# Nonmodular infinite products and a Conjecture of Seo and Yee

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**Abstract**. We will tackle a conjecture of S. Seo and A. J. Yee, which says that the series expansion of $1/(q,-q^3;q^4)_\infty$ has nonnegative coefficients. Our approach relies on an approximation of the generally nonmodular infinite product $1/(q^a;q^M)_\infty$ where $M$ is a positive integer and $a$ is any of $1,2,\ldots,M$.

**Keywords**. Nonmodular infinite product, asymptotic formula, circle method.

**2010MSC**. 11P55, 11P82.

## 1. Introduction

Throughout, we adopt the standard $q$-series notations:

$$(A;q)_\infty := \prod_{k\geq 0}(1-Aq^k)$$

and

$$(A,B,\ldots,C;q)_\infty := (A;q)_\infty(B;q)_\infty\cdots(C;q)_\infty.$$

In their work on the index of seaweed algebras and integer partitions, Seo and Yee [9] proved that an earlier conjecture of Coll, A. Mayers and N. Mayers [4] is equivalent to the following nonnegativity conjecture.

**Conjecture 1.1.** The series expansion of

$$\frac{1}{(q,-q^3;q^4)_\infty} \tag{1.1}$$

has nonnegative coefficients.

As a $q$-hypergeometric proof of this conjecture is notoriously difficult to find, one may give his hope to the approach of deriving an asymptotic formula for the coefficients. If one is also patient enough to compute an explicit bound of the errors, then a direct examination will yield a proof of such nonnegativity. However, it is notable that the infinite product in (1.1) is different from products of Dedekind eta function or Jacobi theta function and indeed it is no more modular. Hence, a Rademacher-type proof fails. Also, if we rewrite this product as

$$\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty(q^6;q^8)_\infty},$$

then the numerator $(q^3;q^4)_\infty$ causes the expiration of Meinardus' powerful approach [8]. One of the few works about asymptotics of nonmodular infinite products is due to Grosswald [5], who absorbed ideas from Lehner [6] and Livingood [7]. In his paper, the infinite product

$$\frac{1}{(q^a;q^M)_\infty} \tag{1.2}$$

with a prime modulus $M$ is considered. However, a closer examination of Grosswald's paper reveals several mistakes, among which at least the calculation of the residue $R_3$ on page 119 of [5] is not robust. Also, a natural question is about the case where the modulus is composite.

Let $M$ be a positive integer and $a$ be any of $1, 2, \ldots, M$. The first goal of this paper is to investigate the asymptotic behavior of

$$\Phi_{a,M}(q) := \log\left(\frac{1}{(q^a;q^M)_\infty}\right) \tag{1.3}$$

when the complex variable $q$ with $|q| < 1$ approaches the unit circle.

**Theorem 1.1.** *Let $X$ be a sufficiently large positive number. Let*

$$q = e^{-\tau+2\pi i h/k} \tag{1.4}$$

*where $1 \le h \le k \le \lfloor\sqrt{2\pi X}\rfloor =: N$ with $(h,k) = 1$ (throughout, $(m,n)$ denotes the greatest common divisor of integers $m$ and $n$) and $\tau = X^{-1} + 2\pi i Y$ with $|Y| \le 1/(kN)$. Let $M$ be a positive integer and $a$ be any of $1, 2, \ldots, M$. If we denote by $b$ the unique integer between 1 and $(k,M)$ such that $b \equiv -ha \pmod{(k,M)}$ and write*

$$b^* = \begin{cases} (k,M) - b & \text{if } b \ne (k,M), \\ (k,M) & \text{if } b = (k,M), \end{cases}$$

*then*

$$\begin{aligned}\log\left(\frac{1}{(q^a;q^M)_\infty}\right) = \frac{1}{\tau}\frac{(k,M)^2}{k^2M}\Bigg(\pi^2\left(\frac{b^2}{(k,M)^2} - \frac{b}{(k,M)} + \frac{1}{6}\right) \\ + 2\pi i\left(-\zeta'\left(-1, \frac{b}{(k,M)}\right) + \zeta'\left(-1, \frac{b^*}{(k,M)}\right)\right)\Bigg) + E \end{aligned}\tag{1.5}$$

*where*

$$|\Re(E)| \ll_{a,M} X^{1/2}\log X. \tag{1.6}$$

*Remark* 1.1. Let $\mathcal{Q}_{h/k}$ be the set of $q$ with respect to $h/k$ defined in Theorem 1.1. For any $q$ with $|q| = -X^{-1}$, we are always able to find an $h/k$ such that $q \in \mathcal{Q}_{h/k}$. This is a direct consequence of the theory of Farey fractions. In fact, if $h/k$ is a Farey fraction of order $N$ and $\xi_+$ (resp. $\xi_-$) denotes the distance from $h/k$ to its right (resp. left) neighboring mediant, then

$$\frac{1}{2kN} \le \xi_\pm \le \frac{1}{kN}.$$

Hence, $\mathbb{R}/\mathbb{Z}$ can be covered by intervals

$$\bigcup_{\substack{1\le h\le k\le N\\(h,k)=1}} \left[\frac{h}{k} - \frac{1}{kN}, \frac{h}{k} + \frac{1}{kN}\right].$$

Equipped with Theorem 1.1, we almost arrive at a proof of Conjecture 1.1.

**Theorem 1.2.** *Let*

$$G(q) := \sum_{n\geq 0} g(n)q^n = \frac{1}{(q, -q^3; q^4)_\infty}. \tag{1.7}$$

*We have, as $n \to \infty$,*

$$g(n) \sim \frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + (-1)^n \frac{\pi^{3/4}\Gamma(3/4)}{2^{11/4}3^{5/8}n^{5/8}} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) \tag{1.8}$$

*where $I_s(x)$ is the modified Bessel function of the first kind. Further, when $n \geq 2.4 \times 10^{14}$, we have $g(n) > 0$.*

Unfortunately, my personal laptop did not support me to verify the coefficients $g(n)$ up to $n = 2.4 \times 10^{14}$. But I deeply believe the validity of their nonnegativity after computing the first 10,000 terms.

**Notation.** We will use many standard notations from analytic number theory. First, the Vinogradov notation $f(x) \ll g(x)$ means that there exists an absolute constant $C$ such that $|f(x)| \leq Cg(x)$. If the constant $C$ depends on some variables, then we attach a subscript and write $f(x) \ll_{\text{variables}} g(x)$.

Also, $\zeta(s)$ and $\zeta(s, a)$ are respectively Riemann zeta function and Hurwitz zeta function. We denote by $\zeta'(s, a)$ the partial derivative of Hurwitz zeta function with respect to $s$, namely,

$$\zeta'(s, a) = \frac{\partial}{\partial s}\zeta(s, a).$$

Finally, $\Gamma(s)$ is the gamma function and $\gamma$ is the Euler–Mascheroni constant.

## 2. Theorem 1.1: Preparation

Recall that

$$\Phi_{a,M}(q) = \log\left(\frac{1}{(q^a; q^M)_\infty}\right) = \sum_{\substack{m\geq 1 \\ m\equiv a \bmod M}} \sum_{\ell\geq 1} \frac{q^{\ell m}}{\ell}. \tag{2.1}$$

Throughout, let us assume $X \geq 16$ and $N = \left\lfloor \sqrt{2\pi X} \right\rfloor$. As in Theorem 1.1, we put

$$q = e^{-\tau} e^{2\pi i h/k} \tag{2.2}$$

where $1 \leq h \leq k \leq N$ with $(h, k) = 1$ and

$$\tau = X^{-1} + 2\pi i Y \tag{2.3}$$

with the restriction

$$|Y| \leq \frac{1}{kN}. \tag{2.4}$$

Now we are going to collect some bounds that will be frequently used in the sequel. First, the assumptions of $X$ and $N$ imply that

$$0.9\sqrt{2\pi X} \leq N \leq \sqrt{2\pi X}. \tag{2.5}$$

Further, $N \leq \sqrt{2\pi X}$ implies that

$$\frac{1}{X} \leq \frac{2\pi}{N^2} \leq \frac{2\pi}{kN}.$$

Hence,

$$|\tau| \le \frac{2\sqrt{2}\pi}{kN}. \tag{2.6}$$

Finally,

$$\Re\left(\frac{1}{\tau}\right) \ge 0.07k^2. \tag{2.7}$$

This is because

$$\begin{aligned}\Re\left(\frac{1}{k^2\tau}\right) &= \frac{X^{-1}}{k^2(X^{-2}+4\pi^2Y^2)}\\ &\ge \frac{X^{-1}}{k^2(X^{-2}+4\pi^2k^{-2}N^{-2})}\\ &= \frac{X^{-1}}{k^2X^{-2}+4\pi^2N^{-2}}\\ &\ge \frac{X^{-1}}{N^2X^{-2}+4\pi^2N^{-2}}\\ &\ge \frac{X^{-1}}{(0.9\sqrt{2\pi X})^2X^{-2}+4\pi^2(0.9\sqrt{2\pi X})^{-2}}\\ &\ge 0.07.\end{aligned}$$

