Linked partition ideals

Combinatory **A**nalysis meeting **C**omputer **A**lgebra

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The Legacy of Ramanujan

@ Penn State

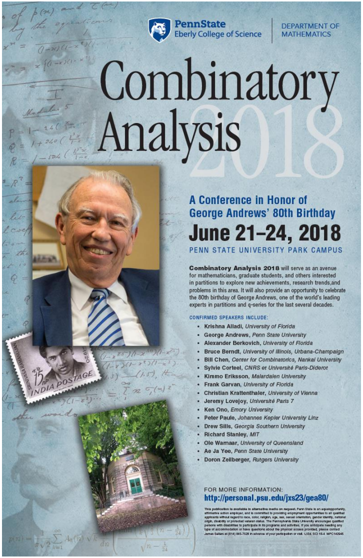
Jun 08, 2024

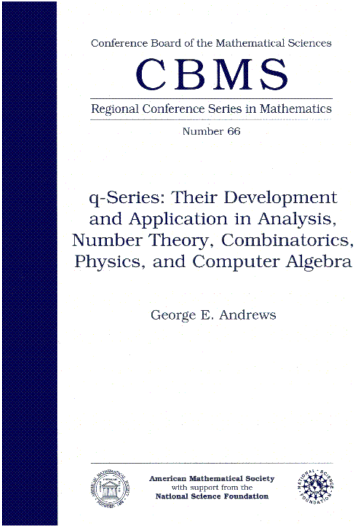
College **A**lgebra

~~**C**ollege **A**lgebra~~

Combinatory **A**nalysis **C**omputer **A**lgebra

Combinatory Analysis & Computer Algebra

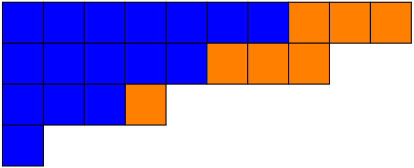


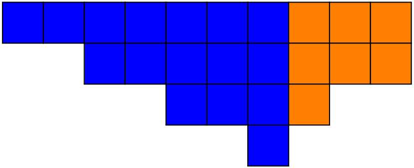


Rogers–Ramanujan

Theorem (First Rogers–Ramanujan Identity)

The number of partitions of a nonnegative integer n into parts congruent to ± 1 *modulo* 5 *is the same as the number of partitions of n such that each two consecutive parts have difference at least* 2.





Theorem (First RR Identity (analytic form))

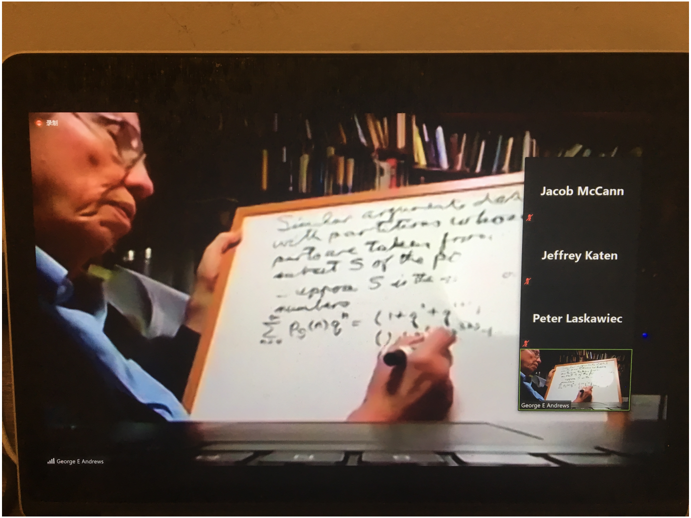
$$\frac{1}{(q, q^4; q^5)_\infty} = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n}.$$

