Asymmetric Rogers–Ramanujan type identities. I. The Andrews–Uncu Conjecture

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Abstract. In this work, we start an investigation of asymmetric Rogers–Ramanujan type identities. The first object is the following unexpected relation

$$\sum_{n\geq 0} \frac{(-1)^n q^{3\binom{n}{2}+4n}(q;q^3)_n}{(q^9;q^9)_n} = \frac{(q^4;q^6)_\infty (q^{12};q^{18})_\infty}{(q^5;q^6)_\infty (q^9;q^{18})_\infty}$$

and its a-generalization. We then use this identity as a key ingredient to confirm a recent conjecture of G. E. Andrews and A. K. Uncu.

Keywords. Rogers–Ramanujan type identity, Andrews–Uncu Conjecture, asymmetry, a-generalization.

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1. Introduction

The Rogers–Ramanujan identities,

$$\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{1.1}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}, \tag{1.2}$$

first appeared in L. J. Rogers' *Second memoir on the expansion of certain infinite products*, which was completely overlooked for over two decades. They bloomed again after S. Ramanujan's rediscovery (without proof) [22], and then Ramanujan and Rogers [23] presented a joint new proof. Independently, I. Schur [27] also published two fundamentally different proofs. The proofs of the Rogers–Ramanujan identities were later perfected by W. N. Bailey [4, 5] through the ingenious Bailey's Lemma, which finally led to L. J. Slater's list [28, 29] of 130 identities of Rogers–Ramanujan type.

As usual, we adopt the q-Pochhammer symbol for $n \in \mathbb{N} \cup \{\infty\}$,

$$(A;q)_n := \prod_{k=0}^{n-1} (1 - Aq^k),$$

$$(A_1, A_2, \ldots, A_m; q)_n := (A_1;q)_n (A_2;q)_n \cdots (A_m;q)_n.$$

Throughout, we always assume that q is a complex variable such that $|q| < 1$. We shall also introduce other complex parameters and they are such that zero denominators are avoided.

In light of (1.1) and (1.2), a Rogers–Ramanujan type identity asserts the equality of a q-summation and an infinite product. Also, on the summation side, we often require that the summand contains a factor in which the power of q is quadratic in the summation variable(s).

Further, we may distinguish Rogers–Ramanujan type identities according to the shape of the summation and product sides:

For the summation side, we mainly focus on whether it is a *single* or *multiple* series.

For the product side, we say that it is *periodic* if there exists a positive integer M such that the infinite product has the form

$$\prod_{m=1}^{M}(q^m;q^M)_\infty^{\delta_m}.$$

Further, for a periodic Rogers–Ramanujan type identity, we say that it is *symmetric* if the powers δ_m satisfy $\delta_m=\delta_{M-m}$ for all $1\le m\le M-1$, and *asymmetric* otherwise.

Most known Rogers–Ramanujan type identities are symmetric. In particular, all 130 identities in Slater's list are single-symmetric, and the infinite products in them may be deduced through Jacobi's triple product identity or the quintuple product identity. There are also many multiple-symmetric identities of Rogers–Ramanujan type, among which the most important one is the generalization of (1.1) and (1.2) due to G. E. Andrews [1] and B. Gordon [15]:

$$\prod_{\substack{n\ge 1\\ n\not\equiv 0,\pm i \pmod{2k+1}}}\frac{1}{1-q^n}=\sum_{n_1,\ldots,n_{k-1}\ge 0}\frac{q^{N_1^2+N_2^2+\cdots N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}},$$

where $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$.

In contrast, few asymmetric Rogers–Ramanujan type identities were recorded. The first such instances are the little Göllnitz identities discovered by H. Göllnitz [14]. They have the analytic form:

$$\sum_{n\ge 0}\frac{q^{n^2+n}(-q^{-1};q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q,q^5,q^6;q^8)_\infty},$$

$$\sum_{n\ge 0}\frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q^2,q^3,q^7;q^8)_\infty}.$$

In fact, they are special cases of an identity due to V. A. Lebesgue [20]:

$$\sum_{n\ge 0}\frac{q^{\binom{n}{2}+n}(a;q)_n}{(q;q)_n}=\frac{(aq;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.3}$$