Given any positive integer $k$, we write

$$K = k\frac{M}{(k,M)}. \tag{2.8}$$

Notice that $M \mid K$. Write in (2.1)

$$\ell = bk+\mu \qquad (1\le \mu \le k)$$

and

$$m = cK+\lambda \qquad (1\le\lambda\le K,\ \lambda\equiv a \bmod M).$$

Then

$$\Phi_{a,M}(q) = \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} e^{\frac{2\pi i h\mu\lambda}{k}} \sum_{b,c\ge 0} \frac{1}{bk+\mu} e^{-(bk+\mu)(cK+\lambda)\tau}.$$

Applying the Mellin transform further gives

$$\begin{aligned}\Phi_{a,M}(q) &= \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} e^{\frac{2\pi i h\mu\lambda}{k}} \sum_{b,c\ge 0} \frac{1}{2\pi i}\int_{(3/2)} \frac{\Gamma(s)}{bk+\mu}\frac{ds}{(bk+\mu)^s(cK+\lambda)^s\tau^s}\\ &= \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} e^{\frac{2\pi i h\mu\lambda}{k}} \frac{1}{2\pi i}\int_{(3/2)} \frac{\Gamma(s)}{\tau^s k^{s+1}K^s}\zeta\left(s,\frac{\lambda}{K}\right)\zeta\left(1+s,\frac{\mu}{k}\right)ds.\end{aligned}$$

Here the path of integration $(\alpha)$ is from $\alpha - i\infty$ to $\alpha + i\infty$.

Recall the functional equation of Hurwitz zeta function:

$$\zeta\left(s,\frac{\lambda}{k}\right) = 2\Gamma(1-s)(2\pi k)^{s-1}\left(\sin\frac{\pi s}{2}\sum_{1\le\nu\le k}\cos\frac{2\pi\lambda\nu}{k}\,\zeta\left(1-s,\frac{\nu}{k}\right)\right.$$

$$+\cos\frac{\pi s}{2}\sum_{1\le\nu\le k}\sin\frac{2\pi\lambda\nu}{k}\,\zeta\Big(1-s,\frac{\nu}{k}\Big)\Bigg). \tag{2.9}$$

If we further put

$$z=\frac{\tau k}{2\pi}, \tag{2.10}$$

then

$$\begin{aligned}\Phi_{a,M}(q)=&\frac{1}{4\pi ikK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi h\mu\lambda}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1+s,\frac{\mu}{k}\big)\,\zeta\big(1-s,\frac{\nu}{K}\big)}{z^s\cos\frac{\pi s}{2}}ds\\ &+\frac{1}{4\pi ikK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi h\mu\lambda}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1+s,\frac{\mu}{k}\big)\,\zeta\big(1-s,\frac{\nu}{K}\big)}{z^s\sin\frac{\pi s}{2}}ds\\ &+\frac{1}{4\pi kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\sin\frac{2\pi h\mu\lambda}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1+s,\frac{\mu}{k}\big)\,\zeta\big(1-s,\frac{\nu}{K}\big)}{z^s\sin\frac{\pi s}{2}}ds\\ &+\frac{1}{4\pi kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\sin\frac{2\pi h\mu\lambda}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1+s,\frac{\mu}{k}\big)\,\zeta\big(1-s,\frac{\nu}{K}\big)}{z^s\cos\frac{\pi s}{2}}ds.\end{aligned} \tag{2.11}$$

Notice that $1\le\lambda\le K$. If $h\lambda_1\equiv h\lambda_2 \pmod k$, then by recalling $h_1\equiv h_2\equiv a \pmod M$ and the fact that $(h,k)=1$, we conclude that $\lambda_1\equiv\lambda_2 \pmod K$. Hence, the $h\lambda$'s give

$$\frac{K}{M}=\frac{k}{(k,M)}$$

residue classes modulo $k$. For each $\lambda$, we denote by $\rho=\rho(\lambda)$ the unique integer between 1 and $k$ such that

$$\rho\equiv -h\lambda \pmod k. \tag{2.12}$$

Then the $\rho$'s are pairwise distinct. Further, if we put

$$M^*=(k,M),$$

then for all $\rho$,

$$\rho\equiv -ha \pmod{M^*}. \tag{2.13}$$

Let us choose $h'$ so that

$$hh'\equiv -1 \pmod k.$$

This is always possible since $(h,k)=1$. Notice that $\lambda\equiv a \pmod M$. Hence, we have the system

$$\begin{cases}\lambda\equiv h'\rho & \pmod k\\ \lambda\equiv a & \pmod M\end{cases}. \tag{2.14}$$

This system is solvable whenever $h'\rho\equiv a \pmod{M^*}$. But this can be ensured by (2.13) and the fact that $hh'\equiv -1 \pmod{M^*}$. We next find, using Euclid's algorithm, integers $\alpha$ and $\beta$ such that

$$\alpha k+\beta M=M^*. \tag{2.15}$$

We therefore have (notice that $\mathrm{lcm}(k,M)=K$)

$$\lambda\equiv a+\beta M\frac{h'\rho-a}{M^*}=\beta h'\frac{M}{M^*}\rho+\alpha a\frac{k}{M^*} \pmod K. \tag{2.16}$$

In (2.11), replacing $s$ by $-s$, reversing the direction of integration path and shifting the path back to $(3/2)$, one has, with $h\lambda$ replaced by $-\rho$,

$$\begin{aligned}\Phi_{a,M}(q) = \frac{1}{4\pi ikK} &\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&- \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&+ \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&- \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\big(1-s,\frac{\mu}{k}\big)\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&- 2\pi i(R_1+R_2+R_3+R_4)\\
=:{}& \Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4 - 2\pi i(R_1+R_2+R_3+R_4) \qquad (2.17)\end{aligned}$$

where $R_*$ comes from the sum of residues of the corresponding integrand inside the stripe $-3/2 < \Re(s) < 3/2$.

In the next two sections, we shall evaluate the integrals $\Upsilon_*$ and the residues $R_*$, respectively. One may conclude Theorem 1.1 directly from (2.17) and the estimations (3.15), (4.11), (4.13), (4.15), (4.17), (4.22) and (4.24).

## 3. Theorem 1.1: The integrals

**3.1. An auxiliary function.** Let us define an auxiliary function

$$\Psi_{a,M}(q) := \log\left(\prod_{\substack{m\ge 1\\ m\equiv -ha \bmod M^*}} \frac{1}{1-e^{2\pi i\alpha a/M}q^m}\right). \qquad (3.1)$$

where $\alpha$ is defined in (2.15). We further write

$$m = bk+\rho \qquad (1\le\rho\le k,\ \rho\equiv -ha \bmod M^*).$$

Also, we put

$$q^* := \exp\left(\frac{2\pi i\beta h'}{k} - \frac{2\pi}{Kz}\right) \qquad (3.2)$$

where $\beta$ is again defined in (2.15). Then

$$\Psi_{a,M}(q^*) = -\sum_{\substack{1\le\rho\le k\\ \rho\equiv -ha \bmod M^*}}\sum_{b\ge 0}\log\left(1-\exp\left(\frac{2\pi i\beta h'}{k}\rho - \frac{2\pi}{Kz}(bk+\rho) + \frac{2\pi i\alpha a}{M}\right)\right).$$

It follows from (2.16) that

$$\exp\left(\frac{2\pi i\lambda}{K}\right) = \exp\left(\frac{2\pi i\beta h' M}{KM^*}\rho + \frac{2\pi i\alpha a k}{KM^*}\right) = \exp\left(\frac{2\pi i\beta h'}{k}\rho + \frac{2\pi i\alpha a}{M}\right).$$

Hence,

$$\begin{aligned}
\Psi_{a,M}(q^*) &= -\sum_{\substack{1\le\rho\le k\\ \rho\equiv -ha \bmod M^*}}\sum_{b\ge 0}\log\left(1-\exp\left(-\frac{2\pi}{Kz}(bk+\rho)+\frac{2\pi i\lambda}{K}\right)\right)\\
&= \sum_{\substack{1\le\rho\le k\\ \rho\equiv -ha \bmod M^*}}\sum_{1\le\nu\le K}\sum_{b,c\ge 0}\frac{1}{cK+\nu}\\
&\quad\times\exp\left((cK+\nu)\left(-\frac{2\pi}{Kz}(bk+\rho)+\frac{2\pi i\lambda}{K}\right)\right)\\
&= \sum_{\substack{1\le\rho\le k\\ \rho\equiv -ha \bmod M^*}}\sum_{1\le\nu\le K}e^{\frac{2\pi i\nu\lambda}{K}}\sum_{b,c\ge 0}\frac{1}{cK+\nu}e^{-(bk+\rho)(cK+\nu)\frac{2\pi}{Kz}}.
\end{aligned}$$

If we substitute $\rho$ back to $\lambda$ and apply Mellin transform and the functional equation of Hurwitz zeta function to $\Psi_{a,M}(q^*)$, then

$$\begin{aligned}
\Psi_{a,M}(q^*) &= \frac{1}{4\pi ikK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&\quad+\frac{1}{4\pi ikK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\sin\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&\quad+\frac{1}{4\pi kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\sin\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&\quad+\frac{1}{4\pi kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&=: J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.3}$$

