Multi-headed Lattice Green Function (N = 4, M = 3)
Find minimal recurrence for the coefficients

In[∘]:=
```
NN = 4;
MM = 3;
```

Generate a sequence from recurrence & initial values
Koutschan’s implementation

In[∘]:=
```
(* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals, Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

In[∘]:=
```
<< RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type ?HolonomicFunctions for help.
```

In[∘]:=
```
ClearAll[z, w, α, β];
```

Import our REC for {r(0), r(2), r(4), ...}

In[ ]:=
```
ClearAll[Seq];
SeqNormalized = (-221 086 792 032 258 663 383 040 - 3 002 581 182 281 579 476 549 632 α -
      18 896 284 453 973 181 469 818 880 α^2 - 73 337 056 136 834 742 984 114 176 α^3 -
      197 017 275 538 043 925 583 364 096 α^4 - 389 745 626 428 476 129 286 291 456 α^5 -
      589 529 476 016 351 811 509 157 888 α^6 - 698 690 177 713 813 455 561 031 680 α^7 -
      659 396 154 092 196 671 988 432 896 α^8 - 500 766 687 956 261 350 615 810 048 α^9 -
      307 887 490 552 535 839 569 608 704 α^10 - 153 616 793 330 862 792 246 296 576 α^11 -
      62 125 104 506 185 984 379 977 728 α^12 - 20 265 270 278 609 884 774 662 144 α^13 -
      5 282 843 409 745 454 510 899 200 α^14 - 1 084 193 901 809 507 676 192 768 α^15 -
      171 154 981 038 855 165 050 880 α^16 - 20 040 031 539 432 857 272 320 α^17 -
      1 638 003 152 561 664 688 128 α^18 - 83 373 097 696 100 352 000 α^19 - 1 988 330 027 074 191 360 α^20)
    Seq[α] + (123 596 648 884 357 621 088 256 + 1 387 410 081 329 207 115 251 712 α +
      7 308 010 505 383 031 273 947 136 α^2 + 24 020 604 752 075 269 740 691 456 α^3 +
      55 262 591 055 735 725 773 815 808 α^4 + 94 607 549 345 038 165 436 006 400 α^5 +
      125 070 786 847 359 746 869 821 440 α^6 + 130 760 992 638 503 780 446 109 696 α^7 +
      109 819 712 522 499 293 630 693 376 α^8 + 74 830 049 897 678 615 099 736 064 α^9 +
      41 599 115 200 046 517 939 601 408 α^10 + 18 902 277 196 351 684 209 803 264 α^11 +
      7 008 965 526 989 775 347 122 176 α^12 + 2 109 519 207 312 665 281 560 576 α^13 +
      510 375 764 108 304 797 663 232 α^14 + 97 744 104 267 386 959 429 632 α^15 +
      14 472 279 363 085 494 386 688 α^16 + 1 596 811 738 769 963 089 920 α^17 +
      123 530 156 260 699 668 480 α^18 + 5 975 058 303 292 538 880 α^19 + 135 920 997 944 524 800 α^20)
    Seq[1 + α] + (-2 413 729 498 666 800 513 024 - 25 435 086 835 865 925 058 560 α -
      125 542 481 225 411 227 975 680 α^2 - 386 097 946 352 750 392 590 336 α^3 -
      830 183 396 028 360 968 208 384 α^4 - 1 327 255 653 860 270 011 465 728 α^5 -
      1 637 850 112 836 596 110 688 256 α^6 - 1 598 197 760 043 557 807 628 288 α^7 -
      1 252 980 911 862 994 173 739 008 α^8 - 797 358 770 338 813 407 952 896 α^9 -
      414 276 959 391 975 941 603 328 α^10 - 176 103 421 096 866 815 410 176 α^11 -
      61 159 515 859 482 838 548 480 α^12 - 17 263 930 413 062 410 149 888 α^13 -
      3 923 295 133 237 310 914 560 α^14 - 706 924 713 366 338 125 824 α^15 -
      98 652 029 401 005 981 696 α^16 - 10 278 087 291 823 325 184 α^17 - 752 234 327 699 226 624 α^18 -
      34 490 272 274 841 600 α^19 - 745 214 176 788 480 α^20) Seq[2 + α] +
   (9 569 617 440 812 835 840 + 97 443 791 378 162 009 856 α + 463 583 339 186 644 316 800 α^2 +
      1 370 837 922 368 778 354 176 α^3 + 2 827 452 328 200 593 850 560 α^4 +
      4 326 575 055 112 730 856 640 α^5 + 5 099 519 612 920 329 528 000 α^6 +
      4 743 666 552 937 883 189 952 α^7 + 3 539 068 890 050 114 722 112 α^8 + 2 139 750 587 880 300 657 856
        α^9 + 1 054 730 779 373 468 537 920 α^10 + 424 824 967 934 147 228 480 α^11 +
      139 643 546 214 642 867 648 α^12 + 37 274 084 807 088 072 384 α^13 + 8 003 802 897 605 020 608 α^14 +
      1 361 866 764 260 304 576 α^15 + 179 386 646 751 384 192 α^16 + 17 635 678 788 631 680 α^17 +
      1 217 772 669 657 600 α^18 + 52 679 537 809 920 α^19 + 1 074 030 451 200 α^20) Seq[3 + α] +
   (-9 051 531 325 562 880 - 90 332 029 095 081 984 α - 420 333 410 362 428 416 α^2 -
      1 213 206 945 955 473 664 α^3 - 2 437 377 188 874 087 136 α^4 - 3 625 291 113 645 770 712 α^5 -
      4 144 688 219 837 114 384 α^6 - 3 731 957 019 300 871 994 α^7 - 2 689 507 840 271 682 912 α^8 -
      1 567 534 832 320 365 967 α^9 - 743 334 125 295 350 476 α^10 - 287 455 002 784 035 524 α^11 -
      90 539 774 552 500 272 α^12 - 23 112 095 925 472 389 α^13 - 4 737 102 973 509 780 α^14 -
      767 930 664 461 310 α^15 - 96 195 146 877 576 α^16 - 8 977 485 504 456 α^17 -
      587 451 930 408 α^18 - 24 041 253 600 α^19 - 462 944 160 α^20) Seq[4 + α];
```

In[∘]:=
```
RecNormalizedOrder = 4;
```

Initial values of {r(0), r(2), r(4), ...}

In[∘]:=
```
SeqListIni = {};

