q-Log-concavity and q-unimodality of Gaussian polynomials and a problem of Andrews and Newman

Shane Chern

Abstract. We answer a nonnegativity problem of G. E. Andrews and D. Newman by the q-unimodality of Gaussian polynomials. Some new considerations of the q-log-concavity and q-unimodality of Gaussian polynomials from a purely partition-theoretic perspective will also be presented.

Keywords. Gaussian polynomial, q-log-concavity, q-unimodality, nonnegativity.

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1. Introduction

The *Gaussian polynomials*, also know as the *Gaussian binomial coefficients*, are given by

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_M}{(q;q)_N (q;q)_{M-N}} & \text{if } 0 \le N \le M, \\ 0 & \text{otherwise,} \end{cases}$$

with the *q-Pochhammer symbol* defined for $n \in \mathbb{N}$,

$$(A;q)_n := \prod_{k=0}^{n-1} (1 - Aq^k).$$

They reduce to the binomial coefficients at the $q \to 1^-$ limit. Gaussian polynomials play a substantial role in combinatorics and special functions. In particular, from the perspective of integer partitions, the Gaussian polynomial $\begin{bmatrix} M+N \\ M \end{bmatrix}$ is the generating function for partitions with at most M parts and largest part at most N; see [1, Chapter 3]. As usual, what we mean by an *integer partition* of a natural number n is a nonincreasing sequence of positive integers that sum to n.

In their recent paper, G. E. Andrews and D. Newman [2] proposed the following problem.

Problem 1.1 (Andrews and Newman [2, p. 9, Question II(1)]). *Prove that all coefficients are nonnegative in the polynomial*

$$\sum_{r=0}^{n} (-1)^r \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix}. \tag{1.1}$$

The motivation of the above polynomial comes from the minimal excludant (mex) statistic of integer partitions. More precisely, it is related to the smallest integer congruent to 2 modulo 4 that is not a part of a partition. Note that the top entry of each Gaussian polynomial in (1.1) is a fixed odd natural number. With some extra calculations, we find that a similar result holds for fixed even top entries. So we uniformly have the following nonnegativity result.

Theorem 1.1. *All coefficients are nonnegative in the polynomial*

$$\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor - r \end{bmatrix}, \tag{1.2}$$

where $\lfloor x \rfloor$ *is the largest integer not exceeding* x.

The polynomials in (1.1) and (1.2) immediately remind us of the q-unimodality of Gaussian polynomials.

Definition 1.1. We say $f(q) \geq_q g(q)$, or equivalently, $g(q) \leq_q f(q)$, if the polynomial $f(q) - g(q)$ has nonnegative coefficients, for $f(q)$ and $g(q)$ two polynomials in $\mathbb{R}[q]$.

Definition 1.2 (q-Unimodality)**.** For $\mathbf{f} = \{f_k(q)\}_{k \geq 0}$ a sequence of polynomials in $\mathbb{R}[q]$, it is *q-unimodal* if there exists an index N such that

$$f_0(q) \leq_q \cdots \leq_q f_N(q) \geq_q f_{N+1}(q) \geq_q \cdots .$$

The q-unimodality of Gaussian polynomials $\{\begin{bmatrix} n \\ k \end{bmatrix}\}_{k \geq 0}$ for fixed $n \geq 0$ was proved by L. M. Butler [3] using ideas from group theory. In fact, what L. M. Butler had proved in [3, p. 772, Theorem] is much stronger. Let $\alpha_\lambda(k;p)$ be the number of subgroups of order p^k in a finite abelian p-group of type λ where λ is a partition of $n \geq k$. Then for fixed n and λ, the sequence $\{\alpha_\lambda(k;p)\}_{0 \leq k \leq n}$ is p-unimodal. In particular, if $\lambda = (1, 1, \ldots, 1)$, then $\alpha_\lambda(k;p) = \begin{bmatrix} n \\ k \end{bmatrix}_p$.

Now, we present a partition-theoretic treatment of L. M. Butler's q-unimodality result.

Theorem 1.2. *For* $n \geq 0$,

$$\begin{bmatrix} n \\ 0 \end{bmatrix} \leq_q \cdots \leq_q \begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix} \geq_q \cdots \geq_q \begin{bmatrix} n \\ n \end{bmatrix}. \tag{1.3}$$

Our alternative proof of the above theorem requires another property of Gaussian polynomials, namely, the q-log-concavity.

