The Seo–Yee conjecture

Nonmodular infinite products, seaweed algebras, and integer partitions

Shane Chern
陈小航

Universität Wien (维也纳大学)

chenxiaohang92@gmail.com

@ Chongqing University
Chongqing
Jul 12, 2024

Leibniz–Bernoulli Correspondence — "*Divulsions*" of integers?

> How many representations are there to write a natural number n as a sum of positive integers if the order of the summands is not taken into account?

LEIBNITII AD BERNOULLIUM.

An unquam conſideraſti numerum diſcerptionum vel divulſionum numeri dati, quot ſcilicet modis poſſit divelli] in partes duas, tres, &c. Videtur mihi ejus determinatio non facilis, & tamen digna quæ habeatur. Dabam *Hanoveræ* 28. Julii 1699.

Deditiſſimus
G. G. Leibnitius.

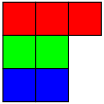
Integer Partitions

Integer partition — A non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$, a given natural number.

- λ_i: *Parts* in the partition;
- n: *Size* of the partition;
- ℓ: *Length* of the partition;
- $p(n)$: Number of partitions of n (☞ *partition function*)

n	$p(n)$	partitions of n
0	1	$\varnothing$
1	1	1
2	2	$2,\ 1+1$
3	3	$3,\ 2+1,\ 1+1+1$
4	5	$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$
5	7	$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,$ $2+1+1+1,\ 1+1+1+1+1$

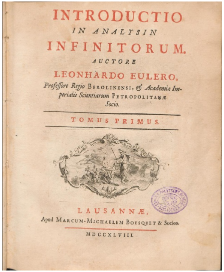
$$7 = 3 + 2 + 2$$



Integer Partitions

Genenrating function (☞ Euler's *Introductio in Analysin Infinitorum*)

$$
\begin{aligned}
\sum_{n\geq 0} p(n)q^n &= (q^{0\cdot 1} + q^{1\cdot 1} + q^{2\cdot 1} + q^{3\cdot 1} + \cdots) \\
&\quad \times (q^{0\cdot 2} + q^{1\cdot 2} + q^{2\cdot 2} + q^{3\cdot 2} + \cdots) \\
&\quad \times (q^{0\cdot 3} + q^{1\cdot 3} + q^{2\cdot 3} + q^{3\cdot 3} + \cdots) \\
&\quad \times \cdots \\
&= \prod_{k\geq 1} (1 + q^k + q^{2k} + q^{3k} + \cdots) \\
&= \prod_{k\geq 1} \frac{1}{1-q^k} = \frac{1}{(q;q)_\infty}.
\end{aligned}
$$


INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, & Academiæ Imperialis Scientiarum PETROPOLITANÆ Socio.
TOMUS PRIMUS.
LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLVIII.

q-Pochhammer symbol $\quad (A;q)_n = \prod_{k=0}^{n-1} (1 - Aq^k)$

Dedekind eta function

$$\eta(\tau) = q^{1/24}(q;q)_\infty$$

with $q = e^{2\pi i\tau}$ where $\tau \in \mathbb{H}$.

Example (☞ *frequency notation*). The partition $3+3+1+1+1$ of 9 corresponds to

$$3+3+1+1+1 = 3\cdot 1 + 0\cdot 2 + 2\cdot 3.$$

Euler's *Pentagonal Number Theorem*

$$\prod_{k\geq 1}(1-q^k) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

Recall that

$$\prod_{k\geq 1} \frac{1}{1-q^k} = \sum_{n\geq 0} p(n) q^n.$$

Hence,

$$(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots) \sum_{n\geq 0} p(n) q^n = 1.$$

Equivalently,

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots.$$

Circle Method

Table IV*: $p(n)$.

