
Multi-headed Lattice Green Function (N = 5, M = 4)

Find minimal recurrence for the coefficients

```
In[1]:= NN = 5;  
MM = 4;
```

Generate a sequence from recurrence & initial values
Koutschan's implementation

```
In[3]:= (* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}  
where inits are the initial values  
{f[0],...,f[d-1]} with d being the order of the recurrence *)  
Clear[UnrollRecurrence];  
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=  
Module[{i, x, vals = inits, rec = rec1},  
If[Head[rec] != Equal, rec = (rec == 0)];  
rec = rec /. n -> n - Max[Cases[rec, f[n + a_.] >= a, Infinity]];  
Do[  
AppendTo[vals, Solve[rec /. n -> i /. f[i] -> x /. f[a_] >= vals[[a + 1]], x][[1, 1, 2]]];  
, {i, Length[inits], bound}];  
Return[vals];  
];
```

```
In[5]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

--> Type ?HolonomicFunctions for help.

```
In[6]:= ClearAll[z, w,  $\alpha$ ,  $\beta$ ];
```

Import our REC for {r(0), r(1), r(2), ...}

```
In[7]:= ClearAll[Seq];  
SeqNormalized =  
(-2 364 822 061 925 891 270 067 722 649 600 000 - 24 311 763 241 480 737 290 507 853 496 320 000  $\alpha$  -  
118 884 714 388 336 585 062 289 753 767 936 000  $\alpha^2$  -  
368 251 136 151 853 255 846 369 719 798 988 800  $\alpha^3$  - 811 793 640 582 985 414 140 746 797 028 474 880  
 $\alpha^4$  - 1 356 499 120 040 750 577 583 138 444 526 223 360  $\alpha^5$  -  
1 786 835 040 377 781 128 110 811 754 937 712 640  $\alpha^6$  -  
1 904 958 007 246 824 509 445 186 467 125 002 240  $\alpha^7$  -  
1 674 545 402 297 600 373 785 511 713 251 000 320  $\alpha^8$  -  
1 230 194 808 706 317 371 163 067 050 208 788 480  $\alpha^9$  -
```

$$\begin{aligned}
& 762\,791\,807\,513\,049\,677\,466\,384\,009\,532\,538\,880\,\alpha^{10} - \\
& 402\,079\,430\,499\,218\,110\,643\,393\,128\,200\,929\,280\,\alpha^{11} - \\
& 181\,085\,303\,893\,806\,582\,831\,390\,648\,576\,245\,760\,\alpha^{12} - \\
& 69\,909\,566\,044\,762\,687\,837\,271\,137\,604\,075\,520\,\alpha^{13} - 23\,174\,037\,389\,797\,607\,720\,091\,614\,796\,840\,960\,\alpha^{14} - \\
& 6\,597\,237\,647\,955\,223\,324\,018\,009\,760\,071\,680\,\alpha^{15} - \\
& 1\,610\,851\,715\,462\,724\,269\,782\,004\,410\,613\,760\,\alpha^{16} - 336\,382\,193\,033\,012\,242\,367\,855\,858\,810\,880\,\alpha^{17} - \\
& 59\,795\,770\,083\,083\,316\,221\,336\,805\,703\,680\,\alpha^{18} - 8\,987\,061\,025\,545\,721\,077\,834\,511\,810\,560\,\alpha^{19} - \\
& 1\,131\,237\,375\,988\,193\,565\,613\,353\,861\,120\,\alpha^{20} - 117\,704\,523\,870\,056\,936\,584\,154\,972\,160\,\alpha^{21} - \\
& 9\,941\,030\,662\,497\,120\,749\,554\,237\,440\,\alpha^{22} - 664\,040\,244\,922\,741\,425\,721\,835\,520\,\alpha^{23} - \\
& 33\,746\,986\,442\,943\,554\,031\,452\,160\,\alpha^{24} - 1\,225\,566\,587\,608\,656\,091\,545\,600\,\alpha^{25} - \\
& 28\,320\,365\,528\,012\,449\,382\,400\,\alpha^{26} - 312\,808\,771\,118\,086\,225\,920\,\alpha^{27}) \text{Seq}[\alpha] + \\
& (-880\,540\,948\,213\,763\,261\,498\,004\,602\,880\,000 - 8\,086\,612\,414\,279\,581\,582\,690\,097\,299\,456\,000\,\alpha - \\
& 35\,535\,843\,625\,080\,580\,938\,628\,852\,403\,404\,800\,\alpha^2 - 99\,482\,199\,073\,846\,865\,130\,149\,987\,053\,731\,840\,\alpha^3 - \\
& 199\,278\,215\,238\,194\,877\,084\,174\,219\,759\,058\,944\,\alpha^4 - \\
& 304\,147\,288\,569\,704\,121\,767\,283\,668\,058\,636\,288\,\alpha^5 - 367\,726\,422\,460\,034\,552\,713\,877\,456\,306\,307\,072\,\alpha^6 - \\
& 361\,508\,986\,147\,801\,089\,153\,130\,211\,095\,805\,952\,\alpha^7 - \\
& 294\,331\,319\,744\,750\,632\,422\,172\,167\,712\,997\,376\,\alpha^8 - 201\,108\,607\,972\,501\,732\,293\,906\,606\,562\,934\,784\,\alpha^9 - \\
& 116\,437\,788\,942\,848\,727\,536\,075\,769\,222\,856\,704\,\alpha^{10} - \\
