The Pennsylvania State University
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THE WORLD OF P AND Q:

CONGRUENCES, IDENTITIES AND ASYMPTOTICS

A Dissertation in
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Abstract

This thesis is devoted to providing some "travel tips" that arise from my personal visit in the world of p(artitions) and q(-series).

In the first part, we will focus on partition congruences, especially from an elementary perspective. We first give a completely elementary proof of an infinite family of congruences modulo powers of 5 for the number of partitions of n into distinct parts. As a by-product, we also consider some eta-quotient representations concerning the Rogers–Ramanujan continued fraction.

In the second part, our attention is turned to identities. The first two chapters in this part are devoted to partition identities — one treats weighted partition rank and crank moments and the other investigates partitions with bounded part differences. Then in a series of three chapters, a general theory of span one linked partition ideals will be presented. We start from several conjectures of Kanade and Russell and then link this theory with directed graphs. A comprehensive example on Gleißberg's identity will finally be discussed. The last chapter in this part will be devoted to analytic identities of Rogers–Ramanujan type with manipulations of basic hypergeometric series heavily involved.

In the third part, asymptotic aspects of integer partitions will be investigated. We first use square-root partitions into distinct parts to illustrate a refined Meinardus-type approach. In the next three chapters, we will focus on modular infinite products that concern either Dedekind eta function or Jacobi theta function with the assistance of Rademacher's circle method. Finally, we will study nonmodular infinite products that are related to a conjecture of Seo and Yee.

In the last part, we will leave for another world of p, that is, the world of $p$atterns in inversion sequences. We mainly focus on two recent conjectures, one of which on 0012-avoidance is due to Lin and Ma and the other on the avoidance of triples of binary relations is due to Lin.

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Added on June 15 in McAllister.

Chapter 1

Introduction

1.1 Notation and Terminology

1.1.1 The Theory of Partitions

The theory of partitions was given to birth in a letter [119] from Leibniz to Bernoulli in September 1674, in which Leibniz asked for the number of representations of a positive integer n as a sum of positive integers, which is now called the number of integer partitions of n, usually denoted by $p(n)$, if the order of the summands is not taken into account.

Definition 1.1.1. An *integer partition* of a positive integer n is a non-increasing sequence of positive integers whose sum equals n. These summands are called *parts* of this partition. We usually use $p(n)$ to denote the number of partitions of n. Conventionally, we also put $p(0) = 1$, which means that 0 has an *empty partition* $\emptyset$ containing no parts.

Notation 1.1.1. Given a partition λ, we denote by $\sharp(\lambda)$ and $|\lambda|$ the number of parts and the sum of parts of λ, respectively.

For example, the partitions of 0, 1, ..., 5 are listed in Table 1.1.

Of course, we may impose or lessen restrictions on the parts of partitions. One important example is that the parts are required to be pairwise distinct.

Definition 1.1.2. A *distinct partition* is a partition such that its parts are pairwise distinct. We will use $p_D(n)$ to denote the number of distinct partitions of n.

In Table 1.2, we will delete those partitions in Table 1.1 with repeated parts and leave all distinct partitions.

Another important variant of partitions is called overpartitions.

Definition 1.1.3. An *overpartition* is a partition where the first occurrence of each distinct part may be overlined. We will use $\overline{p}(n)$ to denote the number of overpartitions of n.

Table 1.1. Partitions of 0, 1, ..., 5

n	$p(n)$	partitions of n
0	1	$\emptyset$
1	1	1
2	2	$2,\ 1+1$
3	3	$3,\ 2+1,\ 1+1+1$
4	5	$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$
5	7	$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,$ $1+1+1+1+1$

Table 1.2. Distinct partitions of 0, 1, ..., 5

n	$p_D(n)$	distinct partitions of n
0	1	$\emptyset$
1	1	1
2	1	$2,\ \cancel{1+1}$
3	2	$3,\ 2+1,\ \cancel{1+1+1}$
4	2	$4,\ 3+1,\ \cancel{2+2},\ \cancel{2+1+1},\ \cancel{1+1+1+1}$
5	3	$5,\ 4+1,\ 3+2,\ \cancel{3+1+1},\ \cancel{2+2+1},\ \cancel{2+1+1+1},$ $\cancel{1+1+1+1+1}$

For example, 4 has 14 overpartitions:

$$4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ \overline{2}+2,$$
$$2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1.$$

The name of overpartitions was given by Corteel and Lovejoy [72] in 2004, but they have already been extensively used by Andrews in 1967 [5], by Joichi and Stanton in 1987 [105] and by Corteel in 2003 [71].

Regarding combinatorial aspects of partitions, an important concept is *Ferrers diagram*, which is also known as *Young diagram*.

Definition 1.1.4. Given an integer partition λ, its *Ferrers diagram* is a diagram of squares aligned in the upper-left corner such that the n-th row has the same number of squares as the n-th part of λ.

Using Ferrers diagrams, we are able to define two other important combinatorial objects.

Definition 1.1.5. Given an integer partition λ, its *conjugate* $\overline{\lambda}$ is the partition whose Ferrers diagram can be obtained by flipping the Ferrers diagram of λ along the main diagonal.

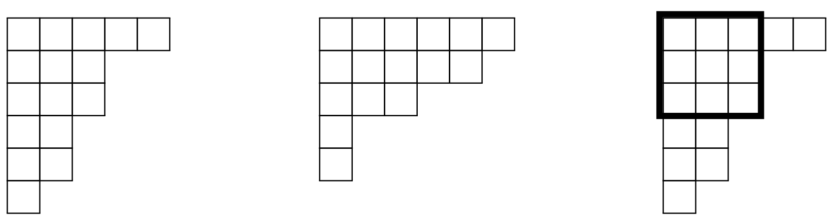
Definition 1.1.6. Given an integer partition λ, its *Durfee square* is the largest square contained in its Ferrers diagram.

For example, the Ferrers diagram of $5+3+3+2+2+1$ is shown in Figure 1.1(a). It can be seen from Figure 1.1(b) that its conjugate is $6+5+3+1+1$. Finally, it has a Durfee square of size 3, which is given in Figure 1.1(c).

Figure 1.1. Ferrers diagram

$\lambda = 5+3+3+2+2+1$ has conjugate $\overline{\lambda} = 6+5+3+1+1$ and a Durfee square of size 3.

(a). λ (b). $\overline{\lambda}$ (c). Durfee square



Of course, the theory of partitions is rather vast. To end this section, I will not hesitate to recommend Andrews' monograph: *The theory of partitions* [12].

1.1.2 The Theory of q-Series

Before going into details of the theory of q-series, it is necessary to introduce some notations.

Notation 1.1.2 (q-Pochhammer symbols)**.** Let $q \in \mathbb{C}$ be such that $|q| < 1$. Let $n \in \mathbb{Z}$.

$$(A;q)_n := \begin{cases} \prod_{k=0}^{n-1} (1 - Aq^k) & \text{if } n \geq 0, \\ 1/(Aq^n;q)_{-n} & \text{if } n < 0, \end{cases}$$

$$(A;q)_\infty := \prod_{k \geq 0} (1 - Aq^k),$$

$$(A_1, A_2 \dots, A_m; q)_n := (A_1; q)_n (A_2; q)_n \cdots (A_m; q)_n,$$
$$(A_1, A_2, \dots, A_m; q)_\infty := (A_1; q)_\infty (A_2; q)_\infty \cdots (A_m; q)_\infty,$$
$$\begin{pmatrix} A_1, A_2, \dots, A_{m_A} \\ B_1, B_2, \dots, B_{m_B} \end{pmatrix}_n := \frac{(A_1; q)_n (A_2; q)_n \cdots (A_{m_A}; q)_n}{(B_1; q)_n (B_2; q)_n \cdots (B_{m_B}; q)_n}$$

and

$$\begin{pmatrix} A_1, A_2, \dots, A_{m_A} \\ B_1, B_2, \dots, B_{m_B} \end{pmatrix}_\infty := \frac{(A_1; q)_\infty (A_2; q)_\infty \cdots (A_{m_A}; q)_\infty}{(B_1; q)_\infty (B_2; q)_\infty \cdots (B_{m_B}; q)_\infty}.$$

With these notations, we may also define the basic hypergeometric (or q-hypergeometric) function ${}_r\phi_s$ and the bilateral basic hypergeometric (or bilateral q-hypergeometric) function ${}_r\psi_s$, which lie in the core of the theory of q-series.

Notation 1.1.3 (q-Hypergeometric function ${}_r\phi_s$)**.**

$${}_r\phi_s \left(\begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_s \end{matrix}; q, z \right) := \sum_{n \geq 0} \left(\begin{matrix} A_1, A_2, \dots, A_r \\ q, B_1, B_2, \dots, B_s \end{matrix}; q \right)_n \left((-1)^n q^{\binom{n}{2}} \right)^{s-r+1} z^n.$$

Notation 1.1.4 (Bilateral q-hypergeometric function ${}_r\psi_s$)**.**

$${}_r\psi_s \left(\begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_s \end{matrix}; q, z \right) := \sum_{n=-\infty}^{\infty} \left(\begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_s \end{matrix}; q \right)_n \left((-1)^n q^{\binom{n}{2}} \right)^{s-r} z^n.$$

Also, the q-binomial coefficients are important.

Notation 1.1.5 (q-Binomial coefficient)**.**

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} & \text{if } 0 \leq m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Roughly speaking, the theory of q-series deals with identities. It seems that the first important monograph is Bailey's 1935 book [32], but according to Richard Askey, the best monograph should be George Gasper's copy of Bailey's book, which is now published as [83]. For a selection of q-series identities, Andrews' chapter (Chapter 17) of the "*NIST handbook of mathematical functions*" [16] provides a good reference.

Let me excerpt several important identities as instances.

Theorem 1.1.1 (Eluer's first sum)**.** *For* $|z| < 1$*,*

$$ {}_1\phi_0\left(\begin{matrix}0\\ -\end{matrix};q,z\right) = \sum_{n\geq 0}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}. \tag{1.1.1}$$

Theorem 1.1.2 (Eluer's second sum)**.** *For* $|z| < 1$*,*

$$ {}_0\phi_0\left(\begin{matrix}-\\ -\end{matrix};q,z\right) = \sum_{n\geq 0}\frac{(-1)^n q^{\binom{n}{2}} z^n}{(q;q)_n} = (z;q)_\infty. \tag{1.1.2}$$

Theorem 1.1.3 (q-Gauß sum)**.** *For* $|c| < |ab|$*,*

$$ {}_2\phi_1\left(\begin{matrix}a,b\\ c\end{matrix};q,\frac{c}{ab}\right) = \sum_{n\geq 0}\left(\begin{matrix}a,b\\ q,c\end{matrix};q\right)_n \left(\frac{c}{ab}\right)^n = \left(\begin{matrix}c/a,c/b\\ c,c/(ab)\end{matrix};q\right)_\infty. \tag{1.1.3}$$

Theorem 1.1.4 (q-Binomial theorem)**.** *For* $N \geq 0$*,*

$$ {}_1\phi_0\left(\begin{matrix}q^{-N}\\ -\end{matrix};q,z\right) = (zq^{-N};q)_N. \tag{1.1.4}$$

1.1.2.1 Generating Functions

Generating functions create a fantastic kingdom where the worlds of p and q meet. Let us begin with the first deep result (in the 16th century) due to Euler [78].

Theorem 1.1.5 (Generating function of $p(n)$)**.** *We have*

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1.1.5}$$

This identity can be understood as follows.

$$\begin{aligned}\frac{1}{(q;q)_\infty} &= \prod_{k\geq 1}\frac{1}{1-q^k}\\ &= \prod_{k\geq 1}\left(1+q^k+q^{2k}+q^{3k}+\cdots\right).\end{aligned}$$

Now the term q^{mk} can be treated in the sense that the part k appears m times. Hence, if we expand the infinite product, the coefficient of q^n exactly enumerate the number of partitions of n.

It is notable that if we expand the reciprocal of the infinite product in (1.1.5), namely, $(q;q)_\infty$, one has

$$(q;q)_\infty = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots .$$

If one pays attention to the powers, then it could be observed that they are simply pentagonal numbers. In [78], Euler conjectured an identity based on this observation, which was proved by himself a couple of years later and is now known as Euler's Pentagonal Number Theorem.

Theorem 1.1.6 (Euler's Pentagonal Number Theorem)**.** *We have*

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{1.1.6}$$

It follows by multiplying (1.1.5) and (1.1.6) that $p(n)$ can be computed recursively by

$$p(n) = p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots .$$

Next, q-series sometimes help us reduce the difficulty of proving partition identities. The simplest example is still due to Euler [78].

Theorem 1.1.7. *For $n \geq 0$, the number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts.*

Proof. Let $p_o(n)$ denote the number of partitions of n into odd parts. Then,

$$\begin{aligned}
\sum_{n\geq 0} p_o(n)q^n &= \prod_{k\geq 1} \frac{1}{1-q^{2k-1}} \\
&= \prod_{k\geq 1} \frac{1}{1-q^{2k-1}} \frac{1-q^{2k}}{1-q^{2k}} \\
&= \prod_{k\geq 1} \frac{1-q^{2k}}{1-q^k} \\
&= \prod_{k\geq 1} \frac{(1-q^k)(1+q^k)}{1-q^k} \\
&= \prod_{k\geq 1} (1+q^k).
\end{aligned}$$

This is just the generating function for distinct partitions. $\square$

Finally, partitions can help us understand or even prove q-series identities. Let us use Euler's first sum as an example. Here we shall prove the special case:

$$\sum_{k\geq 0}\frac{q^k}{(q;q)_k}=\frac{1}{(q;q)_\infty}.$$

Proof. We enumerate partitions with exactly k parts. If we take the conjugate of such a partition, then it has the largest part of size k. Below the largest part, we have a partition with largest part not exceeding k. Hence, such partitions can be enumerated by the generating function

$$\frac{q^k}{(q;q)_k}.$$

Finally, we sum up $k \geq 0$ to arrive at the desired identity. $\square$

1.1.3 The Theory of Numbers

When studying the asymptotic behavior of certain sequences or complex functions, we require the following traditional notations.

First, we have the Vinogradov notations introduced by I.M. Vinogradov in the 1930s.

Notation 1.1.6 (Vinogradov notations)**.** We say $f(x) \ll g(x)$ if there exists an absolute constant C such that $|f(x)| \leq Cg(x)$. If the constant C depends on some variables, then we attach a subscript and write $f(x) \ll_{\text{variables}} g(x)$. Likewise, we write $f(x) \gg g(x)$ if $g(x) \ll f(x)$.

The Bachmann–Landau notations are also useful.

Notation 1.1.7 (Bachmann–Landau notations)**.** The big-O notation[1] is defined in the usual way: $f(x) = O(g(x))$ means that $f(x) \ll g(x)$. Again, subscripts are allowed as the Vinogradov notations. Also, we have the small-o notation: $f(x) = o(g(x))$ means that $\lim f(x)/g(x) = 0$. Further, if $\lim f(x)/g(x) = 1$, then we write $f(x) \sim g(x)$.

1.1.4 The Theory of Patterns

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $[n] := \{1, 2, \ldots, n\}$. One of its most natural encodings is known as its inversion sequence.

[1]Here O stands for "Ordnung", which means "order of" in German; see [31].

Definition 1.1.7. The *inversion sequence* of $\pi = \pi_1\pi_2\cdots\pi_n$ is a sequence $e_1e_2\cdots e_n$ of length n where for each i, e_i is the number of integers larger than i that precede π_i in π. That is,

$$e_i = \sharp\{1 \le j < i : \pi_j > \pi_i\}.$$

We can see from the above definition that for each i, we always have $0 \le e_i \le i-1$. Notice that given two permutations, their inversion sequences are different. Since there are exactly $n!$ permutations of $[n]$, we conclude that there is a bijection between $\mathfrak{S}_n$, the set of permutations of $[n]$, and the set of sequences of length n:

$$\{e_1e_2\cdots e_n : 0 \le e_i \le i-1 \text{ for all } i \in [n]\},$$

as the cardinality of the above set is also $n!$. Therefore, we have the second definition of inversion sequences.

Definition 1.1.8. A sequence $e = e_1e_2\cdots e_n$ of natural numbers is called an *inversion sequence* if $0 \le e_i \le i-1$ for all $i \in [n]$. We usually denote by $\mathbf{I}_n$ the set of inversion sequences of length n.

Let $v, w \in \mathbb{N}^n$ be two words of length n.

Definition 1.1.9. We say v and w are *order isomorphic* if for each k, the k-th smallest entries of v and w occur at the same places. Further, we say the *reduction* of v is a sequence obtained by replacing the k-th smallest entries of v with $k-1$. In particular, v and its reduction are order isomorphic.

Example 1.1.1. $(16, 5, 14, 14, 0, 19, 20, 1, 20, 5)$[2] and $(5, 3, 4, 4, 1, 6, 7, 2, 7, 3)$ are order isomorphic. They both reduce to $(4, 2, 3, 3, 0, 5, 6, 1, 6, 2)$.

Now we turn our attention to patterns in sequences.

Definition 1.1.10. We say a sequence e *contains* a given pattern p if there exists a subsequence of e such that it is order isomorphic to p; otherwise, we say that e *avoids* the pattern p.

Example 1.1.2. The sequence $e = e_1e_2\cdots e_6 = 002030$ does not avoid the pattern 100 since the subsequence $e_3e_4e_6 = 200$ is order isomorphic to 100, but avoids 011 since no subsequences of e are order isomorphic to 011.

[2] Do you realize this sequence may be converted to "Penn State"?

Notation 1.1.8. Let p_1, p_2, $\ldots$, p_m be given patterns. We denote by $\mathbf{I}_n(p_1, p_2, \ldots, p_m)$ the set of inversion sequences of length n that avoid all of the patterns p_1, p_2, $\ldots$, p_m.

It is known that permutations that avoid given patterns have extensive applications in computer science, biology and many other fields; see the monograph of Kitaev [114]. Considering the close connection between permutations and inversion sequences, there are also flourish trends in the study of pattern avoidance in inversion sequences in recent years.

1.1.4.1 Kernel Method

The *kernel method* is a powerful tool to discover a closed form of a generating function if functional equations concerning it is known. In the perspective of Helmut Prodinger [142], this method originated as an exercise in the first volume of Donald Knuth's book "*The Art of Computer Programming*" [115, Exercise 4, §2.2.1, p. 243]. Then it was turned into a method in the work of Banderier er al. [34] on generating trees.

To briefly illustrate this method, I will use Knuth's exercise with the solution provided by Prodinger [142]. I will omit the combinatorial statements while only focus on the generating function.

Theorem 1.1.8. *Let $F(x,q) \in \mathbb{R}[[q]][[x]]$ satisfy the functional equation*

$$F(x,q) = xqF(x,q) + \frac{q}{x}\Big(F(x,q) - F(0,q)\Big) + 1. \tag{1.1.7}$$

Then,

$$F(x,q) = \frac{\dfrac{1-\sqrt{1-4q^2}}{2q} - x}{qx^2 - x + q}. \tag{1.1.8}$$

Proof. We first rewrite the functional equation (1.1.7) as

$$(qx^2 - x + q)F(x,q) = qF(0,q) - x. \tag{1.1.9}$$

Here the coefficient $(qx^2 - x + q)$ on the left-hand side is usually called the *kernel polynomial.*

If we treat the kernel polynomial $(qx^2 - x + q)$ as a polynomial in x, it is easy to compute its two roots

$$r_1(q) = \frac{1-\sqrt{1-4q^2}}{2q} \quad \text{and} \quad r_2(q) = \frac{1+\sqrt{1-4q^2}}{2q}.$$

Now we have

$$F(x,q) = \frac{qF(0,q) - x}{q(x - r_1(q))(x - r_2(q))}.$$

Notice that $r_1(q) = q + O(q^2)$. Therefore, $1/(x - r_1(q))$ has no power series expansion around $(0,0)$. However, as $F(x,q)$ is a formal power series in x and q, we must have that $x - r_1(q)$ is a factor of the numerator and thus $qF(0,q) = r_1(q)$. That is,

$$F(0,q) = \frac{r_1(q)}{q} = \frac{1 - \sqrt{1 - 4q^2}}{2q^2} \tag{1.1.10}$$

and the desired result follows by substituting the above into (1.1.9). □

1.2 State of the Art

1.2.1 Partition Congruences

The theory of partition congruences, which is now a blooming topic, was given to birth when Ramanujan studied the table of the values of $p(n)$ up to $n = 200$, which is calculated by Major MacMahon.

Table 1.3. Values of $p(n)$ for $1 \leq n \leq 20$

n	1	2	3	4	5
$p(n)$	1	2	3	5	7
n	6	7	8	9	10
$p(n)$	11	15	22	30	42
n	11	12	13	14	15
$p(n)$	56	77	101	135	176
n	16	17	18	19	20
$p(n)$	231	297	385	490	627

If one looks at the column where $n \equiv 4 \pmod 5$ in Table 1.3, it can be seen that $p(n)$ is divisible by 5. With such an observation, Ramanujan [146] announced in 1919 the following congruences.

Theorem 1.2.1. *We have*

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.2.1}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{1.2.2}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.2.3}$$

The proofs of the first two congruences were given in [146] while the proof of the last was announced one year later in a short note [147] and was finally published in [148], in which indeed a unified proof of all three congruences was presented.

It is also notable that in [146], Ramanujan actually showed an identity which is regarded as his "Most Beautiful Identity" by both Hardy and MacMahon [151, p. xxxv].

Theorem 1.2.2. *We have*

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{1.2.4}$$

From this identity, (1.2.1) follows as a direct corollary.

There are many directions to generalize Ramanujan's congruences. First, *what happens if one changes the moduli to powers of* 5, 7 *and* 11*?* Such general families of congruences were indeed conjectured by Ramanujan himself in 1919 [150] in which the conjecture for powers of 7 is incorrect. But this was fixed later by Watson [171].

Theorem 1.2.3. *We have, for $\ell \in \{5,7,11\}$ and $\alpha \geq 1$,*

$$p(\ell^\alpha n + \delta_{\alpha,\ell}) \equiv \begin{cases} 0 \pmod{\ell^\alpha} & \ell = 5, 11, \\ 0 \pmod{7^{\lceil\frac{\alpha+1}{2}\rceil}} & \ell = 7, \end{cases} \tag{1.2.5}$$

with $0 \leq \delta_{\alpha,\ell} \leq \ell^\alpha - 1$ such that

$$24\delta_{\alpha,\ell} \equiv 1 \pmod{\ell^\alpha}.$$

Watson [171] was able to prove the cases of powers of 5 and 7 using modular forms. But for the case of powers of 11, he stated that

> "Da die Untersuchung der Aussage über 11^α recht langweilig ist, verschiebe ich den Beweis dieses Falles auf eine spätere Abhandlung."

Nearly thirty years later in 1967, Atkin [28] eventually completed the project of proving the case of powers of 11 with an agreement with Watson's assertion that the proof is "langweilig." Some fifteen years later in the 1980s, elementary proofs of the cases of powers of 5 and 7 were further discovered, respectively by Hirschhorn and Hunt [100] and Garvan [79]. However, elementary proof of the case of powers of 11 is still a mystery.

Another question we could ask about partition congruences is *what happens if one changes the moduli to an arbitrary integer?* This problem was first considered by Atkin in the 1960s with discoveries like

$$p(11^3 \cdot 13n + 237) \equiv 0 \pmod{13}.$$

Along this direction, the most exciting result is obtained by Ono [137] in 2000.

Theorem 1.2.4. *For any positive integer m that is coprime to 6, there exists an arithmetic progression $A_m n + B_m$ such that*

$$p(A_m n + B_m) \equiv 0 \pmod{m}. \tag{1.2.6}$$

Ono also provided examples like

$$p(107^4 \cdot 31n + 30064597) \equiv 0 \pmod{31}.$$

In 2001, Weaver [172] compiled a list of 76,065 Ramanujan-like congruences while the list was extended by Johansson [104] to 22,474,608,014 congruences in 2012.

1.2.2 Rank and Crank of Integer Partitions

In the previous section, we have introduced partition congruences from the analytic side. A natural question is that *can we interpret these congruences, especially* (1.2.1), (1.2.2) *and* (1.2.3)*, combinatorially?* In other words, can we find a combinatorial statistic such that we can split the partitions of, for example, $5n + 4$, into five subclasses of equal size with the statistic satisfying a certain property in each subclass? This idea was first raised in 1944 by Dyson [76], who was an undergraduate at that time.

The first statistic Dyson defined is called *rank*.

Definition 1.2.1. The *rank* of an integer partition is the largest part minus the number of parts.

We list the rank of all partitions of 4, 5 and 6 in Table 1.4. From this Table, one may observe that the partitions of 4 and 5 are divided into five and seven equally numerous subclasses according to the rank modulo 5 and 7. Based on this observation, Dyson

Table 1.4. Ranks of all partitions of 4, 5 and 6

partitions of 4	rank	partitions of 5	rank	partitions of 6	rank
4	3	5	4	6	5
$3+1$	1	$4+1$	2	$5+1$	3
$2+2$	0	$3+2$	1	$4+2$	2
$2+1+1$	-1	$3+1+1$	0	$4+1+1$	1
$1+1+1+1$	-3	$2+2+1$	-1	$3+3$	1
		$2+1+1+1$	-2	$3+2+1$	0
		$1+1+1+1+1$	-4	$3+1+1+1$	-1
				$2+2+2$	-1
				$2+2+1+1$	-2
				$2+1+1+1+1$	-3
				$1+1+1+1+1+1$	-5

made the following conjecture, which was proved by Atkin and Swinnerton-Dyer [30] about ten years later.

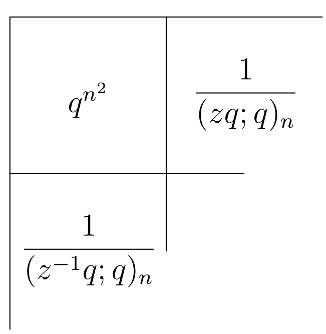
Theorem 1.2.5. *Let $N(k, m, n)$ denote the number of partitions of n whose rank is congruent to k modulo m. Then,*

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4) = \frac{p(5n+4)}{5} \tag{1.2.7}$$

and

$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5) = \frac{p(7n+5)}{7}. \tag{1.2.8}$$

Figure 1.2. Splitting a partition through the Durfee square



Let $N(m,n)$ denote the number of partitions of n whose rank is m. We may define a bivariate generating function

$$\sum_{n\geq 0}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n.$$

This generating function can be easily obtained through the Durfee square. In Figure 1.2, we can see that for any partition λ with Durfee square of size n, there is a partition λ_1 with largest part not exceeding n below the Durfee square. Also, to the left of the Durfee square, there is another partition λ_2 with the number of parts not exceeding n. Taking the conjugate of λ_2, we can see that $\overline{\lambda}_2$ is also a partition with largest part not exceeding n. Finally, the rank of λ of simply the number of parts in $\overline{\lambda}_2$ minus the number of parts in λ_1. We therefore arrive at the following identity.

Theorem 1.2.6. *We have*

$$\sum_{n\geq 0}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n\geq 0}\frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n}. \tag{1.2.9}$$

From this generating function identity, the following symmetry property can be deduced without difficulty.

Theorem 1.2.7. *We have*

$$N(m,n) = N(-m,n). \tag{1.2.10}$$

On the other hand, we see from Table 1.4 that partitions of 6 are not divided into eleven subclasses of equal size according to the rank modulo 11. Therefore, Dyson further conjectured in [76] the existence of another statistic called *crank* such that there exists a unified combinatorial interpretation of all three Ramanujan's congruences (1.2.1), (1.2.2) and (1.2.3) through this statistic. Let me quote Dyson's original words:

> "One is thus led irresistibly to the conclusion that there must be some analogue modulo 11 . . .
>
> I hold in fact:
>
> That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, . . .

. . . . . .

... Whatever the final verdict of posterity may be, I believe the "crank" is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!"

This statistic was found after over four decades by Andrews and Garvan [23] after Garvan's discovery of the vector crank shortly beforehand [80].

Definition 1.2.2. The *crank* of a partition λ is defined by

$$\operatorname{crank}(\lambda) := \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0, \end{cases} \tag{1.2.11}$$

where $\mu(\lambda)$ denotes the number of parts in λ larger than $\omega(\lambda)$.

Table 1.5. Cranks of all partitions of 4, 5 and 6

partitions of 4	crank	partitions of 5	crank	partitions of 6	crank
4	4	5	5	6	6
$3+1$	0	$4+1$	0	$5+1$	0
$2+2$	2	$3+2$	4	$4+2$	4
$2+1+1$	-2	$3+1+1$	-1	$4+1+1$	-1
$1+1+1+1$	-4	$2+2+1$	1	$3+3$	3
		$2+1+1+1$	-3	$3+2+1$	1
		$1+1+1+1+1$	-5	$3+1+1+1$	-3
				$2+2+2$	2
				$2+2+1+1$	-2
				$2+1+1+1+1$	-4
				$1+1+1+1+1+1$	-6

In Table 1.5, the cranks of all partitions of 4, 5 and 6. One can see how the equally numerous subclasses appear according to the crank modulo 5, 7 and 11.

Let $M(m,n)$ denote the number of partitions of n whose rank is m except for $n=1$ where $M(-1,1) = -M(0,1) = M(1,1) = 1$. The following result was due to Andrews and Garvan.

Theorem 1.2.8. *We have*

$$\sum_{n\geq 0}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{1.2.12}$$

We have a similar symmetry property to the rank case.

Theorem 1.2.9. *We have*

$$M(m,n) = M(-m,n). \tag{1.2.13}$$

Finally, Dyson's crank now is not fabled!

Theorem 1.2.10. *Let $M(k,m,n)$ denote the number of partitions of n whose crank is congruent to k modulo m. Then,*

$$M(0,5,5n+4) = M(1,5,5n+4) = \cdots = M(4,5,5n+4) = \frac{p(5n+4)}{5}, \tag{1.2.14}$$

$$M(0,7,7n+5) = M(1,7,7n+5) = \cdots = M(6,7,7n+5) = \frac{p(7n+5)}{7} \tag{1.2.15}$$

and

$$M(0,11,11n+6) = M(1,11,11n+6) = \cdots = M(10,11,11n+6) = \frac{p(11n+6)}{11}. \tag{1.2.16}$$

1.2.3 Identities of Rogers–Ramanujan Type

Another ingenious work of Ramanujan [145], which was claimed in 1913 in his first letter to Hardy, is about the Rogers–Ramanujan identities, which should indeed be attributed to Rogers in a paper [156] that was completely ignored.

Theorem 1.2.11 (First Rogers–Ramanujan identity)**.** *The number of partitions of a non-negative integer n into parts congruent to ± 1 modulo 5 is the same as the number of partitions of n such that each two consecutive parts have difference at least 2.*

Theorem 1.2.12 (Second Rogers–Ramanujan identity)**.** *The number of partitions of a non-negative integer n into parts congruent to ± 2 modulo 5 is the same as the number of partitions of n such that each two consecutive parts have difference at least 2 and such that the smallest part is at least 2.*

Their analytic forms, which are in terms of generating functions, also look nice.

Theorem 1.2.13 (Rogers–Ramanujan identities (analytic form))**.** *We have*

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty} \tag{1.2.17}$$

and

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.2.18}$$

One can observe that in the Rogers–Ramanujan identities, two types of partition sets are considered. One partition set consists of partitions under certain *congruence* conditions. For example, in the first Rogers–Ramanujan identity, we enumerate partitions into parts congruent to ± 1 modulo 5. The other partition set contains partitions under certain *difference* conditions. For example, we require that each two consecutive parts have difference at least 2 in the first Rogers–Ramanujan identity.

More identities of the same flavor were discovered by mathematicians including Schur [160], Gleißberg [84], Gordon [86], Göllnitz [85], Andrews [7] and so forth. Let me excerpt Schur's 1926 identity as an example.

Theorem 1.2.14 (Schur)**.** *Let $A(n)$ denote the number of partitions of n into distinct parts congruent to ± 1 modulo 3.*

Let $B(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 6.

Let $C(n)$ denote the number of partitions of n such that the difference between two consecutive parts is at least 3 and greater than 3 if the smaller part is a multiple of 3.

Then,

$$A(n) = B(n) = C(n). \tag{1.2.19}$$

The most standard proof of Schur's 1926 identity is based on recurrences [6], but there are also combinatorial proofs through bijections [40] and weighted words [3].

In the 1970s, George Andrews [8, 10, 11] further started a systematic study of Rogers–Ramanujan type identities and developed a general theory in which the concept of linked partition ideals was introduced. Unfortunately, this theory was then almost ignored. However, three chapters of this thesis are devoted to give a revisit of Andrews' idea and to make it more feasible.

Finally, it is notable that identities of Rogers–Ramanujan type also have deep connections with other branches of mathematics. One instance is with representations of Lie algebras shown in a series of papers of Lepowsky and Wilson [120–123]. For example, they considered level 3 modules for the affine Lie algebra $\widehat{\mathfrak{sl}_2}$, from which they also obtained a proof of the Rogers–Ramanujan identities. Along this line, Kanade and Russell [108] discovered more identities of Rogers–Ramanujan type by experiments, which are now known as Kanade–Russell conjectures.

