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Vanishing coefficients and identities concerning Ramanujan's parameters

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Abstract. In this paper, we investigate several infinite products with vanishing Taylor coefficients in arithmetic progressions. These infinite products are closely related to Ramanujan's parameters introduced in his Lost Notebook. Also, a handful of new identities involving these parameters will be established.

Keywords. Vanishing coefficients, Rogers–Ramanujan continued fraction, Ramanujan's parameters, Ramanujan's Lost Notebook.

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1. Introduction

Throughout, the following customary q-series notation will be adopted:

$$\begin{aligned}(A;q)_\infty &:= \prod_{k=0}^{\infty}(1-Aq^k),\\ (A_1,A_2,\ldots,A_n;q)_\infty &:= (A_1;q)_\infty(A_2;q)_\infty\cdots(A_n;q)_\infty,\\ \begin{pmatrix}A_1,A_2,\ldots,A_n\\ B_1,B_2,\ldots,B_m\end{pmatrix};q\Bigg)_\infty &:= \frac{(A_1;q)_\infty(A_2;q)_\infty\cdots(A_n;q)_\infty}{(B_1;q)_\infty(B_2;q)_\infty\cdots(B_m;q)_\infty}.\end{aligned}$$

Let $G(q)$ and $H(q)$ be the Rogers–Ramanujan functions defined, respectively, by

$$G(q) := \frac{1}{(q,q^4;q^5)_\infty} \qquad \text{and} \qquad H(q) := \frac{1}{(q^2,q^3;q^5)_\infty}.$$

The Rogers–Ramanujan continued fraction (with a factor of $q^{1/5}$ dropped off)

$$R(q) := \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots}$$

can be represented as the quotient of $H(q)$ and $G(q)$ (see [25]):

$$R(q) = \frac{H(q)}{G(q)} = \left(\begin{matrix}q,q^4\\ q^2,q^3\end{matrix};q^5\right)_\infty.$$

In a recent paper of the authors [10], we established 5-dissection formulas for Ramanujan's parameter (cf. [12, p. 523] or [23, p. 362]):

$$k(q) := qR(q)R(q^2)^2, \tag{1.1}$$

its companion

$$\varkappa(q) := \frac{R(q)^2}{R(q^2)}, \tag{1.2}$$

and their reciprocals. These results may be viewed as analogs of Hirschhorn's 5-dissections of $R(q)$ and $R(q)^{-1}$ in [17]. On the other hand, in connection with $R(q)$, Ramanujan also introduced another two parameters in his lost notebook [2, p. 13]:

$$\mu(q) := qR(q)R(q^4) \qquad \text{and} \qquad \nu(q) := \frac{R(q^{1/2})^2 R(q)}{R(q^2)}.$$

Although the two parameters and their reciprocals do not have simple 5-dissection formulas, our numerical experiment reveals that both $\nu(q^2)$ and $\nu(q^2)^{-1}$ join the shortlist of infinite products with vanishing coefficients in arithmetic progressions. Before stating these results, let us briefly review the history of coefficient-vanishing infinite products.

In 1978, Richmond and Szekeres [24] investigated vanishing coefficients in the following infinite product due to Gordon [15]:

$$\sum_{n=0}^{\infty} c(n)q^n = \left(\begin{matrix} q^3, q^5 \\ q, q^7 \end{matrix}; q^8\right)_\infty \tag{1.3}$$

and its reciprocal. One result shown by them states that $c(4n+3)$ is always zero. This paper then led to the work of Andrews and Bressoud [3], in which it was proved that the general infinite product

$$\left(\begin{matrix} q^r, q^{2k-r} \\ q^{k-r}, q^{k+r} \end{matrix}; q^{2k}\right)_\infty$$

shares the same coefficient-vanishing nature whenever $1 \le r \le k-1$ with k and r coprime and of opposite parity. Other studies of coefficient-vanishing infinite products were then carried out by Alladi and Gordon [1], Mc Laughlin [20, 22], Hirschhorn [19], the second author [26, 27], Baruah and Kaur [5], Dou and Xiao [14] and the authors [8].

Now we are in a position to state the vanishing coefficients in $\nu(q^2)$ and $\nu(q^2)^{-1}$. Let

$$\sum_{n=0}^{\infty} \alpha(n)q^n = \nu(q^2) \qquad \text{and} \qquad \sum_{n=0}^{\infty} \beta(n)q^n = \frac{1}{\nu(q^2)}.$$

Theorem 1.1. *For any $n \ge 0$,*

$$\alpha(10n+3) = \alpha(10n+7) = 0, \tag{1.4}$$

$$\beta(10n+3) = \beta(10n+7) = 0. \tag{1.5}$$

In addition, we find another two infinite products with vanishing coefficients in arithmetic progressions that are closely related to $\varkappa(q)$:

$$\sum_{n=0}^{\infty} \gamma(n)q^n = \frac{(-q, -q^4; q^5)_\infty^2 (q^4, q^6; q^{10})_\infty}{(-q^2, -q^3; q^5)_\infty^2 (q^3, q^7; q^{10})_\infty} = \frac{1}{\varkappa(q)} \left(\begin{matrix} q^2, q^8 \\ q^3, q^7 \end{matrix}; q^{10}\right)_\infty, \tag{1.6}$$

$$\sum_{n=0}^{\infty} \delta(n)q^n = \frac{(-q^2, -q^3; q^5)_\infty^2 (q^2, q^8; q^{10})_\infty}{(-q, -q^4; q^5)_\infty^2 (q, q^9; q^{10})_\infty} = \varkappa(q) \left(\begin{matrix} q^4, q^6 \\ q, q^9 \end{matrix}; q^{10}\right)_\infty. \tag{1.7}$$

Theorem 1.2. *For any $n \ge 0$,*

$$\gamma(5n+4) = 0, \tag{1.8}$$

$$\delta(5n+4) = 0. \tag{1.9}$$

The rest of this paper is organized as follows. We first collect some necessary lemmas in Sect. 2. Next in Sect. 3, several new identities concerning Ramanujan's parameters will be established. These identities play an important role in the proof of Theorem 1.1, which is presented in Sect. 4. We then work on the proof of Theorem 1.2 in Sect. 5. Finally, we conclude in the last section with some remarks.

2. Preliminaries

For notational convenience, we will write throughout

$$J_{a,m} = (q^a, q^{m-a}, q^m; q^m)_\infty \qquad \text{and} \qquad J_m = J_{m,3m} = (q^m; q^m)_\infty.$$

Lemma 2.1.

$$\frac{J_2^{10}}{J_1^4 J_4^4} - \frac{J_{10}^{10}}{J_5^4 J_{20}^4} = \frac{4qJ_2^2 J_5 J_{20}}{J_1 J_4}, \tag{2.1}$$

$$\frac{J_2^4}{J_1^2} - \frac{qJ_{10}^4}{J_5^2} = \frac{J_2 J_5^3}{J_1 J_{10}}. \tag{2.2}$$

Proof. The two identities come from (34.1.20) and (34.1.21) in [18], respectively. □

Lemma 2.2.