& 57\,524\,299\,296\,878\,619\,402\,424\,939\,339\,382\,784\,\alpha^{11} - 24\,367\,165\,878\,769\,872\,656\,509\,536\,747\,061\,248\,\alpha^{12} - \\
& 8\,877\,402\,295\,660\,764\,714\,512\,245\,808\,234\,496\,\alpha^{13} - \\
& 2\,785\,748\,984\,068\,408\,698\,625\,918\,477\,467\,648\,\alpha^{14} - 752\,972\,653\,647\,501\,430\,958\,086\,738\,673\,664\,\alpha^{15} - \\
& 175\,049\,743\,314\,674\,169\,771\,167\,299\,534\,848\,\alpha^{16} - 34\,895\,534\,864\,837\,208\,484\,258\,292\,957\,184\,\alpha^{17} - \\
& 5\,936\,277\,532\,573\,962\,980\,718\,997\,929\,984\,\alpha^{18} - 855\,818\,515\,821\,739\,179\,539\,429\,326\,848\,\alpha^{19} - \\
& 103\,560\,073\,600\,267\,246\,364\,541\,321\,216\,\alpha^{20} - 10\,380\,185\,487\,431\,012\,018\,005\,475\,328\,\alpha^{21} - \\
& 846\,180\,664\,706\,397\,472\,693\,420\,032\,\alpha^{22} - 54\,656\,640\,176\,185\,180\,963\,209\,216\,\alpha^{23} - \\
& 2\,690\,612\,916\,385\,314\,156\,576\,768\,\alpha^{24} - 94\,804\,345\,329\,795\,433\,758\,720\,\alpha^{25} - \\
& 2\,128\,785\,749\,082\,227\,343\,360\,\alpha^{26} - 22\,881\,382\,331\,785\,936\,896\,\alpha^{27}) \text{Seq}[1 + \alpha] + \\
& (664\,078\,540\,666\,702\,251\,488\,371\,015\,680\,000 + 5\,805\,956\,958\,011\,506\,960\,041\,778\,348\,032\,000\,\alpha + \\
& 24\,298\,272\,789\,380\,152\,495\,188\,221\,126\,246\,400\,\alpha^2 + 64\,810\,405\,629\,301\,547\,428\,216\,819\,254\,558\,720\,\alpha^3 + \\
& 123\,755\,374\,367\,469\,269\,296\,809\,845\,353\,611\,264\,\alpha^4 + \\
& 180\,149\,375\,502\,996\,189\,202\,275\,648\,542\,982\,144\,\alpha^5 + 207\,865\,771\,244\,125\,682\,287\,781\,841\,861\,722\,112\,\alpha^6 + \\
& 195\,153\,222\,041\,523\,657\,876\,484\,723\,267\,989\,504\,\alpha^7 + \\
& 151\,846\,270\,858\,495\,120\,363\,896\,477\,860\,167\,680\,\alpha^8 + 99\,230\,231\,828\,276\,421\,932\,960\,434\,682\,314\,752\,\alpha^9 + \\
& 54\,993\,115\,047\,787\,497\,911\,079\,580\,675\,899\,392\,\alpha^{10} + \\
& 26\,028\,017\,908\,489\,825\,928\,212\,462\,245\,453\,824\,\alpha^{11} + 10\,572\,113\,416\,646\,586\,933\,511\,582\,698\,766\,336\,\alpha^{12} + \\
& 3\,696\,722\,231\,163\,815\,760\,173\,082\,026\,344\,448\,\alpha^{13} + \\
& 1\,114\,468\,173\,061\,041\,282\,670\,805\,399\,093\,248\,\alpha^{14} + 289\,688\,969\,845\,746\,113\,335\,461\,572\,931\,584\,\alpha^{15} + \\
& 64\,831\,091\,647\,102\,802\,811\,533\,842\,055\,168\,\alpha^{16} + 12\,454\,053\,009\,083\,005\,771\,527\,566\,163\,968\,\alpha^{17} + \\
& 2\,043\,760\,292\,966\,696\,499\,523\,264\,184\,320\,\alpha^{18} + 284\,532\,912\,366\,921\,324\,027\,166\,588\,928\,\alpha^{19} + \\
& 33\,284\,416\,956\,384\,385\,896\,458\,223\,616\,\alpha^{20} + 3\,228\,606\,478\,351\,534\,833\,828\,626\,432\,\alpha^{21} + \\
& 254\,974\,947\,491\,313\,890\,128\,560\,128\,\alpha^{22} + 15\,972\,126\,457\,377\,261\,067\,698\,176\,\alpha^{23} + \\
& 763\,333\,007\,662\,980\,725\,211\,136\,\alpha^{24} + 26\,138\,887\,552\,462\,651\,129\,856\,\alpha^{25} + \\
& 570\,997\,443\,951\,748\,710\,400\,\alpha^{26} + 5\,976\,795\,675\,008\,958\,464\,\alpha^{27}) \text{Seq}[2 + \alpha] + \\
& (-36\,337\,840\,931\,616\,555\,318\,702\,833\,664\,000 - 310\,343\,693\,247\,202\,072\,877\,171\,431\,833\,600\,\alpha - \\
& 1\,268\,062\,726\,217\,635\,641\,408\,454\,051\,430\,400\,\alpha^2 - 3\,300\,521\,955\,790\,071\,740\,463\,976\,232\,263\,680\,\alpha^3 - \\
& 6\,146\,984\,578\,367\,464\,065\,862\,054\,879\,242\,240\,\alpha^4 - 8\,723\,512\,529\,514\,925\,026\,222\,139\,080\,468\,480\,\alpha^5 - \\
& 9\,808\,817\,646\,565\,897\,068\,529\,809\,213\,239\,808\,\alpha^6 - 8\,970\,447\,157\,798\,999\,809\,214\,350\,039\,412\,224\,\alpha^7 - \\
& 6\,796\,618\,106\,855\,403\,262\,931\,535\,421\,469\,184\,\alpha^8 - 4\,323\,600\,610\,674\,086\,572\,350\,145\,316\,732\,416\,\alpha^9 - \\
& 2\,331\,860\,127\,398\,843\,166\,087\,931\,718\,971\,904\,\alpha^{10} - \\
