Multi-headed Lattice Green Function (N = 5, M = 4)
Find minimal recurrence for the coefficients

In[ ]:=
```
NN = 5;
MM = 4;
```

Generate a sequence from recurrence & initial values
Koutschan’s implementation

In[ ]:=
```
(* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals, Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

In[ ]:=
```
<< RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

In[ ]:=
```
ClearAll[z, w, α, β];
```

Import our REC for {r(0), r(1), r(2), ...}

In[ ]:=
```
ClearAll[Seq];
SeqNormalized =
```

$$\left(-2\,364\,822\,061\,925\,891\,270\,067\,722\,649\,600\,000 - 24\,311\,763\,241\,480\,737\,290\,507\,853\,496\,320\,000\,\alpha - 118\,884\,714\,388\,336\,585\,062\,289\,753\,767\,936\,000\,\alpha^2 - 368\,251\,136\,151\,853\,255\,846\,369\,719\,798\,988\,800\,\alpha^3 - 811\,793\,640\,582\,985\,414\,140\,746\,797\,028\,474\,880\,\alpha^4 - 1\,356\,499\,120\,040\,750\,577\,583\,138\,444\,526\,223\,360\,\alpha^5 - 1\,786\,835\,040\,377\,781\,128\,110\,811\,754\,937\,712\,640\,\alpha^6 - 1\,904\,958\,007\,246\,824\,509\,445\,186\,467\,125\,002\,240\,\alpha^7 - 1\,674\,545\,402\,297\,600\,373\,785\,511\,713\,251\,000\,320\,\alpha^8 - 1\,230\,194\,808\,706\,317\,371\,163\,067\,050\,208\,788\,480\,\alpha^9 -\right.$$

$762 791 807 513 049 677 466 384 009 532 538 880\, \alpha^{10} -$
$402 079 430 499 218 110 643 393 128 200 929 280\, \alpha^{11} -$
$181 085 303 893 806 582 831 390 648 576 245 760\, \alpha^{12} -$
$69 909 566 044 762 687 837 271 137 604 075 520\, \alpha^{13} - 23 174 037 389 797 607 720 091 614 796 840 960$
$\alpha^{14} - 6 597 237 647 955 223 324 018 009 760 071 680\, \alpha^{15} -$
$1 610 851 715 462 724 269 782 004 410 613 760\, \alpha^{16} - 336 382 193 033 012 242 367 855 858 810 880\, \alpha^{17} -$
$59 795 770 083 083 316 221 336 805 703 680\, \alpha^{18} - 8 987 061 025 545 721 077 834 511 810 560\, \alpha^{19} -$
$1 131 237 375 988 193 565 613 353 861 120\, \alpha^{20} - 117 704 523 870 056 936 584 154 972 160\, \alpha^{21} -$
$9 941 030 662 497 120 749 554 237 440\, \alpha^{22} - 664 040 244 922 741 425 721 835 520\, \alpha^{23} -$
$33 746 986 442 943 554 031 452 160\, \alpha^{24} - 1 225 566 587 608 656 091 545 600\, \alpha^{25} -$
$28 320 365 528 012 449 382 400\, \alpha^{26} - 312 808 771 118 086 225 920\, \alpha^{27}\Big)\ \mathrm{Seq}[\alpha] +$
$\Big(-880 540 948 213 763 261 498 004 602 880 000 - 8 086 612 414 279 581 582 690 097 299 456 000\, \alpha -$
$35 535 843 625 080 580 938 628 852 403 404 800\, \alpha^{2} - 99 482 199 073 846 865 130 149 987 053 731 840$
$\alpha^{3} - 199 278 215 238 194 877 084 174 219 759 058 944\, \alpha^{4} -$
$304 147 288 569 704 121 767 283 668 058 636 288\, \alpha^{5} - 367 726 422 460 034 552 713 877 456 306 307 072$
$\alpha^{6} - 361 508 986 147 801 089 153 130 211 095 805 952\, \alpha^{7} -$
$294 331 319 744 750 632 422 172 167 712 997 376\, \alpha^{8} - 201 108 607 972 501 732 293 906 606 562 934 784$
$\alpha^{9} - 116 437 788 942 848 727 536 075 769 222 856 704\, \alpha^{10} -$
$57 524 299 296 878 619 402 424 939 339 382 784\, \alpha^{11} - 24 367 165 878 769 872 656 509 536 747 061 248$
$\alpha^{12} - 8 877 402 295 660 764 714 512 245 808 234 496\, \alpha^{13} -$
$2 785 748 984 068 408 698 625 918 477 467 648\, \alpha^{14} - 752 972 653 647 501 430 958 086 738 673 664\, \alpha^{15} -$
$175 049 743 314 674 169 771 167 299 534 848\, \alpha^{16} - 34 895 534 864 837 208 484 258 292 957 184\, \alpha^{17} -$
$5 936 277 532 573 962 980 718 997 929 984\, \alpha^{18} - 855 818 515 821 739 179 539 429 326 848\, \alpha^{19} -$
$103 560 073 600 267 246 364 541 321 216\, \alpha^{20} - 10 380 185 487 431 012 018 005 475 328\, \alpha^{21} -$
$846 180 664 706 397 472 693 420 032\, \alpha^{22} - 54 656 640 176 185 180 963 209 216\, \alpha^{23} -$
$2 690 612 916 385 314 156 576 768\, \alpha^{24} - 94 804 345 329 795 433 758 720\, \alpha^{25} -$
$2 128 785 749 082 227 343 360\, \alpha^{26} - 22 881 382 331 785 936 896\, \alpha^{27}\Big)\ \mathrm{Seq}[1 + \alpha] +$
$\Big(664 078 540 666 702 251 488 371 015 680 000 + 5 805 956 958 011 506 960 041 778 348 032 000\, \alpha +$
$24 298 272 789 380 152 495 188 221 126 246 400\, \alpha^{2} + 64 810 405 629 301 547 428 216 819 254 558 720$
$\alpha^{3} + 123 755 374 367 469 269 296 809 845 353 611 264\, \alpha^{4} +$
$180 149 375 502 996 189 202 275 648 542 982 144\, \alpha^{5} + 207 865 771 244 125 682 287 781 841 861 722 112$
$\alpha^{6} + 195 153 222 041 523 657 876 484 723 267 989 504\, \alpha^{7} +$
$151 846 270 858 495 120 363 896 477 860 167 680\, \alpha^{8} + 99 230 231 828 276 421 932 960 434 682 314 752$
$\alpha^{9} + 54 993 115 047 787 497 911 079 580 675 899 392\, \alpha^{10} +$
$26 028 017 908 489 825 928 212 462 245 453 824\, \alpha^{11} + 10 572 113 416 646 586 933 511 582 698 766 336$
$\alpha^{12} + 3 696 722 231 163 815 760 173 082 026 344 448\, \alpha^{13} +$
$1 114 468 173 061 041 282 670 805 399 093 248\, \alpha^{14} + 289 688 969 845 746 113 335 461 572 931 584\, \alpha^{15} +$
$64 831 091 647 102 802 811 533 842 055 168\, \alpha^{16} + 12 454 053 009 083 005 771 527 566 163 968\, \alpha^{17} +$
$2 043 760 292 966 696 499 523 264 184 320\, \alpha^{18} + 284 532 912 366 921 324 027 166 588 928\, \alpha^{19} +$
$33 284 416 956 384 385 896 458 223 616\, \alpha^{20} + 3 228 606 478 351 534 833 828 626 432\, \alpha^{21} +$
$254 974 947 491 313 890 128 560 128\, \alpha^{22} + 15 972 126 457 377 261 067 698 176\, \alpha^{23} +$
$763 333 007 662 980 725 211 136\, \alpha^{24} + 26 138 887 552 462 651 129 856\, \alpha^{25} +$
$570 997 443 951 748 710 400\, \alpha^{26} + 5 976 795 675 008 958 464\, \alpha^{27}\Big)\ \mathrm{Seq}[2 + \alpha] +$
$\Big(-36 337 840 931 616 555 318 702 833 664 000 - 310 343 693 247 202 072 877 171 431 833 600\, \alpha -$
$1 268 062 726 217 635 641 408 454 051 430 400\, \alpha^{2} - 3 300 521 955 790 071 740 463 976 232 263 680\, \alpha^{3} -$
$6 146 984 578 367 464 065 862 054 879 242 240\, \alpha^{4} - 8 723 512 529 514 925 026 222 139 080 468 480\, \alpha^{5} -$
$9 808 817 646 565 897 068 529 809 213 239 808\, \alpha^{6} - 8 970 447 157 798 999 809 214 350 039 412 224\, \alpha^{7} -$
$6 796 618 106 855 403 262 931 535 421 469 184\, \alpha^{8} - 4 323 600 610 674 086 572 350 145 316 732 416\, \alpha^{9} -$
$2 331 860 127 398 843 166 087 931 718 971 904\, \alpha^{10} -$
$1 073 804 990 271 736 796 663 841 511 156 224\, \alpha^{11} -$
$424 279 297 446 148 516 898 147 199 947 264\, \alpha^{12} - 144 293 344 557 135 741 340 883 292 465 664\, \alpha^{13} -$

