned}$$

Therefore,

$$\frac{1}{f(-\omega^4,-q^4/\omega^4)}=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,\mathrm{i}\right)\frac{E(q^8)E(q^{12})}{E(q^4)^2E(q^{24})}. \tag{3.6}$$

Taking $a=-\mathrm{i}\,q^2$ and $b=-q^{10}/\mathrm{i}$ in (2.4) yields

$$\begin{aligned}
f(-\mathrm{i}\,q^2,-q^{10}/\mathrm{i})&=f(-q^{16},-q^{32})-\mathrm{i}\,q^2f(-q^8,-q^{40})\\
&=E(q^{16})-\mathrm{i}\,q^2\frac{E(q^8)E(q^{48})^2}{E(q^{16})E(q^{24})}.
\end{aligned} \tag{3.7}$$

On the other hand, if we put $a=-q^2/\mathrm{i}$ and $b=-\mathrm{i}\,q^{10}$ in (2.4), then

$$\begin{aligned}
f(-q^2/\mathrm{i},-\mathrm{i}\,q^{10})&=f(-q^{16},-q^{32})-(q^2/\mathrm{i})f(-q^8,-q^{40})\\
&=E(q^{16})+\mathrm{i}\,q^2\frac{E(q^8)E(q^{48})^2}{E(q^{16})E(q^{24})}.
\end{aligned} \tag{3.8}$$

Finally, (3.1) follows from (3.3) and (3.5)–(3.8). Also, replacing q by $\mathrm{i}\,q$ in (3.1) yields (3.2). □

Now, we turn to prove Theorem 1.2.

Proof of Theorem 1.2. Utilizing (1.8), (1.9) and the fact that

$$\sum_{j=0}^{23}\omega^{kj}=\begin{cases}24, & \text{if } k\equiv 0 \pmod{24},\\ 0, & \text{if } k\not\equiv 0 \pmod{24},\end{cases}$$

we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}p(a,24;n)q^n&=\frac{1}{24}\sum_{j=0}^{23}\omega^{-aj}\sum_{n=0}^{\infty}\sum_{r=-\infty}^{\infty}p(r;n)\omega^{jr}q^n\\&=\frac{1}{24}\frac{E(q^2)^2}{E(q)E(q^4)^2}\sum_{j=0}^{23}\omega^{-aj}G(\omega^j,q),\end{aligned}\tag{3.9}$$

where $G(z,q)$ is defined in (2.11). In light of (2.11) and (2.12),

$$G(\omega^j,q)=\begin{cases}\dfrac{E(q^4)^2}{E(q^2)^2}, & \text{if } j\in\{0,12\},\\ \displaystyle\prod_{k=0}^{\infty}\frac{1}{(1-\sqrt{3}q^{4k+2}+q^{8k+4})}, & \text{if } j\in\{1,11,13,23\},\\ \dfrac{E(q^4)^2E(q^6)E(q^{24})}{E(q^2)E(q^8)E(q^{12})^2}, & \text{if } j\in\{2,10,14,22\},\\ \dfrac{E(q^4)E(q^{16})}{E(q^8)^2}, & \text{if } j\in\{3,9,15,21\},\\ \dfrac{E(q^2)E(q^{12})}{E(q^4)E(q^6)}, & \text{if } j\in\{4,8,16,20\},\\ \displaystyle\prod_{k=0}^{\infty}\frac{1}{(1+\sqrt{3}q^{4k+2}+q^{8k+4})}, & \text{if } j\in\{5,7,17,19\},\\ \dfrac{E(q^2)^2E(q^8)^2}{E(q^4)^4}, & \text{if } j\in\{6,18\}.\end{cases}\tag{3.10}$$

By (3.1), (3.2), (3.9) and (3.10),

$$\begin{aligned}\sum_{n=0}^{\infty}p(a,24;n)q^n=\frac{1}{24}\frac{E(q^2)^2}{E(q)E(q^4)^2}\Bigg\{&(1+(-1)^a)\frac{E(q^4)^2}{E(q^2)^2}\\&+\left(\omega^{-a}+\omega^{-11a}+\omega^{-13a}+\omega^{-23a}\right)\\&\quad\times\left(\frac{E(q^8)E(q^{12})E(q^{16})}{E(q^4)^2E(q^{24})}+\sqrt{3}q^2\frac{E(q^8)^2E(q^{12})E(q^{48})^2}{E(q^4)^2E(q^{16})E(q^{24})^2}\right)\\&+\left(\omega^{-2a}+\omega^{-10a}+\omega^{-14a}+\omega^{-22a}\right)\frac{E(q^4)^2E(q^6)E(q^{24})}{E(q^2)E(q^8)E(q^{12})^2}\\&+\left(\omega^{-3a}+\omega^{-9a}+\omega^{-15a}+\omega^{-21a}\right)\frac{E(q^4)E(q^{16})}{E(q^8)^2}\\&+\left(\omega^{-4a}+\omega^{-8a}+\omega^{-16a}+\omega^{-20a}\right)\frac{E(q^2)E(q^{12})}{E(q^4)E(q^6)}\end{aligned}$$

