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Linked partition ideals and Andrews–Gordon type series for Alladi and Gordon's extension of Schur's identity

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Abstract. Based on the framework of linked partition ideals, we derive some double and triple series of Andrews–Gordon type for partitions in Alladi and Gordon's extension of Schur's identity. We also display similar series for such partitions with additional restrictions on the smallest part. Also, an alternative proof of Alladi and Gordon's extension of Schur's identity is presented.

Keywords. Linked partition ideals, Andrews–Gordon type series, Schur's identity, generating function.

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1. Introduction

A partition of a nonnegative integer n is a weakly decreasing sequence of positive integers that sum to n. These positive integers are called parts of this partition. Given a partition λ, we denote by $|\lambda|$ the sum of all parts in λ, by $\sharp(\lambda)$ the number of parts in λ, and by $\sharp_{a,M}(\lambda)$ the number of parts in λ that are congruent to a modulo M.

One of the most fascinating identities in the theory of partitions is due to Rogers and Ramanujan [13, 14].

Theorem RR. (i). *The number of partitions of n into parts congruent to ± 1 modulo 5 is the same as the number of partitions of n such that every two consecutive parts have difference at least 2.*

(ii). *The number of partitions of n into parts congruent to ± 2 modulo 5 is the same as the number of partitions of n such that every two consecutive parts have difference at least 2 and that the smallest part is greater than 1.*

Here we witness two types of partitions — one concerns partitions whose parts satisfy certain congruence conditions and the other concentrates on partitions with restrictions on the differences of the parts. The next important result of this nature is Schur's 1926 identity [15].

Theorem S. *The number of partitions of n into distinct parts congruent to ± 1 modulo 3 is the same as the number of partitions of n such that every two consecutive parts have difference at least 3 and that no two consecutive multiples of 3 occur as parts.*

Schur's identity was subsequently extended by Gleißberg [10] to arbitrary moduli and then a more general result was discovered by Alladi and Gordon [1].

We know that the Rogers–Ramanujan identities also have analytic counterparts:

$$\prod_{n\geq 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}=\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}, \tag{1.1}$$

$$\prod_{n\geq 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}=\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}. \tag{1.2}$$

Here we adopt the q-Pochhammer symbol for $n\in\mathbb{N}\cup\{\infty\}$,

$$(A;q)_n:=\prod_{k=0}^{n-1}(1-Aq^k).$$

We sometimes say that the generating function for a partition set is a *series of Andrews–Gordon type* if it is of the form

$$\sum_{n_1,\ldots,n_r\geq 0}\frac{(-1)^{L_1(n_1,\ldots,n_r)}q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}}, \tag{1.3}$$

in which L_1 and L_2 are linear forms and Q is a quadratic form in $n_1,\ldots,n_r$. The name of this type of series comes from a multi-summation generalization of the Rogers–Ramanujan identities (1.1) and (1.2) due to Andrews [3]: for $1\leq i\leq k$ and $k\geq 2$,

$$\prod_{\substack{n\geq 1\\ n\not\equiv 0,\pm i \ (\mathrm{mod}\ 2k+1)}}\frac{1}{1-q^n}=\sum_{n_1,\ldots,n_{k-1}\geq 0}\frac{q^{N_1^2+N_2^2+\cdots N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} \tag{1.4}$$

where $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$. This identity is the analytic counterpart of Gordon's partition-theoretic extension of the Rogers–Ramanujan identities in [11].

It was also discovered that the generating function for the partition sets in Schur's identity may be represented as a double or triple series of Andrews–Gordon type. For instance, Andrews, Bringmann and Mahlburg [7] showed by constructing certain q-difference equations that

$$\sum_{\lambda}x^{\sharp(\lambda)}q^{|\lambda|}=\sum_{m,n\geq 0}\frac{(-1)^n q^{(m+3n)^2+\frac{m(m-1)}{2}}x^{m+2n}}{(q;q)_m(q^6;q^6)_n}$$

and Kurşungöz [12] proved by some combinatorial transformations that

$$\sum_{\lambda}x^{\sharp(\lambda)}q^{|\lambda|}$$
$$=\sum_{\substack{n_1\geq 0\\ n_{21},n_{22}\geq 0}}\frac{q^{2n_1^2-n_1+6n_{21}^2-n_{21}+6n_{22}^2+n_{22}+6n_1(n_{21}+n_{22})+12n_{21}n_{22}}x^{n_1+2n_{21}+2n_{22}}}{(q;q)_{n_1}(q^6;q^6)_{n_{21}}(q^6;q^6)_{n_{22}}}$$

where the sums of λ are over the partition set that is under the difference restrictions in Schur's identity. Furthermore, based on their creative technique of weighted words, Alladi and Gordon [1] derived a general Andrews–Gordon type triple series for the partition sets in their extension of Schur's identity.

In the recent work of the author and Li [9] and the author [8], a framework due to Andrews [2, 4, 5], called *linked partition ideals*, was revisited to study the Andrews–Gordon type series for certain partition sets. In particular, we focused on *span one linked partition ideals* and translated this framework into matrix equations and then applied either computer-assisted or q-hypergeometric approaches to solve such equations.

The first object of this paper is to fit Schur's identity and its extension into the framework of linked partition ideals. We will not only reproduce the triple series of Alladi and Gordon in [1] but also derive more triple series of Andrews–Gordon type for such partitions with further restrictions on the smallest part.

Theorem 1.1. *Let $0<\alpha<\beta<M$ be positive integers. We denote by $\mathscr{S}$ the set of partitions into parts congruent to α, β or $\alpha+\beta$ modulo M such that*

(i). *the difference between any two consecutive parts is $\ge M$,*
(ii). *the difference between two consecutive parts is $>M$ if any of them is congruent to $\alpha+\beta$ modulo M.*

We further denote by $\mathscr{S}_S$ the set of partitions in $\mathscr{S}$ whose smallest part is not in the set S of positive integers. Then

(a). *If $\alpha+\beta\le M$, we have*

$$\begin{aligned}\sum_{\lambda\in\mathscr{S}} & x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|}\\ &=\sum_{n_1,n_2,n_3\ge 0}\frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}}\\ &\quad\times q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+\alpha n_1+\beta n_2+(\alpha+\beta)n_3},\end{aligned}\tag{1.5}$$

$$\begin{aligned}\sum_{\lambda\in\mathscr{S}_{\{\alpha\}}} & x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|}\\ &=\sum_{n_1,n_2,n_3\ge 0}\frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}}\\ &\quad\times q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+(\alpha+M)n_1+\beta n_2+(\alpha+\beta)n_3},\end{aligned}\tag{1.6}$$

$$\begin{aligned}\sum_{\lambda\in\mathscr{S}_{\{\alpha,\beta,\alpha+\beta\}}} & x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|}\\ &=\sum_{n_1,n_2,n_3\ge 0}\frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}}\\ &\quad\times q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+(\alpha+M)n_1+(\beta+M)n_2+(\alpha+\beta+M)n_3}.\end{aligned}\tag{1.7}$$

(b). *If $\alpha+\beta>M$, we have*

$$\begin{aligned}\sum_{\lambda\in\mathscr{S}} & x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|}\\ &=\sum_{n_1,n_2,n_3\ge 0}\frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}}\\ &\quad\times q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+\alpha n_1+\beta n_2+(\alpha+\beta-M)n_3},\end{aligned}\tag{1.8}$$

$$\sum_{\lambda\in\mathscr{S}_{\{\alpha+\beta-M\}}} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|}$$

$$= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}} \times q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+\alpha n_1+\beta n_2+(\alpha+\beta)n_3}, \tag{1.9}$$

$$\sum_{\lambda\in\mathscr{S}_{\{\alpha,\alpha+\beta-M\}}} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}} \times q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+(\alpha+M)n_1+\beta n_2+(\alpha+\beta)n_3}. \tag{1.10}$$

We then turn our attention to a special case of Alladi and Gordon's extension. It will be shown that in this case, the generating functions may also be represented as double series of Andrews–Gordon type.