Rogers–Ramanujan

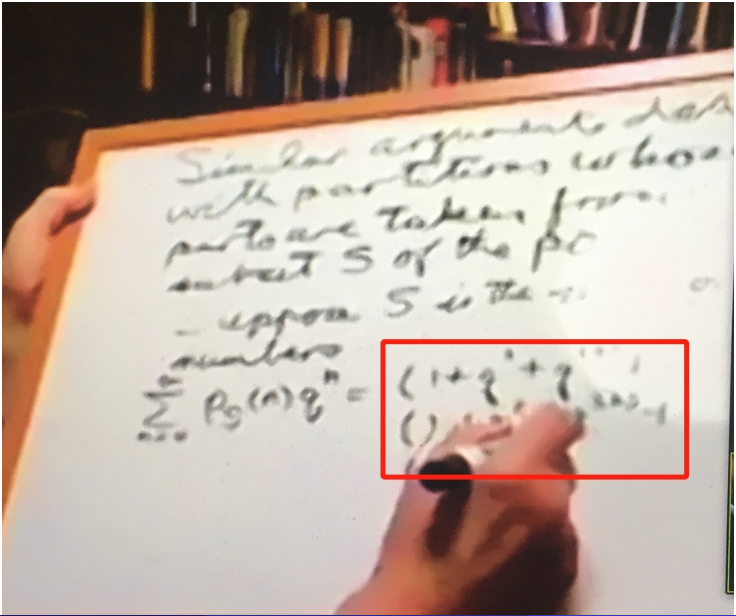
$$
\begin{tikzcd}
q\text{-Prod} \arrow[r, equal, "q\text{-series}"] \arrow[d, leftrightarrow, "\text{G.F.}"', "\text{ESAY!}"] & q\text{-Sum} \arrow[d, leftrightarrow, "\text{HARD!}"', "\text{G.F.}"] \\
\text{Cong. cond.} \arrow[r, Leftrightarrow, "\textit{bijection}"'] & \text{Diff. cond.}
\end{tikzcd}
$$

Rogers–Ramanujan

Taken in Week 1 of Fall 20



Rogers–Ramanujan



Alladi–Schur

Theorem (Schur, 1926)

Let $A(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 6. Let $B(n)$ denote the number of partitions of n into distinct nonmultiples of 3. Let $D(n)$ denote the number of partitions of n of the form $\mu_1 + \mu_2 + \cdots + \mu_s$ where $\mu_i - \mu_{i+1} \geq 3$ with strict inequality if $3 \mid \mu_i$. Then

$$A(n) = B(n) = D(n).$$

Theorem (Alladi, unpublished)

Let $C(n)$ denote the number of partitions of n into odd parts with none appearing more than twice. Then

$$C(n) = A(n) = B(n) = D(n).$$

- $A(n)$ G.F.: $\displaystyle\prod_{k\geq 0} \frac{1}{(1-q^{6k+1})(1-q^{6k+5})}$
- $B(n)$ G.F.: $\displaystyle\prod_{k\geq 0} (1+q^{3k+1})(1+q^{3k+2})$
- $C(n)$ G.F.: $\displaystyle\prod_{k\geq 0} (1+q^{2k+1}+q^{4k+2})$

Alladi–Schur

- $D(n)$ G.F.? **Andrews–Bringmann–Mahlburg (2015):**

$$\sum_{\lambda} x^{\sharp(\lambda)} q^{|\lambda|} \overset{\text{HARD!}}{=} \sum_{n_1,n_2\geq 0} \frac{(-1)^{n_2} q^{3\binom{n_1}{2}+18\binom{n_2}{2}+6n_1n_2+n_1+9n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^6;q^6)_{n_2}}.$$

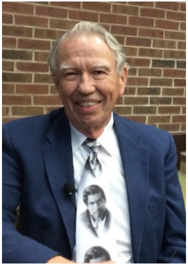
Theorem (Andrews, 2017)

Let $C(m,n)$ denote the number of partitions of n into m odd parts with none appearing more than twice. Let $D(m,n)$ denote the number of partitions of n enumerated by $D(n)$ such that the total number of parts plus the number of even parts equals m. Then

$$C(m,n) = D(m,n).$$

Linked partition ideals

George Andrews — *Linked Partition Ideals*



ADVANCES IN MATHEMATICS 9, 10–51 (1972)

Partition Identities*

GEORGE E. ANDREWS

Department of Mathematics, Massachusetts Institute of Technology and Pennsylvania State University

Linked Partition Ideals are **NOT** ideal for the pencil-and-paper mode!