Further, identities of Rogers–Ramanujan type may even have connections with theories outside of mathematics. For example, in statistical mechanics, the hard hexagon model has the solution, which was due to Baxter [36], involving the Rogers–Ramanujan identities.

1.2.4 Asymptotics

There is a natural injection between partitions of n and $n+1$, that is, we can append 1 as a part to each partition of n so that a partition of $n+1$ is constructed. This implies that $\{p(n)\}$ is an increasing sequence for $n \geq 1$. The next question is *how large $p(n)$ is?* In other words, *is there an asymptotic formula or even an exact formula for $p(n)$?*

This question was first treated by Hardy and Ramanujan [96] in 1918 using a method which is now called the *Hardy–Littlewood circle method*.

Theorem 1.2.15. *As $n \to \infty$,*

$$p(n) \sim \frac{1}{4\sqrt{3}} n^{-1} e^{\frac{2\pi\sqrt{n}}{\sqrt{6}}}. \tag{1.2.20}$$

One could imagine how shocked MacMahon was when Ramanujan presented him the value of $p(200)$. Let me simply quote a piece of lines from the movie "*The Man Who Knew Infinity*."

> (M stands for MacMahon and R stands for Ramanujan.)
>
> M: Well, here we are. $p(200)$, the moment of truth ... Well, you first. What is your formula given you?
>
> R: Three billion nine hundred and seventy two thousand nine hundred and ninety eight million.
>
> M: My God! You are close [*silent for 5 seconds*] within two percent. Well, I will be damned.

Well, what is the exact value of $p(200)$ then? The answer is

$$p(200) = 3,972,999,029,388.$$

There are a number of ways to study the asymptotic behavior of a sequence, all starting with the generating function. Recall that the generating function of $p(n)$ is

$$P(q) = \frac{1}{(q;q)_\infty} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

The easiest method is Ingham's Tauberian theorem, which is stated as follows.

Theorem 1.2.16 (Ingham [101])**.** *Let $f(q) = \sum_{n\geq 0} a(n)q^n$ be a power series with weakly increasing nonnegative coefficients and radius of convergence equal to 1. If there are constants $A > 0$ and $\lambda, \alpha \in \mathbb{R}$ such that*

$$f\left(e^{-t}\right) \sim \lambda t^\alpha e^{\frac{A}{t}}$$

as $t \to 0^+$, then

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}$$

as $n \to \infty$.

Since $p(n)$ is non-decreasing, it is only necessary to study the asymptotics of $P(q)$ as $q \to 1^-$ along the real line.

But it is not always the case that the non-decreasing condition is satisfied for a given sequence. In such cases, one should continue with some more complicated calculations. Let us still use $p(n)$ to illustrate. The basic idea here is Cauchy's integral formula. Recall that

$$p(n) = \frac{1}{2\pi i} \oint_{\mathcal{C}:|q|=r} \frac{P(q)}{q^{n+1}}\, dq,$$

where the contour $\mathcal{C}$ is inside the unit disc and the contour integral is taken counter-clockwise.

Since

$$P(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots},$$

one can see that $P(q)$ has a dominant pole at $q = 1$. Hence, if the radius of the contour $\mathcal{C}$ inside the unit disc is taken to approach 1, then the main contribution comes from the arc close to 1. This is essentially the principle of Wright's circle method.

When we study the asymptotics, one important function that always appears is the *modified Bessel function of the first kind* or the *I-Bessel function.*

Definition 1.2.3. The *modified Bessel function of the first kind* is defined by

$$I_s(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+s+1)} \left(\frac{x}{2}\right)^{2m+s}, \tag{1.2.21}$$

where $\Gamma(z)$ is the gamma function.

Its asymptotic behavior is also well known.

Theorem 1.2.17 (Cf. [2, p. 377, (9.7.1)])**.** *For fixed s, when $|\arg x| < \pi/2$,*

$$I_s(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4s^2-1}{8x} + \frac{(4s^2-1)(4s^2-9)}{2!(8x)^2} - \cdots\right). \tag{1.2.22}$$

After considering the dominant pole at $q = 1$, one could even move further by noticing that the pole of $P(q)$ at -1 is $1/2$ as "important" as the pole at 1, the pole at primitive cube roots of unity is $1/3$ as "important," and so on. Hence, we could focus on the asymptotic behavior of $P(q)$ on arcs inside the unit disc that is close to a rational point on the unit circle, that is, a point of the form $\exp(2\pi ih/k)$. Based on this idea along with other techniques, Rademacher [143] eventually arrived at an exact formula as follows.

Theorem 1.2.18. *We have*

$$p(n) = \frac{1}{2\sqrt{2}\pi} \sum_{k\geq 1} A_k(n)\sqrt{k}\, \frac{d}{dn}\left(\frac{2}{\sqrt{n-\frac{1}{24}}} \sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)\right), \tag{1.2.23}$$

where

$$A_k(n) = \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} e^{\pi i(s(h,k)-2nh/k)}$$

with $s(h,k)$ being the Dedekind sum defined by

$$s(d,c) := \sum_{n \bmod c} \left(\!\left(\frac{dn}{c}\right)\!\right)\left(\!\left(\frac{n}{c}\right)\!\right)$$

where

$$((x)) := \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Let me end this section with an example given on page 70 of [12].

Example 1.2.1. Let us use the first 8 terms in Rademacher's formula to estimate $p(200)$:

$$\begin{array}{r} +\,3,972,998,993,185.896 \\ +\,36,282.978 \\ -\,87.584 \\ +\,5.147 \\ +\,1.424 \\ +\,0.071 \\ +\,0.000 \\ +\,0.044 \\ \hline 3,972,999,029,387.975 \end{array}$$

Eureka! We are only .025 away from the exact value!

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Part I
Congruences

Outline

- Chapter 2 is devoted to an elementary proof of an infinite family of congruences modulo powers of 5 for $p_D(n)$, the number of partitions of n into distinct parts.
- Chapter 3 is devoted to some eta-quotient representations concerning the Rogers–Ramanujan continued fraction.

Chapter 2

Partitions into Distinct Parts Modulo Powers of 5

This chapter comes from

- S. Chern and M. D. Hirschhorn, Partitions into distinct parts modulo powers of 5, *Ann. Comb.* **23** (2019), no. 3-4, 659–682. Also in: *George E. Andrews—80 Years of Combinatory Analysis*, 305–328, Birkhäuser/Springer, Cham, 2021. (Ref. [64])

2.1 Introduction and Main Result

Let $p_D(n)$ denote the number of partitions of n into distinct parts. Then

$$\sum_{n\geq 0} p_D(n)q^n = (-q;q)_\infty = \frac{E(q^2)}{E(q)} \tag{2.1.1}$$

where

$$E(q) = (q;q)_\infty.$$

Like the ordinary partition function $p(n)$, $p_D(n)$ also enjoys an infinite family of congruences modulo powers of 5. Namely,

$$p_D\left(5^{2\alpha+1}n + \frac{5^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod{5^\alpha}. \tag{2.1.2}$$

It should be noted that this congruence, which is due to Rødseth [154] and independently to Gordon and Hughes [87], requires the theory of modular forms in its proof. On the other hand, the congruence for $p(n)$ stated as follows,

$$p\Big(5^\alpha n + \delta_\alpha\Big) \equiv 0 \pmod{5^\alpha} \tag{2.1.3}$$

where

$$\delta_\alpha = \begin{cases} \dfrac{19 \times 5^\alpha + 1}{24} & \text{if } \alpha \text{ is odd,} \\ \dfrac{23 \times 5^\alpha + 1}{24} & \text{if } \alpha \text{ is even,} \end{cases} \tag{2.1.4}$$

which was conjectured by Ramanujan [150] in 1919, was first demonstrated by Watson [171] in 1938, again, with the help of modular forms. However, in 1981, it was shown by Hirschhorn and Hunt [100] that (2.1.3) could also be achieved by a purely elementary approach, which is based on a modular equation of degree 5.

Now a natural question arises:

Can we prove (2.1.2) by an elementary method, or even using the words of Mike Hirschhorn, by only "high School algebra, but taken somewhat further?"

The goal of this chapter is to settle this problem in affirmative. Our result can be stated as follows.

Theorem 2.1.1. *For $\alpha \geq 1$,*

$$\sum_{n\geq 0} p_D\left(5^{2\alpha-1}n + \frac{5^{2\alpha}-1}{24}\right) q^n = \gamma \sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\zeta^{i-1}, \tag{2.1.5}$$

$$\sum_{n\geq 0} p_D\left(5^{2\alpha}n + \frac{5^{2\alpha}-1}{24}\right) q^n = \delta \sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\zeta^{i-1} \tag{2.1.6}$$

where

$$\gamma = \frac{E(q^2)^2E(q^5)^3}{E(q)^4E(q^{10})}, \quad \delta = \frac{E(q^2)^3E(q^5)^4}{E(q)^5E(q^{10})^2}, \quad \zeta = q\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)} \tag{2.1.7}$$

and where the coefficient vectors $\mathbf{x}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, \cdots)$ are given recursively by

$$\mathbf{x}_1 = (1, 0, \cdots), \tag{2.1.8}$$

and for $\alpha \geq 1$,

$$\mathbf{x}_{2\alpha} = \mathbf{x}_{2\alpha-1}A \tag{2.1.9}$$

and

$$\mathbf{x}_{2\alpha+1} = \mathbf{x}_{2\alpha} B, \tag{2.1.10}$$

where A is the matrix $(\alpha_{i,j})_{i,j\geq 1}$ and B is the matrix $(\beta_{i,j})_{i,j\geq 1}$ where the $\alpha_{i,j}$ and $\beta_{i,j}$ are given by

$$\sum_{i,j\geq 1} \alpha_{i,j} x^i y^j = \frac{N_\alpha}{D'} \tag{2.1.11}$$

and

$$\sum_{i,j\geq 1} \beta_{i,j} x^i y^j = \frac{N_\beta}{D'} \tag{2.1.12}$$

where

$$\begin{aligned} N_\alpha = {} & (y + 160y^2 + 2800y^3 + 16000y^4 + 32000y^5)x \\ & + (180y^2 + 3000y^3 + 16800y^4 + 32000y^5)x^2 \\ & + (75y^2 + 1215y^3 + 6600y^4 + 12000y^5)x^3 \\ & + (14y^2 + 220y^3 + 1150y^4 + 2000y^5)x^4 \\ & + (y^2 + 15y^3 + 75y^4 + 125y^5)x^5, \end{aligned} \tag{2.1.13}$$

$$\begin{aligned} N_\beta = {} & (5y + 660y^2 + 14400y^3 + 120000y^4 + 448000y^5 + 640000y^6)x \\ & + (y + 680y^2 + 14900y^3 + 123200y^4 + 456000y^5 + 640000y^6)x^2 \\ & + (265y^2 + 5785y^3 + 47500y^4 + 174000y^5 + 240000y^6)x^3 \\ & + (46y^2 + 1000y^3 + 8150y^4 + 29500y^5 + 40000y^6)x^4 \\ & + (3y^2 + 65y^3 + 525y^4 + 1875y^5 + 2500y^6)x^5 \end{aligned} \tag{2.1.14}$$

and

$$\begin{aligned} D' = 1 & - (205y + 4300y^2 + 34000y^3 + 120000y^4 + 160000y^5)x \\ & - (215y + 4475y^2 + 35000y^3 + 122000y^4 + 160000y^5)x^2 \\ & - (85y + 1750y^2 + 13525y^3 + 46000y^4 + 60000y^5)x^3 \\ & - (15y + 305y^2 + 2325y^3 + 7875y^4 + 10000y^5)x^4 \end{aligned}$$

$$- (y + 20y^2 + 150y^3 + 500y^4 + 625y^5)x^5. \tag{2.1.15}$$

Furthermore, for $\alpha \geq 1$,

$$x_{2\alpha+1,i} \equiv 0 \pmod{5^\alpha}, \tag{2.1.16}$$
$$x_{2\alpha+2,i} \equiv 0 \pmod{5^\alpha}, \tag{2.1.17}$$

from which it follows that for $\alpha \geq 1$,

$$p_D\left(5^{2\alpha+1}n + \frac{5^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod{5^\alpha}, \tag{2.1.18}$$
$$p_D\left(5^{2\alpha+2}n + \frac{5^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod{5^\alpha}. \tag{2.1.19}$$

(*Of course,* (2.1.19) *is a special case of* (2.1.18).)

2.2 Preliminaries

Let

$$R(q) = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)_\infty, \quad \chi(-q) = (q;q^2)_\infty = \frac{E(q)}{E(q^2)}.$$

Then ([98, (8.1.1)])

$$E(q) = E(q^{25})\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right), \tag{2.2.1}$$

([98, (8.4.4)])

$$\begin{aligned}\frac{1}{E(q)} = \frac{E(q^{25})^5}{E(q^5)^6}\Bigg(&\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \\ &-3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\Bigg),\end{aligned} \tag{2.2.2}$$

([98, (40.2.3)])

$$R(q^2) - R(q)^2 = 2q\left(\begin{matrix} q,\ q,\ q^9, q^9 \\ q^3, q^5, q^5, q^7 \end{matrix}; q^{10}\right)_\infty, \tag{2.2.3}$$

([98, (40.2.4)])

$$R(q^2)+R(q)^2=2\begin{pmatrix} q,\ q^4, q^6, q^9 \\ q^2, q^5, q^5, q^8 \end{pmatrix}_\infty, \tag{2.2.4}$$

([98, (41.1.3)])

$$1-qR(q)R(q^2)^2=\begin{pmatrix} q,\ q^4, q^5, q^5, q^6, q^9 \\ q^2, q^3, q^3, q^7, q^7, q^8 \end{pmatrix}_\infty, \tag{2.2.5}$$

([98, (41.1.2)])

$$1+qR(q)R(q^2)^2=\begin{pmatrix} q^2, q^2, q^5, q^5, q^8, q^8 \\ q,\ q^4, q^4, q^6, q^6, q^9 \end{pmatrix}_\infty, \tag{2.2.6}$$

([98, (34.8.4)])

$$\frac{E(q^2)^4}{E(q)^2}-q\frac{E(q^{10})^4}{E(q^5)^2}=\frac{E(q^2)E(q^5)^3}{E(q)E(q^{10})} \tag{2.2.7}$$

and ([98, (34.8.3)])

$$\frac{E(q^5)^4}{E(q^{10})^2}-\frac{E(q)^4}{E(q^2)^2}=4q\frac{E(q)E(q^{10})^3}{E(q^2)E(q^5)}. \tag{2.2.8}$$

We require the following results.

Lemma.

$$\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}=4q\frac{\chi(-q)}{\chi(-q^5)^5}, \tag{2.2.9}$$

$$\frac{R(q^2)-R(q)^2}{R(q^2)+R(q)^2}=qR(q)R(q^2)^2, \tag{2.2.10}$$

$$\frac{1}{R(q)R(q^2)^2}-q^2R(q)R(q^2)^2=\frac{\chi(-q^5)^5}{\chi(-q)}, \tag{2.2.11}$$

$$\frac{1-qR(q)R(q^2)^2}{1+qR(q)R(q^2)^2}=\frac{R(q)^2}{R(q^2)}, \tag{2.2.12}$$

$$\frac{R(q)}{R(q^2)^3}+q^2\frac{R(q^2)^3}{R(q)}=\frac{\chi(-q^5)^5}{\chi(-q)}-2q+4q^2\frac{\chi(-q)}{\chi(-q^5)^5}, \tag{2.2.13}$$

$$\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)=\frac{\chi(-q^5)^5}{\chi(-q)}+2q+4q^2\frac{\chi(-q)}{\chi(-q^5)^5}, \tag{2.2.14}$$

$$\frac{\chi(-q^5)^5}{\chi(-q)}+q=\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4} \tag{2.2.15}$$

and

$$1-4q\frac{\chi(-q)}{\chi(-q^5)^5}=\frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4}. \tag{2.2.16}$$

Proof of (2.2.9). If we multiply (2.2.3) by (2.2.4) and divide by $R(q)^2R(q^2)$, we find that

$$\begin{aligned}
\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)} &= \frac{(R(q^2)-R(q)^2)(R(q^2)+R(q)^2)}{R(q)^2R(q^2)}\\
&= \frac{2q\left(\begin{matrix}q,\ q,\ q^9,q^9\\ q^3,q^5,q^5,q^7\end{matrix};q^{10}\right)_\infty\cdot 2\left(\begin{matrix}q,\ q^4,q^6,q^9\\ q^2,q^5,q^5,q^8\end{matrix};q^{10}\right)_\infty}{\left(\begin{matrix}q,\ q,\ q^4,q^4,q^6,q^6,q^9,q^9,q^2,q^8\\ q^2,q^2,q^3,q^3,q^7,q^7,q^8,q^8,q^4,q^6\end{matrix};q^{10}\right)_\infty}\\
&= 4q\left(\begin{matrix}q,\ q^3,q^5,q^7,q^9\\ q^5,q^5,q^5,q^5,q^5\end{matrix};q^{10}\right)_\infty\\
&= 4q\frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty^5}\\
&= 4q\frac{\chi(-q)}{\chi(-q^5)^5}.
\end{aligned}$$

□

Proof of (2.2.10). If we divide (2.2.3) by (2.2.4), we obtain

$$\begin{aligned}
\frac{R(q^2)-R(q)^2}{R(q^2)+R(q)^2} &= q\frac{\begin{pmatrix} q,\ q,\ q^9,q^9 \\ q^3,q^5,q^5,q^7 \end{pmatrix};q^{10}\Big)_\infty}{\begin{pmatrix} q,\ q^4,q^6,q^9 \\ q^2,q^5,q^5,q^8 \end{pmatrix};q^{10}\Big)_\infty} \\
&= q\left(\begin{matrix} q,\ q^2,q^8,q^9 \\ q^3,q^4,q^6,q^7 \end{matrix};q^{10}\right)_\infty \\
&= q\left(\begin{matrix} q,\ q^4,q^6,q^9,q^2,q^2,q^8,q^8 \\ q^2,q^3,q^7,q^8,q^4,q^4,q^6,q^6 \end{matrix};q^{10}\right)_\infty \\
&= qR(q)R(q^2)^2.
\end{aligned}$$

□

Proof of (2.2.11). If we multiply (2.2.5) by (2.2.6) and divide by $R(q)R(q^2)^2$, we find

$$\begin{aligned}
\frac{1}{R(q)R(q^2)^2} - q^2R(q)R(q^2)^2 &= \frac{(1-qR(q)R(q^2)^2)(1+qR(q)R(q^2)^2)}{R(q)R(q^2)^2} \\
&= \frac{\left(\begin{matrix} q,\ q^4,q^5,q^5,q^6,q^9 \\ q^2,q^3,q^3,q^7,q^7,q^8 \end{matrix};q^{10}\right)_\infty \left(\begin{matrix} q^2,q^2,q^5,q^5,q^8,q^8 \\ q,\ q^4,q^4,q^6,q^6,q^9 \end{matrix};q^{10}\right)_\infty}{\left(\begin{matrix} q,\ q^4,q^6,q^9 \\ q^2,q^3,q^7,q^8 \end{matrix};q^{10}\right)_\infty \left(\begin{matrix} q^2,q^2,q^8,q^8 \\ q^4,q^4,q^6,q^6 \end{matrix};q^{10}\right)_\infty} \\
&= \left(\begin{matrix} q^5,q^5,q^5,q^5,q^5 \\ q,\ q^3,q^5,q^7,q^9 \end{matrix};q^{10}\right)_\infty \\
&= \frac{\chi(-q^5)^5}{\chi(-q)}.
\end{aligned}$$

□

Proof of (2.2.12). If we divide (2.2.5) by (2.2.6) we obtain

$$\frac{1-qR(q)R(q^2)^2}{1+qR(q)R(q^2)^2} = \frac{\left(\begin{matrix} q,\ q^4,q^5,q^5,q^6,q^9 \\ q^2,q^3,q^3,q^7,q^7,q^9 \end{matrix};q^{10}\right)_\infty}{\left(\begin{matrix} q^2,q^2,q^5,q^5,q^8,q^8 \\ q,\ q^4,q^4,q^6,q^6,q^9 \end{matrix};q^{10}\right)_\infty}$$

$$= \left(\begin{matrix} q,\ q,\ q^4, q^4, q^6, q^6, q^9, q^9, q^4, q^6 \\ q^2, q^2, q^3, q^3, q^7, q^7, q^8, q^8, q^2, q^8 \end{matrix} ; q^{10} \right)_\infty$$
$$= \frac{R(q)^2}{R(q^2)}.$$

□

Proof of (2.2.13). Note that (2.2.10) is equivalent to (2.2.12), because they are both equivalent to

$$R(q^2) - R(q)^2 = qR(q)^3 R(q^2)^2 + qR(q)R(q^2)^3. \tag{2.2.17}$$

If we divide (2.2.17) by $R(q)R(q^2)^3$ and rearrange, we find that

$$\frac{R(q)}{R(q^2)^3} = \frac{1}{R(q)R(q^2)^2} - q\frac{R(q)^2}{R(q^2)} - q, \tag{2.2.18}$$

while if we divide (2.2.17) by $R(q)^2$, rearrange and multiply by q, we obtain

$$q^2\frac{R(q^2)^3}{R(q)} = -q^2 R(q)R(q^2)^2 + q\frac{R(q^2)}{R(q)^2} - q. \tag{2.2.19}$$

If we add (2.2.18) and (2.2.19), we obtain

$$\begin{aligned} \frac{R(q)}{R(q^2)^3} + q^2\frac{R(q^2)^3}{R(q)} &= \left(\frac{1}{R(q)R(q^2)^2} - q^2 R(q)R(q^2)^2 \right) - 2q + q\left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} \right) \\ &= \frac{\chi(-q^5)^5}{\chi(-q)} - 2q + 4q^2\frac{\chi(-q)}{\chi(-q^5)^5}. \end{aligned}$$

□

Proof of (2.2.14). If we multiply (2.2.9) by (2.2.11) and add (2.2.13), we find that

$$\begin{aligned} \frac{1}{R(q)^3 R(q^2)} + q^2 R(q)^3 R(q^2) &= \left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} \right) \left(\frac{1}{R(q)R(q^2)^2} - q^2 R(q)R(q^2)^2 \right) \\ &\quad + \left(\frac{R(q)}{R(q^2)^3} + q^2\frac{R(q^2)^3}{R(q)} \right) \\ &= 4q\frac{\chi(-q)}{\chi(-q^5)^5} \cdot \frac{\chi(-q^5)^5}{\chi(-q)} \end{aligned}$$

$$+\left(\frac{\chi(-q^5)^5}{\chi(-q)}-2q+4q^2\frac{\chi(-q)}{\chi(-q^5)^5}\right)$$
$$=\frac{\chi(-q^5)^5}{\chi(-q)}+2q+4q^2\frac{\chi(-q)}{\chi(-q^5)^5}.$$

□

Proof of (2.2.15).

$$\begin{aligned}\frac{\chi(-q^5)^5}{\chi(-q)}+q&=\frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5}+q\\&=\frac{E(q^5)^2}{E(q^{10})^4}\left(\frac{E(q^2)E(q^5)^3}{E(q)E(q^{10})}+q\frac{E(q^{10})^4}{E(q^5)^2}\right)\\&=\frac{E(q^5)^2}{E(q^{10})^4}\cdot\frac{E(q^2)^4}{E(q)^2}.\end{aligned}$$

□

Proof of (2.2.16).

$$\begin{aligned}1-4q\frac{\chi(-q)}{\chi(-q^5)^5}&=1-4q\frac{E(q)E(q^{10})^5}{E(q^2)E(q^5)^5}\\&=\frac{E(q^{10})^2}{E(q^5)^4}\left(\frac{E(q^5)^4}{E(q^{10})^2}-4q\frac{E(q)E(q^{10})^3}{E(q^2)E(q^5)}\right)\\&=\frac{E(q^{10})^2}{E(q^5)^4}\cdot\frac{E(q)^4}{E(q^2)^2}.\end{aligned}$$

□

2.3 The Work of Baruah and Begum

It is fair to mention that our idea is motivated by a recent work of Baruah and Begum [35], in which the following results were shown.

$$\sum_{n\geq 0}p_D(5n+1)q^n=\frac{E(q^2)^2E(q^5)^3}{E(q)^4E(q^{10})},\tag{2.3.1}$$

$$\begin{aligned}\sum_{n\geq 0} p_D(25n+1)q^n &= \frac{E(q^2)^3E(q^5)^4}{E(q)^5E(q^{10})^2}\\ &\times\left(1+160q\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)+2800q^2\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)^2\right.\\ &\left.+16000q^3\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)^3+32000q^4\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)^4\right),\end{aligned}\tag{2.3.2}$$

as well as the corresponding result for $\sum_{n\geq 0} p_D(125n+26)q^n$.

Now let us reprove (2.3.1). We have

$$\begin{aligned}\sum_{n\geq 0} p_D(n)q^n = (-q;q)_\infty &= \frac{E(q^2)}{E(q)}\\ &= \frac{E(q^{25})^5}{E(q^5)^6}\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4\right.\\ &\qquad\left.-3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\right)\\ &\times E(q^{50})\left(\frac{1}{R(q^{10})}-q^2-q^4R(q^{10})\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n\geq 0} p_D(5n+1)q^n\\ &\quad= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\\ &\quad\times\left(\left(\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)\right)-5q-2q\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)\right)\\ &\quad= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\\ &\quad\times\left(\left(\frac{\chi(-q^5)^5}{\chi(-q)}+2q+4q^2\frac{\chi(-q)}{\chi(-q^5)^5}\right)-5q-2q\cdot 4q\frac{\chi(-q)}{\chi(-q^5)^5}\right)\\ &\quad= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\left(\frac{\chi(-q^5)^5}{\chi(-q)}-3q-4q^2\frac{\chi(-q)}{\chi(-q^5)^5}\right)\\ &\quad= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\left(\frac{\chi(-q^5)^5}{\chi(-q)}+q\right)\left(1-4q\frac{\chi(-q)}{\chi(-q^5)^5}\right)\\ &\quad= \left(\frac{E(q^5)^5E(q^{10})}{E(q)^6}\right)\left(\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4}\right)\left(\frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4}\right)\end{aligned}$$

$$= \frac{E(q^2)^2 E(q^5)^3}{E(q)^4 E(q^{10})}.$$

$\square$

Note that (2.3.1) is the case $\alpha = 1$ of (2.1.5).

2.4 The Modular Equation

We obtain the modular equation for ζ.

Let $\zeta(q^5) = Z$.

Theorem 2.4.1.

$$\begin{aligned}
&\zeta^5 - (205Z + 4300Z^2 + 34000Z^3 + 120000Z^4 + 160000Z^5)\zeta^4 \\
&\quad - (215Z + 4475Z^2 + 35000Z^3 + 122000Z^4 + 160000Z^5)\zeta^3 \\
&\quad - (85Z + 1750Z^2 + 13525Z^3 + 46500Z^4 + 60000Z^5)\zeta^2 \\
&\quad - (15Z + 305Z^2 + 2325Z^3 + 7875Z^4 + 10000Z^5)\zeta \\
&\quad - (Z + 20Z^2 + 150Z^3 + 500Z^4 + 625Z^5) = 0.
\end{aligned} \tag{2.4.1}$$

Proof. Let H be the huffing operator, given by

$$H\left(\sum_n a(n)q^n\right) = \sum_n a(5n)q^{5n}. \tag{2.4.2}$$

We can show, using extremely lengthy but elementary calculations (see §2.9), that

$$H(\zeta) = 41Z + 860Z^2 + 6800Z^3 + 24000Z^4 + 32000Z^5, \tag{2.4.3}$$

$$\begin{aligned}
H(\zeta^2) &= 86Z + 10195Z^2 + 366600Z^3 + 6534800Z^4 + 68384000Z^5 + 450720000Z^6 \\
&\quad + 1907200000Z^7 + 5056000000Z^8 + 7680000000Z^9 + 5120000000Z^{10},
\end{aligned} \tag{2.4.4}$$

$$\begin{aligned}
H(\zeta^3) &= 51Z + 27495Z^2 + 2836265Z^3 + 128688900Z^4 + 3343692000Z^5 \\
&\quad + 56283680000Z^6 + 656205600000Z^7 + 5502096000000Z^8 \\
&\quad + 33821312000000Z^9 + 153192960000000Z^{10} \\
&\quad + 506956800000000Z^{11} + 1195008000000000Z^{12}
\end{aligned}$$

$$
\begin{aligned}
&+ 1904640000000000Z^{13} + 1843200000000000Z^{14} \\
&+ 819200000000000Z^{15},
\end{aligned}
\tag{2.4.5}
$$

$$
\begin{aligned}
H(\zeta^4) = {} & 12Z + 32674Z^2 + 8579260Z^3 + 831492275Z^4 + 42958434000Z^5 \\
&+ 1396773180000Z^6 + 31314949600000Z^7 + 511802288800000Z^8 \\
&+ 6319880448000000Z^9 + 60349364480000000Z^{10} \\
&+ 452174745600000000Z^{11} + 2679038592000000000Z^{12} \\
&+ 12574269440000000000Z^{13} + 46561935360000000000Z^{14} \\
&+ 134544588800000000000Z^{15} + 297365504000000000000Z^{16} \\
&+ 485949440000000000000Z^{17} + 553779200000000000000Z^{18} \\
&+ 393216000000000000000Z^{19} + 131072000000000000000Z^{20}
\end{aligned}
\tag{2.4.6}
$$

and

$$
\begin{aligned}
H(\zeta^5) = {} & Z + 21370Z^2 + 13932050Z^3 + 2684902125Z^4 + 251131688125Z^5 \\
&+ 14097638650000Z^6 + 532547945100000Z^7 + 14515766554000000Z^8 \\
&+ 298883447380000000Z^9 + 4797842366000000000Z^{10} \\
&+ 61395781800000000000Z^{11} + 636255683040000000000Z^{12} \\
&+ 5398601306880000000000Z^{13} + 37772239436800000000000Z^{14} \\
&+ 218755840000000000000000Z^{15} + 1049457704960000000000000Z^{16} \\
&+ 4160657715200000000000000Z^{17} + 13552680960000000000000000Z^{18} \\
&+ 35909189632000000000000000Z^{19} + 76195266560000000000000000Z^{20} \\
&+ 126438604800000000000000000Z^{21} + 158138368000000000000000000Z^{22} \\
&+ 140247040000000000000000000Z^{23} + 78643200000000000000000000Z^{24} \\
&+ 20971520000000000000000000Z^{25}.
\end{aligned}
\tag{2.4.7}
$$

Let η be a fifth root of unity other than 1, and for $i = 0,\ 1,\ 2,\ 3,\ 4$ define

$$
\zeta_i = \zeta(\eta^i q). \tag{2.4.8}
$$

Then the power sums $\pi_1, \cdots, \pi_5$ of the ζ_i are given by

$$\begin{aligned}
&\pi_1 = \zeta_0 + \cdots + \zeta_4 = 5H(\zeta),\\
&\pi_2 = \zeta_0^2 + \cdots + \zeta_4^2 = 5H(\zeta^2),\\
&\cdots\\
&\pi_5 = \zeta_0^5 + \cdots + \zeta_4^5 = 5H(\zeta^5).
\end{aligned} \tag{2.4.9}$$

From (2.4.9) we obtain the symmetric functions $\sigma_1, \cdots, \sigma_5$ of the ζ_i,

$$\begin{aligned}
\sigma_1 &= \sum_i \zeta_i = \pi_1\\
&= 205Z + 4300Z^2 + 34000Z^3 + 120000Z^4 + 160000Z^5,\\
\sigma_2 &= \sum_{i<j} \zeta_i\zeta_j = \frac{1}{2}(\pi_1\sigma_1 - \pi_2)\\
&= -215Z - 4475Z^2 - 35000Z^3 - 122000Z^4 - 160000Z^5,\\
\sigma_3 &= \sum_{i<j<k} \zeta_i\zeta_j\zeta_k = \frac{1}{3}(\pi_1\sigma_2 - \pi_2\sigma_1 + \pi_3)\\
&= 85Z + 1750Z^2 + 13525Z^3 + 46000Z^4 + 60000Z^5,\\
\sigma_4 &= \sum_{i<j<k<l} \zeta_i\zeta_j\zeta_k\zeta_l = \frac{1}{4}(\pi_1\sigma_3 - \pi_2\sigma_2 + \pi_3\sigma_1 - \pi_4)\\
&= -15Z - 305Z^2 - 2325Z^3 - 7875Z^4 - 10000Z^5,\\
\sigma_5 &= \zeta_0\zeta_1\cdots\zeta_4 = \frac{1}{5}(\pi_1\sigma_4 - \pi_2\sigma_3 + \pi_3\sigma_2 - \pi_4\sigma_1 + \pi_5)\\
&= Z + 20Z^2 + 150Z^3 + 500Z^4 + 625Z^5.
\end{aligned} \tag{2.4.10}$$

Now, $\zeta_0, \cdots, \zeta_4$ are the roots of

$$\begin{aligned}
&(X-\zeta_0)(X-\zeta_1)(X-\zeta_2)(X-\zeta_3)(X-\zeta_4)\\
&= X^5 - \sigma_1X^4 + \sigma_2X^3 - \sigma_3X^2 + \sigma_4X - \sigma_5 = 0,
\end{aligned} \tag{2.4.11}$$

or,

$$\begin{aligned}
X^5 &- (205Z + 4300Z^2 + 34000Z^3 + 120000Z^4 + 160000Z^5)X^4\\
&- (215Z + 4475Z^2 + 35000Z^3 + 122000Z^4 + 160000Z^5)X^3\\
&- (85Z + 1750Z^2 + 13525Z^3 + 46500Z^4 + 60000Z^5)X^2
\end{aligned}$$

$$
\begin{aligned}
&- (15Z + 305Z^2 + 2325Z^3 + 7875Z^4 + 10000Z^5)X \\
&- (Z + 20Z^2 + 150Z^3 + 500Z^4 + 625Z^5) = 0. \qquad (2.4.12)
\end{aligned}
$$

In particular, ζ is a root, and we obtain (2.4.1). $\square$

Remark. It is truly remarkable, amazing even, that although $\pi_1, \cdots \pi_5$ are polynomials of degree up to 25, $\sigma_1, \cdots, \sigma_5$ are of degree 5.

