Multi-headed Lattice Green Function (N = 5, M = 2)
Find Minimal REC

```
In[•]:= NN = 5;
MM = 2;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)} \, d\theta_1 \ldots d\theta_N$$

$$R_{M,N}(z) := P_{M,N}\left(2^M \binom{N}{M} z\right) \text{ and } R_{M,N}(z) = \sum_{n\geq 0} r_{M,N}(n) z^n$$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define $\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n\geq 0} \tilde{r}_{M,N}(n) z^n = \sum_{n\geq 0} r_{M,N}(2n) z^n$

Our goal is to find:

Case 1. *M* even and *M* ≠ *N*:
- minimal recurrences (REC) for *r*(*n*).

Case 2. *M* odd or *M* = *N*:
- minimal recurrences (REC) for $\tilde{r}(n)$.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan’s implementation).

```
In[•]:= (* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals, Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Load RISC packages.

In[ ]:=
```
<< RISC`HolonomicFunctions`
<< RISC`Asymptotics`
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan


```
--> Type  ?HolonomicFunctions  for help.
```

Asymptotics Package version 0.3
written by Manuel Kauers

Package GeneratingFunctions version 0.9 written by Christian Mallinger

Guess Package version 0.52
written by Manuel Kauers

We start by importing known ODE for $R(z)$.

Note that the ODE in Koutschan (2013, pp. 11-12, Thm 3) is for $P(z) = R\left(z \Big/ \binom{N}{M} 2^M\right)$.

```
In[]:= ODEDiv2 = ToOrePolynomial[
   30 * (27 000 000 000 + 84 037 500 000 * z - 346 865 625 000 * z^2 - 55 567 000 000 * z^3 +
        187 923 165 625 * z^4 + 36 477 006 875 * z^5 + 21 336 230 625 * z^6 + 19 123 388 575 * z^7 +
        6 925 739 310 * z^8 + 1 443 544 710 * z^9 + 163 913 184 * z^10 + 7 525 440 * z^11) * P[z] +
     10 * (- 189 000 000 000 + 4 816 462 500 000 * z - 7 268 326 875 000 * z^2 -
        21 210 430 812 500 * z^3 + 2 664 478 321 875 * z^4 + 3 711 617 481 250 * z^5 -
        135 661 728 250 * z^6 + 689 643 286 650 * z^7 + 607 021 304 825 * z^8 + 209 673 119 160 * z^9 +
        40 678 130 502 * z^10 + 4 143 853 440 * z^11 + 167 064 768 * z^12) * Derivative[1][P][z] +
     5 * (- 3 240 000 000 000 + 5 055 750 000 000 * z + 44 457 862 500 000 * z^2 -
        133 825 053 750 000 * z^3 - 110 925 736 437 500 * z^4 + 13 367 806 743 750 * z^5 -
        6 199 228 765 625 * z^6 - 8 282 515 456 375 * z^7 + 1 646 226 060 075 * z^8 +
        2 287 368 823 475 * z^9 + 810 956 145 330 * z^10 + 149 186 684 934 * z^11 +
        13 819 981 248 * z^12 + 496 679 040 * z^13) * Derivative[2][P][z] +
     5 * z * (- 13 162 500 000 000 + 45 343 125 000 000 * z + 40 530 375 000 000 * z^2 -
        190 176 960 000 000 * z^3 - 77 498 059 625 000 * z^4 - 3 649 915 059 375 * z^5 -
        26 918 293 320 000 * z^6 - 13 545 524 756 500 * z^7 - 465 440 555 100 * z^8 +
        1 350 059 072 325 * z^9 + 524 857 986 060 * z^10 + 92 744 995 638 * z^11 +
        7 892 060 544 * z^12 + 255 864 960 * z^13) * Derivative[3][P][z] +
     10 * z^2 * (- 5 568 750 000 000 + 23 905 125 000 000 * z + 3 393 646 875 000 * z^2 -
        39 702 348 750 000 * z^3 - 7 716 298 734 375 * z^4 - 3 779 011 321 875 * z^5 -
        7 801 785 421 250 * z^6 - 3 351 125 770 500 * z^7 - 382 134 335 775 * z^8 +
        148 313 757 125 * z^9 + 68 439 921 540 * z^10 + 11 725 276 842 * z^11 +
        923 795 772 * z^12 + 27 279 720 * z^13) * Derivative[4][P][z] +
     8 * z^3 * (5 + z) * (- 354 375 000 000 + 1 774 828 125 000 * z - 503 550 000 000 * z^2 -
        1 289 447 109 375 * z^3 + 254 876 515 625 * z^4 - 266 627 903 125 * z^5 -
        304 623 830 625 * z^6 - 87 265 479 875 * z^7 - 4 878 146 975 * z^8 + 3 939 663 705 * z^9 +
        1 048 560 285 * z^10 + 97 471 734 * z^11 + 3 057 210 * z^12) * Derivative[5][P][z] +
     16 * (- 5 + z) * (- 1 + z) * z^4 * (5 + z)^2 * (10 + z) * (15 + z) * (5 + 3 * z) *
      (- 675 000 + 3 465 000 * z - 1 053 375 * z^2 + 933 650 * z^3 +
        449 735 * z^4 + 144 776 * z^5 + 15 678 * z^6) * Derivative[6][P][z] /.
    {Derivative[k_][P][z] → Der[z]^k} /. {P[z] → 1}];
```

Process the data.

Write the ODE in terms of the operators D and θ.