$\mathscr{D}$: $\mu_1 + \mu_2 + \cdots + \mu_s$

- $\mu_i - \mu_{i+1} \geq 3$;
- $\mu_i - \mu_{i+1} > 3$ if $3 \mid \mu_i$.

Example. We decompose each partition in $\mathscr{D}$ into blocks $B_0, B_1, \ldots$ such that all parts between $3i+1$ and $3i+3$ fall into block B_i.

$$4+7+12+17+20+24$$
$$\Downarrow$$
$$()+(4)+(7)+(12)+()+(17)+(20)+(24)$$
$$\Downarrow$$
$$\phi^0(\varnothing)+\phi^3(1)+\phi^6(1)+\phi^9(3)+\phi^{12}(\varnothing)+\phi^{15}(2)+\phi^{18}(2)+\phi^{21}(3)$$
$$\Downarrow$$
$$\varnothing \to 1 \to 1 \to 3 \to \varnothing \to 2 \to 2 \to 3$$

We define operators ϕ^ℓ with $\ell \geq 0$ for partitions by adding ℓ to each part of the partition. In particular, $\phi^\ell(\varnothing) = \varnothing$ for all $\ell \geq 0$.

Linked partition ideals

$\mathscr{D}$: $\mu_1 + \mu_2 + \cdots + \mu_s$

- $\mu_i - \mu_{i+1} \geq 3$;
- $\mu_i - \mu_{i+1} > 3$ if $3 \mid \mu_i$.

From the decomposition:

- Finite set of partitions $\Pi = \{\pi_1 = \varnothing, \pi_2 = (1), \pi_3 = (2), \pi_4 = (3)\}$.
- Further requirements:
  - $\pi_1 \to \{\pi_1, \pi_2, \pi_3, \pi_4\}$. *If $\phi^{-3i}(B_i)$ is $\pi_1 = \varnothing$, then $\phi^{-3(i+1)}(B_{i+1})$ can be any among $\{\pi_1, \pi_2, \pi_3, \pi_4\}$.*
  - $\pi_2 \to \{\pi_1, \pi_2, \pi_3, \pi_4\}$.
  - $\pi_3 \to \{\pi_1, \pi_3, \pi_4\}$. $\boxed{(3i+2) \to (3(i+1)+1) ✗}$
  - $\pi_4 \to \{\pi_1\}$. $\boxed{(3i+3) \to (3(i+1)+1) \textit{ or } (3(i+1)+2) \textit{ or } (3(i+1)+3) ✗}$

Linked partition ideals

Assume that we are given

- a finite set $\Pi = \{\pi_1, \pi_2, \ldots, \pi_K\}$ of integer partitions with $\pi_1 = \varnothing$, the empty partition,
- a *map of linking sets*, $\mathcal{L} : \Pi \to P(\Pi)$, the power set of Π, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\varnothing) = \Pi$ and $\pi_1 = \varnothing \in \mathcal{L}(\pi_k)$ for any $1 \leq k \leq K$,
- and a positive integer T, called the *modulus*, which is greater than or equal to the largest part among all partitions in Π.

Linked partition ideals

Consider

- an infinite chain of partitions in Π:

$$\lambda_0 \to \lambda_1 \to \cdots \to \lambda_N \to \pi_1 \to \pi_1 \to \cdots$$

  ending with a series of empty partitions, such that $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i;
- an integer partition λ by

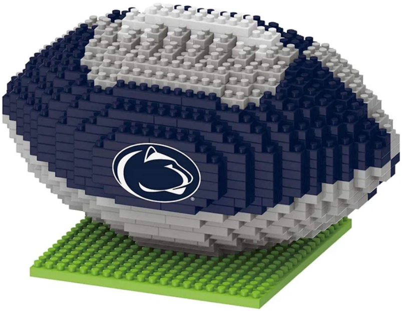
$$\lambda = \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \phi^{2T}(\lambda_2) \oplus \cdots \oplus \phi^{NT}(\lambda_N),$$

  where $\mu \oplus \nu$ is the partition constructed by collecting all parts in partitions μ and ν, and $\phi^m(\mu)$ is the partition obtained by adding m to each part of μ.