Recently, S. Corteel, C. D. Savage and A. V. Sills [10] also discovered two single-asymmetric Rogers–Ramanujan type identities:

$$\sum_{n\ge 0}\frac{q^{3\binom{n}{2}+n}(q^2;q^6)_n}{(q;q)_{3n}}=\frac{1}{(q;q^3)_\infty(q^5;q^6)_\infty},$$

$$\sum_{n\geq 0}\frac{q^{3\binom{n}{2}+2n}(q^4;q^6)_n}{(q;q)_{3n+1}}=\frac{1}{(q^2;q^3)_\infty(q;q^6)_\infty},$$

which follow from a specialization of Heine's q-Gauß summation:

$$\sum_{n\geq 0}\frac{(-1)^n a^n q^{\binom{n}{2}}(a;q)_n}{(a^2;q)_n(q;q)_n}=\frac{(a;q)_\infty}{(a^2;q)_\infty}. \tag{1.4}$$

See [10, p. 62]. For multiple-asymmetric Rogers–Ramanujan type identities, an important source is some "Mod 12" Conjectures due to S. Kanade and M. C. Russell [17] and M. C. Russell [26]. Their original conjectures are partition-theoretic and the corresponding analytic identities were later established by K. Kurşungöz [19], Kanade and Russell themselves [18], and S. Chern and Z. Li [8]. Finally, these conjectures were proved by K. Bringmann, C. Jennings-Shaffer and K. Mahlburg [7] and H. Rosengren [25]. For example, the analytic form of [17, Conjecture I_5] is (see [7, (1.15)])

$$\sum_{n_1,n_2,n_3\geq 0}\frac{q^{\binom{n_1}{2}+6\binom{n_2}{2}+9\binom{n_3}{2}+2n_1n_2+6n_2n_3+3n_3n_1+n_1+4n_2+7n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}$$
$$=\frac{1}{(q;q^3)_\infty(q^3,q^6,q^{11};q^{12})_\infty}.$$

Recently, G. E. Andrews and A. K. Uncu [3, Theorem 1.1] proved one more multiple-asymmetric identity of Rogers–Ramanujan type:

$$\sum_{m,n\geq 0}\frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+m+6n}}{(q;q)_m(q^3;q^3)_n}=\frac{1}{(q;q^3)_\infty}. \tag{1.5}$$

They also discovered a similar conjectural identity.

Conjecture 1.1 (The Andrews–Uncu Conjecture [3, Conjecture 1.2])**.**

$$\sum_{m,n\geq 0}\frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+2m+7n}}{(q;q)_m(q^3;q^3)_n}=\frac{1}{(q^2,q^3;q^6)_\infty}.$$

There are two objects in this paper. First, we establish the following unexpected single-asymmetric Rogers–Ramanujan type identity.

Theorem 1.1.

$$\sum_{n\geq 0}\frac{(-1)^n q^{3\binom{n}{2}+4n}(q;q^3)_n}{(q^9;q^9)_n}=\frac{(q^4;q^6)_\infty(q^{12};q^{18})_\infty}{(q^5;q^6)_\infty(q^9;q^{18})_\infty}. \tag{1.6}$$

In fact, this identity is the $(a,q)\mapsto(q,q^3)$ case of its a-generalization.

Theorem 1.2.

$$\sum_{n\geq 0}\frac{(a;q)_n(a^{-1}q^2;q^2)_n}{(a^2q;q^2)_n(q^3;q^3)_n}(-1)^n a^n q^{\binom{n}{2}+n}=\frac{(aq;q^2)_\infty(a^3q^3;q^6)_\infty}{(a^2q;q^2)_\infty(q^3;q^6)_\infty}. \tag{1.7}$$

Using (1.6) as a key ingredient, the second part of this paper is devoted to a proof of the Andrews–Uncu Conjecture.

Theorem 1.3. *The Andrews–Uncu Conjecture is true. That is,*

$$\sum_{m,n\geq 0} \frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+2m+7n}}{(q;q)_m(q^3;q^3)_n} = \frac{1}{(q^2,q^3;q^6)_\infty}. \tag{1.8}$$

Remarkably, Andrews and Uncu [3] connected (1.5) with overpartitions whose consecutive parts avoid certain patterns. Recall that an *overpartition* [9] is a partition where the first occurrence of each distinct part may be overlined. In light of [3, Theorem 5.1] with $k=i=1$, we may also combinatorially interpret (1.8) as follows.