$$
\begin{aligned}
&42\,304\,696\,119\,152\,808\,149\,756\,544\,291\,840\,\alpha^{14} - 10\,693\,366\,157\,119\,575\,923\,154\,101\,714\,944\,\alpha^{15} - \\
&2\,327\,102\,570\,668\,214\,059\,453\,238\,664\,192\,\alpha^{16} - 434\,708\,874\,971\,795\,823\,099\,840\,116\,736\,\alpha^{17} - \\
&69\,373\,988\,097\,051\,870\,247\,906\,934\,784\,\alpha^{18} - 9\,393\,304\,762\,567\,159\,143\,035\,764\,736\,\alpha^{19} - \\
&1\,068\,815\,757\,774\,279\,757\,481\,902\,080\,\alpha^{20} - 100\,861\,570\,825\,855\,881\,262\,923\,776\,\alpha^{21} - \\
&7\,750\,770\,733\,439\,394\,600\,976\,384\,\alpha^{22} - 472\,551\,963\,878\,997\,639\,561\,216\,\alpha^{23} - \\
&21\,986\,541\,883\,647\,884\,001\,280\,\alpha^{24} - 733\,188\,729\,988\,561\,502\,208\,\alpha^{25} - \\
&15\,602\,375\,112\,618\,147\,840\,\alpha^{26} - 159\,149\,910\,074\,064\,896\,\alpha^{27}\Big)\ \mathrm{Seq}[3+\alpha] + \\
&\Big(-1\,737\,772\,868\,400\,007\,324\,872\,130\,560\,000 - 14\,528\,609\,204\,414\,291\,845\,066\,255\,564\,800\,\alpha - \\
&58\,083\,087\,258\,852\,534\,411\,685\,975\,019\,520\,\alpha^{2} - 147\,846\,850\,915\,658\,722\,383\,612\,355\,430\,400\,\alpha^{3} - \\
&269\,164\,023\,324\,400\,460\,962\,054\,275\,740\,928\,\alpha^{4} - 373\,240\,816\,513\,597\,979\,905\,593\,440\,661\,888\,\alpha^{5} - \\
&409\,908\,879\,949\,766\,514\,326\,399\,060\,864\,064\,\alpha^{6} - 366\,016\,393\,873\,249\,701\,940\,597\,734\,061\,344\,\alpha^{7} - \\
&270\,676\,671\,846\,416\,971\,917\,873\,052\,917\,920\,\alpha^{8} - 168\,013\,318\,310\,785\,666\,403\,759\,927\,887\,584\,\alpha^{9} - \\
&88\,393\,926\,598\,940\,439\,065\,183\,725\,045\,600\,\alpha^{10} - 39\,697\,363\,634\,496\,672\,642\,069\,844\,386\,912\,\alpha^{11} - \\
&15\,293\,672\,611\,896\,263\,618\,803\,193\,519\,136\,\alpha^{12} - 5\,070\,491\,874\,452\,377\,148\,797\,920\,831\,072\,\alpha^{13} - \\
&1\,449\,002\,022\,519\,967\,409\,403\,051\,116\,512\,\alpha^{14} - 356\,957\,682\,436\,813\,381\,749\,659\,746\,304\,\alpha^{15} - \\
&75\,700\,244\,148\,872\,939\,301\,421\,992\,640\,\alpha^{16} - 13\,779\,371\,789\,456\,905\,170\,877\,563\,840\,\alpha^{17} - \\
&2\,142\,685\,081\,818\,193\,152\,012\,367\,872\,\alpha^{18} - 282\,685\,926\,147\,777\,894\,282\,083\,328\,\alpha^{19} - \\
&31\,341\,335\,886\,140\,485\,043\,322\,880\,\alpha^{20} - 2\,881\,942\,426\,887\,984\,021\,438\,464\,\alpha^{21} - \\
&215\,812\,414\,752\,103\,173\,455\,872\,\alpha^{22} - 12\,823\,036\,513\,484\,289\,343\,488\,\alpha^{23} - \\
&581\,508\,878\,853\,457\,575\,936\,\alpha^{24} - 18\,903\,053\,117\,719\,314\,432\,\alpha^{25} - \\
&392\,186\,219\,850\,629\,120\,\alpha^{26} - 3\,900\,964\,176\,134\,144\,\alpha^{27}\Big)\ \mathrm{Seq}[4+\alpha] + \\
&\Big(36\,446\,102\,109\,669\,030\,849\,285\,120\,000 + 301\,794\,930\,778\,773\,719\,063\,321\,856\,000\,\alpha + \\
&1\,194\,401\,836\,156\,084\,887\,609\,064\,224\,000\,\alpha^{2} + 3\,008\,156\,975\,709\,477\,795\,289\,491\,275\,520\,\alpha^{3} + \\
&5\,415\,770\,546\,395\,539\,670\,222\,530\,489\,360\,\alpha^{4} + 7\,422\,453\,554\,874\,065\,600\,190\,474\,289\,032\,\alpha^{5} + \\