Notice that

$$\Upsilon_1 = J_1 \qquad\text{and}\qquad \Upsilon_3 = J_3.$$

Further,

$$2(J_1+J_3) = \Psi_{a,M}(q^*)+\Psi_{M-a,M}(q^*).\tag{3.4}$$

**3.2. Estimations concerning Hurwitz zeta function.** Recall (see, for instance, [2, (25.11.9)]) that for $\Re(s)>1$ and $0<\alpha\le 1$,

$$\zeta(1-s,\alpha)=\frac{2\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{1}{n^s}\cos\left(\frac{1}{2}\pi s-2n\pi\alpha\right).$$

This implies that for $0<\alpha\le 1$, we have a uniform bound

$$|\zeta(-0.5+it,\alpha)|\le\frac{2\Gamma(3/2)\zeta(3/2)\cosh(\pi|t|/2)}{(2\pi)^{3/2}}.\tag{3.5}$$

It also follows from [1, Theorem 12.23] with some simple calculations that, uniformly for $|t|\ge 3$ and $0<\alpha\le 1$,

$$|\zeta(-0.5+it,\alpha)|\le 11|t|^{3/2}.\tag{3.6}$$

Finally, we have, for $0 < \alpha \le 1$,

$$|\zeta(2.5+it,\alpha)\,| \le \alpha^{-5/2} + \zeta(5/2). \tag{3.7}$$

**Lemma 3.1.** *Let $z$ be a complex number with $\Re(z) > 0$. Let $0 < \alpha, \beta \le 1$. Define integrals*

$$\mathcal{I}_+(z) := \int_{(3/2)} z^s \zeta(1+s,\alpha)\,\zeta(1-s,\beta)\left(\frac{1}{\cos\frac{\pi s}{2}} + \frac{1}{i\sin\frac{\pi s}{2}}\right) ds \tag{3.8}$$

*and*

$$\mathcal{I}_-(z) := \int_{(3/2)} z^s \zeta(1+s,\alpha)\,\zeta(1-s,\beta)\left(\frac{1}{\cos\frac{\pi s}{2}} - \frac{1}{i\sin\frac{\pi s}{2}}\right) ds. \tag{3.9}$$

*Then if $\Im(z) \le 0$, we have*

$$|\mathcal{I}_+(z)| \le 7.23|z|^{3/2}\big(\alpha^{5/2} + \zeta(5/2)\big), \tag{3.10}$$

*while if $\Im(z) \ge 0$, we have*

$$|\mathcal{I}_-(z)| \le 7.23|z|^{3/2}\big(\alpha^{5/2} + \zeta(5/2)\big). \tag{3.11}$$

*Proof.* Let us write $s = 3/2 + it$ as the path of integration is the vertical line $\Re(s) = 3/2$. We have

$$|z^s| = |z|^{3/2} e^{-\operatorname{Arg}(z)t}.$$

Also,

$$\left|\frac{1}{\cos\frac{\pi s}{2}} + \frac{1}{i\sin\frac{\pi s}{2}}\right| = \frac{2e^{-\frac{\pi}{2}t}}{|\sin(\pi s)|}.$$

Hence, for $z$ with $\Im(z) \le 0$ (recall that $\Re(z) > 0$ so that $-\pi/2 < \operatorname{Arg}(z) \le 0$), we have

$$|z^s|\left|\frac{1}{\cos\frac{\pi s}{2}} + \frac{1}{i\sin\frac{\pi s}{2}}\right| \le 2|z|^{3/2}\frac{e^{\frac{\pi}{2}|t|}}{|\sin(\pi s)|}.$$

It follows that

$$\begin{aligned}|\mathcal{I}_+(z)| &\le 2|z|^{3/2}\big((\alpha^{5/2} + \zeta(5/2)\big)\int_{-\infty}^{\infty} |\zeta(-0.5-it,\beta)|\,\frac{e^{\frac{\pi}{2}|t|}}{|\sin(\pi(1.5+it))|}dt\\ &\le 7.23|z|^{3/2}\big(\alpha^{5/2} + \zeta(5/2)\big).\end{aligned}$$

Similar arguments also apply to $\mathcal{I}_-(z)$ if $\Im(z) \ge 0$. $\square$

**3.3. Bounding the integrals.** Recall that

$$z = \frac{\tau k}{2\pi}.$$

For $\Upsilon_2$ and $\Upsilon_4$, we define

$$\Upsilon_* \pm J_* := \begin{cases}\Upsilon_* + J_* & \text{if } \Im(z) \ge 0,\\ \Upsilon_* - J_* & \text{if } \Im(z) < 0.\end{cases} \tag{3.12}$$

It follows from Lemma 3.1 that

$$|\Upsilon_* \pm J_*| \le \frac{1}{4\pi kK}\frac{kK}{M}\sum_{1\le\nu\le K} 7.23\,|z|^{3/2}\left(\left(\frac{K}{\nu}\right)^{5/2} + \zeta\left(\frac{5}{2}\right)\right)$$

$$\begin{aligned}
&\le \frac{1}{4\pi M} \cdot 7.23\, |z|^{3/2} \cdot 2\zeta(5/2) K^{5/2} \\
&\le \frac{7.23\, \zeta(5/2)}{2\pi M} \left|\frac{\tau k}{2\pi}\right|^{3/2} \left(k \frac{M}{(k,M)}\right)^{5/2} \\
&\le \frac{7.23\, \zeta(5/2)}{2\pi M} \left(\frac{\sqrt{2}}{N}\right)^{3/2} \left(\frac{M}{(k,M)}\right)^{5/2} N^{5/2} \qquad \text{(by (2.6))} \\
&\le \frac{7.23\, \zeta(5/2)\, 2^{3/4}}{2\pi M} \left(\frac{M}{(k,M)}\right)^{5/2} \sqrt{2\pi X} \\
&\le 6.51 \frac{M^{3/2}}{(k,M)^{5/2}} X^{1/2} \\
&\ll X^{1/2}. \qquad (3.13)
\end{aligned}$$

Finally, we bound

$$\begin{aligned}
|\Re(\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4)| &\le |\Re(\Upsilon_1+\Upsilon_3)| + |\Re(\Upsilon_2+\Upsilon_4)| \\
&\le |\Re(J_1+J_3)| + |\Re(J_2+J_4)| + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4| \\
&\le |\Re(\Psi_{a,M}(q^*))| + 2|\Re(J_1+J_3)| + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4| \\
&\le 2|\Re(\Psi_{a,M}(q^*))| + |\Re(\Psi_{M-a,M}(q^*))| \\
&\quad + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4|.
\end{aligned}$$

Recall from (3.2) that

$$q^* = \exp\left(\frac{2\pi i\beta h'}{k} - \frac{2\pi}{Kz}\right).$$

Hence,

$$|q^*| = \exp\left(\Re\left(-\frac{2\pi}{Kz}\right)\right) = \exp\left(-4\pi^2 \frac{(k,M)}{M} \Re\left(\frac{1}{k^2\tau}\right)\right).$$

By (2.7), we have

$$|q^*| \le \exp\left(-4\pi^2 \frac{(k,M)}{M} \cdot 0.07\right) \ll 1. \qquad (3.14)$$

We further have, by some simple partition-theoretic arguments that, for any $a = 1, 2, \ldots, M$,

$$\begin{aligned}
e^{|\Re(\Psi_{a,M}(q^*))|} &\le \prod_{\substack{m\ge 1 \\ m\equiv -ha \bmod M^*}} \frac{1}{1-|q^*|^m} \le \frac{1}{(|q^*|;|q^*|)_\infty} \\
&= \exp\left(-\sum_{\ell\ge 1} \log(1-|q^*|^\ell)\right) = \exp\left(\sum_{\ell\ge 1}\sum_{m\ge 1} \frac{|q^*|^{\ell m}}{m}\right) \\
&\le \exp\left(\sum_{n\ge 1} n|q^*|^n\right) = \exp\left(\frac{|q^*|}{(1-|q^*|)^2}\right).
\end{aligned}$$

In consequence,

$$|\Re(\Psi_{a,M}(q^*))| \le \frac{e^{-0.28\pi^2 \frac{(k,M)}{M}}}{\left(1-e^{-0.28\pi^2 \frac{(k,M)}{M}}\right)^2} \ll 1.$$

It turns out that

$$\begin{aligned}|\Re(\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4)| &\le 3\cdot \frac{e^{-0.28\pi^2 \frac{(k,M)}{M}}}{\left(1-e^{-0.28\pi^2 \frac{(k,M)}{M}}\right)^2} + 2\cdot 6.51\frac{M^{3/2}}{(k,M)^{5/2}}X^{1/2}\\ &\le \frac{3e^{-0.28\pi^2 \frac{(k,M)}{M}}}{\left(1-e^{-0.28\pi^2 \frac{(k,M)}{M}}\right)^2} + 13.02\frac{M^{3/2}}{(k,M)^{5/2}}X^{1/2}\\ &\ll X^{1/2}. \end{aligned} \tag{3.15}$$

## 4. Theorem 1.1: The residues

**4.1. Some lemmas.** We first require some finite summation formulas of Hurwitz zeta function, which follow from the first two aligned formulas on page 587 of [3].