$$\frac{J_5^2 J_{10}}{J_{1,5}^2 J_{4,10}} + \frac{J_5^2 J_{10}}{J_{2,5}^2 J_{2,10}} = \frac{2J_{10}^4}{J_{1,5} J_{2,10}^2 J_5}, \tag{2.3}$$

$$\frac{J_5^2 J_{10}}{J_{1,5}^2 J_{4,10}} - \frac{J_5^2 J_{10}}{J_{2,5}^2 J_{2,10}} = \frac{2qJ_{10}^4}{J_{2,5} J_{4,10}^2 J_5}, \tag{2.4}$$

$$\frac{J_5 J_{10}^2}{J_{1,5} J_{2,10}^2} - \frac{qJ_5 J_{10}^2}{J_{2,5} J_{4,10}^2} = \frac{J_5^4}{J_{2,5}^2 J_{2,10} J_{10}}, \tag{2.5}$$

$$\frac{J_5 J_{20}}{J_{1,5} J_{4,20}} - \frac{qJ_5 J_{20}}{J_{2,5} J_{8,20}} = \frac{J_{10}^5}{J_2 J_5^2 J_{20}^2}, \tag{2.6}$$

$$\frac{J_5 J_{20}}{J_{1,5} J_{4,20}} + \frac{qJ_5 J_{20}}{J_{2,5} J_{8,20}} = \frac{J_2^5}{J_1^2 J_2 J_4^2}, \tag{2.7}$$

$$\frac{J_5^2 J_{10} J_{20}}{J_{1,5}^2 J_{2,10} J_{8,20}} - \frac{J_5^2 J_{10} J_{20}}{J_{2,5}^2 J_{4,10} J_{4,20}} = \frac{2qJ_5 J_{20}^3}{J_1 J_4 J_{10}^2}, \tag{2.8}$$

$$\frac{J_5^2 J_{10} J_{20}}{J_{1,5}^2 J_{2,10} J_{8,20}} + \frac{J_5^2 J_{10} J_{20}}{J_{2,5}^2 J_{4,10} J_{4,20}} = \frac{2J_4 J_5 J_{20}}{J_1 J_2 J_{10}}. \tag{2.9}$$

Proof. The identities (2.3)–(2.7) are, respectively, (17.4.10), (17.4.11), (17.4.13), (17.4.9), and (17.7.8) in [18]. The last two identities can be found in [16, Theorem 3.1(v) and (vi)]. □

Lemma 2.3.

$$\frac{J_{2,5} J_{4,10}^2}{J_{1,5} J_{2,10}^2} - \frac{q^2 J_{1,5} J_{2,10}^2}{J_{2,5} J_{4,10}^2} = \frac{J_2 J_5^5}{J_1 J_{10}^5}, \tag{2.10}$$

$$\frac{J_{2,5}^2 J_{2,10}}{J_{1,5}^2 J_{4,10}} - \frac{J_{1,5}^2 J_{4,10}}{J_{2,5}^2 J_{2,10}} = \frac{4qJ_1 J_{10}^5}{J_2 J_5^5}, \tag{2.11}$$

$$\frac{J_{2,5}^3 J_{4,10}}{J_{1,5}^3 J_{2,10}} + \frac{q^2 J_{1,5}^3 J_{2,10}}{J_{2,5}^3 J_{4,10}} = \frac{4q^2 J_1 J_{10}^5}{J_2 J_5^5} + \frac{J_2 J_5^5}{J_1 J_{10}^5} + 2q, \tag{2.12}$$

$$\frac{J_{1,5} J_{4,10}^3}{J_{2,5} J_{2,10}^3} + \frac{q^2 J_{2,5} J_{2,10}^3}{J_{1,5} J_{4,10}^3} = \frac{J_2 J_5^5}{J_1 J_{10}^5} + \frac{4q^2 J_1 J_{10}^5}{J_2 J_5^5} - 2q. \tag{2.13}$$

Proof. For (2.10), see [18, Eq. (41.1.4)] or [11, Theorem 3.5]. One may further deduce (2.11) from (41.1.4) and (40.1.5) in [18]. Finally, (2.12) and (2.13) are consequences of Theorem 1.1 in [9]; see also [4, Eqs. (1.21) and (1.22)] or [7, Eqs. (9.3) and (9.4)]. □

Lemma 2.4.

$$\frac{J_5^8 J_{10}^2}{J_{1,5}^2 J_{2,5}^6 J_{2,10}^2} - \frac{J_5^8 J_{10}^2}{J_{1,5}^6 J_{2,5}^2 J_{4,10}^2} + \frac{J_5^8 J_{10}^2}{J_{1,5}^4 J_{2,5}^4 J_{2,10} J_{4,10}} = \frac{J_{10}^6}{J_{2,10}^3 J_{4,10}^3}. \tag{2.14}$$

Proof. This identity follows from Theorem 3.2 of [10]. □

Lemma 2.5.

$$\begin{aligned}\frac{J_{1,5}^2 J_{4,10}}{J_{2,5}^2 J_{2,10}} &= \frac{J_{10,50} J_{20,50}^3 J_{25}^8}{J_{5,25}^2 J_{10,25}^6 J_{50}^4} - \frac{2q J_{10,50} J_{20,50}^2 J_{25}^5}{J_{5,25}^4 J_{10,25}^3 J_{50}} + \frac{4q^2 J_{10,50} J_{20,50}^2 J_{25}^5}{J_{5,25}^3 J_{10,25}^4 J_{50}} \\ &\quad - \frac{4q^3 J_{20,50}^2 J_{25}^2 J_{50}^2}{J_{5,25}^3 J_{10,25}^3} + \frac{2q^4 J_{10,50}^2 J_{20,50} J_{25}^5}{J_{5,25}^3 J_{10,25}^4 J_{50}},\end{aligned} \tag{2.15}$$

$$\begin{aligned}\frac{J_{2,5}^2 J_{2,10}}{J_{1,5}^2 J_{4,10}} &= \frac{J_{10,50}^3 J_{20,50} J_{25}^8}{J_{5,25}^6 J_{10,25}^2 J_{50}^4} + \frac{2q J_{10,50} J_{20,50}^2 J_{25}^5}{J_{5,25}^4 J_{10,25}^3 J_{50}} - \frac{4q^7 J_{10,50}^2 J_{25}^2 J_{50}^2}{J_{5,25}^3 J_{10,25}^3} \\ &\quad - \frac{4q^3 J_{10,50}^2 J_{20,50} J_{25}^5}{J_{5,25}^4 J_{10,25}^3 J_{50}} - \frac{2q^4 J_{10,50}^2 J_{20,50} J_{25}^5}{J_{5,25}^3 J_{10,25}^4 J_{50}}.\end{aligned} \tag{2.16}$$

Proof. The two identities are equivalent to (1.7) and (1.8) in [10]. □

3. Identities involving Ramanujan's parameters

In this section, we present several identities involving Ramanujan's parameters $k(q)$, $\mu(q)$ and $\nu(q)$, some of which will be used in the proof of Theorem 1.1. To the best of our knowledge, these identities seem to be new. It is worth pointing out that Gugg [16] also established a number of modular identities for $k(q)$, $\mu(q)$, and $\nu(q)$.

First, we notice that (2.10) and (2.11) can be restated in terms of $k(q)$ and $\varkappa(q)$:

$$\frac{1}{k(q)} - k(q) = \frac{J_2 J_5^5}{q J_1 J_{10}^5},$$

$$\frac{1}{\varkappa(q)} - \varkappa(q) = \frac{4q J_1 J_{10}^5}{J_2 J_5^5}.$$

We have analogs for $\mu(q)$ and $\nu(q^2)$.