Theorem 1.2. *Let $\mathscr{S}^\star$ denote the partition set $\mathscr{S}$ in Theorem 1.1 with M even and $\beta=\alpha+\frac{M}{2}$. We also denote by $\mathscr{S}^\star_S$ the set of partitions in $\mathscr{S}^\star$ whose smallest part is not in the set S of positive integers. Then*

(a). *If $0<\alpha\leq\frac{M}{4}$, we have*

$$\sum_{\lambda\in\mathscr{S}^\star} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2\geq 0} \frac{x^{n_1+n_2}y^{n_2}q^{M\binom{n_1}{2}+2M\binom{n_2}{2}+Mn_1n_2+\alpha n_1+(2\alpha+\frac{M}{2})n_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{n_1}(q^M;q^M)_{n_2}}, \tag{1.11}$$

$$\sum_{\lambda\in\mathscr{S}^\star_{\{\alpha\}}} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2\geq 0} \frac{x^{n_1+n_2}y^{n_2}q^{M\binom{n_1}{2}+2M\binom{n_2}{2}+Mn_1n_2+(\alpha+\frac{M}{2})n_1+(2\alpha+\frac{M}{2})n_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{n_1}(q^M;q^M)_{n_2}}, \tag{1.12}$$

$$\sum_{\lambda\in\mathscr{S}^\star_{\{\alpha,\alpha+\frac{M}{2},2\alpha+\frac{M}{2}\}}} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2\geq 0} \frac{x^{n_1+n_2}y^{n_2}q^{M\binom{n_1}{2}+2M\binom{n_2}{2}+Mn_1n_2+(\alpha+M)n_1+(2\alpha+\frac{3M}{2})n_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{n_1}(q^M;q^M)_{n_2}}. \tag{1.13}$$

(b). *If $\frac{M}{4}<\alpha<\frac{M}{2}$, we have*

$$\sum_{\lambda\in\mathscr{S}^\star} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2\geq 0} \frac{x^{n_1+n_2}y^{n_2}q^{M\binom{n_1}{2}+2M\binom{n_2}{2}+Mn_1n_2+\alpha n_1+(2\alpha-\frac{M}{2})n_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{n_1}(q^M;q^M)_{n_2}}, \tag{1.14}$$

$$\sum_{\lambda\in\mathscr{S}^\star_{\{2\alpha-\frac{M}{2}\}}} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|} = \sum_{n_1,n_2\geq 0} \frac{x^{n_1+n_2}y^{n_2}q^{M\binom{n_1}{2}+2M\binom{n_2}{2}+Mn_1n_2+\alpha n_1+(2\alpha+\frac{M}{2})n_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{n_1}(q^M;q^M)_{n_2}}, \tag{1.15}$$

$$\sum_{\lambda\in\mathscr{S}^\star_{\{\alpha,2\alpha-\frac{M}{2}\}}} x^{\sharp(\lambda)}y^{\sharp_{\alpha+\beta,M}(\lambda)}q^{|\lambda|}$$
$$=\sum_{n_1,n_2\geq 0}\frac{x^{n_1+n_2}y^{n_2}q^{M\binom{n_1}{2}+2M\binom{n_2}{2}+Mn_1n_2+(\alpha+\frac{M}{2})n_1+(2\alpha+\frac{M}{2})n_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{n_1}(q^M;q^M)_{n_2}}. \tag{1.16}$$

Lastly, we investigate the partition sets that are restricted by the congruence conditions in Schur's identity and its extension with the assistance of linked partition ideals. Double and triple series of Andrews–Gordon type will be deduced for such partition sets.

Theorem 1.3. *Let $0<\alpha<\beta<M$ be positive integers. We denote by $\mathscr{C}$ the set of partitions into distinct parts congruent to α or β modulo M. Then*

$$\sum_{\lambda\in\mathscr{C}} x^{\sharp(\lambda)}q^{|\lambda|}=(-xq^\alpha;q^M)_\infty(-xq^\beta;q^M)_\infty$$
$$=\sum_{n_1,n_2,n_3\geq 0}\frac{x^{n_1+n_2+2n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}}$$
$$\times q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+\alpha n_1+\beta n_2+(\alpha+\beta)n_3}. \tag{1.17}$$

Further, if $\mathscr{C}^\star$ denotes the partition set $\mathscr{C}$ with M even and $\beta=\alpha+\frac{M}{2}$, then

$$\sum_{\lambda\in\mathscr{C}^\star} x^{\sharp(\lambda)}q^{|\lambda|}=(-xq^\alpha;q^M)_\infty(-xq^{\alpha+\frac{M}{2}};q^M)_\infty$$
$$=\sum_{n_1,n_2\geq 0}\frac{x^{n_1+2n_2}q^{M\binom{n_1}{2}+2M\binom{n_2}{2}+Mn_1n_2+\alpha n_1+(2\alpha+\frac{M}{2})n_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{n_1}(q^M;q^M)_{n_2}}. \tag{1.18}$$

Taking $y=x$ in (1.5) and (1.9), and comparing these triple series with (1.17), we are led to an alternative proof of Alladi and Gordon's extension of Schur's identity.

Corollary 1.4 (Alladi and Gordon (modified))**.** *Let $0<\alpha<\beta<M$ be positive integers. Let the partition sets $\mathscr{S}$ and $\mathscr{C}$ be as in Theorems 1.1 and 1.3, respectively.*

Let $A(n,k)$ denote the number of partitions of n into k parts in $\mathscr{C}$.

Let $B(n,k)$ denote the number of partitions of n into k parts (with parts congruent to $\alpha+\beta$ modulo M counted twice) in $\mathscr{S}$ if $\alpha+\beta\leq M$, and with an additional restriction that $\alpha+\beta-M$ is not the smallest part if $\alpha+\beta>M$.

Then
$$A(n,k)=B(n,k).$$

This paper is organized as follows. In the next section, we give a review of the framework of span one linked partition ideals and especially the idea in [8]. Then in Section 3, we study the partition set $\mathscr{S}$ and show Theorems 1.1 and 1.2. Next, Section 4 is devoted to an investigation of the partition set $\mathscr{C}$ and a proof of Theorem 1.3. Finally, we close this paper with some remarks.

2. Linked partition ideals and q-multi-summations

2.1. Span one linked partition ideals. In this section, we give a brief review of span one linked partition ideals. Details may be found in [6, Chapter 8] or [8,9].

First, let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_K\}$ be a finite set of integer partitions with $\pi_1 = \emptyset$, the empty partition. We also assign a *map of linking sets*, $\mathcal{L} : \Pi \to P(\Pi)$, the power set of Π, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\emptyset) = \Pi$ and $\pi_1 = \emptyset \in \mathcal{L}(\pi_k)$ for any $1 \le k \le K$. Let T, which we call the *modulus*, be a positive integer that is greater than or equal to the largest part among all partitions in Π.