$$+\left(\omega^{-5a}+\omega^{-7a}+\omega^{-17a}+\omega^{-19a}\right)$$
$$\times\left(\frac{E(q^8)E(q^{12})E(q^{16})}{E(q^4)^2E(q^{24})}-\sqrt{3}q^2\frac{E(q^8)^2E(q^{12})E(q^{48})^2}{E(q^4)^2E(q^{16})E(q^{24})^2}\right)$$
$$\left.+\left(\mathrm{i}^a+(-\mathrm{i})^a\right)\frac{E(q^2)^2E(q^8)^2}{E(q^4)^4}\right\}. \tag{3.11}$$

Theorem 1.2 follows from (3.11) and the fact that $\omega=\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{6}-\sqrt{2}}{4}\mathrm{i}$. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 based on Theorems 1.1 and 1.2 along with a result due to Sussman [11].

In [11], applying the standard circle method due to Rademacher [8], Sussman obtained an exact series of $g(n)$ that is given by the generating function

$$\sum_{n\geq 0}g(n)q^n=\prod_{j=1}^{J}E(q^{m_j})^{\delta_j}, \tag{4.1}$$

where $\mathbf{m}=(m_1,\ldots,m_J)$ is a sequence of distinct positive integers and $\mathbf{d}=(\delta_1,\ldots,\delta_J)$ is a sequence of nonzero integers such that $\sum_{j=1}^{J}\delta_j<0$.

To state Sussman's result, we first fix some notation. Let k be a positive integer. We define

$$\Sigma:=-\frac{1}{2}\sum_{j=1}^{J}\delta_j,\qquad \Omega:=\sum_{j=1}^{J}\delta_j m_j,$$
$$\Delta(k):=-\sum_{j=1}^{J}\frac{\delta_j\gcd^2(m_j,k)}{m_j},\qquad \Pi(k):=\prod_{j=1}^{J}\left(\frac{m_j}{\gcd(m_j,k)}\right)^{-\frac{\delta_j}{2}}.$$

Further, for an integer h such that $\gcd(h,k)=1$, we define

$$\omega_{h,k}:=\exp\left(-\pi\mathrm{i}\sum_{j=1}^{J}\delta_j\cdot s\left(\frac{m_jh}{\gcd(m_j,k)},\frac{k}{\gcd(m_j,k)}\right)\right),$$

where $s(d,c)$ is the Dedekind sum defined by

$$s(d,c):=\sum_{n \bmod c}\left(\left(\frac{dn}{c}\right)\right)\left(\left(\frac{n}{c}\right)\right)$$

with

$$((x)):=\begin{cases}x-\lfloor x\rfloor-1/2 & \text{if } x\notin\mathbb{Z},\\ 0 & \text{if } x\in\mathbb{Z}.\end{cases}$$

Let $L=\operatorname{lcm}(m_1,\ldots,m_J)$. We divide the set $\{1,2,\ldots,L\}$ into two disjoint subsets:

$$\mathcal{L}_{>0}:=\{1\leq\ell\leq L\ :\ \Delta(\ell)>0\},$$
$$\mathcal{L}_{\leq 0}:=\{1\leq\ell\leq L\ :\ \Delta(\ell)\leq 0\}.$$

Theorem 4.1 (Sussman)**.** *If $\Sigma > 0$ and the inequality*

$$\min_{1\le j\le J}\left(\frac{\gcd^2(m_j,\ell)}{m_j}\right) \ge \frac{\Delta(\ell)}{24} \tag{4.2}$$

holds for all $1 \le \ell \le L$, then for positive integers $n > -\Omega/24$,

$$\begin{aligned} g(n) = 2\pi &\sum_{\ell\in\mathcal{L}_{>0}} \Pi(\ell)\left(\frac{24n+\Omega}{\Delta(\ell)}\right)^{-\frac{\Sigma+1}{2}} \\ &\times \sum_{\substack{k\ge 1\\ k\equiv \ell \bmod L}} \frac{1}{k} I_{\Sigma+1}\left(\frac{\pi}{6k}\sqrt{\Delta(\ell)(24n+\Omega)}\right) \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}}\omega_{h,k}, \end{aligned} \tag{4.3}$$

where $I_s(x)$ is the modified Bessel function of the first kind.