```
In[]:= ODENormalizedinD = - DFiniteSubstitute[{ODEDiv2},
     {z → w * 2^MM * Binomial[NN, MM]}, Algebra → OreAlgebra[Der[w]]][[1]];

In[]:= ODENormalizedinTheta = ChangeOreAlgebra[w ** ODENormalizedinD, OreAlgebra[Euler[w]]];
```

Then transform the above to a REC for $r(n)$ and write it explicitly.

```
In[]:= RECNormalizedinS = DFiniteDE2RE[{ODENormalizedinD}, {w}, {α}][[1]];

In[]:= RECNormalizedinSOrder = OrePolynomialDegree[RECNormalizedinS, S[α]]

Out[]= 13

In[]:= ClearAll[Seq];
       SeqNormalized = ApplyOreOperator[RECNormalizedinS, Seq[α]];
```

Compute the initial values of $r(n)$ and then verify the REC numerically.

```
In[]:= SeqListIni = {1};

sympoly = SymmetricPolynomial[MM, Table[Indexed[ξ, i] + Indexed[ξ, i]^-1, {i, 1, NN}]];

MAX = 15;

sympolypower = 1;

For[n = 1, n ≤ MAX, n++,
  sympolypower = Expand[sympolypower * sympoly];
  p = Coefficient[Expand[sympolypower * Product[Indexed[ξ, i], {i, 1, NN}]],
    Product[Indexed[ξ, i], {i, 1, NN}]];
  SeqListIni = Append[SeqListIni, p];
 ];

SeqListIni

seq[n_] := SeqListIni[[n + 1]];

Out[]= {1, 0, 40, 480, 11 880, 281 280, 7 506 400, 210 268 800, 6 166 993 000,
  187 069 411 200, 5 833 030 976 640, 186 014 056 166 400, 6 044 435 339 896 800,
  199 561 060 892 793 600, 6 679 216 425 794 140 800, 226 213 441 773 789 550 080}

In[]:= Table[SeqNormalized /. {Seq → seq, α → n}, {n, 0, MAX - RECNormalizedinSOrder}]

Out[]= {0, 0, 0}
```

Generate a list of $r(n)$.

```
In[]:= Bound = 200;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Guess a Minimal REC for $r(n)$.

SeqfromRECGuess gives the REC in Theorem 6.2! (To be displayed at the end of this notebook)

REC: Order 7

ODE: Order 12

```
In[]:= RECGuessTmp = GuessMinRE[Take[SeqList, 200], Seq[α]];
DenominatorsLCM = LCM[Sequence @@
    Denominator[Flatten[CoefficientList[RECGuessTmp /. {Seq[k_] → w^(k-α)}, {α, w}]]]];

In[]:= RECGuessinS = ToOrePolynomial[RECGuessTmp * DenominatorsLCM /. {Seq[k_] → S[α]^(k-α)}];

In[]:= RECGuessinSOrder = OrePolynomialDegree[RECGuessinS, S[α]]

Out[]= 7

In[]:= ODEfromRECGuessOrder = Max[Exponent[OrePolynomialListCoefficients[
    α^Max[Exponent[OrePolynomialListCoefficients[RECGuessinS]/.{α→α^-1},α]] * RECGuessinS], α]]
```