1...	1	51...	239943	101...	214481126	151...	45060624582
2...	2	52...	281589	102...	241265379	152...	49686288421
3...	3	53...	329931	103...	271248950	153...	54770336324
4...	5	54...	386155	104...	304801365	154...	60356673280
5...	7	55...	451276	105...	342325709	155...	66493182097
6...	11	56...	526823	106...	384276336	156...	73232243759
7...	15	57...	614154	107...	431149389	157...	80630964769
8...	22	58...	715220	108...	483502844	158...	88751778802
9...	30	59...	831820	109...	541946240	159...	97662728555
10...	42	60...	966467	110...	607163746	160...	107438159466
11...	56	61...	1121505	111...	679903203	161...	118159068427
12...	77	62...	1300156	112...	761002156	162...	129913904637
13...	101	63...	1505499	113...	851376628	163...	142798995930
14...	135	64...	1741630	114...	952050665	164...	156919475295
15...	176	65...	2012558	115...	1064144451	165...	172389800255
16...	231	66...	2323520	116...	1188908248	166...	189334822579
17...	297	67...	2679689	117...	1327710076	167...	207890420102
18...	385	68...	3087735	118...	1482074143	168...	228204732751
19...	490	69...	3554345	119...	1653668665	169...	250438925115
20...	627	70...	4087968	120...	1844349560	170...	274768617130
21...	792	71...	4697205	121...	2056148051	171...	301384802048
22...	1002	72...	5392783	122...	2291320912	172...	330495499613
23...	1255	73...	6185689	123...	2552338241	173...	362326859895
24...	1575	74...	7089500	124...	2841940500	174...	397125074750
25...	1958	75...	8118264	125...	3163127352	175...	435157697830
26...	2436	76...	9289091	126...	3519222692	176...	476715857290
27...	3010	77...	10619863	127...	3913864295	177...	522115831195
28...	3718	78...	12132164	128...	4351078600	178...	571701605655
29...	4565	79...	13848650	129...	4835271870	179...	625846753120
30...	5604	80...	15796476	130...	5371315400	180...	684957390936
31...	6842	81...	18004327	131...	5964539504	181...	749474411781
32...	8349	82...	20506255	132...	6620830889	182...	819876908323
33...	10143	83...	23338469	133...	7346629512	183...	896684817527
34...	12310	84...	26543660	134...	8149040695	184...	980462880430
35...	14883	85...	30167357	135...	9035836076	185...	1071823774337
36...	17977	86...	34262962	136...	10015581680	186...	1171432692373
37...	21637	87...	38887673	137...	11097645016	187...	1280011042268
38...	26015	88...	44108109	138...	12292341831	188...	1398341745571
39...	31185	89...	49995925	139...	13610949895	189...	1527273599625
40...	37338	90...	56634173	140...	15065878135	190...	1667727404093
41...	44583	91...	64112359	141...	16670689208	191...	1820701100652
42...	53174	92...	72533807	142...	18440293320	192...	1987276856363
43...	63261	93...	82010177	143...	20390982757	193...	2168627105469
44...	75175	94...	92669720	144...	22540654445	194...	2366022741845
45...	89134	95...	104651419	145...	24908858009	195...	2580840212973
46...	105558	96...	118114304	146...	27517052599	196...	2814570987591
47...	124754	97...	133230930	147...	30388671978	197...	3068829878530
48...	147273	98...	150198136	148...	33549419497	198...	3345365983698
49...	173525	99...	169229875	149...	37027355200	199...	3646072432125
50...	204226	100...	190569292	150...	40853235313	200...	3972999029388

MacMahon's Table (in 1910s)

$$p(200) = 3,972,999,029,388$$

Theorem (Hardy–Ramanujan, 1918)

As $n \to \infty$,

$$p(n) \sim \frac{1}{4\sqrt{3}} n^{-1} e^{\frac{2\pi\sqrt{n}}{\sqrt{6}}}.$$

Circle Method

The Man Who Knew Infinity (1:11:30):

(M stands for MacMahon and R stands for Ramanujan.)

M: Well, here we are. p(200)*, the moment of truth ...Well, you first. What has your formula given you?*
R: Three trillion and nine hundred and seventy two thousand nine hundred and ninety eight million.
*M: My God! You are close [*silent for 5 seconds*] within two percent. Well, I will be damned.*

Circle Method

- **Cauchy's integral formula**: Suppose $\mathcal{C}$ is a simple closed curve and the function $f(z)$ is analytic on a region containing $\mathcal{C}$ and its interior. if $\mathcal{C}$ is oriented counterclockwise, then for any z_0 inside $\mathcal{C}$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

- $$P(q) := \frac{1}{(q;q)_\infty} = \prod_{k\geq 1} \frac{1}{1-q^k} = \sum_{n\geq 0} p(n) q^n.$$

- $$p(n) = \frac{1}{2\pi i} \oint_{\mathcal{C}:|q|=r} \frac{P(q)}{q^{n+1}}\, dq,$$

  where $0 < r < 1$ (we will choose r to be close to 1 **so that the contour is close to the *poles* of $P(q)$**).

Circle Method

- $$P(q) = \frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3}\frac{1}{1-q^4}\frac{1}{1-q^5}\frac{1}{1-q^6}\cdots$$

  has poles at roots of unity.