& 1\,073\,804\,990\,271\,736\,796\,663\,841\,511\,156\,224\,\alpha^{11} - \\
& 424\,279\,297\,446\,148\,516\,898\,147\,199\,947\,264\,\alpha^{12} - 144\,293\,344\,557\,135\,741\,340\,883\,292\,465\,664\,\alpha^{13} -
\end{aligned}$$

$$\begin{aligned}
& 42\,304\,696\,119\,152\,808\,149\,756\,544\,291\,840\,\alpha^{14} - 10\,693\,366\,157\,119\,575\,923\,154\,101\,714\,944\,\alpha^{15} - \\
& 2\,327\,102\,570\,668\,214\,059\,453\,238\,664\,192\,\alpha^{16} - 434\,708\,874\,971\,795\,823\,099\,840\,116\,736\,\alpha^{17} - \\
& 69\,373\,988\,097\,051\,870\,247\,906\,934\,784\,\alpha^{18} - 9\,393\,304\,762\,567\,159\,143\,035\,764\,736\,\alpha^{19} - \\
& 1\,068\,815\,757\,774\,279\,757\,481\,902\,080\,\alpha^{20} - 100\,861\,570\,825\,855\,881\,262\,923\,776\,\alpha^{21} - \\
& 7\,750\,770\,733\,439\,394\,600\,976\,384\,\alpha^{22} - 472\,551\,963\,878\,997\,639\,561\,216\,\alpha^{23} - \\
& 21\,986\,541\,883\,647\,884\,001\,280\,\alpha^{24} - 733\,188\,729\,988\,561\,502\,208\,\alpha^{25} - \\
& 15\,602\,375\,112\,618\,147\,840\,\alpha^{26} - 159\,149\,910\,074\,064\,896\,\alpha^{27}) \text{ Seq}[3 + \alpha] + \\
& (-1\,737\,772\,868\,400\,007\,324\,872\,130\,560\,000 - 14\,528\,609\,204\,414\,291\,845\,066\,255\,564\,800\,\alpha - \\
& 58\,083\,087\,258\,852\,534\,411\,685\,975\,019\,520\,\alpha^2 - 147\,846\,850\,915\,658\,722\,383\,612\,355\,430\,400\,\alpha^3 - \\
& 269\,164\,023\,324\,400\,460\,962\,054\,275\,740\,928\,\alpha^4 - 373\,240\,816\,513\,597\,979\,905\,593\,440\,661\,888\,\alpha^5 - \\
& 409\,908\,879\,949\,766\,514\,326\,399\,060\,864\,064\,\alpha^6 - 366\,016\,393\,873\,249\,701\,940\,597\,734\,061\,344\,\alpha^7 - \\
& 270\,676\,671\,846\,416\,971\,917\,873\,052\,917\,920\,\alpha^8 - 168\,013\,318\,310\,785\,666\,403\,759\,927\,887\,584\,\alpha^9 - \\
& 88\,393\,926\,598\,940\,439\,065\,183\,725\,045\,600\,\alpha^{10} - 39\,697\,363\,634\,496\,672\,642\,069\,844\,386\,912\,\alpha^{11} - \\
& 15\,293\,672\,611\,896\,263\,618\,803\,193\,519\,136\,\alpha^{12} - 5\,070\,491\,874\,452\,377\,148\,797\,920\,831\,072\,\alpha^{13} - \\
& 1\,449\,002\,022\,519\,967\,409\,403\,051\,116\,512\,\alpha^{14} - 356\,957\,682\,436\,813\,381\,749\,659\,746\,304\,\alpha^{15} - \\
& 75\,700\,244\,148\,872\,939\,301\,421\,992\,640\,\alpha^{16} - 13\,779\,371\,789\,456\,905\,170\,877\,563\,840\,\alpha^{17} - \\
& 2\,142\,685\,081\,818\,193\,152\,012\,367\,872\,\alpha^{18} - 282\,685\,926\,147\,777\,894\,282\,083\,328\,\alpha^{19} - \\
& 31\,341\,335\,886\,140\,485\,043\,322\,880\,\alpha^{20} - 2\,881\,942\,426\,887\,984\,021\,438\,464\,\alpha^{21} - \\
& 215\,812\,414\,752\,103\,173\,455\,872\,\alpha^{22} - 12\,823\,036\,513\,484\,289\,343\,488\,\alpha^{23} - \\
& 581\,508\,878\,853\,457\,575\,936\,\alpha^{24} - 18\,903\,053\,117\,719\,314\,432\,\alpha^{25} - \\
& 392\,186\,219\,850\,629\,120\,\alpha^{26} - 3\,900\,964\,176\,134\,144\,\alpha^{27}) \text{ Seq}[4 + \alpha] + \\
& (36\,446\,102\,109\,669\,030\,849\,285\,120\,000 + 301\,794\,930\,778\,773\,719\,063\,321\,856\,000\,\alpha + \\
& 1\,194\,401\,836\,156\,084\,887\,609\,064\,224\,000\,\alpha^2 + 3\,008\,156\,975\,709\,477\,795\,289\,491\,275\,520\,\alpha^3 + \\
& 5\,415\,770\,546\,395\,539\,670\,222\,530\,489\,360\,\alpha^4 + 7\,422\,453\,554\,874\,065\,600\,190\,474\,289\,032\,\alpha^5 + \\
& 8\,052\,206\,383\,842\,449\,223\,124\,682\,104\,644\,\alpha^6 + 7\,098\,162\,826\,794\,167\,361\,280\,152\,144\,294\,\alpha^7 + \\
& 5\,179\,144\,111\,408\,801\,590\,076\,035\,892\,950\,\alpha^8 + 3\,169\,950\,795\,733\,038\,711\,522\,140\,215\,280\,\alpha^9 + \\
& 1\,643\,499\,248\,947\,095\,475\,104\,215\,404\,004\,\alpha^{10} + 726\,910\,788\,718\,026\,537\,302\,273\,862\,144\,\alpha^{11} + \\
& 275\,635\,972\,025\,251\,416\,199\,969\,761\,656\,\alpha^{12} + 89\,889\,728\,147\,001\,421\,773\,544\,625\,132\,\alpha^{13} + \\