**Lemma 4.1.** *For any $\theta = 1,2,\dots,k$,*

$$\sum_{1\le\alpha\le k} \cos\frac{2\pi\alpha\theta}{k}\zeta\Big(0,\frac{\alpha}{k}\Big) = -\frac{1}{2} \tag{4.1}$$

*and*

$$\sum_{1\le\alpha\le k} \cos\frac{2\pi\alpha\theta}{k}\zeta\Big(2,\frac{\alpha}{k}\Big) = \frac{\pi^2}{6}(6\theta^2-6k\theta+k^2). \tag{4.2}$$

*For any $\theta = 1,2,\dots,k-1$,*

$$\sum_{1\le\alpha\le k} \sin\frac{2\pi\alpha\theta}{k}\zeta\Big(0,\frac{\alpha}{k}\Big) = \frac{1}{2\pi}\left(\frac{\Gamma'}{\Gamma}\left(1-\frac{\theta}{k}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{\theta}{k}\right)\right) = \frac{1}{2}\cot\frac{\pi\theta}{k} \tag{4.3}$$

*and*

$$\sum_{1\le\alpha\le k} \sin\frac{2\pi\alpha\theta}{k}\zeta\Big(2,\frac{\alpha}{k}\Big) = 2\pi k^2\left(\zeta'\left(-1,\frac{\theta}{k}\right)-\zeta'\left(-1,1-\frac{\theta}{k}\right)\right). \tag{4.4}$$

We also need three finite summation formulas of the digamma function due to Gauß (cf. [10]).

**Lemma 4.2.** *For any $\theta = 1,2,\dots,k-1$,*

$$\sum_{1\le\alpha\le k} \cos\frac{2\pi\alpha\theta}{k}\frac{\Gamma'}{\Gamma}\Big(\frac{\alpha}{k}\Big) = k\log\left(2\sin\frac{\pi\theta}{k}\right) \tag{4.5}$$

*and*

$$\sum_{1\le\alpha\le k} \sin\frac{2\pi\alpha\theta}{k}\frac{\Gamma'}{\Gamma}\Big(\frac{\alpha}{k}\Big) = \frac{\pi}{2}(2\theta-k). \tag{4.6}$$

*Further,*

$$\sum_{1\le\alpha\le k}\frac{\Gamma'}{\Gamma}\Big(\frac{\alpha}{k}\Big)=-k(\gamma+\log k). \tag{4.7}$$

Next, it is easy to compute that

$$\sum_{|\Re(s)|\le 3/2}\mathrm{Res}_s\,\frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\cos\frac{\pi s}{2}}=\mathrm{Res}_{s=0}(*)+\mathrm{Res}_{s=-1}(*)+\mathrm{Res}_{s=1}(*)$$
$$=-\log z-\frac{\Gamma'}{\Gamma}\Big(\frac{\mu}{k}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{\nu}{K}\Big)+\frac{2\zeta\big(2,\frac{\mu}{k}\big)\,\zeta\big(0,\frac{\nu}{K}\big)}{\pi z}-\frac{2z\zeta\big(0,\frac{\mu}{k}\big)\,\zeta\big(2,\frac{\nu}{K}\big)}{\pi}$$

and

$$\sum_{|\Re(s)|\le 3/2}\mathrm{Res}_s\,\frac{\zeta\big(1-s,\frac{\mu}{k}\big)\,\zeta\big(1+s,\frac{\nu}{K}\big)}{z^{-s}\sin\frac{\pi s}{2}}=\mathrm{Res}_{s=0}(*)$$
$$=-\frac{\pi}{12}-\frac{(\log z)^2}{\pi}-\frac{2\log z}{\pi}\frac{\Gamma'}{\Gamma}\Big(\frac{\mu}{k}\Big)+\frac{2\log z}{\pi}\frac{\Gamma'}{\Gamma}\Big(\frac{\nu}{K}\Big)$$
$$+\frac{2}{\pi}\frac{\Gamma'}{\Gamma}\Big(\frac{\mu}{k}\Big)\frac{\Gamma'}{\Gamma}\Big(\frac{\nu}{K}\Big)+\frac{2}{\pi}\gamma_1\Big(\frac{\mu}{k}\Big)+\frac{2}{\pi}\gamma_1\Big(\frac{\nu}{K}\Big)$$

where $\gamma_1(\alpha)$ is the generalized Stieltjes constant.

Finally, recall from (2.12) that $\rho$ is the unique integer between 1 and $k$ such that $\rho\equiv -h\lambda \pmod k$. Hence,

$$\lambda=K \iff \rho=k. \tag{4.8}$$

Further, (2.13) says $\rho\equiv -ha \pmod{M^*}$. Recall also that $b$ is the unique integer between 1 and $M^*$ such that

$$b\equiv -ha \pmod{M^*}. \tag{4.9}$$

Then the following two summations represent the same thing:

$$\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}(*)\equiv\sum_{\substack{1\le\rho\le k\\ \rho\equiv b \bmod M^*}}(*).$$

**4.2. Evaluation of $R_1$.** We have

$$R_1=\frac{1}{4\pi ikK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}$$
$$\times\left(\frac{2\zeta\big(2,\frac{\mu}{k}\big)\,\zeta\big(0,\frac{\nu}{K}\big)}{\pi z}-\frac{2z\zeta\big(0,\frac{\mu}{k}\big)\,\zeta\big(2,\frac{\nu}{K}\big)}{\pi}\right).$$

First,

$$R_{11}:=\frac{1}{z}\frac{1}{2i\pi^2kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{1\le\mu\le k}\cos\frac{2\pi\mu\rho}{k}\zeta\Big(2,\frac{\mu}{k}\Big)\sum_{1\le\nu\le K}\cos\frac{2\pi\nu\lambda}{K}\zeta\Big(0,\frac{\nu}{K}\Big)$$
$$=\frac{1}{z}\frac{1}{2i\pi^2kK}\sum_{\substack{1\le\rho\le k\\ \rho\equiv b \bmod M^*}}\frac{\pi^2}{6}(6\rho^2-6k\rho+k^2)\cdot\Big(-\frac{1}{2}\Big)$$

$$
\begin{aligned}
&= \frac{1}{z}\frac{1}{2i\pi^2 kK}\cdot\frac{\pi^2}{6}\frac{k}{M^*}(6b^2-6bM^*+(M^*)^2)\cdot\left(-\frac{1}{2}\right)\\
&= -\frac{2\pi}{\tau k}\frac{1}{24iKM^*}(6b^2-6bM^*+(M^*)^2)\\
&= -\frac{2\pi}{\tau k}\frac{1}{24ikM}(6b^2-6bM^*+(M^*)^2). \qquad (4.10)
\end{aligned}
$$

Hence,

$$
-2\pi i R_{11} = \frac{1}{\tau}\frac{\pi^2}{6k^2M}(6b^2-6b(k,M)+(k,M)^2). \qquad (4.11)
$$

Also,

$$
\begin{aligned}
R_{12} &:= -z\frac{1}{2i\pi^2kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{1\le\mu\le k}\cos\frac{2\pi\mu\rho}{k}\zeta\Big(0,\frac{\mu}{k}\Big)\sum_{1\le\nu\le K}\cos\frac{2\pi\nu\lambda}{K}\zeta\Big(2,\frac{\nu}{K}\Big)\\
&= -z\frac{1}{2i\pi^2kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\left(-\frac{1}{2}\right)\cdot\frac{\pi^2}{6}(6\lambda^2-6k\lambda+k^2)\\
&= -z\frac{1}{2i\pi^2kK}\cdot\left(-\frac{1}{2}\right)\cdot\frac{\pi^2}{6}\frac{K}{M}(6a^2-6aM+M^2)\\
&= \frac{\tau k}{2\pi}\frac{1}{24ikM}(6a^2-6aM+M^2)\\
&= \tau\frac{1}{48i\pi M}(6a^2-6aM+M^2). \qquad (4.12)
\end{aligned}
$$

Hence, recalling that $a=1,2,\ldots,M$, we have

$$
\begin{aligned}
|-2\pi i R_{12}| &= |\tau|\frac{|6a^2-6aM+M^2|}{24M}\\
&\le \frac{2\sqrt{2}\pi}{kN}\cdot\frac{M^2}{24M}\\
&\le \frac{2\sqrt{2}\pi}{k\cdot 0.9\sqrt{2\pi X}}\cdot\frac{M}{24}.
\end{aligned}
$$

In consequence,

$$
|-2\pi i R_{12}| \le 0.17\frac{M}{k}X^{-1/2} \ll X^{-1/2}. \qquad (4.13)
$$

**4.3. Evaluation of $R_2$.** We have

$$
R_2 = -\frac{1}{4\pi ikK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\cdot\frac{2}{\pi}\frac{\Gamma'}{\Gamma}\Big(\frac{\mu}{k}\Big)\frac{\Gamma'}{\Gamma}\Big(\frac{\nu}{K}\Big).
$$