- The pole at $q = 1$ is dominant;
  The pole at -1 is $1/2$ as "important" as the pole at 1;
  The pole at primitive cubic roots of unity is $1/3$ as "important";
  ...

Circle Method

- Divide the contour into two parts: one close to $q = 1$ and the other away from $q = 1$. The former gives a main contribution to the contour integral.

- In analytic number theory, we call the arcs in the contour integral that makes a dominant contribution the **major arcs**, and the rest the **minor arcs**.

- For the evaluation for the major arcs, we may utilize the **modular transformation** of the Dedekind eta function.

Circle Method

Hans Rademacher

ON THE PARTITION FUNCTION $p(n)$.

By HANS RADEMACHER.

[Received 30 November, 1936.—Read 10 December, 1936.]

University of Pennsylvania,
Philadelphia.

Theorem (Rademacher, 1937)

$$p(n) = \frac{1}{2\sqrt{2}\pi} \sum_{k \geq 1} A_k(n) \sqrt{k} \, \frac{d}{dn} \left(\frac{2}{\sqrt{n - \frac{1}{24}}} \sinh \left(\frac{\pi}{k} \sqrt{\frac{2}{3} \left(n - \frac{1}{24} \right)} \right) \right),$$

where $A_k(n) = \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} e^{\pi i (s(h,k) - 2nh/k)}$ *with* $s(h,k)$ *the Dedekind sum.*

Circle Method

Let us use the first 8 terms in Rademacher's formula to estimate $p(200)$:

$$\begin{array}{r} +\,3,972,998,993,185.896 \\ +\,36,282.978 \\ -\,87.584 \\ +\,5.147 \\ +\,1.424 \\ +\,0.071 \\ +\,0.000 \\ +\,0.044 \\ \hline 3,972,999,029,387.975 \end{array}$$

Eureka! We are only $.025$ away from the exact value

$$p(200) = 3,972,999,029,388.$$

Circle Method

$$\prod_{j=1}^{J}(q^{m_j};q^{m_j})_\infty^{\delta_j}$$

$$\prod_{j=1}^{J}(q^{r_j},q^{m_j-r_j};q^{m_j})_\infty^{\delta_j}$$

S. Chern, Asymptotics for the Fourier coefficients of eta-quotients, *J. Number Theory* **199** (2019), 168–191.

S. Chern, Asymptotics for the Taylor coefficients of certain infinite products, *Ramanujan J.* **55** (2021), no. 3, 987–1014.

Seo–Yee Conjecture

Conjecture (Seo–Yee, 2019)

The series expansion of

$$\frac{1}{(q,-q^3;q^4)_\infty} = \prod_{k\geq 0} \frac{1}{1-q^{4k+1}} \frac{1}{1+q^{4k+3}}$$

has nonnegative coefficients.

Do the coefficients in the series expansion count something?

☞ **Seaweed Algebra!**

Seo–Yee Conjecture

Conjecture (Seo–Yee, 2019)

The series expansion of

$$\frac{1}{(q,-q^3;q^4)_\infty} = \prod_{k\geq 0} \frac{1}{1-q^{4k+1}} \frac{1}{1+q^{4k+3}}$$

has nonnegative coefficients.

Do the coefficients in the series expansion count something?

☞ **Integer Partitions!**

Seo–Yee Conjecture

- Let λ and μ be two partitions of n. E.g. $\lambda = (3, 2, 1, 1)$ and $\mu = (4, 3)$ are partitions of 7.
- The meander associated to λ and μ:

- The index $\mathrm{ind}_\mu(\lambda) := 2C + P - 1$. Here C and P count the number of cycles and paths in the meander. (Note. Each isolated vertex is treated as a path).

$$\mathrm{ind}_\mu(\lambda) = 2 \times 0 + 2 - 1 = 1.$$

- The case where $\mu = (n)$ corresponds to the **maximal parabolic seaweed algebra**.

Seo–Yee Conjecture

- Let λ and μ be two partitions of n. E.g. $\lambda = (3, 2, 1, 1)$ and $\mu = (4, 3)$ are partitions of 7.
- The meander associated to λ and μ:

- The index $\operatorname{ind}_\mu(\lambda) := 2C + P - 1$. Here C and P count the number of cycles and paths in the meander. (Note. Each isolated vertex is treated as a path).

$$\operatorname{ind}_\mu(\lambda) = 2 \times 0 + 2 - 1 = 1.$$

- The case where $\mu = (n)$ corresponds to the **maximal parabolic seaweed algebra**.

