Multi-headed Lattice Green Function (N = 4, M = 3)

```
In[]:= NN = 4;
MM = 3;
```

Generate a sequence from recurrence & initial values

Koutschan's implementation

```
In[]:= (* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] ⧴ a, Infinity]];
   Do[
    AppendTo[vals, Solve[rec /. n → i /. f[i] → x /. f[a_] ⧴ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Marathon begins...

```
In[]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

We work on $\tilde{r}(n) := r(2n)$.

```
In[]:= ClearAll[k1, k2, k3, k4, z, w, α, β];

In[]:= k4 = α - k1 - k2 - k3;
summand = Binomial[2 α, 2 k1] Binomial[2 α - 2 k1, 2 k2]
   Binomial[2 α - 2 k1 - 2 k2, 2 k3] Binomial[2 (α - k1), α - k1]
   Binomial[2 (α - k2), α - k2] Binomial[2 (α - k3), α - k3] Binomial[2 (α - k4), α - k4];
```

Apply “Creative Telescoping”.

In[ ]:=
```
Timing[ann0 = Annihilator[summand, {S[k1], S[k2], S[k3], S[α]}];]
```
Out[ ]= {0.046875, Null}

In[ ]:=
```
Timing[ann1 = FindCreativeTelescoping[ann0, S[k1] - 1][[1]];]
```
Out[ ]= {37.2969, Null}

In[ ]:=
```
Timing[ann2 = FindCreativeTelescoping[ann1, S[k2] - 1][[1]];]
```
Out[ ]= {347.047, Null}

In[ ]:=
```
Timing[ann3 = FindCreativeTelescoping[ann2, S[k3] - 1][[1]];]
```
Out[ ]= {291.984, Null}

Recurrence for $\tilde{r}(n)$

In[∘]:=
```
RECNormalized = ann3;
ToOrePolynomial[RECNormalized]
```

Out[∘]=

$$
\begin{aligned}
\{ \big(& -9\,051\,531\,325\,562\,880 - 90\,332\,029\,095\,081\,984\,\alpha - \\
& 420\,333\,410\,362\,428\,416\,\alpha^{2} - 1\,213\,206\,945\,955\,473\,664\,\alpha^{3} - 2\,437\,377\,188\,874\,087\,136\,\alpha^{4} - \\
& 3\,625\,291\,113\,645\,770\,712\,\alpha^{5} - 4\,144\,688\,219\,837\,114\,384\,\alpha^{6} - 3\,731\,957\,019\,300\,871\,994\,\alpha^{7} - \\
& 2\,689\,507\,840\,271\,682\,912\,\alpha^{8} - 1\,567\,534\,832\,320\,365\,967\,\alpha^{9} - 743\,334\,125\,295\,350\,476\,\alpha^{10} - \\
& 287\,455\,002\,784\,035\,524\,\alpha^{11} - 90\,539\,774\,552\,500\,272\,\alpha^{12} - 23\,112\,095\,925\,472\,389\,\alpha^{13} - \\
& 4\,737\,102\,973\,509\,780\,\alpha^{14} - 767\,930\,664\,461\,310\,\alpha^{15} - 96\,195\,146\,877\,576\,\alpha^{16} - \\
& 8\,977\,485\,504\,456\,\alpha^{17} - 587\,451\,930\,408\,\alpha^{18} - 24\,041\,253\,600\,\alpha^{19} - 462\,944\,160\,\alpha^{20} \big)\, S_{\alpha}^{4} + \\
\big(& 9\,569\,617\,440\,812\,835\,840 + 97\,443\,791\,378\,162\,009\,856\,\alpha + 463\,583\,339\,186\,644\,316\,800\,\alpha^{2} + \\
& 1\,370\,837\,922\,368\,778\,354\,176\,\alpha^{3} + 2\,827\,452\,328\,200\,593\,850\,560\,\alpha^{4} + 4\,326\,575\,055\,112\,730\,856\,640\,\alpha^{5} + \\
& 5\,099\,519\,612\,920\,329\,528\,000\,\alpha^{6} + 4\,743\,666\,552\,937\,883\,189\,952\,\alpha^{7} + 3\,539\,068\,890\,050\,114\,722\,112\,\alpha^{8} + \\
& 2\,139\,750\,587\,880\,300\,657\,856\,\alpha^{9} + 1\,054\,730\,779\,373\,468\,537\,920\,\alpha^{10} + 424\,824\,967\,934\,147\,228\,480\,\alpha^{11} + \\
& 139\,643\,546\,214\,642\,867\,648\,\alpha^{12} + 37\,274\,084\,807\,088\,072\,384\,\alpha^{13} + 8\,003\,802\,897\,605\,020\,608\,\alpha^{14} + \\
& 1\,361\,866\,764\,260\,304\,576\,\alpha^{15} + 179\,386\,646\,751\,384\,192\,\alpha^{16} + 17\,635\,678\,788\,631\,680\,\alpha^{17} + \\
& 1\,217\,772\,669\,657\,600\,\alpha^{18} + 52\,679\,537\,809\,920\,\alpha^{19} + 1\,074\,030\,451\,200\,\alpha^{20} \big)\, S_{\alpha}^{3} + \\
\big(& -2\,413\,729\,498\,666\,800\,513\,024 - 25\,435\,086\,835\,865\,925\,058\,560\,\alpha - 125\,542\,481\,225\,411\,227\,975\,680\,\alpha^{2} - \\
& 386\,097\,946\,352\,750\,392\,590\,336\,\alpha^{3} - 830\,183\,396\,028\,360\,968\,208\,384\,\alpha^{4} - \\
& 1\,327\,255\,653\,860\,270\,011\,465\,728\,\alpha^{5} - 1\,637\,850\,112\,836\,596\,110\,688\,256\,\alpha^{6} - \\
& 1\,598\,197\,760\,043\,557\,807\,628\,288\,\alpha^{7} - 1\,252\,980\,911\,862\,994\,173\,739\,008\,\alpha^{8} - \\
& 797\,358\,770\,338\,813\,407\,952\,896\,\alpha^{9} - 414\,276\,959\,391\,975\,941\,603\,328\,\alpha^{10} - \\
& 176\,103\,421\,096\,866\,815\,410\,176\,\alpha^{11} - 61\,159\,515\,859\,482\,838\,548\,480\,\alpha^{12} - \\
& 17\,263\,930\,413\,062\,410\,149\,888\,\alpha^{13} - 3\,923\,295\,133\,237\,310\,914\,560\,\alpha^{14} - \\
& 706\,924\,713\,366\,338\,125\,824\,\alpha^{15} - 98\,652\,029\,401\,005\,981\,696\,\alpha^{16} - 10\,278\,087\,291\,823\,325\,184\,\alpha^{17} - \\
& 752\,234\,327\,699\,226\,624\,\alpha^{18} - 34\,490\,272\,274\,841\,600\,\alpha^{19} - 745\,214\,176\,788\,480\,\alpha^{20} \big)\, S_{\alpha}^{2} + \\
\big(& 123\,596\,648\,884\,357\,621\,088\,256 + 1\,387\,410\,081\,329\,207\,115\,251\,712\,\alpha + \\
& 7\,308\,010\,505\,383\,031\,273\,947\,136\,\alpha^{2} + 24\,020\,604\,752\,075\,269\,740\,691\,456\,\alpha^{3} + \\
& 55\,262\,591\,055\,735\,725\,773\,815\,808\,\alpha^{4} + 94\,607\,549\,345\,038\,165\,436\,006\,400\,\alpha^{5} + \\
& 125\,070\,786\,847\,359\,746\,869\,821\,440\,\alpha^{6} + 130\,760\,992\,638\,503\,780\,446\,109\,696\,\alpha^{7} + \\
& 109\,819\,712\,522\,499\,293\,630\,693\,376\,\alpha^{8} + 74\,830\,049\,897\,678\,615\,099\,736\,064\,\alpha^{9} + \\
& 41\,599\,115\,200\,046\,517\,939\,601\,408\,\alpha^{10} + 18\,902\,277\,196\,351\,684\,209\,803\,264\,\alpha^{11} + \\
& 7\,008\,965\,526\,989\,775\,347\,122\,176\,\alpha^{12} + 2\,109\,519\,207\,312\,665\,281\,560\,576\,\alpha^{13} + \\
& 510\,375\,764\,108\,304\,797\,663\,232\,\alpha^{14} + 97\,744\,104\,267\,386\,959\,429\,632\,\alpha^{15} + \\
& 14\,472\,279\,363\,085\,494\,386\,688\,\alpha^{16} + 1\,596\,811\,738\,769\,963\,089\,920\,\alpha^{17} + \\
& 123\,530\,156\,260\,699\,668\,480\,\alpha^{18} + 5\,975\,058\,303\,292\,538\,880\,\alpha^{19} + 135\,920\,997\,944\,524\,800\,\alpha^{20} \big)\, S_{\alpha} + \\
\big(& -221\,086\,792\,032\,258\,663\,383\,040 - 3\,002\,581\,182\,281\,579\,476\,549\,632\,\alpha - \\
& 18\,896\,284\,453\,973\,181\,469\,818\,880\,\alpha^{2} - 73\,337\,056\,136\,834\,742\,984\,114\,176\,\alpha^{3} - \\
& 197\,017\,275\,538\,043\,925\,583\,364\,096\,\alpha^{4} - 389\,745\,626\,428\,476\,129\,286\,291\,456\,\alpha^{5} - \\
& 589\,529\,476\,016\,351\,811\,509\,157\,888\,\alpha^{6} - 698\,690\,177\,713\,813\,455\,561\,031\,680\,\alpha^{7} - \\
& 659\,396\,154\,092\,196\,671\,988\,432\,896\,\alpha^{8} - 500\,766\,687\,956\,261\,350\,615\,810\,048\,\alpha^{9} - \\
& 307\,887\,490\,552\,535\,839\,569\,608\,704\,\alpha^{10} - 153\,616\,793\,330\,862\,792\,246\,296\,576\,\alpha^{11} - \\
& 62\,125\,104\,506\,185\,984\,379\,977\,728\,\alpha^{12} - 20\,265\,270\,278\,609\,884\,774\,662\,144\,\alpha^{13} - \\
& 5\,282\,843\,409\,745\,454\,510\,899\,200\,\alpha^{14} - 1\,084\,193\,901\,809\,507\,676\,192\,768\,\alpha^{15} - \\
& 171\,154\,981\,038\,855\,165\,050\,880\,\alpha^{16} - 20\,040\,031\,539\,432\,857\,272\,320\,\alpha^{17} - \\
& 1\,638\,003\,152\,561\,664\,688\,128\,\alpha^{18} - 83\,373\,097\,696\,100\,352\,000\,\alpha^{19} - 1\,988\,330\,027\,074\,191\,360\,\alpha^{20} \big) \}
\end{aligned}
$$