We collect all such partitions λ constructed as above and call this partition set a *span one linked partition ideal*, denoted by $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L} \rangle, T)$.

Linked partition ideals

We Are playing with LEGOs!



Linked partition ideals

Define for any partition λ,

- $|\lambda|$: its size (aka. sum of all parts);
- $\sharp(\lambda)$: its length (aka. the number of parts);
- $s(\lambda)$: a statistic of $\lambda \in \mathscr{I}$ such that

$$s(\lambda) = s(\phi^T(\lambda)) \text{ and } s(\lambda) = s(\lambda_0) + s(\lambda_1) + \cdots + s(\lambda_N).$$

For each $1 \leq k \leq K$, we write

$$G_k(x) := \sum_{\substack{\lambda \in \mathscr{I} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y^{s(\lambda)} q^{|\lambda|}.$$

Then these generating functions satisfy a **system of q-difference equations**:

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_K(x) \end{pmatrix} = \mathscr{M} \cdot \begin{pmatrix} G_1(xq^T) \\ G_2(xq^T) \\ \vdots \\ G_K(xq^T) \end{pmatrix}$$

Alladi–Andrews

Theorem (Andrews–C.–Li, 2022)

$$\sum_{\lambda\in\mathscr{D}} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} q^{|\lambda|} = \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3} x^{n_1+n_2+2n_3} y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}}$$
$$\times\, q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

Corollary

$$\prod_{n\geq 0}(1+xq^{2n+1}+x^2q^{4n+2}) = \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3} x^{n_1+2n_2+3n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}}$$
$$\times\, q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

Alladi–Andrews

$$\begin{pmatrix} A_1(x) \\ A_2(x) \\ A_3(x) \end{pmatrix} = \mathscr{M}. \begin{pmatrix} A_1(xq^6) \\ A_2(xq^6) \\ A_3(xq^6) \end{pmatrix}$$

where

$$\mathscr{M} = \begin{pmatrix} 1 + xq + xyq^2 + xq^3 + xyq^4 + x^2yq^5 & xq^5 + x^2q^6 + x^2yq^7 & xyq^6 + x^2yq^7 + x^2y^2q^8 \\ 1 + xyq^2 + xq^3 + xyq^4 & xq^5 + x^2yq^7 & xyq^6 + x^2y^2q^8 \\ 1 + xyq^4 & xq^5 & xyq^6 \end{pmatrix}.$$

$$\boxed{\sum_{\lambda \in \mathscr{D}} x^{\sharp(\lambda)} y^{\sharp_{0,2}(\lambda)} q^{|\lambda|} = A_1(x)}$$

Alladi–Andrews

An algorithm of **C.–Li (Discrete Math., 2020)**: Given a q-difference system

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_k(x) \end{pmatrix} = \begin{pmatrix} p_{1,1}(x) & p_{1,2}(x) & \cdots & p_{1,k}(x) \\ p_{2,1}(x) & p_{2,2}(x) & \cdots & p_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{k,1}(x) & p_{k,2}(x) & \cdots & p_{k,k}(x) \end{pmatrix} \begin{pmatrix} F_1(xq^m) \\ F_2(xq^m) \\ \vdots \\ F_k(xq^m) \end{pmatrix},$$

can we determine the q-difference equation satisfied by $F_1(x)$?