Corollary 1.4. *The number of partitions of N into red and green parts with each green part congruent to $2,3$ modulo 6 equals the number of overpartitions of N such that the smallest overlined part is at least 2 and such that the consecutive parts are not of the forms $\overline{j}+\overline{(j-1)}$ and $j+\overline{(j-1)}+(j-1)$.*

Example. When $N=5$, the 14 colored partitions in the first class are

$$\begin{gathered}5_r,\ 4_r+1_r,\ 3_r+2_r,\ 3_r+2_g,\ 3_g+2_r,\ 3_g+2_g,\ 3_r+1_r+1_r,\ 3_g+1_r+1_r,\\ 2_r+2_r+1_r,\ 2_r+2_g+1_r,\ 2_g+2_g+1_r,\ 2_r+1_r+1_r+1_r,\\ 2_g+1_r+1_r+1_r,\ 1_r+1_r+1_r+1_r+1_r,\end{gathered}$$

and the 14 overpartitions in the second class are

$$\begin{gathered}5,\ \overline{5},\ 4+1,\ \overline{4}+1,\ 3+2,\ \overline{3}+2,\ 3+\overline{2},\ 3+1+1,\ \overline{3}+1+1,\\ 2+2+1,\ \overline{2}+2+1,\ 2+1+1+1,\ \overline{2}+1+1+1,\\ 1+1+1+1+1.\end{gathered}$$

2. The a-generalization

In this section, we prove the a-generalization of (1.6) given in Theorem 1.2. We first observe that, as functions in a, both sides of (1.7) are analytic on the set

$$\mathbb{C}\backslash\{\pm q^{-\frac{1}{2}},\pm q^{-\frac{3}{2}},\pm q^{-\frac{5}{2}},\ldots\}.$$

Here we shall notice that for nonnegative integers n,

$$(a^{-1}q^2;q^2)_n a^n = \prod_{k=1}^{n}(a-q^{2k}).$$

We start by showing that (1.7) is true for a on the open disk $\mathscr{D} := \{a\in\mathbb{C} : |a| < |q|^{-\frac{1}{2}}\}$. Now, we find that the set $\{q^{2M} : M\in\mathbb{Z}_{\geq 0}\}$ has an accumulation point 0 which is on the disc $\mathscr{D}$. Therefore, we know from the identity theorem for holomorphic functions that it suffices to prove (1.7) for $a=q^{2M}$ with M nonnegative integers. Let

$$S_M := \sum_{n\geq 0} \frac{(q^{2M};q)_n(q^{2-2M};q^2)_n}{(q^{4M+1};q^2)_n(q^3;q^3)_n}(-1)^n q^{\binom{n}{2}+n+2Mn}. \tag{2.1}$$

We remark that these summations are terminating.

Our next task is to derive a recurrence for S_M. This can be done automatically by the *Mathematica* package `qZeil` implemented by P. Paule and A. Riese [21]. The package can be downloaded at the website of the Research Institute for Symbolic Computation (RISC) of Johannes Kepler University:

`https://www3.risc.jku.at/research/combinat/software/ergosum/index.html`

To begin with, we import this package.

```
<< RISC`qZeil`
```

The recurrence for S_M can be computed by the following codes:

```
ClearAll[n, M];
summand = (
qPochhammer[q^(2 M), q, n] qPochhammer[q^(2 - 2 M),
    q^2, n])/(
qPochhammer[q^(4 M + 1), q^2, n] qPochhammer[q^3, q
    ^3, n]) (-1)^
n q^(2 M*n) q^(Binomial[n, 2] + n);
qZeil[summand, {n, 0, Infinity}, M, 1]
```

The output gives us

$$\mathrm{SUM}[M] == \frac{q^2(1-q^{-3+4M})(1-q^{-1+4M})\,\mathrm{SUM}[-1+M]}{(1-q^{-1+2M})^2(q^2+q^{4M}+q^{1+2M})}$$

Thus, for $M \geq 1$,

$$\begin{aligned}\frac{S_M}{S_{M-1}} &= \frac{q^2(1-q^{-3+4M})(1-q^{-1+4M})}{(1-q^{-1+2M})^2(q^2+q^{4M}+q^{1+2M})}\\ &= \frac{(1-q^{4M-3})(1-q^{4M-1})}{(1-q^{2M-1})(1-q^{6M-3})}.\end{aligned}$$

Recalling that $S_0 = 1$, we have

$$\begin{aligned}S_M &= \frac{(q;q^2)_{2M}}{(q;q^2)_M(q^3;q^6)_M}\\ &= \frac{(q^{2M+1};q^2)_\infty(q^{6M+3};q^6)_\infty}{(q^{4M+1};q^2)_\infty(q^3;q^6)_\infty}.\end{aligned}$$

This is the right-hand side of (1.7) with $a = q^{2M}$.