Hence, with (4.8),

$$
\begin{aligned}
R_2 &= -\frac{1}{2i\pi^2kK}\sum_{\substack{1\le\lambda<K\\ \lambda\equiv a \bmod M}}\sum_{1\le\mu\le k}\cos\frac{2\pi\mu\rho}{k}\frac{\Gamma'}{\Gamma}\Big(\frac{\mu}{k}\Big)\sum_{1\le\nu\le K}\sin\frac{2\pi\nu\lambda}{K}\frac{\Gamma'}{\Gamma}\Big(\frac{\nu}{K}\Big)\\
&= -\frac{1}{2i\pi^2kK}\sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}}k\log\Big(2\sin\frac{\pi\rho}{k}\Big)\cdot\frac{\pi}{2}(2\lambda-K)
\end{aligned}
$$

$$= -\frac{1}{4i\pi K} \sum_{\substack{1\le\rho<k \\ \rho\equiv b \bmod M^*}} (2\lambda-K)\log\left(2\sin\frac{\pi\rho}{k}\right). \tag{4.14}$$

Notice that for $0 \le x \le \pi/2$, we have

$$|\log(2\sin x)| \le \frac{\pi \log 2}{2x}.$$

Hence,

$$\begin{aligned} |-2\pi i R_2| &\le \frac{1}{2K}\cdot 2 \sum_{1\le\rho<k} K \cdot \frac{\pi\log 2}{2}\frac{k}{\pi\rho} \\ &= \frac{\log 2}{2} k \sum_{1\le\rho<k}\frac{1}{\rho} \\ &\le \frac{\log 2}{2} k(\log k+\gamma) \\ &\le \frac{\log 2}{2} N(\log N+\gamma) \\ &\le \frac{\log 2}{2}\sqrt{2\pi X}(\log\sqrt{2\pi X}+\gamma). \end{aligned}$$

In consequence,

$$|-2\pi i R_2| \le 1.3X^{1/2} + 0.44X^{1/2}\log X \ll X^{1/2}\log X. \tag{4.15}$$

**4.4. Evaluation of $R_3$.** We have

$$R_3 = \frac{1}{4\pi k K}\sum_{\substack{1\le\lambda\le K \\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k \\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\cdot\frac{2}{\pi}\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right).$$

Hence, with (4.8),

$$\begin{aligned} R_3 &= \frac{1}{2\pi^2 kK}\sum_{\substack{1\le\lambda<K \\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k}\sin\frac{2\pi\mu\rho}{k}\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)\sum_{1\le\nu\le K}\sin\frac{2\pi\nu\lambda}{K}\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right) \\ &= \frac{1}{2\pi^2 kK}\sum_{\substack{1\le\rho<k \\ \rho\equiv b \bmod M^*}} \frac{\pi}{2}(2\rho-k)\cdot\frac{\pi}{2}(2\lambda-K) \\ &= \frac{1}{8kK}\sum_{\substack{1\le\rho<k \\ \rho\equiv b \bmod M^*}} (2\rho-k)(2\lambda-K). \end{aligned} \tag{4.16}$$

In consequence,

$$|-2\pi i R_3| \le 2\pi\cdot\frac{1}{8kK}\cdot\frac{k}{M^*}kK = \frac{\pi k}{4M^*} \le \frac{\pi N}{4M^*} \le \frac{\pi\sqrt{2\pi X}}{4M^*}.$$

Namely,

$$|-2\pi i R_3| \le \frac{1.97}{(k,M)}X^{1/2} \ll X^{1/2}. \tag{4.17}$$

**4.5. Evaluation of $R_4$.** We have

$$R_4 = -\frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}$$
$$\times\left(\frac{2\zeta\big(2,\frac{\mu}{k}\big)\,\zeta\big(0,\frac{\nu}{K}\big)}{\pi z} - \frac{2z\zeta\big(0,\frac{\mu}{k}\big)\,\zeta\big(2,\frac{\nu}{K}\big)}{\pi}\right).$$

First,

$$\begin{aligned}
R_{41} &:= -\frac{1}{z}\frac{1}{2\pi^2 kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} \sin\frac{2\pi\mu\rho}{k}\zeta\Big(2,\frac{\mu}{k}\Big) \sum_{1\le\nu\le K} \cos\frac{2\pi\nu\lambda}{K}\zeta\Big(0,\frac{\nu}{K}\Big)\\
&= -\frac{1}{z}\frac{1}{2\pi^2 kK} \sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}} 2\pi k^2\left(\zeta'\left(-1,\frac{\rho}{k}\right) - \zeta'\left(-1,1-\frac{\rho}{k}\right)\right)\cdot\left(-\frac{1}{2}\right)\\
&= \frac{1}{z}\frac{k}{2\pi K} \sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}} \left(\zeta'\left(-1,\frac{\rho}{k}\right) - \zeta'\left(-1,1-\frac{\rho}{k}\right)\right).
\end{aligned}$$

If $b = M^*$, then both $\rho$ and $k-\rho$ run through all multiples of $M^*$ within the range $[1,k)$, and hence

$$R_{41} = 0. \tag{4.18}$$

We further notice that if $d \mid k$ and $1 \le c \le d$, then for any $s \ne 1$,

$$\sum_{\substack{1\le\ell\le k\\ \ell\equiv c \bmod d}} \zeta\Big(s,\frac{\ell}{k}\Big) = \left(\frac{k}{d}\right)^s \zeta\Big(s,\frac{c}{d}\Big) \tag{4.19}$$

Hence,

$$\sum_{\substack{1\le\ell<k\\ \ell\equiv c \bmod d}} \zeta'\left(s,\frac{\ell}{k}\right) = \left(\frac{k}{d}\right)^s \zeta\Big(s,\frac{c}{d}\Big)\log(k/d) + \left(\frac{k}{d}\right)^s \zeta'\left(s,\frac{c}{d}\right). \tag{4.20}$$

Since $M^* = (k,M)$ divides $k$, it follows that if $b \ne M^*$ (and hence $\rho \ne k$), then

$$\begin{aligned}
R_{41} &= \frac{1}{z}\frac{k}{2\pi K}\Bigg(\left(\frac{M^*}{k}\right)\zeta\Big(-1,\frac{b}{M^*}\Big)\log\frac{k}{M^*} + \left(\frac{M^*}{k}\right)\zeta'\left(-1,\frac{b}{M^*}\right)\\
&\qquad - \left(\frac{M^*}{k}\right)\zeta\Big(-1,\frac{M^*-b}{M^*}\Big)\log\frac{k}{M^*} - \left(\frac{M^*}{k}\right)\zeta'\left(-1,\frac{M^*-b}{M^*}\right)\Bigg)\\
&= \frac{1}{z}\frac{(k,M)^2}{M}\frac{1}{2\pi k}\left(\zeta'\left(-1,\frac{b}{M^*}\right) - \zeta'\left(-1,\frac{M^*-b}{M^*}\right)\right)\\
&= \frac{1}{\tau}\frac{(k,M)^2}{M}\frac{1}{k^2}\left(\zeta'\left(-1,\frac{b}{(k,M)}\right) - \zeta'\left(-1,\frac{(k,M)-b}{(k,M)}\right)\right).
\end{aligned} \tag{4.21}$$

It turns out that

$$-2\pi i R_{41} = \begin{cases} 0 & \text{if } b = (k,M), \\ -\frac{1}{\tau}\frac{(k,M)^2}{M}\frac{2\pi i}{k^2}\left(\zeta'\left(-1, \frac{b}{(k,M)}\right) - \zeta'\left(-1, \frac{(k,M)-b}{(k,M)}\right)\right) & \text{if } b \neq (k,M). \end{cases} \tag{4.22}$$

Also,

$$\begin{aligned} R_{42} &:= z\frac{1}{2\pi^2 kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{1\le\mu\le k}\sin\frac{2\pi\mu\rho}{k}\zeta\Big(0,\frac{\mu}{k}\Big)\sum_{1\le\nu\le K}\cos\frac{2\pi\nu\lambda}{K}\zeta\Big(2,\frac{\nu}{K}\Big) \\ &= z\frac{1}{2\pi^2 kK}\sum_{\substack{1\le\lambda<K\\ \lambda\equiv a \bmod M}}\frac{1}{2}\cot\frac{\pi\rho}{k}\cdot\frac{\pi^2}{6}(6\lambda^2-6K\lambda+K^2) \\ &= z\frac{1}{24kK}\sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}}(6\lambda^2-6K\lambda+K^2)\cot\frac{\pi\rho}{k} \\ &= \tau\frac{1}{48\pi K}\sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}}(6\lambda^2-6K\lambda+K^2)\cot\frac{\pi\rho}{k}. \end{aligned} \tag{4.23}$$

Notice that for $1 \le \lambda \le K$,

$$|6\lambda^2 - 6K\lambda + K^2| \le K^2$$

and for $0 < x \le \pi/2$,

$$|\cot x| \le \frac{1}{x}.$$

Hence,

$$\begin{aligned} |-2\pi i R_{42}| &= |\tau|\frac{1}{24K}\left|\sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}}(6\lambda^2-6K\lambda+K^2)\cot\frac{\pi\rho}{k}\right| \\ &\le \frac{2\sqrt{2}\pi}{kN}\frac{1}{24K}\cdot 2K^2\sum_{1\le\ell<k}\frac{k}{\pi\ell} \\ &= \frac{\sqrt{2}}{6N}\frac{kM}{(k,M)}(\log k+\gamma) \\ &\le \frac{\sqrt{2}}{6N}\frac{NM}{(k,M)}(\log N+\gamma) \\ &\le \frac{\sqrt{2}}{6}\frac{M}{(k,M)}(\log\sqrt{2\pi X}+\gamma). \end{aligned}$$

In consequence,

$$|-2\pi i R_{42}| \le 0.12\frac{M}{(k,M)}\log X + 0.36\frac{M}{(k,M)} \ll \log X. \tag{4.24}$$

## 5. Explicit bounds of $G(q)$

Recall that

$$G(q) = \frac{(q^3;q^4)_\infty}{(q;q^4)_\infty (q^6;q^8)_\infty}.$$

The goal of this section is the following uniform bound of $|G(q)|$ when $q$ is away from $\pm 1$.

