MATH 3070 – THEORY OF NUMBERS

Homework 2

Due: Thursday, Sep 29, 2022 (in class)

1. Prove Theorem 3.2.

(i). $a \equiv b \pmod{m}$ if and only if $a - b \equiv 0 \pmod{m}$;

(ii). If $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$, then

$$a_1 + a_2 \equiv b_1 + b_2 \pmod{m},$$
$$a_1 a_2 \equiv b_1 b_2 \pmod{m};$$

(iii). If $a \equiv b \pmod{m}$, then for any positive integer k,

$$a^k \equiv b^k \pmod{m};$$

(iv). If $f(x)$ is a polynomial with integer coefficients, and $u \equiv v \pmod{m}$, then

$$f(u) \equiv f(v) \pmod{m}.$$

(Hint: Write $f(x)$ as $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with all a_i integers.)

2. Let $\phi(n)$ be the Euler totient function.

(i). Compute $\phi(1)$ and $\phi(2)$.

(ii). Let p be a prime and α be a positive integer. Prove that $\phi(p^\alpha)$ is odd only if $p = 2$ and $\alpha = 1$.

(iii). Prove that for integers $n \geq 3$, $\phi(n)$ is even.

3. Let n be a positive integer, and write n in the canonical form $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$.

(i). Prove that for $n \geq 2$,

$$\frac{n}{\phi(n)} = \prod_{i=1}^{r} \frac{p_i}{p_i - 1}.$$

(ii). Prove that if n is odd with $n \geq 3$, then $\frac{n}{\phi(n)}$ is not an integer, i.e., $\phi(n) \nmid n$. (You may use the results in Question 2.)

(iii). Prove that if n has two distinct odd prime factors, then $\phi(n) \nmid n$.

(iv). Prove that if $\phi(n) \mid n$, then $n = 1$ or 2^α $(\alpha \geq 1)$ or $2^\alpha 3^\beta$ $(\alpha, \beta \geq 1)$.