Out[ ]= 12

We may also write this REC explicitly.

```
In[ ]:= ClearAll[Seq];
SeqfromRECGuess = ApplyOreOperator[RECGuessinS, Seq[α]];
```

```
In[ ]:= SeqfromRECGuessList =
  UnrollRecurrence[SeqfromRECGuess, Seq[α], Take[SeqList, RECGuessinSOrder], 200];
```

Prove the minimal REC for *r*(*n*).

```
In[ ]:= RECCompare = DFinitePlus[{RECNormalizedinS}, {RECGuessinS}][[1]];
```

```
In[ ]:= RECCompareOrder = LeadingExponent[RECCompare][[1]]
```

Out[ ]= 13

```
In[ ]:= CheckNum = RECCompareOrder + 20;
Take[SeqList, CheckNum] - Take[SeqfromRECGuessList, CheckNum]
```

Out[ ]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

Display the REC in Theorem 6.2

```
In[ ]:= SeqfromRECGuess
```

Out[•]=

$$
\begin{aligned}
&\big(42\,140\,738\,676\,326\,400\,000 + 157\,842\,901\,249\,818\,624\,000\,\alpha + 257\,331\,505\,709\,737\,574\,400\,\alpha^{2} + \\
&\quad 243\,764\,108\,399\,982\,673\,920\,\alpha^{3} + 150\,397\,023\,447\,243\,816\,960\,\alpha^{4} + 63\,968\,341\,924\,254\,842\,880\,\alpha^{5} + \\
&\quad 19\,301\,263\,998\,729\,584\,640\,\alpha^{6} + 4\,174\,508\,253\,346\,529\,280\,\alpha^{7} + 643\,779\,101\,841\,162\,240\,\alpha^{8} + \\
&\quad 69\,168\,932\,868\,587\,520\,\alpha^{9} + 4\,922\,454\,740\,828\,160\,\alpha^{10} + 208\,614\,614\,630\,400\,\alpha^{11} + 3\,986\,266\,521\,600\,\alpha^{12}\big) \\
&\ \mathrm{Seq}[\alpha] + \big(118\,427\,858\,324\,029\,440\,000 + 355\,246\,559\,316\,108\,902\,400\,\alpha + \\
&\quad 481\,552\,669\,599\,250\,186\,240\,\alpha^{2} + 390\,301\,079\,007\,991\,857\,152\,\alpha^{3} + \\
&\quad 210\,764\,527\,991\,633\,575\,936\,\alpha^{4} + 79\,918\,506\,618\,774\,847\,488\,\alpha^{5} + 21\,826\,970\,852\,964\,532\,224\,\alpha^{6} + \\
&\quad 4\,327\,696\,049\,218\,387\,968\,\alpha^{7} + 618\,429\,092\,691\,574\,784\,\alpha^{8} + 62\,134\,020\,238\,999\,552\,\alpha^{9} + \\
&\quad 4\,167\,373\,533\,741\,056\,\alpha^{10} + 167\,578\,215\,383\,040\,\alpha^{11} + 3\,056\,137\,666\,560\,\alpha^{12}\big)\ \mathrm{Seq}[1+\alpha] + \\
&\big(62\,676\,619\,662\,919\,680\,000 + 168\,213\,967\,990\,385\,049\,600\,\alpha + 205\,820\,392\,167\,964\,974\,080\,\alpha^{2} + \\
&\quad 151\,791\,584\,110\,964\,534\,272\,\alpha^{3} + 75\,137\,340\,688\,642\,841\,600\,\alpha^{4} + 26\,295\,232\,911\,598\,126\,080\,\alpha^{5} + \\
&\quad 6\,670\,149\,766\,003\,083\,264\,\alpha^{6} + 1\,235\,525\,904\,487\,723\,008\,\alpha^{7} + 165\,841\,646\,014\,996\,480\,\alpha^{8} + \\
&\quad 15\,729\,900\,132\,270\,080\,\alpha^{9} + 1\,000\,638\,108\,860\,416\,\alpha^{10} + 38\,329\,059\,901\,440\,\alpha^{11} + 668\,530\,114\,560\,\alpha^{12}\big) \\
&\ \mathrm{Seq}[2+\alpha] + \big(1\,794\,185\,247\,360\,768\,000 + 4\,260\,839\,636\,091\,043\,840\,\alpha + 4\,649\,746\,903\,477\,813\,888\,\alpha^{2} + \\