We then consider an infinite chain of partitions in Π:

$$\lambda_0 \to \lambda_1 \to \cdots \to \lambda_N \to \pi_1 \to \pi_1 \to \cdots$$

ending with a series of empty partitions, such that $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i. This chain then uniquely determines an integer partition λ by

$$\lambda = \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \phi^{2T}(\lambda_2) \oplus \cdots \oplus \phi^{NT}(\lambda_N),$$

where $\mu \oplus \nu$ is the partition constructed by collecting all parts in partitions μ and ν, and $\phi^m(\mu)$ is the partition obtained by adding m to each part of μ.

Conversely, given such a partition λ, we may decompose it into blocks B_0, B_1, $\ldots$, B_N such that all parts between $iT+1$ and $(i+1)T$ fall into block B_i. Applying ϕ^{-iT} to block B_i then leads to the above λ_0, λ_1, $\ldots$, λ_N.

We collect all such partitions λ constructed as above and call this partition set a *span one linked partition ideal*, denoted by $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, T)$.

Now let us assume that $s(\lambda)$ is a statistic of $\lambda \in \mathscr{I}$ such that

$$s(\lambda) = s(\phi^T(\lambda))$$

and

$$s(\lambda) = s(\lambda_0) + s(\lambda_1) + \cdots + s(\lambda_N).$$

Notice that each π_k is in $\mathscr{I}$ since we may treat it as the partition constructed by $\pi_k \to \pi_1 \to \pi_1 \to \cdots$.

For each $1 \le k \le K$, we write

$$G_k(x) := \sum_{\substack{\lambda \in \mathscr{I} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y^{s(\lambda)} q^{|\lambda|}.$$

Then

$$\begin{aligned} G_k(x) &= x^{\sharp(\pi_k)} y^{s(\pi_k)} q^{|\pi_k|} \sum_{j : \pi_j \in \mathcal{L}(\pi_k)} \sum_{\substack{\lambda' \in \mathscr{I} \\ \lambda'_0 = \pi_j}} (xq^T)^{\sharp(\lambda')} y^{s(\lambda')} q^{|\lambda'|} \\ &= x^{\sharp(\pi_k)} y^{s(\pi_k)} q^{|\pi_k|} \sum_{j : \pi_j \in \mathcal{L}(\pi_k)} G_j(xq^T). \end{aligned}$$

Therefore,

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_K(x) \end{pmatrix} = \mathscr{W}(x).\mathscr{A}. \begin{pmatrix} G_1(xq^T) \\ G_2(xq^T) \\ \vdots \\ G_K(xq^T) \end{pmatrix} \tag{2.1}$$

where the diagonal matrix $\mathscr{W}(x)$ is given by

$$\operatorname{diag}(x^{\sharp(\pi_1)} y^{s(\pi_1)} q^{|\pi_1|}, \ldots, x^{\sharp(\pi_K)} y^{s(\pi_K)} q^{|\pi_K|})$$

and the zero-one matrix $\mathscr{A}$ is given by

$$\mathscr{A}_{i,j} = \begin{cases} 1 & \text{if } \pi_j \in \mathcal{L}(\pi_i), \\ 0 & \text{if } \pi_j \notin \mathcal{L}(\pi_i). \end{cases}$$

Let us write

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_K(x) \end{pmatrix} = \mathscr{A} . \begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_K(x) \end{pmatrix}. \tag{2.2}$$

Then

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_K(x) \end{pmatrix} = \mathscr{A} . \mathscr{W}(x) . \mathscr{A} . \begin{pmatrix} G_1(xq^T) \\ G_2(xq^T) \\ \vdots \\ G_K(xq^T) \end{pmatrix},$$

and therefore,

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_K(x) \end{pmatrix} = \mathscr{A} . \mathscr{W}(x) . \begin{pmatrix} F_1(xq^T) \\ F_2(xq^T) \\ \vdots \\ F_K(xq^T) \end{pmatrix}. \tag{2.3}$$

Notice that we have $G_k(0) = 1$ if $k = 1$, and 0 otherwise. This is because the only non-vanishing term in $G_k(0)$ comes from the empty partition, which is exclusively counted by $G_1(x)$. Also, all entries in the first column of $\mathscr{A}$ are 1 since $\pi_1 \in \mathcal{L}(\pi_k)$ for all k. Hence, $F_1(0) = \cdots = F_K(0) = 1$.

Recall that $F_k(x) \in \mathbb{C}[[q]][[x]]$. If we treat (2.3) as a matrix equation over $\mathbb{C}[[q]][[x]]$, then [8, Proposition 15] ensures the uniqueness of $F_k(x)$.

2.2. A key recurrence. The second half of [8] focuses on matrix equations (2.3) whose solution is a special series of Andrews–Gordon type. A crucial ingredient is a recurrence relation given in [8, Theorem 18]. Our object here is to give a refinement of that recurrence.

Let R be a fixed positive integer. We fix a symmetric matrix $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathrm{Mat}_{R\times R}(\mathbb{N})$ and a vector $\underline{\mathbf{A}} = (A_r) \in \mathbb{N}_{>0}^R$. We also fix J vectors $\underline{\boldsymbol{\gamma_j}} = (\gamma_{j,r}) \in \mathbb{N}_{\geq 0}^R$ for $j = 1, 2, \ldots, J$. Let $x_1, x_2, \ldots, x_J$ and q be indeterminates such that the following q-multi-summation $H(\underline{\boldsymbol{\beta}}) = H(\beta_1, \ldots, \beta_R)$ for $\underline{\boldsymbol{\beta}} \in \mathbb{Z}^R$ converges:

$$\begin{aligned} H(\underline{\boldsymbol{\beta}}) := \sum_{n_1,\ldots,n_R \geq 0} & \frac{x_1^{\sum_{r=1}^R \gamma_{1,r} n_r} \cdots x_J^{\sum_{r=1}^R \gamma_{J,r} n_r}}{(q^{A_1}; q^{A_1})_{n_1} \cdots (q^{A_R}; q^{A_R})_{n_R}} \\ & \times q^{\sum_{r=1}^R \alpha_{r,r}\binom{n_r}{2} + \sum_{1\leq i<j\leq R} \alpha_{i,j} n_i n_j + \sum_{r=1}^R \beta_r n_r}. \end{aligned} \tag{2.4}$$

Lemma 2.1. *For $1 \leq r \leq R$, we have*

$$\begin{aligned} H(\beta_1, \ldots, \beta_r, \ldots, \beta_R) &= H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R) \\ &\quad + x_1^{\gamma_{1,r}} \cdots x_J^{\gamma_{J,r}} q^{\beta_r} H(\beta_1 + \alpha_{r,1}, \ldots, \beta_r + \alpha_{r,r}, \ldots, \beta_R + \alpha_{r,R}). \end{aligned} \tag{2.5}$$