Now we have proved (1.7) for $a \in \{q^{2M} : M \in \mathbb{Z}_{\geq 0}\}$, and thus for $a \in \mathscr{D} = \{a \in \mathbb{C} : |a| < |q|^{-\frac{1}{2}}\}$ by the previous argument. Finally, the principle of analytic continuation allows us to extend (1.7) to the set $\mathbb{C}\backslash\{\pm q^{-\frac{1}{2}}, \pm q^{-\frac{3}{2}}, \pm q^{-\frac{5}{2}}, \ldots\}$, and finishes the proof.

3. The Andrews–Uncu Conjecture

3.1. A contour integral representation. Throughout, all contour integrals are over a positively oriented contour separating 0 from all poles of the integrand.

We recall a general basic contour integral formula given in [13, p. 127, (4.10.9)] with $m = 0$.

$$\begin{aligned}&\oint \frac{(a_1z,\ldots,a_Mz,b_1/z,\ldots b_L/z;q)_\infty}{(c_1z,\ldots c_Nz,d_1/z,\ldots,d_L/z;q)_\infty}\frac{\mathrm{d}\,z}{2\pi \mathrm{i} z}\\ &\quad= \frac{(b_1c_1,\ldots,b_Lc_1,a_1/c_1,\ldots,a_M/c_1;q)_\infty}{(q,d_1c_1,\ldots,d_Lc_1,c_2/c_1,\ldots,c_N/c_1;q)_\infty}\end{aligned}$$

$$\times {}_{M+L}\phi_{N+L-1}\left(\begin{matrix} d_1c_1,\ldots,d_Lc_1,qc_1/a_1,\ldots,qc_1/a_M \\ b_1c_1,\ldots,b_Lc_1,qc_1/c_2,\ldots,qc_1/c_N\end{matrix};q,u(qc_1)^{N-M}\right)$$
$$+\operatorname{idem}(c_1;c_2,\ldots,c_N), \tag{3.1}$$

where $u=a_1\cdots a_M/(c_1\cdots c_N)$ and $\operatorname{idem}(c_1;c_2,\ldots,c_N)$ stands for the sum of the $N-1$ expressions obtained from the preceding expression by interchanging c_1 with each c_k $(k=2,\ldots,N)$.

Notice that in (3.1), we shall assume that $M<N$, or $M=N$ with an additional constraint that $|u|<1$.

Let H be the huffing operator, given by

$$H\left(\sum_n a(n)q^n\right):=\sum_n a(3n)q^{3n}. \tag{3.2}$$

Theorem 3.1. *Let $\omega=\mathrm{e}^{2\pi i/3}$. Then*

$$\oint\frac{(q^6z,q^3z,1/z;q^3)_\infty}{(q^4z,\omega q^4z,\omega^2q^4z;q^3)_\infty}\frac{\mathrm{d}\,z}{2\pi\mathrm{i}z}$$
$$=H\left(\frac{(q^4,q^2,q^{-1};q^3)_\infty}{(q^9;q^9)_\infty}\sum_{n\ge0}\frac{(-1)^nq^{3\binom{n}{2}+4n}(q;q^3)_n}{(q^9;q^9)_n}\right). \tag{3.3}$$

Proof. In (3.1), we replace $q\mapsto q^3$ and choose $M=2$, $N=3$ and $L=1$ with

$$(a_1,a_2)\mapsto(q^6,q^3),$$
$$(b_1)\mapsto(1),$$
$$(c_1,c_2,c_3)\mapsto(q^4,\omega q^4,\omega^2q^4),$$
$$(d_1)\mapsto(0).$$