**Theorem 5.1.** *Let $\mathcal{Q}_{h/k}$ be as in Remark 1.1. For any $q$ (with $|q| = e^{-1/X}$) not in $\mathcal{Q}_{1/1}$ and $\mathcal{Q}_{1/2}$, we have, if $X \geq 3.4 \times 10^7$, then*

$$|G(q)| \leq \exp\left( \left( \frac{\pi^2}{48} - \frac{1}{100} \right) X \right). \tag{5.1}$$

*Further, if $q = e^{-\tau + 2\pi i h/k}$ with $\tau = X^{-1} + 2\pi i Y$ is in $\mathcal{Q}_{1/1}$ or $\mathcal{Q}_{1/2}$, then* (5.1) *still holds under the assumption $X \geq 3.4 \times 10^7$ provided that $|Y| \geq 1/(2\pi X)$.*

Notice that $\tau = X^{-1} + 2\pi i Y$. Hence,

$$\tau^{-1} = \frac{X^{-1}}{X^{-2} + 4\pi^2 Y^2} - i \frac{2\pi Y}{X^{-2} + 4\pi^2 Y^2}. \tag{5.2}$$

In the sequel, we write $b$ as $b(h,a,k,M)$ to avoid confusion. We also write for convenience

$$\begin{aligned} \mathfrak{M}_{a,M} := \frac{1}{\tau} \frac{(k,M)^2}{k^2 M} \Bigg( & \pi^2 \left( \frac{b^2}{(k,M)^2} - \frac{b}{(k,M)} + \frac{1}{6} \right) \\ & + 2\pi i \left( -\zeta'\left(-1, \frac{b}{(k,M)}\right) + \zeta'\left(-1, \frac{b^*}{(k,M)}\right) \right) \Bigg), \end{aligned} \tag{5.3}$$

which is the main term in (1.5). Further,

$$\mathfrak{M}_G := \mathfrak{M}_{1,4} - \mathfrak{M}_{3,4} + \mathfrak{M}_{6,8} \tag{5.4}$$

denotes the main term of $\log G(q)$ whereas

$$E_G := \log G(q) - \mathfrak{M}_G \tag{5.5}$$

denotes the error term.

### 5.1. Case 1: $k \in 2\mathbb{Z} + 1$.

Notice that $(k,4) = 1$. Hence, we always have $b(h,1,k,4) = b(h,3,k,4) = 1$. Also, $(k,8) = 1$. Then $b(h,6,k,8) = 1$. It is not hard to compute that

$$\mathfrak{M}_G = \frac{1}{\tau} \frac{\pi^2}{48k^2}. \tag{5.6}$$

It follows from (5.2) that

$$\Re(\mathfrak{M}_G) \leq \frac{\pi^2}{48k^2} X. \tag{5.7}$$

We may also compute from the bounds (3.15), (4.13), (4.15), (4.17) and (4.24) that

$$|\Re(E_G)| \leq 1.32 X^{1/2} \log X + 512.74 X^{1/2} + 1.92 \log X + 42.74 + 2.72 X^{-1/2}. \tag{5.8}$$

**5.2. Case 2: $k \in 4\mathbb{Z}+2$.** Notice that $(k,4)=2$. Since $(h,k)=1$, so $h$ is odd. Hence, we always have $b(h,1,k,4)=b(h,3,k,4)=1$. Also, $(k,8)=1$. We have $b(h,6,k,8)=2$. It is not hard to compute that

$$\mathfrak{M}_G = \frac{1}{\tau}\frac{\pi^2}{12k^2}. \tag{5.9}$$

It follows from (5.2) that

$$\Re(\mathfrak{M}_G) \le \frac{\pi^2}{12k^2}X. \tag{5.10}$$

For the error term $E_G$, we have

$$|\Re(E_G)| \le 1.32X^{1/2}\log X + 95.77X^{1/2} + 0.96\log X + 11.61 + 2.72X^{-1/2}. \tag{5.11}$$

**5.3. Case 3: $k \in 8\mathbb{Z}+4$.** Notice that $(k,4)=4$. If $h \equiv 1 \pmod 4$, then $b(h,1,k,4)=3$ and $b(h,3,k,4)=1$. If $h\equiv 3 \pmod 4$, then $b(h,1,k,4)=1$ and $b(h,3,k,4)=3$. Hence,

$$\mathfrak{M}_{1,4}-\mathfrak{M}_{3,4} = \frac{1}{\tau}\frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)$$

where

$$\chi(h) = \begin{cases} 1 & \text{if } h\equiv 1 \pmod 4, \\ -1 & \text{if } h \equiv 3 \pmod 4. \end{cases}$$

Also, $(k,8)=4$. Since $(h,k)=1$, so $h$ is odd. Hence, we have $b(h,6,k,8)=2$. It follows that

$$\mathfrak{M}_{6,8} = -\frac{1}{\tau}\frac{\pi^2}{6k^2}.$$

Hence,

$$\mathfrak{M}_G = \frac{1}{\tau}\left(-\frac{\pi^2}{6k^2}+\frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)\right). \tag{5.12}$$

It follows from (5.2) that

$$\begin{aligned}
\Re(\mathfrak{M}_G) &= -\frac{\pi^2}{6k^2}\frac{X^{-1}}{X^{-2}+4\pi^2Y^2} \\
&\quad + \frac{16\pi\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)\frac{2\pi Y}{X^{-2}+4\pi^2Y^2} \\
&\le \frac{1}{k^2}\cdot\frac{-\frac{\pi^2}{6}X^{-1}+16\pi\left(\zeta'\left(-1,\frac14\right)-\zeta'\left(-1,\frac34\right)\right)2\pi|Y|}{X^{-2}+4\pi^2|Y|^2} \\
&= \frac{\pi^2}{6k^2}\cdot\frac{-X^{-1}+192\left(\zeta'\left(-1,\frac14\right)-\zeta'\left(-1,\frac34\right)\right)|Y|}{X^{-2}+4\pi^2|Y|^2}.
\end{aligned}$$

We next show that

$$\Re(\mathfrak{M}_G) \le \frac{2.94}{k^2}X. \tag{5.13}$$

It suffices to prove that

$$\frac{\pi^2}{6k^2}\cdot\frac{-X^{-1}+192\left(\zeta'\left(-1,\frac14\right)-\zeta'\left(-1,\frac34\right)\right)|Y|}{X^{-2}+4\pi^2|Y|^2} \le \frac{2.94}{k^2}X.$$

Namely,

$$70.56X|Y|^2 - 192\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)|Y| + \left(\frac{17.64}{\pi^2}+1\right)X^{-1} \geq 0.$$

Notice that on the left-hand side if we replace $|Y|$ by $t$ and treat it as a quadratic function of real $t$, then it reaches the minimum when

$$t = \frac{192\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)}{2\times 70.56X}.$$

Further, the minimum is

$$-70.56X \times \left(\frac{192\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)}{2\times 70.56X}\right)^2 + \left(\frac{17.64}{\pi^2}+1\right)X^{-1} \geq 0.01X^{-1} \geq 0.$$

Hence, (5.13) holds.

