Programs for “A note on balancing binomial coefficients”

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In “A note on balancing binomial coefficients” [1], I proved that equation

$$\binom{1}{5}+\binom{2}{5}+\cdots+\binom{x-1}{5}=\binom{x+1}{5}+\cdots+\binom{y}{5}, \tag{1}$$

where $y > x > 5$, has only one integral solution $(x, y) = (14, 15)$. Here I will describe the programs I used in that note. If without declaration, the code is written in Magma.

First, after some substitutions, we have $2u^3 - v^3 - 10u^2 + 8v^2 + 8u - 12v = 0$. Set $u = U/W$ and $v = V/W$, we then use the following code:

```
PP<U,V,W>:=ProjectiveSpace(Rationals(),2);
C:=Curve(PP,2*U^3-V^3-10*U^2*W+8*V^2*W+8*U*W^2-12*V*
   W^2);
P0:=C![0,0,1];
E,phi:=EllipticCurve(C,P0);
E,phi;
EE,pp:=SimplifiedModel(E);
EE,pp;
```

Now we obtain the minimal Weierstrass model

$$E : Y^2 = X^3 - X^2 - 30X + 81,$$

where

$$(X, Y) = \left(\frac{-4u - 17v + 58}{2u - 3v}, \frac{-146u^2 - 5v^2 + 686u - 188v}{(2u - 3v)^2}\right),$$

and

$$(u, v) = \left(\frac{18X^2 + 28X - 176}{8X^2 + 3XY - 31X - 17Y + 67}, \frac{12X^2 - 44X - 58Y + 270}{8X^2 + 3XY - 31X - 17Y + 67}\right).$$

To obtain the Mordell-Weil group $E(\mathbf{Q})$, we use

```
E:=EllipticCurve([0,-1,0,-30,81]);
MordellWeilGroup(E);
Generators(E);
```

We therefore have $E(\mathbf{Q}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ with generators $P_1 = (3, -3)$, $P_2 = (-6, 3)$ and $P_3 = (11, 31)$.

We then use the function `RealPeriod` to compute the fundamental real period ω of E. Compared with [2], we have $\omega = 2 *$ `RealPeriod(E)` $= 5.832948\ldots$. To compute $\phi(P_1)$, we use the function `EllipticLogarithm`. We then have $\phi(P_1) = -1 *$ `EllipticLogarithm(P1)/RealPeriod(E)` mod 1, where we set `P1:=E![3,-3,1]`.

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Likewise, we get $\phi(P_2)$ and $\phi(P_3)$. It seems that MAGMA is helpless to compute $\phi(Q_0)$, we thus use the following SAGE code:

```
sage: E=EllipticCurve([0,-1,0,-30,81])
sage: E.period_lattice().real_period()
sage: K.<a>=NumberField(x^3-2)
sage: EQ=EllipticCurve(K,[0,-1,0,-30,81])
sage: Q=EQ(7+2*a+3*a^2,-17-15*a-8*a^2)
sage: Q.elliptic_logarithm()
```

Note that here `E.period_lattice().real_period()` = `RealPeriod(E)`. Similarly, we have $\phi(Q_0) = -1 *$ `Q.elliptic_logarithm()/E.period_lattice().real_period()` mod 1.

To compute the Néron-Tate height pairing matrix, `HeightPairingMatrix(E)` is used. We then use the MATHEMATICA function `Eigenvalues` to get the least eigenvalue. Next, the Silverman's bound is given by `SilvermanBound(E)`.

Finally, to find all integral solutions of (1), we use the following code:

```
for i:=-11 to 11 do
  for j:=-11 to 11 do
    for k:=-11 to 11 do
      P:=i*P1+j*P2+k*P3;
      X:=P[1];Y:=P[2];
      if (8*X^2+3*X*Y-31*X-17*Y+67) ne 0 then
        u:=(18*X^2+28*X-176)/(8*X^2+3*X*Y-31*X-17*Y
            +67);
        v:=(12*X^2-44*X-58*Y+270)/(8*X^2+3*X*Y-31*X
            -17*Y+67);
        if (IsIntegral(u)) and (IsIntegral(v)) then
          print i,j,k,X,Y,u,v;
        end if;
      end if;
    end for;
  end for;
end for;
```

References

1. S. Chern, A note on balancing binomial coefficients, *Proc. Japan Acad. Ser. A Math. Sci.* **91** (2015), no. 8, 110–111.
2. R. J. Stroeker and B. M. M. de Weger, Solving elliptic Diophantine equations: the general cubic case, *Acta Arith.*, **87** (1999), no. 4, 339–365.

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