Then

$$\oint\frac{(q^6z,q^3z,1/z;q^3)_\infty}{(q^4z,\omega q^4z,\omega^2q^4z;q^3)_\infty}\frac{\mathrm{d}\,z}{2\pi\mathrm{i}z}$$
$$=\frac{(q^4,q^2,q^{-1};q^3)_\infty}{(q^3,\omega,\omega^2;q^3)_\infty}{}_2\phi_2\left(\begin{matrix}0,q\\ \omega q^3,\omega^2q^3\end{matrix};q^3,q^4\right)$$
$$+\frac{(\omega q^4,\omega^2q^2,\omega^2q^{-1};q^3)_\infty}{(q^3,\omega,\omega^2;q^3)_\infty}{}_2\phi_2\left(\begin{matrix}0,\omega q\\ \omega q^3,\omega^2q^3\end{matrix};q^3,\omega q^4\right)$$
$$+\frac{(\omega^2q^4,\omega q^2,\omega q^{-1};q^3)_\infty}{(q^3,\omega,\omega^2;q^3)_\infty}{}_2\phi_2\left(\begin{matrix}0,\omega^2q\\ \omega q^3,\omega^2q^3\end{matrix};q^3,\omega^2q^4\right).$$

Now, let

$$F(q):=\frac{(q^4,q^2,q^{-1};q^3)_\infty}{(q^3,\omega,\omega^2;q^3)_\infty}{}_2\phi_2\left(\begin{matrix}0,q\\ \omega q^3,\omega^2q^3\end{matrix};q^3,q^4\right)$$
$$=\frac{1}{3}\frac{(q^4,q^2,q^{-1};q^3)_\infty}{(q^9;q^9)_\infty}\sum_{n\ge0}\frac{(-1)^nq^{3\binom{n}{2}+4n}(q;q^3)_n}{(q^9;q^9)_n}. \tag{3.4}$$

We observe that

$$\frac{(\omega q^4,\omega^2q^2,\omega^2q^{-1};q^3)_\infty}{(q^3,\omega,\omega^2;q^3)_\infty}{}_2\phi_2\left(\begin{matrix}0,\omega q\\ \omega q^3,\omega^2q^3\end{matrix};q^3,\omega q^4\right)=F(\omega q)$$

and

$$\frac{(\omega^2 q^4, \omega q^2, \omega q^{-1}; q^3)_\infty}{(q^3, \omega, \omega^2; q^3)_\infty} {}_2\phi_2 \left(\begin{matrix} 0, \omega^2 q \\ \omega q^3, \omega^2 q^3 \end{matrix} ; q^3, \omega^2 q^4 \right) = F(\omega^2 q).$$

Thus,

$$\begin{aligned} \oint \frac{(q^6 z, q^3 z, 1/z; q^3)_\infty}{(q^4 z, \omega q^4 z, \omega^2 q^4 z; q^3)_\infty} \frac{\mathrm{d}\, z}{2\pi \mathrm{i} z} &= F(q) + F(\omega q) + F(\omega^2 q) \\ &= 3H\big(F(q)\big). \end{aligned}$$

This gives our desired result by recalling (3.4). □

3.2. 3-Dissections. By (1.6), we have

$$\begin{aligned} &\frac{(q^4, q^2, q^{-1}; q^3)_\infty}{(q^9; q^9)_\infty} \sum_{n\ge 0} \frac{(-1)^n q^{3\binom{n}{2}+4n} (q; q^3)_n}{(q^9; q^9)_n} \\ &\quad = \frac{(q^4, q^2, q^{-1}; q^3)_\infty}{(q^9; q^9)_\infty} \frac{(q^4; q^6)_\infty (q^{12}; q^{18})_\infty}{(q^5; q^6)_\infty (q^9; q^{18})_\infty} \\ &\quad = (q^{-1}, q^2, q^2, q^4, q^4, q^7; q^6)_\infty \frac{(q^{12}; q^{18})_\infty}{(q^9; q^9)_\infty (q^9; q^{18})_\infty} \\ &\quad = (1 - q^{-1})(q^5, q^7; q^6)_\infty \frac{(q^2; q^2)_\infty^2}{(q^6; q^6)_\infty^2} \frac{(q^{12}; q^{18})_\infty}{(q^9; q^9)_\infty (q^9; q^{18})_\infty} \\ &\quad = -q^{-1} (q, q^5; q^6)_\infty (q^2; q^2)_\infty^2 \frac{(q^{12}; q^{18})_\infty}{(q^6; q^6)_\infty^2 (q^9; q^9)_\infty (q^9; q^{18})_\infty} \\ &\quad = -q^{-1} (q; q)_\infty (q^2; q^2)_\infty \frac{(q^{12}; q^{18})_\infty}{(q^3; q^3)_\infty (q^6; q^6)_\infty (q^9; q^9)_\infty (q^9; q^{18})_\infty}. \end{aligned} \tag{3.5}$$

Let us recall Ramanujan's classical theta functions

$$\begin{aligned} \phi(-q) &:= \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}, \\ \psi(q) &:= \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}. \end{aligned}$$

Their 3-dissections are well-known.