MAX = 20;

For[n = 0, n ≤ MAX, n++,
  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence @@ (2 coord[[i]])] *
     Product[Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];
 ];

SeqListIni

seq[n_] := SeqListIni[[n + 1]];
```

Out[∘]=
```
{1, 32, 6048, 2 451 200, 1 391 236 000, 921 422 380 032, 663 895 856 219 904, 505 041 413 866 868 736,
 399 445 932 990 555 902 880, 325 440 143 503 901 735 429 120, 271 445 584 301 606 582 663 031 808,
 230 773 066 339 125 955 854 130 661 376, 199 326 200 240 673 646 611 787 771 995 904,
 174 478 237 021 099 598 812 491 315 604 889 600, 154 480 035 620 813 053 446 642 174 412 128 768 000,
 138 129 336 609 134 098 952 004 475 839 318 761 472 000,
 124 577 089 053 969 968 356 059 653 140 361 638 344 938 400,
 113 209 463 052 287 193 655 237 025 876 331 530 870 707 737 600,
 103 573 496 015 054 055 969 039 980 718 499 533 706 000 571 520 000,
 95 328 837 240 197 678 160 114 853 748 204 677 385 026 223 109 120 000,
 88 215 610 025 056 975 283 519 690 346 309 846 200 279 286 296 474 496 000}
```

Verify recurrence by initial values

In[∘]:=
```
Table[SeqNormalized /. {Seq → seq, α → n}, {n, 0, MAX - RecNormalizedOrder}]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Generate more terms in the sequence
SeqList[[n]] = r(2n)