&8\,052\,206\,383\,842\,449\,223\,124\,682\,104\,644\,\alpha^{6} + 7\,098\,162\,826\,794\,167\,361\,280\,152\,144\,294\,\alpha^{7} + \\
&5\,179\,144\,111\,408\,801\,590\,076\,035\,892\,950\,\alpha^{8} + 3\,169\,950\,795\,733\,038\,711\,522\,140\,215\,280\,\alpha^{9} + \\
&1\,643\,499\,248\,947\,095\,475\,104\,215\,404\,004\,\alpha^{10} + 726\,910\,788\,718\,026\,537\,302\,273\,862\,144\,\alpha^{11} + \\
&275\,635\,972\,025\,251\,416\,199\,969\,761\,656\,\alpha^{12} + 89\,889\,728\,147\,001\,421\,773\,544\,625\,132\,\alpha^{13} + \\
&25\,251\,994\,806\,501\,150\,584\,061\,125\,784\,\alpha^{14} + 6\,111\,409\,098\,652\,595\,993\,659\,452\,026\,\alpha^{15} + \\
&1\,272\,483\,225\,563\,071\,816\,917\,699\,490\,\alpha^{16} + 227\,273\,250\,419\,552\,627\,170\,585\,084\,\alpha^{17} + \\
&34\,655\,941\,701\,831\,856\,557\,922\,624\,\alpha^{18} + 4\,480\,880\,404\,407\,427\,210\,024\,320\,\alpha^{19} + \\
&486\,585\,842\,769\,876\,461\,484\,032\,\alpha^{20} + 43\,798\,304\,089\,562\,788\,663\,296\,\alpha^{21} + \\
&3\,208\,710\,131\,027\,557\,023\,744\,\alpha^{22} + 186\,416\,522\,833\,559\,945\,216\,\alpha^{23} + 8\,261\,380\,192\,874\,790\,912\,\alpha^{24} + \\
&262\,301\,388\,296\,421\,376\,\alpha^{25} + 5\,312\,632\,953\,241\,600\,\alpha^{26} + 51\,561\,082\,388\,480\,\alpha^{27}\Big)\ \mathrm{Seq}[5+\alpha] + \\
&\Big(154\,404\,486\,709\,237\,819\,219\,968\,000 + 1\,265\,327\,918\,255\,018\,927\,110\,348\,800\,\alpha + \\
&4\,953\,641\,658\,930\,095\,511\,385\,751\,040\,\alpha^{2} + 12\,335\,446\,851\,783\,544\,166\,937\,390\,720\,\alpha^{3} + \\
&21\,947\,702\,123\,383\,074\,616\,990\,244\,544\,\alpha^{4} + 29\,712\,684\,443\,300\,038\,100\,072\,561\,760\,\alpha^{5} + \\
&31\,824\,626\,177\,807\,101\,870\,129\,360\,368\,\alpha^{6} + 27\,684\,339\,638\,906\,598\,652\,692\,786\,888\,\alpha^{7} + \\
&19\,923\,668\,408\,873\,674\,929\,361\,243\,572\,\alpha^{8} + 12\,021\,754\,897\,932\,453\,908\,473\,126\,194\,\alpha^{9} + \\
&6\,141\,402\,912\,303\,808\,338\,721\,284\,327\,\alpha^{10} + 2\,675\,090\,519\,652\,464\,763\,702\,625\,995\,\alpha^{11} + \\
&998\,451\,712\,547\,824\,111\,144\,656\,513\,\alpha^{12} + 320\,337\,381\,856\,256\,276\,567\,115\,789\,\alpha^{13} + \\
&88\,485\,146\,094\,830\,787\,771\,471\,525\,\alpha^{14} + 21\,045\,641\,782\,461\,353\,200\,898\,049\,\alpha^{15} + \\
&4\,304\,140\,182\,149\,530\,399\,276\,227\,\alpha^{16} + 754\,678\,659\,252\,915\,954\,749\,073\,\alpha^{17} + \\
&112\,910\,766\,050\,133\,819\,763\,020\,\alpha^{18} + 14\,316\,213\,223\,182\,938\,203\,068\,\alpha^{19} + \\
&1\,523\,679\,350\,645\,560\,062\,336\,\alpha^{20} + 134\,345\,128\,624\,663\,841\,280\,\alpha^{21} + \\
&9\,635\,762\,018\,738\,626\,560\,\alpha^{22} + 547\,760\,583\,383\,666\,688\,\alpha^{23} + 23\,739\,371\,943\,886\,848\,\alpha^{24} + \\
&736\,693\,272\,182\,784\,\alpha^{25} + 14\,575\,541\,944\,320\,\alpha^{26} + 138\,110\,042\,112\,\alpha^{27}\Big)\ \mathrm{Seq}[6+\alpha];
\end{aligned}
$$

```
In[ ]:= RecNormalizedOrder = 6;
```