In[∘]:=
```
RECNormalizedinS = RECNormalized[[1]];
ToOrePolynomial[RECNormalizedinS]
```

Out[∘]=

$$
\begin{aligned}
&\big(-9 051 531 325 562 880 - 90 332 029 095 081 984\,\alpha - \\
&\quad 420 333 410 362 428 416\,\alpha^{2} - 1 213 206 945 955 473 664\,\alpha^{3} - 2 437 377 188 874 087 136\,\alpha^{4} - \\
&\quad 3 625 291 113 645 770 712\,\alpha^{5} - 4 144 688 219 837 114 384\,\alpha^{6} - 3 731 957 019 300 871 994\,\alpha^{7} - \\
&\quad 2 689 507 840 271 682 912\,\alpha^{8} - 1 567 534 832 320 365 967\,\alpha^{9} - 743 334 125 295 350 476\,\alpha^{10} - \\
&\quad 287 455 002 784 035 524\,\alpha^{11} - 90 539 774 552 500 272\,\alpha^{12} - 23 112 095 925 472 389\,\alpha^{13} - \\
&\quad 4 737 102 973 509 780\,\alpha^{14} - 767 930 664 461 310\,\alpha^{15} - 96 195 146 877 576\,\alpha^{16} - \\
&\quad 8 977 485 504 456\,\alpha^{17} - 587 451 930 408\,\alpha^{18} - 24 041 253 600\,\alpha^{19} - 462 944 160\,\alpha^{20}\big)\,S_{\alpha}^{4} + \\
&\big(9 569 617 440 812 835 840 + 97 443 791 378 162 009 856\,\alpha + 463 583 339 186 644 316 800\,\alpha^{2} + \\
&\quad 1 370 837 922 368 778 354 176\,\alpha^{3} + 2 827 452 328 200 593 850 560\,\alpha^{4} + 4 326 575 055 112 730 856 640\,\alpha^{5} + \\
&\quad 5 099 519 612 920 329 528 000\,\alpha^{6} + 4 743 666 552 937 883 189 952\,\alpha^{7} + 3 539 068 890 050 114 722 112\,\alpha^{8} + \\
&\quad 2 139 750 587 880 300 657 856\,\alpha^{9} + 1 054 730 779 373 468 537 920\,\alpha^{10} + 424 824 967 934 147 228 480\,\alpha^{11} + \\
&\quad 139 643 546 214 642 867 648\,\alpha^{12} + 37 274 084 807 088 072 384\,\alpha^{13} + 8 003 802 897 605 020 608\,\alpha^{14} + \\
&\quad 1 361 866 764 260 304 576\,\alpha^{15} + 179 386 646 751 384 192\,\alpha^{16} + 17 635 678 788 631 680\,\alpha^{17} + \\
&\quad 1 217 772 669 657 600\,\alpha^{18} + 52 679 537 809 920\,\alpha^{19} + 1 074 030 451 200\,\alpha^{20}\big)\,S_{\alpha}^{3} + \\
&\big(-2 413 729 498 666 800 513 024 - 25 435 086 835 865 925 058 560\,\alpha - 125 542 481 225 411 227 975 680\,\alpha^{2} - \\
&\quad 386 097 946 352 750 392 590 336\,\alpha^{3} - 830 183 396 028 360 968 208 384\,\alpha^{4} - \\
&\quad 1 327 255 653 860 270 011 465 728\,\alpha^{5} - 1 637 850 112 836 596 110 688 256\,\alpha^{6} - \\
&\quad 1 598 197 760 043 557 807 628 288\,\alpha^{7} - 1 252 980 911 862 994 173 739 008\,\alpha^{8} - \\
&\quad 797 358 770 338 813 407 952 896\,\alpha^{9} - 414 276 959 391 975 941 603 328\,\alpha^{10} - \\
&\quad 176 103 421 096 866 815 410 176\,\alpha^{11} - 61 159 515 859 482 838 548 480\,\alpha^{12} - \\
&\quad 17 263 930 413 062 410 149 888\,\alpha^{13} - 3 923 295 133 237 310 914 560\,\alpha^{14} - \\
&\quad 706 924 713 366 338 125 824\,\alpha^{15} - 98 652 029 401 005 981 696\,\alpha^{16} - 10 278 087 291 823 325 184\,\alpha^{17} - \\
&\quad 752 234 327 699 226 624\,\alpha^{18} - 34 490 272 274 841 600\,\alpha^{19} - 745 214 176 788 480\,\alpha^{20}\big)\,S_{\alpha}^{2} + \\
&\big(123 596 648 884 357 621 088 256 + 1 387 410 081 329 207 115 251 712\,\alpha + \\
&\quad 7 308 010 505 383 031 273 947 136\,\alpha^{2} + 24 020 604 752 075 269 740 691 456\,\alpha^{3} + \\
&\quad 55 262 591 055 735 725 773 815 808\,\alpha^{4} + 94 607 549 345 038 165 436 006 400\,\alpha^{5} + \\
&\quad 125 070 786 847 359 746 869 821 440\,\alpha^{6} + 130 760 992 638 503 780 446 109 696\,\alpha^{7} + \\
&\quad 109 819 712 522 499 293 630 693 376\,\alpha^{8} + 74 830 049 897 678 615 099 736 064\,\alpha^{9} + \\
&\quad 41 599 115 200 046 517 939 601 408\,\alpha^{10} + 18 902 277 196 351 684 209 803 264\,\alpha^{11} + \\
&\quad 7 008 965 526 989 775 347 122 176\,\alpha^{12} + 2 109 519 207 312 665 281 560 576\,\alpha^{13} + \\
&\quad 510 375 764 108 304 797 663 232\,\alpha^{14} + 97 744 104 267 386 959 429 632\,\alpha^{15} + \\
&\quad 14 472 279 363 085 494 386 688\,\alpha^{16} + 1 596 811 738 769 963 089 920\,\alpha^{17} + \\
&\quad 123 530 156 260 699 668 480\,\alpha^{18} + 5 975 058 303 292 538 880\,\alpha^{19} + 135 920 997 944 524 800\,\alpha^{20}\big)\,S_{\alpha} + \\
&\big(-221 086 792 032 258 663 383 040 - 3 002 581 182 281 579 476 549 632\,\alpha - \\
&\quad 18 896 284 453 973 181 469 818 880\,\alpha^{2} - 73 337 056 136 834 742 984 114 176\,\alpha^{3} - \\
&\quad 197 017 275 538 043 925 583 364 096\,\alpha^{4} - 389 745 626 428 476 129 286 291 456\,\alpha^{5} - \\
&\quad 589 529 476 016 351 811 509 157 888\,\alpha^{6} - 698 690 177 713 813 455 561 031 680\,\alpha^{7} - \\
&\quad 659 396 154 092 196 671 988 432 896\,\alpha^{8} - 500 766 687 956 261 350 615 810 048\,\alpha^{9} - \\
&\quad 307 887 490 552 535 839 569 608 704\,\alpha^{10} - 153 616 793 330 862 792 246 296 576\,\alpha^{11} - \\
&\quad 62 125 104 506 185 984 379 977 728\,\alpha^{12} - 20 265 270 278 609 884 774 662 144\,\alpha^{13} - \\
&\quad 5 282 843 409 745 454 510 899 200\,\alpha^{14} - 1 084 193 901 809 507 676 192 768\,\alpha^{15} - \\
&\quad 171 154 981 038 855 165 050 880\,\alpha^{16} - 20 040 031 539 432 857 272 320\,\alpha^{17} - \\
&\quad 1 638 003 152 561 664 688 128\,\alpha^{18} - 83 373 097 696 100 352 000\,\alpha^{19} - 1 988 330 027 074 191 360\,\alpha^{20}\big)
\end{aligned}
$$

In[∘]:=
```
RecNormalizedOrder = OrePolynomialDegree[RECNormalizedinS, S[α]]
```

Out[∘]= 4

Since M = 1 is odd, we only need to work on $R(w^{1/2})$.

ODE for $R(w^{1/2})$.

In[ ]:=
```
RecNormalizedOrder = OrePolynomialDegree[RECNormalizedinS, S[α]]
RECNormalizedinSDetails = First[RECNormalizedinS]
```

Out[ ]= 4

Out[ ]=

$$\{\{-9051531325562880-90332029095081984\,\alpha-420333410362428416\,\alpha^{2}-1213206945955473664\,\alpha^{3}-2437377188874087136\,\alpha^{4}-3625291113645770712\,\alpha^{5}-4144688219837114384\,\alpha^{6}-3731957019300871994\,\alpha^{7}-2689507840271682912\,\alpha^{8}-1567534832320365967\,\alpha^{9}-743334125295350476\,\alpha^{10}-287455002784035524\,\alpha^{11}-90539774552500272\,\alpha^{12}-23112095925472389\,\alpha^{13}-4737102973509780\,\alpha^{14}-767930664461310\,\alpha^{15}-96195146877576\,\alpha^{16}-8977485504456\,\alpha^{17}-587451930408\,\alpha^{18}-24041253600\,\alpha^{19}-462944160\,\alpha^{20},\{4\}\},$$

$$\{9569617440812835840+97443791378162009856\,\alpha+463583339186644316800\,\alpha^{2}+1370837922368778354176\,\alpha^{3}+2827452328200593850560\,\alpha^{4}+4326575055112730856640\,\alpha^{5}+5099519612920329528000\,\alpha^{6}+4743666552937883189952\,\alpha^{7}+3539068890050114722112\,\alpha^{8}+2139750587880300657856\,\alpha^{9}+1054730779373468537920\,\alpha^{10}+424824967934147228480\,\alpha^{11}+139643546214642867648\,\alpha^{12}+37274084807088072384\,\alpha^{13}+8003802897605020608\,\alpha^{14}+1361866764260304576\,\alpha^{15}+179386646751384192\,\alpha^{16}+17635678788631680\,\alpha^{17}+1217772669657600\,\alpha^{18}+52679537809920\,\alpha^{19}+1074030451200\,\alpha^{20},\{3\}\},$$

$$\{-2413729498666800513024-25435086835865925058560\,\alpha-125542481225411227975680\,\alpha^{2}-386097946352750392590336\,\alpha^{3}-830183396028360968208384\,\alpha^{4}-1327255653860270011465728\,\alpha^{5}-1637850112836596110688256\,\alpha^{6}-1598197760043557807628288\,\alpha^{7}-1252980911862994173739008\,\alpha^{8}-797358770338813407952896\,\alpha^{9}-414276959391975941603328\,\alpha^{10}-176103421096866815410176\,\alpha^{11}-61159515859482838548480\,\alpha^{12}-17263930413062410149888\,\alpha^{13}-3923295133237310914560\,\alpha^{14}-706924713366338125824\,\alpha^{15}-98652029401005981696\,\alpha^{16}-10278087291823325184\,\alpha^{17}-752234327699226624\,\alpha^{18}-34490272274841600\,\alpha^{19}-745214176788480\,\alpha^{20},\{2\}\},$$

