MATH 3070
Theory of Numbers

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What's NUMBER THEORY?

We are expected to learn the properties of

- *integers* $(0, \pm 1, \pm 2, \ldots)$
  - especially *primes* $(2, 3, 5, 7, 11, \ldots)$
- as well as mathematical objects made out of integers, e.g., *rationals*
- and generalizations of the integers, e.g., *algebraic integers*

Examples

Let's start with

$$9 + 16 = 25;$$

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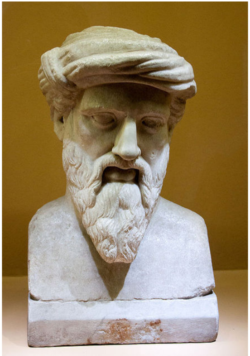
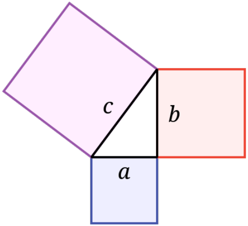
Something trickier

$$3^2 + 4^2 = 5^2.$$

An instance of the Pythagorean theorem.





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- B. What integers can be written as $x^2 + y^2$ with x and y integers? And how many such representations?

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- A. Can we determine all its integer solutions?
- B. What integers can be written as $x^2 + y^2$ with x and y integers? And how many such representations?
- C. What happens if we replace the square with an n-th power with $n \geq 3$

  $$x^n + y^n = z^n?$$

  Do we still have integer solutions?

Examples

A. All integer solutions of

$$x^2 + y^2 = z^2.$$

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Theorem

All integer solutions of

$$x^2 + y^2 = z^2$$

can be parameterized as

$$x = k \cdot (r^2 - s^2), \quad y = k \cdot 2rs, \quad z = k \cdot (r^2 + s^2).$$

Examples

B. Representation of

$$m = x^2 + y^2.$$

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Theorem (Pierre de Fermat)

A square-free integers m is representable as $x^2 + y^2$ with x and y integers if and only if n has no prime factors of the form $4k + 3$.

Examples

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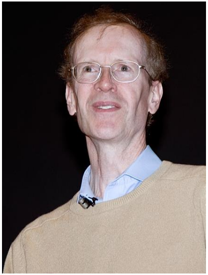
Theorem (Fermat's last theorem, proved by Andrew Wiles)

There is no integer solution with $x, y, z \neq 0$ to

$$x^n + y^n = z^n$$

for $n \geq 3$.



Number-theoretic problems

A. Multiplicative problems

- Divisors
- Primes, composites
- Arithmetic functions

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E.g.,

- *Prime number theorem*: The number of primes $\leq x$.
- *Gauss circle problem*: The number of integer lattice points there are in a circle centered at the origin and with radius r.

Number-theoretic problems

B. Additive problems

- Representation of integers

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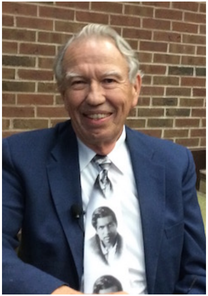
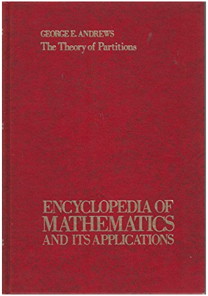
- Representation of integers

E.g.,

- *Sum of two squares*: Representation of $n = x^2 + y^2$.
- *Integer partitions*: Representation of n as a sum of nonincreasing positive integers.

$$5 = 4 + 1 = 3 + 2 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$





Number-theoretic problems

C. Diophantine equations

- Integer solutions to polynomial equations

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E.g.,

- *Fermat's last theorem*: $x^n + y^n = z^n$.
- *Pell's equation*: $x^2 + dy^2 = 1$ with d a non-square positive integer.
- *Sum of three cubes*: $x^3 + y^3 + z^3 = 33$.

  $8866128975287528^3 + (-8778405442862239)^3 + (-2736111468807040)^3 = 33.$

  This is the first known solution to the above Diophantine equation, discoved by Andrew Booker in 2019.

Number-theoretic problems

D. Diophantine approximations

- Approximation of real numbers by rational numbers

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E.g.,

- *The best Diophantine approximation*: Given a real number α, find the rational number p/q such that

$$\left|\alpha - \frac{p}{q}\right| \leq \left|\alpha - \frac{p'}{q'}\right|$$

for every rational number p'/q' with $0 < q' \leq q$.