& 25\,251\,994\,806\,501\,150\,584\,061\,125\,784\,\alpha^{14} + 6\,111\,409\,098\,652\,595\,993\,659\,452\,026\,\alpha^{15} + \\
& 1\,272\,483\,225\,563\,071\,816\,917\,699\,490\,\alpha^{16} + 227\,273\,250\,419\,552\,627\,170\,585\,084\,\alpha^{17} + \\
& 34\,655\,941\,701\,831\,856\,557\,922\,624\,\alpha^{18} + 4\,480\,880\,404\,407\,427\,210\,024\,320\,\alpha^{19} + \\
& 486\,585\,842\,769\,876\,461\,484\,032\,\alpha^{20} + 43\,798\,304\,089\,562\,788\,663\,296\,\alpha^{21} + \\
& 3\,208\,710\,131\,027\,557\,023\,744\,\alpha^{22} + 186\,416\,522\,833\,559\,945\,216\,\alpha^{23} + 8\,261\,380\,192\,874\,790\,912\,\alpha^{24} + \\
& 262\,301\,388\,296\,421\,376\,\alpha^{25} + 5\,312\,632\,953\,241\,600\,\alpha^{26} + 51\,561\,082\,388\,480\,\alpha^{27}) \text{ Seq}[5 + \alpha] + \\
& (154\,404\,486\,709\,237\,819\,219\,968\,000 + 1\,265\,327\,918\,255\,018\,927\,110\,348\,800\,\alpha + \\
& 4\,953\,641\,658\,930\,095\,511\,385\,751\,040\,\alpha^2 + 12\,335\,446\,851\,783\,544\,166\,937\,390\,720\,\alpha^3 + \\
& 21\,947\,702\,123\,383\,074\,616\,990\,244\,544\,\alpha^4 + 29\,712\,684\,443\,300\,038\,100\,072\,561\,760\,\alpha^5 + \\
& 31\,824\,626\,177\,807\,101\,870\,129\,360\,368\,\alpha^6 + 27\,684\,339\,638\,906\,598\,652\,692\,786\,888\,\alpha^7 + \\
& 19\,923\,668\,408\,873\,674\,929\,361\,243\,572\,\alpha^8 + 12\,021\,754\,897\,932\,453\,908\,473\,126\,194\,\alpha^9 + \\
& 6\,141\,402\,912\,303\,808\,338\,721\,284\,327\,\alpha^{10} + 2\,675\,090\,519\,652\,464\,763\,702\,625\,995\,\alpha^{11} + \\
& 998\,451\,712\,547\,824\,111\,144\,656\,513\,\alpha^{12} + 320\,337\,381\,856\,256\,276\,567\,115\,789\,\alpha^{13} + \\
& 88\,485\,146\,094\,830\,787\,771\,471\,525\,\alpha^{14} + 21\,045\,641\,782\,461\,353\,200\,898\,049\,\alpha^{15} + \\
& 4\,304\,140\,182\,149\,530\,399\,276\,227\,\alpha^{16} + 754\,678\,659\,252\,915\,954\,749\,073\,\alpha^{17} + \\
& 112\,910\,766\,050\,133\,819\,763\,020\,\alpha^{18} + 14\,316\,213\,223\,182\,938\,203\,068\,\alpha^{19} + \\
& 1\,523\,679\,350\,645\,560\,062\,336\,\alpha^{20} + 134\,345\,128\,624\,663\,841\,280\,\alpha^{21} + \\
& 9\,635\,762\,018\,738\,626\,560\,\alpha^{22} + 547\,760\,583\,383\,666\,688\,\alpha^{23} + 23\,739\,371\,943\,886\,848\,\alpha^{24} + \\
& 736\,693\,272\,182\,784\,\alpha^{25} + 14\,575\,541\,944\,320\,\alpha^{26} + 138\,110\,042\,112\,\alpha^{27}) \text{ Seq}[6 + \alpha] ;
\end{aligned}$$

In[9]:= RecNormalizedOrder = 6;

Initial values of $\{r(0), r(1), r(2), \dots\}$

```
In[10]:= SeqListIni = {};

MAX = 10;

For[n = 0, n ≤ MAX, n++,
  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence@@(2 coord[[i]])] *
    Product[Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];

  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n + (1 - NN) / 2 &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence@@(2 coord[[i]] + 1)] *
    Product[Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];
];

SeqListIni

seq[n_] := SeqListIni[[n + 1]];
```

```
Out[13]= {1, 0, 80, 0, 58320, 933120, 107360000, 4032000000, 305742850000,
  16007947200000, 1092754448110080, 66052872139161600, 4433464272394080000,
  287105556124600012800, 19441756158387587481600, 1307659624636945150771200,
  89869341860254106893314000, 6191536013119541254794624000,
  431788153780445031117712736000, 30259578124053738011950295040000,
  2137643722042861014846923875678720, 151778757062056398402787590848716800}
```

Verify recurrence by initial values

```
In[15]:= Table[SeqNormalized /. {Seq → seq, α → n}, {n, 0, 2 MAX - RecNormalizedOrder}]

Out[15]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Generate more terms in the sequence

$$\text{SeqList}[[n]] = r(n)$$

```
In[21]:= Bound = 500;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Let's guess (and prove!) a shorter recurrence.