Proof. We have (recall that $\underline{\boldsymbol{\alpha}}$ is symmetric so that $\alpha_{i,j}=\alpha_{j,i}$ for $1\le i,j\le R$)

$$\begin{aligned}
&H(\beta_1,\ldots,\beta_r,\ldots,\beta_R)-H(\beta_1,\ldots,\beta_r+A_r,\ldots,\beta_R)\\
&\quad=\sum_{n_1,\ldots,n_R\ge 0}\frac{q^{\sum_i \alpha_{i,i}n_i(n_i-1)/2}q^{\sum_{i<j}\alpha_{i,j}n_in_j}q^{\sum_i\beta_in_i}(1-q^{n_rA_r})}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}\cdots(q^{A_R};q^{A_R})_{n_R}}\\
&\qquad\times x_1^{\sum_i\gamma_{1,i}n_i}\cdots x_J^{\sum_i\gamma_{J,i}n_i}\\
&\quad=\sum_{\substack{n_1,\ldots,n_R\ge 0\\ n_r\ge 1}}\frac{q^{\sum_i \alpha_{i,i}n_i(n_i-1)/2}q^{\sum_{i<j}\alpha_{i,j}n_in_j}q^{\sum_i\beta_in_i}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r-1}\cdots(q^{A_R};q^{A_R})_{n_R}}\\
&\qquad\times x_1^{\sum_i\gamma_{1,i}n_i}\cdots x_J^{\sum_i\gamma_{J,i}n_i}\\
&\quad=x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r}\sum_{n_1,\ldots,n_R\ge 0}\frac{q^{\sum_i \alpha_{i,i}n_i(n_i-1)/2}q^{\sum_{i<j}\alpha_{i,j}n_in_j}q^{\sum_i(\beta_i+\alpha_{r,i})n_i}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}\cdots(q^{A_R};q^{A_R})_{n_R}}\\
&\qquad\times x_1^{\sum_i\gamma_{1,i}n_i}\cdots x_J^{\sum_i\gamma_{J,i}n_i}\\
&\quad=x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r}H(\beta_1+\alpha_{r,1},\ldots,\beta_r+\alpha_{r,r},\ldots,\beta_R+\alpha_{r,R}).
\end{aligned}$$

The desired identity therefore follows. $\square$

As remarked in [8], the recurrence relation (2.5) can be illustrated by a binary tree as shown in Figure 1. (Notice that in such a binary tree, if (2.5) is applied to the coordinate β_r, then it will be displayed in boldface.)

FIGURE 1. Node $H(\beta_1,\ldots,\beta_r,\ldots,\beta_R)$ and its children

$$\begin{array}{ccc}
 & H(\beta_1,\ldots,\boldsymbol{\beta_r},\ldots,\beta_R) & \\
\swarrow^{1} & & \searrow^{x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r}}\\
H(\beta_1,\ldots,\beta_r+A_r,\ldots,\beta_R) & & H(\beta_1+\alpha_{r,1},\ldots,\beta_r+\alpha_{r,r},\ldots,\beta_R+\alpha_{r,R})
\end{array}$$

3. Partition set $\mathscr{S}$

3.1. Triple series.

3.1.1. *Case of $\alpha+\beta\le M$.* We first claim that when $\alpha+\beta\le M$, the partition set $\mathscr{S}$ is a span one linked partition ideal $\mathscr{I}(\langle\Pi_{\mathscr{S}_1},\mathcal{L}_{\mathscr{S}_1}\rangle,M)$ with $\Pi_{\mathscr{S}_1}=\{\pi_1=\emptyset,\pi_2=(\alpha),\pi_3=(\beta),\pi_4=(\alpha+\beta)\}$ and

$$\begin{cases}
\mathcal{L}_{\mathscr{S}_1}(\pi_1)=\{\pi_1,\pi_2,\pi_3,\pi_4\},\\
\mathcal{L}_{\mathscr{S}_1}(\pi_2)=\{\pi_1,\pi_2,\pi_3,\pi_4\},\\
\mathcal{L}_{\mathscr{S}_1}(\pi_3)=\{\pi_1,\pi_3,\pi_4\},\\
\mathcal{L}_{\mathscr{S}_1}(\pi_4)=\{\pi_1\}.
\end{cases}$$

To see this, we decompose each partition in $\mathscr{S}$ into blocks $B_0,B_1,\ldots$ such that all parts between $iM+1$ and $(i+1)M$ fall into block B_i. Notice that there is at most one part in each block B_i. Otherwise, if two parts fall into the same block, then their difference is smaller than M. Further, applying ϕ^{-iM} to block B_i gives a partition in $\Pi_{\mathscr{S}_1}$. Next, if the resulting partition is $\pi_1=\emptyset$ or $\pi_2=(\alpha)$, then the resulting partition of block B_{i+1} may be any among $\Pi_{\mathscr{S}_1}$. If the resulting partition is $\pi_3=(\beta)$, then the resulting partition of block B_{i+1} cannot be $\pi_2=(\alpha)$. Otherwise, there are two parts $\beta+iM$ and $\alpha+(i+1)M$ in the original partition and their difference is $\alpha-\beta+M<M$. Lastly, if the resulting partition is $\pi_4=(\alpha+\beta)$, then the resulting partition of block B_{i+1} cannot be $\pi_2=(\alpha)$, $\pi_3=(\beta)$ or $\pi_4=(\alpha+\beta)$. Otherwise, the difference of parts in the original partition coming from blocks B_i and B_{i+1} is at most $(\alpha+\beta+(i+1)M)-(\alpha+\beta+iM)=M$, which violates the second condition in the definition of $\mathscr{S}$. Conversely, it is straightforward to verify that all partitions in $\mathscr{I}(\langle\Pi_{\mathscr{S}_1},\mathcal{L}_{\mathscr{S}_1}\rangle,M)$ are in $\mathscr{S}$. Thus, $\mathscr{S}=\mathscr{I}(\langle\Pi_{\mathscr{S}_1},\mathcal{L}_{\mathscr{S}_1}\rangle,M)$.

Let $s(\lambda)$ count the number of parts in λ that are congruent to $\alpha+\beta$ modulo M. Then (2.3) becomes

$$\begin{pmatrix}F_1(x)\\F_2(x)\\F_3(x)\\F_4(x)\end{pmatrix}=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&0&1&1\\1&0&0&0\end{pmatrix}\cdot\begin{pmatrix}1&&&\\&xq^{\alpha}&&\\&&xq^{\beta}&\\&&&xyq^{\alpha+\beta}\end{pmatrix}\cdot\begin{pmatrix}F_1(xq^M)\\F_2(xq^M)\\F_3(xq^M)\\F_4(xq^M)\end{pmatrix}. \tag{3.1}$$

We choose $\underline{\boldsymbol{\alpha}}=\begin{pmatrix}M&M&M\\M&M&M\\M&M&2M\end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}}=(1,1,1)$, $\underline{\boldsymbol{\gamma_2}}=(0,0,1)$ and $\underline{\mathbf{A}}=(M,M,M)$ in (2.4) and write $x_1=x$ and $x_2=y$. To distinguish with the double series in Theorem 1.2, we use H_T for H.