Proofs: Why do we need PROOFS?

For Natural Sciences, especially Experimental Sciences, nobody can prove that a phenomenon or a rule is real in general.

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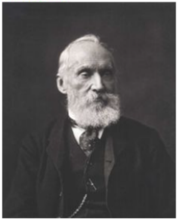
QUESTION. Will *Newtonian mechanics* expire in the scale of the UNIVERSE or ATOMS?

Proofs: Why do we need PROOFS?

Lord Kelvin's two CLOUDS in physics

Clouds on the Horizon

" Beauty and clearness of theory… Overshadowed by two clouds…"



Lord Kelvin

Baltimore Lectures

Johns Hopkins University

1900

The two clouds:

Failure of the Michelson – Morley experiment
$\rightarrow$ Einstein's Relativity

Failure of classical electrodynamics to describe thermal radiation
$\rightarrow$ Quantum Mechanics

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There are statements which can neither be proved nor disproved in an axiomatic system.

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Theorem (Gödel's incompleteness theorem)

There are statements which can neither be proved nor disproved in an axiomatic system.

But what can be proved or disproved is already very vast!

Proofs: Why do we care about PROOFS?

The existence of *large counterexamples*!

- The GCD (greatest common divisor) of $n^{17} + 9$ and $(n+1)^{17} + 9$:

$$\gcd(1^{17} + 9, 2^{17} + 9) = \gcd(10, 131081) = 1;$$
$$\gcd(2^{17} + 9, 3^{17} + 9) = \gcd(131081, 129140172) = 1;$$
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Is it true for all positive integers n that

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NO! But the *first* counterexample appears when

$n = 8424432925592889329288197322308900672459420460792433.$

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$$\pi(x) := \text{the number of primes} \leq x,$$
$$\mathrm{li}(x) := \int_0^x \frac{dt}{\log t}.$$

Prime number theorem. $\pi(x) \sim \mathrm{li}(x)$. I.e.,

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{li}(x)} = 1.$$

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  — It is negative for all small x.

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  - Skewes proved that such x is smaller than

    $$e^{e^{e^{e^{7.705}}}}.$$

  - It is believed that such x is around 10^{316}.

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A BELIEF IS *NEVER* A PROOF.

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$$\begin{aligned} 2S_n &= (1+n) + (2 + (n-1)) + \cdots + (n+1) \\ &= (n+1) + (n+1) + \cdots (n+1) \qquad [n \text{ copies of } (n+1)] \\ &= n(n+1). \end{aligned}$$

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Proof.

- Is the statement TRUE for $n = 1$?
  $$1^2 = 1 = \frac{1(1+1)(2 \times 1 + 1)}{6}.$$

$\square$

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$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof.

- Assume that the statement is true for some $n = k \geq 1$:

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

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$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)(k+2)(2(k+1)+1)}{6}. \end{aligned}$$

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- Conclude that the statement is true for all positive integers n.

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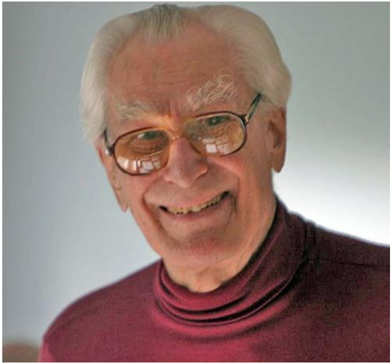
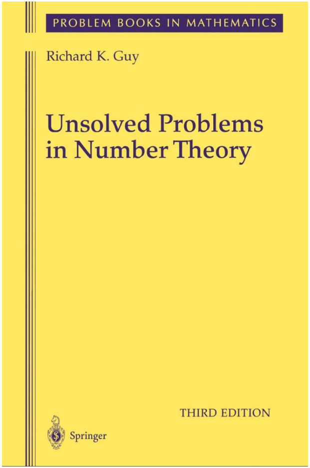
- Assume that no boxes contain at least 2 balls.
- Then the total number of balls is $\leq N \times 1 = N$.
- But there are $N+1$ balls, thereby leading to a contradiction.
- So our assumption is false — There must be some box with at least 2 balls.

$\square$

Unsolved Problems in Number Theory

Richard K. Guy, *Unsolved Problems in Number Theory, Third edition*, Springer-Verlag, New York, 2004.





We will switch back to the traditional
"chalk-and-blackboard"
style in the rest of this semester.