In[19]:= << RISC`Guess`

Package GeneratingFunctions version 0.9 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Guess Package version 0.52
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[24]:= SeqGuess = GuessMinRE[Take[SeqList, 500], Seq[α]]

Out[24]:=
$$\left(-\frac{839\,014\,160\,464\,878\,334\,200\,000}{49} - \frac{60\,378\,917\,161\,327\,444\,738\,417\,500}{343} \alpha - \frac{295\,253\,382\,097\,523\,870\,722\,179\,000}{343} \alpha^2 - \frac{914\,561\,589\,936\,050\,911\,362\,890\,700}{1372} \alpha^3 - \frac{32\,257\,781\,051\,913\,317\,606\,478\,802\,245}{5488} \alpha^4 - \frac{13\,475\,608\,031\,359\,817\,361\,664\,160\,535}{1372} \alpha^5 - \frac{40\,572\,825\,849\,119\,393\,669\,101\,437\,045}{3136} \alpha^6 - \frac{302\,784\,920\,890\,621\,266\,276\,866\,033\,415}{21\,952} \alpha^7 - \frac{532\,323\,647\,275\,797\,644\,864\,880\,788\,565}{43\,904} \alpha^8 - \frac{55\,866\,913\,963\,786\,414\,517\,133\,215\,505}{6272} \alpha^9 - \frac{242\,484\,985\,196\,765\,161\,344\,650\,271\,585}{43\,904} \alpha^{10} - \frac{127\,817\,608\,674\,118\,712\,947\,652\,539\,635}{43\,904} \alpha^{11} - \frac{57\,565\,467\,800\,714\,678\,074\,960\,914\,795}{43\,904} \alpha^{12} - \frac{22\,223\,652\,535\,998\,880\,970\,405\,304\,465}{43\,904} \alpha^{13} - \frac{1\,052\,404\,012\,670\,862\,457\,642\,855\,635}{6272} \alpha^{14} - \frac{299\,600\,766\,861\,108\,295\,078\,187\,205}{6272} \alpha^{15} - \frac{128\,018\,992\,381\,312\,391\,740\,640\,355}{10\,976} \alpha^{16} - \frac{26\,733\,254\,832\,666\,098\,464\,954\,365}{10\,976} \alpha^{17} - \frac{21\,214\,914\,493\,292\,327\,516\,610}{44\,639\,215\,000\,200\,873\,006\,555} \alpha^{18} - \frac{44\,639\,215\,000\,200\,873\,006\,555}{686} \alpha^{19} - \frac{401\,351\,202\,097\,745\,724\,240}{292\,322\,347\,238\,801\,262\,240} \alpha^{20} - \frac{292\,322\,347\,238\,801\,262\,240}{343} \alpha^{21} - \frac{24\,688\,816\,722\,475\,292\,160}{1\,649\,161\,788\,132\,641\,280} \alpha^{22} - \frac{1\,649\,161\,788\,132\,641\,280}{83\,811\,547\,465\,482\,240} \alpha^{23} - \frac{83\,811\,547\,465\,482\,240}{434\,818\,220\,851\,200} \alpha^{24} - \frac{1\,435\,395\,686\,400}{2\,264\,924\,160} \alpha^{25} - \frac{2\,264\,924\,160}{7} \alpha^{26} - \frac{2\,264\,924\,160}{7} \alpha^{27} \Big) \text{Seq}[\alpha] +$$

$$\left(-\frac{2\,186\,847\,101\,186\,124\,636\,476\,250}{343} - \frac{40\,166\,638\,365\,783\,202\,562\,183\,625}{686} \alpha \right)$$

$$\begin{aligned}
& \frac{706\,033\,778\,688\,919\,165\,504\,551\,975}{2744} \alpha^2 - \frac{3\,953\,067\,424\,847\,557\,748\,562\,097\,035}{5488} \alpha^3 - \\
& \frac{63\,348\,838\,563\,981\,017\,139\,490\,197\,423}{43\,904} \alpha^4 - \frac{386\,743\,276\,684\,702\,710\,173\,649\,683\,709}{175\,616} \alpha^5 - \\
& \frac{467\,588\,326\,085\,452\,464\,693\,549\,418\,521}{175\,616} \alpha^6 - \frac{1\,838\,729\,787\,942\,510\,422\,531\,790\,217\,569}{702\,464} \alpha^7 - \\
& \frac{1\,497\,046\,507\,490\,797\,080\,597\,799\,518\,397}{702\,464} \alpha^8 - \frac{1\,022\,891\,275\,901\,803\,244\,496\,188\,387\,873}{702\,464} \alpha^9 - \\
& \frac{592\,233\,220\,127\,607\,867\,106\,505\,173\,863}{702\,464} \alpha^{10} - \frac{292\,583\,716\,313\,062\,639\,375\,940\,650\,123}{702\,464} \alpha^{11} - \\
& \frac{123\,937\,814\,731\,698\,977\,948\,555\,179\,581}{702\,464} \alpha^{12} - \frac{45\,152\,803\,017\,480\,289\,278\,728\,463\,787}{702\,464} \alpha^{13} - \\
& \frac{14\,169\,052\,042\,991\,173\,800\,791\,007\,881}{702\,464} \alpha^{14} - \frac{478\,727\,120\,493\,253\,981\,881\,514\,701}{87\,808} \alpha^{15} - \\
& \frac{445\,174\,518\,113\,897\,119\,575\,925\,953}{351\,232} \alpha^{16} - \frac{44\,371\,967\,143\,805\,451\,055\,219\,387}{175\,616} \alpha^{17} - \\