We want to show that

$$\begin{aligned}F_1(x)=F_2(x)&=H_T(\alpha,\beta,\alpha+\beta),\\F_3(x)&=H_T(\alpha+M,\beta,\alpha+\beta),\\F_4(x)&=H_T(\alpha+M,\beta+M,\alpha+\beta+M).\end{aligned}$$

It suffices to prove that

$$\begin{aligned}H_T(\alpha,\beta,\alpha+\beta)&=H_T(\alpha+M,\beta+M,\alpha+\beta+M)\\&\quad+xq^{\alpha}H_T(\alpha+M,\beta+M,\alpha+\beta+M)\\&\quad+xq^{\beta}H_T(\alpha+2M,\beta+M,\alpha+\beta+M)\\&\quad+xyq^{\alpha+\beta}H_T(\alpha+2M,\beta+2M,\alpha+\beta+2M)\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}H_T(\alpha+M,\beta,\alpha+\beta)&=H_T(\alpha+M,\beta+M,\alpha+\beta+M)\\&\quad+xq^{\beta}H_T(\alpha+2M,\beta+M,\alpha+\beta+M)\\&\quad+xyq^{\alpha+\beta}H_T(\alpha+2M,\beta+2M,\alpha+\beta+2M).\end{aligned}\tag{3.3}$$

We make use of Lemma 2.1 repeatedly. First (notice again that the coordinate will be displayed in boldface if (2.5) is applied to it),

$$\begin{aligned}H_T(\alpha+M,\boldsymbol{\beta},\alpha+\beta)&=H_T(\alpha+M,\beta+M,\boldsymbol{\alpha+\beta})\\&\quad+xq^{\beta}H_T(\alpha+2M,\beta+M,\alpha+\beta+M)\\&=H_T(\alpha+M,\beta+M,\alpha+\beta+M)\\&\quad+xyq^{\alpha+\beta}H_T(\alpha+2M,\beta+2M,\alpha+\beta+2M)\end{aligned}$$

$$+ xq^{\beta} H_T(\alpha + 2M, \beta + M, \alpha + \beta + M).$$

This is (3.3). To see (3.2), we further notice that

$$\begin{aligned} H_T(\boldsymbol{\alpha}, \beta, \alpha + \beta) &= H_T(\alpha + M, \beta, \alpha + \beta) \\ &\quad + xq^{\alpha} H_T(\alpha + M, \beta + M, \alpha + \beta + M) \end{aligned}$$

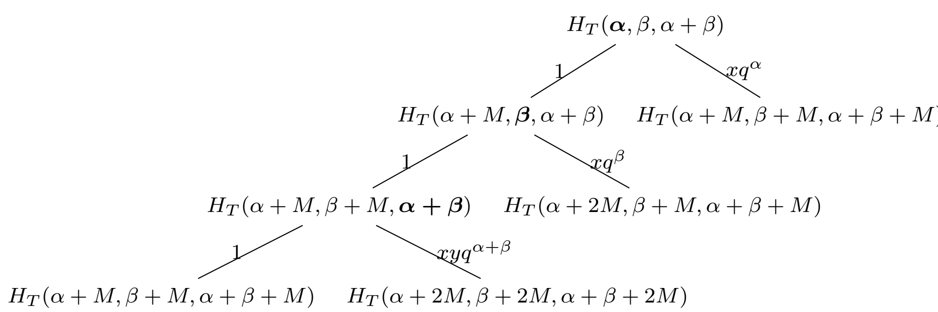
and then use (3.3). We remark that the above proof can be represented by the binary tree in Figure 2.

Finally, by (2.2), we conclude that

$$\begin{aligned} \sum_{\lambda \in \mathscr{S}} x^{\sharp(\lambda)} y^{\sharp_{\alpha+\beta,M}(\lambda)} q^{|\lambda|} &= G_1(x) + G_2(x) + G_3(x) + G_4(x) = F_1(x), \\ \sum_{\lambda \in \mathscr{S}_{\{\alpha\}}} x^{\sharp(\lambda)} y^{\sharp_{\alpha+\beta,M}(\lambda)} q^{|\lambda|} &= G_1(x) + G_3(x) + G_4(x) = F_3(x), \\ \sum_{\lambda \in \mathscr{S}_{\{\alpha,\beta,\alpha+\beta\}}} x^{\sharp(\lambda)} y^{\sharp_{\alpha+\beta,M}(\lambda)} q^{|\lambda|} &= G_1(x) = F_4(x). \end{aligned}$$

The first part of Theorem 1.1 therefore follows.

FIGURE 2. The binary tree for Section 3.1.1



3.1.2. *Case of* $\alpha + \beta > M$. It is also easy to verify that when $\alpha + \beta > M$, the partition set $\mathscr{S}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi_{\mathscr{S}_{\mathrm{II}}}, \mathcal{L}_{\mathscr{S}_{\mathrm{II}}} \rangle, M)$ with $\Pi_{\mathscr{S}_{\mathrm{II}}} = \{\pi_1 = \emptyset, \pi_2 = (\alpha), \pi_3 = (\beta), \pi_4 = (\alpha + \beta - M)\}$ and

$$\begin{cases} \mathcal{L}_{\mathscr{S}_{\mathrm{II}}}(\pi_1) = \{\pi_1, \pi_2, \pi_3, \pi_4\}, \\ \mathcal{L}_{\mathscr{S}_{\mathrm{II}}}(\pi_2) = \{\pi_1, \pi_2, \pi_3\}, \\ \mathcal{L}_{\mathscr{S}_{\mathrm{II}}}(\pi_3) = \{\pi_1, \pi_3\}, \\ \mathcal{L}_{\mathscr{S}_{\mathrm{II}}}(\pi_4) = \{\pi_1, \pi_2, \pi_3\}. \end{cases}$$

Let $s(\lambda)$ count the number of parts in λ that are congruent to $\alpha + \beta$ modulo M. Then (2.3) becomes

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq^{\alpha} & & \\ & & xq^{\beta} & \\ & & & xyq^{\alpha+\beta-M} \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^M) \\ F_2(xq^M) \\ F_3(xq^M) \\ F_4(xq^M) \end{pmatrix}. \tag{3.4}$$

We still choose $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} M & M & M \\ M & M & M \\ M & M & 2M \end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}} = (1,1,1)$, $\underline{\boldsymbol{\gamma_2}} = (0,0,1)$ and $\underline{\mathbf{A}} = (M,M,M)$ in (2.4) and write $x_1 = x$ and $x_2 = y$. We will also use H_T for H.

We want to show that

$$\begin{aligned} F_1(x) &= H_T(\alpha,\beta,\alpha+\beta-M), \\ F_2(x) = F_4(x) &= H_T(\alpha,\beta,\alpha+\beta), \\ F_3(x) &= H_T(\alpha+M,\beta,\alpha+\beta). \end{aligned}$$

It suffices to prove that

$$\begin{aligned} H_T(\alpha,\beta,\alpha+\beta-M) &= H_T(\alpha+M,\beta+M,\alpha+\beta) \\ &\quad + xq^{\alpha} H_T(\alpha+M,\beta+M,\alpha+\beta+M) \\ &\quad + xq^{\beta} H_T(\alpha+2M,\beta+M,\alpha+\beta+M) \\ &\quad + xyq^{\alpha+\beta-M} H_T(\alpha+M,\beta+M,\alpha+\beta+M), \\ H_T(\alpha,\beta,\alpha+\beta) &= H_T(\alpha+M,\beta+M,\alpha+\beta) \\ &\quad + xq^{\alpha} H_T(\alpha+M,\beta+M,\alpha+\beta+M) \\ &\quad + xq^{\beta} H_T(\alpha+2M,\beta+M,\alpha+\beta+M), \\ H_T(\alpha+M,\beta,\alpha+\beta) &= H_T(\alpha+M,\beta+M,\alpha+\beta) \\ &\quad + xq^{\beta} H_T(\alpha+2M,\beta+M,\alpha+\beta+M). \end{aligned}$$

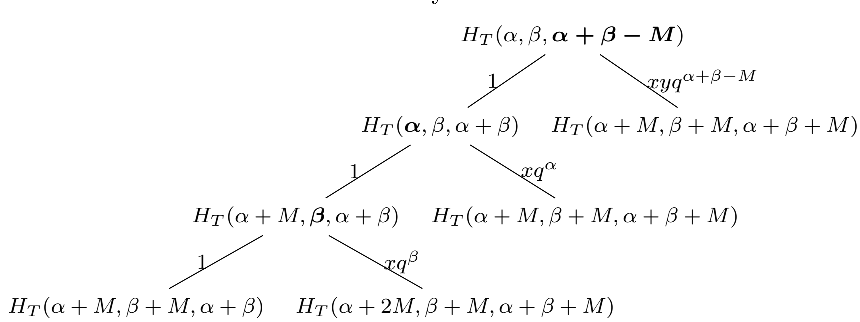
These can be shown by the binary tree in Figure 3.