Idea. Making substitutions to reduce this q-difference system as

$$\begin{pmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_{\ell-1}(x) \\ u_\ell(x) \end{pmatrix} = \begin{pmatrix} r_{1,1}(x) & 1 & 0 & 0 & \cdots & 0 \\ r_{2,1}(x) & r_{2,2}(x) & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{\ell-1,1}(x) & r_{\ell-1,2}(x) & \cdots & \cdots & r_{\ell-1,\ell-1}(x) & 1 \\ r_{\ell,1}(x) & r_{\ell,2}(x) & \cdots & \cdots & r_{\ell,\ell-1}(x) & r_{\ell,\ell}(x) \end{pmatrix} \begin{pmatrix} u_1(xq^m) \\ u_2(xq^m) \\ \vdots \\ u_{\ell-1}(xq^m) \\ u_\ell(xq^m) \end{pmatrix}.$$

Alladi–Andrews

- q-Difference equation for $A_1(x)$:

$$\begin{aligned}
0 &= \left[1 + x(q^7 + yq^8)\right] A_1(x) \\
&\quad - \left[1 + x(q + q^3 + q^5 + q^7 + yq^2 + yq^4 + yq^6 + yq^8)\right. \\
&\qquad + x^2(q^6 + q^8 + q^{10} + yq^5 + 2yq^7 + 2yq^9 + 2yq^{11} + yq^{13} + y^2q^8 + y^2q^{10} + y^2q^{12}) \\
&\qquad \left. + x^3(yq^{12} + yq^{14} + y^2q^{13} + y^2q^{15})\right] A_1(xq^6) \\
&\quad + \left[x^2yq^{15} + x^3(-q^{21} + yq^{16} + y^2q^{17} - y^3q^{24})\right. \\
&\qquad + x^4(-q^{22} - yq^{23} + y^2q^{30} - y^3q^{25} - y^4q^{26}) \\
&\qquad \left. + x^5(y^2q^{31} + y^3q^{32})\right] A_1(xq^{12}).
\end{aligned}$$

- Let $A_1(x) = \sum_{M\geq 0} a(M)x^M$. For any $M \geq 0$,

$$\begin{aligned}
0 &= q^{12M}(y^2q^{31} + y^3q^{32})a(M) \\
&\quad + q^{12(M+1)}(-q^{22} - yq^{23} + y^2q^{30} - y^3q^{25} - y^4q^{26})a(M+1) \\
&\quad + \left[-q^{6(M+2)}(yq^{12} + yq^{14} + y^2q^{13} + y^2q^{15})\right. \\
&\qquad \left. + q^{12(M+2)}(-q^{21} + yq^{16} + y^2q^{17} - y^3q^{24})\right] a(M+2) \\
&\quad + \left[-q^{6(M+3)}(q^6 + q^8 + q^{10} + yq^5 + 2yq^7 + 2yq^9 + 2yq^{11} + yq^{13} + y^2q^8 + y^2q^{10} + y^2q^{12})\right. \\
&\qquad \left. + q^{12(M+3)}yq^{15}\right] a(M+3) \\
&\quad + \left[(q^7 + yq^8) - q^{6(M+4)}(q + q^3 + q^5 + q^7 + yq^2 + yq^4 + yq^6 + yq^8)\right] a(M+4) \\
&\quad + \left[1 - q^{6(M+5)}\right] a(M+5).
\end{aligned}$$

Alladi–Andrews

- Assume the ansatz that $A_1(x)$ can be represented in the form:

$$\sum_{n_1,\ldots,n_r\geq 0} \frac{(-1)^{L_1(n_1,\ldots,n_r)} q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}}.$$

- Compute initial coefficients:

$$\begin{aligned}
a(0) &= 1,\\
a(1) &= \frac{q(1+yq)}{1-q^2},\\
a(2) &= \frac{q^5(q-q^7+y+yq^2-yq^4-yq^{10}+y^2q^3-y^2q^9)}{(1-q^2)(1-q^4)(1-q^6)},\\
a(3) &= \frac{q^{12}(1+yq)(q^3+y+yq^2-yq^4+yq^8+y^2q^5)}{(1-q^2)(1-q^4)(1-q^6)}.
\end{aligned}$$