Lemma 3.2. *We have*

$$\phi(-q) = \frac{(q^9; q^9)_\infty^2}{(q^{18}; q^{18})_\infty} - 2q \frac{(q^3; q^3)_\infty (q^{18}; q^{18})_\infty^2}{(q^6; q^6)_\infty (q^9; q^9)_\infty}, \tag{3.6}$$

$$\psi(q) = \frac{(q^6; q^6)_\infty (q^9; q^9)_\infty^2}{(q^3; q^3)_\infty (q^{18}; q^{18})_\infty} + q \frac{(q^{18}; q^{18})_\infty^2}{(q^9; q^9)_\infty}. \tag{3.7}$$

Proof. See [16, (14.3.2) and (14.3.3)]. □

Theorem 3.3. *We have*

$$H\left(\frac{(q^4, q^2, q^{-1}; q^3)_\infty}{(q^9; q^9)_\infty} \sum_{n\ge 0} \frac{(-1)^n q^{3\binom{n}{2}+4n} (q; q^3)_n}{(q^9; q^9)_n} \right) = \frac{1}{(q^3; q^3)_\infty (q^6, q^9; q^{18})_\infty}. \tag{3.8}$$

Proof. We conclude from (3.5) that

$$
\begin{aligned}
&H\left(\frac{(q^4,q^2,q^{-1};q^3)_\infty}{(q^9;q^9)_\infty}\sum_{n\geq 0}\frac{(-1)^n q^{3\binom{n}{2}+4n}(q;q^3)_n}{(q^9;q^9)_n}\right)\\
&\quad = -\frac{(q^{12};q^{18})_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty(q^9;q^9)_\infty(q^9;q^{18})_\infty}H\big(q^{-1}(q;q)_\infty(q^2;q^2)_\infty\big)\\
&\quad = -\frac{(q^{12};q^{18})_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty(q^9;q^9)_\infty(q^9;q^{18})_\infty}H\big(q^{-1}\phi(-q)\psi(q)\big)\\
&\quad = \frac{(q^{12};q^{18})_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty(q^9;q^9)_\infty(q^9;q^{18})_\infty}\cdot(q^9;q^9)_\infty(q^{18};q^{18})_\infty.
\end{aligned}
$$

This simplifies to our desired result. □

3.3. Proof of the Andrews–Uncu Conjecture. Now we complete the proof of the Andrews–Uncu Conjecture. Recall Euler's summations [2, (2.2.5) and (2.2.6)]:

$$
\begin{aligned}
\frac{1}{(z;q)_\infty} &= \sum_{n\geq 0}\frac{z^n}{(q;q)_n},\\
(z;q)_\infty &= \sum_{n\geq 0}\frac{(-1)^n z^n q^{\binom{n}{2}}}{(q;q)_n},
\end{aligned}
$$

and Jacobi's triple product identity [2, (2.2.10)]:

$$
(q,z,q/z;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\binom{n}{2}}.
$$

Notice that

$$
\begin{aligned}
&\sum_{m,n\geq 0}\frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+2m+7n}}{(q;q)_m(q^3;q^3)_n}\\
&\quad = \oint \sum_{m\geq 0}\frac{(-1)^m q^{\binom{m}{2}+2m}z^m}{(q;q)_m}\sum_{n\geq 0}\frac{q^{4n}z^{3n}}{(q^3;q^3)_n}\sum_{\ell=-\infty}^{\infty}(-1)^\ell q^{\binom{\ell}{2}}z^{-\ell}\frac{\mathrm{d}\,z}{2\pi\mathrm{i}z}\\
&\quad = \oint \frac{(q^2z;q)_\infty(q,qz,1/z;q)_\infty}{(q^4z^3;q^3)_\infty}\frac{\mathrm{d}\,z}{2\pi\mathrm{i}z}\\
&\quad = (q;q)_\infty\oint\frac{(q^2z,qz,1/z;q)_\infty}{(q^{4/3}z,\omega q^{4/3}z,\omega^2q^{4/3}z;q)_\infty}\frac{\mathrm{d}\,z}{2\pi\mathrm{i}z},
\end{aligned}
$$