Initial values of {r(0), r(1), r(2), ...}

In[∘]:=
```
SeqListIni = {};

MAX = 10;

For[n = 0, n ≤ MAX, n++,
  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence @@ (2 coord[[i]])] *
     Product[Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];

  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n + (1 - NN) / 2 &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence @@ (2 coord[[i]] + 1)] *
     Product[Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];
 ];

SeqListIni

seq[n_] := SeqListIni[[n + 1]];
```

Out[∘]=
```
{1, 0, 80, 0, 58 320, 933 120, 107 360 000, 4 032 000 000, 305 742 850 000,
 16 007 947 200 000, 1 092 754 448 110 080, 66 052 872 139 161 600, 4 433 464 272 394 080 000,
 287 105 556 124 600 012 800, 19 441 756 158 387 587 481 600, 1 307 659 624 636 945 150 771 200,
 89 869 341 860 254 106 893 314 000, 6 191 536 013 119 541 254 794 624 000,
 431 788 153 780 445 031 117 712 736 000, 30 259 578 124 053 738 011 950 295 040 000,
 2 137 643 722 042 861 014 846 923 875 678 720, 151 778 757 062 056 398 402 787 590 848 716 800}
```

Verify recurrence by initial values

In[∘]:=
```
Table[SeqNormalized /. {Seq → seq, α → n}, {n, 0, 2 MAX - RecNormalizedOrder}]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Generate more terms in the sequence
SeqList[[n]] = r(n)

In[∘]:=
```
Bound = 500;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Let’s guess (and prove!) a shorter recurrence.

In[ ]:= `<< RISC`Guess``

Package GeneratingFunctions version 0.9 written by Christian Mallinger
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Johannes Kepler University, Linz, Austria

Guess Package version 0.52
written by Manuel Kauers
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Johannes Kepler University, Linz, Austria