In[∘]:=
```
Bound = 200;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Let’s guess (and prove!) a shorter recurrence.

In[∘]:=
```
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan

```
--> Type  ?HolonomicFunctions  for help.
```

Package GeneratingFunctions version 0.9 written by Christian Mallinger

Guess Package version 0.52
written by Manuel Kauers

In[ ]:= **SeqGuess = GuessMinRE[Take[SeqList, 100], Seq[α]]**

Out[ ]=

$$\left(\frac{21\,933\,213\,495\,263\,756\,288}{45\,927}+\frac{10\,425\,629\,105\,144\,373\,182\,464\,\alpha}{1\,607\,445}+\frac{118\,101\,777\,837\,332\,384\,186\,368\,\alpha^{2}}{2\,893\,401}+\frac{2\,291\,783\,004\,276\,085\,718\,253\,568\,\alpha^{3}}{14\,467\,005}+\frac{6\,156\,789\,860\,563\,872\,674\,480\,128\,\alpha^{4}}{14\,467\,005}+\frac{12\,179\,550\,825\,889\,879\,040\,196\,608\,\alpha^{5}}{14\,467\,005}+\frac{6\,140\,932\,041\,836\,998\,036\,553\,728\,\alpha^{6}}{4\,822\,335}+\frac{1\,455\,604\,536\,903\,778\,032\,418\,816\,\alpha^{7}}{964\,467}+\frac{2\,943\,732\,830\,768\,735\,142\,805\,504\,\alpha^{8}}{2\,066\,715}+\frac{15\,648\,958\,998\,633\,167\,206\,744\,064\,\alpha^{9}}{14\,467\,005}+\frac{9\,621\,484\,079\,766\,744\,986\,550\,272\,\alpha^{10}}{14\,467\,005}+\frac{4\,800\,524\,791\,589\,462\,257\,696\,768\,\alpha^{11}}{14\,467\,005}+\frac{1\,467\,429\,717\,171\,815\,579\,648\,\alpha^{12}}{10\,935}+\frac{70\,365\,521\,800\,728\,766\,578\,688\,\alpha^{13}}{1\,607\,445}+\frac{174\,697\,202\,703\,222\,702\,080\,\alpha^{14}}{15\,309}+\frac{179\,264\,864\,717\,180\,502\,016\,\alpha^{15}}{76\,545}+\frac{13\,206\,402\,857\,936\,355\,328\,\alpha^{16}}{35\,721}+\frac{73\,633\,272\,852\,119\,552\,\alpha^{17}}{1701}+\frac{1\,432\,983\,357\,620\,224\,\alpha^{18}}{405}+\frac{34\,037\,615\,820\,800\,\alpha^{19}}{189}+4\,294\,967\,296\,\alpha^{20}\right)\mathrm{Seq}[\alpha]+$$

$$\left(-\frac{61\,307\,861\,549\,780\,566\,016}{229\,635}-\frac{4\,817\,396\,115\,726\,413\,594\,624\,\alpha}{1\,607\,445}-\frac{228\,375\,328\,293\,219\,727\,310\,848\,\alpha^{2}}{14\,467\,005}-\frac{750\,643\,898\,502\,352\,179\,396\,608\,\alpha^{3}}{14\,467\,005}-\frac{1\,726\,955\,970\,491\,741\,430\,431\,744\,\alpha^{4}}{14\,467\,005}-\frac{197\,099\,061\,135\,496\,177\,991\,680\,\alpha^{5}}{964\,467}-\frac{781\,692\,417\,795\,998\,417\,936\,384\,\alpha^{6}}{2\,893\,401}-\frac{4\,086\,281\,019\,953\,243\,138\,940\,928\,\alpha^{7}}{14\,467\,005}-\frac{381\,318\,446\,258\,678\,102\,884\,352\,\alpha^{8}}{1\,607\,445}-\frac{334\,062\,722\,757\,493\,817\,409\,536\,\alpha^{9}}{2\,066\,715}-\frac{1\,299\,972\,350\,001\,453\,685\,612\,544\,\alpha^{10}}{14\,467\,005}-\right.$$