where we still put $\omega = \mathrm{e}^{2\pi i/3}$. We further observe that the above contour integral is indeed the integral in (3.3) with q replaced by $q^{1/3}$. By (3.3) and (3.8), we conclude that

$$
\begin{aligned}
\sum_{m,n\geq 0}\frac{(-1)^n q^{2\binom{m}{2}+9\binom{n}{2}+3mn+2m+7n}}{(q;q)_m(q^3;q^3)_n} &= (q;q)_\infty\cdot\frac{1}{(q;q)_\infty(q^2,q^3;q^6)_\infty}\\
&= \frac{1}{(q^2,q^3;q^6)_\infty}.
\end{aligned}
$$

4. Conclusion

The entire surprise comes from the single-asymmetric Rogers–Ramanujan type identity (1.6), which is *completely weird*, as pointed out by George Andrews in a personal communication. Although its a-generalization (1.7) presents a similar look to the Lebesgue identity (1.3) or the Corteel–Savage–Sills identity (1.4), the unexpectedness rises as two extra q-shifted factorials are attached to the summand in (1.7) in comparison to the original (1.6). It is also natural to ask if there are other instances that behave as "unreasonable" as (1.6). A thorough computer search will be carried out in a forthcoming project.

In addition, as pointed out by Wadim Zudilin in a personal communication, (1.7) gives a q-analog of the following identity [11, p. 27, eq. (xiv)]:

$$ {}_2F_1\left(\begin{matrix}2A,-A+1\\ 2A+\frac{1}{2}\end{matrix};-\frac{1}{3}\right)=\frac{\sqrt{\pi}}{3^A}\frac{\Gamma(2A+\frac{1}{2})}{\Gamma(A+\frac{1}{2})^2}. \tag{4.1} $$

The right hand side of the above can be rewritten by a standard manipulation of the Gamma function:

$$ \begin{aligned}\frac{\sqrt{\pi}}{3^A}\frac{\Gamma(2A+\frac{1}{2})}{\Gamma(A+\frac{1}{2})^2}&=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{2A}{3}+\frac{1}{6})\Gamma(\frac{2A}{3}+\frac{1}{2})\Gamma(\frac{2A}{3}+\frac{5}{6})}{\Gamma(A+\frac{1}{2})\Gamma(\frac{A}{3}+\frac{1}{6})\Gamma(\frac{A}{3}+\frac{1}{2})\Gamma(\frac{A}{3}+\frac{5}{6})}\\ &=\prod_{n\geq 0}\frac{(n+A+\frac{1}{2})(n+\frac{A}{3}+\frac{1}{6})(n+\frac{A}{3}+\frac{1}{2})(n+\frac{A}{3}+\frac{5}{6})}{(n+\frac{1}{2})(n+\frac{2A}{3}+\frac{1}{6})(n+\frac{2A}{3}+\frac{1}{2})(n+\frac{2A}{3}+\frac{5}{6})},\end{aligned} $$

in which the last equality follows from [30, p. 247]. It is remarkable that in this limiting case (for $q\to 1^-$), the argument of the ${}_2F_1$ series is $-\frac{1}{3}$. There are several evaluations of ${}_2F_1$ series sharing a similar nature. One example is [12, eq. (2.8.53)]:

$$ {}_2F_1\left(\begin{matrix}-A,-A+\frac{1}{2}\\ 2A+\frac{3}{2}\end{matrix};-\frac{1}{3}\right)=\left(\frac{8}{9}\right)^{2A}\frac{\Gamma(\frac{4}{3})\Gamma(2A+\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(2A+\frac{4}{3})}. $$

In fact, Ebisu [11] developed an algorithm to produce strange Γ-evaluations of hypergeometric series, and his approach is perfectly outlined in [6, p. 2]. Therefore, it might be more promising to establish new a-parametric identities such as (1.7) from a hypergeometric perspective, rather than looking directly at the q-hypergeometric side.

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