$$\{123596648884357621088256+1387410081329207115251712\,\alpha+7308010505383031273947136\,\alpha^{2}+24020604752075269740691456\,\alpha^{3}+55262591055735725773815808\,\alpha^{4}+94607549345038165436006400\,\alpha^{5}+125070786847359746869821440\,\alpha^{6}+130760992638503780446109696\,\alpha^{7}+109819712522499293630693376\,\alpha^{8}+74830049897678615099736064\,\alpha^{9}+41599115200046517939601408\,\alpha^{10}+18902277196351684209803264\,\alpha^{11}+7008965526989775347122176\,\alpha^{12}+2109519207312665281560576\,\alpha^{13}+510375764108304797663232\,\alpha^{14}+97744104267386959429632\,\alpha^{15}+14472279363085494386688\,\alpha^{16}+1596811738769963089920\,\alpha^{17}+123530156260699668480\,\alpha^{18}+5975058303292538880\,\alpha^{19}+135920997944524800\,\alpha^{20},\{1\}\},$$

$$\{-221086792032258663383040-3002581182281579476549632\,\alpha-18896284453973181469818880\,\alpha^{2}-73337056136834742984114176\,\alpha^{3}-197017275538043925583364096\,\alpha^{4}-389745626428476129286291456\,\alpha^{5}-589529476016351811509157888\,\alpha^{6}-698690177713813455561031680\,\alpha^{7}-659396154092196671988432896\,\alpha^{8}-500766687956261350615810048\,\alpha^{9}-307887490552535839569608704\,\alpha^{10}-153616793330862792246296576\,\alpha^{11}-62125104506185984379977728\,\alpha^{12}-20265270278609884774662144\,\alpha^{13}-5282843409745454510899200\,\alpha^{14}-1084193901809507676192768\,\alpha^{15}-171154981038855165050880\,\alpha^{16}-20040031539432857272320\,\alpha^{17}-1638003152561664688128\,\alpha^{18}-83373097696100352000\,\alpha^{19}-1988330027074191360\,\alpha^{20},\{0\}\}\}$$

In[•]:=
```
ODENormalizedinTheta = Sum[
   w^(RecNormalizedOrder-RECNormalizedinSDetails[[i,2]][[1]]) ** Expand[RECNormalizedinSDetails[[i, 1]] /.
      {α → Euler[w] - RECNormalizedinSDetails[[i, 2]][[1]]}],
   {i, 1, Length@RECNormalizedinSDetails}];
ToOrePolynomial[ODENormalizedinTheta]
```

Out[•]=

$$
\begin{aligned}
&(-462944160 + 1074030451200\, w - 745214176788480\, w^2 + \\
&\quad 135920997944524800\, w^3 - 1988330027074191360\, w^4)\ \theta_w^{20} + \\
&(12994279200 - 11762289262080\, w - 4681705203302400\, w^2 + \\
&\quad 3256638344402042880\, w^3 - 83373097696100352000\, w^4)\ \theta_w^{19} + \\
&(-167666903208 + 51631086044160\, w - 7966755614490624\, w^2 + \\
&\quad 35829038107601141760\, w^3 - 1638003152561664688128\, w^4)\ \theta_w^{18} + \\
&(1318589548920 - 108893971347840\, w + 7355561668116480\, w^2 + \\
&\quad 240053958283634933760\, w^3 - 20040031539432857272320\, w^4)\ \theta_w^{17} + \\
&(-7064975486952 + 83127349930752\, w + 35117658989494272\, w^2 + \\
&\quad 1095299451033923616768\, w^3 - 171154981038855165050880\, w^4)\ \theta_w^{16} + \\
&(27307686312450 + 88420190673600\, w + 19325639654129664\, w^2 + \\
&\quad 3605430253433068191744\, w^3 - 1084193901809507676192768\, w^4)\ \theta_w^{15} + \\
&(-78641843797260 - 217024094610432\, w - 41404494399897600\, w^2 + \\
&\quad 8848339391569956175872\, w^3 - 5282843409745454510899200\, w^4)\ \theta_w^{14} + \\
&(171828669178107 + 93489033189888\, w - 51550211830063104\, w^2 + \\
&\quad 16497260099262994710528\, w^3 - 20265270278609884774662144\, w^4)\ \theta_w^{13} + \\
&(-287324585519724 + 103202291096064\, w + 6453114945896448\, w^2 + \\
&\quad 23584876575260902686720\, w^3 - 62125104506185984379977728\, w^4)\ \theta_w^{12} + \\
&(368147728552924 - 104110563382912\, w + 37207925339652096\, w^2 + \\
&\quad 25892926506337036402688\, w^3 - 153616793330862792246296576\, w^4)\ \theta_w^{11} + \\
&(-359429453456796 + 3827954332672\, w + 14670744167645184\, w^2 + \\
&\quad 21691593131730745688064\, w^3 - 307887490552535839569608704\, w^4)\ \theta_w^{10} + \\
&(263938631647633 + 22253218369408\, w - 6345062028312576\, w^2 + \\
&\quad 13639004338019037347840\, w^3 - 500766687956261350615810048\, w^4)\ \theta_w^{9} + \\
&(-142547108499972 - 5728248485248\, w - 6570482528403456\, w^2 + \\
&\quad 6224475099794557108224\, w^3 - 659396154092196671988432896\, w^4)\ \theta_w^{8} + \\
&(54596939279110 + 1385515822464\, w - 1783233893277696\, w^2 + \\
&\quad 1920210063103086559232\, w^3 - 698690177713813455561031680\, w^4)\ \theta_w^{7} + \\
&(-13951372518072 - 530584935168\, w + 141943281106944\, w^2 + \\
&\quad 327744477473545912320\, w^3 - 589529476016351811509157888\, w^4)\ \theta_w^{6} + \\
&(2120315034504 - 553252773632\, w + 174703264874496\, w^2 + 686902664765374464\, w^3 - \\
&\quad 389745626428476129286291456\, w^4)\ \theta_w^{5} + (-144091306368 + 162268262144\, w - \\
&\quad 5802923261952\, w^2 - 10454767853003341824\, w^3 - 197017275538043925583364096\, w^4)\ \theta_w^{4} + \\
&(41954472128\, w - 9164740706304\, w^2 - 609564618675191808\, w^3 - \\
&\quad 73337056136834742984114176\, w^4)\ \theta_w^{3} + (-6630653312\, w + 1109426503680\, w^2 + \\
&\quad 479050054452117504\, w^3 - 18896284453973181469818880\, w^4)\ \theta_w^{2} + \\
&(-1073410944\, w + 241224450048\, w^2 + 105228107796971520\, w^3 - 3002581182281579476549632\, w^4) \\
&\ \theta_w + (211189248\, w - 65671004160\, w^2 + 7093249848115200\, w^3 - 221086792032258663383040\, w^4)
\end{aligned}
$$

In[•]:=
```
ODENormalizedinD =
  ChangeOreAlgebra[ToOrePolynomial[w^-1 ** ODENormalizedinTheta], OreAlgebra[Der[w]]];
ToOrePolynomial[ODENormalizedinD]
```