$$\frac{590\,696\,162\,385\,990\,131\,556\,352\,\alpha^{11}}{14\,467\,005}-\frac{2\,704\,076\,206\,400\,376\,291\,328\,\alpha^{12}}{178\,605}-$$

$$\frac{7\,324\,719\,469\,835\,643\,338\,752\,\alpha^{13}}{1\,607\,445}-\frac{196\,904\,229\,980\,055\,863\,296\,\alpha^{14}}{178\,605}-$$

$$\frac{113\,129\,750\,309\,475\,647\,488\,\alpha^{15}}{535\,815}-\frac{797\,634\,444\,614\,500\,352\,\alpha^{16}}{25\,515}-\frac{5\,867\,180\,110\,118\,912\,\alpha^{17}}{1701}-$$

$$\frac{151\,295\,997\,771\,776\,\alpha^{18}}{567}-\frac{348\,479\,553\,536\,\alpha^{19}}{27}-293\,601\,280\,\alpha^{20}\Bigg)\,\mathrm{Seq}[1+\alpha]+$$

$$\Bigg(\frac{44\,343\,942\,876\,741\,632}{8505}+\frac{17\,663\,254\,747\,129\,114\,624\,\alpha}{321\,489}+\frac{261\,546\,835\,886\,273\,391\,616\,\alpha^{2}}{964\,467}+$$

$$\frac{4\,021\,853\,607\,841\,149\,922\,816\,\alpha^{3}}{4\,822\,335}+\frac{411\,797\,319\,458\,512\,385\,024\,\alpha^{4}}{229\,635}+\frac{1\,975\,082\,818\,244\,449\,421\,824\,\alpha^{5}}{688\,905}+$$

$$\frac{5\,686\,979\,558\,460\,403\,162\,112\,\alpha^{6}}{1\,607\,445}+\frac{5\,549\,297\,777\,929\,020\,165\,376\,\alpha^{7}}{1\,607\,445}+$$

$$\frac{13\,051\,884\,498\,572\,855\,976\,448\,\alpha^{8}}{4\,822\,335}+\frac{8\,305\,820\,524\,362\,639\,666\,176\,\alpha^{9}}{4\,822\,335}+$$

$$\frac{68\,498\,174\,502\,641\,524\,736\,\alpha^{10}}{76\,545}+\frac{1\,834\,410\,636\,425\,695\,993\,856\,\alpha^{11}}{4\,822\,335}+$$

$$\frac{4\,719\,098\,445\,947\,749\,888\,\alpha^{12}}{35\,721}+\frac{2\,220\,155\,660\,116\,050\,688\,\alpha^{13}}{59\,535}+\frac{100\,907\,796\,636\,762\,112\,\alpha^{14}}{11\,907}+$$

$$\frac{272\,733\,299\,909\,852\,672\,\alpha^{15}}{178\,605}+\frac{12\,686\,732\,176\,055\,296\,\alpha^{16}}{59\,535}+\frac{62\,941\,451\,669\,504\,\alpha^{17}}{2835}+$$

$$\frac{1\,535\,523\,074\,048\,\alpha^{18}}{945}+\frac{4\,693\,626\,880\,\alpha^{19}}{63}+1\,609\,728\,\alpha^{20}\Bigg)\,\mathrm{Seq}[2+\alpha]+$$

$$\Bigg(-\frac{949\,366\,809\,604\,448}{45\,927}-\frac{338\,346\,497\,840\,840\,312\,\alpha}{1\,607\,445}-\frac{2\,897\,395\,869\,916\,526\,980\,\alpha^{2}}{2\,893\,401}-$$