& \frac{471\,772\,951\,489\,604\,551\,054\,557}{10\,976} \alpha^{18} - \frac{34\,007\,172\,418\,504\,523\,417\,927}{5488} \alpha^{19} - \\
& \frac{257\,194\,224\,994\,051\,571\,599}{343} \alpha^{20} - \frac{25\,779\,469\,528\,374\,602\,442}{343} \alpha^{21} - \\
& \frac{300\,216\,059\,140\,647\,264}{49} \alpha^{22} - \frac{135\,741\,234\,263\,244\,224}{343} \alpha^{23} - \frac{6\,682\,209\,462\,884\,352}{343} \alpha^{24} - \\
& \frac{33\,635\,591\,229\,440}{49} \alpha^{25} - \frac{107\,895\,848\,960}{7} \alpha^{26} - 165\,675\,008 \alpha^{27} \Big) \text{Seq}[1 + \alpha] + \\
& \left(\frac{1\,649\,256\,896\,641\,607\,710\,441\,875}{343} + \frac{32\,958\,285\,910\,807\,944\,252\,614\,625}{784} \alpha + \right. \\
& \frac{1\,103\,458\,985\,345\,938\,205\,500\,476\,525}{6272} \alpha^2 + \frac{20\,602\,673\,094\,845\,310\,029\,416\,662\,615}{43\,904} \alpha^3 + \\
& \frac{78\,681\,548\,034\,330\,539\,256\,292\,880\,601}{87\,808} \alpha^4 + \frac{523\,592\,632\,484\,061\,654\,815\,021\,764\,971}{401\,408} \alpha^5 + \\
& \frac{4\,229\,039\,942\,303\,989\,304\,357\,540\,727\,981}{15\,881\,609\,866\,660\,453\,928\,750\,384\,380\,533} \alpha^6 + \frac{2\,809\,856}{11\,239\,424} \alpha^7 + \\
& \frac{772\,330\,072\,319\,005\,942\,606\,081\,532\,085}{2\,018\,844\,234\,787\,524\,860\,289\,722\,385\,301} \alpha^8 + \frac{2\,809\,856}{3\,177\,248\,279\,844\,949\,454\,127\,497\,832\,697} \alpha^9 + \\
& \frac{1\,678\,256\,684\,807\,968\,075\,899\,645\,406\,369}{4\,214\,784} \alpha^{10} + \frac{16\,859\,136}{16\,116\,429\,927\,995\,151\,019\,169\,756\,323} \alpha^{11} + \\
& \frac{161\,317\,648\,569\,436\,446\,129\,021\,952\,801}{2\,107\,392} \alpha^{12} + \frac{602\,112}{70\,724\,846\,153\,746\,609\,701\,040\,423\,079} \alpha^{13} + \\
& \frac{68\,021\,739\,078\,432\,695\,475\,513\,024\,847}{8429\,568} \alpha^{14} + \frac{33\,718\,272}{54\,295\,362\,239\,654\,566\,177\,488\,343} \alpha^{15} + \\
& \frac{494\,621\,976\,067\,373\,678\,676\,863\,419}{1053\,696} \alpha^{16} + \frac{602\,112}{1085\,406\,922\,786\,412\,521\,465\,937} \alpha^{17} + \\
& \frac{7\,796\,326\,801\,173\,006\,055\,920\,655}{526\,848} \alpha^{18} + \frac{1085\,406\,922\,786\,412\,521\,465\,937}{526\,848} \alpha^{19} +
\end{aligned}$$

$$\begin{aligned}
& \frac{3\,967\,811\,698\,482\,559\,430\,177\,\alpha^{20}}{16\,464} + \frac{24\,054\,992\,782\,708\,516\,969\,\alpha^{21}}{1029} + \\
& \frac{1\,899\,711\,396\,480\,455\,176\,\alpha^{22}}{1029} + \frac{119\,001\,615\,474\,948\,742\,\alpha^{23}}{1029} + \frac{812\,467\,727\,983\,616\,\alpha^{24}}{147} + \\
& \frac{567\,783\,931\,264\,\alpha^{25}}{3} + \frac{12\,403\,097\,600\,\alpha^{26}}{3} + \frac{129\,826\,816\,\alpha^{27}}{3} \Big) \text{Seq}[2 + \alpha] + \\
& \left(- \frac{3\,300\,425\,296\,203\,663\,718\,696\,125}{12\,544} - \frac{789\,244\,825\,356\,043\,683\,057\,585\,225\,\alpha}{351\,232} - \right. \\
& \frac{6\,449\,700\,552\,457\,863\,573\,244\,496\,925\,\alpha^2}{702\,464} - \frac{134\,298\,582\,185\,468\,413\,918\,618\,824\,555\,\alpha^3}{5\,619\,712} - \\
& \frac{1\,000\,485\,771\,218\,662\,771\,136\,402\,161\,335\,\alpha^4}{22\,478\,848} - \frac{1\,419\,842\,534\,100\,736\,495\,153\,342\,949\,295\,\alpha^5}{22\,478\,848} - \\
& \frac{6\,385\,948\,988\,649\,672\,570\,657\,427\,873\,203\,\alpha^6}{89\,915\,392} - \frac{5\,840\,134\,868\,358\,723\,834\,123\,925\,806\,909\,\alpha^7}{89\,915\,392} - \\
& \frac{4\,424\,881\,579\,983\,986\,499\,304\,385\,040\,019\,\alpha^8}{89\,915\,392} - \frac{8\,444\,532\,442\,722\,825\,336\,621\,377\,571\,743\,\alpha^9}{269\,746\,176} - \\
& \frac{4\,554\,414\,311\,325\,865\,558\,765\,491\,638\,617\,\alpha^{10}}{699\,091\,790\,541\,495\,310\,328\,021\,817\,159\,\alpha^{11}} - \\
