

Further results on biases in integer partitions

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Abstract. Let $p_{a,b,m}(n)$ be the number of integer partitions of n with more parts congruent to a modulo m than parts congruent to b modulo m . We prove that $p_{a,b,m}(n) \geq p_{b,a,m}(n)$ whenever $1 \leq a < b \leq m$. We also propose some conjectures concerning series with nonnegative coefficients in their expansions.

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1. Introduction

In analogy to *Chebyshev's bias* [3] concerning the excess of the number of primes of the form $4k + 3$ over the number of primes of the form $4k + 1$, B. Kim, E. Kim and Lovejoy [5] introduced a phenomenon called *parity bias* for integer partitions.

Theorem 1.1 (B. Kim, E. Kim and Lovejoy). *Let $p_o(n)$ (resp. $p_e(n)$) denote the number of integer partitions of n with more odd parts than even parts (resp. with more even parts than odd parts). Then*

$$p_o(n) \geq p_e(n).$$

This phenomenon is called “parity bias” for integer partitions.

Recently, B. Kim and E. Kim [4] went on to investigate this phenomenon in a more general setting. Let us first adopt their notation.

Definition 1.1. We denote by $p_{a,b,m}(n)$ the number of partitions of n with more parts congruent to a modulo m than parts congruent to b modulo m .

Making use of the above notation, we have $p_o(n) = p_{1,2,2}(n)$ and $p_e(n) = p_{2,1,2}(n)$ and therefore arrive at the inequality $p_{1,2,2}(n) \geq p_{2,1,2}(n)$ from Theorem 1.1. Similar phenomena shown in [4] also include inequalities as follows.

Theorem 1.2 (B. Kim and E. Kim). *Let $m \geq 2$ be an integer. Then*

$$\begin{aligned} p_{1,m,m}(n) &\geq p_{m,1,m}(n), \\ p_{1,m-1,m}(n) &\geq p_{m-1,1,m}(n). \end{aligned}$$

Our object here is to extend the above results for general $p_{a,b,m}(n)$.

Theorem 1.3. *Let $m \geq 2$ be an integer. For any two integers a and b with $1 \leq a < b \leq m$, we have*

$$p_{a,b,m}(n) \geq p_{b,a,m}(n). \tag{1.1}$$

We separate this theorem into two cases. First, we prove the case $(a,b) \neq (1,2)$ using q -series manipulations. Then we provide an injective proof for $(a,b) = (1,2)$.

2. Case $(a, b) \neq (1, 2)$

Let us first recall the notation of q -Pochhammer symbols: for $n \in \mathbb{N} \cup \{\infty\}$,

$$(A; q)_n := \prod_{k=0}^{n-1} (1 - Aq^k),$$

$$(A_1, A_2, \dots, A_m; q)_n := (A_1; q)_n (A_2; q)_n \cdots (A_m; q)_n.$$

Next, given an integer partition λ , we denote by $|\lambda|$ the sum of parts in λ and by $\#_{a,m}(\lambda)$ the number of parts in λ that are congruent to a modulo m . Let \mathcal{P} be the set of integer partitions.

Our starting point is the following trivial trivariate generating function:

$$\sum_{\lambda \in \mathcal{P}} x^{\#_{a,m}(\lambda)} y^{\#_{b,m}(\lambda)} q^{|\lambda|} = \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \frac{1}{(xq^a, yq^b; q^m)_\infty}, \quad (2.1)$$

provided that $1 \leq a, b \leq m$ and $a \neq b$.

We are then led to the following lemma.

Lemma 2.1. *Let $1 \leq a, b \leq m$ and $a \neq b$. We have*

$$\sum_{n \geq 0} p_{a,b,m}(n) q^n = \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{\substack{i, j \geq 0 \\ i > j}} \frac{q^{ai+bj}}{(q^m; q^m)_i (q^m; q^m)_j}. \quad (2.2)$$

Proof. Recall Euler's first identity [2, p. 19, (2.2.5)]:

$$\frac{1}{(z; q)_\infty} = \sum_{n \geq 0} \frac{z^n}{(q; q)_n}. \quad (2.3)$$

Setting $y = x^{-1}$ in (2.1) yields

$$\begin{aligned} & \sum_{\lambda \in \mathcal{P}} x^{\#_{a,m}(\lambda) - \#_{b,m}(\lambda)} q^{|\lambda|} \\ &= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \frac{1}{(xq^a, x^{-1}q^b; q^m)_\infty} \\ &= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{i \geq 0} \frac{x^i q^{ai}}{(q^m; q^m)_i} \sum_{j \geq 0} \frac{x^{-j} q^{bj}}{(q^m; q^m)_j} \\ & \quad (\text{by (2.3) with } q \mapsto q^m \text{ and } z \mapsto xq^a \text{ and } z \mapsto x^{-1}q^b, \text{ respectively}) \\ &= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{i, j \geq 0} \frac{x^{i-j} q^{ai+bj}}{(q^m; q^m)_i (q^m; q^m)_j}. \end{aligned}$$