For the error term $E_G$, we have

$$|\Re(E_G)| \leq 1.32X^{1/2}\log X + 21.1X^{1/2} + 0.48\log X + 3.22 + 2.72X^{-1/2}. \tag{5.14}$$

**5.4. Case 4: $k \in 8\mathbb{Z}$.** As in *Case 3*, we still have

$$\mathfrak{M}_{1,4} - \mathfrak{M}_{3,4} = \frac{1}{\tau}\frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right).$$

Also, $(k,8)=8$. If $h \equiv 1 \pmod 4$, then $b(h,6,k,8)=2$. If $h\equiv 3 \pmod 4$, then $b(h,6,k,8)=6$. Hence,

$$\mathfrak{M}_{6,8} = \frac{1}{\tau}\left(-\frac{\pi^2}{6k^2} - \frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)\right).$$

In consequence,

$$\mathfrak{M}_G = -\frac{1}{\tau}\frac{\pi^2}{6k^2}. \tag{5.15}$$

Further,

$$\Re(\mathfrak{M}_G) < 0. \tag{5.16}$$

For the error term $E_G$, we have

$$|\Re(E_G)| \leq 1.32X^{1/2}\log X + 13.27X^{1/2} + 0.36\log X + 1.73 + 2.72X^{-1/2}. \tag{5.17}$$

*Proof of Theorem 5.1.* We have

$$\log|G(q)| = \Re(\log G(q)) \leq \Re(\mathfrak{M}_G) + |\Re(E_G)|.$$

The first part simply follows from some direct computation by taking into account of the bounds for $\Re(\mathfrak{M}_G)$ and $|\Re(E_G)|$. For the second part, we notice by (5.2) that, when $|Y| \geq 1/(2\pi X)$,

$$\Re(\tau^{-1}) \leq \frac{X}{2}.$$

Whenever $q$ is in $\mathcal{Q}_{1/1}$ or $\mathcal{Q}_{1/2}$, we apply (5.6) and (5.9) to obtain the bound

$$\Re(\mathfrak{M}_G) \leq \frac{\pi^2}{48}\frac{X}{2}.$$

Hence, (5.1) follows by inserting the contribution of the error term and carrying on the routine computation. □

## 6. Precise approximations of $G(q)$ near the dominant poles

Recall that

$$G(q) = \frac{1}{(q, -q^3; q^4)_\infty}. \tag{6.1}$$

From the analysis in the previous section, we know that $G(q)$ indeed has dominant poles at $q = \pm 1$. In fact, if $q = e^{-\tau + 2\pi i h/k}$ is in $\mathcal{Q}_{1/1}$ or $\mathcal{Q}_{1/2}$, then (5.6) and (5.9) tell us that $\log G(q)$ is dominated by $\pi^2/(48\tau)$ while the coefficient $\pi^2/48$ is the largest comparing with that for other $\mathcal{Q}_{h/k}$. Now we want to give some more precise approximations of $\log G(q)$ near the dominant poles.

**Theorem 6.1.** *Let $\tau = X^{-1} + 2\pi i Y$ with $|Y| \le 1/(2\pi X)$. Then*

$$\log G(e^{-\tau}) = \frac{\pi^2}{48}\frac{1}{\tau} - \frac{1}{4}\log\tau - \frac{3}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma\left(\frac{1}{4}\right) + E_+ \tag{6.2}$$

*where*

$$|E_+| \le 0.66 X^{-3/4}. \tag{6.3}$$

*Further,*

$$\log G(-e^{-\tau}) = \frac{\pi^2}{48}\frac{1}{\tau} + \frac{1}{4}\log\tau - \frac{1}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma\left(\frac{3}{4}\right) + E_- \tag{6.4}$$

*where*

$$|E_-| \le 0.82 X^{-3/4}. \tag{6.5}$$

*Proof.* We deduce from (6.1) with the help of Mellin transform that

$$\begin{aligned}
\log G(e^{-\tau}) &= \sum_{m\ge 0}\sum_{\ell\ge 1}\left(\frac{e^{-(4m+1)\ell\tau}}{\ell} + \frac{(-1)^\ell e^{-(4m+3)\ell\tau}}{\ell}\right)\\
&= \frac{1}{2\pi i}\int_{(3/2)} \tau^{-s}\Gamma(s)\sum_{m\ge 0}\sum_{\ell\ge 1}\ell^{-s-1}\left(\frac{1}{(4m+1)^s} + \frac{(-1)^\ell}{(4m+3)^s}\right)ds\\
&= \frac{1}{2\pi i}\int_{(3/2)} (4\tau)^{-s}\Gamma(s)\zeta(s+1)\left(\zeta\left(s, \frac{1}{4}\right) - (1-2^{-s})\zeta\left(s, \frac{3}{4}\right)\right)ds\\
&=: \frac{1}{2\pi i}\int_{(3/2)} \Theta_+(s)ds.
\end{aligned}$$

Now one may shift the path of integration to $(-3/4)$ by taking into consideration of the residues of $\Theta_+(s)$ inside the stripe $-3/4 < \Re(s) < 3/2$. Hence,

$$\log G(e^{-\tau}) = \sum_{-3/4<\Re(s)<3/2} \operatorname{Res}_s \Theta_+(s) + \frac{1}{2\pi i}\int_{(-3/4)} \Theta_+(s)ds.$$

Notice that $\Theta_+(s)$ has two singularities respectively at $s = 0$ and $1$ when $-3/4 < \Re(s) < 3/2$. We compute that

$$\operatorname{Res}_{s=1}\Theta_+(s) = \frac{\pi^2}{48}\frac{1}{\tau}$$

and

$$\operatorname{Res}_{s=0}\Theta_+(s) = -\frac{1}{4}\log(4\tau) + \zeta'\left(0, \frac{1}{4}\right) - (\log 2)\zeta\left(0, \frac{3}{4}\right)$$

$$= -\frac{1}{4}\log(4\tau) + \log\Gamma\left(\frac{1}{4}\right) - \frac{1}{2}\log(2\pi) + \frac{1}{4}\log 2$$
$$= -\frac{1}{4}\log\tau - \frac{3}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma\left(\frac{1}{4}\right).$$

Further, recalling that $\tau = X^{-1} + 2\pi i Y$ where $|Y| \le 1/(2\pi X)$, we have $|\operatorname{Arg}(\tau)| \le \pi/4$. Since for $\Re(s) = -3/4$,

$$|\tau^{-s}| = \exp\left(\frac{3}{4}\log|\tau| + \Im(s)\operatorname{Arg}(\tau)\right) \le |\tau|^{3/4} e^{|\Im(s)|\pi/4},$$

it follows that

$$\begin{aligned}
|E_+| &= \left|\frac{1}{2\pi i}\int_{(-3/4)} \Theta_+(s)ds\right| \\
&= \left|\frac{1}{2\pi i}\int_{(-3/4)} (4\tau)^{-s}\Gamma(s)\zeta(s+1)\left(\zeta\left(s,\frac{1}{4}\right) - (1-2^{-s})\zeta\left(s,\frac{3}{4}\right)\right)ds\right| \\
&\le |\tau|^{3/4}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} 4^{3/4} e^{|t|\pi/4}\left|\Gamma\left(-\frac{3}{4}+it\right)\right|\left|\zeta\left(\frac{1}{4}+it\right)\right| \\
&\quad\times\left(\left|\zeta\left(-\frac{3}{4}+it,\frac{1}{4}\right)\right| + (1+2^{3/4})\left|\zeta\left(-\frac{3}{4}+it,\frac{3}{4}\right)\right|\right)dt \\
&\le 0.507|\tau|^{3/4}.
\end{aligned}$$

We also have

$$|\tau| = \sqrt{X^{-2} + 4\pi^2 Y^2} \le \sqrt{2}X^{-1}.$$

Hence,

$$|E_+| \le 0.66X^{-3/4}.$$

For $\log G(-e^{-\tau})$, we simply notice that

$$\log G(-e^{-\tau}) = \frac{1}{2\pi i}\int_{(3/2)} (4\tau)^{-s}\Gamma(s)\zeta(s+1)\left(\zeta\left(s,\frac{3}{4}\right) - (1-2^{-s})\zeta\left(s,\frac{1}{4}\right)\right)ds.$$

The rest follows from similar calculations. □

## 7. Applying the circle method

The proof of Theorem 1.2 is simply an exercise of the circle method. We first put

$$X = \sqrt{\frac{48n}{\pi^2}}. \tag{7.1}$$

Since it is assumed that $X \ge 3.4 \times 10^7$ as in Theorem 5.1, one has

$$n \ge 2.4 \times 10^{14}. \tag{7.2}$$

Recall that Cauchy's integral formula indicates that

$$\begin{aligned}
g(n) &= \frac{1}{2\pi i}\int_{|q|=e^{-1/X}} \frac{G(q)}{q^{n+1}}dq \\
&= e^{n/X}\int_{-\frac{1}{2\pi X}}^{1-\frac{1}{2\pi X}} G\big(e^{-(X^{-1}+2\pi i t)}\big)e^{2\pi i n t}\,dt. \qquad (7.3)
\end{aligned}$$

Now we separate the interval $[-\frac{1}{2\pi X},\, 1-\frac{1}{2\pi X}]$ into three (disjoint) subintervals:

$$I_1 := \left[-\frac{1}{2\pi X},\, \frac{1}{2\pi X}\right],$$
$$I_2 := \left[\frac{1}{2}-\frac{1}{2\pi X},\, \frac{1}{2}+\frac{1}{2\pi X}\right]$$

and

$$I_3 := \left[-\frac{1}{2\pi X},\, 1-\frac{1}{2\pi X}\right] - I_1 - I_2.$$

Before evaluating (7.3) for each subinterval, we fix the notation that $\mathfrak{O}(x)$ means an expression $E$ such that $|E| \le x$. We also write for $j = 1, 2, 3$,

$$g_j(n) := e^{n/X} \int_{I_j} G\big(e^{-(X^{-1}+2\pi i t)}\big) e^{2\pi i n t}\, dt.$$

First,

$$\begin{aligned} g_1(n) &= e^{n/X} \int_{-\frac{1}{2\pi X}}^{\frac{1}{2\pi X}} G\big(e^{-(X^{-1}+2\pi i t)}\big) e^{2\pi i n t}\, dt \\ &= \frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} e^{n\tau} G(e^{-\tau})\, d\tau. \end{aligned}$$