& \frac{118\,381\,500\,403\,501\,260\,295\,241\,964\,271\,\alpha^{12}}{281\,822\,938\,588\,155\,744\,806\,412\,680\,597\,\alpha^{13}} - \\
& \frac{13\,771\,059\,934\,620\,054\,736\,248\,875\,095\,\alpha^{14}}{652\,671\,274\,238\,255\,366\,403\,448\,591\,\alpha^{15}} - \\
& \frac{757\,520\,368\,056\,059\,264\,144\,934\,461\,\alpha^{16}}{70\,753\,397\,619\,107\,393\,082\,656\,269\,\alpha^{17}} - \\
& \frac{705\,708\,700\,531\,533\,510\,822\,621\,\alpha^{18}}{143\,330\,455\,971\,788\,927\,353\,451\,\alpha^{19}} - \\
& \frac{339\,767\,379\,053\,204\,777\,235\,\alpha^{20}}{48\,094\,544\,804\,504\,337\,913\,\alpha^{21}} - \\
& \frac{230\,990\,968\,151\,074\,487\,\alpha^{22}}{2\,347\,190\,995\,410\,073\,\alpha^{23}} - \frac{163\,812\,502\,351\,760\,\alpha^{24}}{163\,812\,502\,351\,760\,\alpha^{24}} - \\
& \frac{260\,127\,701\,216\,\alpha^{25}}{49} - \frac{790\,794\,240\,\alpha^{26}}{7} - \frac{3\,457\,024\,\alpha^{27}}{3} \Big) \text{Seq}[3 + \alpha] + \\
& \left(- \frac{4\,419\,384\,939\,575\,213\,940\,613\,125}{351\,232} - \frac{73\,896\,327\,740\,551\,207\,708\,059\,975\,\alpha}{702\,464} - \right. \\
& \frac{4\,726\,813\,741\,768\,598\,178\,034\,340\,415\,\alpha^2}{24\,063\,615\,057\,887\,161\,846\,291\,073\,475\,\alpha^3} - \\
& \frac{350\,473\,988\,703\,646\,433\,544\,341\,504\,871\,\alpha^4}{971\,981\,293\,004\,161\,406\,004\,149\,585\,057\,\alpha^5} - \\
& \frac{2\,134\,942\,083\,071\,700\,595\,449\,995\,108\,667\,\alpha^6}{3\,812\,670\,769\,513\,017\,728\,547\,893\,063\,139\,\alpha^7} - \\
& \frac{8\,458\,645\,995\,200\,530\,372\,433\,532\,903\,685\,\alpha^8}{1\,750\,138\,732\,404\,017\,358\,372\,499\,248\,829\,\alpha^9} - \\
& \frac{2\,762\,310\,206\,216\,888\,720\,786\,991\,407\,675\,\alpha^{10}}{413\,514\,204\,526\,007\,006\,688\,227\,545\,697\,\alpha^{11}} - \\
& \frac{4\,315\,938\,816}{1\,438\,646\,272}
\end{aligned}$$

$$\begin{aligned}
& \frac{159\,309\,089\,707\,252\,746\,029\,199\,932\,491}{1\,438\,646\,272} \alpha^{12} - \frac{7\,545\,374\,813\,173\,180\,280\,949\,286\,951}{205\,520\,896} \alpha^{13} - \\
& \frac{45\,281\,313\,203\,748\,981\,543\,845\,347\,391}{4\,315\,938\,816} \alpha^{14} - \frac{3\,112\,423\,988\,881\,255\,072\,454\,483}{1\,204\,224} \alpha^{15} - \\
& \frac{168\,973\,759\,260\,877\,096\,654\,959\,805}{308\,281\,344} \alpha^{16} - \frac{71\,767\,561\,403\,421\,381\,098\,320\,645}{719\,323\,136} \alpha^{17} - \\
& \frac{348\,744\,316\,702\,179\,875\,002\,013}{22\,478\,848} \alpha^{18} - \frac{92\,020\,158\,251\,229\,783\,294\,949}{44\,957\,696} \alpha^{19} - \\
& \frac{956\,461\,666\,447\,158\,357\,035}{4\,214\,784} \alpha^{20} - \frac{6\,282\,135\,940\,307\,582\,357}{301\,056} \alpha^{21} - \frac{205\,814\,757\,110\,694\,097}{131\,712} \alpha^{22} - \\
& \frac{4\,076\,333\,527\,083\,171}{43\,904} \alpha^{23} - \frac{2\,888\,385\,846\,483}{686} \alpha^{24} - \frac{13\,413\,211\,503}{98} \alpha^{25} - \frac{59\,632\,960}{21} \alpha^{26} - \frac{84\,736}{3} \alpha^{27} \Big) \\
& \text{Seq}[4 + \alpha] + \left(\frac{1\,482\,995\,691\,311\,402\,622\,448\,125}{5\,619\,712} + \frac{49\,120\,268\,681\,441\,035\,003\,795\,875}{22\,478\,848} \alpha + \right. \\
& \frac{1\,555\,210\,724\,161\,568\,864\,074\,302\,375}{179\,830\,784} \alpha^2 + \frac{3\,916\,871\,062\,121\,715\,879\,283\,191\,765}{179\,830\,784} \alpha^3 + \\
& \frac{112\,828\,553\,049\,907\,076\,462\,969\,385\,195}{2\,877\,292\,544} \alpha^4 + \frac{309\,268\,898\,119\,752\,733\,341\,269\,762\,043}{5\,754\,585\,088} \alpha^5 + \\
& \frac{671\,017\,198\,653\,537\,435\,260\,390\,175\,387}{11\,509\,170\,176} \alpha^6 + \frac{1\,183\,027\,137\,799\,027\,893\,546\,692\,024\,049}{23\,018\,340\,352} \alpha^7 + \\