2.5 Some Important Recurrences and Generating Functions

Let U be the unitizing operator, given by

$$
U\left(\sum_n a(n)q^n\right) = \sum_n a(5n)q^n. \qquad (2.5.1)
$$

It follows from (2.4.1) that for $i \geq 6$, $u_i = U(\zeta^i)$ satisfies the recurrence

$$
\begin{aligned}
u_i &= (205\zeta + 4300\zeta^2 + 34000\zeta^3 + 120000\zeta^4 + 160000\zeta^5)u_{i-1} \\
&+ (215\zeta + 4475\zeta^2 + 35000\zeta^3 + 122000\zeta^4 + 160000\zeta^5)u_{i-2} \\
&+ (85\zeta + 1750\zeta^2 + 13525\zeta^3 + 46500\zeta^4 + 60000\zeta^5)u_{i-3} \\
&+ (15\zeta + 305\zeta^2 + 2325\zeta^3 + 7875\zeta^4 + 10000\zeta^5)u_{i-4} \\
&+ (\zeta + 20\zeta^2 + 150\zeta^3 + 500\zeta^4 + 625\zeta^5)u_{i-5}. \qquad (2.5.2)
\end{aligned}
$$

The recurrence (2.5.2), together with the five initial values $u_1, u_2 \cdots, u_5$, which can be read off from (2.4.3)–(2.4.7) by replacing Z by ζ, gives

$$
\sum_{i\geq 1} u_i x^i = \frac{N}{D} \qquad (2.5.3)
$$

where

$$
\begin{aligned}
N &= (41\zeta + 860\zeta^2 + 6800\zeta^3 + 24000\zeta^4 + 32000\zeta^5)x \\
&+ (86\zeta + 1790\zeta^2 + 14000\zeta^3 + 48800\zeta^4 + 64000\zeta^5)x^2 \\
&+ (51\zeta + 1050\zeta^2 + 8115\zeta^3 + 27900\zeta^4 + 36000\zeta^5)x^3 \\
&+ (12\zeta + 244\zeta^2 + 1869\zeta^3 + 6300\zeta^4 + 8000\zeta^5)x^4 \\
&+ (\zeta + 20\zeta^2 + 150\zeta^3 + 500\zeta^4 + 625\zeta^5)x^5 \qquad (2.5.4)
\end{aligned}
$$

and

$$
\begin{aligned}
D = 1 &- (205\zeta + 4300\zeta^2 + 34000\zeta^3 + 120000\zeta^4 + 160000\zeta^5)x \\
&- (215\zeta + 4475\zeta^2 + 35000\zeta^3 + 122000\zeta^4 + 160000\zeta^5)x^2 \\
&- (85\zeta + 1750\zeta^2 + 13525\zeta^3 + 46500\zeta^4 + 60000\zeta^5)x^3 \\
&- (15\zeta + 305\zeta^2 + 2325\zeta^3 + 7875\zeta^4 + 10000\zeta^5)x^4 \\
&- (\zeta + 20\zeta^2 + 150\zeta^3 + 500\zeta^4 + 625\zeta^5)x^5. \qquad (2.5.5)
\end{aligned}
$$

From (2.5.3)–(2.5.5) we deduce that for $i \geq 1$,

$$U(\zeta^i) = u_i = \sum_{j=1}^{5i} \mu_{i,j}\zeta^j \qquad (2.5.6)$$

where the $\mu_{i,j}$ are given by

$$\sum_{i=1}^{\infty}\sum_{j=1}^{5i} \mu_{i,j}x^i y^j = \frac{N'}{D'} \qquad (2.5.7)$$

where

$$
\begin{aligned}
N' = &(41y + 860y^2 + 6800y^3 + 24000y^4 + 32000y^5)x \\
&+ (86y + 1790y^2 + 14000y^3 + 48800y^4 + 64000y^5)x^2 \\
&+ (51y + 1050y^2 + 8115y^3 + 27900y^4 + 36000y^5)x^3 \\
&+ (12y + 244y^2 + 1869y^3 + 6300y^4 + 8000y^5)x^4 \\
&+ (y + 20y^2 + 150y^3 + 500y^4 + 625y^5)x^5 \qquad (2.5.8)
\end{aligned}
$$

and

$$
\begin{aligned}
D' = 1 &- (205y + 4300y^2 + 34000y^3 + 120000y^4 + 160000y^5)x \\
&- (215y + 4475y^2 + 35000y^3 + 122000y^4 + 160000y^5)x^2 \\
&- (85y + 1750y^2 + 13525y^3 + 46500y^4 + 60000y^5)x^3 \\
&- (15y + 305y^2 + 2325y^3 + 7875y^4 + 10000y^5)x^4 \\
&- (y + 20y^2 + 150y^3 + 500y^4 + 625y^5)x^5. \qquad (2.5.9)
\end{aligned}
$$

More importantly, if we multiply (2.4.1) by γ and apply the operator U, we see that

$v_i = U(\gamma\zeta^{i-1})$ satisfy the recurrence (2.5.2) (with v for u).

Also, using the same sort of calculations as in §2.4 (see §2.9 Appendix),

$$v_1 = U(\gamma) = \delta(1 + 160\zeta + 2800\zeta^2 + 16000\zeta^3 + 32000\zeta^4), \tag{2.5.10}$$

$$\begin{aligned} v_2 = U(\gamma\zeta) = \delta(&385\zeta + 40100\zeta^2 + 1312800\zeta^3 + 20912000\zeta^4 + 189920000\zeta^5 \\ &+ 1043200000\zeta^6 + 3456000000\zeta^7 + 6400000000\zeta^8 + 5120000000\zeta^9), \end{aligned} \tag{2.5.11}$$

$$\begin{aligned} v_3 = U(\gamma\zeta^2) = \delta(&290\zeta + 119015\zeta^2 + 11235600\zeta^3 + 476348000\zeta^4 + 11537760000\zeta^5 \\ &+ 179434400000\zeta^6 + 1908992000000\zeta^7 + 14377472000000\zeta^8 \\ &+ 77783040000000\zeta^9 + 301644800000000\zeta^{10} + 821248000000000\zeta^{11} \\ &+ 1495040000000000\zeta^{12} + 1638400000000000\zeta^{13} + 819200000000000\zeta^{14}), \end{aligned} \tag{2.5.12}$$

$$\begin{aligned} v_4 = U(\gamma\zeta^3) = \delta(&99\zeta + 157795\zeta^2 + 36522125\zeta^3 + 3308569500\zeta^4 \\ &+ 161943150000\zeta^5 + 4995603800000\zeta^6 + 105933588800000\zeta^7 \\ &+ 1628976896000000\zeta^8 + 18797435520000000\zeta^9 + 166360908800000000\zeta^{10} \\ &+ 1143762304000000000\zeta^{11} + 6142300160000000000\zeta^{12} \\ &+ 25729781760000000000\zeta^{13} + 83330457600000000000\zeta^{14} \\ &+ 204857344000000000000\zeta^{15} + 370032640000000000000\zeta^{16} \\ &+ 463667200000000000000\zeta^{17} + 360448000000000000000\zeta^{18} \\ &+ 131072000000000000000\zeta^{19}) \end{aligned} \tag{2.5.13}$$

and

$$\begin{aligned} v_5 = U(\gamma\zeta^4) = \delta(&16\zeta + 118090\zeta^2 + 63835100\zeta^3 + 11315760375\zeta^4 \\ &+ 1002222145000\zeta^5 + 53778439200000\zeta^6 + 1946392973200000\zeta^7 \\ &+ 50789296612000000\zeta^8 + 998696483520000000\zeta^9 + 15256932894400000000\zeta^{10} \\ &+ 185007570368000000000\zeta^{11} + 1807671489280000000000\zeta^{12} \\ &+ 14376293539840000000000\zeta^{13} + 93630345523200000000000\zeta^{14} \\ &+ 500636522496000000000000\zeta^{15} + 2195582095360000000000000\zeta^{16} \\ &+ 7860788428800000000000000\zeta^{17} + 22768123904000000000000000\zeta^{18} \end{aligned}$$

$$+ 52564656128000000000000000\zeta^{19} + 94522572800000000000000000\zeta^{20}$$
$$+ 127664128000000000000000000\zeta^{21} + 121896960000000000000000000\zeta^{22}$$
$$+ 73400320000000000000000000\zeta^{23} + 209715200000000000000000000\zeta^{24}). \tag{2.5.14}$$

It follows that for $i \geq 1$,

$$U(\gamma\zeta^{i-1}) = \delta \sum_{j=1}^{5i} \alpha_{i,j}\zeta^{j-1} \tag{2.5.15}$$

where

$$\sum_{i=1}^{\infty} \sum_{j=1}^{5i} \alpha_{i,j} x^i y^j = \frac{N_\alpha}{D'} \tag{2.5.16}$$

where

$$\begin{aligned} N_\alpha = {} & (y + 160y^2 + 2800y^3 + 16000y^4 + 32000y^5)x \\ & + (180y^2 + 3000y^3 + 16800y^4 + 32000y^5)x^2 \\ & + (75y^2 + 1215y^3 + 6600y^4 + 12000y^5)x^3 \\ & + (14y^2 + 220y^3 + 1150y^4 + 2000y^5)x^4 \\ & + (y^2 + 15y^3 + 75y^4 + 125y^5)x^5 \end{aligned} \tag{2.5.17}$$

and D' is given in (2.5.9).

Similarly, if we multiply (2.4.1) by $q^{-1}\delta$ and apply the operator U, we see that $w_i = U(q^{-1}\delta\zeta^{i-1})$ satisfy (2.5.2) (with w for u).

Also,

$$w_1 = U(q^{-1}\delta) = \gamma(5 + 660\zeta + 14400\zeta^2 + 120000\zeta^3 + 448000\zeta^4 + 640000\zeta^5), \tag{2.5.18}$$

$$\begin{aligned} w_2 = U(q^{-1}\delta\zeta) = \gamma(&1 + 1705\zeta + 171700\zeta^2 + 6083200\zeta^3 + 110016000\zeta^4 \\ & + 178080000\zeta^5 + 797120000\zeta^6 + 34688000000\zeta^7 + 94720000000\zeta^8 \\ & + 148480000000\zeta^9 + 1024000000000\zeta^{10}), \end{aligned} \tag{2.5.19}$$

$$w_3 = U(q^{-1}\delta\zeta^2) = \gamma(1545\zeta + 523885\zeta^2 + 48836000\zeta^3 + 2157580000\zeta^4$$

$$+ 55972480000\zeta^5 + 950485600000\zeta^6 + 11233328000000\zeta^7$$
$$+ 95713408000000\zeta^8 + 598718720000000\zeta^9 + 2762265600000000\zeta^{10}$$
$$+ 9317888000000000\zeta^{11} + 22405120000000000\zeta^{12} + 36454400000000000\zeta^{13}$$
$$+ 36044800000000000\zeta^{14} + 16384000000000000\zeta^{15}),$$
$$(2.5.20)$$

$$w_4 = U(q^{-1}\delta\zeta^3) = \gamma(686\zeta + 753625\zeta^2 + 161075075\zeta^3 + 14497246500\zeta^4$$
$$+ 727863490000\zeta^5 + 23458401400000\zeta^6 + 526452595200000\zeta^7$$
$$+ 8658501792000000\zeta^8 + 107918950400000000\zeta^9 + 1042082905600000000\zeta^{10}$$
$$+ 7904596864000000000\zeta^{11} + 47450048000000000000\zeta^{12}$$
$$+ 225774243840000000000\zeta^{13} + 847926476800000000000\zeta^{14}$$
$$+ 2486042624000000000000\zeta^{15} + 5577277440000000000000\zeta^{16}$$
$$+ 9255321600000000000000\zeta^{17} + 10715136000000000000000\zeta^{18}$$
$$+ 7733248000000000000000\zeta^{19} + 2621440000000000000000\zeta^{20})$$
$$(2.5.21)$$

and

$$w_5 = U(q^{-1}\delta\zeta^4) = \gamma(163\zeta + +630970\zeta^2 + 295013300\zeta^3$$
$$+ 50030923625\zeta^4 + 4413689785000\zeta^5 + 240963519250000\zeta^6$$
$$+ 8992052284600000\zeta^7 + 244243690752000000\zeta^8 + 5037514186320000000\zeta^9$$
$$+ 81262009334400000000\zeta^{10} + 1047144506208000000000\zeta^{11}$$
$$+ 10942698476160000000000\zeta^{12} + 93715045227520000000000\zeta^{13}$$
$$+ 662259232256000000000000\zeta^{14} + 3875774510080000000000000\zeta^{15}$$
$$+ 18796453150720000000000000\zeta^{16} + 75357109452800000000000000\zeta^{17}$$
$$+ 248290942976000000000000000\zeta^{18} + 665623035904000000000000000\zeta^{19}$$
$$+ 1429384069120000000000000000\zeta^{20} + 2401107968000000000000000000\zeta^{21}$$
$$+ 3040870400000000000000000000\zeta^{22} + 2731540480000000000000000000\zeta^{23}$$
$$+ 1551892480000000000000000000\zeta^{24} + 419430400000000000000000000\zeta^{25}).$$
$$(2.5.22)$$

It follows that for $i \geq 1$,

$$U(q^{-1}\delta\zeta^{i-1}) = \gamma \sum_{j=1}^{5i+1} \beta_{i,j}\zeta^{j-1} \tag{2.5.23}$$

where

$$\sum_{i=1}^{\infty}\sum_{j=1}^{5i+1} \beta_{i,j}x^i y^j = \frac{N_\beta}{D'} \tag{2.5.24}$$

where

$$\begin{aligned}
N_\beta = {} & (5y + 660y^2 + 14400y^3 + 120000y^4 + 448000y^5 + 640000y^6)x \\
& + (y + 680y^2 + 14900y^3 + 123200y^4 + 456000y^5 + 640000y^6)x^2 \\
& + (265y^2 + 5785y^3 + 47500y^4 + 174000y^5 + 240000y^6)x^3 \\
& + (46y^2 + 1000y^3 + 8150y^4 + 29500y^5 + 40000y^6)x^4 \\
& + (3y^2 + 65y^3 + 525y^4 + 1875y^5 + 2500y^6)x^5
\end{aligned} \tag{2.5.25}$$

and D' is given in (2.5.9).

2.6 Proof of the First Part of Theorem 2.1.1

The first part of Theorem 2.1.1 follows by a simple induction from (2.3.1), (2.5.15) and (2.5.23), as we now demonstrate.

We know that (2.1.5) is true for $\alpha = 1$.

Suppose (2.1.5) is true for some $\alpha \geq 1$.

Then

$$\sum_{n\geq 0} p_D\left(5^{2\alpha-1}n + \frac{5^{2\alpha}-1}{24}\right)q^n = \gamma \sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\zeta^{i-1}. \tag{2.6.1}$$

If we apply the operator U to (2.6.1) and use (2.5.15), we find

$$\begin{aligned}
\sum_{n\geq 0} p_D\left(5^{2\alpha-1}(5n) + \frac{5^{2\alpha}-1}{24}\right)q^n &= \sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}U(\gamma\zeta^{i-1}) \\
&= \sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\delta\sum_{j=1}^{5i}\alpha_{i,j}\zeta^{j-1}
\end{aligned}$$

$$= \delta \sum_{j=1}^{(5^{2\alpha+1}-5)/24} \left(\sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\alpha_{i,j} \right) \zeta^{j-1}$$
$$= \delta \sum_{j=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,j}\zeta^{j-1},$$

or,

$$\sum_{n\geq 0} p_D \left(5^{2\alpha}n + \frac{5^{2\alpha}-1}{24}\right) q^n = \delta \sum_{j=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,j}\zeta^{j-1},$$

which is (2.1.6).

Now suppose (2.1.6) is true for some $\alpha \geq 1$.

Then

$$\sum_{n\geq 0} p_D \left(5^{2\alpha}n + \frac{5^{2\alpha}-1}{24}\right) q^{n-1} = q^{-1}\delta \sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\zeta^{i-1}. \tag{2.6.2}$$

If we apply the operator U to (2.6.2) and use (2.5.23), we find

$$\sum_{n\geq 0} p_D \left(5^{2\alpha}(5n+1) + \frac{5^{2\alpha}-1}{24}\right) q^n = \sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i} U(q^{-1}\delta\zeta^{i-1})$$
$$= \sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\gamma \sum_{j=1}^{5i+1} \beta_{i,j}\zeta^{j-1}$$
$$= \gamma \sum_{j=1}^{(5^{2\alpha+2}-1)/24} \left(\sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\beta_{i,j} \right) \zeta^{j-1}$$
$$= \gamma \sum_{j=1}^{(5^{2\alpha+2}-1)/24} x_{2\alpha+1,j}\zeta^{j-1},$$

or,

$$\sum_{n\geq 0} p_D \left(5^{2\alpha+1}n + \frac{5^{2\alpha+2}-1}{24}\right) q^n = \gamma \sum_{j=1}^{(5^{2\alpha+2}-1)/24} x_{2\alpha+1,j}\zeta^{j-1},$$

which is (2.1.5) with $\alpha+1$ for α. $\square$

2.7 Proof of the Second Part of Theorem 2.1.1

Let $\nu(n)$ denote the (highest) power of 5 that divides n.

We prove the following theorem.

Theorem 2.7.1.

$$\nu(\alpha_{i,j}) \geq \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \tag{2.7.1}$$

$$\nu(\beta_{i,j}) \geq \left\lfloor \frac{5j-i-1}{6} \right\rfloor. \tag{2.7.2}$$

Proof. Let $\lambda_{i,j} = \nu(\alpha_{i,j})$, $\rho_{i,j} = \left\lfloor \dfrac{5j-i-1}{6} \right\rfloor$.

Observe that from the recurrence (2.5.2), for $i, j \geq 6$,

$$\begin{aligned}\lambda_{i,j} \geq \min(&\lambda_{i-1,j-1}+1, \lambda_{i-1,j-2}+2, \lambda_{i-1,j-3}+3, \lambda_{i-1,j-4}+4, \lambda_{i-1,j-5}+4\\ &\lambda_{i-2,j-1}+1, \lambda_{i-2,j-2}+2, \lambda_{i-2,j-3}+4, \lambda_{i-2,j-4}+3, \lambda_{i-2,j-5}+4,\\ &\lambda_{i-3,j-1}+1, \lambda_{i-3,j-2}+3, \lambda_{i-3,j-3}+2, \lambda_{i-3,j-4}+3, \lambda_{i-3,j-5}+4,\\ &\lambda_{i-4,j-1}+1, \lambda_{i-4,j-2}+1, \lambda_{i-4,j-3}+2, \lambda_{i-4,j-4}+3.\lambda_{i-4,j-5}+4,\\ &\lambda_{i-5,j-1}+0, \lambda_{i-5,j-2}+1, \lambda_{i-5,j-3}+2, \lambda_{i-5,j-4}+3, \lambda_{i-5,j-5}+4).\end{aligned} \tag{2.7.3}$$

On the other hand,

$$\begin{aligned}\rho_{i,j} = \min(&\rho_{i-1,j-1}+1, \rho_{i-1,j-2}+2, \rho_{i-1,j-3}+3, \rho_{i-1,j-4}+4, \rho_{i-1,j-5}+4\\ &\rho_{i-2,j-1}+1, \rho_{i-2,j-2}+2, \rho_{i-2,j-3}+4, \rho_{i-2,j-4}+3, \rho_{i-2,j-5}+4,\\ &\rho_{i-3,j-1}+1, \rho_{i-3,j-2}+3, \rho_{i-3,j-3}+2, \rho_{i-3,j-4}+3, \rho_{i-3,j-5}+4,\\ &\rho_{i-4,j-1}+1, \rho_{i-4,j-2}+1, \rho_{i-4,j-3}+2, \rho_{i-4,j-4}+3, \rho_{i-4,j-5}+4,\\ &\rho_{i-5,j-1}+0, \rho_{i-5,j-2}+1, \rho_{i-5,j-3}+2, \rho_{i-5,j-4}+3, \rho_{i-5,j-5}+4).\end{aligned} \tag{2.7.4}$$

For, the right side of (2.7.4)

$$\begin{aligned}= \min\Bigg(&\left\lfloor \frac{5j-i+1}{6} \right\rfloor, \left\lfloor \frac{5j-i+2}{6} \right\rfloor, \left\lfloor \frac{5u-i+3}{6} \right\rfloor, \left\lfloor \frac{5j-i+4}{6} \right\rfloor, \left\lfloor \frac{5j-i-1}{6} \right\rfloor\\ &\left\lfloor \frac{5j-i+2}{6} \right\rfloor, \left\lfloor \frac{5j-i+3}{6} \right\rfloor, \left\lfloor \frac{5j-i+10}{6} \right\rfloor, \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \left\lfloor \frac{5j-i}{6} \right\rfloor,\\ &\left\lfloor \frac{5j-i+3}{6} \right\rfloor, \left\lfloor \frac{5j-i+10}{6} \right\rfloor, \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \left\lfloor \frac{5j-i}{6} \right\rfloor, \left\lfloor \frac{5j-i+1}{6} \right\rfloor,\end{aligned}$$

$$
\left\lfloor\frac{5j-i+4}{6}\right\rfloor, \left\lfloor\frac{5j-i-1}{6}\right\rfloor, \left\lfloor\frac{5j-i}{6}\right\rfloor, \left\lfloor\frac{5j-i+1}{6}\right\rfloor, \left\lfloor\frac{5j-i+3}{6}\right\rfloor,
$$
$$
\left\lfloor\frac{5j-i-1}{6}\right\rfloor, \left\lfloor\frac{5j-i}{6}\right\rfloor, \left\lfloor\frac{5j-i+1}{6}\right\rfloor, \left\lfloor\frac{5j-i+2}{6}\right\rfloor, \left\lfloor\frac{5j-i+3}{6}\right\rfloor\Bigg)
$$
$$
= \left\lfloor\frac{5j-i-1}{6}\right\rfloor = \rho_{i,j}.
$$

The values of $\lambda_{i,j} - \rho_{i,j}$ for $1 \le i \le 5$ and for $1 \le j \le 5$ are given in the following tables. Note that they are all non-negative. (We use $\bullet$ for ∞.)

	j																										
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	
i 1	0	0	0	0	0	$\bullet$	$\cdots$																				
2	$\bullet$	0	0	0	0	0	0	0	1	0	$\bullet$	$\cdots$															
3	$\bullet$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\bullet$	$\cdots$										
4	$\bullet$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\bullet$	$\cdots$					
5	$\bullet$	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	1	1	1	0	0	$\bullet$	$\cdots$

(2.7.5)

	i																						
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	
j 1	0	$\bullet$	$\cdots$																				
2	0	0	0	0	0	0	$\bullet$	$\cdots$															
3	0	0	0	0	0	0	0	1	0	0	0	$\bullet$	$\cdots$										
4	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	$\bullet$	$\cdots$					
5	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	$\bullet$	$\cdots$

(2.7.6)

(2.7.1) follows from (2.7.3)–(2.7.6) by induction.

The proof of (2.7.2) is essentially the same as that of (2.7.1). The boundary values are given by the following tables.

$$
\begin{array}{cc|ccccccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & j & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\
i & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & \cdots \\
 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \cdots \\
 & 3 & \bullet & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \cdots \\
 & 4 & \bullet & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \cdots \\
 & 5 & \bullet & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & \bullet & \cdots
\end{array}
\tag{2.7.7}
$$

$$
\begin{array}{cc|ccccccccccccccccccccccc}
 & i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\
j & 1 & 1 & 0 & \bullet & \cdots \\
 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \bullet & \cdots \\
 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \bullet & \cdots \\
 & 4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \bullet & \cdots \\
 & 5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \bullet & \cdots
\end{array}
\tag{2.7.8}
$$

$\square$

Theorem 2.7.2. *For $\alpha \geq 0$,*

$$\nu(x_{2\alpha+1,1}) \geq \alpha, \quad \nu(x_{2\alpha+1,i}) \geq \alpha + \left\lfloor \frac{5i-8}{6} \right\rfloor \text{ for } i \geq 2, \tag{2.7.9}$$

$$\nu(x_{2\alpha+2,i}) \geq \alpha + \left\lfloor \frac{5i-2}{6} \right\rfloor. \tag{2.7.10}$$

Proof. If we replace $\nu(A)$ by $\left(\left\lfloor \frac{5j-i-1}{6} \right\rfloor\right)_{i,j\geq 1}$ and $\nu(B)$ by $\left(\left\lfloor \frac{5j-i-1}{6} \right\rfloor\right)_{i,j\geq 1}$ with the exception $\nu(b_{1,1}) = 1$, and we start with $\nu(\mathbf{x}_1) = (0, \infty, \cdots)$, the results follow by induction. $\square$

This completes the proof of Theorem 2.1.1. $\square$

2.8 Calculations

We find that

$$\mathbf{x}_1 = (1,\, 0,\, \cdots), \tag{2.8.1}$$

$$\mathbf{x}_2 = (1,\, 160,\, 2800,\, 16000,\, 32000,\, 0,\, \cdots), \tag{2.8.2}$$

$$\begin{aligned}\mathbf{x}_3 = (&5*33,\, 2^2*5*1039573,\, 2^4*5^2*84358511,\, 2^6*5^3*1519417629,\\ &2^8*5^3*57468885219,\, 2^{10}*5^4*239126250621,\, 2^{20}*5^6*493702983,\\ &2^{16}*5^7*57851635449,\, 2^{17}*5^8*155363323153,\, 2^{22}*5^8*99443868167,\\ &2^{20}*5^9*1277863945093,\, 2^{23}*5^{11}*82117001559,\, 2^{24}*5^{12}*85675198911,\\ &2^{29}*5^{14}*916288433,\, 2^{29}*5^{13}*32357578059,\, 2^{33}*5^{14}*2366343709,\\ &2^{36}*5^{16}*57370733,\, 2^{37}*5^{17}*22998577,\, 2^{36}*5^{18}*30309607,\\ &2^{38}*5^{18}*20313321,\, 2^{40}*5^{19}*2181069,\, 2^{43}*5^{21}*18319,\\ &2^{48}*5^{23}*29,\, 2^{46}*5^{22}*521,\, 2^{49}*5^{22}*37,\, 2^{50}*5^{23},\, 0,\, \cdots),\end{aligned} \tag{2.8.3}$$

in agreement with Baruah and Begum and

$$\nu(\mathbf{x}_1) = (0,\, \infty,\, \cdots), \tag{2.8.4}$$

$$\nu(\mathbf{x}_2) = (0,\, 1,\, 2,\, 3,\, 3,\, \infty,\, \cdots), \tag{2.8.5}$$

$$\begin{aligned}\nu(\mathbf{x}_3) = (&1,\, 1,\, 2,\, 3,\, 3,\, 4,\, 6,\, 7,\, 8,\, 8,\, 9,\, 11,\, 12,\, 14,\, 13,\, 14,\, 16,\, 17,\, 18,\, 18,\, 19,\, 21,\\ &23,\, 22,\, 22,\, 23,\, \infty,\, \cdots).\end{aligned} \tag{2.8.6}$$

2.9 Proof of (2.4.3)

We provide a proof of (2.4.3). The proofs of (2.4.4)–(2.4.7), (2.5.10)–(2.5.14) and (2.5.18)–(2.5.22) are similar but lengthier.

We require the following results.

Lemma. *Let*

$$K = q^{-1}\frac{\chi(-q^5)^5}{\chi(-q)} = q^{-1}\frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5}. \tag{2.9.1}$$

Then

$$\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} = \frac{4}{K}, \tag{2.9.2}$$

$$\frac{1}{R(q)R(q^2)^2} - q^2 R(q) R(q^2)^2 = qK, \tag{2.9.3}$$

$$\frac{R(q)}{R(q^2)^3} + q^2 \frac{R(q^2)^3}{R(q)} = q\left(K - 2 + \frac{4}{K}\right), \tag{2.9.4}$$

$$\frac{1}{R(q)^3 R(q^2)} + q^2 R(q)^3 R(q^2) = q\left(K + 2 + \frac{4}{K}\right), \tag{2.9.5}$$

$$\frac{1}{R(q)^5} - q^2 R(q)^5 = q\left(K + 4 + \frac{8}{K} + \frac{16}{K^2}\right), \tag{2.9.6}$$

$$\frac{R(q^2)}{R(q)^7} + q^2 \frac{R(q)^7}{R(q^2)} = q\left(K + 6 + \frac{20}{K} + \frac{32}{K^2} + \frac{64}{K^3}\right), \tag{2.9.7}$$

$$\frac{1}{R(q)^{10}} + q^4 R(q)^{10} = q^2\left(K^2 + 8K + 34 + \frac{96}{K} + \frac{192}{K^2} + \frac{2546}{K^3} + \frac{256}{K^4}\right), \tag{2.9.8}$$

$$\frac{1}{R(q)^8 R(q^2)} - q^4 R(q)^8 R(q^2) = q^2\left(K^2 + 6K + 20 + \frac{44}{K} + \frac{64}{K^2} + \frac{64}{K^3}\right), \tag{2.9.9}$$

$$\frac{R(q^2)}{R(q)^{12}} - q^4 \frac{R(q)^{12}}{R(q^2)} = q^2\left(K^2 + 10K + 52 + \frac{180}{K} + \frac{448}{K^2} + \frac{832}{K^3} + \frac{1024}{K^4} + \frac{1024}{K^5}\right), \tag{2.9.10}$$

$$K+1=q^{-1}\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4}, \tag{2.9.11}$$

$$1-\frac{4}{K}=\frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4} \tag{2.9.12}$$

and

$$\frac{1}{K-4}=\zeta. \tag{2.9.13}$$

Proofs of (2.9.2)–(2.9.5). (2.9.2) is (2.2.9), (2.9.3) is (2.2.11), (2.9.4) is (2.2.13) and (2.9.5) is (2.2.14). $\square$

Proof of (2.9.6).

$$\begin{aligned}\frac{1}{R(q)^5}-q^2R(q^5)&=\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)\left(\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)\right)\\&\quad+\left(\frac{1}{R(q)R(q^2)^2}-q^2R(q)R(q^2)^2\right)\\&=\frac{4}{K}\cdot q\left(K+2+\frac{4}{K}\right)+qK\\&=q\left(K+4+\frac{8}{K}+\frac{16}{K^2}\right).\end{aligned}$$

$\square$

Proof of (2.9.7).

$$\begin{aligned}\frac{R(q^2)}{R(q)^7}+q^2\frac{R(q)^7}{R(q^2)}&=\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)\left(\frac{1}{R(q)^5}-q^2R(q)^5\right)\\&\quad+\left(\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)\right)\\&=\frac{4}{K}\cdot q\left(K+4+\frac{8}{K}+\frac{16}{K^2}\right)+q\left(K+2+\frac{4}{K}\right)\\&=q\left(K+6+\frac{20}{K}+\frac{32}{K^2}+\frac{64}{K^3}\right).\end{aligned}$$

$\square$

Proof of (2.9.8).

$$\begin{aligned}
\frac{1}{R(q)^{10}} + q^4 R(q)^{10} &= \left(\frac{1}{R(q)^5} - q^2 R(q)^5\right)^2 + 2q^2 \\
&= q^2\left(K + 4 + \frac{8}{K} + \frac{16}{K^2}\right)^2 + 2q^2 \\
&= q^2\left(K^2 + 8K + 34 + \frac{96}{K} + \frac{192}{K^2} + \frac{2546}{K^3} + \frac{256}{K^4}\right).
\end{aligned}$$

$\square$

Proof of (2.9.9).

$$\begin{aligned}
&\frac{1}{R(q)^8 R(q^2)} - q^4 R(q)^8 R(q^2) \\
&\quad= \left(\frac{1}{R(q)^5} - q^2 R(q)^5\right)\left(\frac{1}{R(q)^3 R(q^2)} + q^2 R(q)^3 R(q^2)\right) - q^2\left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)}\right) \\
&\quad= q\left(K + 4 + \frac{8}{K} + \frac{16}{K^2}\right)\cdot q\left(K + 2 + \frac{4}{K}\right) - q^2\left(\frac{4}{K}\right) \\
&\quad= q^2\left(K^2 + 6K + 20 + \frac{44}{K} + \frac{64}{K^2} + \frac{64}{K^3}\right).
\end{aligned}$$

$\square$

Proof of (2.9.10).