Theorem 3.1.

$$\frac{1}{\mu(q)} - \mu(q) = \frac{J_2^3 J_{10}^5}{q J_1 J_4 J_5^3 J_{20}^3}, \tag{3.1}$$

$$\frac{1}{\nu(q^2)} - \nu(q^2) = \frac{4q J_4 J_{20}^3}{J_{10}^4}. \tag{3.2}$$

In addition, Cooper [11] established infinite product representations for $1-k(q)$ and $1+k(q)$:

$$1-k(q)=\frac{J_{1,10}J_{4,10}J_{5,10}}{J_{2,10}J_{3,10}^2}, \tag{3.3}$$

$$1+k(q)=\frac{J_{2,10}^2J_{5,10}}{J_{1,10}J_{4,10}^2}. \tag{3.4}$$

In light of the above two relations and the definition of $k(q)$, one may deduce

$$\frac{1}{\sqrt{k(q)}}-\sqrt{k(q)}=\sqrt{\frac{R(q)^3}{q}}\frac{J_{5,10}}{J_{1,10}}, \tag{3.5}$$

$$\frac{1}{\sqrt{k(q)}}+\sqrt{k(q)}=\frac{1}{\sqrt{qR(q)^3}}\frac{J_{5,10}}{J_{3,10}}. \tag{3.6}$$

These two identities can be found in [13, Eqs. (1.12) and (1.13)] or [18, Eqs. (41.1.7) and (41.1.8)]. Interestingly, we may find analogous relations concerning Ramanujan's parameters $\mu(q)$ and $\nu(q)$.

Theorem 3.2.

$$\frac{1}{\sqrt{\mu(q)}}-\sqrt{\mu(q)}=\sqrt{\frac{J_1J_4J_{10}^{10}}{qJ_2^2J_5^5J_{20}^5}}, \tag{3.7}$$

$$\frac{1}{\sqrt{\mu(q)}}+\sqrt{\mu(q)}=\sqrt{\frac{J_2^8}{qJ_1^3J_4^3J_5J_{20}}}, \tag{3.8}$$

$$\frac{1}{\sqrt{\nu(q^2)}}-\sqrt{\nu(q^2)}=2q\sqrt{\frac{J_2J_{20}^5}{J_4J_{10}^5}}, \tag{3.9}$$

$$\frac{1}{\sqrt{\nu(q^2)}}+\sqrt{\nu(q^2)}=2\sqrt{\frac{J_4^3J_{20}}{J_2J_{10}^3}}, \tag{3.10}$$

Further, it is easy to observe that $\nu(q^2)=\varkappa(q)\varkappa(q^2)$. Hence, (3.2) can be rewritten as follows:

$$\frac{1}{\varkappa(q)\varkappa(q^2)}-\varkappa(q)\varkappa(q^2)=\frac{4qJ_4J_{20}^3}{J_{10}^4}.$$

For $k(q)$, we have an identity of similar nature.

Theorem 3.3.

$$\frac{k(q)}{k(q^2)}-\frac{k(q^2)}{k(q)}=\frac{J_1J_5^3}{qJ_{10}^4}. \tag{3.11}$$

Finally, we have two identities related to the Rogers–Ramanujan continued fraction.

Theorem 3.4.

$$\frac{R(q)}{R(q^2)R(q^4)}-\frac{q^2R(q^2)R(q^4)}{R(q)}=\frac{J_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}. \tag{3.12}$$

Also,

$$\left(\frac{1}{R(q)R(q^4)}+q^2R(q)R(q^4)\right)-\left(\frac{R(q)}{R(q^2)R(q^4)}-\frac{q^2R(q^2)R(q^4)}{R(q)}\right)=2q. \tag{3.13}$$

We organize the proofs as follows. First, Theorem 3.4 will be established. We then give a proof of Theorem 3.1. Next, the proof of Theorem 3.2 will be presented. Finally, we will show Theorem 3.3.

Proof of Theorem 3.4. From [18, Eq. (17.4.9)], we have

$$\frac{J_5J_{20}}{J_{1,5}J_{4,20}}-\frac{qJ_5J_{20}}{J_{2,5}J_{8,20}}=\frac{J_{10}^5}{J_2J_5^2J_{20}^2}.$$

Squaring both sides gives

$$\frac{J_5^2J_{20}^2}{J_{1,5}^2J_{4,20}^2}+\frac{q^2J_5^2J_{20}^2}{J_{2,5}^2J_{8,20}^2}=\frac{2qJ_5^2J_{20}^2}{J_{1,5}J_{2,5}J_{4,20}J_{8,20}}+\frac{J_{10}^{10}}{J_2^2J_5^4J_{20}^4}.$$

We then divide both sides by $J_5^2J_{20}^2/(J_{1,5}J_{2,5}J_{4,20}J_{8,20})$. Therefore,

$$\frac{J_{2,5}J_{8,20}}{J_{1,5}J_{4,20}}+\frac{q^2J_{1,5}J_{4,20}}{J_{2,5}J_{8,20}}=2q+\frac{J_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}. \tag{3.14}$$

On the other hand, it follows from (2.10) and (2.11) that

$$\begin{aligned}
\frac{4qJ_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}&=\frac{4qJ_1J_{10}^5}{J_2J_5^5}\cdot\frac{J_4J_{10}^5}{J_2J_{20}^5}\\
&=\left(\frac{J_{2,5}^2J_{2,10}}{J_{1,5}^2J_{4,10}}-\frac{J_{1,5}^2J_{4,10}}{J_{2,5}^2J_{2,10}}\right)\cdot\left(\frac{J_{4,10}J_{8,20}^2}{J_{2,10}J_{4,20}^2}-\frac{q^4J_{2,10}J_{4,20}^2}{J_{4,10}J_{8,20}^2}\right)\\
&=\left(\frac{J_{2,5}^2J_{8,20}^2}{J_{1,5}^2J_{4,20}^2}+\frac{q^4J_{1,5}^2J_{4,20}^2}{J_{2,5}^2J_{8,20}^2}\right)-\left(\frac{J_{1,5}^2J_{4,10}^2J_{8,20}^2}{J_{2,5}^2J_{2,10}^2J_{4,20}^2}+\frac{q^4J_{2,5}^2J_{2,10}^2J_{4,20}^2}{J_{1,5}^2J_{4,10}^2J_{8,20}^2}\right)\\
&=\left(\frac{J_{2,5}J_{8,20}}{J_{1,5}J_{4,20}}+\frac{q^2J_{1,5}J_{4,20}}{J_{2,5}J_{8,20}}\right)^2-\left(\frac{J_{1,5}J_{4,10}J_{8,20}}{J_{2,5}J_{2,10}J_{4,20}}-\frac{q^2J_{2,5}J_{2,10}J_{4,20}}{J_{1,5}J_{4,10}J_{8,20}}\right)^2-4q^2.
\end{aligned}$$

Substituting (3.14) into the above gives

$$\frac{4qJ_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}=\left(\frac{J_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}\right)^2+\frac{4qJ_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}-\left(\frac{J_{1,5}J_{4,10}J_{8,20}}{J_{2,5}J_{2,10}J_{4,20}}-\frac{q^2J_{2,5}J_{2,10}J_{4,20}}{J_{1,5}J_{4,10}J_{8,20}}\right)^2,$$

namely,

$$\left(\frac{J_{1,5}J_{4,10}J_{8,20}}{J_{2,5}J_{2,10}J_{4,20}}-\frac{q^2J_{2,5}J_{2,10}J_{4,20}}{J_{1,5}J_{4,10}J_{8,20}}\right)^2=\left(\frac{J_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}\right)^2.$$

We, therefore, obtain (3.12) by equating the constant term. Further, (3.13) follows from (3.12) and (3.14). □

Proof of Theorem 3.1. We first restate (3.1) and (3.2) as

$$\frac{J_{2,5}J_{8,20}}{J_{1,5}J_{4,20}}-\frac{q^2J_{1,5}J_{4,20}}{J_{2,5}J_{8,20}}=\frac{J_2^3J_{10}^5}{J_1J_4J_5^3J_{20}^3}, \tag{3.15}$$

$$\frac{J_{2,5}^2J_{4,10}J_{4,20}}{J_{1,5}^2J_{2,10}J_{8,20}}-\frac{J_{1,5}^2J_{2,10}J_{8,20}}{J_{2,5}^2J_{4,10}J_{4,20}}=\frac{4qJ_4J_{20}^3}{J_{10}^4}. \tag{3.16}$$