Seo–Yee Conjecture

- $\mathcal{O}$: The set of partitions into odd parts.
- $o(n)$: The number of $\lambda \in \mathcal{O}$ of size n such that $\mathrm{ind}_{(n)}(\lambda)$ is odd.
- $e(n)$: The number of $\lambda \in \mathcal{O}$ of size n such that $\mathrm{ind}_{(n)}(\lambda)$ is even.

Conjecture (Coll–Mayers–Mayers, 2018)

$$\sum_{n\geq 0} |o(n) - e(n)| q^n \stackrel{?}{=} \frac{1}{(q, -q^3; q^4)_\infty}.$$

This conjecture is true up to sign.

Theorem (Seo–Yee, 2019)

$$\sum_{n\geq 0} (-1)^{\lceil \frac{n}{2} \rceil} (o(n) - e(n)) q^n = \frac{1}{(q, -q^3; q^4)_\infty}.$$

Seo–Yee Conjecture

Theorem (C., 2023, *Adv. Math.*)

Let

$$G(q) := \sum_{n\geq 0} g(n)q^n = \frac{1}{(q,-q^3;q^4)_\infty}.$$

We have, as $n\to\infty$,

$$g(n) \sim \frac{\pi^{\frac{1}{4}}\Gamma(\frac{1}{4})}{2^{\frac{9}{4}}3^{\frac{3}{8}}n^{\frac{3}{8}}} I_{-\frac{3}{4}}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + (-1)^n \frac{\pi^{\frac{3}{4}}\Gamma(\frac{3}{4})}{2^{\frac{11}{4}}3^{\frac{5}{8}}n^{\frac{5}{8}}} I_{-\frac{5}{4}}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right),$$

where $I_s(x)$ is the modified Bessel function of the first kind. Further, when $n\geq 2.4\times 10^{14}$, we have $g(n)>0$.

Why is the Seo–Yee Conjecture difficult?

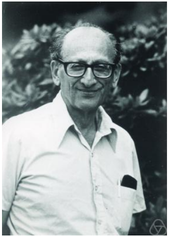
- The infinite product is different from products of Dedekind eta function or Jacobi theta function and indeed it is no longer modular. Hence, a Rademacher-type exact formula is out of reach.
- If we rewrite the product as

$$\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty(q^6;q^8)_\infty},$$

then the numerator $(q^3;q^4)_\infty$ prevents us using a powerful approach of Meinardus, which treats

$$\prod_{k\geq 1}\frac{1}{(1-q^k)^{\delta_k}} \qquad (\delta_k \geq 0).$$

Seo–Yee Conjecture



Emil Grosswald

TRANSACTIONS
OF THE
AMERICAN MATHEMATICAL SOCIETY

VOLUME 89
SEPTEMBER TO DECEMBER, 1958

SOME THEOREMS CONCERNING PARTITIONS(1)

BY
EMIL GROSSWALD

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

$$\frac{1}{(q^a;q^p)_\infty} = \prod_{k\geq 0} \frac{1}{1-q^{kp+a}} \qquad (p \text{ prime})$$

- Incorrect evaluation of residues
- Mistakes on the uniformness of error terms
- Prime moduli

Seo–Yee Conjecture

Let M be a positive integer and a be any of $1, 2, \ldots, M$. We shall investigate the asymptotic behavior of

$$\Phi_{a,M}(q) := \log\left(\frac{1}{(q^a; q^M)_\infty}\right)$$

when the complex variable q with $|q| < 1$ approaches an arbitrary root of unity:

- $|q| \to 1^-$;
- $\mathrm{Arg}(q) \approx \frac{2\pi h}{k}$.

$$\boxed{q := e^{-\tau + \frac{2\pi i h}{k}}}$$

- $\tau := X^{-1} + 2\pi i Y$ with $|Y| \le \frac{1}{kN}$;
- $1 \le h \le k \le \lfloor \sqrt{2\pi X} \rfloor =: N$ with $(h, k) = 1$.

Why do we choose $|Y| \leq \frac{1}{kN}$?