$$\begin{aligned}
&\frac{R(q^2)}{R(q)^{12}} - q^4\frac{R(q)^{12}}{R(q^2)} \\
&\quad= \left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)}\right)\left(\frac{1}{R(q)^{10}} + q^4 R(q)^{10}\right) + \left(\frac{1}{R(q)^8 R(q^2)} - q^4 R(q)^8 R(q^2)\right) \\
&\quad= \frac{4}{K}\cdot q^2\left(K^2 + 8K + 34 + \frac{96}{K} + \frac{192}{K^2} + \frac{256}{K^3} + \frac{256}{K^3}\right) \\
&\qquad\quad + q^2\left(K^2 + 6K + 20 + \frac{44}{K} + \frac{64}{K^2} + \frac{64}{K^3}\right) \\
&\quad= q^2\left(K^2 + 10K + 52 + \frac{180}{K} + \frac{448}{K^2} + \frac{832}{K^3} + \frac{1024}{K^4} + \frac{1024}{K^5}\right).
\end{aligned}$$

$\square$

Proofs of (2.9.11) *and* (2.9.12). (2.9.11) is (2.2.15) and (2.9.12) is (2.2.16). □

Proof of (2.9.13).

$$K-4=K\left(1-\frac{4}{K}\right)=q^{-1}\frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5}\cdot\frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4}=q^{-1}\frac{E(q)^3E(q^5)}{E(q^2)E(q^{10})^3}=\frac{1}{\zeta},$$

from which the result follows. □

Proof of (2.4.3). We start by noting that (2.4.3) is equivalent to

$$U(\zeta)=41\zeta+860\zeta^2+6800\zeta^3+24000\zeta^4+32000\zeta^5. \tag{2.9.14}$$

We have

$$\begin{aligned}
&U(\zeta)=U\left(q\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)\\
&=\frac{E(q^2)^3}{E(q)}U\left(q\frac{E(q^2)}{E(q)^3}\right)\\
&=\frac{E(q^2)^3}{E(q)}U\left(qE(q^{50})\left(\frac{1}{R(q^{10})}-q^2-q^4R(q^{10})\right)\right.\\
&\qquad\times\left(\frac{E(q^{25})^5}{E(q^5)^6}\right)^3\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4\right.\\
&\qquad\qquad\left.\left.-3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\right)^3\right)\\
&=\frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\left(51q\left(\frac{1}{R(q)^8R(q^2)}-q^4R(q)^8R(q^2)\right)\right.\\
&\qquad-9q\left(\frac{1}{R(q)^{10}}+q^4R(q)^{10}\right)-q\left(\frac{R(q^2)}{R(q)^{12}}-q^4\frac{R(q)^{12}}{R(q^2)}\right)\\
&\qquad+153q^2\left(\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)\right)-177q^2\left(\frac{1}{R(q)^5}-q^2R(q)^5\right)\\
&\qquad\left.-78q^2\left(\frac{R(q^2)}{R(q)^7}+q^2\frac{R(q)^7}{R(q^2)}\right)-219q^3\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)-71q^3\right)\\
&=\frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\\
&\times\left(51q\cdot q^2\left(K^2+6K+20+\frac{44}{K}+\frac{64}{K^2}+\frac{64}{K^3}\right)\right.\\
&\qquad-9q\cdot q^2\left(K^2+8K+34+\frac{96}{K}+\frac{192}{K^2}+\frac{256}{K^3}+\frac{256}{K^4}\right)
\end{aligned}$$

$$
\begin{aligned}
&-q\cdot q^2\left(K^2+10K+52+\frac{180}{K}+\frac{448}{K^2}+\frac{832}{K^3}+\frac{1024}{K^4}+\frac{1024}{K^5}\right)\\
&+153q^2\cdot q\left(K+2+\frac{4}{K}\right)-177q^2\cdot q\left(K+4+\frac{8}{K}+\frac{16}{K^2}\right)\\
&-78q^2\cdot q\left(K+6+\frac{20}{K}+\frac{32}{K^2}+\frac{64}{K^3}\right)-219q^3\left(\frac{4}{K}\right)-71q^3\Big)\\
=\ & q^3\frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\\
\times\ & \frac{(K+1)^2(K-4)}{K^5}(41K^4+204K^3+416K^2+384K+256)\\
=\ & q^3\frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\cdot\frac{(K+1)^2(K-4)}{K^5}\\
\times\ & (41(K-4)^4+860(K-4)^3+6800(K-4)^2+24000(K-4)+32000)\\
=\ & q^3\frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\cdot\frac{(K+1)^2(K-4)^6}{K^5}\\
\times\ & \left(\frac{41}{K-4}+\frac{860}{(K-4)^2}+\frac{6800}{(K-4)^3}+\frac{24000}{(K-4)^4}+\frac{32000}{(K-4)^5}\right)\\
=\ & q^3\left(\frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\right)\left(q^{-1}\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4}\right)^2\left(q^{-1}\frac{E(q)^3E(q^5)}{E(q^2)E(q^{10})^3}\right)^6\\
\times\ & \left(q\frac{E(q)E(q^{10})^5}{E(q^2)E(q^5)^5}\right)^5(41\zeta+860\zeta^2+6800\zeta^3+24000\zeta^4+32000\zeta^5)\\
=\ & 41\zeta+860\zeta^2+6800\zeta^3+24000\zeta^4+32000\zeta^5.
\end{aligned}
$$

□

2.10 Endnotes

Using a similar argument, I [56] obtained an elementary proof of an infinite family of congruences modulo powers of 5 for $g(n)$ given by

$$\sum_{n\geq 0} g(n)q^n=\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{2.10.1}$$

Theorem 2.10.1. *For $\alpha\geq 1$ and $n\geq 0$,*

$$g\left(5^{2\alpha-1}n+\frac{5^{2\alpha}-1}{6}\right)\equiv 0 \pmod{5^\alpha}. \tag{2.10.2}$$

Further, $g(n)$ is closely related to the number of 1-shell totally symmetric plane partitions of n, denoted by $s(n)$. As a consequence, we have the following result.

Theorem 2.10.2. *For $\alpha \geq 1$ and $n \geq 0$,*

$$s\left(2 \cdot 5^{2\alpha-1}n + 5^{2\alpha-1}\right) \equiv 0 \pmod{5^\alpha}. \tag{2.10.3}$$

For a detailed description, see [56].

2.11 References

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Chapter 3

Eta-quotient Representations and Rogers–Ramanujan Continued Fraction

This chapter comes from

- S. Chern and D. Tang, The Rogers–Ramanujan continued fraction and related eta-quotient representations, *Bull. Aust. Math. Soc.* **103** (2021), no. 2, 248–259. (Ref. [66])

As one might have already seen from the previous chapter, in proceeding in the same manner with proofs of (2.4.4)–(2.4.7), (2.5.10)–(2.5.14) and (2.5.18)–(2.5.22), we encounter terms of the form

$$P(\alpha,\beta) := \frac{1}{q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}} + (-1)^{\alpha+\beta}q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta} \tag{3.0.1}$$

with $\alpha \geq 0$. Hence, it is necessary to take further investigation. But before moving forward, let us first review some background materials.

3.1 Background and Results

Recall that

$$R(q) = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)_{\infty} \tag{3.1.1}$$

is indeed the infinite product form of the Rogers–Ramanujan continued fraction that was discovered by Rogers [155], independently by Ramanujan [149], and also independently by Schur [159]. In the literature, the Rogers–Ramanujan continued fraction often refers to the generalized continued fraction

$$\frac{q^{1/5}}{1} \genfrac{}{}{0pt}{}{}{+} \frac{q}{1} \genfrac{}{}{0pt}{}{}{+} \frac{q^2}{1} \genfrac{}{}{0pt}{}{}{+} \frac{q^3}{1} \genfrac{}{}{0pt}{}{}{+} \cdots,$$

but here we will discard the factor of $q^{1/5}$, that is, we define

$$R(q) := \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots}.$$

It is known (see, for example, [98, p. 145]) that (3.1.1) holds.

In the past, modular equations for the Rogers–Ramanujan continued fraction have been studied extensively by many mathematicians, including Rogers and Ramanujan themselves [20, 149, 150, 157, 170]. For example, [98, (40.1.10)] states that

$$\Big(R(q^2) - R(q)^2\Big)\Big(1 + qR(q)R(q^2)^2\Big) = 2qR(q)R(q^2)^3 \tag{3.1.2}$$

and [98, (40.1.12)] states that

$$\Big(R(q^3) - R(q)^3\Big)\Big(1 + q^2R(q)R(q^3)^3\Big) = 3qR(q)^2R(q^3)^2. \tag{3.1.3}$$

Now let us turn our attention to $P(\alpha, \beta)$ defined in (3.0.1) for $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$.

Theorem 3.1.1. *Let K be as in* (2.9.1)*, namely,*

$$K = q^{-1}\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{10};q^{10})_\infty^5}. \tag{3.1.4}$$

Then the following recurrence relations hold:

$$P(\alpha, \beta+1) = 4K^{-1}P(\alpha,\beta) + P(\alpha, \beta-1) \tag{3.1.5}$$

and

$$P(\alpha+2, \beta) = KP(\alpha+1,\beta) + P(\alpha,\beta). \tag{3.1.6}$$

We also have initial values:

$$P(0,0) = 2, \tag{3.1.7}$$

$$P(0,1) = \frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} = 4K^{-1}, \tag{3.1.8}$$

$$P(1,0) = \frac{1}{qR(q)R(q^2)^2} - qR(q)R(q^2)^2 = K \tag{3.1.9}$$

and

$$P(1,-1)=\frac{R(q)}{qR(q^2)^3}+\frac{qR(q^2)^3}{R(q)}=4K^{-1}-2+K. \tag{3.1.10}$$

Interestingly, we also have an analog with $R(q^3)$ involved. Let us define, for $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$,

$$Q(\alpha,\beta):=\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}+(-1)^\alpha q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}. \tag{3.1.11}$$

Theorem 3.1.2. *Let*

$$S=q^{-2}\frac{(q;q)_\infty^3(q^3;q^3)_\infty^3}{(q^5;q^5)_\infty^3(q^{15};q^{15})_\infty^3} \tag{3.1.12}$$

and

$$T=q^{-2}\frac{(q^3;q^3)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{15};q^{15})_\infty^5}. \tag{3.1.13}$$

Then the following recurrence relations hold:

$$Q(\alpha,\beta+1)=\Big(2+9T^{-1}\Big)Q(\alpha,\beta)-Q(\alpha,\beta-1) \tag{3.1.14}$$

and

$$Q(\alpha+2,\beta)=\left(-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T+\frac{3}{2}\right)Q(\alpha+1,\beta)+Q(\alpha,\beta). \tag{3.1.15}$$

We also have initial values:

$$Q(0,0)=2, \tag{3.1.16}$$

$$Q(0,1)=\frac{R(q^3)}{R(q)^3}+\frac{R(q)^3}{R(q^3)}=2+9T^{-1}, \tag{3.1.17}$$

$$Q(1,0)=\frac{1}{qR(q)^2R(q^3)}-qR(q)^2R(q^3)=-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T+\frac{3}{2} \tag{3.1.18}$$

and

$$Q(1,-1)=\frac{R(q)}{qR(q^3)^2}-\frac{qR(q^3)^2}{R(q)}=-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T-\frac{3}{2}. \tag{3.1.19}$$

Remark 3.1.1. Let us take a look at the initial values in Theorems 3.1.1 and 3.1.2. We find that (3.1.8) is (2.2.9), (3.1.9) is (2.2.11) and (3.1.10) is (2.2.13). Also, (3.1.17) is due to Gugg [91]. However, the two complicated identities (3.1.18) and (3.1.19) appear to be novel.

Remark 3.1.2. It follows from (3.1.16), (3.1.17) and the recurrence relation (3.1.14) that

$$Q(0,-1) = \frac{R(q)^3}{R(q^3)} + \frac{R(q^3)}{R(q)^3} = 2 + 9T^{-1}. \tag{3.1.20}$$

Also, Gugg [91, Theorem 5.1] proved that

$$Q(2,-1) = \frac{1}{q^2R(q)R(q^3)^3} + q^2R(q)R(q^3)^3 = -2 + T. \tag{3.1.21}$$

Therefore, we deduce from (3.1.18)–(3.1.21) and (3.1.15) the following relation between S and T.

Corollary 3.1.3. *We have*

$$81 + 144T + 46T^2 - 16T^3 + T^4 - 18ST - 2ST^3 + S^2T^2 = 0. \tag{3.1.22}$$

It follows from (3.1.7) and (3.1.8) together with the recurrence relation (3.1.5) that for each $\beta \in \mathbb{Z}$, we can represent $P(0,\beta)$ in terms of K. Likewise, we have similar representations for $P(1,\beta)$ for each $\beta \in \mathbb{Z}$. Finally, the recurrence relation (3.1.6) reveals that for each $\alpha \geq 2$ and $\beta \in \mathbb{Z}$, we have $P(\alpha,\beta) \in \mathbb{Z}[K,K^{-1}]$. In Table 3.1, we list the representations of $P(\alpha,\beta)$ in terms of K with $0 \leq \alpha \leq 2$ and $-3 \leq \beta \leq 3$. Similar arguments can be applied to $Q(\alpha,\beta)$ to show that for each $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, $Q(\alpha,\beta) \in \mathbb{Q}[S,T,T^{-1}]$. Since the representations of $Q(\alpha,\beta)$ are much lengthier, we will not list them concretely like Table 3.1.

Let $\mathbb{H}$ be the upper half complex plane, and put $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$. For any positive integer N, let $\Gamma_0(N)$ be the Hecke congruence subgroup of level N defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \ c \equiv 0 \pmod{N} \right\}.$$

Let $K_0(N)$ be the field of meromorphic functions on the compact Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$. It follows from Newman [135] that K is in $K_0(10)$, and S and T are both in $K_0(15)$. Hence, we have the following results.

Table 3.1. Representations of $P(\alpha,\beta)$ in $\mathbb{Z}[K,K^{-1}]$

$\beta \backslash \alpha$	0	1
-3	$-64K^{-3}-12K^{-1}$	$64K^{-3}-32K^{-2}+20K^{-1}-6+K$
-2	$16K^{-2}+2$	$-16K^{-2}+8K^{-1}-4+K$
-1	$-4K^{-1}$	$4K^{-1}-2+K$
0	2	K
1	$4K^{-1}$	$4K^{-1}+2+K$
2	$16K^{-2}+2$	$16K^{-2}+8K^{-1}+4+K$
3	$64K^{-3}+12K^{-1}$	$64K^{-3}+32K^{-2}+20K^{-1}+6+K$

$\beta \backslash \alpha$	2
-3	$-64K^{-3}+64K^{-2}-44K^{-1}+20-6K+K^2$
-2	$16K^{-2}-16K^{-1}+10-4K+K^2$
-1	$-4K^{-1}+4-2K+K^2$
0	$2+K^2$
1	$4K^{-1}+4+2K+K^2$
2	$16K^{-2}+16K^{-1}+10+4K+K^2$
3	$64K^{-3}+64K^{-2}+44K^{-1}+20+6K+K^2$

Corollary 3.1.4. *For any $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, $P(\alpha,\beta) \in \mathbb{Z}[K,K^{-1}]$ and hence $P(\alpha,\beta) \in K_0(10)$.*

Corollary 3.1.5. *For any $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, $Q(\alpha,\beta) \in \mathbb{Q}[S,T,T^{-1}]$ and hence $Q(\alpha,\beta) \in K_0(15)$.*

3.2 Proofs of the Recurrences

We shall prove the following identities, from which the recurrence relations (3.1.5), (3.1.6), (3.1.14) and (3.1.15) follow immediately.

$$P(\alpha,\beta)P(0,1) = P(\alpha,\beta+1) - P(\alpha,\beta-1), \tag{3.2.1}$$

$$P(\alpha+1,\beta)P(1,0) = P(\alpha+2,\beta) - P(\alpha,\beta), \tag{3.2.2}$$

$$Q(\alpha,\beta)Q(0,1)=Q(\alpha,\beta+1)+Q(\alpha,\beta-1) \tag{3.2.3}$$

and

$$Q(\alpha+1,\beta)Q(1,0)=Q(\alpha+2,\beta)-Q(\alpha,\beta). \tag{3.2.4}$$

Proof of (3.2.1) *and* (3.2.2). It follows from (2.9.1) that

$$\begin{aligned}
&P(\alpha,\beta)P(0,1)\\
&=\left(\frac{1}{q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}+(-1)^{\alpha+\beta}q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\right)\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)\\
&=\left(\frac{1}{q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}\frac{R(q^2)}{R(q)^2}-(-1)^{\alpha+\beta}q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\frac{R(q)^2}{R(q^2)}\right)\\
&\quad-\left(\frac{1}{q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}\frac{R(q)^2}{R(q^2)}-(-1)^{\alpha+\beta}q^{\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\frac{R(q^2)}{R(q)^2}\right)\\
&=\left(\frac{1}{q^{\alpha}R(q)^{\alpha+2(\beta+1)}R(q^2)^{2\alpha-(\beta+1)}}+(-1)^{\alpha+(\beta+1)}q^{\alpha}R(q)^{\alpha+2(\beta+1)}R(q^2)^{2\alpha-(\beta+1)}\right)\\
&\quad-\left(\frac{1}{q^{\alpha}R(q)^{\alpha+2(\beta-1)}R(q^2)^{2\alpha-(\beta-1)}}+(-1)^{\alpha+(\beta-1)}q^{\alpha}R(q)^{\alpha+2(\beta-1)}R(q^2)^{2\alpha-(\beta-1)}\right)\\
&=P(\alpha,\beta+1)-P(\alpha,\beta-1).
\end{aligned}$$

This is (3.2.1). Also, (3.2.2) follows by a similar argument. □

Proof of (3.2.3) *and* (3.2.4). It follows from (3.1.11) that

$$\begin{aligned}
&Q(\alpha,\beta)Q(0,1)\\
&=\left(\frac{1}{q^{\alpha}R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}+(-1)^{\alpha}q^{\alpha}R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\right)\left(\frac{R(q^3)}{R(q)^3}+\frac{R(q)^3}{R(q^3)}\right)\\
&=\left(\frac{1}{q^{\alpha}R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}\frac{R(q^3)}{R(q)^3}+(-1)^{\alpha}q^{\alpha}R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\frac{R(q)^3}{R(q^3)}\right)\\
&\quad+\left(\frac{1}{q^{\alpha}R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}\frac{R(q)^3}{R(q^3)}+(-1)^{\alpha}q^{\alpha}R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\frac{R(q^3)}{R(q)^3}\right)\\
&=\left(\frac{1}{q^{\alpha}R(q)^{2\alpha+3(\beta+1)}R(q^3)^{\alpha-(\beta+1)}}+(-1)^{\alpha}q^{\alpha}R(q)^{2\alpha+3(\beta+1)}R(q^3)^{\alpha-(\beta+1)}\right)\\
&\quad+\left(\frac{1}{q^{\alpha}R(q)^{2\alpha+3(\beta-1)}R(q^3)^{\alpha-(\beta-1)}}+(-1)^{\alpha}q^{\alpha}R(q)^{2\alpha+3(\beta-1)}R(q^3)^{\alpha-(\beta-1)}\right)\\
&=Q(\alpha,\beta+1)+Q(\alpha,\beta-1).
\end{aligned}$$

This is (3.2.3). Likewise, one may derive (3.2.4). $\square$

3.3 Proofs of (3.1.18) and (3.1.19)

As we have seen in Remark 3.1.1, the only (and true!) difficulty is proving (3.1.18) and (3.1.19). Let us begin with an interesting relation between $Q(1,0)$ and $Q(1,-1)$.

Lemma 3.3.1. *We have*

$$Q(1,0)-Q(1,-1)=\left(\frac{1}{qR(q)^2R(q^3)}-qR(q)^2R(q^3)\right)-\left(\frac{R(q)}{qR(q^3)^2}-\frac{qR(q^3)^2}{R(q)}\right)=3. \tag{3.3.1}$$

Proof. Notice that

$$\text{LHS of (3.3.1)}=\frac{\Big(R(q^3)-R(q)^3\Big)\Big(1+q^2R(q)R(q^3)^3\Big)}{qR(q)^2R(q^3)^2}=3,$$

in the last identity of which we use the modular equation (3.1.3). Therefore, (3.3.1) follows. $\square$

Lemma 3.3.1 implies that if one of (3.1.18) and (3.1.19) is proved, then the other follows automatically.

Now recall that $K_0(N)$ is the field of meromorphic functions on the compact Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$. Further, the U-operator is defined in (2.5.1). A standard result [25, pp. 80–82] states that for any positive integer N, if $f\in K_0(5N)$, we have $U(f)\in K_0(N)$.

Our proof of (3.1.18) relies on a surprisingly neat 5-dissection identity as follows.

Lemma 3.3.2. *We have*

$$U\left(\frac{E(q^3)^2}{E(q)}\right)=\frac{E(q^3)^3E(q^5)^2}{E(q)^3E(q^{15})}. \tag{3.3.2}$$

Proof. It follows from Newman [135] that

$$q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3}\in K_0(15)$$

and

$$q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\in K_0(75).$$

If we compare the Fourier expansions of

$$q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3} \quad \text{and} \quad U\left(q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\right),$$

which are both in $K_0(15)$, it can be observed that

$$U\left(q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\right) = q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3},$$

from which (3.3.2) follows. □

Now we move to prove (3.1.18). It follows from the 5-dissection identities for $E(q)$ and $1/E(q)$, namely, (2.2.1) and (2.2.2), that

$$\begin{aligned}\frac{E(q^3)^2}{E(q)} &= \frac{E(q^{25})^5E(q^{75})^2}{E(q^5)^6}\\ &\times\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4\right.\\ &\qquad\left.-3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\right)\left(\frac{1}{R(q^{15})}-q^3-q^6R(q^{15})\right)^2,\end{aligned}$$

from which we extract

$$\begin{aligned}U\left(\frac{E(q^3)^2}{E(q)}\right) = \frac{q^2E(q^5)^5E(q^{15})^2}{E(q)^6}&\left(\left(\frac{1}{q^2R(q)^4R(q^3)^2}+q^2R(q)^4R(q^3)^2\right)\right.\\ &-4\left(\frac{1}{qR(q)^2R(q^3)}-qR(q)^2R(q^3)\right)-3\left(\frac{R(q)}{qR(q^3)^2}-\frac{qR(q^3)^2}{R(q)}\right)\\ &\left.+2\left(\frac{R(q^3)}{R(q)^3}+\frac{R(q)^3}{R(q^3)}\right)-5\right).\end{aligned}$$

Hence,

$$\frac{E(q^3)^3E(q^5)^2}{E(q)^3E(q^{15})} = \frac{q^2E(q^5)^5E(q^{15})^2}{E(q)^6}\big(Q(2,0)-4Q(1,0)-3Q(1,-1)+2Q(0,1)-5\big),$$

that is,

$$S = Q(2,0)-4Q(1,0)-3Q(1,-1)+2Q(0,1)-5.$$

It follows from (3.2.4) and (3.1.16) that

$$Q(2,0)=Q(1,0)^2+Q(0,0)=Q(1,0)^2+2$$

and from (3.2.4) and (3.1.21) that

$$Q(1,-1)Q(1,0)=Q(2,-1)-Q(0,-1)=-9T^{-1}-4+T.$$

Also, (3.3.1) states that

$$Q(1,0)-Q(1,-1)=3.$$

Therefore,

$$\begin{aligned}
S &= \Big(Q(1,0)^2+2\Big)-4Q(1,0)-3Q(1,-1)+2Q(0,1)-5\\
&= Q(1,0)\Big(Q(1,-1)+3\Big)-4Q(1,0)-3\Big(Q(1,0)-3\Big)+2Q(0,1)-3\\
&= -4Q(1,0)+Q(1,0)Q(1,-1)+2Q(0,1)+6\\
&= -4Q(1,0)+\Big(-9T^{-1}-4+T\Big)+2\Big(2+9T^{-1}\Big)+6\\
&= -4Q(1,0)+9T^{-1}+6+T,
\end{aligned}$$

from which (3.1.18) follows. Further, (3.1.19) follows from (3.1.18) and (3.3.1).

3.4 Endnotes

Mike Hirschhorn emailed me on Aug 28, 2019 with a beautiful bivariate generating function identity:

$$\sum_{\alpha,\beta\geq 0} P(\alpha,\beta)x^\alpha y^\beta = \frac{2-Kx-4K^{-1}y+(K+2+4K^{-1})xy}{(1-Kx-x^2)(1-4K^{-1}y-y^2)} \tag{3.4.1}$$

where K is as in (2.9.1). Analogously, one could obtain

$$\sum_{\alpha,\beta\geq 0} P(\alpha,-\beta)x^\alpha y^\beta = \frac{2-Kx+4K^{-1}y+(K-2+4K^{-1})xy}{(1-Kx-x^2)(1+4K^{-1}y-y^2)}. \tag{3.4.2}$$

A direct application of the "series expansion" command in most computer algebra systems such as *Mathematica* to the above relations makes it easier to find the expression of $P(\alpha,\beta)$; see the discussion in [56, §2.1].

3.5 References

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Part II Identities

Outline

- Chapter 4 is devoted to weighted partition rank and crank moments that are closely related to Andrews–Beck type congruences.
- Chapter 5 is devoted to partitions with bounded part differences in which both analytic and combinatorial aspects will be discussed.
- Chapters 6–8 are devoted to a general theory of span one linked partition ideals.
- Chapter 9 is devoted to analytic identities of Rogers–Ramanujan type based on basic hypergeometric transformation formulas.

Chapter 4

Weighted Partition Rank and Crank Moments

This chapter comes from

- S. Chern, Weighted partition rank and crank moments. I. Andrews–Beck type congruences, to appear in *Proceedings of the Conference in Honor of Bruce Berndt*. (Ref. [58])
- S. Chern, Weighted partition rank and crank moments. II. Odd-order moments, to appear in *Ramanujan J.* (Ref. [59])
- S. Chern, Weighted partition rank and crank moments. III. A list of Andrews–Beck type congruences modulo 5, 7, 11 and 13, to appear in *Int. J. Number Theory*. (Ref. [60])

4.1 Introduction

4.1.1 Rank and Crank of an Integer Partition

Let us first recall the definition of rank and crank of an integer partition λ. We use $\sharp(\lambda)$, $\omega(\lambda)$ and $\ell(\lambda)$ to denote the number of parts in λ, the number of ones in λ and the largest part in λ, respectively.

The rank of λ is defined by Dyson [76]:

$$\operatorname{rank}(\lambda) := \ell(\lambda) - \sharp(\lambda),$$

namely, the largest part minus the number of parts in λ. On the other hand, Andrews and Garvan [23] defined the crank of a partition λ by

$$\operatorname{crank}(\lambda) := \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0, \end{cases}$$

where $\mu(\lambda)$ denotes the number of parts in λ larger than $\omega(\lambda)$.

The two partition statistics were introduced to combinatorially interpret Ramanujan's celebrated congruences:

$$p(5n+4) \equiv 0 \pmod{5}, \tag{4.1.1}$$

$$p(7n+5) \equiv 0 \pmod{7} \tag{4.1.2}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{4.1.3}$$

Now let $N(m,n)$ (resp. $M(m,n)$) count the number of partitions of n whose rank (resp. crank) is m. Further, we shall put $M(-1,1) = -M(0,1) = M(1,1) = 1$ and $M(m,1) = 0$ otherwise.

4.1.2 Ordinary and Symmetrized Rank and Crank Moments

In general, there are two types of rank and crank moments attracting broad research interest. The first type, which is due to Atkin and Garvan [29], is the most natural. Let us define k-th ordinary rank and crank moments respectively by

$$N_k(n) := \sum_{m=-\infty}^{\infty} m^k N(m,n) = \sum_{\lambda \vdash n} \mathrm{rank}^k(\lambda) \tag{4.1.4}$$

and

$$M_k(n) := \sum_{m=-\infty}^{\infty} m^k M(m,n) = \sum_{\lambda \vdash n} \mathrm{crank}^k(\lambda). \tag{4.1.5}$$

In light of the symmetry property that $N(m,n) = N(-m,n)$ and $M(m,n) = M(-m,n)$, we see that the odd order moments are all zero. For the even order moments, Atkin and Garvan [29] showed that the generating functions of $M_k(n)$ are related to quasimodular forms, while Bringmann, Garvan and Mahlburg [42] showed that the generating functions of $N_k(n)$ are related to quasimock theta functions.

On the other hand, Andrews [14] defined the k-th symmetrized rank moment by

$$\eta_k(n) := \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m,n) = \sum_{\lambda \vdash n} \binom{\mathrm{rank}(\lambda) + \lfloor \frac{k-1}{2} \rfloor}{k}. \tag{4.1.6}$$

As a crank analog, Garvan [82] defined the k-th symmetrized crank moment by

$$\mu_k(n) := \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m,n) = \sum_{\lambda \vdash n} \binom{\mathrm{crank}(\lambda) + \lfloor \frac{k-1}{2} \rfloor}{k}. \tag{4.1.7}$$

It was shown that $\eta_k(n) = \mu_k(n) = 0$ when k is odd. Further, the generating functions of the even order symmetrized moments $\eta_{2k}(n)$ and $\mu_{2k}(n)$ can be nicely formulated (cf. [14, 82]):

$$\sum_{n\geq 1} \eta_{2k}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n-1)/2+kn} \frac{1+q^n}{(1-q^n)^{2k}} \tag{4.1.8}$$

and

$$\sum_{n\geq 1} \mu_{2k}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(n-1)/2+kn} \frac{1+q^n}{(1-q^n)^{2k}}. \tag{4.1.9}$$

4.1.3 Main Results

Recall that $\mathcal{P}$ denotes the set of integer partitions. Parallel to (4.1.4), (4.1.5), (4.1.6) and (4.1.7), we define the weighted k-th ordinary and symmetrized rank and crank moments by

$$N_k^\sharp(n) := \sum_{\lambda \vdash n} \sharp(\lambda)\, \mathrm{rank}^k(\lambda), \tag{4.1.10}$$

$$M_k^\omega(n) := \sum_{\lambda \vdash n} \omega(\lambda)\, \mathrm{crank}^k(\lambda), \tag{4.1.11}$$

$$\eta_k^\sharp(n) := \sum_{\lambda \vdash n} \sharp(\lambda) \binom{\mathrm{rank}(\lambda) + \lfloor \frac{k-1}{2} \rfloor}{k} \tag{4.1.12}$$

and

$$\mu_k^\omega(n) := \sum_{\lambda \vdash n} \omega(\lambda) \binom{\mathrm{crank}(\lambda) + \lfloor \frac{k-1}{2} \rfloor}{k}. \tag{4.1.13}$$

For the weighted rank moments, we have relations as follows.

Theorem 4.1.1. *Let k be a positive integer. We have*

$$\sum_{n\geq 1} N_{2k-1}^\sharp(n) q^n = -\frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} \frac{P_{2k}(q^n)}{(1-q^n)^{2k}}, \tag{4.1.14}$$

where $P_k(u)$ is defined recursively by $P_1(u) = 1$ and for $k \geq 1$,

$$P_{k+1}(u) = \Big(1 - u + (k+1)u\Big) P_k(u) + \Big(u - u^2\Big) P_k'(u), \tag{4.1.15}$$

where $P_k'(u)$ as usual denotes $\mathrm{d}P_k(u)/\mathrm{d}u$. In particular,

$$N_{2k-1}^{\sharp}(n) = -\frac{1}{2} N_{2k}(n). \tag{4.1.16}$$

Remark 4.1.1. It is worth pointing out that the polynomials $P_k(u)$ satisfy the exponential generating function

$$\mathscr{P}(u,t) := \sum_{k \geq 1} P_k(u) \frac{t^k}{k!} = -\frac{e^{ut} - e^t}{e^{ut} - ue^t}. \tag{4.1.17}$$

To see this, we translate the recurrence (4.1.15) into the functional equation

$$\frac{\partial}{\partial t} \mathscr{P}(u,t) - 1 = \mathscr{P}(u,t) + ut \frac{\partial}{\partial t} \mathscr{P}(u,t) + \Big(u - u^2\Big) \frac{\partial}{\partial u} \mathscr{P}(u,t).$$

Solving the above PDE with the boundary condition $\mathscr{P}(u,0) = 0$ yields (4.1.17).

Theorem 4.1.2. *Let k be a positive integer. We have*

$$\sum_{n \geq 1} \eta_{2k-1}^{\sharp}(n) q^n = -\frac{k}{(q;q)_\infty} \sum_{n \geq 1} (-1)^{n-1} q^{n(3n-1)/2 + kn} \frac{1 + q^n}{(1 - q^n)^{2k}}. \tag{4.1.18}$$

In particular,

$$\eta_{2k-1}^{\sharp}(n) = -k \cdot \eta_{2k}(n). \tag{4.1.19}$$

We also have crank analogs.