Let us square both sides of (3.14) and subtract $4q^2$. Then

$$\left(\frac{J_{2,5}J_{8,20}}{J_{1,5}J_{4,20}}-\frac{q^2J_{1,5}J_{4,20}}{J_{2,5}J_{8,20}}\right)^2=\frac{4qJ_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}+\left(\frac{J_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}\right)^2. \tag{3.17}$$

It follows from (2.1) that

$$\begin{aligned}\frac{4qJ_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}&=\frac{J_1^2J_4^2J_{10}^{10}}{J_2^4J_5^6J_{20}^6}\cdot\frac{4qJ_2^2J_5J_{20}}{J_1J_4}=\frac{J_1^2J_4^2J_{10}^{10}}{J_2^4J_5^6J_{20}^6}\left(\frac{J_2^{10}}{J_1^4J_4^4}-\frac{J_{10}^{10}}{J_5^4J_{20}^4}\right)\\&=\left(\frac{J_2^3J_{10}^5}{J_1J_4J_5^3J_{20}^3}\right)^2-\left(\frac{J_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}\right)^2.\end{aligned}$$

Substituting the above into (3.17) yields

$$\left(\frac{J_{2,5}J_{8,20}}{J_{1,5}J_{4,20}}-\frac{q^2J_{1,5}J_{4,20}}{J_{2,5}J_{8,20}}\right)^2=\left(\frac{J_2^3J_{10}^5}{J_1J_4J_5^3J_{20}^3}\right)^2.$$

We, therefore, obtain (3.15).

Next, we prove (3.16). Multiplying (2.10) by (3.12) gives

$$\begin{aligned}\frac{J_4J_{10}^5}{J_2J_{20}^5}&=\frac{J_1J_4J_{10}^{10}}{J_2^2J_5^5J_{20}^5}\cdot\frac{J_2J_5^5}{J_1J_{10}^5}\\&=\left(\frac{J_{1,5}J_{4,10}J_{8,20}}{J_{2,5}J_{2,10}J_{4,20}}-\frac{q^2J_{2,5}J_{2,10}J_{4,20}}{J_{1,5}J_{4,10}J_{8,20}}\right)\cdot\left(\frac{J_{2,5}J_{4,10}^2}{J_{1,5}J_{2,10}^2}-\frac{q^2J_{1,5}J_{2,10}^2}{J_{2,5}J_{4,10}^2}\right)\\&=\left(\frac{J_{4,10}^3J_{8,20}}{J_{2,10}^3J_{4,20}}+\frac{q^4J_{2,10}^3J_{4,20}}{J_{4,10}^3J_{8,20}}\right)-q^2\left(\frac{J_{2,5}^2J_{4,10}J_{4,20}}{J_{1,5}^2J_{2,10}J_{8,20}}+\frac{J_{1,5}^2J_{2,10}J_{8,20}}{J_{2,5}^2J_{4,10}J_{4,20}}\right).\end{aligned}$$

We further substitute (2.12) with q replaced by q^2 into the above. Thus,

$$\frac{J_{2,5}^2J_{4,10}J_{4,20}}{J_{1,5}^2J_{2,10}J_{8,20}}+\frac{J_{1,5}^2J_{2,10}J_{8,20}}{J_{2,5}^2J_{4,10}J_{4,20}}=\frac{4q^2J_2J_{20}^5}{J_4J_{10}^5}+2,$$

from which we obtain

$$\begin{aligned}\left(\frac{J_{2,5}^2J_{4,10}J_{4,20}}{J_{1,5}^2J_{2,10}J_{8,20}}-\frac{J_{1,5}^2J_{2,10}J_{8,20}}{J_{2,5}^2J_{4,10}J_{4,20}}\right)^2&=\left(\frac{J_{2,5}^2J_{4,10}J_{4,20}}{J_{1,5}^2J_{2,10}J_{8,20}}+\frac{J_{1,5}^2J_{2,10}J_{8,20}}{J_{2,5}^2J_{4,10}J_{4,20}}\right)^2-4\\&=\frac{16q^2J_2J_{20}^5}{J_4J_{10}^5}+\frac{16q^4J_2^2J_{20}^{10}}{J_4^2J_{10}^{10}}.\end{aligned}$$

We further rewrite (2.2) as follows:

$$1+\frac{q^2J_2J_{20}^5}{J_4J_{10}^5}=\frac{J_4^3J_{20}}{J_2J_{10}^3}.$$

Hence,

$$\begin{aligned}\left(\frac{J_{2,5}^2J_{4,10}J_{4,20}}{J_{1,5}^2J_{2,10}J_{8,20}}-\frac{J_{1,5}^2J_{2,10}J_{8,20}}{J_{2,5}^2J_{4,10}J_{4,20}}\right)^2&=\frac{16q^2J_2J_{20}^5}{J_4J_{10}^5}\left(1+\frac{q^2J_2J_{20}^5}{J_4J_{10}^5}\right)\\&=\frac{16q^2J_2J_{20}^5}{J_4J_{10}^5}\cdot\frac{J_4^3J_{20}}{J_2J_{10}^3}=\left(\frac{4qJ_4J_{20}^3}{J_{10}^4}\right)^2,\end{aligned}$$

from which we obtain (3.16). □

Proof of Theorem 3.2. It follows from (2.6), (2.7), and the definition of $\mu(q)$ that

$$\frac{1}{\sqrt{\mu(q)}} \pm \sqrt{\mu(q)} = \frac{1 \pm \mu(q)}{\sqrt{\mu(q)}} = \frac{\dfrac{J_5 J_{20}}{J_{1,5} J_{4,20}} \pm \dfrac{q J_5 J_{20}}{J_{2,5} J_{8,20}}}{\dfrac{J_5 J_{20}}{J_{1,5} J_{4,20}}} \sqrt{\frac{\dfrac{J_5 J_{20}}{J_{1,5} J_{4,20}}}{\dfrac{q J_5 J_{20}}{J_{2,5} J_{8,20}}}}.$$

Therefore, (3.7) and (3.8) hold true after simplification.

Similarly, by (2.8) and (2.9) and the definition of $\nu(q)$, we have

$$\frac{1}{\sqrt{\nu(q^2)}} \pm \sqrt{\nu(q^2)} = \frac{1 \pm \nu(q^2)}{\sqrt{\nu(q^2)}} = \frac{\dfrac{J_5^2 J_{10} J_{20}}{J_{1,5}^2 J_{2,10} J_{8,20}} \pm \dfrac{J_5^2 J_{10} J_{20}}{J_{2,5}^2 J_{4,10} J_{4,20}}}{\dfrac{J_5^2 J_{10} J_{20}}{J_{1,5}^2 J_{2,10} J_{8,20}}} \sqrt{\frac{\dfrac{J_5^2 J_{10} J_{20}}{J_{1,5}^2 J_{2,10} J_{8,20}}}{\dfrac{J_5^2 J_{10} J_{20}}{J_{2,5}^2 J_{4,10} J_{4,20}}}},$$

from which we obtain (3.9) and (3.10). □

Remark 3.1. Two remarks on Theorem 3.2 are necessary. First, taking the product of (3.7) and (3.8) (resp. (3.9) and (3.10)) yields (3.1) (resp. (3.2)). From this perspective, we give alternative proofs of (3.1) and (3.2). Second, following a similar argument with (2.3) and (2.4) applied, we may obtain identities involving $\varkappa(q)$ of the same nature:

$$\frac{1}{\sqrt{\varkappa(q)}} - \sqrt{\varkappa(q)} = 2q\sqrt{\frac{J_1^2 J_2 J_{10}^7}{J_5^4 J_{2,5}^2 J_{4,10}^4}},$$
$$\frac{1}{\sqrt{\varkappa(q)}} + \sqrt{\varkappa(q)} = 2\sqrt{\frac{J_{2,5}^2 J_{4,10}^4 J_{10}^3}{J_2^3 J_5^6}}.$$