- From $a(1)$, it is natural to expect summations of the form:

$$\sum_{n_1\geq 0} \frac{q^{?}x^{n_1}}{(q^2;q^2)_{n_1}} \quad \text{and} \quad \sum_{n_2\geq 0} \frac{q^{?}x^{n_2}y^{n_2}}{(q^2;q^2)_{n_2}}.$$

- From $a(2)$, it is also highly possible that an extra summation is needed:

$$\sum_{n_3\geq 0} \frac{(-1)^{?}q^{?}x^{2n_3}y^{n_3}}{(q^6;q^6)_{n_3}}.$$

Alladi–Andrews

- Guess(?)

$$A_1(x) \overset{?}{=} \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3} x^{n_1+n_2+2n_3} y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}} \times q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

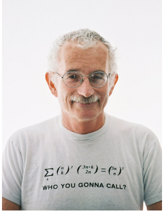
- Prove(!)

$$\boxed{a(M) = \tilde{a}(M)}$$

where

$$\sum_{M\geq 0} \tilde{a}(M) x^M = \sum_{n_1,n_2,n_3\geq 0} \frac{(-1)^{n_3} x^{n_1+n_2+2n_3} y^{n_2+n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^6;q^6)_{n_3}} \times q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

Wilf–Zeilberger Algorithm



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RATIONAL FUNCTIONS CERTIFY COMBINATORIAL IDENTITIES

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Alladi–Andrews

- `qMultiSum` implemented by Riese.

```
In[ ]:= (********************
 Computing the recurrence for ã(M) using the qMultiSum package.
  ********************)

ClearAll[M, n1, n2, n3, U1, U2, U3, y];
U1 = 1;
U2 = 2;
U3 = 9;
n1 = M - n2 - 2 n3;
summand = ((-1)^n3 q^(4 Binomial[n1,2]+4 Binomial[n2,2]+18 Binomial[n3,2]+2 n1 n2+6 n2 n3+6 n3 n1+U1 n1+U2 n2+U3 n3) y^(n2+n3)) /
   (qPochhammer[q^2, q^2, n1] qPochhammer[q^2, q^2, n2] qPochhammer[q^6, q^6, n3]);
stru = qFindStructureSet[summand, {M}, {n2, n3}, {2}, {2, 2}, {2, 2}, qProtocol → True]
rec = qFindRecurrence[summand, {M}, {n2, n3}, {2}, {2, 2}, {2, 2}, qProtocol → True,
  StructSet → stru[[1]]]
sumrec = qSumRecurrence[rec]
```

Alladi–Andrews

Out[ ]=

$$
\begin{aligned}
\{&q^{24+12M} y^2 \left(1 + q^{22+6M} + 2 q y + q^{23+6M} y + q^2 y^2 + q^{24+6M} y^2\right) \mathrm{SUM}[M] - \\
&q^{27+12M} (1 + q y) \left(1 + q^{22+6M} + q y + q^2 y^2 - q^8 y^2 + q^{24+6M} y^2 + q^3 y^3 + q^4 y^4 + q^{26+6M} y^4\right) \mathrm{SUM}[1 + M] + \\
&q^{17+6M} \left(q^{15+6M} - q^{21+6M} - y - q^2 y + 2 q^{16+6M} y - 2 q^{22+6M} y - q^{24+6M} y + q^{38+12M} y - \right. \\
&\quad 2 q y^2 - 2 q^3 y^2 + 3 q^{17+6M} y^2 - 2 q^{23+6M} y^2 - q^{25+6M} y^2 + q^{39+12M} y^2 - q^2 y^3 - q^4 y^3 + \\
&\quad \left. 2 q^{18+6M} y^3 - 2 q^{24+6M} y^3 - q^{26+6M} y^3 + q^{40+12M} y^3 + q^{19+6M} y^4 - q^{25+6M} y^4\right) \mathrm{SUM}[2 + M] - \\
&q^{17+6M} \left(1 - q + q^2\right) \left(1 + q + q^2\right) (1 + q y) \left(1 + q^{20+6M} + q y + q^3 y + q^{21+6M} y + q^2 y^2 + q^{22+6M} y^2\right) \\
&\mathrm{SUM}[3 + M] - \left(-1 + q^{4+M}\right) \left(1 + q^{4+M}\right) \left(1 - q^{4+M} + q^{8+2M}\right) \left(1 + q^{4+M} + q^{8+2M}\right) \\
&\left(1 + q^{16+6M} + 2 q y + q^{17+6M} y + q^2 y^2 + q^{18+6M} y^2\right) \mathrm{SUM}[4 + M] == 0\}
\end{aligned}
$$