Finally, by (2.2), we conclude that

$$\begin{aligned} \sum_{\lambda\in\mathscr{S}} x^{\sharp(\lambda)} y^{\sharp_{\alpha+\beta,M}(\lambda)} q^{|\lambda|} &= G_1(x)+G_2(x)+G_3(x)+G_4(x) = F_1(x), \\ \sum_{\lambda\in\mathscr{S}_{\{\alpha+\beta-M\}}} x^{\sharp(\lambda)} y^{\sharp_{\alpha+\beta,M}(\lambda)} q^{|\lambda|} &= G_1(x)+G_2(x)+G_3(x) = F_2(x), \\ \sum_{\lambda\in\mathscr{S}_{\{\alpha,\alpha+\beta-M\}}} x^{\sharp(\lambda)} y^{\sharp_{\alpha+\beta,M}(\lambda)} q^{|\lambda|} &= G_1(x)+G_3(x) = F_3(x). \end{aligned}$$

The second part of Theorem 1.1 therefore follows.

Figure 3. The binary tree for Section 3.1.2



3.2. Double series. Recall that the partition set $\mathscr{S}^\star$ is the M even and $\beta = \alpha + \frac{M}{2}$ case of $\mathscr{S}$.

3.2.1. *Case of* $0 < \alpha \leq \frac{M}{4}$. In this case, we have $\alpha + \beta = 2\alpha + \frac{M}{2} \leq M$. Taking $\beta = \alpha + \frac{M}{2}$ in (3.1) yields

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq^{\alpha} & & \\ & & xq^{\alpha+\frac{M}{2}} & \\ & & & xyq^{2\alpha+\frac{M}{2}} \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^M) \\ F_2(xq^M) \\ F_3(xq^M) \\ F_4(xq^M) \end{pmatrix}.$$

We choose $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} M & M \\ M & 2M \end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}} = (1,1)$, $\underline{\boldsymbol{\gamma_2}} = (0,1)$ and $\underline{\mathbf{A}} = (\frac{M}{2}, M)$ in (2.4) and write $x_1 = x$ and $x_2 = y$. Now we use H_D for H.

Then the first part of Theorem 1.2 is equivalent to

$$\begin{aligned} F_1(x) = F_2(x) &= H_D(\alpha, 2\alpha + \tfrac{M}{2}), \\ F_3(x) &= H_D(\alpha + \tfrac{M}{2}, 2\alpha + \tfrac{M}{2}), \\ F_4(x) &= H_D(\alpha + M, 2\alpha + \tfrac{3M}{2}). \end{aligned}$$

Namely,

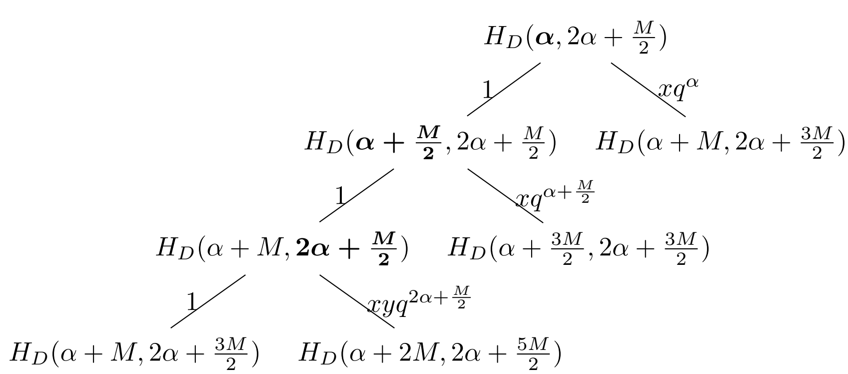
$$\begin{aligned} H_D(\alpha, 2\alpha + \tfrac{M}{2}) = H_D&(\alpha + M, 2\alpha + \tfrac{3M}{2}) \\ &+ xq^{\alpha} H_D(\alpha + M, 2\alpha + \tfrac{3M}{2}) \\ &+ xq^{\alpha+\frac{M}{2}} H_D(\alpha + \tfrac{3M}{2}, 2\alpha + \tfrac{3M}{2}) \\ &+ xyq^{2\alpha+\frac{M}{2}} H_D(\alpha + 2M, 2\alpha + \tfrac{5M}{2}) \end{aligned}$$

and

$$\begin{aligned} H_D(\alpha + \tfrac{M}{2}, 2\alpha + \tfrac{M}{2}) = H_D&(\alpha + M, 2\alpha + \tfrac{3M}{2}) \\ &+ xq^{\alpha+\frac{M}{2}} H_D(\alpha + \tfrac{3M}{2}, 2\alpha + \tfrac{3M}{2}) \\ &+ xyq^{2\alpha+\frac{M}{2}} H_D(\alpha + 2M, 2\alpha + \tfrac{5M}{2}). \end{aligned}$$

These can be shown by the binary tree in Figure 4.

FIGURE 4. The binary tree for Section 3.2.1



3.2.2. *Case of* $\frac{M}{4} < \alpha < \frac{M}{2}$. In this case, we have $\alpha+\beta = 2\alpha + \frac{M}{2} > M$. Taking $\beta = \alpha + \frac{M}{2}$ in (3.4) yields

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq^{\alpha} & & \\ & & xq^{\alpha+\frac{M}{2}} & \\ & & & xyq^{2\alpha-\frac{M}{2}} \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^M) \\ F_2(xq^M) \\ F_3(xq^M) \\ F_4(xq^M) \end{pmatrix}.$$

We still choose $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} M & M \\ M & 2M \end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}} = (1,1)$, $\underline{\boldsymbol{\gamma_2}} = (0,1)$ and $\underline{\mathbf{A}} = (\frac{M}{2}, M)$ in (2.4) and write $x_1 = x$ and $x_2 = y$. We also use H_D for H.