&\quad 3\,082\,953\,754\,682\,083\,328\,\alpha^{3} + 1\,382\,952\,049\,413\,254\,272\,\alpha^{4} + 442\,032\,317\,052\,873\,728\,\alpha^{5} + \\
&\quad 103\,190\,706\,316\,889\,344\,\alpha^{6} + 17\,720\,524\,544\,509\,952\,\alpha^{7} + 2\,220\,812\,336\,954\,368\,\alpha^{8} + \\
&\quad 198\,014\,286\,036\,992\,\alpha^{9} + 11\,919\,389\,769\,728\,\alpha^{10} + 434\,786\,795\,520\,\alpha^{11} + 7\,266\,631\,680\,\alpha^{12}\big)\ \mathrm{Seq}[3+\alpha] + \\
&\big(-3\,522\,851\,180\,688\,416\,000 - 8\,446\,568\,365\,407\,735\,680\,\alpha - 9\,248\,095\,565\,260\,356\,576\,\alpha^{2} - \\
&\quad 6\,114\,775\,140\,268\,882\,576\,\alpha^{3} - 2\,719\,484\,985\,845\,017\,792\,\alpha^{4} - 857\,108\,315\,069\,629\,104\,\alpha^{5} - \\
&\quad 196\,310\,820\,429\,867\,616\,\alpha^{6} - 32\,924\,151\,546\,376\,000\,\alpha^{7} - 4\,013\,146\,001\,886\,336\,\alpha^{8} - \\
&\quad 346\,719\,870\,364\,160\,\alpha^{9} - 20\,154\,401\,039\,360\,\alpha^{10} - 707\,739\,648\,000\,\alpha^{11} - 11\,354\,112\,000\,\alpha^{12}\big)\ \mathrm{Seq}[4+\alpha] + \\
&\big(-458\,904\,717\,778\,020\,000 - 1\,056\,134\,626\,035\,848\,800\,\alpha - 1\,109\,896\,707\,061\,337\,856\,\alpha^{2} - \\
&\quad 704\,344\,314\,090\,018\,780\,\alpha^{3} - 300\,647\,030\,233\,781\,612\,\alpha^{4} - 90\,944\,593\,157\,694\,708\,\alpha^{5} - \\
&\quad 19\,993\,089\,019\,041\,540\,\alpha^{6} - 3\,218\,776\,240\,146\,608\,\alpha^{7} - 376\,681\,142\,235\,984\,\alpha^{8} - \\
&\quad 31\,252\,297\,558\,272\,\alpha^{9} - 1\,745\,103\,671\,296\,\alpha^{10} - 58\,889\,994\,240\,\alpha^{11} - 908\,328\,960\,\alpha^{12}\big)\ \mathrm{Seq}[5+\alpha] + \\
&\big(-1\,106\,658\,753\,555\,600 - 2\,330\,306\,062\,592\,328\,\alpha - 2\,249\,741\,897\,564\,436\,\alpha^{2} - 1\,317\,143\,965\,540\,014\,\alpha^{3} - \\
&\quad 520\,970\,340\,108\,810\,\alpha^{4} - 146\,691\,130\,015\,168\,\alpha^{5} - 30\,156\,685\,922\,334\,\alpha^{6} - 4\,561\,556\,620\,082\,\alpha^{7} - \\
&\quad 503\,951\,197\,636\,\alpha^{8} - 39\,663\,617\,640\,\alpha^{9} - 2\,111\,344\,496\,\alpha^{10} - 68\,259\,840\,\alpha^{11} - 1\,013\,760\,\alpha^{12}\big)\ \mathrm{Seq}[6+\alpha] + \\
&\big(836\,209\,651\,013\,100 + 1\,823\,470\,291\,632\,528\,\alpha + 1\,811\,702\,917\,816\,029\,\alpha^{2} + 1\,084\,613\,257\,235\,718\,\alpha^{3} + \\
&\quad 435\,833\,439\,807\,171\,\alpha^{4} + 123\,860\,858\,052\,324\,\alpha^{5} + 25\,531\,982\,914\,119\,\alpha^{6} + 3\,847\,089\,898\,422\,\alpha^{7} + \\
&\quad 420\,608\,699\,769\,\alpha^{8} + 32\,547\,074\,928\,\alpha^{9} + 1\,692\,297\,492\,\alpha^{10} + 53\,095\,680\,\alpha^{11} + 760\,320\,\alpha^{12}\big)\ \mathrm{Seq}[7+\alpha]
\end{aligned}
$$