Remark 4.1. We also frequently make use of the asymptotic expansion of $I_s(x)$ (see [1, p. 377, (9.7.1)]): for fixed s, when $|\arg x| < \frac{\pi}{2}$,

$$I_s(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4s^2-1}{8x} + \frac{(4s^2-1)(4s^2-9)}{2!(8x)^2} - \cdots\right). \tag{4.4}$$

Remark 4.2. In [6], Chern considered the case where $\Sigma \le 0$ in (4.1) and obtained a similar asymptotic formula for $g(n)$.

Let us define, for $i = 1, \ldots, 5$,

$$\sum_{n=0}^{\infty} s_i(n)q^n = S_i(q),$$

where $S_i(q)$'s are as defined in Theorem 1.1. We first know from a famous result due to Hardy and Ramanujan [7] that, as $n \to \infty$,

$$s_1(n) = p(n) \sim \frac{1}{4\cdot 3^{1/2}\cdot n}\exp\left(2\pi\sqrt{\frac{n}{6}}\right). \tag{4.5}$$

We next show that, as $n \to \infty$,

$$s_2(n) \sim \frac{43^{1/2}}{2^5\cdot 3^{1/2}\cdot n}\exp\left(\frac{\pi}{4}\sqrt{\frac{43n}{6}}\right), \tag{4.6}$$

$$s_3(n) \sim \frac{7^{1/2}}{2^3\cdot 3^{1/2}\cdot n}\exp\left(\frac{\pi}{2}\sqrt{\frac{7n}{6}}\right), \tag{4.7}$$

$$s_4(n) \sim \frac{13^{1/2}}{2^3\cdot 3^{1/2}\cdot n}(-1)^n\cos\left(\frac{n\pi}{2}+\frac{\pi}{8}\right)\exp\left(\frac{\pi}{2}\sqrt{\frac{13n}{6}}\right), \tag{4.8}$$

$$s_5(n) \sim \frac{43^{1/2}}{2^{11/2}\cdot 3^{1/2}\cdot n}\exp\left(\frac{\pi}{4}\sqrt{\frac{43n}{6}}\right). \tag{4.9}$$

We only prove (4.6) and (4.8) as instances. The rest can be shown analogously by Sussman's result (4.3).

First, we show (4.6). In (4.1), let us put

$$\mathbf{m} = (1,2,4,8,16,32,64), \quad \mathbf{d} = (-1,2,-3,1,-2,3,-1).$$

Thus, we have $\Sigma = \frac{1}{2}$ and $\Omega = -1$. Also, $L = 64$. We compute that

$$\begin{aligned}\mathcal{L}_{>0} = \{&1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24,\\ &25, 27, 28, 29, 31, 33, 35, 36, 37, 39, 40, 41, 43, 44, 45,\\ &47, 48, 49, 51, 52, 53, 55, 56, 57, 59, 60, 61, 63, 64\}.\end{aligned}$$

We next verify that the assumption (4.2) is satisfied. Then, it can be computed that when $k = 1$, the I-Bessel term has the largest order, which is

$$I_{3/2}\left(\frac{\sqrt{43}\pi}{48}\sqrt{24n-1}\right).$$

Further, when $k = 1$, we have

$$\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}}\omega_{h,k} = 1.$$

It follows from (4.3), with (4.4) utilized, that

$$s_2(n) \sim \frac{43^{1/2}}{2^5\cdot 3^{1/2}\cdot n}\exp\left(\frac{\pi}{4}\sqrt{\frac{43n}{6}}\right).$$

For (4.8), we put

$$\mathbf{m} = (1, 2, 4, 8), \quad \mathbf{d} = (-1, 4, -6, 2)$$

in (4.1). Thus, $\Sigma = \frac{1}{2}$ and $\Omega = -1$. Further, $L = 8$. We compute that

$$\mathcal{L}_{>0} = \{1, 3, 4, 5, 7, 8\}.$$

We next verify that the assumption (4.2) is satisfied. Then, it can be computed that when $k = 4$, the I-Bessel term has the largest order, which is

$$I_{3/2}\left(\frac{\sqrt{13}\pi}{24}\sqrt{24n-1}\right).$$

Further, when $k = 4$, we have

$$\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}}\omega_{h,k} = 2(-1)^n\cos\left(\frac{n\pi}{2}+\frac{\pi}{8}\right).$$