Then the second part of Theorem 1.2 is equivalent to

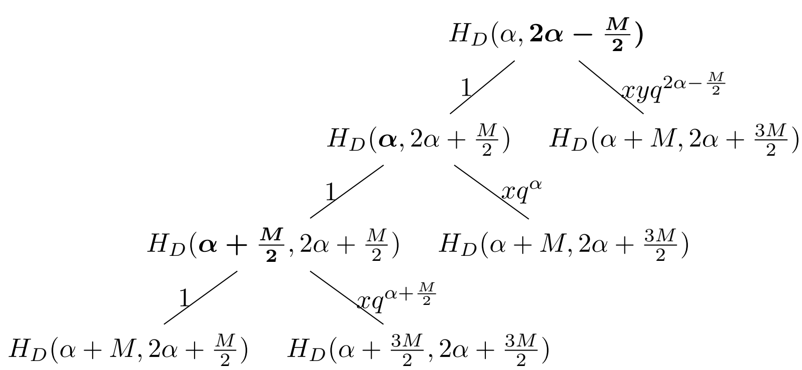
$$\begin{aligned} F_1(x) &= H_D(\alpha, 2\alpha - \tfrac{M}{2}), \\ F_2(x) = F_4(x) &= H_D(\alpha, 2\alpha + \tfrac{M}{2}), \\ F_3(x) &= H_D(\alpha + \tfrac{M}{2}, 2\alpha + \tfrac{M}{2}). \end{aligned}$$

Namely,

$$\begin{aligned} H_D(\alpha, 2\alpha - \tfrac{M}{2}) &= H_D(\alpha + M, 2\alpha + \tfrac{M}{2}) \\ &\quad + xq^{\alpha} H_D(\alpha + M, 2\alpha + \tfrac{3M}{2}) \\ &\quad + xq^{\alpha+\frac{M}{2}} H_D(\alpha + \tfrac{3M}{2}, 2\alpha + \tfrac{3M}{2}) \\ &\quad + xyq^{2\alpha-\frac{M}{2}} H_D(\alpha + M, 2\alpha + \tfrac{3M}{2}), \\ H_D(\alpha, 2\alpha + \tfrac{M}{2}) &= H_D(\alpha + M, 2\alpha + \tfrac{M}{2}) \\ &\quad + xq^{\alpha} H_D(\alpha + M, 2\alpha + \tfrac{3M}{2}) \\ &\quad + xq^{\alpha+\frac{M}{2}} H_D(\alpha + \tfrac{3M}{2}, 2\alpha + \tfrac{3M}{2}), \\ H_D(\alpha + \tfrac{M}{2}, 2\alpha + \tfrac{M}{2}) &= H_D(\alpha + M, 2\alpha + \tfrac{M}{2}) \\ &\quad + xq^{\alpha+\frac{M}{2}} H_D(\alpha + \tfrac{3M}{2}, 2\alpha + \tfrac{3M}{2}). \end{aligned}$$

These can be shown by the binary tree in Figure 5.

Figure 5. The binary tree for Section 3.2.2



4. Partition set $\mathscr{C}$

It is straightforward to verify that the partition set $\mathscr{C}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi_{\mathscr{C}}, \mathcal{L}_{\mathscr{C}}\rangle, M)$ with $\Pi_{\mathscr{C}} = \{\pi_1 = \emptyset, \pi_2 = (\alpha), \pi_3 = (\beta), \pi_4 = (\alpha, \beta)\}$ and

$$\mathcal{L}_{\mathscr{C}}(\pi_1) = \mathcal{L}_{\mathscr{C}}(\pi_2) = \mathcal{L}_{\mathscr{C}}(\pi_3) = \mathcal{L}_{\mathscr{C}}(\pi_4) = \{\pi_1, \pi_2, \pi_3, \pi_4\}.$$

Here we do not need the statistic $s(\lambda)$. Thus, we may further take $y = 1$ in (2.3). Then

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq^{\alpha} & & \\ & & xq^{\beta} & \\ & & & x^2q^{\alpha+\beta} \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^M) \\ F_2(xq^M) \\ F_3(xq^M) \\ F_4(xq^M) \end{pmatrix}. \tag{4.1}$$

4.1. Triple series. We choose $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} M & M & M \\ M & M & M \\ M & M & 2M \end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}} = (1, 1, 2)$ and $\underline{\mathbf{A}} = (M, M, M)$ in (2.4) and write $x_1 = x$. We will use H'_T for H.

We want to show that

$$F_1(x) = F_2(x) = F_3(x) = F_4(x) = H'_T(\alpha, \beta, \alpha + \beta).$$

It suffices to prove that

$$\begin{aligned} H'_T(\alpha, \beta, \alpha + \beta) &= H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M) \\ &\quad + xq^{\alpha} H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M) \\ &\quad + xq^{\beta} H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M) \\ &\quad + x^2q^{\alpha+\beta} H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M). \end{aligned} \tag{4.2}$$

By the binary tree in Figure 6(a), we have

$$\begin{aligned} H'_T(\alpha, \beta, \alpha + \beta) &= H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M) \\ &\quad + xq^{\alpha} H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M) \\ &\quad + x^2q^{\alpha+\beta} H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M) \\ &\quad + xq^{\beta} H'_T(\alpha + 2M, \beta + M, \alpha + \beta + 2M) \\ &\quad + x^2q^{\alpha+\beta+M} H'_T(\alpha + 2M, \beta + 2M, \alpha + \beta + 3M). \end{aligned} \tag{4.3}$$

Further, for the last two terms on the right hand side of the above, we see from Figure 6(b) that

$$\begin{aligned} H'_T(\alpha + M, \beta + M, \alpha + \beta + 2M) &= H'_T(\alpha + 2M, \beta + M, \alpha + \beta + 2M) \\ &\quad + xq^{\alpha+M} H'_T(\alpha + 2M, \beta + 2M, \alpha + \beta + 3M). \end{aligned} \tag{4.4}$$

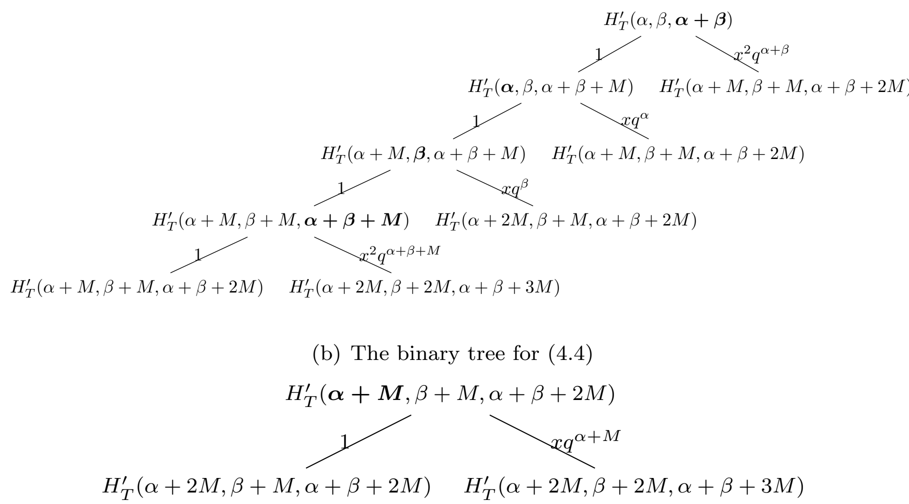
Therefore, (4.2) holds true.