Theorem 4.1.3. *Let k be a positive integer. We have*

$$\sum_{n \geq 1} M_{2k-1}^{\omega}(n) q^n = -\frac{1}{(q;q)_\infty} \sum_{n \geq 1} (-1)^{n-1} q^{n(n+1)/2} \frac{P_{2k}(q^n)}{(1 - q^n)^{2k}}, \tag{4.1.20}$$

where $P_k(u)$ is as in Theorem 4.1.1. In particular,

$$M_{2k-1}^{\omega}(n) = -\frac{1}{2} M_{2k}(n). \tag{4.1.21}$$

Theorem 4.1.4. *Let k be a positive integer. We have*

$$\sum_{n\geq 1} \mu^{\omega}_{2k-1}(n) q^n = -\frac{k}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(n-1)/2+kn} \frac{1+q^n}{(1-q^n)^{2k}}. \tag{4.1.22}$$

In particular,

$$\mu^{\omega}_{2k-1}(n) = -k \cdot \mu_{2k}(n). \tag{4.1.23}$$

4.2 Warm-up: The First Moment

Let us warm up with the first moment case.

Theorem 4.2.1. *We have*

$$\sum_{\lambda\in\mathcal{P}} \sharp(\lambda)\,\mathrm{rank}(\lambda) q^{|\lambda|} = -\sum_{n\geq 1} \frac{q^{n^2}}{(q;q)_n^2} \sum_{m=1}^{n} \frac{q^m}{(1-q^m)^2}. \tag{4.2.1}$$

It follows that

$$\sum_{\lambda\vdash n} \sharp(\lambda)\,\mathrm{rank}(\lambda) = -\frac{1}{2} N_2(n). \tag{4.2.2}$$

Remark 4.2.1. It is worth pointing out that the following generating function identity for $N_2(n)$ is used most frequently.

$$\sum_{n\geq 0} N_2(n) q^n = -\frac{2}{(q;q)_\infty} \sum_{n\geq 1} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}.$$

See Eq. (3.4) in [15].

Proof. Recall that it was shown in [18] that

$$\mathcal{N}(x,z;q) := \sum_{n\geq 0}\sum_{\lambda\vdash n} x^{\sharp(\lambda)} z^{\mathrm{rank}(\lambda)} q^n = \sum_{n\geq 0} \frac{x^n q^{n^2}}{(zq;q)_n (xq/z;q)_n}. \tag{4.2.3}$$

We first apply the operator $[\partial/\partial x]_{x=1}$ to (4.2.3).

$$\begin{aligned}&\sum_{n\geq 0}\sum_{\lambda\vdash n} \sharp(\lambda) z^{\mathrm{rank}(\lambda)} q^n \\ &\quad= \sum_{n\geq 0} \left[\frac{\partial}{\partial x} \frac{x^n q^{n^2}}{(zq;q)_n (xq/z;q)_n}\right]_{x=1}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0}\left[\frac{x^n q^{n^2}}{(zq;q)_n (xq/z;q)_n}\frac{\partial}{\partial x}\log\left(\frac{x^n}{(xq/z;q)_n}\right)\right]_{x=1} \\
&= \sum_{n\geq 0}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}\left[\frac{\partial}{\partial x}\left(n\log x - \sum_{m=1}^{n}\log(1-xq^m/z)\right)\right]_{x=1} \\
&= \sum_{n\geq 0}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}\left[\frac{n}{x}+\sum_{m=1}^{n}\frac{q^m}{z-xq^m}\right]_{x=1} \\
&= \sum_{n\geq 1}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}\left(n+\sum_{m=1}^{n}\frac{q^m}{z-q^m}\right). \qquad (4.2.4)
\end{aligned}
$$

We next make the following easy observation: for any $n \geq 1$ where n can also be ∞,

$$
\left[\frac{\partial}{\partial z}\log\left(\frac{1}{(zq;q)_n(q/z;q)_n}\right)\right]_{z=1} = \left[\sum_{m=1}^{n}\left(\frac{q^m}{1-zq^m}+\frac{q^m}{zq^m-z^2}\right)\right]_{z=1} = 0. \qquad (4.2.5)
$$

Applying the operator $[\partial/\partial z]_{z=1}$ to (4.2.4) and using (4.2.5) yields

$$
\begin{aligned}
&\sum_{n\geq 0}\sum_{\lambda\vdash n}\sharp(\lambda)\,\mathrm{rank}(\lambda)q^n \\
&\quad= \sum_{n\geq 1}\left[\frac{\partial}{\partial z}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}\left(n+\sum_{m=1}^{n}\frac{q^m}{z-q^m}\right)\right]_{z=1} \\
&\quad= \sum_{n\geq 1}\left[\frac{nq^{n^2}}{(zq;q)_n(q/z;q)_n}\frac{\partial}{\partial z}\log\left(\frac{1}{(zq;q)_n(q/z;q)_n}\right)\right]_{z=1} \\
&\qquad+\sum_{n\geq 1}\sum_{m=1}^{n}\left[\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}\frac{q^m}{z-q^m}\frac{\partial}{\partial z}\log\left(\frac{1}{(zq;q)_n(q/z;q)_n(z-q^m)}\right)\right]_{z=1} \\
&\quad= -\sum_{n\geq 1}\frac{q^{n^2}}{(q;q)_n^2}\sum_{m=1}^{n}\frac{q^m}{(1-q^m)^2}. \qquad (4.2.6)
\end{aligned}
$$

This is the first part of Theorem 4.2.1.

If one applies the operator $\left[\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\right)\right]_{z=1}$ to the generating function

$$
\sum_{n\geq 0}\sum_{\lambda\vdash n} z^{\mathrm{rank}(\lambda)}q^n = \sum_{n\geq 0}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n},
$$

then one shall find that

$$
\sum_{n\geq 0} N_2(n)q^n = \sum_{n\geq 0}\sum_{\lambda\vdash n}\mathrm{rank}(\lambda)^2 q^n
$$

$$
\begin{aligned}
&= \sum_{n\geq 0}\left[\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}\right)\right]_{z=1}\\
&= \sum_{n\geq 0}\left[\frac{\partial}{\partial z}\frac{zq^{n^2}}{(zq;q)_n(q/z;q)_n}\sum_{m=1}^{n}\left(\frac{q^m}{1-zq^m}+\frac{q^m}{zq^m-z^2}\right)\right]_{z=1}\\
&= 2\sum_{n\geq 1}\frac{q^{n^2}}{(q;q)_n^2}\sum_{m=1}^{n}\frac{q^m}{(1-q^m)^2}. \qquad (4.2.7)
\end{aligned}
$$

This combining with (4.2.6) gives the second part of Theorem 4.2.1. □

Theorem 4.2.2. *We have*

$$\sum_{\lambda\in\mathcal{P}}\omega(\lambda)\,\mathrm{crank}(\lambda)q^{|\lambda|} = -\frac{1}{(q;q)_\infty}\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}. \qquad (4.2.8)$$

It follows that

$$\sum_{\lambda\vdash n}\omega(\lambda)\,\mathrm{crank}(\lambda) = -\frac{1}{2}M_2(n). \qquad (4.2.9)$$

Remark 4.2.2. It was shown by means of a relation due to Dyson [77] that

$$M_2(n) = 2np(n).$$

It turns out that

$$\sum_{\lambda\vdash n}\omega(\lambda)\,\mathrm{crank}(\lambda) = -np(n). \qquad (4.2.10)$$

Proof. As in [23], we have

$$
\begin{aligned}
\mathcal{M}(x,z;q) := \sum_{n\geq 0}\sum_{\lambda\vdash n}x^{\omega(\lambda)}z^{\mathrm{crank}(\lambda)}q^n &= \frac{1-q}{(zq;q)_\infty}+\sum_{j\geq 1}\frac{x^jq^jz^{-j}}{(q^2;q)_{j-1}(zq^{j+1};q)_\infty}\\
&= \frac{1-q}{(zq;q)_\infty}\sum_{j\geq 0}\frac{(zq;q)_j}{(q;q)_j}\left(\frac{xq}{z}\right)^j\\
&= \frac{(1-q)(xq^2;q)_\infty}{(zq;q)_\infty(xq/z;q)_\infty}. \qquad (4.2.11)
\end{aligned}
$$

Here in the last identity we use the q-binomial theorem (see Theorem 2.1 in [12]):

$$\sum_{n\geq 0}\frac{(a;q)_nt^n}{(q;q)_n} = \frac{(at;q)_\infty}{(t;q)_\infty}.$$

If we take $x=1$ in (4.2.11), then we recover the bivariate generating function in [23].

Now we apply the operator $[\partial/\partial x]_{x=1}$ to (4.2.11).

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{\lambda\vdash n}\omega(\lambda)z^{\mathrm{crank}(\lambda)}q^n\\
&\quad=\left[\frac{\partial}{\partial x}\frac{(1-q)(xq^2;q)_\infty}{(zq;q)_\infty(xq/z;q)_\infty}\right]_{x=1}\\
&\quad=\left[\frac{(1-q)(xq^2;q)_\infty}{(zq;q)_\infty(xq/z;q)_\infty}\frac{\partial}{\partial x}\log\left(\frac{(xq^2;q)_\infty}{(xq/z;q)_\infty}\right)\right]_{x=1}\\
&\quad=\frac{(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}\left[\frac{\partial}{\partial x}\sum_{n\geq 1}\Big(\log(1-xq^{n+1})-\log(1-xq^n/z)\Big)\right]_{x=1}\\
&\quad=\frac{(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}\sum_{n\geq 1}\left(-\frac{q^{n+1}}{1-q^{n+1}}+\frac{q^n/z}{1-q^n/z}\right). \qquad (4.2.12)
\end{aligned}$$

We then apply the operator $[\partial/\partial z]_{z=1}$ to (4.2.12) and use (4.2.5) to deduce

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{\lambda\vdash n}\omega(\lambda)\,\mathrm{crank}(\lambda)q^n\\
&\quad=\left[\frac{\partial}{\partial z}\frac{(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}\sum_{n\geq 1}\left(-\frac{q^{n+1}}{1-q^{n+1}}+\frac{q^n/z}{1-q^n/z}\right)\right]_{z=1}\\
&\quad=\sum_{n\geq 1}\left[\frac{(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}\cdot\frac{q^n/z}{1-q^n/z}\cdot\frac{1}{q^n-z}\right]_{z=1}\\
&\quad=-\frac{1}{(q;q)_\infty}\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}. \qquad (4.2.13)
\end{aligned}$$

To prove the second part of Theorem 4.2.2, we apply the operator $\left[\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\right)\right]_{z=1}$ to the generating function

$$\sum_{n\geq 0}\sum_{\lambda\vdash n}z^{\mathrm{crank}(\lambda)}q^n=\frac{(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}.$$

Then

$$\begin{aligned}
&\sum_{n\geq 0}M_2(n)q^n=\sum_{n\geq 0}\sum_{\lambda\vdash n}\mathrm{crank}(\lambda)^2q^n\\
&\quad=\left[\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\frac{(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}\right)\right]_{z=1}\\
&\quad=\left[\frac{\partial}{\partial z}\frac{z(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}\sum_{m\geq 1}\left(\frac{q^m}{1-zq^m}+\frac{q^m}{zq^m-z^2}\right)\right]_{z=1}
\end{aligned}$$

$$= \frac{2}{(q;q)_\infty} \sum_{m\geq 1} \frac{q^m}{(1-q^m)^2}. \tag{4.2.14}$$

This combining with (4.2.13) gives the second part of Theorem 4.2.2. □

Finally, let $\operatorname{spt}(n)$ denote the total number of appearances of the smallest parts in all of the partitions of n. In [15], it was shown that

$$\operatorname{spt}(n) = \frac{1}{2}M_2(n) - \frac{1}{2}N_2(n).$$

In view of (4.2.2) and (4.2.9), we immediately obtain the following interesting relation.

Corollary 4.2.3. *For $n \geq 0$,*

$$\operatorname{spt}(n) = \sum_{\lambda\vdash n} \sharp(\lambda)\operatorname{rank}(\lambda) - \sum_{\lambda\vdash n} \omega(\lambda)\operatorname{crank}(\lambda). \tag{4.2.15}$$

4.3 General Odd Moments

Now we are in a position to prove Theorems 4.1.1–4.1.4.

4.3.1 Rank

We require a reformulation of $\mathcal{N}(x,z;q)$ shown in [18]:

$$\begin{aligned}\mathcal{N}(x,z;q) = 1 + \frac{1}{(xq;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} x^n \frac{(xq;q)_n}{(q;q)_{n-1}} \\ \times \left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z\left(1-\frac{xq^n}{z}\right)} \right).\end{aligned} \tag{4.3.1}$$

It is convenient to define two auxiliary functions

$$\alpha(z;Q) := \frac{1}{Q(1-zQ)} \quad \text{and} \quad \beta(x,z;Q) := \frac{x}{z\left(1-\frac{xQ}{z}\right)}.$$

To study the weighted k-th ordinary rank moment, we require the following family of operators for $k \geq 1$:

$$\mathcal{D}_k(f(z)) := \begin{cases} \frac{\partial}{\partial z} f(z) & \text{if } k = 1, \\ \frac{\partial}{\partial z} z\mathcal{D}_{k-1}(f(z)) & \text{if } k > 1. \end{cases} \tag{4.3.2}$$

We first show that $\alpha(z;Q)$ and $\beta(x,z;Q)$ satisfy the following proposition.

Proposition 4.3.1. *For each positive integer k, there is a polynomial P_k with integer coefficients such that*

$$\mathcal{D}_k(\alpha(z;Q)) = \frac{P_k(zQ)}{(1-zQ)^{k+1}} \tag{4.3.3}$$

and

$$\mathcal{D}_k(\beta(x,z;Q)) = (-1)^k \frac{xP_k\left(\frac{xQ}{z}\right)}{z^2\left(1-\frac{xQ}{z}\right)^{k+1}}. \tag{4.3.4}$$

Further, P_k satisfies $P_1(u) = 1$ and for $k \geq 1$,

$$P_{k+1}(u) = \Big(1 - u + (k+1)u\Big)P_k(u) + \Big(u - u^2\Big)P_k'(u). \tag{4.3.5}$$

Proof. It is not hard to compute that

$$\mathcal{D}_1(\alpha(z;Q)) = \frac{\partial}{\partial z}\alpha(z;Q) = \frac{1}{(1-zQ)^2}$$

and

$$\mathcal{D}_1(\beta(x,z;Q)) = \frac{\partial}{\partial z}\beta(x,z;Q) = -\frac{x}{z^2\left(1-\frac{xQ}{z}\right)^2}.$$

Let us assume that the proposition is true for some $k \geq 1$. Then

$$\begin{aligned}
\mathcal{D}_{k+1}(\alpha(z;Q)) &= \frac{\partial}{\partial z} z\mathcal{D}_k(\alpha(z;Q)) \\
&= \frac{\partial}{\partial z}\frac{zP_k(zQ)}{(1-zQ)^{k+1}} \\
&= \frac{\Big(1 - zQ + (k+1)zQ\Big)P_k(zQ) + \Big(zQ - z^2Q^2\Big)P_k'(zQ)}{(1-zQ)^{k+2}}.
\end{aligned}$$

Likewise, we have

$$\mathcal{D}_{k+1}(\beta(x,z;Q))$$

$$= (-1)^{k+1} \frac{x\left(\left(1 - \frac{xQ}{z} + (k+1)\frac{xQ}{z}\right)P_k\left(\frac{xQ}{z}\right) + \left(\frac{xQ}{z} - \frac{x^2Q^2}{z^2}\right)P_k'\left(\frac{xQ}{z}\right)\right)}{z^2\left(1 - \frac{xQ}{z}\right)^{k+2}}.$$

The proposition follows by induction on k. $\square$

Next, for $k \geq 1$, we write

$$\Big[\mathcal{D}_{2k-1}(\alpha(z;Q))\Big]_{z=1} = \frac{P_{2k-1}(Q)}{(1-Q)^{2k}} =: \alpha_{2k-1}(Q)$$

and

$$\Big[\mathcal{D}_{2k-1}(\beta(x,z;Q))\Big]_{z=1} = -\frac{xP_{2k-1}(xQ)}{(1-xQ)^{2k}} =: \beta_{2k-1}(x;Q).$$

Noticing that $\alpha_{2k-1}(Q) + \beta_{2k-1}(1;Q) = 0$, we may factor out $(1-x)$ from $\alpha_{2k-1}(Q) + \beta_{2k-1}(x;Q)$ for all $k \geq 1$. Let us write

$$\alpha_{2k-1}(Q) + \beta_{2k-1}(x;Q) = (1-x)F_{2k-1}(x;Q).$$

Applying the trivial identity

$$\left[\frac{\partial}{\partial x}(1-x)f(x)\right]_{x=1} = -f(1), \tag{4.3.6}$$

we deduce that

$$\begin{aligned}
F_{2k-1}(1;Q) &= -\left[\frac{\partial}{\partial x}\Big(\alpha_{2k-1}(Q) + \beta_{2k-1}(x;Q)\Big)\right]_{x=1} \\
&= \left[\frac{\partial}{\partial x}\frac{xP_{2k-1}(xQ)}{(1-xQ)^{2k}}\right]_{x=1} \\
&= \left[\frac{(1-xQ+2kxQ)P_{2k-1}(xQ) + (xQ - x^2Q^2)P_{2k-1}'(xQ)}{(1-xQ)^{2k+1}}\right]_{x=1} \\
&= \frac{(1-Q+2kQ)P_{2k-1}(Q) + (Q-Q^2)P_{2k-1}'(Q)}{(1-Q)^{2k+1}} \\
&= \frac{P_{2k}(Q)}{(1-Q)^{2k+1}}.
\end{aligned}$$

For $k \geq 1$, it follows from (4.3.1) that

$$\begin{aligned}
\sum_{\lambda\in\mathcal{P}} \operatorname{rank}^{2k-1}(\lambda) x^{\sharp(\lambda)} q^{|\lambda|} &= \Big[\mathcal{D}_{2k-1}\Big(\mathcal{N}(x,z;q)\Big)\Big]_{z=1} \\
&= \frac{1}{(xq;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} x^n \frac{(xq;q)_n}{(q;q)_{n-1}} \\
&\quad \times \left[\mathcal{D}_{2k-1}\left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z\Big(1-\frac{xq^n}{z}\Big)}\right)\right]_{z=1} \\
&= \frac{1}{(xq;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} x^n \frac{(xq;q)_n}{(q;q)_{n-1}} \\
&\quad \times \Big(\alpha_{2k-1}(q^n) + \beta_{2k-1}(x;q^n)\Big) \\
&= \frac{1-x}{(xq;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} x^n \frac{(xq;q)_n}{(q;q)_{n-1}} \\
&\quad \times F_{2k-1}(x;q^n).
\end{aligned}$$

Applying (4.3.6) again, we have

$$\begin{aligned}
\sum_{\lambda\in\mathcal{P}} \sharp(\lambda) \operatorname{rank}^{2k-1}(\lambda) q^{|\lambda|} &= \left[\frac{\partial}{\partial x} \sum_{\lambda\in\mathcal{P}} \operatorname{rank}^{2k-1}(\lambda) x^{\sharp(\lambda)} q^{|\lambda|}\right]_{x=1} \\
&= -\left[\frac{1}{(xq;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} x^n \frac{(xq;q)_n}{(q;q)_{n-1}} \right. \\
&\quad \left. \times F_{2k-1}(x;q^n)\right]_{x=1} \\
&= -\frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} (1-q^n) F_{2k-1}(1;q^n) \\
&= -\frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} (1-q^n) \frac{P_{2k}(q^n)}{(1-q^n)^{2k+1}}.
\end{aligned}$$

We therefore arrive at (4.1.14).

Further,

$$\begin{aligned}
\sum_{\lambda\in\mathcal{P}} \operatorname{rank}^{2k}(\lambda) q^{|\lambda|} &= \Big[\mathcal{D}_{2k}\Big(\mathcal{N}(1,z;q)\Big)\Big]_{z=1} \\
&= \frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n+1)/2} (1-q^n)
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\mathcal{D}_{2k}\left(\frac{1}{q^n(1-zq^n)}+\frac{1}{z\left(1-\frac{q^n}{z}\right)}\right)\right]_{z=1}\\
&=\frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n+1)/2}(1-q^n)\\
&\times\left[\frac{P_{2k}(zq^n)}{(1-zq^n)^{2k+1}}+\frac{P_{2k}\left(\frac{q^n}{z}\right)}{z^2\left(1-\frac{q^n}{z}\right)^{2k+1}}\right]_{z=1}\\
&=\frac{2}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n+1)/2}(1-q^n)\frac{P_{2k}(q^n)}{(1-q^n)^{2k+1}}.
\end{aligned}
$$

It turns out that

$$
\sum_{\lambda\in\mathcal{P}}\sharp(\lambda)\operatorname{rank}^{2k-1}(\lambda)q^{|\lambda|}=-\frac{1}{2}\sum_{\lambda\in\mathcal{P}}\operatorname{rank}^{2k}(\lambda)q^{|\lambda|}
$$

and hence

$$
\sum_{\lambda\vdash n}\sharp(\lambda)\operatorname{rank}^{2k-1}(\lambda)=-\frac{1}{2}\sum_{\lambda\vdash n}\operatorname{rank}^{2k}(\lambda),
$$

which is (4.1.16).

Let us turn to the weighted k-th symmetrized rank moment

$$
\eta_k^\sharp(n)=\sum_{\lambda\vdash n}\sharp(\lambda)\binom{\operatorname{rank}(\lambda)+\lfloor\frac{k-1}{2}\rfloor}{k}.
$$

We first study the k-th derivatives

$$
\frac{\partial^k}{\partial z^k}\frac{z^{\lfloor\frac{k-1}{2}\rfloor}}{k!}\alpha(z;Q)
$$

and

$$
\frac{\partial^k}{\partial z^k}\frac{z^{\lfloor\frac{k-1}{2}\rfloor}}{k!}\beta(x,z;Q).
$$

Proposition 4.3.2. *For any positive integer k, we have*

$$
\frac{\partial^k}{\partial z^k}\frac{z^{\lfloor\frac{k-1}{2}\rfloor}}{k!}\alpha(z;Q)=\frac{Q^{\lfloor\frac{k}{2}\rfloor}}{(1-zQ)^{k+1}} \tag{4.3.7}
$$

and

$$\frac{\partial^k}{\partial z^k}\frac{z^{\lfloor\frac{k-1}{2}\rfloor}}{k!}\beta(x,z;Q) = (-1)^k\frac{x^{\lfloor\frac{k+1}{2}\rfloor}Q^{\lfloor\frac{k-1}{2}\rfloor}}{(z-xQ)^{k+1}}. \tag{4.3.8}$$

Proof. The desired results follow directly from Leibniz's rule. □

Let $k \geq 1$. We write

$$\left[\frac{\partial^{2k-1}}{\partial z^{2k-1}}\frac{z^{k-1}}{(2k-1)!}\alpha(z;Q)\right]_{z=1} = \frac{Q^{k-1}}{(1-Q)^{2k}} =: \tilde{\alpha}_{2k-1}(Q)$$

and

$$\left[\frac{\partial^{2k-1}}{\partial z^{2k-1}}\frac{z^{k-1}}{(2k-1)!}\beta(x,z;Q)\right]_{z=1} = -\frac{x^kQ^{k-1}}{(1-xQ)^{2k}} =: \tilde{\beta}_{2k-1}(x;Q).$$

Noticing again that $\tilde{\alpha}_{2k-1}(Q)+\tilde{\beta}_{2k-1}(1;Q)=0$, we may factor out $(1-x)$ from $\tilde{\alpha}_{2k-1}(Q)+\tilde{\beta}_{2k-1}(x;Q)$ for all $k \geq 1$. Hence, we write

$$\tilde{\alpha}_{2k-1}(Q)+\tilde{\beta}_{2k-1}(x;Q) = (1-x)\tilde{F}_{2k-1}(x;Q).$$

It follows from (4.3.6) that

$$\begin{aligned}\tilde{F}_{2k-1}(1;Q) &= -\left[\frac{\partial}{\partial x}\Big(\tilde{\alpha}_{2k-1}(Q)+\tilde{\beta}_{2k-1}(x;Q)\Big)\right]_{x=1}\\ &= \left[\frac{\partial}{\partial x}\frac{x^kQ^{k-1}}{(1-xQ)^{2k}}\right]_{x=1}\\ &= \frac{kQ^{k-1}(1+Q)}{(1-Q)^{2k+1}}.\end{aligned}$$

We know from (4.3.1) that

$$\begin{aligned}&\sum_{\lambda\in\mathcal{P}}\binom{\operatorname{rank}(\lambda)+k-1}{2k-1}x^{\sharp(\lambda)}q^{|\lambda|}\\ &\quad= \left[\frac{\partial^{2k-1}}{\partial z^{2k-1}}\frac{z^{k-1}}{(2k-1)!}\mathcal{N}(x,z;q)\right]_{z=1}\\ &\quad= \frac{1}{(xq;q)_\infty}\sum_{n\geq1}(-1)^{n-1}q^{n(3n+1)/2}x^n\frac{(xq;q)_n}{(q;q)_{n-1}}\end{aligned}$$

$$\times\left[\frac{\partial^{2k-1}}{\partial z^{2k-1}}\frac{z^{k-1}}{(2k-1)!}\left(\frac{1}{q^n(1-zq^n)}+\frac{x}{z\left(1-\frac{xq^n}{z}\right)}\right)\right]_{z=1}$$
$$=\frac{1}{(xq;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n+1)/2}x^n\frac{(xq;q)_n}{(q;q)_{n-1}}\left(\tilde{\alpha}_{2k-1}(q^n)+\tilde{\beta}_{2k-1}(x;q^n)\right)$$
$$=\frac{1-x}{(xq;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n+1)/2}x^n\frac{(xq;q)_n}{(q;q)_{n-1}}\tilde{F}_{2k-1}(x;q^n).$$

Applying (4.3.6) again, we have

$$\sum_{\lambda\in\mathcal{P}}\sharp(\lambda)\binom{\mathrm{rank}(\lambda)+k-1}{2k-1}q^{|\lambda|}$$
$$=\left[\frac{\partial}{\partial x}\sum_{\lambda\in\mathcal{P}}\binom{\mathrm{rank}(\lambda)+k-1}{2k-1}x^{\sharp(\lambda)}q^{|\lambda|}\right]_{x=1}$$
$$=-\left[\frac{1}{(xq;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n+1)/2}x^n\frac{(xq;q)_n}{(q;q)_{n-1}}\tilde{F}_{2k-1}(x;q^n)\right]_{x=1}$$
$$=-\frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n+1)/2}(1-q^n)\tilde{F}_{2k-1}(1;q^n)$$
$$=-\frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n+1)/2}(1-q^n)\frac{kq^{(k-1)n}(1+q^n)}{(1-q^n)^{2k+1}}$$
$$=-\frac{k}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n-1)/2+kn}\frac{1+q^n}{(1-q^n)^{2k}}.$$

This proves (4.1.18). Finally, (4.1.19) follows in light of (4.1.8).

4.3.2 Crank

Analogous to (4.3.1), we need to reformulate $\mathcal{M}(x,z;q)$.

Recall the limiting form of Jackson's theorem (cf. [9, Theorem 3.2]):

$$
\begin{aligned}
&{}_6\phi_5\left(\begin{matrix} w, q\sqrt{w}, -q\sqrt{w}, a, b, c \\ \sqrt{w}, -\sqrt{w}, wq/a, wq/b, wq/c\end{matrix}; q, \frac{wq}{abc}\right)\\
&\quad=\frac{(wq,q)_\infty(wq/ab,q)_\infty(wq/ac,q)_\infty(wq/bc,q)_\infty}{(wq/a,q)_\infty(wq/b,q)_\infty(wq/c,q)_\infty(wq/abc,q)_\infty}. \qquad (4.3.9)
\end{aligned}
$$

If we let $w \to x$, $a \to z$, $b \to x/z$ and $c \to \infty$, then (4.3.9) becomes

$$\frac{(q;q)_\infty (xq;q)_\infty}{(zq;q)_\infty (xq/z;q)_\infty} = 1 + \sum_{n\geq 1} (-1)^n q^{n(n+1)/2} \frac{(xq;q)_{n-1}(1-xq^{2n})(1-z)(1-x/z)}{(q;q)_n (1-zq^n)(1-xq^n/z)}. \tag{4.3.10}$$

Next, notice that

$$\begin{aligned}\frac{(1-xq^{2n})(1-z)(1-x/z)}{(1-zq^n)(1-xq^n/z)} &= \frac{1-xq^{2n}}{q^n} - (1-q^n)(1-xq^n) \\ &\quad \times \left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z\left(1-\frac{xq^n}{z}\right)}\right). \end{aligned} \tag{4.3.11}$$

Substituting (4.3.11) into (4.3.10) yields

$$\begin{aligned}\frac{(q;q)_\infty (xq;q)_\infty}{(zq;q)_\infty (xq/z;q)_\infty} &= 1 + \sum_{n\geq 1} (-1)^n q^{n(n-1)/2} \frac{(xq;q)_{n-1}(1-xq^{2n})}{(q;q)_n} \\ &\quad + \sum_{n\geq 1} (-1)^{n-1} q^{n(n+1)/2} \frac{(xq;q)_n}{(q;q)_{n-1}} \\ &\quad \times \left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z\left(1-\frac{xq^n}{z}\right)}\right). \end{aligned} \tag{4.3.12}$$

Letting $n \to \infty$ in the following terminating very-well-poised ${}_4\phi_3$ series (see [83, (2.3.4)]):

$${}_4\phi_3\left(\begin{matrix} w, q\sqrt{w}, -q\sqrt{w}, q^{-n} \\ \sqrt{w}, -\sqrt{w}, wq^{n+1} \end{matrix}; q, q^n\right) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases} \tag{4.3.13}$$

we have

$$1 + \sum_{n\geq 1} (-1)^n q^{n(n-1)/2} \frac{(xq;q)_{n-1}(1-xq^{2n})}{(q;q)_n} = 0. \tag{4.3.14}$$

Substituting (4.3.14) into (4.3.12) yields

$$\begin{aligned}\frac{(q;q)_\infty (xq;q)_\infty}{(zq;q)_\infty (xq/z;q)_\infty} &= \sum_{n\geq 1} (-1)^{n-1} q^{n(n+1)/2} \frac{(xq;q)_n}{(q;q)_{n-1}} \\ &\quad \times \left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z\left(1-\frac{xq^n}{z}\right)}\right). \end{aligned} \tag{4.3.15}$$

Finally, we deduce from (4.2.11) the following result.

Theorem 4.3.3. *We have*

$$\mathcal{M}(x,z;q) = \frac{1-q}{(1-xq)(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(n+1)/2} \frac{(xq;q)_n}{(q;q)_{n-1}} \times \left(\frac{1}{q^n(1-zq^n)} + \frac{x}{z\left(1-\frac{xq^n}{z}\right)} \right). \tag{4.3.16}$$

Now notice that the reformulation (4.3.16) of $\mathcal{M}(x,z;q)$ also involves the auxiliary functions

$$\alpha(z;Q) = \frac{1}{Q(1-zQ)} \quad \text{and} \quad \beta(x,z;Q) = \frac{x}{z\left(1-\frac{xQ}{z}\right)}$$

defined in §4.3.1. We therefore may carry out the same procedure to prove Theorems 4.1.3 and 4.1.4. The details are omitted.

4.3.3 Remark

It is worth mentioning more about the necessity of the odd order in the weighted moments. Let us use $N_k^\sharp(n)$ as an example. As we have seen, to obtain

$$\sum_{n\geq 0} N_k^\sharp(n) q^n = \sum_{\lambda\in\mathcal{P}} \sharp(\lambda)\,\mathrm{rank}^k(\lambda) q^{|\lambda|},$$

the last step is to apply the operator $[\partial/\partial x]_{x=1}$ to

$$\mathcal{N}_k(x;q) := \sum_{\lambda\in\mathcal{P}} \mathrm{rank}^k(\lambda) x^{\sharp(\lambda)} q^{|\lambda|}.$$

Our trick here is that by noticing that $\mathcal{N}_k(1;q)=0$ when k is odd due to the symmetry property of the rank function, one may factor out $(1-x)$ from $\mathcal{N}_k(x;q)$ so that (4.3.6) can be applied. However, when k is even, we fail to get the factor $(1-x)$ as $\mathcal{N}_k(1;q)$ is not identical to zero and hence the aforementioned trick cannot be used.

4.4 Andrews–Beck Type Congruences

It is fair to describe the origional motivation of this project.

Recall that the rank and crank statistics interpret Ramanujan's congruences (4.1.1)–(4.1.3) in the following way. Let $N(k,m,n)$ (resp. $M(m,n)$) count the number of partitions of n whose rank (resp. crank) is congruent to k modulo m.

First, Atkin and Swinnerton-Dyer [30] proved that for $0 \leq i \leq 4$,

$$N(i, 5, 5n+4) = \frac{1}{5}p(5n+4)$$

and that for $0 \leq i \leq 6$,

$$N(i, 7, 7n+5) = \frac{1}{7}p(7n+5).$$

On the other hand, Andrews and Garvan [23] showed that for $0 \leq i \leq 4$,

$$M(i, 5, 5n+4) = \frac{1}{5}p(5n+4),$$

that for $0 \leq i \leq 6$,

$$M(i, 7, 7n+5) = \frac{1}{7}p(7n+5)$$

and that for $0 \leq i \leq 10$,

$$M(i, 11, 11n+6) = \frac{1}{11}p(11n+6).$$

One shall see how equally numerous subclasses occur.