Proof of Theorem 3.3. To prove (3.11), we first restate it as follows:

$$\frac{J_{1,5} J_{2,10} J_{8,20}^2}{J_{2,5} J_{4,10} J_{4,20}^2} - \frac{q^2 J_{2,5} J_{4,10} J_{4,20}^2}{J_{1,5} J_{2,10} J_{8,20}^2} = \frac{J_1 J_5^3}{J_{10}^4}. \tag{3.18}$$

According to (2.11) and (3.12), one has

$$\begin{aligned}\frac{4q^2 J_1 J_{10}^5}{J_2 J_5^5} &= \frac{4q^2 J_2 J_{20}^5}{J_4 J_{10}^5} \cdot \frac{J_1 J_4 J_{10}^{10}}{J_2^2 J_5^5 J_{20}^5} \\ &= \left(\frac{J_{1,5} J_{4,10} J_{8,20}}{J_{2,5} J_{2,10} J_{4,20}} - \frac{q^2 J_{2,5} J_{2,10} J_{4,20}}{J_{1,5} J_{4,10} J_{8,20}}\right) \cdot \left(\frac{J_{4,10}^2 J_{4,20}}{J_{2,10}^2 J_{8,20}} - \frac{J_{2,10}^2 J_{8,20}}{J_{4,10}^2 J_{4,20}}\right) \\ &= \left(\frac{J_{1,5} J_{4,10}^3}{J_{2,5} J_{2,10}^3} + \frac{q^2 J_{2,5} J_{2,10}^3}{J_{1,5} J_{4,10}^3}\right) - \left(\frac{J_{1,5} J_{2,10} J_{8,20}^2}{J_{2,5} J_{4,10} J_{4,20}^2} + \frac{q^2 J_{2,5} J_{4,10} J_{4,20}^2}{J_{1,5} J_{2,10} J_{8,20}^2}\right).\end{aligned}$$

We then substitute (2.13) into the above and conclude

$$\frac{J_{1,5} J_{2,10} J_{8,20}^2}{J_{2,5} J_{4,10} J_{4,20}^2} + \frac{q^2 J_{2,5} J_{4,10} J_{4,20}^2}{J_{1,5} J_{2,10} J_{8,20}^2} = \frac{J_2 J_5^5}{J_1 J_{10}^5} - 2q,$$

from which we obtain

$$\left(\frac{J_{1,5} J_{2,10} J_{8,20}^2}{J_{2,5} J_{4,10} J_{4,20}^2} - \frac{q^2 J_{2,5} J_{4,10} J_{4,20}^2}{J_{1,5} J_{2,10} J_{8,20}^2}\right)^2 = \left(\frac{J_{1,5} J_{2,10} J_{8,20}^2}{J_{2,5} J_{4,10} J_{4,20}^2} + \frac{q^2 J_{2,5} J_{4,10} J_{4,20}^2}{J_{1,5} J_{2,10} J_{8,20}^2}\right)^2 - 4q^2$$

$$= \frac{J_2^2 J_5^{10}}{J_1^2 J_{10}^{10}} - \frac{4q J_2 J_5^5}{J_1 J_{10}^5}.$$

Replacing q by $-q$ in (2.1), we find that

$$\frac{J_2 J_5^5}{J_1 J_{10}^5} - 4q = \frac{J_1^3 J_5}{J_2 J_{10}^3}.$$

Hence,

$$\left(\frac{J_{1,5} J_{2,10} J_{8,20}^2}{J_{2,5} J_{4,10} J_{4,20}^2} - \frac{q^2 J_{2,5} J_{4,10} J_{4,20}^2}{J_{1,5} J_{2,10} J_{8,20}^2} \right)^2 = \frac{J_2 J_5^5}{J_1 J_{10}^5} \left(\frac{J_2 J_5^5}{J_1 J_{10}^5} - 4q \right)$$
$$= \frac{J_2 J_5^5}{J_1 J_{10}^5} \cdot \frac{J_1^3 J_5}{J_2 J_{10}^3} = \left(\frac{J_1 J_5^3}{J_{10}^4} \right)^2,$$

from which (3.18) follows. □

4. Proof of Theorem 1.1

We first require a corollary of (3.2).

Corollary 4.1. *For any $n \geq 0$,*

$$\alpha(5n+2) = \beta(5n+2), \tag{4.1}$$
$$\alpha(5n+3) = \beta(5n+3). \tag{4.2}$$

Proof. Notice that (3.2) can be restated as

$$\sum_{n=0}^{\infty} \beta(n) q^n - \sum_{n=0}^{\infty} \alpha(n) q^n = \frac{4q J_4 J_{20}^3}{J_{10}^4}.$$

Euler's Pentagonal Number Theorem [18, Eq. (1.6.1)] tells us that there are no terms of the form q^{5n+3} and q^{5n+4} in the expansion of $J_1 = (q;q)_\infty$. Hence, there are no terms of the form q^{5n+2} and q^{5n+3} in the expansion of qJ_4. Our desired results, therefore, hold. □

The relations (4.1) and (4.2) clearly imply that if we can prove one of (1.4) and (1.5), the other holds automatically. Here, we will show (1.4). Our proof relies on the following result.

Theorem 4.2.

$$\sum_{n=0}^{\infty} \alpha(5n+2) q^n = \frac{2 J_{4,20} J_{8,20}^2 J_{10}^5}{J_{2,10}^4 J_{4,10}^3 J_{20}}, \tag{4.3}$$
$$\sum_{n=0}^{\infty} \alpha(5n+3) q^n = -\frac{2q J_{4,20}^2 J_{8,20} J_{10}^5}{J_{2,10}^3 J_{4,10}^4 J_{20}}. \tag{4.4}$$

For their proofs, we begin with an auxiliary identity.

Lemma 4.3. *Let*

$$Z(q) := \frac{2q J_5^5 J_{10}^8}{J_{1,5}^3 J_{2,5}^4 J_{4,10}^4 J_{4,20} J_{20}} - \frac{2q J_5^8 J_{10}^2 J_{20}^2}{J_{1,5}^4 J_{2,5}^4 J_{2,10} J_{4,10} J_{4,20} J_{8,20}} + \frac{2q^2 J_5^8 J_{10}^2 J_{20}^2}{J_{1,5}^5 J_{2,5}^3 J_{4,10}^2 J_{8,20}^2}. \tag{4.5}$$

Then

$$Z(q)=0. \tag{4.6}$$

Proof. We know from (3.13) that

$$\begin{aligned}
0&=\frac{J_5^8J_{10}^2J_{20}^2}{J_{1,5}^4J_{2,5}^4J_{2,10}J_{4,10}J_{4,20}J_{8,20}}\\
&\quad\times\left(\left(\frac{J_{2,5}J_{8,20}}{J_{1,5}J_{4,20}}+\frac{q^2J_{1,5}J_{4,20}}{J_{2,5}J_{8,20}}\right)-\left(\frac{J_{1,5}J_{4,10}J_{8,20}}{J_{2,5}J_{2,10}J_{4,20}}-\frac{q^2J_{2,5}J_{2,10}J_{4,20}}{J_{1,5}J_{4,10}J_{8,20}}\right)-2q\right)\\
&=\frac{J_5^6}{J_{1,5}^3J_{2,5}^3}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}-\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\left(\frac{J_{10}J_{20}^2}{J_{2,10}J_{4,20}^2}-\frac{q^2J_{10}J_{20}^2}{J_{4,10}J_{8,20}^2}\right)\\
&\quad-\frac{2qJ_5^8J_{10}^2J_{20}^2}{J_{1,5}^4J_{2,5}^4J_{2,10}J_{4,10}J_{4,20}J_{8,20}}+\frac{2q^2J_5^8J_{10}^2J_{20}^2}{J_{1,5}^5J_{2,5}^3J_{4,10}^2J_{8,20}^2}.
\end{aligned}$$