Noticing that $p_{a,b,m}(n)$ counts the number of partitions λ of n such that $\#_{a,m}(\lambda) > \#_{b,m}(\lambda)$, we must single out terms in the above with positive exponents in x and therefore terms with $i - j > 0$. The desired result immediately follows. \square

Now, we are in a position to prove Theorem 1.3 for $(a, b) \neq (1, 2)$.

Proof of Theorem 1.3 for $(a, b) \neq (1, 2)$. Recall that $1 \leq a < b \leq m$. The following is a simple consequence of Lemma 2.1:

$$\sum_{n \geq 0} (p_{a,b,m}(n) - p_{b,a,m}(n)) q^n$$

$$\begin{aligned}
&= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{\substack{i, j \geq 0 \\ i > j}} \left(\frac{q^{ai+bj}}{(q^m; q^m)_i (q^m; q^m)_j} - \frac{q^{bi+aj}}{(q^m; q^m)_i (q^m; q^m)_j} \right) \\
&= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{\substack{i, j \geq 0 \\ i > j}} \frac{q^{ai+bj} (1 - q^{a(j-i)+b(i-j)})}{(q^m; q^m)_i (q^m; q^m)_j} \\
&= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{a(j+k)+bj} (1 - q^{(b-a)k})}{(q^m; q^m)_j (q^m; q^m)_{j+k}}. \quad (\text{with } k = i - j)
\end{aligned}$$

We then consider two subcases.

Subcase I. $a \neq 1$. Noticing that $(b-a)k$ is always a positive integer, we may factor $1 - q^{(b-a)k}$ as $(1-q)(1+q+q^2+\dots+q^{(b-a)k-1})$. Thus,

$$\begin{aligned}
&\sum_{n \geq 0} (p_{a,b,m}(n) - p_{b,a,m}(n)) q^n \\
&= \frac{(1-q)(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{a(j+k)+bj} (1+q+q^2+\dots+q^{(b-a)k-1})}{(q^m; q^m)_j (q^m; q^m)_{j+k}}.
\end{aligned}$$

Apparently, the Taylor expansion of the double series in the above has nonnegative coefficients. For the infinite product in the above, we have, as $2 \leq a < b \leq m$,

$$\frac{(1-q)(q^a, q^b; q^m)_\infty}{(q; q)_\infty} = \frac{(q^a, q^b; q^m)_\infty}{(q^2; q)_\infty},$$

which also has nonnegative coefficients in its series expansion. We therefore conclude that $p_{a,b,m}(n) \geq p_{b,a,m}(n)$ for $a \neq 1$.

Subcase II. $a = 1$ and $b \neq 2$. We have

$$\begin{aligned}
&\sum_{n \geq 0} (p_{1,b,m}(n) - p_{b,1,m}(n)) q^n \\
&= \frac{(q, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{(j+k)+bj} (1 - q^{(b-1)k})}{(q^m; q^m)_j (q^m; q^m)_{j+k}}.
\end{aligned}$$

Notice that $b > a = 1$. This time we should factor $1 - q^{(b-1)k}$ as $(1 - q^{b-1})(1 + q^{b-1} + \dots + q^{(b-1)(k-1)})$. Thus,

$$\begin{aligned}
&\sum_{n \geq 0} (p_{1,b,m}(n) - p_{b,1,m}(n)) q^n \\
&= \frac{(1 - q^{b-1})(q, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{(j+k)+bj} (1 + q^{b-1} + \dots + q^{(b-1)(k-1)})}{(q^m; q^m)_j (q^m; q^m)_{j+k}}.
\end{aligned}$$

Similarly, the double series in the above can be expanded as a nonnegative series in q . Also, as $b \neq 2$, we have $1 < b-1 < b \leq m$. This implies that the infinite product part in the above is also a nonnegative series in q . Therefore, $p_{1,b,m}(n) \geq p_{b,1,m}(n)$ for $b \neq 2$. \square

3. Case $(a, b) = (1, 2)$

When $(a, b) = (1, 2)$, it looks like a q -theoretic proof is painfully difficult. Therefore, we consider this case in a combinatorial manner. First, for $d \in \mathbb{Z}$, we define

$$\mathcal{P}_d(n) = \mathcal{P}_d^{(m)}(n) := \{\lambda \in \mathcal{P} : |\lambda| = n \text{ and } \sharp_{1,m}(\lambda) - \sharp_{2,m}(\lambda) = d\}.$$

Then

$$p_{1,2,m}(n) = \sum_{d \geq 1} \text{card } \mathcal{P}_d(n), \quad (3.1)$$

$$p_{2,1,m}(n) = \sum_{d \leq -1} \text{card } \mathcal{P}_d(n), \quad (3.2)$$

Our object is to show the following inequalities, from which our desired result $p_{1,2,m}(n) \geq p_{2,1,m}(n)$ follows as a direct consequence if we make use of the above two relations.