In a private communication between George Beck and George Andrews, Beck made a number of new conjectures along this line. Instead of considering the $N(m, k, n)$ and $M(m, k, n)$ functions, Beck studied the total number of parts in the partitions of n with rank congruent to m modulo k, which is defined by $NT(m, k, n)$, and the total number of ones in the partitions of n with crank congruent to m modulo k, which is defined by $M_\omega(m, k, n)$. Let me record one example that was proved later by Andrews in [18].

Theorem 4.4.1. *If $i = 1$ or 4, then for $n \geq 0$,*

$$\begin{aligned} &NT(1, 5, 5n+i) + 2NT(2, 5, 5n+i) \\ &- 2NT(3, 5, 5n+i) - NT(4, 5, 5n+i) \equiv 0 \pmod{5}. \end{aligned} \tag{4.4.1}$$

But the arithmetic properties of $N_2(n)$ are extensively studied. For example, (1.14) and (1.15) of [14] state that

$$N_2(5n+1 \text{ or } 4) \equiv 0 \pmod{5}$$

and

$$N_2(7n+1 \text{ or } 5) \equiv 0 \pmod 7.$$

It is also trivial to see that

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}} \sharp(\lambda)\,\mathrm{rank}(\lambda) q^{|\lambda|} \equiv \sum_{n\geq 0} \Big(&NT(1,5,n) + 2NT(2,5,n)\\ &- 2NT(3,5,n) - NT(4,5,n)\Big) q^n \pmod 5\end{aligned}$$

and

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}} \sharp(\lambda)\,\mathrm{rank}(\lambda) q^{|\lambda|} \equiv \sum_{n\geq 0} \Big(&NT(1,7,n) + 2NT(2,7,n)\\ &+ 3NT(3,7,n) - 3NT(4,7,n)\\ &- 2NT(5,7,n) - NT(6,7,n)\Big) q^n \pmod 7.\end{aligned}$$

We therefore arrive at both (4.4.1) and the below through the fact that

$$\sum_{\lambda\vdash n} \sharp(\lambda)\,\mathrm{rank}(\lambda) = -\frac{1}{2} N_2(n).$$

Theorem 4.4.2. *If $i = 1$ or 5, then for $n \geq 0$,*

$$\begin{aligned}&NT(1,7,7n+i) + 2NT(2,7,7n+i)\\ &+ 3NT(3,7,7n+i) - 3NT(4,7,7n+i)\\ &- 2NT(5,7,7n+i) - NT(6,7,7n+i) \equiv 0 \pmod 7.\end{aligned} \tag{4.4.2}$$

In fact, utilizing the relations between ordinary and weighted rank and crank moments, I [60] am able to discover over 70 congruences modulo 5, 7, 11 and 13 involving $NT(r,k,n)$ and $M_\omega(r,k,n)$. Through a computer search, it is believed that the list below is to some extent complete for these moduli (it should be noted that a handful of unlisted congruences could be generated by congruences in the main theorems; see remarks below each theorem).

Theorem 4.4.3. *Let*

$$NT[a_1,a_2](n) := \sum_{r=1}^{2} a_r\Big(NT(r,5,n) - NT(5-r,5,n)\Big)$$

and

$$M_\omega[a_1,a_2](n) := \sum_{r=1}^{2} a_r\Big(M_\omega(r,5,n) - M_\omega(5-r,5,n)\Big).$$

Then (i).

$$NT[1,2](5n+1) \equiv 0 \pmod{5}, \tag{4.4.3-1}$$
$$NT[1,2](5n+4) \equiv 0 \pmod{5}; \tag{4.4.3-2}$$

(ii).

$$M_\omega[1,2](5n) \equiv 0 \pmod{5}, \tag{4.4.4-1}$$
$$M_\omega[1,2](5n+4) \equiv 0 \pmod{5}; \tag{4.4.4-2}$$

(iii).

$$\begin{aligned} NT[0,1](5n) &\equiv M_\omega[0,1](5n) \equiv M_\omega[1,3](5n) \equiv M_\omega[2,0](5n) \\ &\equiv M_\omega[3,2](5n) \equiv M_\omega[4,4](5n) \pmod{5}, \end{aligned} \tag{4.4.5-1}$$
$$NT[0,1](5n+1) \equiv M_\omega[0,1](5n+1) \pmod{5}, \tag{4.4.5-2}$$
$$NT[1,0](5n+1) \equiv M_\omega[0,3](5n+1) \pmod{5}, \tag{4.4.5-3}$$
$$NT[0,1](5n+2) \equiv M_\omega[2,0](5n+2) \pmod{5}, \tag{4.4.5-4}$$
$$NT[1,0](5n+2) \equiv M_\omega[0,3](5n+2) \pmod{5}, \tag{4.4.5-5}$$
$$NT[1,3](5n+3) \equiv M_\omega[1,3](5n+3) \pmod{5}, \tag{4.4.5-6}$$
$$\begin{aligned} NT[0,1](5n+4) &\equiv M_\omega[0,1](5n+4) \equiv M_\omega[1,3](5n+4) \equiv M_\omega[2,0](5n+4) \\ &\equiv M_\omega[3,2](5n+4) \equiv M_\omega[4,4](5n+4) \pmod{5}, \end{aligned} \tag{4.4.5-7}$$
$$\begin{aligned} NT[1,0](5n+4) &\equiv M_\omega[0,3](5n+4) \equiv M_\omega[1,0](5n+4) \equiv M_\omega[2,2](5n+4) \\ &\equiv M_\omega[3,4](5n+4) \equiv M_\omega[4,1](5n+4) \pmod{5}. \end{aligned} \tag{4.4.5-8}$$

Remark 4.4.1. It should be pointed out that one may derive more congruences from

(4.4.5-2) and (4.4.5-3). For example,

$$NT[1,1](5n+1) \equiv M_\omega[0,4](5n+1) \pmod 5,$$

which comes from

$$\begin{aligned} NT[1,1](5n+1) &\equiv NT[0,1](5n+1) + NT[1,0](5n+1) \\ &\equiv M_\omega[0,1](5n+1) + M_\omega[0,3](5n+1) \\ &\equiv M_\omega[0,4](5n+1) \pmod 5. \end{aligned}$$

Similarly, more congruences could be derived from (4.4.5-4) and (4.4.5-5), and from (4.4.5-7) and (4.4.5-8). Also, in (4.4.5-1), we have $M_\omega[0,1](5n) \equiv M_\omega[1,3](5n) \equiv \cdots \pmod 5$. This is a consequence of (4.4.4-1) by noticing that

$$M_\omega[1,3](5n) \equiv M_\omega[0,1](5n) + M_\omega[1,2](5n) \equiv M_\omega[0,1](5n) \pmod 5.$$

Similar arguments could be applied to (4.4.5-7) and (4.4.5-8) with the help of (4.4.4-2).

We notice that (4.4.5-1) and (4.4.5-7) imply [47, (4.10)], and (4.4.5-3) and (4.4.5-5) imply [47, (4.12)].

Theorem 4.4.4. *Let*

$$NT[a_1,a_2,a_3](n) := \sum_{r=1}^{3} a_r\Big(NT(r,7,n) - NT(7-r,7,n)\Big)$$

and

$$M_\omega[a_1,a_2,a_3](n) := \sum_{r=1}^{3} a_r\Big(M_\omega(r,7,n) - M_\omega(7-r,7,n)\Big).$$

Then (i).

$$NT[0,1,4](7n) \equiv 0 \pmod 7, \tag{4.4.6-1}$$
$$NT[0,1,4](7n+1) \equiv 0 \pmod 7, \tag{4.4.6-2}$$
$$NT[1,0,2](7n+1) \equiv 0 \pmod 7, \tag{4.4.6-3}$$
$$NT[1,0,2](7n+3) \equiv 0 \pmod 7, \tag{4.4.6-4}$$
$$NT[1,0,2](7n+4) \equiv 0 \pmod 7, \tag{4.4.6-5}$$
$$NT[0,1,4](7n+5) \equiv 0 \pmod 7, \tag{4.4.6-6}$$

$$NT[1,0,2](7n+5) \equiv 0 \pmod 7; \tag{4.4.6-7}$$

(ii).

$$M_\omega[0,1,4](7n) \equiv 0 \pmod 7, \tag{4.4.7-1}$$
$$M_\omega[1,0,2](7n) \equiv 0 \pmod 7, \tag{4.4.7-2}$$
$$M_\omega[0,1,4](7n+1) \equiv 0 \pmod 7, \tag{4.4.7-3}$$
$$M_\omega[1,0,2](7n+2) \equiv 0 \pmod 7, \tag{4.4.7-4}$$
$$M_\omega[1,3,0](7n+3) \equiv 0 \pmod 7, \tag{4.4.7-5}$$
$$M_\omega[0,1,4](7n+4) \equiv 0 \pmod 7, \tag{4.4.7-6}$$
$$M_\omega[0,1,4](7n+5) \equiv 0 \pmod 7, \tag{4.4.7-7}$$
$$M_\omega[1,0,2](7n+5) \equiv 0 \pmod 7, \tag{4.4.7-8}$$
$$M_\omega[1,0,2](7n+6) \equiv 0 \pmod 7. \tag{4.4.7-9}$$

Remark 4.4.2. Linear combinations of (4.4.6-2) and (4.4.6-3) imply more congruences. For example, $1 \times$ (4.4.6-2) $+ 1 \times$ (4.4.6-3) gives

$$NT[1,1,6](7n+1) \equiv 0 \pmod 7,$$

which is the $i = 1$ case of [18, Theorem 1.2]. More congruences could be derived from linear combinations of (4.4.6-6) and (4.4.6-7), of (4.4.7-1) and (4.4.7-2), and of (4.4.7-7) and (4.4.7-8).

We notice that [47, (4.15) and (4.16)] are shown in Part (ii).

Theorem 4.4.5. *Let*

$$NT[a_1,a_2,a_3,a_4,a_5](n) := \sum_{r=1}^{5} a_r\Big(NT(r,11,n) - NT(11-r,11,n)\Big)$$

and

$$M_\omega[a_1,a_2,a_3,a_4,a_5](n) := \sum_{r=1}^{5} a_r\Big(M_\omega(r,11,n) - M_\omega(11-r,11,n)\Big).$$

We also adopt the notation

$$M_\omega \begin{bmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1, & c_2, & c_3, & c_4, & c_5 \end{bmatrix} (n) := \begin{bmatrix} M_\omega[a_1, a_2, a_3, a_4, a_5](n) \\ M_\omega[b_1, b_2, b_3, b_4, b_5](n) \\ \vdots \\ M_\omega[c_1, c_2, c_3, c_4, c_5](n) \end{bmatrix}.$$

Then (i).

$$NT[0,1,4,10,9](11n) \equiv 0 \pmod{11}, \tag{4.4.8-1}$$

$$NT[1,8,5,9,4](11n+1) \equiv 0 \pmod{11}, \tag{4.4.8-2}$$

$$NT[1,3,7,3,3](11n+6) \equiv 0 \pmod{11}; \tag{4.4.8-3}$$

(ii).

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-1}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \end{bmatrix} (11n+1) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-2}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+2) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-3}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+3) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-4}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 5, & 0 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+4) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-5}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 5, & 0 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+5) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-6}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+6) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-7}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 8, & 0 \end{bmatrix} (11n+7) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-8}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 2, & 0 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 8, & 0 \end{bmatrix} (11n+8) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-9}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 1, & 0, & 6 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+9) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}, \tag{4.4.9-10}$$

$$M_\omega \begin{bmatrix} 0, & 0, & 0, & 1, & 8 \\ 0, & 1, & 0, & 0, & 4 \\ 1, & 0, & 0, & 0, & 2 \end{bmatrix} (11n+10) \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \pmod{11}. \tag{4.4.9-11}$$

Remark 4.4.3. Each of (4.4.9-1)–(4.4.9-11) may lead to more Andrews–Beck type congruences modulo 11 for M_ω.

We notice that (4.4.8-2) is [47, (4.6)] and (4.4.8-3) is [47, (4.5)].

Theorem 4.4.6. *Let*

$$NT[a_1, a_2, a_3, a_4, a_5, a_6](n) := \sum_{r=1}^{6} a_r \Big(NT(r, 13, n) - NT(13-r, 13, n) \Big)$$

and

$$M_\omega[a_1, a_2, a_3, a_4, a_5, a_6](n) := \sum_{r=1}^{6} a_r \Big(M_\omega(r, 13, n) - M_\omega(13-r, 13, n) \Big).$$

Then (i).

$$NT[0,1,4,12,10,3](13n) \equiv 0 \pmod{13}, \tag{4.4.10-1}$$

$$NT[1,1,6,0,0,3](13n+1) \equiv 0 \pmod{13}, \tag{4.4.10-2}$$

$$NT[0,0,1,9,6,8](13n+2) \equiv 0 \pmod{13}, \tag{4.4.10-3}$$

$$NT[1,0,3,9,1,11](13n+3) \equiv 0 \pmod{13}, \tag{4.4.10-4}$$

$$NT[1,5,8,7,12,12](13n+5) \equiv 0 \pmod{13}, \tag{4.4.10-5}$$
$$NT[1,2,8,0,7,11](13n+6) \equiv 0 \pmod{13}, \tag{4.4.10-6}$$
$$NT[1,12,8,7,10,7](13n+7) \equiv 0 \pmod{13}, \tag{4.4.10-7}$$
$$NT[1,6,11,8,0,0](13n+9) \equiv 0 \pmod{13}, \tag{4.4.10-8}$$
$$NT[1,9,4,5,10,7](13n+10) \equiv 0 \pmod{13}; \tag{4.4.10-9}$$

(ii).

$$M_\omega[1,2,3,4,5,6](13n) \equiv 0 \pmod{13}. \tag{4.4.11-1}$$

Remark 4.4.4. We notice that (4.4.10-2) is [47, (4.7)] and (4.4.10-4) is [47, (4.8), corrected].

Proofs of the above congruences also rely on relations between ordinary rank and crank moments $N_{2s}(n)$ and $M_{2s}(n)$ derived by Atkin and Garvan [29]. See [60] for details.

4.5 References

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Chapter 5

Partitions with Bounded Part Differences

This chapter comes from

- S. Chern, A curious identity and its applications to partitions with bounded part differences, *New Zealand J. Math.* **47** (2017), 23–26. (Ref. [49])
- S. Chern, An overpartition analogue of partitions with bounded differences between largest and smallest parts, *Discrete Math.* **340** (2017), no. 12, 2834–2839. (Ref. [50])
- S. Chern, On a conjecture of George Beck, *Int. J. Number Theory* **14** (2018), no. 3, 647–651. (Ref. [51])
- S. Chern, On a conjecture of George Beck. II, *Math. Student* **88** (2019), no. 1-2, 159–164. (Ref. [52])
- S. Chern and A. J. Yee, Overpartitions with bounded part differences, *European J. Combin.* **70** (2018), 317–324. (Ref. [68])

5.1 Introduction

In a paper of Andrews, Beck and Robbins [19], they considered partitions where the difference between largest and smallest parts is a fixed integer t. Let $p(n,t)$ be the number of such partitions of n. We have, for example, $p(4,1)=1$ since 4 has only one such partition: $2+1+1$. In fact, Andrews et al. showed that $p(n,0)=d(n)$ and $p(n,1)=n-d(n)$ where $d(n)$ denotes the number of divisors of n. For $t\geq 2$, they obtained the following generating function

$$\sum_{n\geq 1} p(n,t)q^n=\frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})}-\frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q;q)_t}+\frac{q^t}{(1-q^{t-1})(q;q)_t}. \quad (5.1.1)$$

Motivated by their work, Breuer and Kronholm [41] studied the number of partitions of n with the difference between largest and smallest parts bounded by t, denoted by

$p_t(n)$, and they showed that the generating function is

$$\sum_{n\geq 1} p_t(n)q^n = \frac{1}{1-q^t}\left(\frac{1}{(q;q)_t} - 1\right). \tag{5.1.2}$$

The proof of Breuer and Kronholm has a geometric flavour, and their main tool used in the proof is polyhedral cones.

Subsequently, Chapman [48] also provided a simpler proof, which involves q-series manipulations.

In this chapter, we will further consider other types of partition with bounded part differences. In fact, their generating functions fit into a general framework.

5.2 A General Identity

Let t be a fixed positive integer. Assume that α, β, q are complex variables with $|q| < 1$, $q \neq 0$, $\alpha \neq \beta q$ and $(\beta q; q)_t \neq 0$. We define the following sum

$$S(\alpha, \beta; q; t) := \sum_{r\geq 1} \frac{(1-\alpha q^r)(1-\alpha q^{r+1})\cdots(1-\alpha q^{r+t-2})}{(1-\beta q^r)(1-\beta q^{r+1})\cdots(1-\beta q^{r+t})} q^r. \tag{5.2.1}$$

We have the following identity.

Theorem 5.2.1. *We have*

$$S(\alpha, \beta; q; t) = \frac{q}{(\beta q - \alpha)(1-q^t)}\left(\frac{(\alpha;q)_t}{(\beta q;q)_t} - 1\right). \tag{5.2.2}$$

First let us recall two basic hypergeometric series identities.

Lemma 5.2.2 (First q-Chu–Vandermonde Sum [16, Eq. (17.6.2)])**.** *We have*

$${}_2\phi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{5.2.3}$$

Lemma 5.2.3 (q-Analogue of the Kummer–Thomae–Whipple Transformation [83, p. 72, Eq. (3.2.7)])**.** *We have*

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right) = \frac{(e/a;q)_\infty (de/bc;q)_\infty}{(e;q)_\infty (de/abc;q)_\infty} {}_3\phi_2\left(\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a}\right). \tag{5.2.4}$$

Proof of Theorem 5.2.1. We have

$$\begin{aligned}
&S(\alpha,\beta;q;t) \qquad\qquad (5.2.5)\\
&\quad= \sum_{r\ge 1} \frac{(1-\alpha q^r)(1-\alpha q^{r+1})\cdots(1-\alpha q^{r+t-2})}{(1-\beta q^r)(1-\beta q^{r+1})\cdots(1-\beta q^{r+t})} q^r\\
&\quad= \sum_{r\ge 1} \frac{(\alpha;q)_{r+t-1}(\beta;q)_r}{(\alpha;q)_r(\beta;q)_{r+t+1}} q^r\\
&\quad= \sum_{r\ge 0} \frac{(\alpha;q)_{r+t}(\beta;q)_{r+1}}{(\alpha;q)_{r+1}(\beta;q)_{r+t+2}} q^{r+1}\\
&\quad= \frac{q(1-\beta)(\alpha;q)_t}{(1-\alpha)(\beta;q)_{t+2}} \sum_{r\ge 0} \frac{(q;q)_r(\beta q;q)_r(\alpha q^t;q)_r}{(q;q)_r(\alpha q;q)_r(\beta q^{t+2};q)_r} q^r\\
&\quad= \frac{q(\alpha q;q)_{t-1}}{(\beta q;q)_{t+1}}\ {}_3\phi_2\left(\begin{matrix} q,\beta q,\alpha q^t \\ \alpha q,\beta q^{t+2}\end{matrix}; q, q\right)\\
&\quad= \frac{q(\alpha q;q)_{t-1}}{(\beta q;q)_{t+1}} \frac{(\beta q^{t+1};q)_\infty (q^2;q)_\infty}{(\beta q^{t+2};q)_\infty (q;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} q,\alpha/\beta,q^{1-t} \\ \alpha q, q^2\end{matrix}; q, \beta q^{t+1}\right) \qquad \text{(by Eq. (5.2.4))}\\
&\quad= \frac{q(\alpha q;q)_{t-1}}{(1-q)(\beta q;q)_t} \sum_{r\ge 0} \frac{(\alpha/\beta;q)_r(q^{1-t};q)_r}{(\alpha q;q)_r(q^2;q)_r} \left(\beta q^{t+1}\right)^r\\
&\quad= \frac{q(\alpha q;q)_{t-1}}{(1-q)(\beta q;q)_t} \frac{(1-\alpha)(1-q)}{\beta q^{t+1}\left(1-\frac{\alpha}{\beta q}\right)(1-q^{-t})} \sum_{r\ge 0} \frac{\left(\frac{\alpha}{\beta q};q\right)_{r+1}(q^{-t};q)_{r+1}}{(\alpha;q)_{r+1}(q;q)_{r+1}} \left(\beta q^{t+1}\right)^{r+1}\\
&\quad= \frac{q}{(\beta q-\alpha)(q^t-1)} \frac{(\alpha;q)_t}{(\beta q;q)_t} \left({}_2\phi_1\left(\begin{matrix} \frac{\alpha}{\beta q}, q^{-t} \\ \alpha\end{matrix}; q, \beta q^{t+1}\right) - 1\right)\\
&\quad= \frac{q}{(\beta q-\alpha)(q^t-1)} \frac{(\alpha;q)_t}{(\beta q;q)_t} \left(\frac{(\beta q;q)_t}{(\alpha;q)_t} - 1\right) \qquad \text{(by Eq. (5.2.3))}\\
&\quad= \frac{q}{(\beta q-\alpha)(1-q^t)} \left(\frac{(\alpha;q)_t}{(\beta q;q)_t} - 1\right).
\end{aligned}$$

□

Let us see how to make use of Theorem 5.2.1 to recover (5.1.2). Note that the generating function for partitions counted by $p_t(n)$ with smallest part equal to r is

$$\frac{q^r}{(1-q^r)(1-q^{r+1})\cdots(1-q^{r+t})}.$$

It follows that

$$\sum_{n\geq 1} p_t(n)q^n = \sum_{r\geq 1} \frac{q^r}{(1-q^r)(1-q^{r+1})\cdots(1-q^{r+t})} = S(0,1;q;t).$$

Hence, by Theorem 5.2.1, we have

$$\sum_{n\geq 1} p_t(n)q^n = \frac{1}{1-q^t}\left(\frac{1}{(q;q)_t}-1\right).$$

Analogously, we have the following results.

Theorem 5.2.4. *Let $pd_t(n)$ count the number of partitions of n in which all parts are distinct and the difference between largest and smallest parts is at most t.*

Then

$$\sum_{n\geq 1} pd_t(n)q^n = \frac{1}{1-q^{t+1}}\left((-q;q)_{t+1}-1\right). \tag{5.2.6}$$

Theorem 5.2.5. *Let $po_t(n)$ count the number of partitions of n in which all parts are odd and the difference between largest and smallest parts is at most t.*

Then

$$\sum_{n\geq 1} po_{2t}(n)q^n = \frac{1}{1-q^{2t}}\left(\frac{1}{(q;q^2)_t}-1\right). \tag{5.2.7}$$

5.3 Overpartitions

A more intriguing problem is about overpartitions with bounded differences. Let $g_t(m,n)$ count the number of overpartitions of n in which there are exactly m overlined parts, the difference between largest and smallest parts is at most t, and if the difference between largest and smallest parts is exactly t, then the largest part cannot be overlined. Then

$$\begin{aligned}\sum_{n\geq 1}\sum_{m\geq 0} g_t(m,n)z^m q^n &= \sum_{r\geq 1} \frac{(1+z)q^r}{1-q^r}\frac{1+zq^{r+1}}{1-q^{r+1}}\cdots\frac{1+zq^{r+t-1}}{1-q^{r+t-1}}\frac{1}{1-q^{r+t}}\\ &= (1+z)S(-zq,q;q;t).\end{aligned}$$

The following result immediately follows from Theorem 5.2.1.

Theorem 5.3.1. *We have*

$$\sum_{n\geq 1}\sum_{m\geq 0} g_t(m,n)z^m q^n = \frac{1}{1-q^t}\left(\frac{(-zq;q)_t}{(q;q)_t}-1\right). \tag{5.3.1}$$

Next, we will turn to the combinatorial aspect.

5.3.1 A Semi-Combinatorial Proof

We show the $z = 1$ case of (5.3.1) from the viewpoint of over q-binomial coefficients.

Let $g_t(n)$ count the number of overpartitions of n in which the difference between largest and smallest parts is at most t, and if the difference between largest and smallest parts is exactly t, then the largest part cannot be overlined.

Letting $z = 1$ in (5.3.1) yields the following identity.

Theorem 5.3.2. *We have*

$$\sum_{n\geq 1} g_t(n)q^n = \frac{1}{1-q^t}\left(\frac{(-q;q)_t}{(q;q)_t} - 1\right). \tag{5.3.2}$$

Recall that the q-binomial coefficient

$$\begin{bmatrix} M+N \\ N \end{bmatrix} = \begin{bmatrix} M+N \\ N \end{bmatrix}_q$$

is the generating function for partitions where the largest part is at most M and the number of parts is at most N. In a paper of Dousse and Kim [75], they introduced the over q-binomial coefficient, denoted by

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} = \overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q, \tag{5.3.3}$$

which is an overpartition analog of q-binomial coefficient defined as the generating function for overpartitions where the largest part is at most M and the number of parts is at most N. They showed that for positive integers M and N,

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} = \sum_{k=0}^{\min(M,N)} q^{\binom{k+1}{2}} \frac{(q;q)_{M+N-k}}{(q;q)_k(q;q)_{M-k}(q;q)_{N-k}}. \tag{5.3.4}$$

Of course, if we agree that the number of such overpartitions of 0 is one, then this identity also holds for $M = 0$ or $N = 0$.

Over q-binomial coefficients have many properties similar to those of the standard

q-binomial coefficients. For example, the following recurrence relation

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} = \overline{\begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-1 \\ N \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-2 \\ N-1 \end{bmatrix}} \tag{5.3.5}$$

holds for any positive integers M and N (see [75, (1.1)]). In fact, it can be proved combinatorially.

Now we denote by $\overline{P}_t(q)$ the generating function of overpartitions in which the difference between largest and smallest parts is at most t.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an overpartition of n with exactly ℓ parts, $\lambda_\ell = r \geq 1$, and $\lambda_1 \leq r + t$. Then $\mu = (\lambda_1 - r, \ldots, \lambda_{\ell-1} - r)$ is an overpartition of $n - \ell r$ with at most $\ell - 1$ parts and greatest part $\leq t$. Note that the first occurrence of the smallest part of λ can be either overlined or not. Hence the generating function for such overpartitions is

$$2q^{\ell r} \overline{\begin{bmatrix} t+\ell-1 \\ t \end{bmatrix}},$$

and hence

$$\overline{P}_t(q) = 2\sum_{\ell\geq 1}\sum_{r\geq 1} q^{\ell r} \overline{\begin{bmatrix} t+\ell-1 \\ t \end{bmatrix}} = 2\sum_{\ell\geq 1} \frac{q^\ell}{1-q^\ell} \overline{\begin{bmatrix} t+\ell-1 \\ t \end{bmatrix}}. \tag{5.3.6}$$

We remark that this identity also holds for $t = 0$.

On the other hand, overpartitions where the difference between largest and smallest parts is at most t can be divided into three disjoint cases:

(i) The largest part is at most t;

(ii) The largest part is greater than t, the difference between largest and smallest parts is exactly t, and the first occurrence of the smallest part is overlined;

(iii) Otherwise.

For *Case* (i), one readily sees the generating function is

$$\frac{(-q;q)_t}{(q;q)_t} - 1.$$

For *Case* (ii), its generating function is

$$2\sum_{r\geq 1} \frac{q^r}{1-q^r} \frac{1+q^{r+1}}{1-q^{r+1}} \cdots \frac{1+q^{r+t-1}}{1-q^{r+t-1}} \frac{q^{r+t}}{1-q^{r+t}} = \frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2}.$$

Finally, let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an overpartition of *Case* (iii) with $\lambda_1 = r + t$ (and so $r \geq 1$). We note that $\mu = (\lambda_1 - r, \ldots, \lambda_\ell - r)$ is an overpartition of $|\lambda| - \ell r$ with at most ℓ parts and largest part being exactly t. Hence the generating function is

$$\sum_{\ell\geq 1}\sum_{r\geq 1} q^{\ell r}\left(\overline{\begin{bmatrix} t+\ell \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+\ell-1 \\ t-1 \end{bmatrix}}\right) = \sum_{\ell\geq 1}\frac{q^\ell}{1-q^\ell}\left(\overline{\begin{bmatrix} t+\ell \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+\ell-1 \\ t-1 \end{bmatrix}}\right).$$

We therefore have

$$\overline{P}_t(q) = \left(\frac{(-q;q)_t}{(q;q)_t} - 1\right) + \frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2} + \sum_{\ell\geq 1}\frac{q^\ell}{1-q^\ell}\left(\overline{\begin{bmatrix} t+\ell \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+\ell-1 \\ t-1 \end{bmatrix}}\right).$$

Now we take $M \to \ell$ and $N \to t$ in (5.3.5) and rewrite it as

$$\overline{\begin{bmatrix} t+\ell \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+\ell-1 \\ t-1 \end{bmatrix}} = q^t\left(\overline{\begin{bmatrix} t+\ell-1 \\ t \end{bmatrix}} + \overline{\begin{bmatrix} t+\ell-2 \\ t-1 \end{bmatrix}}\right).$$

We then multiply both sides by $q^\ell/(1-q^\ell)$ and sum over ℓ

$$\begin{aligned}
&\sum_{\ell\geq 1}\frac{q^\ell}{1-q^\ell}\left(\overline{\begin{bmatrix} t+\ell \\ t \end{bmatrix}} - \overline{\begin{bmatrix} t+\ell-1 \\ t-1 \end{bmatrix}}\right) \\
&\quad = q^t\left(\sum_{\ell\geq 1}\frac{q^\ell}{1-q^\ell}\overline{\begin{bmatrix} t+\ell-1 \\ t \end{bmatrix}} + \sum_{\ell\geq 1}\frac{q^\ell}{1-q^\ell}\overline{\begin{bmatrix} t+\ell-2 \\ t-1 \end{bmatrix}}\right).
\end{aligned}$$

From the foregoing argument, we therefore have

$$\overline{P}_t(q) - \left(\frac{(-q;q)_t}{(q;q)_t} - 1\right) - \frac{\overline{P}_t(q) - \overline{P}_{t-1}(q)}{2} = q^t\frac{\overline{P}_t(q) + \overline{P}_{t-1}(q)}{2}.$$

Hence,

$$\frac{\overline{P}_t(q) + \overline{P}_{t-1}(q)}{2} = \frac{1}{1-q^t}\left(\frac{(-q;q)_t}{(q;q)_t} - 1\right). \tag{5.3.7}$$

Finally, we observe that

$$\overline{P}_t(q) + \overline{P}_{t-1}(q) = 2\sum_{r\geq 1}\frac{q^r}{1-q^r}\frac{1+q^{r+1}}{1-q^{r+1}}\cdots\frac{1+q^{r+t-1}}{1-q^{r+t-1}}\left(\frac{1+q^{r+t}}{1-q^{r+t}} + 1\right)$$

$$
\begin{aligned}
&= 4\sum_{r\geq 1}\frac{q^r}{1-q^r}\frac{1+q^{r+1}}{1-q^{r+1}}\cdots\frac{1+q^{r+t-1}}{1-q^{r+t-1}}\frac{1}{1-q^{r+t}} \\
&= 2\sum_{n\geq 1} g_t(n)q^n. \qquad (5.3.8)
\end{aligned}
$$

Theorem 5.3.2 therefore follows from (5.3.7) and (5.3.8).

5.3.2 A Real Combinatorial Proof

As one might realize, our combinatorial proof of (5.3.2) is kind of cheating as many q-series manipulations are still involved. This is why I call it a semi-combinatorial proof. Our next task is to prove not only (5.3.2) but (5.3.1) in a completely combinatorial manner.

Let $\sharp(\lambda)$ be the number of parts of a partition or an overpartition λ. When λ is an overpartition, we use $o(\lambda)$ to count the number of overlined parts in λ. We write parts in weakly decreasing order.

For a positive integer t, we denote by $\overline{\mathcal{P}}_t$ the set of (nonempty) overpartitions with parts less than or equal to t and no parts equal to t overlined, and by $\overline{\mathcal{G}}_t$ the set of (nonempty) overpartitions with the difference between largest and smallest parts at most t and the largest part not overlined when the difference between largest and smallest parts is exactly t. Also, $\overline{\mathcal{B}}_t$ denotes the set of bipartitions where the first subpartition, which can be an empty partition, consists of only parts equal to t, none overlined, and the second subpartition is a nonempty overpartition with parts less than or equal to t.

5.3.2.1 Partition Sets $\overline{\mathcal{G}}_t$ and $\overline{\mathcal{P}}_t$

We first construct a weight preserving map ϕ from $\overline{\mathcal{G}}_t$ to $\overline{\mathcal{P}}_t$.

For an overpartition $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ in $\overline{\mathcal{G}}_t$, let $s(\pi) = \lfloor \pi_\ell / t \rfloor$, where $\lfloor a \rfloor$ denotes the largest integer not exceeding a, and let $k(\pi)$ be the positive integer k such that $\pi_k \geq (s(\pi)+1)t$ and $\pi_{k+1} < (s(\pi)+1)t$. If there is no such k, then we let $k(\pi) = 0$.