$$\frac{6\,119\,812\,153\,432\,046\,224\,\alpha^{3}}{2\,066\,715}-\frac{17\,671\,577\,051\,253\,711\,566\,\alpha^{4}}{2\,893\,401}-\frac{27\,041\,094\,094\,454\,567\,854\,\alpha^{5}}{2\,893\,401}-$$

$$\frac{10\,623\,999\,193\,584\,019\,850\,\alpha^{6}}{964\,467}-\frac{49\,413\,193\,259\,769\,616\,562\,\alpha^{7}}{4\,822\,335}-$$

$$\frac{15\,799\,414\,687\,723\,726\,438\,\alpha^{8}}{2\,066\,715}-\frac{66\,867\,205\,871\,259\,395\,558\,\alpha^{9}}{14\,467\,005}-\frac{6\,592\,067\,371\,084\,178\,362\,\alpha^{10}}{2\,893\,401}-$$

$$\frac{2\,655\,156\,049\,588\,420\,178\,\alpha^{11}}{2\,893\,401}-\frac{161\,624\,474\,785\,466\,282\,\alpha^{12}}{535\,815}-\frac{18\,489\,129\,368\,595\,274\,\alpha^{13}}{229\,635}-$$

$$\frac{9\,263\,660\,761\,116\,922\,\alpha^{14}}{535\,815}-\frac{1\,576\,234\,680\,856\,834\,\alpha^{15}}{535\,815}-\frac{69\,207\,811\,246\,676\,\alpha^{16}}{178\,605}-$$

$$\frac{64\,798\,937\,348\,\alpha^{17}}{1701}-\frac{213\,070\,160\,\alpha^{18}}{81}-\frac{21\,506\,768\,\alpha^{19}}{189}-2320\,\alpha^{20}\Bigg)\,\mathrm{Seq}[3+\alpha]+$$

$$\Bigg(\frac{128\,281\,339\,648}{6561}+\frac{6\,401\,079\,159\,232\,\alpha}{32\,805}+\frac{1\,876\,488\,439\,117\,984\,\alpha^{2}}{2\,066\,715}+\frac{37\,912\,717\,061\,108\,552\,\alpha^{3}}{14\,467\,005}+$$

$$\frac{76\,168\,037\,152\,315\,223\,\alpha^{4}}{14\,467\,005}+\frac{16\,783\,755\,155\,767\,457\,\alpha^{5}}{2\,143\,260}+\frac{259\,043\,013\,739\,819\,649\,\alpha^{6}}{28\,934\,010}+$$

$$\frac{1\,865\,978\,509\,650\,435\,997\,\alpha^{7}}{231\,472\,080}+\frac{28\,015\,706\,669\,496\,697\,\alpha^{8}}{4\,822\,335}+\frac{1\,567\,534\,832\,320\,365\,967\,\alpha^{9}}{462\,944\,160}+$$

$$\frac{185\,833\,531\,323\,837\,619\,\alpha^{10}}{115\,736\,040}+\frac{71\,863\,750\,696\,008\,881\,\alpha^{11}}{115\,736\,040}+\frac{2\,587\,442\,116\,841\,\alpha^{12}}{13\,230}+$$
$$\frac{2\,568\,010\,658\,385\,821\,\alpha^{13}}{51\,438\,240}+\frac{2\,924\,137\,637\,969\,\alpha^{14}}{285\,768}+\frac{406\,312\,520\,879\,\alpha^{15}}{244\,944}+$$
$$\left.\frac{148\,449\,300\,737\,\alpha^{16}}{714\,420}+\frac{659\,721\,157\,\alpha^{17}}{34\,020}+\frac{14\,389\,867\,\alpha^{18}}{11\,340}+\frac{9815\,\alpha^{19}}{189}+\alpha^{20}\right)\text{Seq}[4+\alpha]$$

Okay, the order of this recurrence is the same as what we have computed by creative telescoping; both are 4. So no need to continue.