In view of (2.4) and (2.5), we further find that

$$\begin{aligned}
&\frac{2qJ_5^5J_{10}^8}{J_{1,5}^3J_{2,5}^4J_{4,10}^4J_{4,20}J_{20}}\\
&\qquad=\frac{J_5^6}{J_{1,5}^3J_{2,5}^3}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}-\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\left(\frac{J_{10}J_{20}^2}{J_{2,10}J_{4,20}^2}-\frac{q^2J_{10}J_{20}^2}{J_{4,10}J_{8,20}^2}\right).
\end{aligned}$$

Our desired result, therefore, follows. □

It follows from (2.15) that

$$\begin{aligned}
\sum_{n=0}^{\infty}\alpha(n)q^n&=\frac{J_{1,5}^2J_{2,10}J_{8,20}}{J_{2,5}^2J_{4,10}J_{4,20}}=\frac{J_{1,5}^2J_{4,10}}{J_{2,5}^2J_{2,10}}\cdot\frac{J_{2,10}^2J_{8,20}}{J_{4,10}^2J_{4,20}}\\
&=\left(\frac{J_{10,50}J_{20,50}^3J_{25}^8}{J_{5,25}^2J_{10,25}^6J_{50}^4}-\frac{2qJ_{10,50}J_{20,50}^2J_{25}^5}{J_{5,25}^4J_{10,25}^3J_{50}}+\frac{4q^2J_{10,50}J_{20,50}^2J_{25}^5}{J_{5,25}^3J_{10,25}^4J_{50}}\right.\\
&\qquad\left.-\frac{4q^3J_{20,50}^2J_{25}^2J_{50}^2}{J_{5,25}^3J_{10,25}^3}+\frac{2q^4J_{10,50}^2J_{20,50}J_{25}^5}{J_{5,25}^3J_{10,25}^4J_{50}}\right)\\
&\quad\times\left(\frac{J_{20,100}J_{40,100}^3J_{50}^8}{J_{10,50}^2J_{20,50}^6J_{100}^4}-\frac{2q^2J_{20,100}J_{40,100}^2J_{50}^5}{J_{10,50}^4J_{20,50}^3J_{100}}+\frac{4q^4J_{20,100}J_{40,100}^2J_{50}^5}{J_{10,50}^3J_{20,50}^4J_{100}}\right.\\
&\qquad\left.-\frac{4q^6J_{40,100}^2J_{50}^2J_{100}^2}{J_{10,50}^3J_{20,50}^3}+\frac{2q^8J_{20,100}^2J_{40,100}J_{50}^5}{J_{10,50}^3J_{20,50}^4J_{100}}\right).
\end{aligned}\tag{4.7}$$

In the sequel, let us define for $0\le i\le 4$,

$$A_i(q):=\sum_{n=0}^{\infty}\alpha(5n+i)q^n\quad\text{and}\quad B_i(q):=\sum_{n=0}^{\infty}\beta(5n+i)q^n. \tag{4.8}$$

Proof of (4.3). We deduce from (4.7) that

$$\begin{aligned}
A_2(q)&=\frac{4J_{4,20}J_{8,20}^3J_5^5J_{10}^7}{J_{1,5}^3J_{2,5}^4J_{2,10}J_{4,10}^4J_{20}^4}-\frac{2J_{4,20}J_{8,20}^2J_5^8J_{10}}{J_{1,5}^2J_{2,5}^6J_{2,10}^3J_{20}}-\frac{16qJ_{4,20}J_{8,20}^2J_5^2J_{10}^7}{J_{1,5}^3J_{2,5}^3J_{2,10}^3J_{4,10}^2J_{20}}\\
&\quad+\frac{8qJ_{8,20}^2J_5^5J_{10}J_{20}^2}{J_{1,5}^4J_{2,5}^3J_{2,10}^2J_{4,10}}+\frac{4q^2J_{4,20}^2J_{8,20}J_5^5J_{10}^4}{J_{1,5}^3J_{2,5}^4J_{2,10}J_{4,10}^3J_{20}}.
\end{aligned}$$

With the aid of (2.3) and (2.4), it follows that

$$\begin{aligned}\frac{J_{2,10}J_{10}J_{20}^3}{J_{4,20}^2J_{8,20}^3}\cdot A_2(q)=&\frac{4J_5^5J_{10}^8}{J_{1,5}^3J_{2,5}^4J_{4,10}^4J_{4,20}J_{20}}-\frac{2J_5^8J_{10}^2J_{20}^2}{J_{1,5}^2J_{2,5}^6J_{2,10}^2J_{4,20}J_{8,20}}\\&-\frac{4J_5^4J_{20}^2}{J_{1,5}^2J_{2,5}^2J_{4,20}J_{8,20}}\left(\frac{J_5^4J_{10}^2}{J_{1,5}^4J_{4,10}^2}-\frac{J_5^4J_{10}^2}{J_{2,5}^4J_{2,10}^2}\right)\\&+\frac{4qJ_5^5J_{10}^3J_{20}}{J_{1,5}^4J_{2,5}^3J_{4,10}J_{8,20}}\left(\frac{J_{10}^2J_{20}}{J_{2,10}^2J_{8,20}}+\frac{J_{10}^2J_{20}}{J_{4,10}^2J_{4,20}}\right)\\&+\frac{2qJ_5^6J_{10}J_{20}^2}{J_{1,5}^3J_{2,5}^3J_{4,10}J_{8,20}^2}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}-\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right).\end{aligned}$$

We then tactfully rewrite the second last term on the right-hand side of the above using (2.3) and (2.4):

$$\begin{aligned}&\frac{4qJ_5^5J_{10}^3J_{20}}{J_{1,5}^4J_{2,5}^3J_{4,10}J_{8,20}}\left(\frac{J_{10}^2J_{20}}{J_{2,10}^2J_{8,20}}+\frac{J_{10}^2J_{20}}{J_{4,10}^2J_{4,20}}\right)\\&\quad=\frac{2qJ_5^6J_{10}J_{20}^2}{J_{1,5}^3J_{2,5}^3J_{4,10}J_{8,20}^2}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}+\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\\&\qquad+\frac{2J_5^6J_{10}J_{20}^2}{J_{1,5}^4J_{2,5}^2J_{4,10}J_{4,20}J_{8,20}}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}-\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{J_{2,10}J_{10}J_{20}^3}{J_{4,20}^2J_{8,20}^3}\cdot A_2(q)=&\frac{4J_5^5J_{10}^8}{J_{1,5}^3J_{2,5}^4J_{4,10}^4J_{4,20}J_{20}}-\frac{2J_5^8J_{10}^2J_{20}^2}{J_{1,5}^2J_{2,5}^6J_{2,10}^2J_{4,20}J_{8,20}}\\&-\frac{4J_5^4J_{20}^2}{J_{1,5}^2J_{2,5}^2J_{4,20}J_{8,20}}\left(\frac{J_5^4J_{10}^2}{J_{1,5}^4J_{4,10}^2}-\frac{J_5^4J_{10}^2}{J_{2,5}^4J_{2,10}^2}\right)\\&+\frac{2qJ_5^6J_{10}J_{20}^2}{J_{1,5}^3J_{2,5}^3J_{4,10}J_{8,20}^2}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}+\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\\&+\frac{2J_5^6J_{10}J_{20}^2}{J_{1,5}^4J_{2,5}^2J_{4,10}J_{4,20}J_{8,20}}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}-\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\\&+\frac{2qJ_5^6J_{10}J_{20}^2}{J_{1,5}^3J_{2,5}^3J_{4,10}J_{8,20}^2}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}-\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right).\end{aligned}$$