Out[•]=

$$
\begin{aligned}
&\left(-462\,944\,160\,w^{19}+1\,074\,030\,451\,200\,w^{20}-745\,214\,176\,788\,480\,w^{21}+\right.\\
&\quad\left.135\,920\,997\,944\,524\,800\,w^{22}-1\,988\,330\,027\,074\,191\,360\,w^{23}\right)D_w^{20}+\\
&\left(-74\,965\,111\,200\,w^{18}+192\,303\,496\,465\,920\,w^{19}-146\,272\,398\,793\,113\,600\,w^{20}+\right.\\
&\quad\left.29\,081\,627\,953\,861\,754\,880\,w^{21}-461\,155\,802\,840\,196\,710\,400\,w^{22}\right)D_w^{19}+\\
&\left(-5\,202\,294\,868\,008\,w^{17}+14\,875\,706\,944\,788\,480\,w^{18}-12\,489\,770\,566\,538\,625\,024\,w^{19}+\right.\\
&\quad\left.2\,723\,275\,837\,780\,776\,714\,240\,w^{20}-47\,061\,876\,032\,982\,774\,448\,128\,w^{21}\right)D_w^{18}+\\
&\left(-203\,819\,073\,205\,104\,w^{16}+655\,783\,767\,376\,778\,880\,w^{17}-612\,603\,089\,533\,997\,678\,592\,w^{18}+\right.\\
&\quad\left.147\,501\,966\,974\,486\,210\,150\,400\,w^{19}-2\,794\,824\,649\,678\,835\,630\,997\,504\,w^{20}\right)D_w^{17}+\\
&\left(-5\,001\,226\,425\,658\,440\,w^{15}+18\,306\,656\,980\,258\,689\,792\,w^{16}-19\,202\,743\,354\,671\,793\,078\,272\,w^{17}+\right.\\
&\quad\left.5\,146\,544\,985\,708\,072\,919\,891\,968\,w^{18}-107\,660\,112\,982\,493\,684\,803\,043\,328\,w^{19}\right)D_w^{16}+\\
&\left(-80\,641\,068\,598\,599\,390\,w^{14}+340\,239\,613\,976\,441\,668\,800\,w^{15}-405\,100\,023\,115\,873\,988\,100\,096\,w^{16}+\right.\\
&\quad\left.121\,971\,635\,240\,045\,989\,010\,079\,744\,w^{17}-2\,839\,475\,218\,176\,867\,727\,854\,010\,368\,w^{18}\right)D_w^{15}+\\
&\left(-872\,703\,241\,861\,969\,674\,w^{13}+4\,311\,525\,183\,500\,445\,098\,688\,w^{14}-5\,902\,027\,984\,569\,263\,988\,621\,312\,w^{15}+\right.\\
&\quad\left.2\,018\,138\,387\,302\,900\,705\,926\,316\,032\,w^{16}-52\,770\,397\,017\,901\,063\,461\,691\,981\,824\,w^{17}\right)D_w^{14}+\\
&\left(-6\,366\,201\,528\,121\,595\,037\,w^{12}+37\,545\,367\,858\,272\,602\,568\,576\,w^{13}-60\,011\,414\,666\,097\,279\,384\,231\,936\right.\\
&\quad\left.w^{14}+23\,606\,729\,723\,866\,558\,476\,858\,163\,200\,w^{15}-700\,907\,453\,839\,646\,763\,506\,642\,976\,768\,w^{16}\right)D_w^{13}+\\
&\left(-31\,027\,608\,572\,532\,373\,242\,w^{11}+223\,768\,960\,215\,229\,648\,725\,120\,w^{12}-\right.\\
&\quad 425\,614\,688\,416\,395\,084\,511\,199\,232\,w^{13}+195\,625\,766\,326\,656\,902\,331\,532\,050\,432\,w^{14}-\\
&\quad\left.6\,681\,307\,863\,118\,339\,077\,139\,435\,880\,448\,w^{15}\right)D_w^{12}+\\
&\left(-98\,881\,827\,781\,497\,028\,883\,w^{10}+899\,365\,453\,411\,224\,156\,717\,824\,w^{11}-\right.\\
&\quad 2\,084\,449\,922\,721\,487\,491\,294\,388\,224\,w^{12}+1\,141\,035\,342\,454\,777\,936\,230\,311\,788\,544\,w^{13}-\\
&\quad\left.45\,534\,565\,615\,007\,366\,976\,336\,010\,674\,176\,w^{14}\right)D_w^{11}+\\
&\left(-198\,748\,699\,184\,251\,529\,945\,w^{9}+2\,374\,402\,491\,145\,199\,536\,758\,528\,w^{10}-\right.\\
&\quad 6\,912\,790\,902\,992\,229\,424\,498\,188\,288\,w^{11}+4\,615\,775\,091\,365\,575\,764\,678\,244\,302\,848\,w^{12}-\\
&\quad\left.219\,405\,971\,131\,563\,215\,731\,262\,968\,823\,808\,w^{13}\right)D_w^{10}+\\
&\left(-238\,356\,473\,042\,241\,578\,933\,w^{8}+3\,954\,601\,150\,260\,295\,922\,992\,384\,w^{9}-\right.\\
&\quad 15\,055\,619\,592\,618\,910\,812\,504\,735\,744\,w^{10}+12\,644\,731\,266\,525\,561\,957\,422\,062\,370\,816\,w^{11}-\\
&\quad\left.733\,436\,718\,426\,867\,880\,182\,934\,631\,612\,416\,w^{12}\right)D_w^{9}+\\
&\left(-156\,746\,734\,043\,191\,941\,744\,w^{7}+3\,913\,264\,083\,302\,258\,224\,204\,160\,w^{8}-\right.\\
&\quad 20\,587\,539\,385\,686\,319\,645\,432\,823\,808\,w^{9}+22\,651\,372\,043\,999\,278\,395\,778\,913\,009\,664\,w^{10}-\\
&\quad\left.1\,653\,857\,883\,689\,296\,996\,879\,332\,150\,345\,728\,w^{11}\right)D_w^{8}+\\
&\left(-49\,465\,481\,212\,866\,682\,046\,w^{6}+2\,102\,891\,820\,107\,692\,542\,282\,176\,w^{7}-\right.\\
&\quad 16\,556\,984\,634\,589\,414\,962\,986\,115\,072\,w^{8}+25\,238\,473\,276\,001\,192\,289\,367\,232\,086\,016\,w^{9}-\\
&\quad\left.2\,417\,418\,519\,249\,182\,761\,165\,403\,690\,369\,024\,w^{10}\right)D_w^{7}+\\
&\left(-5\,964\,969\,020\,906\,764\,302\,w^{5}+533\,033\,188\,793\,304\,247\,234\,240\,w^{6}-\right.\\
&\quad 7\,105\,725\,581\,720\,168\,938\,660\,798\,464\,w^{7}+16\,274\,369\,982\,084\,662\,657\,223\,271\,907\,328\,w^{8}-\\
&\quad\left.2\,165\,048\,902\,677\,000\,850\,493\,343\,596\,019\,712\,w^{9}\right)D_w^{6}+\\
&\left(-175\,283\,952\,870\,087\,306\,w^{4}+50\,150\,805\,801\,726\,654\,633\,984\,w^{5}-\right.\\
&\quad 1\,397\,870\,274\,609\,249\,273\,305\,800\,704\,w^{6}+5\,455\,959\,012\,981\,718\,985\,262\,536\,589\,312\,w^{7}-\\
&\quad\left.1\,095\,482\,437\,481\,951\,767\,446\,918\,411\,583\,488\,w^{8}\right)D_w^{5}+\\
&\left(-377\,146\,251\,281\,256\,w^{3}+1\,093\,551\,108\,123\,109\,857\,792\,w^{4}-97\,311\,503\,831\,612\,216\,775\,671\,808\,w^{5}+\right.\\
&\quad\left.803\,438\,910\,384\,939\,975\,156\,855\,472\,128\,w^{6}-277\,253\,611\,907\,637\,068\,611\,200\,347\,013\,120\,w^{7}\right)D_w^{4}+\\
&\left(-886\,384\,872\,w^{2}+1\,594\,934\,124\,304\,732\,032\,w^{3}-1\,441\,603\,071\,070\,395\,421\,851\,648\,w^{4}+\right.\\
&\quad\left.38\,880\,386\,600\,050\,208\,935\,199\,637\,504\,w^{5}-28\,872\,578\,749\,579\,666\,829\,317\,924\,454\,400\,w^{6}\right)D_w^{3}+\\
&\left(401\,904\,w+2\,113\,703\,650\,944\,w^{2}-1\,206\,864\,135\,815\,188\,119\,552\,w^{3}+\right.\\
&\quad\left.346\,594\,153\,367\,562\,739\,771\,244\,544\,w^{4}-874\,026\,053\,327\,930\,086\,187\,461\,509\,120\,w^{5}\right)D_w^{2}+\\
&\left(-6\,599\,664+4\,183\,782\,912\,w-613\,485\,376\,634\,880\,w^{2}+123\,596\,641\,791\,107\,772\,973\,056\,w^{3}-\right.
\end{aligned}
$$

$3\,680\,835\,933\,875\,140\,884\,493\,762\,560\,w^4\big)\,D_w +$
$\big(211\,189\,248 - 65\,671\,004\,160\,w + 7\,093\,249\,848\,115\,200\,w^2 - 221\,086\,792\,032\,258\,663\,383\,040\,w^3\big)$

In[ ]:=
```
ODENormalized = {ODENormalizedinD};
ToOrePolynomial[ODENormalized]
```

Out[ ]=
$$\begin{aligned}
&\{\big(-462\,944\,160\,w^{19} + 1\,074\,030\,451\,200\,w^{20} - 745\,214\,176\,788\,480\,w^{21} +\\
&\quad 135\,920\,997\,944\,524\,800\,w^{22} - 1\,988\,330\,027\,074\,191\,360\,w^{23}\big)\,D_w^{20} +\\
&\big(-74\,965\,111\,200\,w^{18} + 192\,303\,496\,465\,920\,w^{19} - 146\,272\,398\,793\,113\,600\,w^{20} +\\
&\quad 29\,081\,627\,953\,861\,754\,880\,w^{21} - 461\,155\,802\,840\,196\,710\,400\,w^{22}\big)\,D_w^{19} +\\
&\big(-5\,202\,294\,868\,008\,w^{17} + 14\,875\,706\,944\,788\,480\,w^{18} - 12\,489\,770\,566\,538\,625\,024\,w^{19} +\\
&\quad 2\,723\,275\,837\,780\,776\,714\,240\,w^{20} - 47\,061\,876\,032\,982\,774\,448\,128\,w^{21}\big)\,D_w^{18} +\\
&\big(-203\,819\,073\,205\,104\,w^{16} + 655\,783\,767\,376\,778\,880\,w^{17} - 612\,603\,089\,533\,997\,678\,592\,w^{18} +\\
&\quad 147\,501\,966\,974\,486\,210\,150\,400\,w^{19} - 2\,794\,824\,649\,678\,835\,630\,997\,504\,w^{20}\big)\,D_w^{17} +\\
&\big(-5\,001\,226\,425\,658\,440\,w^{15} + 18\,306\,656\,980\,258\,689\,792\,w^{16} - 19\,202\,743\,354\,671\,793\,078\,272\,w^{17} +\\
&\quad 5\,146\,544\,985\,708\,072\,919\,891\,968\,w^{18} - 107\,660\,112\,982\,493\,684\,803\,043\,328\,w^{19}\big)\,D_w^{16} +\\
&\big(-80\,641\,068\,598\,599\,390\,w^{14} + 340\,239\,613\,976\,441\,668\,800\,w^{15} - 405\,100\,023\,115\,873\,988\,100\,096\,w^{16} +\\
&\quad 121\,971\,635\,240\,045\,989\,010\,079\,744\,w^{17} - 2\,839\,475\,218\,176\,867\,727\,854\,010\,368\,w^{18}\big)\,D_w^{15} +\\
&\big(-872\,703\,241\,861\,969\,674\,w^{13} + 4\,311\,525\,183\,500\,445\,098\,688\,w^{14} - 5\,902\,027\,984\,569\,263\,988\,621\,312\,w^{15} +\\
&\quad 2\,018\,138\,387\,302\,900\,705\,926\,316\,032\,w^{16} - 52\,770\,397\,017\,901\,063\,461\,691\,981\,824\,w^{17}\big)\,D_w^{14} +\\
&\big(-6\,366\,201\,528\,121\,595\,037\,w^{12} + 37\,545\,367\,858\,272\,602\,568\,576\,w^{13} - 60\,011\,414\,666\,097\,279\,384\,231\,936\\
&\quad w^{14} + 23\,606\,729\,723\,866\,558\,476\,858\,163\,200\,w^{15} - 700\,907\,453\,839\,646\,763\,506\,642\,976\,768\,w^{16}\big)\,D_w^{13} +\\
&\big(-31\,027\,608\,572\,532\,373\,242\,w^{11} + 223\,768\,960\,215\,229\,648\,725\,120\,w^{12} -\\
&\quad 425\,614\,688\,416\,395\,084\,511\,199\,232\,w^{13} + 195\,625\,766\,326\,656\,902\,331\,532\,050\,432\,w^{14} -\\
&\quad 6\,681\,307\,863\,118\,339\,077\,139\,435\,880\,448\,w^{15}\big)\,D_w^{12} +\\
&\big(-98\,881\,827\,781\,497\,028\,883\,w^{10} + 899\,365\,453\,411\,224\,156\,717\,824\,w^{11} -\\
&\quad 2\,084\,449\,922\,721\,487\,491\,294\,388\,224\,w^{12} + 1\,141\,035\,342\,454\,777\,936\,230\,311\,788\,544\,w^{13} -\\
&\quad 45\,534\,565\,615\,007\,366\,976\,336\,010\,674\,176\,w^{14}\big)\,D_w^{11} +\\
&\big(-198\,748\,699\,184\,251\,529\,945\,w^{9} + 2\,374\,402\,491\,145\,199\,536\,758\,528\,w^{10} -\\
&\quad 6\,912\,790\,902\,992\,229\,424\,498\,188\,288\,w^{11} + 4\,615\,775\,091\,365\,575\,764\,678\,244\,302\,848\,w^{12} -\\
&\quad 219\,405\,971\,131\,563\,215\,731\,262\,968\,823\,808\,w^{13}\big)\,D_w^{10} +\\
&\big(-238\,356\,473\,042\,241\,578\,933\,w^{8} + 3\,954\,601\,150\,260\,295\,922\,992\,384\,w^{9} -\\
&\quad 15\,055\,619\,592\,618\,910\,812\,504\,735\,744\,w^{10} + 12\,644\,731\,266\,525\,561\,957\,422\,062\,370\,816\,w^{11} -\\
&\quad 733\,436\,718\,426\,867\,880\,182\,934\,631\,612\,416\,w^{12}\big)\,D_w^{9} +\\
&\big(-156\,746\,734\,043\,191\,941\,744\,w^{7} + 3\,913\,264\,083\,302\,258\,224\,204\,160\,w^{8} -\\
&\quad 20\,587\,539\,385\,686\,319\,645\,432\,823\,808\,w^{9} + 22\,651\,372\,043\,999\,278\,395\,778\,913\,009\,664\,w^{10} -\\
&\quad 1\,653\,857\,883\,689\,296\,996\,879\,332\,150\,345\,728\,w^{11}\big)\,D_w^{8} +\\
&\big(-49\,465\,481\,212\,866\,682\,046\,w^{6} + 2\,102\,891\,820\,107\,692\,542\,282\,176\,w^{7} -\\
&\quad 16\,556\,984\,634\,589\,414\,962\,986\,115\,072\,w^{8} + 25\,238\,473\,276\,001\,192\,289\,367\,232\,086\,016\,w^{9} -\\
&\quad 2\,417\,418\,519\,249\,182\,761\,165\,403\,690\,369\,024\,w^{10}\big)\,D_w^{7} +\\
&\big(-5\,964\,969\,020\,906\,764\,302\,w^{5} + 533\,033\,188\,793\,304\,247\,234\,240\,w^{6} -\\
&\quad 7\,105\,725\,581\,720\,168\,938\,660\,798\,464\,w^{7} + 16\,274\,369\,982\,084\,662\,657\,223\,271\,907\,328\,w^{8} -\\
&\quad 2\,165\,048\,902\,677\,000\,850\,493\,343\,596\,019\,712\,w^{9}\big)\,D_w^{6} + \big(-175\,283\,952\,870\,087\,306\,w^{4} +\\
&\quad 50\,150\,805\,801\,726\,654\,633\,984\,w^{5} - 1\,397\,870\,274\,609\,249\,273\,305\,800\,704\,w^{6} +\\
&\quad 5\,455\,959\,012\,981\,718\,985\,262\,536\,589\,312\,w^{7} - 1\,095\,482\,437\,481\,951\,767\,446\,918\,411\,583\,488\,w^{8}\big)\,D_w^{5} +\\
&\big(-377\,146\,251\,281\,256\,w^{3} + 1\,093\,551\,108\,123\,109\,857\,792\,w^{4} - 97\,311\,503\,831\,612\,216\,775\,671\,808\,w^{5} +\\
&\quad 803\,438\,910\,384\,939\,975\,156\,855\,472\,128\,w^{6} - 277\,253\,611\,907\,637\,068\,611\,200\,347\,013\,120\,w^{7}\big)\,D_w^{4} +\\
&\big(-886\,384\,872\,w^{2} + 1\,594\,934\,124\,304\,732\,032\,w^{3} - 1\,441\,603\,071\,070\,395\,421\,851\,648\,w^{4} +\\
&\quad 38\,880\,386\,600\,050\,208\,935\,199\,637\,504\,w^{5} - 28\,872\,578\,749\,579\,666\,829\,317\,924\,454\,400\,w^{6}\big)\,D_w^{3} +
\end{aligned}$$