Definition 1.3 (q-Log-concavity)**.** For $\mathbf{f} = \{f_k(q)\}_{k \geq 0}$ a sequence of polynomials in $\mathbb{R}[q]$, it is *q-log-concave* if for all $k \geq 1$, we have $f_k(q)^2 \geq_q f_{k-1}(q) f_{k+1}(q)$. Further, $\mathbf{f}$ is *strongly q-log-concave* if for all $\ell \geq k \geq 1$, we have $f_k(q) f_\ell(q) \geq_q f_{k-1}(q) f_{\ell+1}(q)$.

Consider the sequence $\{\begin{bmatrix} n \\ k \end{bmatrix}\}_{n \geq 0}$ for any fixed nonnegative integer k. B. E. Sagan [6, p. 297, Corollary 3.4(2)] showed by the theory of symmetric functions that it is strongly q-log-concave. The strong q-log-concavity was also stated in [4, p. 62, Section 5] with a partition-theoretic proof. We will use this fact to reprove the q-unimodality of Gaussian polynomials.

On the other hand, it was shown independently by L. M. Butler [4, p. 59, Theorem 4.2], C. Krattenthaler [5, p. 334, Theorem 1] and B. E. Sagan [6, p. 292, Theorem 2.3] that the sequence $\{\begin{bmatrix} n \\ k \end{bmatrix}\}_{k \geq 0}$ is strongly q-log-concave for any fixed nonnegative integer n. Here L. M. Butler's proof is based on partitions and B. E. Sagan's proof is inductive. Moreover, by some set-theoretic constructions, C. Krattenthaler proved the following stronger result: for $a \geq b \geq 0$ and $\ell \geq k \geq 0$, one has

$$\begin{bmatrix} a \\ k \end{bmatrix}\begin{bmatrix} b \\ \ell \end{bmatrix} \geq_q \begin{bmatrix} a \\ k-1 \end{bmatrix}\begin{bmatrix} b \\ \ell+1 \end{bmatrix}. \tag{1.4}$$

We will prove a parallel result of (1.4). Note that letting $\ell = k$ in (1.5) below also yields the strong q-log-concavity of $\{\begin{bmatrix} n \\ k \end{bmatrix}\}_{n\geq 0}$ for fixed k.

Theorem 1.3. *For nonnegative integers a, b, k, ℓ with $\ell \geq k$ and $a + \ell \geq b + k$,*

$$\begin{bmatrix} a \\ k \end{bmatrix}\begin{bmatrix} b \\ \ell \end{bmatrix} \geq_q \begin{bmatrix} a+1 \\ k \end{bmatrix}\begin{bmatrix} b-1 \\ \ell \end{bmatrix}. \tag{1.5}$$

It is worth pointing out that all our proofs use only partition-theoretic ideas.

2. q-Log-concavity

We prove Theorem 1.3 in this section. The main idea follows from [4]. To begin with, we shall recall that the Gaussian polynomial $\begin{bmatrix} m+n \\ m \end{bmatrix}$ is the generating function for partitions with at most m parts and largest part at most n.

Let $\mathscr{P}$ denote the set of integer partitions. Let $\mathscr{P}(m, n)$ denote the set of partitions whose Ferrers diagram fits inside an $m \times n$ rectangle, i.e. partitions with at most m parts and largest part at most n. We are to give a size-preserving injection ϕ from $\mathscr{P}(k, a-k+1) \times \mathscr{P}(\ell, b-\ell-1)$ to $\mathscr{P}(k, a-k) \times \mathscr{P}(\ell, b-\ell)$. Here what we mean by size-preserving is that if $\phi(\lambda, \pi) = (\mu, \nu)$, then $|\lambda| + |\pi| = |\mu| + |\nu|$. With such an injection, (1.5) follows as a direct consequence.