Notice that for $|x| \le 1$,

$$e^x = 1 + \mathfrak{O}(2|x|).$$

Applying (6.2) yields

$$g_1(n) = \frac{\big(1+\mathfrak{O}(1.32X^{-3/4})\big)\Gamma(1/4)}{2^{3/4}\pi^{1/2}} \frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right) d\tau. \quad (7.4)$$

We then separate the integral as

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right) d\tau \\ &= \frac{1}{2\pi i}\left(\int_\Gamma - \int_{-\infty-i\frac{1}{X}}^{\frac{1}{X}-i\frac{1}{X}} + \int_{-\infty+i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}}\right) \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right) d\tau \\ &=: J_{11} + J_{12} + J_{13} \end{aligned}$$

where

$$\Gamma := (-\infty - iX^{-1}) \to (X^{-1} - iX^{-1}) \to (X^{-1} + iX^{-1}) \to (-\infty + iX^{-1}) \quad (7.5)$$

is a Hankel contour. To evaluate $J_{11}$, we make the change of variables

$$\tau = \sqrt{\frac{\pi^2}{48n}}\, w.$$

Then

$$J_{11} = \left(\frac{\pi^2}{48n}\right)^{3/8} \frac{1}{2\pi i} \int_{\tilde\Gamma} w^{-\frac{1}{4}} \exp\left(\sqrt{\frac{\pi^2 n}{48}}\left(\frac{1}{w} + w\right)\right) dw$$

where $\tilde{\Gamma}$ is the new contour. Recalling the contour integral representation of $I_s(x)$:

$$I_s(x) = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} w^{-s-1} e^{\frac{x}{2}\left(w+\frac{1}{w}\right)}\, dw,$$

we conclude

$$J_{11} = \frac{\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).$$

To bound $J_{12}$, we put $\tau = x - iX^{-1}$. Then

$$J_{12} = \frac{1}{2\pi i}\int_{-\infty}^{X^{-1}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right)\, dx.$$

Since $|\tau| \geq X^{-1}$, we have

$$|\tau|^{-1/4} \leq X^{1/4}.$$

Also,

$$|e^{n\tau}| = e^{nx}.$$

Further,

$$\left| e^{\frac{\pi^2}{48}\frac{1}{\tau}} \right| = e^{\frac{\pi^2}{48}\frac{x}{x^2+X^{-2}}} \leq e^{\frac{\pi^2}{96}X}.$$

Hence,

$$\begin{aligned} |J_{12}| &\leq \frac{1}{2\pi}\cdot X^{1/4} e^{\frac{\pi^2}{96}X} \int_{-\infty}^{X^{-1}} e^{nx}\, dx \\ &= \frac{1}{2\pi}\cdot X^{1/4} e^{\frac{\pi^2}{96}X}\cdot \frac{1}{n} e^{n/X} \\ &= \frac{3^{1/8}}{2^{1/2}\pi^{5/4}n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right). \end{aligned}$$

One may carry out a similar argument to obtain

$$|J_{13}| \leq \frac{3^{1/8}}{2^{1/2}\pi^{5/4}n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right).$$

In consequence,

$$\begin{aligned} \frac{1}{2\pi i}\int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right)\, d\tau &= \frac{\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) \\ &\quad + \mathfrak{O}\left(\frac{2^{1/2}3^{1/8}}{\pi^{5/4}n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right)\right). \end{aligned}$$

Recalling (7.4), we have

$$g_1(n) = \frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + E_{g_1} \tag{7.6}$$

where

$$\begin{aligned} |E_{g_1}| \leq \frac{\Gamma(1/4)}{2^{3/4}\pi^{1/2}} \Bigg( &\frac{1.32\pi^{3/2}}{2^3 3^{3/4} n^{3/4}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) \\ &+ \left(1 + \frac{1.32\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}}\right) \frac{2^{1/2}3^{1/8}}{\pi^{5/4}n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right) \Bigg) \end{aligned}$$

$$\ll n^{-3/4} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right). \tag{7.7}$$

On the other hand,

$$\begin{aligned} g_2(n) &= (-1)^n e^{n/X} \int_{-\frac{1}{2\pi X}}^{\frac{1}{2\pi X}} G\big(-e^{-(X^{-1}+2\pi i t)}\big) e^{2\pi i n t}\, dt \\ &= \frac{(-1)^n}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} e^{n\tau} G(-e^{-\tau})\, d\tau. \end{aligned}$$

It follows from (6.4) that

$$g_2(n) = (-1)^n \frac{\big(1+\mathfrak{O}(1.64X^{-3/4})\big)\Gamma(3/4)}{2^{1/4}\pi^{1/2}} \frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau}+n\tau\right) d\tau. \tag{7.8}$$

Similarly, we separate the integral as

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau}+n\tau\right) d\tau \\ &= \frac{1}{2\pi i}\left(\int_{\Gamma} - \int_{-\infty-i\frac{1}{X}}^{\frac{1}{X}-i\frac{1}{X}} + \int_{-\infty+i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}}\right) \tau^{\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau}+n\tau\right) d\tau \\ &=: J_{21}+J_{22}+J_{23} \end{aligned}$$

where the Hankel contour $\Gamma$ is as in (7.5). One may compute by the same argument that

$$J_{21} = \frac{\pi^{5/4}}{2^{5/2}3^{5/8}n^{5/8}} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).$$

To bound $J_{22}$, we still put $\tau = x - iX^{-1}$. Noticing that

$$|\tau|^{1/4} = (x^2+X^{-2})^{1/8} \le |x|^{1/4} + X^{-1/4},$$

we have

$$\begin{aligned} |J_{22}| &\le \frac{1}{2\pi}\cdot e^{\frac{\pi^2}{96}X} \int_{-\infty}^{X^{-1}} e^{nx}\big(|x|^{1/4}+X^{-1/4}\big)\, dx \\ &\le \frac{1}{2\pi}\cdot e^{\frac{\pi^2}{96}X} \int_{-\infty}^{0} e^{nx}(-x)^{1/4}\, dx + \frac{1}{2\pi}\cdot e^{\frac{\pi^2}{96}X} \int_{-\infty}^{X^{-1}} e^{nx}\cdot 2X^{-1/4}\, dx \\ &= \frac{\Gamma(5/4)}{2\pi n^{5/4}} \exp\left(\frac{\pi}{8}\sqrt{\frac{n}{3}}\right) + \frac{1}{2^{1/2}3^{1/8}\pi^{3/4}n^{9/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right). \end{aligned}$$

Likewise,

$$|J_{23}| \le \frac{\Gamma(5/4)}{2\pi n^{5/4}} \exp\left(\frac{\pi}{8}\sqrt{\frac{n}{3}}\right) + \frac{1}{2^{1/2}3^{1/8}\pi^{3/4}n^{9/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right).$$

In consequence,

$$g_2(n) = (-1)^n \frac{\pi^{3/4}\Gamma(3/4)}{2^{11/4}3^{5/8}n^{5/8}} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + E_{g_2} \tag{7.9}$$

where

$$
\begin{aligned}
|E_{g_2}| \le{}& \frac{\Gamma(3/4)}{2^{1/4}\pi^{1/2}}\left(\frac{1.64\pi^2}{2^4 3^1 n} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right)\right.\\
&+\left(1+\frac{1.64\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}}\right)\\
&\left.\times 2\left(\frac{\Gamma(5/4)}{2\pi n^{5/4}}\exp\left(\frac{\pi}{8}\sqrt{\frac{n}{3}}\right)+\frac{1}{2^{1/2}3^{1/8}\pi^{3/4}n^{9/8}}\exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right)\right)\right)\\
\ll{}& n^{-1} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right). \qquad (7.10)
\end{aligned}
$$

*Remark* 7.1. It is necessary to point out that $g_2(n)$ has an absolute size of

$$\text{constant} \times n^{-5/8} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right),$$

while from (7.7),

$$E_{g_1} \ll n^{-3/4} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).$$

Since the two $I$-Bessel functions have the same order, we conclude that $E_{g_1}$ is negligible comparing with $g_2(n)$.

Finally,

$$g_3(n) := e^{n/X}\int_{I_3} G\big(e^{-(X^{-1}+2\pi i t)}\big)e^{2\pi i n t}\,dt.$$

Hence, by Theorem 5.1,

$$
\begin{aligned}
|g_3(n)| &\le e^{n/X}\int_{I_3}\exp\left(\left(\frac{\pi^2}{48}-\frac{1}{100}\right)X\right)dt\\
&\le \exp\left(\frac{n}{X}+\left(\frac{\pi^2}{48}-\frac{1}{100}\right)X\right).
\end{aligned}
$$

Namely,

$$|g_3(n)| \le \exp\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}-\frac{\sqrt{3n}}{25\pi}\right). \qquad (7.11)$$

The asymptotic formula (1.8) follows from (7.6), (7.9) and (7.11). Further, a simple calculation reveals that when $n \ge 2.4\times 10^{14}$, the sign of $g(n)$ depends only on the leading term

$$\frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right),$$

which is of course positive.

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