Rearranging terms then yields

$$\begin{aligned}&\frac{J_{2,10}J_{10}J_{20}^3}{J_{4,20}^2J_{8,20}^3}\cdot A_2(q)\\&=\left(\frac{4J_5^5J_{10}^8}{J_{1,5}^3J_{2,5}^4J_{4,10}^4J_{4,20}J_{20}}-\frac{4J_5^8J_{10}^2J_{20}^2}{J_{1,5}^4J_{2,5}^4J_{2,10}J_{4,10}J_{4,20}J_{8,20}}+\frac{4qJ_5^8J_{10}^2J_{20}^2}{J_{1,5}^5J_{2,5}^3J_{4,10}^2J_{4,20}^2}\right)\\&+\left(\frac{2J_5^8J_{10}^2J_{20}^2}{J_{1,5}^2J_{2,5}^6J_{2,10}^2J_{4,20}J_{8,20}}-\frac{2J_5^8J_{10}^2J_{20}^2}{J_{1,5}^6J_{2,5}^2J_{4,10}^2J_{4,20}J_{8,20}}+\frac{2J_5^8J_{10}^2J_{20}^2}{J_{1,5}^4J_{2,5}^4J_{2,10}J_{4,10}J_{4,20}J_{8,20}}\right).\end{aligned}$$

By virtue of (2.14) and (4.5), we have

$$A_2(q)=\frac{2J_{4,20}J_{8,20}^2J_{10}^5}{J_{2,10}^4J_{4,10}^3J_{20}}+\frac{2J_{4,20}^2J_{8,20}^3}{qJ_{2,10}J_{10}J_{20}^3}\cdot Z(q).$$

Finally, (4.3) follows by making use of (4.6). □

Proof of (4.4). We deduce from (4.7) that

$$\begin{aligned}A_3(q)=&-\frac{4J_{4,20}J_{8,20}^3J_5^2J_{10}^{10}}{J_{1,5}^3J_{2,5}^3J_{2,10}^2J_{4,10}^4J_{20}^4}+\frac{4J_{4,20}J_{8,20}^2J_5^5J_{10}^4}{J_{1,5}^4J_{2,5}^3J_{2,10}^3J_{4,10}J_{20}}+\frac{8qJ_{4,20}J_{8,20}^2J_5^5J_{10}^4}{J_{1,5}^3J_{2,5}^4J_{2,10}J_{4,10}^3J_{20}}\\&-\frac{16qJ_{8,20}^2J_5^5J_{10}J_{20}^2}{J_{1,5}^3J_{2,5}^4J_{2,10}^2J_{4,10}}+\frac{2qJ_{4,20}^2J_{8,20}J_5^8J_{10}}{J_{1,5}^2J_{2,5}^6J_{2,10}^2J_{4,10}J_{20}}.\end{aligned}$$

Applying (2.4) to the third term and (2.3) to the fourth term of the above gives

$$A_3(q)=-\frac{4J_{4,20}J_{8,20}^3J_5^2J_{10}^{10}}{J_{1,5}^3J_{2,5}^3J_{2,10}^2J_{4,10}^4J_{20}^4}+\frac{4J_{4,20}J_{8,20}^2J_5^5J_{10}^4}{J_{1,5}^4J_{2,5}^3J_{2,10}^3J_{4,10}J_{20}}-\frac{J_{4,20}^2J_{8,20}^2}{J_{4,10}J_{10}J_{20}^2}\cdot F(q), \tag{4.9}$$

where

$$\begin{aligned}F(q)=&\frac{4J_5^6J_{10}J_{20}}{J_{1,5}^3J_{2,5}^3J_{2,10}J_{4,20}}\left(-\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}+\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\\&+\frac{8qJ_5^5J_{10}^3}{J_{1,5}^3J_{2,5}^4J_{2,10}}\left(\frac{J_{10}^2J_{20}}{J_{2,10}^2J_{8,20}}+\frac{J_{10}^2J_{20}}{J_{4,10}^2J_{4,20}}\right)-\frac{2qJ_5^8J_{10}^2J_{20}}{J_{1,5}^2J_{2,5}^6J_{2,10}^2J_{8,20}}.\end{aligned}$$

Applying (2.3) and (2.4) again, we find that

$$\begin{aligned}F(q)=&\frac{4J_5^6J_{10}J_{20}}{J_{1,5}^3J_{2,5}^3J_{2,10}J_{4,20}}\left(-\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}+\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\\&+\frac{4qJ_5^6J_{10}J_{20}}{J_{1,5}^2J_{2,5}^4J_{2,10}J_{8,20}}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}+\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)\\&+\frac{4J_5^6J_{10}J_{20}}{J_{1,5}^3J_{2,5}^3J_{2,10}J_{4,20}}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}-\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right)-\frac{2qJ_5^8J_{10}^2J_{20}}{J_{1,5}^2J_{2,5}^6J_{2,10}^2J_{8,20}}.\end{aligned}$$

After simplification,

$$\begin{aligned}F(q)=&\left(\frac{2qJ_5^8J_{10}^2J_{20}}{J_{1,5}^2J_{2,5}^6J_{2,10}^2J_{8,20}}+\frac{2qJ_5^8J_{10}^2J_{20}}{J_{1,5}^4J_{2,5}^4J_{2,10}J_{4,10}J_{8,20}}-\frac{2qJ_5^8J_{10}^2J_{20}}{J_{1,5}^6J_{2,5}^2J_{4,10}^2J_{8,20}}\right)\\&+\frac{2qJ_5^6J_{10}J_{20}}{J_{1,5}^4J_{2,5}^2J_{4,10}J_{8,20}}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}+\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right).\end{aligned}$$

Substituting (2.14) into the above and combining (4.9),

$$\begin{aligned}A_3(q)=&-\frac{2qJ_{4,20}^2J_{8,20}J_{10}^5}{J_{2,10}^3J_{4,10}^4J_{20}}-\frac{4J_{4,20}J_{8,20}^3J_5^2J_{10}^{10}}{J_{1,5}^3J_{2,5}^3J_{2,10}^2J_{4,10}^4J_{20}^4}+\frac{4J_{4,20}J_{8,20}^2J_5^5J_{10}^4}{J_{1,5}^4J_{2,5}^3J_{2,10}^3J_{4,10}J_{20}}\\&-\frac{2qJ_{4,20}^2J_{8,20}J_5^6}{J_{1,5}^4J_{2,5}^2J_{4,10}^2J_{20}}\left(\frac{J_5^2J_{10}}{J_{1,5}^2J_{4,10}}+\frac{J_5^2J_{10}}{J_{2,5}^2J_{2,10}}\right).\end{aligned}$$

Applying (2.3) to the last line of the above,

$$A_3(q) = -\frac{2qJ_{4,20}^2 J_{8,20} J_{10}^5}{J_{2,10}^3 J_{4,10}^4 J_{20}} - \frac{2J_{2,5} J_{4,20}^2 J_{8,20}^3 J_{10}^2}{qJ_{2,10}^2 J_5^3 J_{20}^3} \cdot Z(q).$$

Finally, utilizing (4.6) yields (4.4). □

At the end of this section, we complete the proof of (1.4).