A covering of $\mathbb{R}/\mathbb{Z}$

Let $\mathcal{Q}_{h/k}$ be the set of q with respect to h/k as defined before, that is,

$$\mathcal{Q}_{h/k} := \left\{ e^{-\frac{1}{X}+2\pi i\left(\frac{h}{k}-Y\right)} : |Y| \leq \tfrac{1}{kN} \right\}.$$

For any q with $|q| = e^{-\frac{1}{X}}$, we are always able to find a certain h/k such that $q \in \mathcal{Q}_{h/k}$. This is a direct consequence of Dirichlet's approximation theorem, asserting that $\mathbb{R}/\mathbb{Z}$ can be covered by intervals

$$\bigcup_{\substack{1\leq h\leq k\leq N\\ (h,k)=1}} \left[\frac{h}{k}-\frac{1}{kN}, \frac{h}{k}+\frac{1}{kN}\right].$$

Seo–Yee Conjecture

Theorem (C., 2023, *Adv. Math.*)

Let $X \geq 16$ *be a sufficiently large positive number. Let* $q := e^{-\tau + \frac{2\pi i h}{k}}$ *where* $1 \leq h \leq k \leq \lfloor \sqrt{2\pi X} \rfloor =: N$ *with* $(h,k) = 1$ *and* $\tau := X^{-1} + 2\pi i Y$ *with* $|Y| \leq \frac{1}{kN}$. *Let* M *be a positive integer and* a *be any of* $1, 2, \ldots, M$. *If we denote by* b *the unique integer between* 1 *and* (k,M) *such that* $b \equiv -ha \pmod{(k,M)}$ *and write*

$$b^* := \begin{cases} (k,M) - b & \text{if } b \neq (k,M), \\ (k,M) & \text{if } b = (k,M), \end{cases}$$

then

$$\log\left(\frac{1}{(q^a;q^M)_\infty}\right) = \frac{1}{\tau}\frac{(k,M)^2}{k^2 M}\Bigg[\pi^2\left(\frac{b^2}{(k,M)^2} - \frac{b}{(k,M)} + \frac{1}{6}\right) + 2\pi i\left(-\zeta'\left(-1, \frac{b}{(k,M)}\right) + \zeta'\left(-1, \frac{b^*}{(k,M)}\right)\right)\Bigg] + E,$$

where

$$|\Re(E)| \ll_M X^{\frac{1}{2}} \log X.$$

Seo–Yee Conjecture

- $$\Phi_{a,M}(q) = \log\left(\frac{1}{(q^a;q^M)_\infty}\right) = \sum_{\substack{m\ge 1\\ m\equiv a \bmod M}} \sum_{\ell\ge 1} \frac{q^{\ell m}}{\ell}.$$
- Classify ℓ and m with the same contribution to $e^{\frac{2\pi i h \ell m}{k}}$. Recall that
$$q = e^{-\tau + \frac{2\pi i h}{k}}.$$
- Write
$$\ell = rk + \mu \quad (1 \le \mu \le k) \quad \& \quad m = tK + \lambda \quad (1 \le \lambda \le K,\ \lambda \equiv a \bmod M),$$
where $K = k\frac{M}{(k,M)}$. Then
$$\Phi_{a,M}(q) = \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} e^{\frac{2\pi i h \mu\lambda}{k}} \sum_{r,t\ge 0} \frac{1}{rk+\mu} e^{-(rk+\mu)(tK+\lambda)\tau}.$$

Seo–Yee Conjecture

- Applying the inverse Mellin transform $e^{-t} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)t^{-s}\,ds$,

$$\Phi_{a,M}(q) = \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} e^{\frac{2\pi i h\mu\lambda}{k}} \frac{1}{2\pi i}\int_{(\frac{3}{2})} \frac{\Gamma(s)}{\tau^s k^{s+1} K^s}\zeta\left(s,\frac{\lambda}{K}\right)\zeta\left(1+s,\frac{\mu}{k}\right)ds.$$

Here the path of integration (α) is from $\alpha - i\infty$ to $\alpha + i\infty$.

- Recall the functional equation of Hurwitz zeta function

$$\zeta\left(s,\frac{\lambda}{\kappa}\right) = 2\Gamma(1-s)(2\pi\kappa)^{s-1}\left(\sin\frac{\pi s}{2}\sum_{1\le\nu\le\kappa}\cos\frac{2\pi\lambda\nu}{\kappa}\,\zeta\left(1-s,\frac{\nu}{\kappa}\right)\right.$$
$$\left.+\cos\frac{\pi s}{2}\sum_{1\le\nu\le\kappa}\sin\frac{2\pi\lambda\nu}{\kappa}\,\zeta\left(1-s,\frac{\nu}{\kappa}\right)\right),$$

and Euler's reflection formula for the Gamma function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} = \frac{\pi}{2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}}.$$

Seo–Yee Conjecture

- Let $z = \frac{\tau k}{2\pi}$.