$\left(401904\,w + 2113703650944\,w^2 - 1206864135815188119552\,w^3 + 346594153367562739771244544\,w^4 - 874026053327930086187461509120\,w^5\right) D_w^2 + \left(-6599664 + 4183782912\,w - 613485376634880\,w^2 + 123596641791107772973056\,w^3 - 3680835933875140884493762560\,w^4\right) D_w + \left(211189248 - 65671004160\,w + 7093249848115200\,w^2 - 221086792032258663383040\,w^3\right) \}$

ODE for $R(z)$

In[ ]:=
```
ODE = -DFiniteSubstitute[
    ToOrePolynomial[ODENormalized], {w → z^2}, Algebra → OreAlgebra[Der[z]]];
ToOrePolynomial[
 ODE]
```

Out[•]=

$$
\begin{aligned}
&\{(-14\,467\,005\,z^{18}+33\,563\,451\,600\,z^{20}-\\
&\quad 23\,287\,943\,024\,640\,z^{22}+4\,247\,531\,185\,766\,400\,z^{24}-62\,135\,313\,346\,068\,480\,z^{26})\,D_z^{20}+\\
&(-1\,936\,588\,500\,z^{17}+5\,641\,912\,725\,120\,z^{19}-4\,717\,315\,749\,888\,000\,z^{21}+\\
&\quad 1\,010\,570\,821\,820\,743\,680\,z^{23}-17\,016\,528\,141\,759\,283\,200\,z^{25})\,D_z^{19}+\\
&(-108\,852\,307\,326\,z^{16}+406\,851\,523\,097\,040\,z^{18}-416\,070\,075\,723\,337\,728\,z^{20}+\\
&\quad 105\,864\,003\,406\,135\,296\,000\,z^{22}-2\,069\,771\,412\,396\,904\,022\,016\,z^{24})\,D_z^{18}+\\
&(-3\,370\,597\,495\,398\,z^{15}+16\,579\,706\,882\,412\,240\,z^{17}-21\,099\,493\,183\,605\,387\,264\,z^{19}+\\
&\quad 6\,457\,867\,074\,946\,659\,778\,560\,z^{21}-148\,037\,475\,911\,201\,433\,059\,328\,z^{23})\,D_z^{17}+\\
&(-63\,442\,079\,388\,162\,z^{14}+423\,506\,991\,425\,655\,456\,z^{16}-684\,438\,016\,140\,445\,704\,192\,z^{18}+\\
&\quad 255\,707\,448\,463\,908\,142\,055\,424\,z^{20}-6\,949\,217\,109\,055\,229\,631\,922\,176\,z^{22})\,D_z^{16}+\\
&(-756\,101\,749\,033\,170\,z^{13}+7\,112\,961\,381\,085\,706\,880\,z^{15}-14\,955\,124\,960\,933\,692\,604\,416\,z^{17}+\\
&\quad 6\,937\,083\,672\,214\,807\,630\,577\,664\,z^{19}-226\,339\,521\,441\,947\,695\,466\,938\,368\,z^{21})\,D_z^{15}+\\
&(-5\,769\,894\,282\,015\,168\,z^{12}+80\,195\,348\,439\,151\,482\,816\,z^{14}-225\,851\,865\,846\,831\,614\,214\,144\,z^{16}+\\
&\quad 132\,698\,207\,626\,476\,044\,213\,551\,104\,z^{18}-5\,274\,131\,783\,990\,056\,768\,858\,226\,688\,z^{20})\,D_z^{14}+\\
&(-27\,913\,964\,873\,414\,310\,z^{11}+609\,144\,620\,514\,858\,304\,128\,z^{13}-2\,381\,824\,280\,655\,404\,284\,723\,200\,z^{15}+\\
&\quad 1\,815\,233\,892\,580\,905\,502\,245\,912\,576\,z^{17}-89\,384\,278\,176\,874\,780\,470\,515\,073\,024\,z^{19})\,D_z^{13}+\\
&(-83\,070\,016\,297\,204\,617\,z^{10}+3\,085\,887\,338\,422\,919\,479\,728\,z^{12}-17\,525\,197\,548\,821\,959\,865\,708\,544\,z^{14}+\\
&\quad 17\,829\,800\,370\,673\,184\,633\,851\,478\,016\,z^{16}-1\,109\,562\,633\,354\,790\,604\,422\,552\,485\,888\,z^{18})\,D_z^{12}+\\
&(-143\,902\,495\,168\,313\,096\,z^{9}+10\,181\,926\,240\,408\,659\,812\,096\,z^{11}-\\
&\quad 89\,017\,658\,336\,288\,361\,564\,733\,440\,z^{13}+125\,249\,290\,649\,077\,278\,795\,741\,790\,208\,z^{15}-\\
&\quad 10\,086\,690\,414\,943\,482\,035\,219\,808\,124\,928\,z^{17})\,D_z^{11}+\\
&(-132\,153\,403\,830\,323\,420\,z^{8}+21\,021\,235\,664\,879\,460\,322\,576\,z^{10}-\\
&\quad 305\,777\,797\,698\,084\,890\,587\,385\,856\,z^{12}+622\,058\,590\,161\,851\,463\,805\,772\,496\,896\,z^{14}-\\
&\quad 66\,716\,129\,497\,734\,268\,326\,710\,005\,989\,376\,z^{16})\,D_z^{10}+(-54\,553\,427\,074\,589\,192\,z^{7}+\\
&\quad 25\,492\,972\,749\,524\,213\,713\,936\,z^{9}-687\,639\,745\,093\,042\,281\,248\,894\,976\,z^{11}+\\
&\quad 2\,142\,211\,936\,926\,982\,182\,052\,098\,473\,984\,z^{13}-316\,993\,324\,024\,628\,089\,414\,708\,262\,600\,704\,z^{15})\,D_z^{9}+\\
&(-7\,239\,432\,885\,699\,105\,z^{6}+16\,418\,645\,655\,121\,036\,333\,744\,z^{8}-964\,606\,409\,385\,249\,211\,639\,984\,128\,z^{10}+\\
&\quad 4\,969\,184\,054\,263\,554\,934\,655\,494\,914\,048\,z^{12}-1\,060\,795\,861\,065\,475\,294\,630\,056\,432\,762\,880\,z^{14})\,D_z^{8}+\\
&(-116\,656\,718\,681\,240\,z^{5}+4\,725\,708\,417\,817\,253\,034\,016\,z^{7}-785\,382\,744\,164\,715\,800\,031\,436\,800\,z^{9}+\\
&\quad 7\,448\,919\,993\,019\,426\,073\,438\,144\,430\,080\,z^{11}-2\,430\,575\,307\,489\,529\,542\,445\,037\,321\,715\,712\,z^{13})\,D_z^{7}+\\
&(-110\,857\,036\,479\,z^{4}+430\,864\,911\,968\,597\,888\,864\,z^{6}-331\,954\,375\,958\,956\,680\,821\,710\,848\,z^{8}+\\
&\quad 6\,800\,429\,385\,541\,013\,459\,919\,103\,328\,256\,z^{10}-3\,665\,475\,304\,708\,148\,593\,614\,758\,799\,736\,832\,z^{12})\,D_z^{6}+\\
&(-103\,516\,281\,519\,z^{3}+4\,656\,655\,446\,583\,618\,656\,z^{5}-60\,387\,543\,771\,900\,375\,434\,674\,176\,z^{7}+\\
&\quad 3\,466\,038\,353\,484\,433\,408\,049\,820\,991\,488\,z^{9}-3\,441\,783\,898\,363\,727\,194\,430\,187\,107\,254\,272\,z^{11})\,D_z^{5}+\\
&(-90\,434\,619\,318\,z^{2}+4\,184\,555\,863\,957\,776\,z^{4}-3\,276\,368\,523\,439\,339\,752\,935\,424\,z^{6}+\\
&\quad 862\,257\,619\,454\,123\,494\,550\,016\,098\,304\,z^{8}-1\,857\,196\,793\,061\,249\,965\,887\,449\,401\,917\,440\,z^{10})\,D_z^{4}+\\
&(127\,575\,050\,166\,z+1\,554\,589\,276\,043\,616\,z^{3}-19\,172\,000\,847\,765\,540\,962\,304\,z^{5}+\\
&\quad 83\,501\,711\,699\,211\,811\,743\,753\,830\,400\,z^{7}-510\,013\,753\,832\,678\,716\,887\,333\,548\,851\,200\,z^{9})\,D_z^{3}+\\
&(-108\,128\,894\,976+98\,024\,888\,529\,072\,z^{2}-9\,427\,736\,705\,193\,676\,800\,z^{4}+\\
&\quad 2\,021\,310\,927\,068\,045\,324\,725\,518\,336\,z^{6}-58\,389\,788\,180\,617\,217\,723\,335\,830\,405\,120\,z^{8})\,D_z^{2}+\\
&(-29\,477\,789\,298\,864\,z-623\,607\,705\,592\,197\,120\,z^{3}+3\,696\,452\,037\,464\,427\,665\,031\,168\,z^{5}-\\
&\quad 1\,917\,027\,759\,993\,090\,528\,209\,975\,377\,920\,z^{7})\,D_z+(6\,920\,249\,278\,464-2\,151\,907\,464\,314\,880\,z^{2}+\\
&\quad 232\,431\,611\,023\,038\,873\,600\,z^{4}-7\,244\,572\,001\,313\,051\,881\,735\,454\,720\,z^{6})\}
\end{aligned}
$$