Finally, (1.17) follows by recalling (2.2),

$$\sum_{\lambda \in \mathscr{C}} x^{\sharp(\lambda)} q^{|\lambda|} = G_1(x) + G_2(x) + G_3(x) + G_4(x) = F_1(x).$$

FIGURE 6. The binary trees for Section 4.1

(a) The binary tree for (4.3)



4.2. Double series. Now we consider the case of M even and $\beta=\alpha+\frac{M}{2}$. Then (4.1) becomes

$$\begin{pmatrix}F_1(x)\\F_2(x)\\F_3(x)\\F_4(x)\end{pmatrix}=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\cdot\begin{pmatrix}1&&&\\&xq^{\alpha}&&\\&&xq^{\alpha+\frac{M}{2}}&\\&&&x^2q^{2\alpha+\frac{M}{2}}\end{pmatrix}\cdot\begin{pmatrix}F_1(xq^M)\\F_2(xq^M)\\F_3(xq^M)\\F_4(xq^M)\end{pmatrix}.$$

We choose $\underline{\boldsymbol{\alpha}}=\begin{pmatrix}M&M\\M&2M\end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}}=(1,2)$ and $\underline{\mathbf{A}}=(\frac{M}{2},M)$ in (2.4) and write $x_1=x$. We will use H'_D for H.

Then (1.18) is equivalent to

$$F_1(x)=F_2(x)=F_3(x)=F_4(x)=H'_D(\alpha,2\alpha+\tfrac{M}{2}).$$

Namely,

$$\begin{aligned}H'_D(\alpha,2\alpha+\tfrac{M}{2})&=H'_D(\alpha+M,2\alpha+\tfrac{5M}{2})\\&\quad+xq^{\alpha}H'_D(\alpha+M,2\alpha+\tfrac{5M}{2})\\&\quad+xq^{\alpha+\frac{M}{2}}H'_D(\alpha+M,2\alpha+\tfrac{5M}{2})\\&\quad+x^2q^{2\alpha+\frac{M}{2}}H'_D(\alpha+M,2\alpha+\tfrac{5M}{2}).\end{aligned}\tag{4.5}$$

By the binary tree in Figure 7(a), we have

$$\begin{aligned}H'_D(\alpha,2\alpha+\tfrac{M}{2})&=H'_D(\alpha+M,2\alpha+\tfrac{5M}{2})\\&\quad+xq^{\alpha}H'_D(\alpha+M,2\alpha+\tfrac{5M}{2})\\&\quad+x^2q^{2\alpha+\frac{M}{2}}H'_D(\alpha+M,2\alpha+\tfrac{5M}{2})\\&\quad+xq^{\alpha+\frac{M}{2}}H'_D(\alpha+\tfrac{3M}{2},2\alpha+\tfrac{5M}{2})\end{aligned}$$

$$+ x^2 q^{2\alpha+\frac{3M}{2}} H'_D(\alpha+2M, 2\alpha+\tfrac{7M}{2}). \tag{4.6}$$

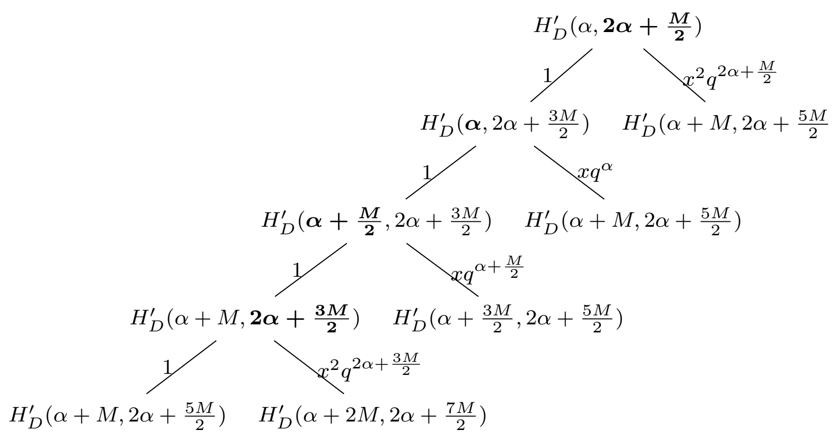
Further, for the last two terms on the right hand side of the above, we see from Figure 7(b) that

$$\begin{aligned} H'_D(\alpha+M, 2\alpha+\tfrac{5M}{2}) &= H'_D(\alpha+\tfrac{3M}{2}, 2\alpha+\tfrac{5M}{2}) \\ &\quad + xq^{\alpha+M} H'_D(\alpha+2M, 2\alpha+\tfrac{7M}{2}). \end{aligned} \tag{4.7}$$

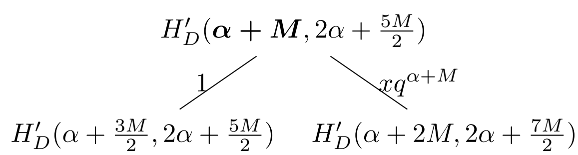
Therefore, (4.5) holds true.

Figure 7. The binary trees for Section 4.2

(a) The binary tree for (4.6)



(b) The binary tree for (4.7)



5. Final remarks

In Section 3, we have provided unified proofs of Theorems 1.1 and 1.2 within the framework of linked partition ideals. On the other hand, as pointed out by the referee, once the triple series in Theorem 1.1 are established, the double series in Theorem 1.2 may be derived through a q-hypergeometric technique. Therefore, the bonds between these triple and double series can be witnessed from a different point of view. Now, we close this paper by giving such an instance of showing (1.11) from (1.5) when M is even and $\beta = \alpha + \frac{M}{2}$. To begin with, we make the change of variables $N_1 = n_1 + n_2$ and $N_2 = n_3$. Then

$$\sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3} y^{n_3}}{(q^M;q^M)_{n_1}(q^M;q^M)_{n_2}(q^M;q^M)_{n_3}}$$
$$\times\, q^{M\binom{n_1}{2}+M\binom{n_2}{2}+2M\binom{n_3}{2}+Mn_1n_2+Mn_2n_3+Mn_3n_1+\alpha n_1+(\alpha+\frac{M}{2})n_2+(2\alpha+\frac{M}{2})n_3}$$

$$
\begin{aligned}
&= \sum_{N_1,N_2\geq 0} \frac{x^{N_1+N_2}y^{N_2}q^{M\binom{N_1}{2}+2M\binom{N_2}{2}+MN_1N_2+\alpha N_1+(2\alpha+\frac{M}{2})N_2}}{(q^M;q^M)_{N_1}(q^M;q^M)_{N_2}} \sum_{n_2=0}^{N_1} \begin{bmatrix} N_1 \\ n_2 \end{bmatrix}_{q^M} q^{\frac{M}{2}n_2}\\
&= \sum_{N_1,N_2\geq 0} \frac{x^{N_1+N_2}y^{N_2}q^{M\binom{N_1}{2}+2M\binom{N_2}{2}+MN_1N_2+\alpha N_1+(2\alpha+\frac{M}{2})N_2}}{(q^M;q^M)_{N_1}(q^M;q^M)_{N_2}} (-q^{\frac{M}{2}};q^{\frac{M}{2}})_{N_1}\\
&= \sum_{N_1,N_2\geq 0} \frac{x^{N_1+N_2}y^{N_2}q^{M\binom{N_1}{2}+2M\binom{N_2}{2}+MN_1N_2+\alpha N_1+(2\alpha+\frac{M}{2})N_2}}{(q^{\frac{M}{2}};q^{\frac{M}{2}})_{N_1}(q^M;q^M)_{N_2}}.
\end{aligned}
$$

Here we make use of the following evaluation of the Rogers–Szegő polynomials (see Exercise 5 in Chapter 3 of [6]):

$$
\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} q^n = (-q;q)_N.
$$

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