$$
\begin{aligned}
&\Phi_{a,M}(q)\\
&= \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi h\mu\lambda}{k} \cos\frac{2\pi\nu\lambda}{K} \int_{(\frac{3}{2})} \frac{\zeta\left(1+s,\frac{\mu}{k}\right)\zeta\left(1-s,\frac{\nu}{K}\right)}{z^s\cos\frac{\pi s}{2}} ds\\
&\quad + \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi h\mu\lambda}{k} \sin\frac{2\pi\nu\lambda}{K} \int_{(\frac{3}{2})} \frac{\zeta\left(1+s,\frac{\mu}{k}\right)\zeta\left(1-s,\frac{\nu}{K}\right)}{z^s\sin\frac{\pi s}{2}} ds\\
&\quad + \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi h\mu\lambda}{k} \sin\frac{2\pi\nu\lambda}{K} \int_{(\frac{3}{2})} \frac{\zeta\left(1+s,\frac{\mu}{k}\right)\zeta\left(1-s,\frac{\nu}{K}\right)}{z^s\sin\frac{\pi s}{2}} ds\\
&\quad + \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi h\mu\lambda}{k} \cos\frac{2\pi\nu\lambda}{K} \int_{(\frac{3}{2})} \frac{\zeta\left(1+s,\frac{\mu}{k}\right)\zeta\left(1-s,\frac{\nu}{K}\right)}{z^s\cos\frac{\pi s}{2}} ds.
\end{aligned}
$$

- Replacing s by $-s$, reversing the direction of the integration path and shifting the path back to $(\frac{3}{2})$, one has, with $\rho \equiv -h\lambda \pmod{k}$,

$$\begin{aligned}
\Phi_{a,M}(q) &= \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&\quad - \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&\quad + \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&\quad - \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&\quad - 2\pi i(R_1+R_2+R_3+R_4)\\
&=: \Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4 - 2\pi i(R_1+R_2+R_3+R_4),
\end{aligned}$$

where the R_j's come from the sum of residues of the corresponding integrand inside the strip $-\frac{3}{2} < \Re(s) < \frac{3}{2}$.

Residues

- The residues are essentially from
$$\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\operatorname{trig}\frac{\pi s}{2}},$$
where the trig function is $\cos$ or $\sin$:
$$\mathcal{R}_{\cos}:=\sum_{|\Re(s)|<\frac{3}{2}}\operatorname{Res}_s\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\cos\frac{\pi s}{2}}=\operatorname*{Res}_{s=0}(*)+\operatorname*{Res}_{s=-1}(*)+\operatorname*{Res}_{s=1}(*)$$
$$=-\log z-\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right)+\frac{2\zeta\left(2,\frac{\mu}{k}\right)\zeta\left(0,\frac{\nu}{K}\right)}{\pi z}-\frac{2z\zeta\left(0,\frac{\mu}{k}\right)\zeta\left(2,\frac{\nu}{K}\right)}{\pi},$$
and that
$$\mathcal{R}_{\sin}:=\sum_{|\Re(s)|<\frac{3}{2}}\operatorname{Res}_s\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\sin\frac{\pi s}{2}}=\operatorname*{Res}_{s=0}(*)$$
$$=-\frac{\pi}{12}-\frac{(\log z)^2}{\pi}-\frac{2\log z}{\pi}\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)+\frac{2\log z}{\pi}\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right)$$
$$+\frac{2}{\pi}\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right)+\frac{2}{\pi}\gamma_1\left(\frac{\mu}{k}\right)+\frac{2}{\pi}\gamma_1\left(\frac{\nu}{K}\right).$$

Residues

- $R_1 = R_{11} + R_{12} + R_{13} + R_{14}$:

$$-2\pi i R_{11} = \frac{1}{\tau}\frac{\pi^2}{6k^2M}\big(6b^2 - 6b(k,M) + (k,M)^2\big),$$

  and

$$|\Re(-2\pi i R_{12})| \le \tfrac{1}{24}MX^{-1} \ll X^{-1},$$
$$|\Re(-2\pi i R_{13})| \le \tfrac{1}{2}\log X + 0.92 \ll \log X,$$
$$|\Re(-2\pi i R_{14})| \le \tfrac{1}{2}\tfrac{M}{(k,M)} \ll 1.$$