We now define a map $\phi : \overline{\mathcal{G}}_t \to \overline{\mathcal{P}}_t$ as follows. For an overpartition $\pi \in \overline{\mathcal{G}}_t$, let $\sharp(\pi) = \ell$, $s(\pi) = s$ and $k(\pi) = k$. Then

$$
\begin{aligned}
\phi\colon &(\pi_1, \pi_2, \ldots, \pi_\ell) \\
&\mapsto (\underbrace{t, t, t, \ldots, t}_{\substack{s(\ell-k)+(s+1)k \\ \text{times}}}, \pi_{k+1} - st, \ldots, \pi_\ell - st, \pi_1 - (s+1)t, \ldots, \pi_k - (s+1)t),
\end{aligned}
$$

where all the parts equal to t are not overlined, and if π_i is overlined, then $\pi_i - st$ (or $\pi_i - (s+1)t$ depending on the value of i) is overlined. In other words, ϕ takes π to $(t, t, \dots, t, a_1, \dots, a_\ell)$ where $a_1, \dots, a_\ell$ are $\pi_1, \dots, \pi_\ell$ reduced modulo t, cyclically permuted to make them weakly decreasing.

Here we note that there may be parts equal to 0 in $\phi(\pi)$. If there are any parts equal to 0, then we delete them so that $\phi(\pi)$ has positive parts only.

Theorem 5.3.3. *ϕ is a weight preserving map from $\overline{\mathcal{G}}_t$ to $\overline{\mathcal{P}}_t$.*

Proof. Since $\pi_1 - \pi_\ell \le t$, $s = \lfloor \pi_\ell / t \rfloor$, and $\pi_k \ge (s+1)t > \pi_{k+1}$, we have

$$t > \pi_{k+1} - st \ge \cdots \ge \pi_\ell - st \ge \pi_1 - (s+1)t \ge \cdots \ge \pi_k - (s+1)t.$$

Thus the parts of $\phi(\pi)$ are less than or equal to t, and if there are overlined parts, they are less than t.

We now show that no more than one part of the same size is overlined. Since π is an overpartition, at most one part of the same size is overlined in π. Hence, of $\pi_1 - st, \dots, \pi_k - st$, if there are overlined parts, then they must be of different sizes. For the same reason, of $\pi_{k+1} - (s+1)t, \dots, \pi_\ell - (s+1)t$, overlined parts must be of different sizes. Thus, if $\pi_\ell - st > \pi_1 - (s+1)t$, then it is clear that all the overlined parts of $\phi(\pi)$ have different sizes.

Let us suppose that $\pi_\ell - st = \pi_1 - (s+1)t$. Then, we have $\pi_1 - \pi_\ell = t$. By the definition of $\overline{\mathcal{G}}_t$, we know that all the parts equal to π_1 are not overlined. Thus, for parts in $\phi(\pi)$ that are equal to $\pi_\ell - st = \pi_1 - (s+1)t$, either the first occurrence or none may be overlined. Therefore, $\phi(\pi) \in \overline{\mathcal{P}}_t$.

We also note that the map ϕ preserves the weight of π, that is, $|\phi(\pi)| = |\pi|$. $\square$

As we see in the following example, the map ϕ is not a bijection.

Example 5.3.1. Let $t = 3$, $\pi = (7, \overline{4})$ and $\tilde{\pi} = (\overline{4}, 4, 3)$. Then

$$s(\pi) = 1, \quad k(\pi) = 1, \quad \phi(\pi) = (3, 3, 3, \overline{1}, 1), \quad |\phi(\pi)| = |\pi| = 11;$$
$$s(\tilde{\pi}) = 1, \quad k(\tilde{\pi}) = 0, \quad \phi(\tilde{\pi}) = (3, 3, 3, \overline{1}, 1), \quad |\phi(\tilde{\pi})| = |\tilde{\pi}| = 11.$$

However, ϕ is a surjection since $\overline{\mathcal{P}}_t$ is a subset of $\overline{\mathcal{G}}_t$ and $\phi(\pi) = \pi$ for any $\pi \in \overline{\mathcal{P}}_t$. So, we will count how many pre-images each $\mu \in \overline{\mathcal{P}}_t$ has under ϕ.

Let $\pi \in \overline{\mathcal{G}}_t$. We describe how to recover π from $\phi(\pi)$. First, note that it is clear from the definition of $s(\pi)$ and $k(\pi)$ that $\pi_i - (s(\pi)+1)t$ and $\pi_j - s(\pi)t$ are the remainders of π_i and π_j when divided by t for $1 \leq i \leq k(\pi)$ and $j > k(\pi)$. If the remainders are equal to 0, then they are deleted in $\phi(\pi)$. Thus if we know the number of such deleted remainders, we can determine $\sharp(\pi)$. Also, one of the deleted remainders may have been overlined.

We then need to find $s(\pi)$ and $k(\pi)$, where $s(\pi)$ is the quotient of the smallest part of π when divided by t and $k(\pi)$ counts the number of parts whose quotients are equal to $s(\pi) + 1$. Therefore, once we have $\sharp(\pi)$, $k(\pi)$, and $s(\pi)$ along with the information on existence of an overlined deleted remainder, it is clear that we can recover π. Thus possible choices for $\sharp(\pi)$, $k(\pi)$, and $s(\pi)$ with having a deleted remainder overlined or not will determine the number of pre-images under ϕ.

In the following lemma, we will see the range for $\sharp(\pi)$. For any $\mu \in \overline{\mathcal{P}}_t$, we use $m(\mu) = m_t(\mu)$ to count the number of parts of μ equal to t.

Lemma 5.3.4. *Let π be a nonempty overpartition in $\overline{\mathcal{G}}_t$ and $\mu = \phi(\pi)$ in $\overline{\mathcal{P}}_t$. Then we have*

(i) $\sharp(\pi) \leq \sharp(\mu)$;

(ii) $\sharp(\pi) \geq \sharp(\mu) - m(\mu) + \delta_{\sharp(\mu),m(\mu)}$, *where $\delta_{\sharp(\mu),m(\mu)}$ is the Kronecker delta.*

Proof. First, (i) is almost trivial. Under ϕ, each part of π splits into its residue modulo t and as many t's as the quotient, i.e., each part π_i contributes $\lceil \pi_i/t \rceil$ to the number of parts of μ. Thus $\sharp(\pi) \leq \sharp(\mu)$.

Next, we prove (ii). If all of the parts of μ are t, i.e., $\sharp(\mu) = m(\mu)$, then

$$\sharp(\mu) - m(\mu) + \delta_{\sharp(\mu),m(\mu)} = 1 \leq \ell,$$

where the last inequality follows from the fact that π is nonempty.

We now suppose that μ has a part not equal to t, i.e, $\sharp(\mu) - m(\mu) \geq 1$. From the definition of ϕ, we know that the parts of μ not equal to t are the positive remainders of the parts of π, so at most ℓ parts of μ are not equal to t. Hence

$$\sharp(\mu) - m(\mu) + \delta_{\sharp(\mu),m(\mu)} = \sharp(\mu) - m(\mu) \leq \ell.$$

This completes the proof of (ii). $\square$

It follows from Lemma 5.3.4 that

$$\delta_{\sharp(\mu),m(\mu)} \leq \sharp(\pi) - \Big(\sharp(\mu) - m(\mu)\Big) \leq m(\mu), \tag{5.3.9}$$

where $\sharp(\pi) - \Big(\sharp(\mu) - m(\mu)\Big)$ is the number of multiples of t in π.

Lemma 5.3.5. *Let n be a fixed positive integer, and n' a fixed nonnegative integer. Then the following system of equations*

$$\begin{cases} x + y & = n, \\ s\,x + (s+1)y & = n' \end{cases} \tag{5.3.10}$$

has exactly one simultaneous solution $(x, y, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

Proof. We readily see that $y = n' - s\,n$. Also, since $x > 0$ and $y \geq 0$, it follows from the first equation that $0 \leq y < n$. Hence

$$\frac{n'}{n} - 1 < s \leq \frac{n'}{n},$$

from which it follows that $s = \lfloor n'/n \rfloor$. Therefore, there is only one solution (x, y, s). $\square$

We are now ready to determine how many pre-images an overpartition in $\overline{\mathcal{P}}_t$ has.

Theorem 5.3.6. *Let μ be a nonempty overpartition in $\overline{\mathcal{P}}_t$.*

(i) *If $\sharp(\mu) = m(\mu)$, then there are exactly $2m(\mu)$ pre-images in $\overline{\mathcal{G}}_t$ under ϕ. Moreover, of those pre-images, exactly $m(\mu)$ pre-images have no overlined parts, and the other $m(\mu)$ pre-images have the first occurrence of the smallest parts overlined.*

(ii) *If $\sharp(\mu) > m(\mu)$, then there are exactly $2m(\mu) + 1$ pre-images in $\overline{\mathcal{G}}_t$ under ϕ. Moreover, of those pre-images, exactly $m(\mu) + 1$ pre-images have the same number of overlined parts as μ and the other $m(\mu)$ pre-images have one more overlined part than μ does.*

Proof. Let π be a pre-image of μ. By Lemma 5.3.4, we know that

$$\sharp(\mu) - m(\mu) + \delta_{\sharp(\mu), m(\mu)} \leq \sharp(\pi) \leq \sharp(\mu). \tag{5.3.11}$$

Hence, for any integer ℓ in this range, we want to know how many $\pi \in \overline{\mathcal{G}}_t$ with $\sharp(\pi) = \ell$ can be pre-images of μ.

In order for π to be a pre-image of μ with $\sharp(\pi) = \ell$, $s(\pi)$ and $k(\pi)$ must satisfy

$$s(\pi)(\ell - k(\pi)) + (s(\pi) + 1)k(\pi) = m(\mu). \tag{5.3.12}$$

By the definition of $k(\pi)$, it should be less than $\sharp(\pi)$, i.e., $\ell - k(\pi) > 0$. Thus, (5.3.12) is equivalent to that $\Big(\ell - k(\pi), k(\pi), s(\pi)\Big)$ is a solution to (5.3.10) with $n = \ell$ and $n' = m(\mu)$, which is unique.

(i) Suppose that $\sharp(\mu) = m(\mu)$. By (5.3.11), there are $m(\mu)$ choices for ℓ. For a fixed ℓ, $k(\pi)$ and $s(\pi)$ are uniquely determined as seen above. With these $\Big(\ell, k(\pi), s(\pi)\Big)$, we can construct π, in which parts are multiples of t differing by at most t and there are no overlined parts.

For each π, by having the first occurrence of the smallest parts overlined, we obtain a different pre-image. Therefore, the total number of pre-images must be equal to $2m(\mu)$ as claimed. Also, $m(\mu)$ pre-images have no overlined parts and the other $m(\mu)$ pre-images have one overlined smallest part.

(ii) Suppose that $\sharp(\mu) > m(\mu)$. By (5.3.11), there are $\Big(m(\mu) + 1\Big)$ choices for ℓ. For a fixed ℓ, $k(\pi)$ and $s(\pi)$ are uniquely determined. With these $\Big(\ell, k(\pi), s(\pi)\Big)$, we can construct π, in which no multiples of t are overlined.

Note that from the construction of ϕ, $\sharp(\mu) - m(\mu)$ counts the nonzero residues of the parts of π modulo t. So, if $\sharp(\pi) > \sharp(\mu) - m(\mu)$, then π must have multiples of t as parts. For such π, by having the first occurrence of the smallest multiples of t overlined, we obtain a different pre-image.

Therefore, the total number of pre-images must be equal to $\Big(2m(\mu) + 1\Big)$ as claimed. Also, $\Big(m(\mu) + 1\Big)$ pre-images have the same number of overlined parts as μ and the other $m(\mu)$ pre-images have one more overlined part than μ does. $\square$

Theorem 5.3.6 yields

$$\sum_{\pi \in \overline{\mathcal{G}}_t} z^{o(\pi)} q^{|\pi|} = \sum_{\mu \in \overline{\mathcal{P}}_t} \Big(\Big(1 - \delta_{\sharp(\mu), m(\mu)}\Big) + (1+z) m(\mu)\Big) z^{o(\mu)} q^{|\mu|}. \tag{5.3.13}$$

In the following example, we present how to find all the pre-images π of μ.

Example 5.3.2. Let $t = 3$.

(i) Let $\mu = (3, 3, 3)$. Since $\sharp(\mu) = m(\mu) = 3$, by Lemma 5.3.4

$$1 \le \sharp(\pi) \le 3.$$

By solving (5.3.12), we have $\Big(\sharp(\pi), k(\pi), s(\pi)\Big) = (1, 0, 3), (2, 1, 1), (3, 0, 1)$, which yield

$$(9), (\overline{9}),$$

$$(6,3),(6,\overline{3}),$$
$$(3,3,3),(\overline{3},3,3),$$

respectively. There are $2m(\mu)$ pre-images.

(ii) Let $\mu=(3,3,3,\overline{1},1)$. Since $\sharp(\mu)=5$ and $m(\mu)=3$, by Lemma 5.3.4

$$2\leq\sharp(\pi)\leq 5.$$

By solving (5.3.12), we have $\Big(\sharp(\pi),k(\pi),s(\pi)\Big)=(2,1,1),(3,0,1),(4,3,0),(5,3,0)$, which yield

$$\begin{aligned}&(7,\overline{4}),\\&(\overline{4},4,3),(\overline{4},4,\overline{3}),\\&(4,3,3,\overline{1}),(4,\overline{3},3,\overline{1}),\\&(3,3,3,\overline{1},1),(\overline{3},3,3,\overline{1},1),\end{aligned}$$

respectively. Thus, there are $2m(\mu)+1$ pre-images.

5.3.2.2 Partition Sets $\overline{\mathcal{P}}_t$ and $\overline{\mathcal{B}}_t$

We next construct a weight preserving map ψ from $\overline{\mathcal{P}}_t$ to $\overline{\mathcal{B}}_t$.

Let us recall the definition of $\overline{\mathcal{B}}_t$, from which it is clear that

$$\begin{aligned}\sum_{\beta\in\overline{\mathcal{B}}_t} z^{o(\beta)}q^{|\beta|}&=(1+q^t+q^{2t}+\cdots)\left(\frac{(-zq;q)_t}{(q;q)_t}-1\right)\\&=\frac{1}{1-q^t}\left(\frac{(-zq;q)_t}{(q;q)_t}-1\right),\end{aligned}\tag{5.3.14}$$

where $o(\beta)$ denotes the number of overlined parts in β, which is indeed the number of overlined parts in the second subpartition of β.

We now construct a map $\psi:\overline{\mathcal{B}}_t\to\overline{\mathcal{P}}_t$ as follows:

(1) First collect all parts equal to t in both subpartitions and replace an overlined t by a non-overlined t;

(2) and then append the remaining parts in the second subpartition to the parts collected in (1).

For example, $[(3),(3,3,\overline{1},1)]$ and $[(3),(\overline{3},3,\overline{1},1)]$ are both mapped to $(3,3,3,\overline{1},1)$ under ψ.

Let $\mu \in \overline{\mathcal{P}}_t$. Suppose that $\sharp(\mu) = m(\mu)$, i.e., μ has parts equal to t only. Then, its pre-image β must be a bipartition of this form

$$[(\underbrace{t,\ldots,t}_{m(\mu)-x}),(\underbrace{t,\ldots,t}_{x})]$$

for some $x > 0$ with either the first occurrence or none of t's in the second subpartition overlined. Thus there are $2m(\mu)$ pre-images of μ in $\overline{\mathcal{B}}_t$ under ψ. Of those pre-images, $m(\mu)$ pre-images have the same number of overlined parts as μ, and the other $m(\mu)$ pre-images have one more overlined part than μ.

Suppose that $\sharp(\mu) > m(\mu)$, i.e., μ has a part not equal to t. Then, its pre-image π must be a bipartition of this form

$$[(\underbrace{t,\ldots,t}_{m(\mu)-x}),(\underbrace{t,\ldots,t}_{x},\mu_{m(\mu)+1},\ldots)]$$

for some $x \geq 0$ with either the first occurrence or none of t's in the second subpartition overlined. Thus there are $2m(\mu)+1$ pre-images of μ in $\overline{\mathcal{B}}_t$ under ψ. Of those pre-images, $(m(\mu)+1)$ pre-images have the same number of overlined parts as μ, and the other $m(\mu)$ pre-images have one more overlined part than μ.

Therefore, it follows from the map ψ that

$$\sum_{\mu\in\overline{\mathcal{P}}_t}\left(\left(1-\delta_{\sharp(\mu),m(\mu)}\right)+(1+z)m(\mu)\right)z^{o(\mu)}q^{|\mu|} = \sum_{\beta\in\overline{\mathcal{B}}_t} z^{o(\beta)}q^{|\beta|}. \tag{5.3.15}$$

By (5.3.13), (5.3.14), and (5.3.15),

$$\sum_{n\geq 1}\sum_{m\geq 0} g_t(m,n)z^m q^n = \sum_{\pi\in\overline{\mathcal{G}}_t} z^{o(\pi)}q^{|\pi|} = \sum_{\beta\in\overline{\mathcal{B}}_t} z^{o(\beta)}q^{|\beta|} = \frac{1}{1-q^t}\left(\frac{(-zq;q)_t}{(q;q)_t}-1\right),$$

which completes the proof of (5.3.1).

5.4 George Beck's Conjecture

Let us turn our attention to a conjecture due to George Beck (cf. A034296 in the OEIS [163]).

Conjecture 5.4.1. The number of gap-free partitions (i.e. partitions with the difference between each pair of consecutive parts being at most 1) of n is also the sum of the smallest parts in the distinct partitions (i.e. partitions with distinct parts) of n with an odd number of parts.

Let $\mathrm{gf}(n)$ denote the number of gap-free partitions of n. This sequence is listed as A034296 in the OEIS [163]. To determine the generating function of $\mathrm{gf}(n)$, we only need the following trivial observation (cf. [17]):

> The conjugates of gap-free partitions are partitions where only the largest part may repeat.

Hence we have

$$\sum_{n\geq 1} \mathrm{gf}(n) q^n = \sum_{t\geq 1} \frac{q^t}{1-q^t}(-q;q)_{t-1}. \tag{5.4.1}$$

Now let us show that, if $\mathrm{sspt}_{\mathscr{D}\mathscr{O}}(n)$ denotes the sum of the smallest parts in the distinct partitions of n with an odd number of parts, then

$$\sum_{n\geq 1} \mathrm{sspt}_{\mathscr{D}\mathscr{O}}(n) q^n = \sum_{t\geq 1} \frac{q^t}{1-q^t}(-q;q)_{t-1}. \tag{5.4.2}$$

The following result is a consequence of (5.4.1) and (5.4.2).

Theorem 5.4.1. *Conjecture 5.4.1 is true.*

To confirm (5.4.2), we require a bivariate generating function identity.

Theorem 5.4.2. *Let $\mathscr{D}$ be the set of distinct partitions. Let $\sigma(\pi)$ denote the smallest part of a partition π, and $\sharp(\pi)$ the number of parts of π.*

Then

$$\sum_{\pi\in\mathscr{D}} \sigma(\pi) z^{\sharp(\pi)} q^{|\pi|} = \sum_{t\geq 1} \frac{q^t}{1-q^t}\Big((-z;q)_t - 1\Big). \tag{5.4.3}$$

Hence,

$$\begin{aligned}
\sum_{n\geq 1} \mathrm{sspt}_{\mathscr{D}\mathscr{O}}(n) q^n &= \frac{1}{2}\sum_{\pi\in\mathscr{D}} \sigma(\pi)\left(1-(-1)^{\sharp(\pi)}\right) q^{|\pi|}\\
&= \frac{1}{2}\left(\sum_{t\geq 1} \frac{q^t}{1-q^t}\Big((-1;q)_t - 1\Big) - \sum_{t\geq 1} \frac{q^t}{1-q^t}\Big((1;q)_t - 1\Big)\right)\\
&= \sum_{t\geq 1} \frac{q^t}{1-q^t}(-q;q)_{t-1},
\end{aligned}$$

which is as desired.

5.4.1 An Analytic Proof of (5.4.3)

We have

$$\begin{aligned}
\sum_{\pi\in\mathscr{D}} x^{\sigma(\pi)} z^{\sharp(\pi)} q^{|\pi|} &= \sum_{r\geq 1} z x^r q^r (1+zq^{r+1})(1+zq^{r+2})\cdots \\
&= z\sum_{r\geq 1} x^r q^r (-zq^{r+1};q)_\infty \\
&= z(-zq;q)_\infty \sum_{r\geq 1} \frac{x^r q^r}{(-zq;q)_r} \\
&= z(-zq;q)_\infty \left({}_2\phi_1 \left(\begin{matrix} 0, q \\ -zq \end{matrix}; q, xq \right) - 1 \right).
\end{aligned}$$

Recall Heine's first transformation [16, Eq. (17.6.6)]:

$$ {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} {}_2\phi_1 \left(\begin{matrix} c/b, z \\ az \end{matrix}; q, b \right). \tag{5.4.4}$$

Then

$$\begin{aligned}
\sum_{\pi\in\mathscr{D}} x^{\sigma(\pi)} z^{\sharp(\pi)} q^{|\pi|} &= \frac{z(q;q)_\infty}{(xq;q)_\infty} {}_2\phi_1 \left(\begin{matrix} -z, xq \\ 0 \end{matrix}; q, q \right) - z(-zq;q)_\infty \\
&= \frac{z(q;q)_\infty}{(xq;q)_\infty} \sum_{n\geq 0} \frac{(-z;q)_n (xq;q)_n q^n}{(q;q)_n} - z(-zq;q)_\infty \\
&= z(q;q)_\infty \sum_{n\geq 0} \frac{(-z;q)_n q^n}{(q;q)_n (xq^{n+1};q)_\infty} - z(-zq;q)_\infty.
\end{aligned}$$

Applying the operator $[\partial/\partial x]_{x=1}$ yields

$$\begin{aligned}
\sum_{\pi\in\mathscr{D}} \sigma(\pi) z^{\sharp(\pi)} q^{|\pi|} &= \left[\frac{\partial}{\partial x} z(q;q)_\infty \sum_{n\geq 0} \frac{(-z;q)_n q^n}{(q;q)_n (xq^{n+1};q)_\infty} \right]_{x=1} \\
&= z(q;q)_\infty \sum_{n\geq 0} \frac{(-z;q)_n q^n}{(q;q)_n} \left[\frac{\partial}{\partial x} \frac{1}{(xq^{n+1};q)_\infty} \right]_{x=1} \\
&= z(q;q)_\infty \sum_{n\geq 0} \frac{(-z;q)_n q^n}{(q;q)_n} \left[\frac{1}{(xq^{n+1};q)_\infty} \sum_{t\geq n+1} \frac{q^t}{1-xq^t} \right]_{x=1} \\
&= z(q;q)_\infty \sum_{n\geq 0} \frac{(-z;q)_n q^n}{(q;q)_\infty} \sum_{t\geq n+1} \frac{q^t}{1-q^t}
\end{aligned}$$

$$
\begin{aligned}
&= z\sum_{t\geq 1}\frac{q^t}{1-q^t}\sum_{n=0}^{t-1}(-z;q)_n q^n\\
&= z\sum_{t\geq 1}\frac{q^t}{1-q^t}\frac{(-z;q)_t-1}{z}\\
&= \sum_{t\geq 1}\frac{q^t}{1-q^t}\Big((-z;q)_t-1\Big).
\end{aligned}
$$

This is (5.4.3).

5.4.2 A Combinatorial Proof of (5.4.3)

It is notable that (5.4.3) looks quite similar to (5.2.6). Hence, it is natural to expect a combinatorial proof analogous to that in §5.3.2.

Our starting point is the following double counting argument.

Let $\Lambda(\pi)$ denote the largest part of a partition π. For a nonnegative integer t, we define

$$\mathscr{D}_t := \Big\{\pi\in\mathscr{D}\ :\ \Lambda(\pi)\geq t+1 \text{ and } \Lambda(\pi)-\sigma(\pi)\leq t\Big\}.$$

Now given any $\pi\in\mathscr{D}$, if $\pi\in\mathscr{D}_t$, then $\Lambda(\pi)-\sigma(\pi)\leq t\leq \Lambda(\pi)-1$ by the definition. Hence, π is exactly contained in the following $\sigma(\pi)$ partition sets: $\mathscr{D}_{\Lambda(\pi)-1}$, $\mathscr{D}_{\Lambda(\pi)-2}$, $\ldots$, $\mathscr{D}_{\Lambda(\pi)-\sigma(\pi)}$. The following statement holds immediately.

Theorem 5.4.3. *We have*

$$\sum_{\pi\in\mathscr{D}}\sigma(\pi)z^{\sharp(\pi)}q^{|\pi|}=\sum_{t\geq 0}\sum_{\pi\in\mathscr{D}_t}z^{\sharp(\pi)}q^{|\pi|}. \tag{5.4.5}$$

One then sees that the remaining task is to study the generating function for $\mathscr{D}_t$ with $t\geq 0$. For convenience, we now consider the generating function for $\mathscr{D}_{t-1}$ with $t\geq 1$.

Let $\mathscr{B}_t$ be the set of partition pairs (μ,ν) where μ is nonempty and its parts all have size t, and ν is a nonempty distinct partition with 0 being allowed as a part and the largest part being at most $t-1$. For example,

$$\Big((5,5,5,5,5),\ (4,2,1,0)\Big)\in\mathscr{B}_5.$$

For $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ in $\mathscr{D}_{t-1}$, we put $s=\lfloor \pi_\ell/t\rfloor$. We also let k be the positive integer such that $\pi_k\geq (s+1)t$ and $\pi_{k+1}<(s+1)t$. If there is no such k, then we let $k=0$.

Now we construct a map $\phi_t : \mathscr{D}_{t-1} \to \mathscr{B}_t$ defined by

$$\begin{aligned}\phi_t \colon & (\pi_1, \pi_2, \ldots, \pi_\ell) \\ & \mapsto \Big((\underbrace{t, t, t, \ldots, t}_{\substack{s(\ell-k)+(s+1)k \\ \text{times}}}),\ (\pi_{k+1} - st, \ldots, \pi_\ell - st, \pi_1 - (s+1)t, \ldots, \pi_k - (s+1)t) \Big).\end{aligned}$$

Note that the second subpartition can be treated as $(\pi_1, \pi_2, \ldots, \pi_\ell)$ reduced modulo t, cyclically permuted such that they are weakly decreasing.

Similar to Theorem 5.3.3, we have

Lemma 5.4.4. *ϕ_t is a weight preserving map from $\mathscr{D}_{t-1}$ to $\mathscr{B}_t$. Furthermore, the number of parts is preserved by the second subpartition of the image.*

Proof. Let $(\mu, \nu) = \phi_t(\pi)$. We first show that μ is nonempty. Since $\pi \in \mathscr{D}_{t-1}$, we have $\pi_1 \geq (t-1) + 1 = t$. Hence we take out at least one t from π_1 to form μ, which implies that μ is not empty.

On the other hand, we know that π is a distinct partition. Since $\pi_1 - \pi_\ell \leq t - 1 < t$, $s = \lfloor \pi_\ell / t \rfloor$, and $\pi_k \geq (s+1)t > \pi_{k+1}$, we have

$$t > \pi_{k+1} - st > \cdots > \pi_\ell - st > \pi_1 - (s+1)t > \cdots > \pi_k - (s+1)t.$$

Note that $\pi_k - (s+1)t$ could be 0 since π_k could be $(s+1)t$. Hence ν satisfies the conditions. It follows that $(\mu, \nu) \in \mathscr{B}_t$.

At last, it is obvious from the definition of ϕ_t that $|\phi_t(\pi)| = |\pi|$ and $\sharp(\nu) = \sharp(\pi)$. $\square$

The rest is different to the argument in §5.3.2. We shall show

Lemma 5.4.5. *ϕ_t is invertible.*

Proof. Let $(\mu, \nu) \in \mathscr{B}_t$. Let the number of t in μ be $r \geq 1$ and let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$. Now we write $r = m\ell + r^*$ with $m \geq 0$ and $0 \leq r^* \leq \ell - 1$ being integers. We construct the inverse $\phi_t^{-1} : \mathscr{B}_t \to \mathscr{D}_{t-1}$ as follows.

$$\phi_t^{-1} \colon (\mu, \nu) \mapsto (\nu_{\ell - r^* + 1} + (m+1)t, \ldots, \nu_\ell + (m+1)t, \nu_1 + mt, \ldots, \nu_{\ell - r^*} + mt).$$

We now show that the image is in $\mathscr{D}_{t-1}$. Recall that $0 \leq \nu_\ell < \cdots < \nu_1 \leq t - 1$. If $r^* \neq 0$, since $\nu_\ell + t > \nu_1$, we have

$$\nu_{\ell - r^* + 1} + (m+1)t > \cdots > \nu_\ell + (m+1)t > \nu_1 + mt > \cdots > \nu_{\ell - r^*} + mt.$$

Notice that $\nu_{\ell-r^*+1}+(m+1)t \geq t$. We further notice that $\nu_{\ell-r^*}$ is not the smallest part of ν, and hence $\nu_{\ell-r^*}>0$. At last, we have $(\nu_{\ell-r^*+1}+(m+1)t)-(\nu_{\ell-r^*}+mt) = t-(\nu_{\ell-r^*}-\nu_{\ell-r^*+1}) \leq t-1$. Hence in this case the image is in $\mathscr{D}_{t-1}$.

If $r^*=0$, then $m \geq 1$ since $r \geq 1$. We have $\nu_1+mt > \cdots > \nu_\ell+mt > 0$ and $\nu_1+mt \geq t$. We also have $(\nu_1+mt)-(\nu_\ell+mt) = \nu_1-\nu_\ell \leq t-1$. Hence the image is also in $\mathscr{D}_{t-1}$.

From the definition of ϕ_t and ϕ_t^{-1}, it is apparent that $\phi_t^{-1}(\phi_t(\pi))=\pi$. Hence ϕ_t is invertible. $\square$

Example 5.4.1. For the partition sets $\mathscr{D}_4$ and $\mathscr{B}_5$, we have

$$(9,7,6,5) \underset{\phi_5^{-1}}{\overset{\phi_5}{\rightleftharpoons}} \Big((5,5,5,5),\ (4,2,1,0)\Big)$$

and

$$(10,9,7,6) \underset{\phi_5^{-1}}{\overset{\phi_5}{\rightleftharpoons}} \Big((5,5,5,5,5),\ (4,2,1,0)\Big).$$

It follows from Lemmas 5.4.4 and 5.4.5 that ϕ_t is a bijection from $\mathscr{D}_{t-1}$ to $\mathscr{B}_t$. Hence, for $t \geq 1$,

$$\sum_{\pi\in\mathscr{D}_{t-1}} z^{\sharp(\pi)}q^{|\pi|} = \sum_{(\mu,\nu)\in\mathscr{B}_t} z^{\sharp(\nu)}q^{|\mu|+|\nu|}. \tag{5.4.6}$$

The generating function for $\mathscr{B}_t$ is easy to get:

$$\sum_{(\mu,\nu)\in\mathscr{B}_t} z^{\sharp(\nu)}q^{|\mu|+|\nu|} = \frac{q^t}{1-q^t}\Big((-z;q)_t-1\Big), \tag{5.4.7}$$

where $q^t/(1-q^t)$ comes from the first subpartition whereas $(-z;q)_t-1$ comes from the second subpartition. Consequently, we have

Theorem 5.4.6. *For $t \geq 1$,*

$$\sum_{\pi\in\mathscr{D}_{t-1}} z^{\sharp(\pi)}q^{|\pi|} = \frac{q^t}{1-q^t}\Big((-z;q)_t-1\Big). \tag{5.4.8}$$

Together with (5.4.5), we have

$$\sum_{\pi\in\mathscr{D}} \sigma(\pi)z^{\sharp(\pi)}q^{|\pi|} = \sum_{t\geq 1}\frac{q^t}{1-q^t}\Big((-z)_t-1\Big),$$

which completes the proof of (5.4.3).

5.5 Endnotes

Quite recently, Bernard Lin [125] refined (5.3.1) and therefore presented a new proof of the general identity (5.2.2) in a combinatorial manner.

5.6 References

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Chapter 6
Span One Linked Partition Ideals: Kanade–Russell Conjectures

This chapter comes from

- S. Chern, Linked partition ideals, directed graphs and q-multi-summations, *Electron. J. Combin.* **27** (2020), no. 3, Paper No. 3.33, 29 pp. (Ref. [54])
- S. Chern and Z. Li, Linked partition ideals and Kanade–Russell conjectures, *Discrete Math.* **343** (2020), no. 7, 111876, 24 pp. (Ref. [65])

In this series of three chapters, we will develop a theory on the Andrews–Gordon type generating function of span one linked partition ideals and related Rogers–Ramanujan type identities.

6.1 Introduction

6.1.1 Rogers–Ramanujan Type Identities and Kanade–Russell Conjectures

Let us warm up with the two Rogers–Ramanujan identities [145, 156], which state as follows.

Theorem 6.1.1 (Rogers–Ramanujan identities)**.** (i). *The number of partitions of a non-negative integer n into parts congruent to ± 1 modulo 5 is the same as the number of partitions of n such that each two consecutive parts have difference at least 2.*

(ii). *The number of partitions of a non-negative integer n into parts congruent to ± 2 modulo 5 is the same as the number of partitions of n such that each two consecutive parts have difference at least 2 and such that the smallest part is at least 2.*

There are a number of identities of the same flavor discovered by Schur [160], Gleißberg [84], Gordon [86], Göllnitz [85] and so forth. Among these Rogers–Ramanujan type identities, two types of partition sets are considered. One partition set is consist of partitions under certain *congruence* condition. For example, in the first Rogers–Ramanujan identity, we enumerate partitions into parts congruent to ± 1 modulo 5. The other partition set contains partitions under certain *difference-at-a-distance* theme. Let us first adopt a definition in [108].