In[ ]:=
```
ODEinD = ODE[[1]];
ToOrePolynomial[ODEinD]
```

Out[ ]=

$$\left(-14\,467\,005\,z^{18}+33\,563\,451\,600\,z^{20}-23\,287\,943\,024\,640\,z^{22}+4\,247\,531\,185\,766\,400\,z^{24}-62\,135\,313\,346\,068\,480\,z^{26}\right)D_z^{20}+$$
$$\left(-1\,936\,588\,500\,z^{17}+5\,641\,912\,725\,120\,z^{19}-4\,717\,315\,749\,888\,000\,z^{21}+1\,010\,570\,821\,820\,743\,680\,z^{23}-17\,016\,528\,141\,759\,283\,200\,z^{25}\right)D_z^{19}+$$
$$\left(-108\,852\,307\,326\,z^{16}+406\,851\,523\,097\,040\,z^{18}-416\,070\,075\,723\,337\,728\,z^{20}+105\,864\,003\,406\,135\,296\,000\,z^{22}-2\,069\,771\,412\,396\,904\,022\,016\,z^{24}\right)D_z^{18}+$$
$$\left(-3\,370\,597\,495\,398\,z^{15}+16\,579\,706\,882\,412\,240\,z^{17}-21\,099\,493\,183\,605\,387\,264\,z^{19}+6\,457\,867\,074\,946\,659\,778\,560\,z^{21}-148\,037\,475\,911\,201\,433\,059\,328\,z^{23}\right)D_z^{17}+$$
$$\left(-63\,442\,079\,388\,162\,z^{14}+423\,506\,991\,425\,655\,456\,z^{16}-684\,438\,016\,140\,445\,704\,192\,z^{18}+255\,707\,448\,463\,908\,142\,055\,424\,z^{20}-6\,949\,217\,109\,055\,229\,631\,922\,176\,z^{22}\right)D_z^{16}+$$
$$\left(-756\,101\,749\,033\,170\,z^{13}+7\,112\,961\,381\,085\,706\,880\,z^{15}-14\,955\,124\,960\,933\,692\,604\,416\,z^{17}+6\,937\,083\,672\,214\,807\,630\,577\,664\,z^{19}-226\,339\,521\,441\,947\,695\,466\,938\,368\,z^{21}\right)D_z^{15}+$$
$$\left(-5\,769\,894\,282\,015\,168\,z^{12}+80\,195\,348\,439\,151\,482\,816\,z^{14}-225\,851\,865\,846\,831\,614\,214\,144\,z^{16}+132\,698\,207\,626\,476\,044\,213\,551\,104\,z^{18}-5\,274\,131\,783\,990\,056\,768\,858\,226\,688\,z^{20}\right)D_z^{14}+$$
$$\left(-27\,913\,964\,873\,414\,310\,z^{11}+609\,144\,620\,514\,858\,304\,128\,z^{13}-2\,381\,824\,280\,655\,404\,284\,723\,200\,z^{15}+1\,815\,233\,892\,580\,905\,502\,245\,912\,576\,z^{17}-89\,384\,278\,176\,874\,780\,470\,515\,073\,024\,z^{19}\right)D_z^{13}+$$
$$\left(-83\,070\,016\,297\,204\,617\,z^{10}+3\,085\,887\,338\,422\,919\,479\,728\,z^{12}-17\,525\,197\,548\,821\,959\,865\,708\,544\,z^{14}+17\,829\,800\,370\,673\,184\,633\,851\,478\,016\,z^{16}-1\,109\,562\,633\,354\,790\,604\,422\,552\,485\,888\,z^{18}\right)D_z^{12}+$$
$$\left(-143\,902\,495\,168\,313\,096\,z^{9}+10\,181\,926\,240\,408\,659\,812\,096\,z^{11}-89\,017\,658\,336\,288\,361\,564\,733\,440\,z^{13}+125\,249\,290\,649\,077\,278\,795\,741\,790\,208\,z^{15}-10\,086\,690\,414\,943\,482\,035\,219\,808\,124\,928\,z^{17}\right)D_z^{11}+$$
$$\left(-132\,153\,403\,830\,323\,420\,z^{8}+21\,021\,235\,664\,879\,460\,322\,576\,z^{10}-305\,777\,797\,698\,084\,890\,587\,385\,856\,z^{12}+622\,058\,590\,161\,851\,463\,805\,772\,496\,896\,z^{14}-66\,716\,129\,497\,734\,268\,326\,710\,005\,989\,376\,z^{16}\right)D_z^{10}+$$
$$\left(-54\,553\,427\,074\,589\,192\,z^{7}+25\,492\,972\,749\,524\,213\,713\,936\,z^{9}-687\,639\,745\,093\,042\,281\,248\,894\,976\,z^{11}+2\,142\,211\,936\,926\,982\,182\,052\,098\,473\,984\,z^{13}-316\,993\,324\,024\,628\,089\,414\,708\,262\,600\,704\,z^{15}\right)D_z^{9}+$$
$$\left(-7\,239\,432\,885\,699\,105\,z^{6}+16\,418\,645\,655\,121\,036\,333\,744\,z^{8}-964\,606\,409\,385\,249\,211\,639\,984\,128\,z^{10}+4\,969\,184\,054\,263\,554\,934\,655\,494\,914\,048\,z^{12}-1\,060\,795\,861\,065\,475\,294\,630\,056\,432\,762\,880\,z^{14}\right)D_z^{8}+$$
$$\left(-116\,656\,718\,681\,240\,z^{5}+4\,725\,708\,417\,817\,253\,034\,016\,z^{7}-785\,382\,744\,164\,715\,800\,031\,436\,800\,z^{9}+7\,448\,919\,993\,019\,426\,073\,438\,144\,430\,080\,z^{11}-2\,430\,575\,307\,489\,529\,542\,445\,037\,321\,715\,712\,z^{13}\right)D_z^{7}+$$
$$\left(-110\,857\,036\,479\,z^{4}+430\,864\,911\,968\,597\,888\,864\,z^{6}-331\,954\,375\,958\,956\,680\,821\,710\,848\,z^{8}+6\,800\,429\,385\,541\,013\,459\,919\,103\,328\,256\,z^{10}-3\,665\,475\,304\,708\,148\,593\,614\,758\,799\,736\,832\,z^{12}\right)D_z^{6}+$$
$$\left(-103\,516\,281\,519\,z^{3}+4\,656\,655\,446\,583\,618\,656\,z^{5}-60\,387\,543\,771\,900\,375\,434\,674\,176\,z^{7}+3\,466\,038\,353\,484\,433\,408\,049\,820\,991\,488\,z^{9}-3\,441\,783\,898\,363\,727\,194\,430\,187\,107\,254\,272\,z^{11}\right)D_z^{5}+$$
$$\left(-90\,434\,619\,318\,z^{2}+4\,184\,555\,863\,957\,776\,z^{4}-3\,276\,368\,523\,439\,339\,752\,935\,424\,z^{6}+862\,257\,619\,454\,123\,494\,550\,016\,098\,304\,z^{8}-1\,857\,196\,793\,061\,249\,965\,887\,449\,401\,917\,440\,z^{10}\right)D_z^{4}+$$
$$\left(127\,575\,050\,166\,z+1\,554\,589\,276\,043\,616\,z^{3}-19\,172\,000\,847\,765\,540\,962\,304\,z^{5}+83\,501\,711\,699\,211\,811\,743\,753\,830\,400\,z^{7}-510\,013\,753\,832\,678\,716\,887\,333\,548\,851\,200\,z^{9}\right)D_z^{3}+$$
$$\left(-108\,128\,894\,976+98\,024\,888\,529\,072\,z^{2}-9\,427\,736\,705\,193\,676\,800\,z^{4}+2\,021\,310\,927\,068\,045\,324\,725\,518\,336\,z^{6}-58\,389\,788\,180\,617\,217\,723\,335\,830\,405\,120\,z^{8}\right)D_z^{2}+$$
$$\left(-29\,477\,789\,298\,864\,z-623\,607\,705\,592\,197\,120\,z^{3}+3\,696\,452\,037\,464\,427\,665\,031\,168\,z^{5}-1\,917\,027\,759\,993\,090\,528\,209\,975\,377\,920\,z^{7}\right)D_z+\left(6\,920\,249\,278\,464-2\,151\,907\,464\,314\,880\,z^{2}+232\,431\,611\,023\,038\,873\,600\,z^{4}-7\,244\,572\,001\,313\,051\,881\,735\,454\,720\,z^{6}\right)$$