- $R_2 = R_{21} + R_{22} + R_{23}$:

$$|\Re(-2\pi i R_{21})| \le 0.44X^{\frac{1}{2}}\log X + 1.3X^{\frac{1}{2}} + 0.25\log X + 0.75 \ll X^{\frac{1}{2}}\log X,$$
$$|\Re(-2\pi i R_{22})| \le \tfrac{1}{2}\log X + 0.92 \ll \log X,$$
$$|\Re(-2\pi i R_{23})| \le \tfrac{1}{4}\log X + \tfrac{1}{2}\tfrac{M}{(k,M)} + \tfrac{1}{2}\log\tfrac{M}{(k,M)} + \log\Gamma\big(\tfrac{(k,M)}{M}\big) + 2.59 \ll \log X.$$

Seo–Yee Conjecture

Residues

- $$|\Re(-2\pi i R_3)| = 0.$$
- $R_4 = R_{41} + R_{42}$:
$$-2\pi i R_{41} = \begin{cases} 0 & \text{if } b = (k, M), \\ -\frac{1}{\tau}\frac{(k,M)^2}{M}\frac{2\pi i}{k^2}\left(\zeta'\left(-1, \frac{b}{(k,M)}\right) - \zeta'\left(-1, \frac{(k,M)-b}{(k,M)}\right)\right) & \text{if } b \neq (k, M), \end{cases}$$
and
$$|\Re(-2\pi i R_{42})| \leq \frac{1}{12}\frac{M}{(k,M)}\log X + 0.25\frac{M}{(k,M)} \ll \log X.$$

Seo–Yee Conjecture

Shifted integrals

- **Trouble arising from $|Y| \leq \frac{1}{kN}$ (where $N = \lfloor \sqrt{2\pi X} \rfloor$):**

$$\int_{(\frac{3}{2})} \frac{\zeta\left(1-s, \frac{\mu}{k}\right) \zeta\left(1+s, \frac{\nu}{K}\right)}{z^{-s} \cos \frac{\pi s}{2}} ds.$$

Write $s = \frac{3}{2} + it$. Then

$$|\text{integrand}| \ll |z|^{\frac{3}{2}} |t|^{C} \exp\left(\left(-\frac{\pi}{2} + |\operatorname{Arg}(z)| \right) |t| \right).$$

Recall that $z = \frac{\tau k}{2\pi}$ and $\tau = X^{-1} + 2\pi i Y$ so that $\operatorname{Arg}(z) = \operatorname{Arg}(\tau)$.

  - **Usual choice of Y:** $|Y| \leq cX^{-1} \Rightarrow |\operatorname{Arg}(z)| \leq \theta < \frac{\pi}{2}$
  - **Our choice of Y:** $|Y| \leq \frac{1}{kN} \Rightarrow |\operatorname{Arg}(z)|$ can be arbitrarily close to $\frac{\pi}{2}$ as $X \to \infty$ for small k

Shifted integrals

- Introduce an auxiliary function

$$\Psi_{a,M}(q^*) := \log\left(\prod_{\substack{m\geq 1\\ m\equiv -ha \bmod M^*}} \frac{1}{1-e^{\frac{2\pi i\alpha a}{M}}(q^*)^m}\right),$$

where $M^* = (k, M)$, α and β are such that $\alpha k + \beta M = M^*$, h' is such that $hh' \equiv -1 \pmod{k}$, and $q^* := \exp\left(\frac{2\pi i\beta h'}{k} - \frac{2\pi}{Kz}\right)$.

Shifted integrals

- $$\begin{aligned}
&\Psi_{a,M}(q^*)\\
&= \frac{1}{4\pi i k K} \sum_{\substack{1\leq\lambda\leq K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\leq\mu\leq k\\ 1\leq\nu\leq K}} \cos\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})}\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&\quad + \frac{1}{4\pi i k K} \sum_{\substack{1\leq\lambda\leq K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\leq\mu\leq k\\ 1\leq\nu\leq K}} \sin\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})}\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&\quad + \frac{1}{4\pi k K} \sum_{\substack{1\leq\lambda\leq K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\leq\mu\leq k\\ 1\leq\nu\leq K}} \sin\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})}\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&\quad + \frac{1}{4\pi k K} \sum_{\substack{1\leq\lambda\leq K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\leq\mu\leq k\\ 1\leq\nu\leq K}} \cos\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(\frac{3}{2})}\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Seo–Yee Conjecture