In[ ]:= `SeqGuess = GuessMinRE[Take[SeqList, 500], Seq[α]]`

Out[ ]=
$$\left(-\frac{839\,014\,160\,464\,878\,334\,200\,000}{49}-\frac{60\,378\,917\,161\,327\,444\,738\,417\,500\,\alpha}{343}-\frac{295\,253\,382\,097\,523\,870\,722\,179\,000\,\alpha^{2}}{343}-\frac{914\,561\,589\,936\,050\,911\,362\,890\,700\,\alpha^{3}}{343}-\frac{32\,257\,781\,051\,913\,317\,606\,478\,802\,245\,\alpha^{4}}{5488}-\frac{13\,475\,608\,031\,359\,817\,361\,664\,160\,535\,\alpha^{5}}{1372}-\frac{40\,572\,825\,849\,119\,393\,669\,101\,437\,045\,\alpha^{6}}{3136}-\frac{302\,784\,920\,890\,621\,266\,276\,866\,033\,415\,\alpha^{7}}{21\,952}-\frac{532\,323\,647\,275\,797\,644\,864\,880\,788\,565\,\alpha^{8}}{43\,904}-\frac{55\,866\,913\,963\,786\,414\,517\,133\,215\,505\,\alpha^{9}}{6272}-\frac{242\,484\,985\,196\,765\,161\,344\,650\,271\,585\,\alpha^{10}}{43\,904}-\frac{127\,817\,608\,674\,118\,712\,947\,652\,539\,635\,\alpha^{11}}{43\,904}-\frac{57\,565\,467\,800\,714\,678\,074\,960\,914\,795\,\alpha^{12}}{43\,904}-\frac{22\,223\,652\,535\,998\,880\,970\,405\,304\,465\,\alpha^{13}}{43\,904}-\frac{1\,052\,404\,012\,670\,862\,457\,642\,855\,635\,\alpha^{14}}{6272}-\frac{299\,600\,766\,861\,108\,295\,078\,187\,205\,\alpha^{15}}{6272}-\frac{128\,018\,992\,381\,312\,391\,740\,640\,355\,\alpha^{16}}{10\,976}-\frac{26\,733\,254\,832\,666\,098\,464\,954\,365\,\alpha^{17}}{10\,976}-\frac{21\,214\,914\,493\,292\,327\,516\,610\,\alpha^{18}}{49}-\frac{44\,639\,215\,000\,200\,873\,006\,555\,\alpha^{19}}{686}-\frac{401\,351\,202\,097\,745\,724\,240\,\alpha^{20}}{49}-\frac{292\,322\,347\,238\,801\,262\,240\,\alpha^{21}}{343}-\frac{24\,688\,816\,722\,475\,292\,160\,\alpha^{22}}{343}-\frac{1\,649\,161\,788\,132\,641\,280\,\alpha^{23}}{343}-\frac{83\,811\,547\,465\,482\,240\,\alpha^{24}}{343}-\frac{434\,818\,220\,851\,200\,\alpha^{25}}{49}-\frac{1\,435\,395\,686\,400\,\alpha^{26}}{7}-2\,264\,924\,160\,\alpha^{27}\right)\mathrm{Seq}[\alpha]+$$
$$\left(-\frac{2\,186\,847\,101\,186\,124\,636\,476\,250}{343}-\frac{40\,166\,638\,365\,783\,202\,562\,183\,625\,\alpha}{686}-\right.$$

$$\frac{706\,033\,778\,688\,919\,165\,504\,551\,975\,\alpha^2}{2744} - \frac{3\,953\,067\,424\,847\,557\,748\,562\,097\,035\,\alpha^3}{5488} -$$
$$\frac{63\,348\,838\,563\,981\,017\,139\,490\,197\,423\,\alpha^4}{43\,904} - \frac{386\,743\,276\,684\,702\,710\,173\,649\,683\,709\,\alpha^5}{175\,616} -$$
$$\frac{467\,588\,326\,085\,452\,464\,693\,549\,418\,521\,\alpha^6}{175\,616} - \frac{1\,838\,729\,787\,942\,510\,422\,531\,790\,217\,569\,\alpha^7}{702\,464} -$$
$$\frac{1\,497\,046\,507\,490\,797\,080\,597\,799\,518\,397\,\alpha^8}{702\,464} - \frac{1\,022\,891\,275\,901\,803\,244\,496\,188\,387\,873\,\alpha^9}{702\,464} -$$
$$\frac{592\,233\,220\,127\,607\,867\,106\,505\,173\,863\,\alpha^{10}}{702\,464} - \frac{292\,583\,716\,313\,062\,639\,375\,940\,650\,123\,\alpha^{11}}{702\,464} -$$
$$\frac{123\,937\,814\,731\,698\,977\,948\,555\,179\,581\,\alpha^{12}}{702\,464} - \frac{45\,152\,803\,017\,480\,289\,278\,728\,463\,787\,\alpha^{13}}{702\,464} -$$
$$\frac{14\,169\,052\,042\,991\,173\,800\,791\,007\,881\,\alpha^{14}}{702\,464} - \frac{478\,727\,120\,493\,253\,981\,881\,514\,701\,\alpha^{15}}{87\,808} -$$
$$\frac{445\,174\,518\,113\,897\,119\,575\,925\,953\,\alpha^{16}}{351\,232} - \frac{44\,371\,967\,143\,805\,451\,055\,219\,387\,\alpha^{17}}{175\,616} -$$
$$\frac{471\,772\,951\,489\,604\,551\,054\,557\,\alpha^{18}}{10\,976} - \frac{34\,007\,172\,418\,504\,523\,417\,927\,\alpha^{19}}{5488} -$$
$$\frac{257\,194\,224\,994\,051\,571\,599\,\alpha^{20}}{343} - \frac{25\,779\,469\,528\,374\,602\,442\,\alpha^{21}}{343} -$$
$$\frac{300\,216\,059\,140\,647\,264\,\alpha^{22}}{49} - \frac{135\,741\,234\,263\,244\,224\,\alpha^{23}}{343} - \frac{6\,682\,209\,462\,884\,352\,\alpha^{24}}{343} -$$
$$\left.\frac{33\,635\,591\,229\,440\,\alpha^{25}}{49} - \frac{107\,895\,848\,960\,\alpha^{26}}{7} - 165\,675\,008\,\alpha^{27}\right) \mathrm{Seq}[1+\alpha] +$$
$$\left(\frac{1\,649\,256\,896\,641\,607\,710\,441\,875}{343} + \frac{32\,958\,285\,910\,807\,944\,252\,614\,625\,\alpha}{784} +\right.$$
$$\frac{1\,103\,458\,985\,345\,938\,205\,500\,476\,525\,\alpha^2}{6272} + \frac{20\,602\,673\,094\,845\,310\,029\,416\,662\,615\,\alpha^3}{43\,904} +$$
$$\frac{78\,681\,548\,034\,330\,539\,256\,292\,880\,601\,\alpha^4}{87\,808} + \frac{523\,592\,632\,484\,061\,654\,815\,021\,764\,971\,\alpha^5}{401\,408} +$$
$$\frac{4\,229\,039\,942\,303\,989\,304\,357\,540\,727\,981\,\alpha^6}{2\,809\,856} + \frac{15\,881\,609\,866\,660\,453\,928\,750\,384\,380\,533\,\alpha^7}{11\,239\,424} +$$
$$\frac{772\,330\,072\,319\,005\,942\,606\,081\,532\,085\,\alpha^8}{702\,464} + \frac{2\,018\,844\,234\,787\,524\,860\,289\,722\,385\,301\,\alpha^9}{2\,809\,856} +$$
$$\frac{1\,678\,256\,684\,807\,968\,075\,899\,645\,406\,369\,\alpha^{10}}{4\,214\,784} + \frac{3\,177\,248\,279\,844\,949\,454\,127\,497\,832\,697\,\alpha^{11}}{16\,859\,136} +$$
$$\frac{161\,317\,648\,569\,436\,446\,129\,021\,952\,801\,\alpha^{12}}{2\,107\,392} + \frac{16\,116\,429\,927\,995\,151\,019\,169\,756\,323\,\alpha^{13}}{602\,112} +$$
$$\frac{68\,021\,739\,078\,432\,695\,475\,513\,024\,847\,\alpha^{14}}{8\,429\,568} + \frac{70\,724\,846\,153\,746\,609\,701\,040\,423\,079\,\alpha^{15}}{33\,718\,272} +$$
$$\frac{494\,621\,976\,067\,373\,678\,676\,863\,419\,\alpha^{16}}{1\,053\,696} + \frac{54\,295\,362\,239\,654\,566\,177\,488\,343\,\alpha^{17}}{602\,112} +$$
$$\frac{7\,796\,326\,801\,173\,006\,055\,920\,655\,\alpha^{18}}{526\,848} + \frac{1\,085\,406\,922\,786\,412\,521\,465\,937\,\alpha^{19}}{526\,848} +$$