- $\tilde{a}(M)$: Order 4
- $a(M)$: Order 5

- Let $d(M) := a(M) - \tilde{a}(M)$.
- qGeneratingFunctions implemented by Koutschan.

```
In[•]:= (*********************
          Computing the recurrence for a(M) - ã(M) using the qGeneratingFunctions package.
          *********************)

        sumrec1 = {Rec == 0};
        sumrec2 = sumrec;
        ClearAll[M, y];
        QREPlus[sumrec1, sumrec2, SUM[M]]
```

Out[•]= $\{-q^{29}(-1+q^M)(1+q^M)(1-q^M+q^{2M})(1+q^M+q^{2M})\,\mathrm{SUM}[M] ==$
$-q^{12M}y^2(1+qy)\,\mathrm{SUM}[-5+M] - q^{3+12M}(-1-qy+q^8y^2-q^3y^3-q^4y^4)\,\mathrm{SUM}[-4+M] +$
$q^{9+6M}(1+qy)(q^{5+6M}+q^{14}y+q^{16}y-q^{6M}y-q^{6+6M}y+q^{7+6M}y^2)\,\mathrm{SUM}[-3+M] +$
$q^{20+6M}(q^3+q^5+q^7+q^2y+2q^4y+2q^6y+2q^8y+q^{10}y-q^{6M}y+q^5y^2+q^7y^2+q^9y^2)\,\mathrm{SUM}[-2+M] -$
$q^{24}(q^{12}-q^{6M}-q^{2+6M}-q^{4+6M}-q^{6+6M})(1+qy)\,\mathrm{SUM}[-1+M]\}$

- $d(M)$: Order 5

Alladi–Andrews

As long as we have verified that

$$d(M) = 0 \qquad \text{for } M = 0, 1, 2, 3, 4,$$

then

$$d(M) = 0 \qquad \text{for all } M \geq 0,$$

so that

$$a(M) = \tilde{a}(M) \qquad \text{for all } M \geq 0,$$

so that

$$A_1(x) = \sum_{n_1, n_2, n_3 \geq 0} \frac{(-1)^{n_3} x^{n_1+n_2+2n_3} y^{n_2+n_3}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} (q^6; q^6)_{n_3}} \times q^{4\binom{n_1}{2}+4\binom{n_2}{2}+18\binom{n_3}{2}+2n_1n_2+6n_2n_3+6n_3n_1+n_1+2n_2+9n_3}.$$

Epilogue

What a successful meeting between

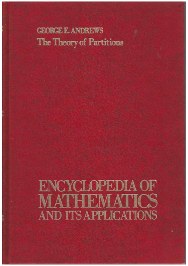
Combinatory **A**nalysis & **C**omputer **A**lgebra!

What a flourishing time of

Partition **A**nalysis

in

PA (Pennsylvania)



George has been bringing to us over the past six decades!

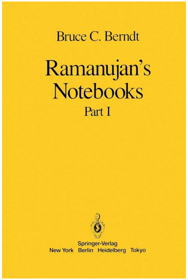
Epilogue

And my adventure in partitions and q-series all starts with the 10 volumes of

Indian **L**egacies

compiled in

ILLINOIS



by Bruce!

Happy Birthday, George and Bruce!