In[ ]:=
```
ODEinTheta = ChangeOreAlgebra[z ** ODEinD, OreAlgebra[Euler[z]]];
ToOrePolynomial[ODEinTheta]
```

Out[ ]=

$$
\begin{aligned}
&\left(-\frac{14\,467\,005}{z}+33\,563\,451\,600\,z-\right.\\
&\quad\left.23\,287\,943\,024\,640\,z^{3}+4\,247\,531\,185\,766\,400\,z^{5}-62\,135\,313\,346\,068\,480\,z^{7}\right)\theta_z^{20}+\\
&\left(\frac{812\,142\,450}{z}-735\,143\,078\,880\,z-292\,606\,575\,206\,400\,z^{3}+203\,539\,896\,525\,127\,680\,z^{5}-\right.\\
&\quad\left.5\,210\,818\,606\,006\,272\,000\,z^{7}\right)\theta_z^{19}+\left(-\frac{20\,958\,362\,901}{z}+6\,453\,885\,755\,520\,z-\right.\\
&\quad\left.995\,844\,451\,811\,328\,z^{3}+4\,478\,629\,763\,450\,142\,720\,z^{5}-204\,750\,394\,070\,208\,086\,016\,z^{7}\right)\theta_z^{18}+\\
&\left(\frac{329\,647\,387\,230}{z}-27\,223\,492\,836\,960\,z+1\,838\,890\,417\,029\,120\,z^{3}+\right.\\
&\quad\left.60\,013\,489\,570\,908\,733\,440\,z^{5}-5\,010\,007\,884\,858\,214\,318\,080\,z^{7}\right)\theta_z^{17}+\\
&\left(-\frac{3\,532\,487\,743\,476}{z}+41\,563\,674\,965\,376\,z+17\,558\,829\,494\,747\,136\,z^{3}+\right.\\
&\quad\left.547\,649\,725\,516\,961\,808\,384\,z^{5}-85\,577\,490\,519\,427\,582\,525\,440\,z^{7}\right)\theta_z^{16}+\\
&\left(\frac{27\,307\,686\,312\,450}{z}+88\,420\,190\,673\,600\,z+19\,325\,639\,654\,129\,664\,z^{3}+\right.\\
&\quad\left.3\,605\,430\,253\,433\,068\,191\,744\,z^{5}-1\,084\,193\,901\,809\,507\,676\,192\,768\,z^{7}\right)\theta_z^{15}+\\
&\left(-\frac{157\,283\,687\,594\,520}{z}-434\,048\,189\,220\,864\,z-82\,808\,988\,799\,795\,200\,z^{3}+\right.\\
&\quad\left.17\,696\,678\,783\,139\,912\,351\,744\,z^{5}-10\,565\,686\,819\,490\,909\,021\,798\,400\,z^{7}\right)\theta_z^{14}+\\
&\left(\frac{687\,314\,676\,712\,428}{z}+373\,956\,132\,759\,552\,z-206\,200\,847\,320\,252\,416\,z^{3}+\right.\\
&\quad\left.65\,989\,040\,397\,051\,978\,842\,112\,z^{5}-81\,061\,081\,114\,439\,539\,098\,648\,576\,z^{7}\right)\theta_z^{13}+\\
&\left(-\frac{2\,298\,596\,684\,157\,792}{z}+825\,618\,328\,768\,512\,z+51\,624\,919\,567\,171\,584\,z^{3}+\right.\\
&\quad\left.188\,679\,012\,602\,087\,221\,493\,760\,z^{5}-497\,000\,836\,049\,487\,875\,039\,821\,824\,z^{7}\right)\theta_z^{12}+\\
&\left(\frac{5\,890\,363\,656\,846\,784}{z}-1\,665\,769\,014\,126\,592\,z+595\,326\,805\,434\,433\,536\,z^{3}+\right.\\
&\quad\left.414\,286\,824\,101\,392\,582\,443\,008\,z^{5}-2\,457\,868\,693\,293\,804\,675\,940\,745\,216\,z^{7}\right)\theta_z^{11}+\\
&\left(-\frac{11\,501\,742\,510\,617\,472}{z}+122\,494\,538\,645\,504\,z+469\,463\,813\,364\,645\,888\,z^{3}+\right.\\
&\quad\left.694\,130\,980\,215\,383\,862\,018\,048\,z^{5}-9\,852\,399\,697\,681\,146\,866\,227\,478\,528\,z^{7}\right)\theta_z^{10}+\\
&\left(\frac{16\,892\,072\,425\,448\,512}{z}+1\,424\,205\,975\,642\,112\,z-406\,083\,969\,812\,004\,864\,z^{3}+\right.\\
&\quad\left.872\,896\,277\,633\,218\,390\,261\,760\,z^{5}-32\,049\,068\,029\,200\,726\,439\,411\,843\,072\,z^{7}\right)\theta_z^{9}+\\
&\left(-\frac{18\,246\,029\,887\,996\,416}{z}-733\,215\,806\,111\,744\,z-841\,021\,763\,635\,642\,368\,z^{3}+\right.
\end{aligned}
$$

$$796\,732\,812\,773\,703\,309\,852\,672\,z^5 - 84\,402\,707\,723\,801\,174\,014\,519\,410\,688\,z^7\Big)\,\theta_z^8 +$$

$$\left(\frac{13\,976\,816\,455\,452\,160}{z} + 354\,692\,050\,550\,784\,z - 456\,507\,876\,679\,090\,176\,z^3 +\right.$$

$$\left.491\,573\,776\,154\,390\,159\,163\,392\,z^5 - 178\,864\,685\,494\,736\,244\,623\,624\,110\,080\,z^7\right)\theta_z^7 +$$

$$\left(-\frac{7\,143\,102\,729\,252\,864}{z} - 271\,659\,486\,806\,016\,z + 72\,674\,959\,926\,755\,328\,z^3 +\right.$$

$$\left.167\,805\,172\,466\,455\,507\,107\,840\,z^5 - 301\,839\,091\,720\,372\,127\,492\,688\,838\,656\,z^7\right)\theta_z^6 +$$

$$\left(\frac{2\,171\,202\,595\,332\,096}{z} - 566\,530\,840\,199\,168\,z + 178\,896\,143\,231\,483\,904\,z^3 +\right.$$

$$\left.703\,388\,328\,719\,743\,451\,136\,z^5 - 399\,099\,521\,462\,759\,556\,389\,162\,450\,944\,z^7\right)\theta_z^5 +$$

$$\left(-\frac{295\,098\,995\,441\,664}{z} + 332\,325\,400\,870\,912\,z - 11\,884\,386\,840\,477\,696\,z^3 -\right.$$

$$\left.21\,411\,364\,562\,950\,844\,055\,552\,z^5 - 403\,491\,380\,301\,913\,959\,594\,729\,668\,608\,z^7\right)\theta_z^4 +$$

$$\left(171\,845\,517\,836\,288\,z - 37\,538\,777\,933\,021\,184\,z^3 - 2\,496\,776\,678\,093\,585\,645\,568\,z^5 -\right.$$
$$\left.300\,388\,581\,936\,475\,107\,262\,931\,664\,896\,z^7\right)\theta_z^3 + \left(-54\,318\,311\,931\,904\,z + 9\,088\,421\,918\,146\,560\,z^3 +\right.$$
$$\left.3\,924\,378\,046\,071\,746\,592\,768\,z^5 - 154\,798\,362\,246\,948\,302\,600\,756\,264\,960\,z^7\right)\theta_z^2 +$$
$$\left(-17\,586\,764\,906\,496\,z + 3\,952\,221\,389\,586\,432\,z^3 + 1\,724\,057\,318\,145\,581\,383\,680\,z^5 -\right.$$
$$\left.49\,194\,290\,090\,501\,398\,143\,789\,170\,688\,z^7\right)\theta_z + \left(6\,920\,249\,278\,464\,z - 2\,151\,907\,464\,314\,880\,z^3 +\right.$$
$$\left.232\,431\,611\,023\,038\,873\,600\,z^5 - 7\,244\,572\,001\,313\,051\,881\,735\,454\,720\,z^7\right)$$

Write recurrence explicitly for $\tilde{r}(n)$

In[ ]:=
```
ClearAll[Seq];
SeqNormalized = ApplyOreOperator[RECNormalizedinS, Seq[α]]
```