Shifted integrals

- $$\Upsilon_1 = J_1 \qquad \text{and} \qquad \Upsilon_3 = J_3.$$
- $$2(J_1 + J_3) = \Psi_{a,M}(q^*) + \Psi_{M-a,M}(q^*).$$
- For Υ_2 and Υ_4, we define
$$\Upsilon_* \pm J_* := \begin{cases} \Upsilon_* + J_* & \text{if } \Im(z) \geq 0, \\ \Upsilon_* - J_* & \text{if } \Im(z) < 0. \end{cases}$$
- $$\begin{aligned} |\Re(\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4)| &\leq |\Re(\Upsilon_1 + \Upsilon_3)| + |\Re(\Upsilon_2 + \Upsilon_4)| \\ &\leq |\Re(J_1 + J_3)| + |\Re(J_2 + J_4)| + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4| \\ &\leq |\Re(\Psi_{a,M}(q^*))| + 2|\Re(J_1 + J_3)| + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4| \\ &\leq 2|\Re(\Psi_{a,M}(q^*))| + |\Re(\Psi_{M-a,M}(q^*))| + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4|. \end{aligned}$$

Shifted integrals

- $$\begin{aligned}|\Re(\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4)| &\le \frac{3e^{-0.28\pi^2\frac{(k,M)}{M}}}{\left(1-e^{-0.28\pi^2\frac{(k,M)}{M}}\right)^2}+13.02\frac{M^{\frac{3}{2}}}{(k,M)^{\frac{5}{2}}}X^{\frac{1}{2}}\\ &\ll X^{\frac{1}{2}}.\end{aligned}$$

Seo–Yee Conjecture

Observation:

$$G(q)=\frac{1}{(q,-q^3;q^4)_\infty}=\frac{1}{1-q}\frac{1}{1+q^3}\frac{1}{1-q^5}\frac{1}{1+q^7}\cdots$$

So $G(q)$ is mainly dominated at $q=\pm 1$.

Major arcs Close to $q=\pm 1$

Minor arcs Away from $q=\pm 1$

Recall that

$$\mathcal{Q}_{h/k}:=\left\{e^{-\frac{1}{X}+2\pi i(\frac{h}{k}-Y)}\ :\ |Y|\le \tfrac{1}{kN}\right\}, \qquad (N:=\lfloor\sqrt{2\pi X}\rfloor).$$

Major arcs "Part of $\mathcal{Q}_{1/1}$" plus "Part of $\mathcal{Q}_{1/2}$"

Minor arcs "Rest of $\mathcal{Q}_{1/1}$ & $\mathcal{Q}_{1/2}$" plus "Other $\mathcal{Q}_{h/k}$"

Seo–Yee Conjecture

Theorem

For any q *with* $|q| = e^{-\frac{1}{X}}$ *such that it is not in* $\mathcal{Q}_{1/1}$ *and* $\mathcal{Q}_{1/2}$*, we have, if* $X \geq 3.4 \times 10^7$*, then*

$$|G(q)| \leq \exp\left(\left(\frac{\pi^2}{48} - \frac{1}{100}\right) X\right).$$

Also, if $q = e^{-\tau + \frac{2\pi i h}{k}}$ *with* $\tau = X^{-1} + 2\pi i Y$ *is in* $\mathcal{Q}_{1/1}$ *or* $\mathcal{Q}_{1/2}$*, then the above bound still holds under the assumption* $X \geq 3.4 \times 10^7$ *provided that* $|Y| \geq \frac{1}{2\pi X}$*.*

Theorem

Let $\tau = X^{-1} + 2\pi i Y$ *with* $|Y| \leq \frac{1}{2\pi X}$. *Then*

$$\log G(e^{-\tau}) = \frac{\pi^2}{48}\frac{1}{\tau} - \frac{1}{4}\log\tau - \frac{3}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma(\tfrac{1}{4}) + E_+,$$

where

$$|E_+| \leq 0.66 X^{-\frac{3}{4}}.$$

Also,

$$\log G(-e^{-\tau}) = \frac{\pi^2}{48}\frac{1}{\tau} + \frac{1}{4}\log\tau - \frac{1}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma(\tfrac{3}{4}) + E_-,$$

where

$$|E_-| \leq 0.82 X^{-\frac{3}{4}}.$$

Outlook

Conjecture (Seo–Yee, 2019)

The series expansion of

$$\frac{1}{(q, -q^{m-1}; q^m)_\infty}$$

has nonnegative coefficients whenever $m \geq 4$.

Conjecture (C., 2018(?))

The series expansion of

$$\frac{(q^{m-1}; q^{2m})_\infty}{(q; q^m)_\infty}$$

has nonnegative coefficients whenever $m \geq 1$.

I formulated this conjecture when reading a paper of Song Heng Chan and Hamza Yesilyurt on Ramanujan's continued fraction $(q^2; q^3)_\infty/(q; q^3)_\infty$.

Thank You!