$$\frac{3\,967\,811\,698\,482\,559\,430\,177\,\alpha^{20}}{16\,464} + \frac{24\,054\,992\,782\,708\,516\,969\,\alpha^{21}}{1029} +$$

$$\frac{1\,899\,711\,396\,480\,455\,176\,\alpha^{22}}{1029} + \frac{119\,001\,615\,474\,948\,742\,\alpha^{23}}{1029} + \frac{812\,467\,727\,983\,616\,\alpha^{24}}{147} +$$

$$\left.\frac{567\,783\,931\,264\,\alpha^{25}}{3} + \frac{12\,403\,097\,600\,\alpha^{26}}{3} + \frac{129\,826\,816\,\alpha^{27}}{3}\right) \mathrm{Seq}[2+\alpha] +$$

$$\left(-\frac{3\,300\,425\,296\,203\,663\,718\,696\,125}{12\,544} - \frac{789\,244\,825\,356\,043\,683\,057\,585\,225\,\alpha}{351\,232} -\right.$$

$$\frac{6\,449\,700\,552\,457\,863\,573\,244\,496\,925\,\alpha^{2}}{702\,464} - \frac{134\,298\,582\,185\,468\,413\,918\,618\,824\,555\,\alpha^{3}}{5\,619\,712} -$$

$$\frac{1\,000\,485\,771\,218\,662\,771\,136\,402\,161\,335\,\alpha^{4}}{22\,478\,848} - \frac{1\,419\,842\,534\,100\,736\,495\,153\,342\,949\,295\,\alpha^{5}}{22\,478\,848} -$$

$$\frac{6\,385\,948\,988\,649\,672\,570\,657\,427\,873\,203\,\alpha^{6}}{89\,915\,392} - \frac{5\,840\,134\,868\,358\,723\,834\,123\,925\,806\,909\,\alpha^{7}}{89\,915\,392} -$$

$$\frac{4\,424\,881\,579\,983\,986\,499\,304\,385\,040\,019\,\alpha^{8}}{89\,915\,392} - \frac{8\,444\,532\,442\,722\,825\,336\,621\,377\,571\,743\,\alpha^{9}}{269\,746\,176} -$$

$$\frac{4\,554\,414\,311\,325\,865\,558\,765\,491\,638\,617\,\alpha^{10}}{269\,746\,176} - \frac{699\,091\,790\,541\,495\,310\,328\,021\,817\,159\,\alpha^{11}}{89\,915\,392} -$$

$$\frac{118\,381\,500\,403\,501\,260\,295\,241\,964\,271\,\alpha^{12}}{38\,535\,168} - \frac{281\,822\,938\,588\,155\,744\,806\,412\,680\,597\,\alpha^{13}}{269\,746\,176} -$$

$$\frac{13\,771\,059\,934\,620\,054\,736\,248\,875\,095\,\alpha^{14}}{44\,957\,696} - \frac{652\,671\,274\,238\,255\,366\,403\,448\,591\,\alpha^{15}}{8\,429\,568} -$$

$$\frac{757\,520\,368\,056\,059\,264\,144\,934\,461\,\alpha^{16}}{44\,957\,696} - \frac{70\,753\,397\,619\,107\,393\,082\,656\,269\,\alpha^{17}}{22\,478\,848} -$$

$$\frac{705\,708\,700\,531\,533\,510\,822\,621\,\alpha^{18}}{1\,404\,928} - \frac{143\,330\,455\,971\,788\,927\,353\,451\,\alpha^{19}}{2\,107\,392} -$$