Proof of (1.4). It is a trivial observation that there are no terms of the form q^{2n+1} in the expansion of the right-hand side of (4.3). Hence, $\alpha(10n+7) = 0$. Similarly, there are no terms of the form q^{2n} in the expansion of the right-hand side of (4.4). This implies that $\alpha(10n+3) = 0$. □

5. Proof of Theorem 1.2

We need two identities that are special cases of a result due to Mc Laughlin [21].

Lemma 5.1.

$$\frac{J_{2,10}}{J_{3,10}} = \frac{J_{5,25}J_{10,25}J_{50}^3}{J_{10,50}^3 J_{20,50}^2} + \frac{q^6 J_{5,25}^2 J_{10,25} J_{50}^6}{J_{10,50}^2 J_{20,50}^4 J_{25}^3} - \frac{q^2 J_{5,25} J_{10,25} J_{50}^3}{J_{10,50}^2 J_{20,50}^3} + \frac{q^3 J_{5,25}^2 J_{50}^3}{J_{10,50}^2 J_{20,50}^3}, \tag{5.1}$$

$$\frac{J_{4,10}}{J_{1,10}} = \frac{J_{10,25}^2 J_{50}^3}{J_{10,50}^3 J_{20,50}^2} + \frac{qJ_{5,25}J_{10,25}J_{50}^3}{J_{10,50}^3 J_{20,50}^2} + \frac{q^2 J_{5,25} J_{10,25}^2 J_{50}^6}{J_{25}^3 J_{10,50}^4 J_{20,50}^2} + \frac{q^3 J_{5,25} J_{10,25} J_{50}^3}{J_{10,50}^2 J_{20,50}^3}. \tag{5.2}$$

Proof. In [21, Proposition 2.1], Mc Laughlin showed that for any positive integer p,

$$\frac{(q, q, az, q/(az); q)_\infty}{(a, q/a, z, q/z; q)_\infty} = \sum_{j=0}^{p-1} z^j \frac{\left(q^p, q^p, aq^j z^p, q^{p-j}/(az^p); q^p\right)_\infty}{(aq^j, q^{p-j}/a, z^p, q^p/z^p; q^p)_\infty}.$$

Taking $(a, z, q) \to (q^5, q^3, q^{10})$ and $p = 5$ then yields (5.1). For (5.2), we need to take $(a, z, q) \to (q^5, q, q^{10})$ and $p = 5$. □

To prove Theorem 1.2, we first multiply (2.16) by (5.1),

$$\begin{aligned}
&\sum_{n=0}^{\infty} \gamma(n) q^n \\
&\quad = \left(\frac{J_{10,50}^3 J_{20,50} J_{25}^8}{J_{5,25}^6 J_{10,25}^2 J_{50}^4} + \frac{2qJ_{10,50}J_{20,50}^2 J_{25}^5}{J_{5,25}^4 J_{10,25}^3 J_{50}} - \frac{4q^7 J_{10,50}^2 J_{25}^2 J_{50}^2}{J_{5,25}^3 J_{10,25}^3} \right. \\
&\qquad \left. - \frac{4q^3 J_{10,50}^2 J_{20,50} J_{25}^5}{J_{5,25}^4 J_{10,25}^3 J_{50}} - \frac{2q^4 J_{10,50}^2 J_{20,50} J_{25}^5}{J_{5,25}^3 J_{10,25}^4 J_{50}} \right) \\
&\qquad \times \left(\frac{J_{5,25}J_{10,25}J_{50}^3}{J_{10,50}^3 J_{20,50}^2} + \frac{q^6 J_{5,25}^2 J_{10,25} J_{50}^6}{J_{10,50}^2 J_{20,50}^4 J_{25}^3} - \frac{q^2 J_{5,25} J_{10,25} J_{50}^3}{J_{10,50}^2 J_{20,50}^3} + \frac{q^3 J_{5,25}^2 J_{50}^3}{J_{10,50}^2 J_{20,50}^3} \right).
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \gamma(5n+4) q^{5n+4}$$

$$
\begin{aligned}
&= \frac{2qJ_{10,50}J_{20,50}^2J_{25}^5}{J_{5,25}^4J_{10,25}^3J_{50}} \cdot \frac{q^3J_{5,25}^2J_{50}^3}{J_{10,50}^2J_{20,50}^3} + \frac{4q^7J_{10,50}^2J_{25}^2J_{50}^2}{J_{5,25}^3J_{10,25}^3} \cdot \frac{q^2J_{5,25}J_{10,25}J_{50}^3}{J_{10,50}^2J_{20,50}^3} \\
&\quad - \frac{4q^3J_{10,50}^2J_{20,50}J_{25}^5}{J_{5,25}^4J_{10,25}^3J_{50}} \cdot \frac{q^6J_{5,25}^2J_{10,25}J_{50}^6}{J_{10,50}^2J_{20,50}^4J_{25}^3} - \frac{2q^4J_{10,50}^2J_{20,50}J_{25}^5}{J_{5,25}^3J_{10,25}^4J_{50}} \cdot \frac{J_{5,25}J_{10,25}J_{50}^3}{J_{10,50}^3J_{20,50}^2} \\
&= 0.
\end{aligned}
$$

It follows that $\gamma(5n+4)$ is always zero.

The proof of (1.9) is analogous. We only need to multiply (2.15) by (5.2) and then expand the product. The details will be omitted.

6. Final remarks

By similar techniques of proving (4.3) and (4.4), we are able to show that

$$
\begin{aligned}
A_1(q) &= -\frac{2J_{4,20}J_{8,20}^2J_5^3J_{10}^2}{J_{2,5}J_{2,10}^3J_{4,10}^3J_{20}}, & A_4(q) &= -\frac{2J_{1,5}J_{4,20}^2J_{8,20}^2J_{10}^9}{J_{2,10}^4J_{4,10}^5J_5^3J_{20}^4}, \\
B_1(q) &= \frac{2J_{1,5}J_{4,20}^2J_{8,20}^2J_{10}^{11}}{J_{2,10}^5J_{4,10}^4J_5^3J_{20}^4}, & B_2(q) &= \frac{2J_{4,20}J_{8,20}^2J_{10}^5}{J_{2,10}^4J_{4,10}^3J_{20}}, \\
B_3(q) &= -\frac{2qJ_{4,20}^2J_{8,20}J_{10}^5}{J_{2,10}^3J_{4,10}^4J_{20}}, & B_4(q) &= -\frac{2J_{4,20}^2J_{8,20}J_5^3J_{10}^2}{J_{1,5}J_{2,10}^3J_{4,10}^3J_{20}},
\end{aligned}
$$

where $A_i(q)$ and $B_i(q)$ are defined as in (4.8). However, it seems that $A_0(q)$ and $B_0(q)$ cannot be simplified to single theta quotients.

Our numerical experiments also reveal the following sign patterns:

$$
\alpha(n) \begin{cases} > 0, & \text{if } n \equiv 0, 2, 5, 9 \pmod{10}, \\ < 0, & \text{if } n \equiv 1, 4, 6, 8 \pmod{10}, \end{cases} \tag{6.1}
$$

$$
\beta(n) \begin{cases} > 0, & \text{if } n \equiv 0, 1, 2 \pmod{10}, \\ < 0, & \text{if } n \equiv 4, 5, 6, 8, 9 \pmod{10}, \end{cases} \tag{6.2}
$$

except for $\alpha(6) = \beta(6) = 0$. However, the sign patterns (6.1) and (6.2) can hardly be derived directly by $A_i(q)$ and $B_i(q)$ $(0 \le i \le 4)$. On the other hand, taking advantage of an asymptotic formula due to the first author [6], one can show the validity of (6.1) and (6.2) for sufficiently large n.

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