Throughout, let $d \geq 1$ be fixed. We construct an involution $\mathfrak{I}_d$ on $\mathscr{P} \times \mathscr{P}$ as follows. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\pi = (\pi_1, \pi_2, \ldots)$ be two partitions in $\mathscr{P}$. Let $T_d(\lambda, \pi) = T$ be the largest index such that $\lambda_T \geq \pi_{T+1} + d$. If no such T exists, let $T_d(\lambda, \pi) = 0$. Define

$$\mathfrak{I}_d(\lambda, \pi) = (\mu, \nu),$$

where

$$\mu = (\pi_1 + d, \ldots, \pi_T + d, \lambda_{T+1}, \lambda_{T+2}, \ldots)$$

and

$$\nu = (\lambda_1 - d, \ldots, \lambda_T - d, \pi_{T+1}, \pi_{T+2}, \ldots).$$

The fact that $\mathfrak{I}_d$ is indeed an involution was shown by L. M. Butler [4]. We briefly reproduce her proof. If $T = 0$, then $\mathfrak{I}_d(\lambda, \pi) = (\lambda, \pi) \in \mathscr{P} \times \mathscr{P}$. If $T > 0$, then since T is the largest index such that $\lambda_T \geq \pi_{T+1} + d$, we have $\lambda_{T+1} < \pi_{T+2} + d$. Hence, $\lambda_{T+1} < \pi_T + d$ so that $\mu \in \mathscr{P}$. Also, $\lambda_T \geq \pi_{T+1} + d$ implies that $\lambda_T - d \geq \pi_{T+1}$ and hence $\nu \in \mathscr{P}$. Hence, $\mathfrak{I}_d(\lambda, \pi) = (\mu, \nu) \in \mathscr{P} \times \mathscr{P}$. Next, $\mathfrak{I}_d(\mathfrak{I}_d(\lambda, \pi)) = (\lambda, \pi)$ comes from the fact that $T_d(\mu, \nu) = T_d(\lambda, \pi)$. Finally, we have $|\lambda| + |\pi| = |\mu| + |\nu|$ by the definition of $\mathfrak{I}_d$.

Now we show that $\mathfrak{I}_{a-b+\ell-k+1}$ (note that $a-b+\ell-k+1 \geq 1$ by our assumption) is the desired size-preserving injection from $\mathscr{P}(k, a-k+1) \times \mathscr{P}(\ell, b-\ell-1)$ to $\mathscr{P}(k, a-k) \times \mathscr{P}(\ell, b-\ell)$. Let $(\lambda, \pi) \in \mathscr{P}(k, a-k+1) \times \mathscr{P}(\ell, b-\ell-1)$ and $T = T_{a-b+\ell-k+1}(\lambda, \pi)$. Write $(\mu, \nu) = \mathfrak{I}_{a-b+\ell-k+1}(\lambda, \pi)$.

Note that $T \leq k \leq \ell$ for there are at most k parts in λ. Also, we know from the definition of the involution $\mathfrak{I}$ that there are at most k (nonzero) parts in μ and at most ℓ (nonzero) parts in ν.

Next, if $T = 0$, then the largest part in μ is

$$\begin{aligned}\lambda_1 &< \pi_2 + (a - b + \ell - k + 1)\\ &\leq (b - \ell - 1) + (a - b + \ell - k + 1)\\ &= a - k.\end{aligned}$$

Further, the largest part in ν is

$$\begin{aligned}\pi_1 &\leq b - \ell - 1\\ &< b - \ell.\end{aligned}$$

Hence, $(\mu, \nu) \in \mathscr{P}(k, a - k) \times \mathscr{P}(\ell, b - \ell)$. If $T > 0$, then the largest part in μ is

$$\begin{aligned}\pi_1 + (a - b + \ell - k + 1) &\leq (b - \ell - 1) + (a - b + \ell - k + 1)\\ &= a - k,\end{aligned}$$

while the largest part in ν is

$$\begin{aligned}\lambda_1 - (a - b + \ell - k + 1) &\leq (a - k + 1) - (a - b + \ell - k + 1)\\ &= b - \ell.\end{aligned}$$

Hence, the image (μ, ν) is also in $\mathscr{P}(k, a - k) \times \mathscr{P}(\ell, b - \ell)$. We therefore conclude that $\mathfrak{I}_{a-b+\ell-k+1}$ is the desired injection.

3. q-Unimodality

By the symmetry of Gaussian polynomials, to establish Theorem 1.2, it suffices to prove that

$$\begin{bmatrix} n \\ m-1 \end{bmatrix} \leq_q \begin{bmatrix} n \\ m \end{bmatrix}$$

for all m with $1 \leq m \leq n/2$.