Theorem 3.1. *Let $m \geq 3$ be an integer. For $k \geq 0$,*

$$\text{card } \mathcal{P}_{-(km+1)}(n) \leq \text{card } \mathcal{P}_{km+2}(n), \quad (3.3)$$

$$\text{card } \mathcal{P}_{-(km+2)}(n) \leq \text{card } \mathcal{P}_{km+1}(n), \quad (3.4)$$

$$\text{card } \mathcal{P}_{-(km+r)}(n) \leq \text{card } \mathcal{P}_{km+r}(n), \quad (3.5)$$

where $3 \leq r \leq m$ in the third inequality.

Proof. We simply construct injections $\mathcal{P}_{-d}(n) \hookrightarrow \mathcal{P}_{d^*}(n)$ for $d = km + r > 0$ with $1 \leq r \leq m$ and

$$d^* = \begin{cases} km + 2 & \text{if } r = 1, \\ km + 1 & \text{if } r = 2, \\ km + r & \text{if } 3 \leq r \leq m. \end{cases}$$

Given any partition λ , we start with the following process.

Process (I). We replace any part in λ that is congruent to 1 modulo m , say $um + 1$, by $um + 2$ and replace any part in λ that is congruent to 2 modulo m , say $vm + 2$, by $vm + 1$. The resulting partition is called λ^* .

Now, if $\lambda \in \mathcal{P}_{-d}(n)$, then $\sharp_{1,m}(\lambda) - \sharp_{2,m}(\lambda) = -d$. Also, trivially,

$$|\lambda^*| = |\lambda| - d = n - d.$$

Thus, to arrive at a partition of size n , we need to append some additional parts that sum to d . We have three subcases.

Subcase I. $3 \leq r \leq m$. We append a part of size d to λ^* and call the new partition λ^{**} . Since $d \not\equiv 1, 2 \pmod{m}$, we have

$$\begin{aligned} \sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= \sharp_{1,m}(\lambda^*) - \sharp_{2,m}(\lambda^*) \\ &= \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) && \text{(by Process (I))} \\ &= -(-d) \\ &= d^*. \end{aligned}$$

Thus, $\lambda^{**} \in \mathcal{P}_{d^*}(n)$.

Subcase II. $r = 1$. Recall that $d = km + 1$. We append a part of size 1 and a part of size km to λ^* and call the new partition λ^{**} . Notice that $km \equiv 0 \not\equiv 1, 2 \pmod{m}$ for $m \geq 3$. Thus,

$$\begin{aligned}
 \sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= (1 + \sharp_{1,m}(\lambda^*)) - \sharp_{2,m}(\lambda^*) \\
 &= 1 + \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) && \text{(by Process (I))} \\
 &= 1 - (-d) \\
 &= km + 2 \\
 &= d^*,
 \end{aligned}$$

which implies that $\lambda^{**} \in \mathcal{P}_{d^*}(n)$.

Subcase III. $r = 2$. Recall that $d = km + 2$. We append a part of size 2 and a part of size km to λ^* and call the new partition λ^{**} . We also have $km \equiv 0 \not\equiv 1, 2 \pmod{m}$ for $m \geq 3$. Thus,

$$\begin{aligned}
 \sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= \sharp_{1,m}(\lambda^*) - (1 + \sharp_{2,m}(\lambda^*)) \\
 &= -1 + \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) && \text{(by Process (I))} \\
 &= -1 - (-d) \\
 &= km + 1 \\
 &= d^*,
 \end{aligned}$$

and therefore, $\lambda^{**} \in \mathcal{P}_{d^*}(n)$.