$$\frac{339\,767\,379\,053\,204\,777\,235\,\alpha^{20}}{43\,904} - \frac{48\,094\,544\,804\,504\,337\,913\,\alpha^{21}}{65\,856} -$$

$$\frac{230\,990\,968\,151\,074\,487\,\alpha^{22}}{4116} - \frac{2\,347\,190\,995\,410\,073\,\alpha^{23}}{686} - \frac{163\,812\,502\,351\,760\,\alpha^{24}}{1029} -$$

$$\left.\frac{260\,127\,701\,216\,\alpha^{25}}{49} - \frac{790\,794\,240\,\alpha^{26}}{7} - \frac{3\,457\,024\,\alpha^{27}}{3}\right) \mathrm{Seq}[3+\alpha] +$$

$$\left(-\frac{4\,419\,384\,939\,575\,213\,940\,613\,125}{351\,232} - \frac{73\,896\,327\,740\,551\,207\,708\,059\,975\,\alpha}{702\,464} -\right.$$

$$\frac{4\,726\,813\,741\,768\,598\,178\,034\,340\,415\,\alpha^{2}}{11\,239\,424} - \frac{24\,063\,615\,057\,887\,161\,846\,291\,073\,475\,\alpha^{3}}{22\,478\,848} -$$

$$\frac{350\,473\,988\,703\,646\,433\,544\,341\,504\,871\,\alpha^{4}}{179\,830\,784} - \frac{971\,981\,293\,004\,161\,406\,004\,149\,585\,057\,\alpha^{5}}{359\,661\,568} -$$

$$\frac{2\,134\,942\,083\,071\,700\,595\,449\,995\,108\,667\,\alpha^{6}}{719\,323\,136} - \frac{3\,812\,670\,769\,513\,017\,728\,547\,893\,063\,139\,\alpha^{7}}{1\,438\,646\,272} -$$

$$\frac{8\,458\,645\,995\,200\,530\,372\,433\,532\,903\,685\,\alpha^{8}}{4\,315\,938\,816} - \frac{1\,750\,138\,732\,404\,017\,358\,372\,499\,248\,829\,\alpha^{9}}{1\,438\,646\,272} -$$

$$\frac{2\,762\,310\,206\,216\,888\,720\,786\,991\,407\,675\,\alpha^{10}}{4\,315\,938\,816} - \frac{413\,514\,204\,526\,007\,006\,688\,227\,545\,697\,\alpha^{11}}{1\,438\,646\,272} -$$

$$\frac{159\,309\,089\,707\,252\,746\,029\,199\,932\,491\,\alpha^{12}}{1\,438\,646\,272} - \frac{7\,545\,374\,813\,173\,180\,280\,949\,286\,951\,\alpha^{13}}{205\,520\,896} -$$
$$\frac{45\,281\,313\,203\,748\,981\,543\,845\,347\,391\,\alpha^{14}}{4\,315\,938\,816} - \frac{3\,112\,423\,988\,881\,255\,072\,454\,483\,\alpha^{15}}{1\,204\,224} -$$
$$\frac{168\,973\,759\,260\,877\,096\,654\,959\,805\,\alpha^{16}}{308\,281\,344} - \frac{71\,767\,561\,403\,421\,381\,098\,320\,645\,\alpha^{17}}{719\,323\,136} -$$
$$\frac{348\,744\,316\,702\,179\,875\,002\,013\,\alpha^{18}}{22\,478\,848} - \frac{92\,020\,158\,251\,229\,783\,294\,949\,\alpha^{19}}{44\,957\,696} -$$
$$\frac{956\,461\,666\,447\,158\,357\,035\,\alpha^{20}}{4\,214\,784} - \frac{6\,282\,135\,940\,307\,582\,357\,\alpha^{21}}{301\,056} - \frac{205\,814\,757\,110\,694\,097\,\alpha^{22}}{131\,712} -$$
$$\left.\frac{4\,076\,333\,527\,083\,171\,\alpha^{23}}{43\,904} - \frac{2\,888\,385\,846\,483\,\alpha^{24}}{686} - \frac{13\,413\,211\,503\,\alpha^{25}}{98} - \frac{59\,632\,960\,\alpha^{26}}{21} - \frac{84\,736\,\alpha^{27}}{3}\right)$$
$$\mathrm{Seq}[4+\alpha] + \left(\frac{1\,482\,995\,691\,311\,402\,622\,448\,125}{5\,619\,712} + \frac{49\,120\,268\,681\,441\,035\,003\,795\,875\,\alpha}{22\,478\,848} +\right.$$
$$\frac{1\,555\,210\,724\,161\,568\,864\,074\,302\,375\,\alpha^{2}}{179\,830\,784} + \frac{3\,916\,871\,062\,121\,715\,879\,283\,191\,765\,\alpha^{3}}{179\,830\,784} +$$
$$\frac{112\,828\,553\,049\,907\,076\,462\,969\,385\,195\,\alpha^{4}}{2\,877\,292\,544} + \frac{309\,268\,898\,119\,752\,733\,341\,269\,762\,043\,\alpha^{5}}{5\,754\,585\,088} +$$
$$\frac{671\,017\,198\,653\,537\,435\,260\,390\,175\,387\,\alpha^{6}}{11\,509\,170\,176} + \frac{1\,183\,027\,137\,799\,027\,893\,546\,692\,024\,049\,\alpha^{7}}{23\,018\,340\,352} +$$
$$\frac{863\,190\,685\,234\,800\,265\,012\,672\,648\,825\,\alpha^{8}}{23\,018\,340\,352} + \frac{28\,303\,132\,104\,759\,274\,210\,019\,109\,065\,\alpha^{9}}{1\,233\,125\,376} +$$
$$\frac{410\,874\,812\,236\,773\,868\,776\,053\,851\,001\,\alpha^{10}}{34\,527\,510\,528} + \frac{473\,249\,211\,404\,965\,193\,556\,167\,879\,\alpha^{11}}{89\,915\,392} +$$
$$\frac{34\,454\,496\,503\,156\,427\,024\,996\,220\,207\,\alpha^{12}}{17\,263\,755\,264} + \frac{22\,472\,432\,036\,750\,355\,443\,386\,156\,283\,\alpha^{13}}{34\,527\,510\,528} +$$
$$\frac{1\,052\,166\,450\,270\,881\,274\,335\,880\,241\,\alpha^{14}}{5\,754\,585\,088} + \frac{3\,055\,704\,549\,326\,297\,996\,829\,726\,013\,\alpha^{15}}{69\,055\,021\,056} +$$
$$\frac{636\,241\,612\,781\,535\,908\,458\,849\,745\,\alpha^{16}}{69\,055\,021\,056} + \frac{56\,818\,312\,604\,888\,156\,792\,646\,271\,\alpha^{17}}{34\,527\,510\,528} +$$
$$\frac{77\,357\,012\,727\,303\,251\,245\,363\,\alpha^{18}}{308\,281\,344} + \frac{11\,668\,959\,386\,477\,675\,026\,105\,\alpha^{19}}{359\,661\,568} +$$
$$\frac{237\,590\,743\,539\,978\,740\,959\,\alpha^{20}}{67\,436\,544} + \frac{509\,187\,873\,065\,043\,581\,\alpha^{21}}{1\,605\,632} + \frac{16\,320\,343\,684\,018\,743\,\alpha^{22}}{702\,464} +$$
$$\left.\frac{1\,422\,245\,199\,841\,003\,\alpha^{23}}{1\,053\,696} + \frac{93\,793\,644\,143\,\alpha^{24}}{1568} + \frac{638\,138\,003\,\alpha^{25}}{336} + \frac{115\,400\,\alpha^{26}}{3} + \frac{1120\,\alpha^{27}}{3}\right)$$
$$\mathrm{Seq}[5+\alpha] + \left(\frac{785\,341\,830\,999\,948\,217\,875}{702\,464} + \frac{51\,486\,324\,798\,788\,205\,041\,925\,\alpha}{5\,619\,712} +\right.$$
$$\frac{3\,225\,027\,121\,699\,280\,931\,891\,765\,\alpha^{2}}{89\,915\,392} + \frac{32\,123\,559\,509\,852\,979\,601\,399\,455\,\alpha^{3}}{359\,661\,568} +$$
$$\frac{114\,310\,948\,559\,286\,846\,963\,490\,857\,\alpha^{4}}{719\,323\,136} + \frac{309\,507\,129\,617\,708\,730\,209\,089\,185\,\alpha^{5}}{1\,438\,646\,272} +$$
$$\frac{663\,013\,045\,370\,981\,288\,961\,028\,341\,\alpha^{6}}{2\,877\,292\,544} + \frac{1\,153\,514\,151\,621\,108\,277\,195\,532\,787\,\alpha^{7}}{5\,754\,585\,088} +$$