It follows from (4.3), with (4.4) utilized, that

$$s_4(n) \sim \frac{13^{1/2}}{2^3\cdot 3^{1/2}\cdot n}(-1)^n\cos\left(\frac{n\pi}{2}+\frac{\pi}{8}\right)\exp\left(\frac{\pi}{2}\sqrt{\frac{13n}{6}}\right).$$

Notice that, for the exponential terms in (4.5)–(4.9), we have, numerically,

$$\begin{aligned}&2\pi\sqrt{\frac{1}{6}} = 2.56\cdots, \quad \frac{\pi}{4}\sqrt{\frac{43}{6}} = 2.10\cdots, \quad \frac{\pi}{2}\sqrt{\frac{7}{6}} = 1.69\cdots,\\ &\frac{\pi}{2}\sqrt{\frac{13}{6}} = 2.31\cdots, \quad \frac{\pi}{4}\sqrt{\frac{43}{6}} = 2.10\cdots.\end{aligned} \tag{4.10}$$

Recall that, for any integer i with $1 \leq i \leq 4$, we have $p(2i, 16, n) = p(16 - 2i, 16, n)$. We conclude from the numerical calculations in (4.10) that

$$p(2i, 16; n) \sim \frac{s_1(n)}{8} = \frac{p(n)}{8}$$

as $n \to \infty$ for any integer i with $0 \leq i \leq 7$, and therefore (1.22) follows when $m = 4$.

We also deduce from the numerical calculations in (4.10) that, for $0 \leq i < 4$,

$$p(4i, 16; n) - p(4i + 2, 16; n) \sim \frac{s_4(n)}{4}$$

as $n \to \infty$. We know from (4.8) that

$$s_4(n) \sim \begin{cases} \dfrac{13^{1/2}}{2^3 \cdot 3^{1/2} \cdot n} \cos\left(\dfrac{\pi}{8}\right) \exp\left(\dfrac{\pi}{2}\sqrt{\dfrac{13n}{6}}\right) & \text{if } n \equiv 0 \pmod 4, \\ \dfrac{13^{1/2}}{2^3 \cdot 3^{1/2} \cdot n} \sin\left(\dfrac{\pi}{8}\right) \exp\left(\dfrac{\pi}{2}\sqrt{\dfrac{13n}{6}}\right) & \text{if } n \equiv 1 \pmod 4, \\ -\dfrac{13^{1/2}}{2^3 \cdot 3^{1/2} \cdot n} \cos\left(\dfrac{\pi}{8}\right) \exp\left(\dfrac{\pi}{2}\sqrt{\dfrac{13n}{6}}\right) & \text{if } n \equiv 2 \pmod 4, \\ -\dfrac{13^{1/2}}{2^3 \cdot 3^{1/2} \cdot n} \sin\left(\dfrac{\pi}{8}\right) \exp\left(\dfrac{\pi}{2}\sqrt{\dfrac{13n}{6}}\right) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Hence, (1.24) and (1.25) hold when $m = 4$. Finally, since

$$\frac{\cos(\pi/8)}{\sin(\pi/8)} = 1 + \sqrt{2},$$

we see that (1.23) is true when $m = 4$.

Next, we prove Theorem 1.3 when $m = 6$. Let us define, for $i = 1, \ldots, 7$,

$$\sum_{n=0}^{\infty} f_i(n) q^n = F_i(q),$$

where $F_i(q)$'s are as defined in Theorem 1.2. Applying Sussman's result (4.3), we have, as $n \to \infty$,

$$f_1(n) = p(n) \sim \frac{1}{4 \cdot 3^{1/2} \cdot n} \exp\left(2\pi\sqrt{\frac{n}{6}}\right), \tag{4.11}$$

$$f_2(n) \sim \frac{37^{1/2}}{2^4 \cdot 3 \cdot n} \exp\left(\frac{\pi}{6}\sqrt{\frac{37n}{2}}\right), \tag{4.12}$$

$$f_3(n) \sim \frac{7^{1/2}}{2^3 \cdot 3^{1/2} \cdot n} \exp\left(\frac{\pi}{2}\sqrt{\frac{7n}{6}}\right), \tag{4.13}$$