Lastly, it is straightforward to verify that the map $\lambda \mapsto \lambda^{**}$ is injective. \square

Proof of Theorem 1.3 for $(a, b) = (1, 2)$. For $m = 2$, see Theorem 1.1 due to B. Kim, E. Kim and Lovejoy. For $m \geq 3$, we have

$$\begin{aligned}
 p_{2,1,m}(n) &= \sum_{d \geq 1} \text{card } \mathcal{P}_{-d}(n) && \text{(by (3.2))} \\
 &= \sum_{k \geq 0} \text{card } \mathcal{P}_{-(km+1)}(n) + \sum_{k \geq 0} \text{card } \mathcal{P}_{-(km+2)}(n) \\
 &\quad + \sum_{3 \leq r \leq m} \sum_{k \geq 0} \text{card } \mathcal{P}_{-(km+r)}(n) \\
 &\leq \sum_{k \geq 0} \text{card } \mathcal{P}_{km+2}(n) + \sum_{k \geq 0} \text{card } \mathcal{P}_{km+1}(n) \\
 &\quad + \sum_{3 \leq r \leq m} \sum_{k \geq 0} \text{card } \mathcal{P}_{km+r}(n) && \text{(by Theorem 3.1)} \\
 &= \sum_{d \geq 1} \text{card } \mathcal{P}_d(n) \\
 &= p_{1,2,m}(n). && \text{(by (3.1))}
 \end{aligned}$$

This is exactly what we need. \square

4. Closing remarks

Following Section 2, the case $(a, b) = (1, 2)$ of Theorem 1.3 is equivalent to the nonnegativity of

$$\frac{(q, q^2; q^m)_\infty}{(q; q)_\infty} \sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j (q^m; q^m)_{j+k}}, \quad (4.1)$$

that is, its series expansion has nonnegative coefficients. Although we do not find a q -theoretic proof of this fact, our numerical calculations indicate the following conjecture.

Conjecture 4.1. For $m \geq 2$, the double series

$$\sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j (q^m; q^m)_{j+k}} \quad (4.2)$$

has nonnegative coefficients in its expansion.

Notice that

$$\sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j (q^m; q^m)_{j+k}} = \sum_{j \geq 0} \frac{q^{3j}}{(q^m; q^m)_j (q^m; q^m)_j} \sum_{k \geq 0} \frac{q^k(1-q^k)}{(q^{(j+1)m}; q^m)_k}.$$

Regarding the inner series, we also have a more surprising conjecture.

Conjecture 4.2. For $m, s \geq 1$,

$$\sum_{k \geq 0} \frac{q^k(1-q^k)}{(q^s; q^m)_k} \quad (4.3)$$

has nonnegative coefficients in its expansion.

Here the case $s = m$ is to some extent easier.

Proof of Conjecture 4.2 for $s = m$. We have

$$\begin{aligned} \sum_{k \geq 0} \frac{q^k(1-q^k)}{(q^m; q^m)_k} &= \sum_{k \geq 0} \frac{q^k}{(q^m; q^m)_k} - \sum_{k \geq 0} \frac{q^{2k}}{(q^m; q^m)_k} \\ &= \frac{1}{(q; q^m)_\infty} - \frac{1}{(q^2; q^m)_\infty} \quad (\text{by (2.3)}) \\ &= \sum_{n \geq 0} \rho_{1,m}(n) q^n - \sum_{n \geq 0} \rho_{2,m}(n) q^n, \end{aligned}$$

where for $i = 1$ or 2 , we denote by $\rho_{i,m}(n)$ the number of partitions of n with parts of the form $km + i$ with $k \geq 0$.

Now we recall a result due to Andrews [1, Theorem 3]:

Let $S = \{a_i\}_{i \geq 1}$ and $T = \{b_i\}_{i \geq 1}$ be two strictly increasing sequences of positive integers such that $b_1 = 1$ and $a_i \geq b_i$ for all i . Then for any $n \geq 0$,

$$\rho_T(n) \geq \rho_S(n),$$

where $\rho_S(n)$ (resp. $\rho_T(n)$) denotes the number of partitions of n into parts taken from S (resp. T).

By the above theorem, we immediately have $\rho_{1,m}(n) \geq \rho_{2,m}(n)$ for all n . Thus, (4.3) is a nonnegative series in q when $s = m$. \square

References

1. G. E. Andrews, On a partition problem of H. L. Alder, *Pacific J. Math.* **36** (1971), 279–284.
2. G. E. Andrews, *The theory of partitions*, Reprint of the 1976 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. xvi+255 pp.
3. P. L. Chebyshev, Lettre de M. le Professeur Tch bychev   M. Fuss sur un nouveaux th or me relatif aux nombres premiers contenus dans les formes $4n + 1$ et $4n + 3$, *Bull. Classe Phys. Acad. Imp. Sci. St. Petersburg* **11** (1853), 208.
4. B. Kim and E. Kim, Biases in integer partitions, *Bull. Aust. Math. Soc.*, in press.
5. B. Kim, E. Kim, and J. Lovejoy, Parity bias in partitions, *European J. Combin.* **89** (2020), 103159, 19 pp.

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