$$\frac{1\,660\,305\,700\,739\,472\,910\,780\,103\,631\,\alpha^{8}}{11\,509\,170\,176}+\frac{2\,003\,625\,816\,322\,075\,651\,412\,187\,699\,\alpha^{9}}{23\,018\,340\,352}+\frac{2\,047\,134\,304\,101\,269\,446\,240\,428\,109\,\alpha^{10}}{46\,036\,680\,704}+\frac{891\,696\,839\,884\,154\,921\,234\,208\,665\,\alpha^{11}}{46\,036\,680\,704}+\frac{332\,817\,237\,515\,941\,370\,381\,552\,171\,\alpha^{12}}{46\,036\,680\,704}+\frac{106\,779\,127\,285\,418\,758\,855\,705\,263\,\alpha^{13}}{46\,036\,680\,704}+\frac{29\,495\,048\,698\,276\,929\,257\,157\,175\,\alpha^{14}}{46\,036\,680\,704}+\frac{7\,015\,213\,927\,487\,117\,733\,632\,683\,\alpha^{15}}{46\,036\,680\,704}+\frac{1\,434\,713\,394\,049\,843\,466\,425\,409\,\alpha^{16}}{46\,036\,680\,704}+\frac{35\,937\,079\,012\,043\,616\,892\,813\,\alpha^{17}}{6\,576\,668\,672}+\frac{9\,409\,230\,504\,177\,818\,313\,585\,\alpha^{18}}{11\,509\,170\,176}+\frac{1\,193\,017\,768\,598\,578\,183\,589\,\alpha^{19}}{11\,509\,170\,176}+\frac{3\,967\,914\,975\,639\,479\,329\,\alpha^{20}}{359\,661\,568}+\frac{87\,464\,276\,448\,348\,855\,\alpha^{21}}{89\,915\,392}+\frac{392\,080\,160\,267\,685\,\alpha^{22}}{5\,619\,712}+\frac{6\,368\,124\,341\,793\,\alpha^{23}}{1\,605\,632}+\frac{15\,093\,086\,207\,\alpha^{24}}{87\,808}+\frac{133\,821\,991\,\alpha^{25}}{25\,088}+\frac{2955\,\alpha^{26}}{28}+\alpha^{27}\Bigg)\,\mathrm{Seq}[6+\alpha]$$

Okay, the order of this recurrence is the same as what we have computed by creative telescoping; both are 6. So no need to continue.