Out[∘]=

$$
\begin{aligned}
&\Big(-221\,086\,792\,032\,258\,663\,383\,040 - 3\,002\,581\,182\,281\,579\,476\,549\,632\,\alpha - \\
&\quad 18\,896\,284\,453\,973\,181\,469\,818\,880\,\alpha^{2} - 73\,337\,056\,136\,834\,742\,984\,114\,176\,\alpha^{3} - \\
&\quad 197\,017\,275\,538\,043\,925\,583\,364\,096\,\alpha^{4} - 389\,745\,626\,428\,476\,129\,286\,291\,456\,\alpha^{5} - \\
&\quad 589\,529\,476\,016\,351\,811\,509\,157\,888\,\alpha^{6} - 698\,690\,177\,713\,813\,455\,561\,031\,680\,\alpha^{7} - \\
&\quad 659\,396\,154\,092\,196\,671\,988\,432\,896\,\alpha^{8} - 500\,766\,687\,956\,261\,350\,615\,810\,048\,\alpha^{9} - \\
&\quad 307\,887\,490\,552\,535\,839\,569\,608\,704\,\alpha^{10} - 153\,616\,793\,330\,862\,792\,246\,296\,576\,\alpha^{11} - \\
&\quad 62\,125\,104\,506\,185\,984\,379\,977\,728\,\alpha^{12} - 20\,265\,270\,278\,609\,884\,774\,662\,144\,\alpha^{13} - \\
&\quad 5\,282\,843\,409\,745\,454\,510\,899\,200\,\alpha^{14} - 1\,084\,193\,901\,809\,507\,676\,192\,768\,\alpha^{15} - \\
&\quad 171\,154\,981\,038\,855\,165\,050\,880\,\alpha^{16} - 20\,040\,031\,539\,432\,857\,272\,320\,\alpha^{17} - \\
&\quad 1\,638\,003\,152\,561\,664\,688\,128\,\alpha^{18} - 83\,373\,097\,696\,100\,352\,000\,\alpha^{19} - 1\,988\,330\,027\,074\,191\,360\,\alpha^{20}\Big) \\
&\mathrm{Seq}[\alpha] + \Big(123\,596\,648\,884\,357\,621\,088\,256 + 1\,387\,410\,081\,329\,207\,115\,251\,712\,\alpha + \\
&\quad 7\,308\,010\,505\,383\,031\,273\,947\,136\,\alpha^{2} + 24\,020\,604\,752\,075\,269\,740\,691\,456\,\alpha^{3} + \\
&\quad 55\,262\,591\,055\,735\,725\,773\,815\,808\,\alpha^{4} + 94\,607\,549\,345\,038\,165\,436\,006\,400\,\alpha^{5} + \\
&\quad 125\,070\,786\,847\,359\,746\,869\,821\,440\,\alpha^{6} + 130\,760\,992\,638\,503\,780\,446\,109\,696\,\alpha^{7} + \\
&\quad 109\,819\,712\,522\,499\,293\,630\,693\,376\,\alpha^{8} + 74\,830\,049\,897\,678\,615\,099\,736\,064\,\alpha^{9} + \\
&\quad 41\,599\,115\,200\,046\,517\,939\,601\,408\,\alpha^{10} + 18\,902\,277\,196\,351\,684\,209\,803\,264\,\alpha^{11} + \\
&\quad 7\,008\,965\,526\,989\,775\,347\,122\,176\,\alpha^{12} + 2\,109\,519\,207\,312\,665\,281\,560\,576\,\alpha^{13} + \\
&\quad 510\,375\,764\,108\,304\,797\,663\,232\,\alpha^{14} + 97\,744\,104\,267\,386\,959\,429\,632\,\alpha^{15} + \\
&\quad 14\,472\,279\,363\,085\,494\,386\,688\,\alpha^{16} + 1\,596\,811\,738\,769\,963\,089\,920\,\alpha^{17} + 123\,530\,156\,260\,699\,668\,480 \\
&\quad \alpha^{18} + 5\,975\,058\,303\,292\,538\,880\,\alpha^{19} + 135\,920\,997\,944\,524\,800\,\alpha^{20}\Big)\ \mathrm{Seq}[1+\alpha] + \\
&\Big(-2\,413\,729\,498\,666\,800\,513\,024 - 25\,435\,086\,835\,865\,925\,058\,560\,\alpha - 125\,542\,481\,225\,411\,227\,975\,680\,\alpha^{2} - \\
&\quad 386\,097\,946\,352\,750\,392\,590\,336\,\alpha^{3} - 830\,183\,396\,028\,360\,968\,208\,384\,\alpha^{4} - \\
&\quad 1\,327\,255\,653\,860\,270\,011\,465\,728\,\alpha^{5} - 1\,637\,850\,112\,836\,596\,110\,688\,256\,\alpha^{6} - \\
&\quad 1\,598\,197\,760\,043\,557\,807\,628\,288\,\alpha^{7} - 1\,252\,980\,911\,862\,994\,173\,739\,008\,\alpha^{8} - \\
&\quad 797\,358\,770\,338\,813\,407\,952\,896\,\alpha^{9} - 414\,276\,959\,391\,975\,941\,603\,328\,\alpha^{10} - \\
&\quad 176\,103\,421\,096\,866\,815\,410\,176\,\alpha^{11} - 61\,159\,515\,859\,482\,838\,548\,480\,\alpha^{12} - \\
&\quad 17\,263\,930\,413\,062\,410\,149\,888\,\alpha^{13} - 3\,923\,295\,133\,237\,310\,914\,560\,\alpha^{14} - \\
&\quad 706\,924\,713\,366\,338\,125\,824\,\alpha^{15} - 98\,652\,029\,401\,005\,981\,696\,\alpha^{16} - 10\,278\,087\,291\,823\,325\,184\,\alpha^{17} - \\
&\quad 752\,234\,327\,699\,226\,624\,\alpha^{18} - 34\,490\,272\,274\,841\,600\,\alpha^{19} - 745\,214\,176\,788\,480\,\alpha^{20}\Big)\ \mathrm{Seq}[2+\alpha] + \\
&\Big(9\,569\,617\,440\,812\,835\,840 + 97\,443\,791\,378\,162\,009\,856\,\alpha + 463\,583\,339\,186\,644\,316\,800\,\alpha^{2} + \\
&\quad 1\,370\,837\,922\,368\,778\,354\,176\,\alpha^{3} + 2\,827\,452\,328\,200\,593\,850\,560\,\alpha^{4} + 4\,326\,575\,055\,112\,730\,856\,640\,\alpha^{5} + \\
&\quad 5\,099\,519\,612\,920\,329\,528\,000\,\alpha^{6} + 4\,743\,666\,552\,937\,883\,189\,952\,\alpha^{7} + 3\,539\,068\,890\,050\,114\,722\,112\,\alpha^{8} + \\
&\quad 2\,139\,750\,587\,880\,300\,657\,856\,\alpha^{9} + 1\,054\,730\,779\,373\,468\,537\,920\,\alpha^{10} + 424\,824\,967\,934\,147\,228\,480\,\alpha^{11} + \\
&\quad 139\,643\,546\,214\,642\,867\,648\,\alpha^{12} + 37\,274\,084\,807\,088\,072\,384\,\alpha^{13} + 8\,003\,802\,897\,605\,020\,608\,\alpha^{14} + \\
&\quad 1\,361\,866\,764\,260\,304\,576\,\alpha^{15} + 179\,386\,646\,751\,384\,192\,\alpha^{16} + 17\,635\,678\,788\,631\,680\,\alpha^{17} + \\
&\quad 1\,217\,772\,669\,657\,600\,\alpha^{18} + 52\,679\,537\,809\,920\,\alpha^{19} + 1\,074\,030\,451\,200\,\alpha^{20}\Big)\ \mathrm{Seq}[3+\alpha] + \\
&\Big(-9\,051\,531\,325\,562\,880 - 90\,332\,029\,095\,081\,984\,\alpha - 420\,333\,410\,362\,428\,416\,\alpha^{2} - \\
&\quad 1\,213\,206\,945\,955\,473\,664\,\alpha^{3} - 2\,437\,377\,188\,874\,087\,136\,\alpha^{4} - 3\,625\,291\,113\,645\,770\,712\,\alpha^{5} - \\
&\quad 4\,144\,688\,219\,837\,114\,384\,\alpha^{6} - 3\,731\,957\,019\,300\,871\,994\,\alpha^{7} - 2\,689\,507\,840\,271\,682\,912\,\alpha^{8} - \\
&\quad 1\,567\,534\,832\,320\,365\,967\,\alpha^{9} - 743\,334\,125\,295\,350\,476\,\alpha^{10} - 287\,455\,002\,784\,035\,524\,\alpha^{11} - \\
&\quad 90\,539\,774\,552\,500\,272\,\alpha^{12} - 23\,112\,095\,925\,472\,389\,\alpha^{13} - 4\,737\,102\,973\,509\,780\,\alpha^{14} - \\
&\quad 767\,930\,664\,461\,310\,\alpha^{15} - 96\,195\,146\,877\,576\,\alpha^{16} - 8\,977\,485\,504\,456\,\alpha^{17} - \\
&\quad 587\,451\,930\,408\,\alpha^{18} - 24\,041\,253\,600\,\alpha^{19} - 462\,944\,160\,\alpha^{20}\Big)\ \mathrm{Seq}[4+\alpha]
\end{aligned}
$$

Initial values of {r(0), r(2), r(4), ...}

In[∘]:=
```
SeqListIni = {};

MAX = 20;

For[n = 0, n ≤ MAX, n++,
  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence @@ (2 coord[[i]])] *
     Product[Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];
 ];

SeqListIni

seq[n_] := SeqListIni[[n + 1]];
```

Out[∘]=
```
{1, 32, 6048, 2 451 200, 1 391 236 000, 921 422 380 032, 663 895 856 219 904, 505 041 413 866 868 736,
 399 445 932 990 555 902 880, 325 440 143 503 901 735 429 120, 271 445 584 301 606 582 663 031 808,
 230 773 066 339 125 955 854 130 661 376, 199 326 200 240 673 646 611 787 771 995 904,
 174 478 237 021 099 598 812 491 315 604 889 600, 154 480 035 620 813 053 446 642 174 412 128 768 000,
 138 129 336 609 134 098 952 004 475 839 318 761 472 000,
 124 577 089 053 969 968 356 059 653 140 361 638 344 938 400,
 113 209 463 052 287 193 655 237 025 876 331 530 870 707 737 600,
 103 573 496 015 054 055 969 039 980 718 499 533 706 000 571 520 000,
 95 328 837 240 197 678 160 114 853 748 204 677 385 026 223 109 120 000,
 88 215 610 025 056 975 283 519 690 346 309 846 200 279 286 296 474 496 000}
```

Verify recurrence by initial values

In[∘]:=
```
Table[SeqNormalized /. {Seq → seq, α → n}, {n, 0, MAX - RecNormalizedOrder}]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Generate more terms in the sequence
SeqList[[n]] = r(2n)

In[∘]:=
```
Bound = 5000;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Asymptotic estimation of SeqList[[n]] = r(2n)

In[∘]:=
```
<< RISC`Asymptotics`
```

Asymptotics Package version 0.3
written by Manuel Kauers

In[∘]:=
```
AsyList = Asymptotics[SeqNormalized, Seq[α]];
N[AsyList]
```

Out[∘]= $\left\{\frac{16.^{\alpha}}{\alpha^2}, \frac{256.^{\alpha}}{\alpha^2}, \frac{1024.^{\alpha}}{\alpha^3}, \frac{1024.^{\alpha}}{\alpha^2}\right\}$

In[∘]:=
```
Ind = Reverse[Table[Floor[Bound / i], {i, 1, 3}]]
Table[N[seq[Ind[[i]]] / (AsyList[[1]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[2]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[3]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[4]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
```

Out[∘]= {1666, 2500, 5000}

Out[∘]= $\{2.806687457612096 \times 10^{3007}, 6.343600724639624 \times 10^{4513}, 1.787780641892824 \times 10^{9029}\}$

Out[∘]= $\{2.422718463768892 \times 10^{1001}, 3.179649140402995 \times 10^{1503}, 4.491598734476526 \times 10^{3008}\}$

Out[∘]= {37.5001, 56.2783, 112.568}

Out[∘]= {0.0225091, 0.0225113, 0.0225136}

Approximate Polya number

In[∘]:=
```
AtOne = N[Sum[seq[n] * (1 / (2^MM Binomial[NN, MM]))^(2 n), {n, 0, Bound}], 11]
N[1 - 1/AtOne, 10]
```

Out[∘]= 1.0452834156

Out[∘]= 0.04332166274