& \frac{863\,190\,685\,234\,800\,265\,012\,672\,648\,825}{28\,303\,132\,104\,759\,274\,210\,019\,109\,065} \alpha^8 + \frac{23\,018\,340\,352}{1\,233\,125\,376} \alpha^9 + \\
& \frac{410\,874\,812\,236\,773\,868\,776\,053\,851\,001}{473\,249\,211\,404\,965\,193\,556\,167\,879} \alpha^{10} + \frac{34\,527\,510\,528}{89\,915\,392} \alpha^{11} + \\
& \frac{34\,454\,496\,503\,156\,427\,024\,996\,220\,207}{22\,472\,432\,036\,750\,355\,443\,386\,156\,283} \alpha^{12} + \frac{17\,263\,755\,264}{34\,527\,510\,528} \alpha^{13} + \\
& \frac{1\,052\,166\,450\,270\,881\,274\,335\,880\,241}{3\,055\,704\,549\,326\,297\,996\,829\,726\,013} \alpha^{14} + \frac{5\,754\,585\,088}{69\,055\,021\,056} \alpha^{15} + \\
& \frac{636\,241\,612\,781\,535\,908\,458\,849\,745}{56\,818\,312\,604\,888\,156\,792\,646\,271} \alpha^{16} + \frac{69\,055\,021\,056}{34\,527\,510\,528} \alpha^{17} + \\
& \frac{77\,357\,012\,727\,303\,251\,245\,363}{11\,668\,959\,386\,477\,675\,026\,105} \alpha^{18} + \frac{308\,281\,344}{359\,661\,568} \alpha^{19} + \\
& \frac{237\,590\,743\,539\,978\,740\,959}{509\,187\,873\,065\,043\,581} \alpha^{20} + \frac{67\,436\,544}{16\,320\,343\,684\,018\,743} \alpha^{21} + \\
& \frac{1\,422\,245\,199\,841\,003}{67\,436\,544} \alpha^{23} + \frac{93\,793\,644\,143}{1\,605\,632} \alpha^{24} + \frac{638\,138\,003}{702\,464} \alpha^{25} + \frac{115\,400}{3} \alpha^{26} + \frac{1120}{3} \alpha^{27} \Big) \\
& \text{Seq}[5 + \alpha] + \left(\frac{785\,341\,830\,999\,948\,217\,875}{702\,464} + \frac{51\,486\,324\,798\,788\,205\,041\,925}{5\,619\,712} \alpha + \right. \\
& \frac{3\,225\,027\,121\,699\,280\,931\,891\,765}{89\,915\,392} \alpha^2 + \frac{32\,123\,559\,509\,852\,979\,601\,399\,455}{359\,661\,568} \alpha^3 + \\
& \frac{114\,310\,948\,559\,286\,846\,963\,490\,857}{309\,507\,129\,617\,708\,730\,209\,089\,185} \alpha^4 + \frac{719\,323\,136}{1\,438\,646\,272} \alpha^5 + \\
& \frac{663\,013\,045\,370\,981\,288\,961\,028\,341}{2\,877\,292\,544} \alpha^6 + \frac{1\,153\,514\,151\,621\,108\,277\,195\,532\,787}{5\,754\,585\,088} \alpha^7 +
\end{aligned}$$

$$\begin{aligned}
& \frac{1\,660\,305\,700\,739\,472\,910\,780\,103\,631\,\alpha^8}{11\,509\,170\,176} + \frac{2\,003\,625\,816\,322\,075\,651\,412\,187\,699\,\alpha^9}{23\,018\,340\,352} + \\
& \frac{2\,047\,134\,304\,101\,269\,446\,240\,428\,109\,\alpha^{10}}{46\,036\,680\,704} + \frac{891\,696\,839\,884\,154\,921\,234\,208\,665\,\alpha^{11}}{46\,036\,680\,704} + \\
& \frac{332\,817\,237\,515\,941\,370\,381\,552\,171\,\alpha^{12}}{46\,036\,680\,704} + \frac{106\,779\,127\,285\,418\,758\,855\,705\,263\,\alpha^{13}}{46\,036\,680\,704} + \\
& \frac{29\,495\,048\,698\,276\,929\,257\,157\,175\,\alpha^{14}}{46\,036\,680\,704} + \frac{7\,015\,213\,927\,487\,117\,733\,632\,683\,\alpha^{15}}{46\,036\,680\,704} + \\
& \frac{1\,434\,713\,394\,049\,843\,466\,425\,409\,\alpha^{16}}{46\,036\,680\,704} + \frac{35\,937\,079\,012\,043\,616\,892\,813\,\alpha^{17}}{6\,576\,668\,672} + \\
& \frac{9\,409\,230\,504\,177\,818\,313\,585\,\alpha^{18}}{11\,509\,170\,176} + \frac{1\,193\,017\,768\,598\,578\,183\,589\,\alpha^{19}}{11\,509\,170\,176} + \\
& \frac{3\,967\,914\,975\,639\,479\,329\,\alpha^{20}}{359\,661\,568} + \frac{87\,464\,276\,448\,348\,855\,\alpha^{21}}{89\,915\,392} + \frac{392\,080\,160\,267\,685\,\alpha^{22}}{5\,619\,712} + \\
& \frac{6\,368\,124\,341\,793\,\alpha^{23}}{1\,605\,632} + \frac{15\,093\,086\,207\,\alpha^{24}}{87\,808} + \frac{133\,821\,991\,\alpha^{25}}{25\,088} + \frac{2955\,\alpha^{26}}{28} + \alpha^{27} \Big) \text{Seq}[6 + \alpha]
\end{aligned}$$

Okay, the order of this recurrence is the same as what we have computed by creative telescoping; both are 6. So no need to continue.