$$f_4(n) \sim \frac{7^{1/2}}{2^2 \cdot 3 \cdot n} \exp\left(\frac{\pi}{3}\sqrt{\frac{7n}{2}}\right), \tag{4.14}$$

$$f_5(n) \sim \frac{19^{1/2}}{2^2 \cdot 3 \cdot n} (-1)^n \cos\left(\frac{n\pi}{2} + \frac{\pi}{8}\right) \exp\left(\frac{\pi}{6}\sqrt{\frac{19n}{2}}\right), \tag{4.15}$$

$$f_6(n) \sim \frac{13^{1/2}}{2^3 \cdot 3^{1/2} \cdot n}(-1)^n \cos\left(\frac{n\pi}{2}+\frac{\pi}{8}\right) \exp\left(\frac{\pi}{2}\sqrt{\frac{13n}{6}}\right), \tag{4.16}$$

$$f_7(n) \sim \frac{37^{1/2}}{2^4 \cdot 3^{3/2} \cdot n} \exp\left(\frac{\pi}{6}\sqrt{\frac{37n}{2}}\right). \tag{4.17}$$

Moreover, we notice that, for the exponential terms in (4.11)–(4.17), we have, numerically,

$$\begin{aligned}
&2\pi\sqrt{\frac{1}{6}} = 2.56\cdots, \quad \frac{\pi}{6}\sqrt{\frac{37}{2}} = 2.25\cdots, \quad \frac{\pi}{2}\sqrt{\frac{7}{6}} = 1.69\cdots,\\
&\frac{\pi}{3}\sqrt{\frac{7}{2}} = 1.95\cdots, \quad \frac{\pi}{6}\sqrt{\frac{19}{2}} = 1.61\cdots, \quad \frac{\pi}{2}\sqrt{\frac{13}{6}} = 2.31\cdots,\\
&\frac{\pi}{6}\sqrt{\frac{37}{2}} = 2.25\cdots.
\end{aligned} \tag{4.18}$$

Recall that, for any integer i with $1 \le i \le 6$, we have $p(2i, 24; n) = p(24 - 2i, 24; n)$. We conclude from the numerical calculations in (4.18) that

$$p(2i, 24; n) \sim \frac{f_1(n)}{12} = \frac{p(n)}{12}$$

as $n \to \infty$ for any integer i with $0 \le i \le 11$, and therefore (1.22) follows when $m = 6$. We also have, for $0 \le i < 6$,

$$p(4i, 24; n) - p(4i+2, 24; n) \sim \frac{f_6(n)}{6}$$

as $n \to \infty$. Hence, in (1.23)–(1.25), the case of $m = 6$ follows by arguments akin to those for the case of $m = 4$.

Therefore, the proof of Theorem 1.3 is completed.

5. Conclusion and conjectures

In this paper, we first establish the generating functions of $p(r, m; n)$ with $m = 16$ and 24 by making use of theta function identities and then prove some inequalities for $p(r, m; n)$ based on their generating functions and Sussman's asymptotic formulas for quotients of Dedekind eta functions. According to the work of Berkovich and Garvan [3], it would be appealing to seek for elementary proofs of Theorem 1.3 with the restriction of "n sufficiently large" removed.

Moreover, based on our numerical calculations, we present the following two conjectures.

Conjecture 5.1. For fixed integers $0 \le i < m$, there always exists a positive integer $N(m, i)$ such that for all $n \ge N(m, i)$,

$$p(4i, 4m; n) > p(4i+2, 4m; n), \text{ if } n \equiv 0, 1 \pmod 4, \tag{5.1}$$

$$p(4i, 4m; n) < p(4i+2, 4m; n), \text{ if } n \equiv 2, 3 \pmod 4, \tag{5.2}$$

$$|p(4i, 4m; 2n) - p(4i+2, 4m; 2n)| > |p(4i, m; 2n+1) - p(4i+2, m; 2n+1)|. \tag{5.3}$$

Conjecture 5.2. For $0 \leq k \leq m$,

$$\lim_{n\to+\infty} \frac{p(2k,4m;n)}{p(n)} = \frac{1}{2m} \tag{5.4}$$

and

$$\lim_{n\to+\infty} \frac{p(4k,4m;2n)-p(4k+2,4m;2n)}{p(4k,4m;2n+1)-p(4k+2,4m;2n+1)} = 1+\sqrt{2}. \tag{5.5}$$

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