Let us consider the partition set $\mathscr{P}(m, n-m)$ containing partitions whose Ferrers diagram fits inside an $m \times (n-m)$ rectangle. Let $\lambda \in \mathscr{P}(m, n-m)$. Since $m \leq n/2$, we know that the size k of the Durfee square (i.e. the largest square that fits inside the Ferrers diagram) of λ is at most m. Further, the part below the Durfee square is a partition contained in an $(m-k) \times k$ rectangle and the part to the right of the Durfee square is a partition contained in a $k \times (n - m - k)$ rectangle. Hence,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{k=0}^{m} q^{k^2} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-m \\ k \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ m-1 \end{bmatrix} = \sum_{k=0}^{m-1} q^{k^2} \begin{bmatrix} m-1 \\ k \end{bmatrix} \begin{bmatrix} n-m+1 \\ k \end{bmatrix},$$

so that

$$\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix} = \sum_{k=0}^{m} q^{k^2} \left(\begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-m \\ k \end{bmatrix} - \begin{bmatrix} m-1 \\ k \end{bmatrix} \begin{bmatrix} n-m+1 \\ k \end{bmatrix} \right). \tag{3.1}$$

Recalling the strong q-log-concavity of the sequence $\{\begin{bmatrix} n \\ k \end{bmatrix}\}_{n\geq 0}$ for fixed $k \geq 0$, we have, for any m with $1 \leq m \leq n/2$,

$$\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n-m \\ k \end{bmatrix} \geq_q \begin{bmatrix} m-1 \\ k \end{bmatrix}\begin{bmatrix} n-m+1 \\ k \end{bmatrix}.$$

In light of (3.1), $\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}$ has nonnegative coefficients.

4. Proof of Theorem 1.1

Note that

$$\begin{aligned}\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^r \begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor - r \end{bmatrix} &= \left(\begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix} - \begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor - 1 \end{bmatrix}\right) \\ &\quad + \left(\begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor - 2 \end{bmatrix} - \begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor - 3 \end{bmatrix}\right) + \cdots \end{aligned} \tag{4.1}$$

We remark that when $\lfloor \frac{n}{2} \rfloor$ is even, there is an extra term $\begin{bmatrix} n \\ 0 \end{bmatrix}$ that cannot be paired but it begins with a plus sign. For each pair on the right-hand side of the above, we know from (1.3) that it is a polynomial with all coefficients nonnegative. Thus, Theorem 1.1 is established.

5. Conclusion

In [2, p. 9, Question II(3)], G. E. Andrews and D. Newman also cried out for a partition-theoretic interpretation of (1.1). We know from the proof of Theorem 1.3 that for $a \geq b$ there is a size-preserving injection from $\mathscr{P}(k, a-k+1) \times \mathscr{P}(k, b-k-1)$ to $\mathscr{P}(k, a-k) \times \mathscr{P}(k, b-k)$. Thus, $\begin{bmatrix} a \\ k \end{bmatrix}\begin{bmatrix} b \\ k \end{bmatrix} - \begin{bmatrix} a+1 \\ k \end{bmatrix}\begin{bmatrix} b-1 \\ k \end{bmatrix}$ corresponds to a subset of $\mathscr{P}(k, a-k) \times \mathscr{P}(k, b-k)$ by excluding the image of $\mathscr{P}(k, a-k+1) \times \mathscr{P}(k, b-k-1)$ under the injection. It turns out that each summand in (3.1),

$$q^{k^2}\left(\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n-m \\ k \end{bmatrix} - \begin{bmatrix} m-1 \\ k \end{bmatrix}\begin{bmatrix} n-m+1 \\ k \end{bmatrix}\right),$$

corresponds to a subset of partitions in $\mathscr{P}(m, n-m)$ with Durfee square of size k. Since the injection is explicitly constructed in Section 2, we also know exactly to which subset of $\mathscr{P}(m, n-m)$, the difference $\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n \\ m-1 \end{bmatrix}$ corresponds. That is to say, we have obtained a clear interpretation of each pair on the right-hand side of (4.1), and thus of (1.2). However, the concrete expression of this interpretation is rather complicated so we would like to ask if there is a simpler correspondence.

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Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 4R2, Canada

E-mail address: chenxiaohang92@gmail.com