Definition 6.1.1. We say that a partition $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ satisfies the *difference at least d at distance k* condition if, for all j, $\lambda_j - \lambda_{j+k} \geq d$.

In this setting, we may paraphrase the corresponding partition set in the first Rogers–Ramanujan identity as the set of partitions with difference at least 2 at distance 1.

In 2014, Kanade and Russell [108] proposed six challenging conjectures on partition identities of Rogers–Ramanujan type. For example, the first of their conjectures reads as follows.

Conjecture 6.1.1 (Kanade–Russell conjecture I_1)**.** The number of partitions of a nonnegative integer n into parts congruent to 1, 3, 6 or 8 modulo 9 is the same as the number of partitions of n with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

It should be remarked that these conjectures are intriguingly related to the representation theory of affine Lie algebra. For a detailed description of the idea behind them, one may refer to Kanade's Ph.D. Thesis [106].

On the other hand, in Russell's Ph.D. Thesis [158], companions to the Kanade–Russell conjectures I_4–I_6 were considered. Further, several more conjectures of the same flavor were proposed in [109]. In particular, among these conjectures (including the six conjectures in [108]), there are eleven of them involving the modulus 12. It is notable that in a very recent paper of Bringmann, Jennings-Shaffer and Mahlburg [43], seven of the modulo 12 conjectures were proved, while the rest were, although not completely proved, simplified to a great extent.

One major difficulty of proving the Kanade–Russell conjectures is that it is not always easy to find generating functions for partitions under certain difference-at-a-distance themes. Fortunately, this problem was settled in two recent papers of Kanade and Russell [109], and Kurşungöz [116], in which different sets of identities (but with some overlap) were demonstrated, respectively. However, their proofs, although different, are both purely combinatorial.

Hence, a natural question arises: *Is it possible to obtain the Andrews–Gordon type generating function for the partitions under certain difference-at-a-distance themes in a more algebraic mannar?*

6.1.2 Span One Linked Partition Ideals

In the 1970s, George Andrews [8, 10, 11] have already started a systematic study of Rogers–Ramanujan type identities and developed a general theory in which the concept of *linked partition ideals* was introduced. However, in this series, we will not go into details of this concept due to its lengthy definition. The interested readers may refer to Chapter 8 of Andrews' monograph: *The theory of partitions* [12].

What we are interested in this paper is a special case of linked partition ideals — the *span one linked partition ideals*. In fact, this special case is enough to cover most partition sets under difference-at-a-distance themes.

Let us first fix some notations.

Let $\mathscr{P}$ be the set of all partitions. We define a map $\phi : \mathscr{P} \to \mathscr{P}$ by sending a partition λ to another partition which is obtained by adding 1 to each part of λ. For example, $\phi(5+3+3+2+1) = 6+4+4+3+2$. Let $\phi^0(\lambda) = \lambda$ and for $n \geq 1$ we recursively define $\phi^n(\lambda) = \phi(\phi^{n-1}(\lambda))$. Hence, $\phi^n(\lambda)$ could be obtained by adding n to each part of λ. Also, for two partitions λ and π, their sum $\lambda \oplus \pi$ is constructed by collecting the parts of λ and π in weakly decreasing order. For example, if $\lambda = 3+2+1+1$ and $\pi = 4+2+2+1+1$, then $\lambda \oplus \pi = 4+3+2+2+2+1+1+1+1$.

Let Π be a finite set of partitions containing the empty partition $\emptyset$. For each partition $\pi \in \Pi$, we define its *linking set* $\mathcal{L}(\pi)$ by a subset of Π containing the empty partition. Also, we require that the linking set of the empty partition, $\mathcal{L}(\emptyset)$, equals Π. It is possible to construct finite chains

$$\lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots \to \lambda_K \tag{6.1.1}$$

such that $\lambda_0 \in \Pi$, $\lambda_K \neq \emptyset$ and for all $1 \leq k \leq K$, $\lambda_k \in \mathcal{L}(\lambda_{k-1})$. We may further extend such a finite chain to an infinite chain ending with a series of empty partitions

$$\mathcal{C} : \lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots \to \lambda_K \to \emptyset \to \emptyset \to \cdots . \tag{6.1.2}$$

Let S be a positive integer no smaller than the largest part among all partitions in Π. The above infinite chain $\mathcal{C}$ uniquely determines a partition by

$$\lambda_0 \oplus \phi^S(\lambda_1) \oplus \phi^{2S}(\lambda_2) \oplus \cdots \oplus \phi^{KS}(\lambda_K) \oplus \phi^{(K+1)S}(\emptyset) \oplus \phi^{(K+2)S}(\emptyset) \oplus \cdots , \tag{6.1.3}$$

which is equivalent to

$$\lambda_0 \oplus \phi^S(\lambda_1) \oplus \phi^{2S}(\lambda_2) \oplus \cdots \oplus \phi^{KS}(\lambda_K). \tag{6.1.4}$$

Let us collect such partitions along with the empty partition $\lambda = \emptyset$ (which corresponds to the infinite chain $\emptyset \to \emptyset \to \cdots$) and obtain a partition set $\mathscr{I} := \mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$. Then $\mathscr{I}$ is called a *span one linked partition ideal*.

Example 6.1.1. In the first Rogers–Ramanujan identity, we consider partitions with difference at least 2 at distance 1. It is not hard to verify that this partition set is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$ where $\Pi = \{\emptyset, 1, 2\}$,[1] the linking sets are

$$\mathcal{L}(\emptyset) = \{\emptyset, 1, 2\}, \quad \mathcal{L}(1) = \{\emptyset, 1, 2\}, \quad \mathcal{L}(2) = \{\emptyset, 2\},$$

and $S = 2$.

Finally, we consider a bivariate generating function for any subset $\mathscr{S}$ of $\mathscr{P}$:

$$G_{\mathscr{S}}(x) = G_{\mathscr{S}}(x, q) := \sum_{\lambda \in \mathscr{S}} x^{\sharp(\lambda)} q^{|\lambda|}. \tag{6.1.5}$$

Let $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$ be a span one linked partition ideal. Let the *S-tail* of a partition λ be the collection of parts $\le S$ in λ. If we further define $\mathscr{I}_\pi$ to be the set of partitions in $\mathscr{I}$ whose S-tail is $\pi \in \Pi$, then (8.4.13) in [12] tells us that

$$\sum_{\mu \in \mathscr{I}_\pi} x^{\sharp(\mu)} q^{|\mu|} = x^{\sharp(\pi)} q^{|\pi|} \sum_{\varpi \in \mathcal{L}(\pi)} \sum_{\nu \in \mathscr{I}_\varpi} (xq^S)^{\sharp(\nu)} q^{|\nu|}. \tag{6.1.6}$$

In other words,

$$G_{\mathscr{I}_\pi}(x) = x^{\sharp(\pi)} q^{|\pi|} \sum_{\varpi \in \mathcal{L}(\pi)} G_{\mathscr{I}_\varpi}(xq^S). \tag{6.1.7}$$

6.1.3 Andrews' Guess

In a private communication between George Andrews, Zhitai Li and me, Andrews provided a basis of "guessing" the generating function for a linked partition ideal:

[1] Here 1 denotes a partition containing one part of size 1 and likewise 2 denotes a partition containing one part of size 2.

Conjecture 6.1.2 (Andrews)**.** Every linked partition ideal has a bivariate generating function of the form

$$\sum_{n_1,\ldots,n_r\geq 0}\frac{(-1)^{L_1(n_1,\ldots,n_r)}q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}x^{L_3(n_1,\ldots,n_r)}}{(q^{B_1};q^{A_1})_{n_1}\cdots(q^{B_r};q^{A_r})_{n_r}}, \tag{6.1.8}$$

in which L_1, L_2 and L_3 are linear forms in $n_1,\ldots,n_r$ and Q is a quadratic form in $n_1,\ldots,n_r$. Here the coefficient of the $x^m q^n$ term is the number of partitions of n in this linked partition ideal with exactly m parts.

This conjecture has numerous pieces of empirical evidence:

1. Recall that in the first Rogers–Ramanujan identity, we consider partitions of n such that each two consecutive parts have difference at least 2. We know that the generating function for such partitions is
$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}.$$
A generalization of the Rogers–Ramanujan identities is due to Gordon (cf. Theorem 7.5 in [12]). In a special case of Gordon's generalization, we deal with partitions of the form $\lambda_1+\lambda_2+\cdots+\lambda_\ell$, where for all j, $\lambda_j-\lambda_{j+k-1}\geq 2$ with $k\geq 2$ fixed. It can be shown that the generating function is
$$\sum_{n_1,n_2,\ldots,n_{k-1}\geq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}},$$
where $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$. Andrews showed that this partition set is a linked partition ideal.

2. In the first Göllnitz–Gordon identity, one studies partitions of the form $\lambda_1+\lambda_2+\cdots+\lambda_\ell$, in which no odd part is repeated, $\lambda_j-\lambda_{j-1}\geq 2$ if λ_j odd and $\lambda_j-\lambda_{j-1}>2$ if λ_j even. It can be shown that the generating function is
$$(-q;q^2)_\infty\sum_{n_1,n_2\geq 0}\frac{(-1)^{n_2}q^{n_1^2+2n_1n_2+n_2}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}}.$$
This partition set is also a linked partition ideal as claimed by Andrews.

With the aid of the above conjecture and necessary computer algebra systems, if we want to find a generating function identity for a linked partition ideal, we are able to single out the promising candidates by running through a number of functions in the above fashion and comparing the series expansions.

6.2 Systems of q-Difference Equations

As we will see in the next section, a crucial point there can be summarized as the following question: *Suppose we have a system of q-difference equations, say,*

$$\begin{cases} F_1(x) = p_{1,1}(x)F_1(xq^S) + p_{1,2}(x)F_2(xq^S) + \cdots + p_{1,k}(x)F_k(xq^S) \\ F_2(x) = p_{2,1}(x)F_1(xq^S) + p_{2,2}(x)F_2(xq^S) + \cdots + p_{2,k}(x)F_k(xq^S) \\ \vdots \\ F_k(x) = p_{k,1}(x)F_1(xq^S) + p_{k,2}(x)F_2(xq^S) + \cdots + p_{k,k}(x)F_k(xq^S) \end{cases}, \tag{6.2.1}$$

where the F's and p's are in x and q, is it possible to deduce a q-difference equation merely involving F_1? Fortunately, an affirmative algorithm is provided by Andrews in the proof of [12, Lemma 8.10]. We would like to translate Andrews' algorithm to the matrix form to make it more transparent.

At first, the system (6.2.1) can be written in the matrix form

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_k(x) \end{pmatrix} = \begin{pmatrix} p_{1,1}(x) & p_{1,2}(x) & \cdots & p_{1,k}(x) \\ p_{2,1}(x) & p_{2,2}(x) & \cdots & p_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{k,1}(x) & p_{k,2}(x) & \cdots & p_{k,k}(x) \end{pmatrix} \begin{pmatrix} F_1(xq^m) \\ F_2(xq^m) \\ \vdots \\ F_k(xq^m) \end{pmatrix}. \tag{6.2.2}$$

Step (1). We put $u_1(x) = F_1(x)$. Then (6.2.2) becomes

$$\begin{pmatrix} u_1(x) \\ F_2(x) \\ \vdots \\ F_k(x) \end{pmatrix} = \begin{pmatrix} p_{1,1}(x) & p_{1,2}(x) & \cdots & p_{1,k}(x) \\ p_{2,1}(x) & p_{2,2}(x) & \cdots & p_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{k,1}(x) & p_{k,2}(x) & \cdots & p_{k,k}(x) \end{pmatrix} \begin{pmatrix} u_1(xq^m) \\ F_2(xq^m) \\ \vdots \\ F_k(xq^m) \end{pmatrix}. \tag{6.2.3}$$

If $p_{1,2}(x) = p_{1,3}(x) = \cdots = p_{1,k}(x) = 0$, then we shall terminate at this place by noticing that

$$u_1(x) = p_{1,1}(x)u_1(xq^m).$$

For Steps (s) with $2 \leq s \leq k$, we proceed iteratively as follows.

Step (s). Supposing that in the $(s-1)$-th Step, we obtain

$$\begin{pmatrix} u_1(x) \\ \vdots \\ u_{s-1}(x) \\ F_s(x) \\ \vdots \\ F_k(x) \end{pmatrix} = \tilde{P}_{s-1} \begin{pmatrix} u_1(xq^m) \\ \vdots \\ u_{s-1}(xq^m) \\ F_s(xq^m) \\ \vdots \\ F_k(xq^m) \end{pmatrix}, \tag{6.2.4}$$

where $\tilde{P}_{s-1}$ is a $k \times k$ matrix with the (i,j)-th entry being $\tilde{p}_{i,j}(x)$.

Since we have arrived at the sth Step, we know that at least one of the $\tilde{p}_{s-1,s}(x)$, $\tilde{p}_{s-1,s+1}(x)$, $\ldots$, $\tilde{p}_{s-1,k}(x)$ is not identically zero. Otherwise, the program should be terminated at the $(s-1)$-th Step. Further, if $\tilde{p}_{s-1,s}(x)$ is identically zero and $\tilde{p}_{s-1,t}(x)$ (for some t with $s+1 \le t \le k$) is not identically zero, (6.2.4) can be rewritten by swapping F_s and F_t. In such a case, $\tilde{P}_{s-1}$ should be rewritten by swapping $\tilde{p}_{s,s}(x)$ and $\tilde{p}_{t,t}(x)$, swapping $\tilde{p}_{s,t}(x)$ and $\tilde{p}_{t,s}(x)$, swapping $\tilde{p}_{i,s}(x)$ and $\tilde{p}_{i,t}(x)$ for $i \neq s,t$, and swapping $\tilde{p}_{s,j}(x)$ and $\tilde{p}_{t,j}(x)$ for $j \neq s,t$. For notational convenience, we simply rename F_s by F_t and F_t by F_s so that the new relation is still of the form (6.2.4) while $\tilde{p}_{s-1,s}(x)$ is not identically zero.

We then make the following substitution

$$u_s(xq^m) = \tilde{p}_{s-1,s}(x)F_s(xq^m) + \tilde{p}_{s-1,s+1}(x)F_{s+1}(xq^m) + \cdots + \tilde{p}_{s-1,k}(x)F_k(xq^m). \tag{6.2.5}$$

Written in the matrix form, we have

$$\begin{pmatrix} u_1(xq^m) \\ u_2(xq^m) \\ \vdots \\ u_{s-1}(xq^m) \\ u_s(xq^m) \\ F_{s+1}(xq^m) \\ \vdots \\ F_k(xq^m) \end{pmatrix} = T(x) \begin{pmatrix} u_1(xq^m) \\ u_2(xq^m) \\ \vdots \\ u_{s-1}(xq^m) \\ F_s(xq^m) \\ F_{s+1}(xq^m) \\ \vdots \\ F_k(xq^m) \end{pmatrix}, \tag{6.2.6}$$

where

$$T(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \tilde{p}_{s-1,s}(x) & \tilde{p}_{s-1,s+1}(x) & \cdots & \tilde{p}_{s-1,k}(x) \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Here all diagonal entries in the $k \times k$ matrix $T(x)$ are 1 except for the sth diagonal entry. In the sth row of $T(x)$, for $s \leq t \leq k$, the (s,t)-th entry is $\tilde{p}_{s-1,t}(x)$. All remaining entries in $T(x)$ are 0.

Since $\tilde{p}_{s-1,s}(x)$ is not identically zero, the matrix $T(x)$ is invertible. In particular, we have

$$T(x)^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\tilde{p}_{s-1,s}(x)} & -\frac{\tilde{p}_{s-1,s+1}(x)}{\tilde{p}_{s-1,s}(x)} & \cdots & -\frac{\tilde{p}_{s-1,k}(x)}{\tilde{p}_{s-1,s}(x)} \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It follows from (6.2.4) and (6.2.6) that

$$\begin{pmatrix} u_1(x) \\ \vdots \\ u_s(x) \\ F_{s+1}(x) \\ \vdots \\ F_k(x) \end{pmatrix} = \tilde{P}_s \begin{pmatrix} u_1(xq^m) \\ \vdots \\ u_s(xq^m) \\ F_{s+1}(xq^m) \\ \vdots \\ F_k(xq^m) \end{pmatrix}, \tag{6.2.7}$$

where

$$\tilde{P}_s = T(xq^{-m})\tilde{P}_{s-1}T(x)^{-1}.$$

Claim 6.2.1. *The matrix $\tilde{P}_s$ obtained above is of the form*

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ s-1 \\ s \\ \vdots \\ k \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & \cdots & s & s+1 & \cdots & k \end{array} \\ \begin{pmatrix} \star & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \star & \star & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \star & \cdots & 1 & 0 & \cdots & 0 \\ \star & \star & \star & \star & \cdots & \star & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \star & \cdots & \star & \star & \cdots & \star \end{pmatrix} \end{array}.$$

More precisely, in row r $(1 \leq r \leq s-1)$ of $\tilde{P}_s$, the $(r, r+1)$-th entry is 1 and the (r, c)-th entries are 0 for all $c > r+1$.

Proof. We argue by induction on s. When $s = 1$, there is nothing to prove. Assuming that the result is true for some $s-1$ and noticing that $\tilde{P}_{s-1}$ is such a matrix obtained in the $(s-1)$-th Step, we know that $\tilde{p}_{r,r+1}(x) = 1$ for all $r \leq s-2$ and that $\tilde{p}_{r,c}(x) = 0$ for all $r \leq s-2$ and $c > r+1$.

It is obvious that the first $s-1$ rows of $T(xq^{-m})\tilde{P}_{s-1}$ are identical to the first $s-1$ rows of $\tilde{P}_{s-1}$. Let the (j, c)-th entry of $T(x)^{-1}$ be $T^{(-1)}_{j,c}(x)$.

For $r \leq s-1$, the (r, c)-th entry of $\tilde{P}_s = T(xq^{-m})\tilde{P}_{s-1}T(x)^{-1}$ is given by

$$\sum_{j=1}^{k} \tilde{p}_{r,j}(x)T^{(-1)}_{j,c}(x).$$

If $c = r+1$, then the only non-zero contribution in the above summation is

$$\begin{aligned} \tilde{p}_{r,r+1}(x)T^{(-1)}_{r+1,r+1}(x) &= \begin{cases} 1 \cdot 1 & \text{if } r \leq s-2 \\ \tilde{p}_{s-1,s}(x) \cdot \frac{1}{\tilde{p}_{s-1,s}(x)} & \text{if } r = s-1 \end{cases} \\ &= 1. \end{aligned}$$

If $c > r+1$, then we first treat the $r = s-1$ case. One has

$$\sum_{j=1}^{k} \tilde{p}_{s-1,j}(x)T^{(-1)}_{j,c}(x) = \tilde{p}_{s-1,s}(x)T^{(-1)}_{s,c}(x) + \tilde{p}_{s-1,c}(x)T^{(-1)}_{c,c}(x)$$

$$= \tilde{p}_{s-1,s}(x) \cdot \left(-\frac{\tilde{p}_{s-1,c}(x)}{\tilde{p}_{s-1,s}(x)} \right) + \tilde{p}_{s-1,c}(x) \cdot 1$$
$$= 0.$$

For $r \le s-2$, we simply notice that $\tilde{p}_{r,j}(x) = 0$ for $j > r+1$ from our assumption and that $T^{(-1)}_{j,c}(x) = 0$ for $j \le r+1$ since $j \le s-1$ and $j \ne c$. $\square$

Let $\tilde{p}^{\mathrm{New}}_{i,j}(x)$ be the (i,j)-th entry of $\tilde{P}_s$. If $\tilde{p}^{\mathrm{New}}_{s,t}(x) = 0$ for all $t \ge s+1$, then we shall stop at this place by noticing with the help of Claim 6.2.1 that

$$\begin{aligned}
u_1(x) &= \tilde{p}^{\mathrm{New}}_{1,1}(x)u_1(xq^m) + u_2(xq^m),\\
u_2(x) &= \tilde{p}^{\mathrm{New}}_{2,1}(x)u_1(xq^m) + \tilde{p}^{\mathrm{New}}_{2,2}(x)u_2(xq^m) + u_3(xq^m),\\
&\vdots\\
u_{s-1}(x) &= \tilde{p}^{\mathrm{New}}_{s-1,1}(x)u_1(xq^m) + \tilde{p}^{\mathrm{New}}_{s-1,2}(x)u_2(xq^m) + \cdots + u_s(xq^m),\\
u_s(x) &= \tilde{p}^{\mathrm{New}}_{s,1}(x)u_1(xq^m) + \tilde{p}^{\mathrm{New}}_{s,2}(x)u_2(xq^m) + \cdots + \tilde{p}^{\mathrm{New}}_{s,s}(x)u_s(xq^m).
\end{aligned}$$

Final setup. Assuming that the above program is terminated after ℓ $(\le k)$ steps, we obtain a new system of q-difference equations

$$\begin{aligned}
u_1(x) &= r_{1,1}(x)u_1(xq^m) + u_2(xq^m),\\
u_2(x) &= r_{2,1}(x)u_1(xq^m) + r_{2,2}(x)u_2(xq^m) + u_3(xq^m),\\
&\vdots\\
u_{\ell-1}(x) &= r_{\ell-1,1}(x)u_1(xq^m) + r_{\ell-1,2}(x)u_2(xq^m) + \cdots + r_{\ell-1,\ell-1}(x)u_{\ell-1}(x) + u_\ell(xq^m),\\
u_\ell(x) &= r_{\ell,1}(x)u_1(xq^m) + r_{\ell,2}(x)u_2(xq^m) + \cdots + r_{\ell,\ell-1}(x)u_{\ell-1}(xq^m) + r_{\ell,\ell}(x)u_\ell(xq^m),
\end{aligned}$$

where the r's are in x and q.

With this new system, a q-difference equation involving merely u_1 can be obtained by simple eliminations. Finally, we recall that $F_1(x)$ is set to be $u_1(x)$ in Step (1).

6.3 Kanade–Russell Conjectures

We may summarize the following four types of partition sets under difference-at-a-distance themes from the Kanade–Russell conjectures.

- Type I:

Partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

- Type II:

Partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is congruent to 2 modulo 3.

- Type III:

Partitions with difference at least 3 at distance 3 such that if parts at distance 2 differ by at most 1, then the sum of the two parts and their intermediate part is congruent to 1 modulo 3.

- Type IV:

Partitions with difference at least 3 at distance 3 such that if parts at distance 2 differ by at most 1, then the sum of the two parts and their intermediate part is congruent to 2 modulo 3.

In this section, we investigate partition sets of types I, II, III and IV under the setting of linked partition ideals.

6.3.1 Partition Set of Type I

Recall that the partition set of type I is the set of partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3. In other words, if $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is in this partition set, then

(i) $\lambda_i - \lambda_{i+2} \geq 3$;

(ii) $\lambda_i - \lambda_{i+1} \leq 1$ implies $\lambda_i + \lambda_{i+1} \equiv 0 \pmod 3$.

Let $\mathscr{I}_{T_{\mathrm{I}}}$ denote the partition set of type I.

Claim 6.3.1. *$\mathscr{I}_{T_{\mathrm{I}}}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ where $S = 3$, and*

$\Pi = \{\pi_1, \pi_2, \dots, \pi_7\}$ *along with the linking sets given as follows.*

Π	*linking set*
$\pi_1 = \emptyset$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_2 = 1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_3 = 2+1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_4 = 3+1$	$\{\pi_1,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_5 = 2$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_6 = 3$	$\{\pi_1,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_7 = 3+3$	$\{\pi_1,\ \pi_6,\ \pi_7\}$

Proof. A straightforward verification tells us that any partition in $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ satisfies distance conditions (i) and (ii) above and hence is in $\mathscr{I}_{T_{\mathrm{I}}}$.

On the other hand, given a partition $\lambda \in \mathscr{I}_{T_{\mathrm{I}}}$, we decompose it as

$$\lambda_0 \oplus \phi^3(\lambda_1) \oplus \phi^{3\cdot 2}(\lambda_2) \oplus \cdots \oplus \phi^{3K}(\lambda_K).$$

Note that for $0 \le k \le K$, $\phi^{3k}(\lambda_k)$ is simply the collection of parts in λ of size between $3k+1$ and $3k+3$. First, to ensure the distance conditions (i) and (ii), we must have $\lambda_k \in \Pi$ for all k. Now we only need to check case by case. For example, if $\lambda_k = \pi_6 = 3$ for some k, then there is only one part of size $3k+3$ between $3k+1$ and $3k+4$. We consider parts of size between $3k+4$ and $3k+6$. The distance conditions (i) and (ii) sieve the following four choices: $\emptyset$, $(3k+5)$, $(3k+6)$ and $(3k+6)+(3k+6)$. Hence, we have four choices for λ_{k+1}: π_1, π_5, π_6 and π_7. For the remaining cases, we may carry out the same argument. Hence, λ is in $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$.

Consequently, $\mathscr{I}_{T_{\mathrm{I}}} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$. $\square$

Let us denote by $H_i(x) = H_i(x,q)$ the generating function of partitions λ in $\mathscr{I}_{T_{\mathrm{I}}}$ with 3-tail equal to π_i for $i = 1, 2, \dots, 7$ where the π_i's are as defined in Lemma 6.3.1.

Following (6.1.7), we have

$$H_1(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.1}$$

$$x^{-1}q^{-1}H_2(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.2}$$

$$x^{-2}q^{-3}H_3(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.3}$$

$$x^{-2}q^{-4}H_4(x) = H_1(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.4}$$

$$x^{-1}q^{-2}H_5(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.5}$$

$$x^{-1}q^{-3}H_6(x) = H_1(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.6}$$

$$x^{-2}q^{-6}H_7(x) = H_1(xq^3) + H_6(xq^3) + H_7(xq^3). \tag{6.3.7}$$

Let $G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x) = G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x,q)$ (resp. $G_{\mathscr{I}_{T_{\mathrm{I},2}}}(x)$, $G_{\mathscr{I}_{T_{\mathrm{I},3}}}(x)$) denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{I}}}$ whose smallest part is at least 1 (resp. 2, 3).

It follows that

$$\begin{aligned} G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x) &= H_1(x) + H_2(x) + H_3(x) + H_4(x) + H_5(x) + H_6(x) + H_7(x) \\ &= H_1(xq^{-3}), && (6.3.8) \\ G_{\mathscr{I}_{T_{\mathrm{I},2}}}(x) &= H_1(x) + H_5(x) + H_6(x) + H_7(x) \\ &= x^{-1}H_6(xq^{-3}), && (6.3.9) \\ G_{\mathscr{I}_{T_{\mathrm{I},3}}}(x) &= H_1(x) + H_6(x) + H_7(x) \\ &= x^{-2}H_7(xq^{-3}). && (6.3.10) \end{aligned}$$

Hence, to determine q-difference equations satisfied by $G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x)$, $G_{\mathscr{I}_{T_{\mathrm{I},2}}}(x)$ and $G_{\mathscr{I}_{T_{\mathrm{I},3}}}(x)$, it suffices to find q-difference equations for $H_1(x)$, $H_6(x)$ and $H_7(x)$, respectively.

We now deduce from (6.3.1), (6.3.2), (6.3.3) and (6.3.5) that

$$H_2(x) = xqH_1(x), \tag{6.3.11}$$

$$H_3(x) = x^2q^3H_1(x), \tag{6.3.12}$$

$$H_5(x) = xq^2H_1(x), \tag{6.3.13}$$

and likewise from (6.3.4) and (6.3.6) that

$$H_4(x) = xqH_6(x). \tag{6.3.14}$$

As a result, the system (6.3.1)–(6.3.7) can be rewritten as

$$H_1(x) = (1 + xq^4 + x^2q^9 + xq^5)H_1(xq^3) + (1 + xq^4)H_6(xq^3) + H_7(xq^3), \tag{6.3.15}$$
$$H_6(x) = (xq^3 + x^2q^8)H_1(xq^3) + xq^3H_6(xq^3) + xq^3H_7(xq^3), \tag{6.3.16}$$
$$H_7(x) = x^2q^6H_1(xq^3) + x^2q^6H_6(xq^3) + x^2q^6H_7(xq^3). \tag{6.3.17}$$

We first use the algorithm in §6.2 to deduce the q-difference equation satisfied by $H_1(x)$ and accordingly $G_{\mathscr{I}_{T_{I,1}}}(x)$.

Step (1). We put $u_1(x) = H_1(x)$. Then

$$\begin{pmatrix} u_1(x) \\ H_6(x) \\ H_7(x) \end{pmatrix} = \tilde{P}_1 \begin{pmatrix} u_1(xq^3) \\ H_6(xq^3) \\ H_7(xq^3) \end{pmatrix}, \tag{6.3.18}$$

where

$$\tilde{P}_1 = \begin{pmatrix} 1 + xq^4 + x^2q^9 + xq^5 & 1 + xq^4 & 1 \\ xq^3 + x^2q^8 & xq^3 & xq^3 \\ x^2q^6 & x^2q^6 & x^2q^6 \end{pmatrix}.$$

Step (2). We put $u_6(x) = (1 + xq^4)H_6(xq^3) + H_7(xq^3)$. Then

$$\begin{pmatrix} u_1(x) \\ u_6(x) \\ H_7(x) \end{pmatrix} = \tilde{P}_2 \begin{pmatrix} u_1(xq^3) \\ u_6(xq^3) \\ H_7(xq^3) \end{pmatrix}, \tag{6.3.19}$$

where

$$\tilde{P}_2 = \begin{pmatrix} 1 + xq^4 + xq^5 + x^2q^9 & 1 & 0 \\ xq^3(1 + xq + xq^3 + xq^5 + x^2q^6) & \frac{xq^3(1+xq+xq^3)}{1+xq^4} & \frac{x^2q^7(1+xq+xq^3)}{1+xq^4} \\ x^2q^6 & \frac{x^2q^6}{1+xq^4} & \frac{x^3q^{10}}{1+xq^4} \end{pmatrix}.$$

Step (3). We put $u_7(x) = \frac{x^2q^7(1+xq+xq^3)}{1+xq^4}H_7(xq^3)$. Then

$$\begin{pmatrix} u_1(x) \\ u_6(x) \\ u_7(x) \end{pmatrix} = \tilde{P}_3 \begin{pmatrix} u_1(xq^3) \\ u_6(xq^3) \\ u_7(xq^3) \end{pmatrix}, \tag{6.3.20}$$

where

$$\tilde{P}_3 = \begin{pmatrix} 1+xq^4+xq^5+x^2q^9 & 1 & 0 \\ xq^3(1+xq+xq^3+xq^5+x^2q^6) & \frac{xq^3(1+xq+xq^3)}{1+xq^4} & 1 \\ \frac{x^4q^7(1+x+xq^{-2})}{1+xq} & \frac{x^4q^7(1+x+xq^{-2})}{(1+xq)(1+xq^4)} & \frac{x^3q^4(1+x+xq^{-2})}{(1+xq)(1+xq+xq^3)} \end{pmatrix}.$$

For convenience, we write

$$u_1(x) = r_{1,1}(x)u_1(xq^3) + u_6(xq^3), \tag{6.3.21}$$

$$u_6(x) = r_{6,1}(x)u_1(xq^3) + r_{6,6}(x)u_6(xq^3) + u_7(xq^3), \tag{6.3.22}$$

$$u_7(x) = r_{7,1}(x)u_1(xq^3) + r_{7,6}(x)u_5(xq^3) + r_{7,7}u_6(xq^3), \tag{6.3.23}$$

where the coefficients are rational functions in x and q given by $\tilde{P}_3$.

Noting from (6.3.8) that

$$G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x) = H_1(xq^{-3}) = u_1(xq^{-3}), \tag{6.3.24}$$

we may eliminate $u_6(x)$ by (6.3.21)

$$u_6(x) = G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x) - r_{1,1}(xq^{-3})G_{\mathscr{I}_{T_{\mathrm{I},1}}}(xq^3). \tag{6.3.25}$$

Substituting (6.3.25) into (6.3.22), we may eliminate $u_7(x)$

$$\begin{aligned} u_7(x) = G_{\mathscr{I}_{T_{\mathrm{I},1}}}(xq^{-3}) &- \Big(r_{1,1}(xq^{-6}) + r_{6,6}(xq^{-3})\Big)\, G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x) \\ &+ \Big(r_{1,1}(xq^{-3})r_{6,6}(xq^{-3}) - r_{6,1}(xq^{-3})\Big)\, G_{\mathscr{I}_{T_{\mathrm{I},1}}}(xq^3). \end{aligned} \tag{6.3.26}$$

Substituting (6.3.24), (6.3.25) and (6.3.26) into (6.3.23), we arrive at, after simplification, the following q-difference equation for $G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x)$.

Theorem 6.3.2. *It holds that*

$$\begin{aligned} p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{I},1}}}(xq^3) &+ p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{I},1}}}(xq^6) \\ &+ p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{I},1}}}(xq^9) = 0, \end{aligned} \tag{6.3.27}$$

where

$$p_0(x,q) = 1 + x(q^4+q^6),$$

$$
\begin{aligned}
p_3(x,q) &= -1 - x(q+q^2+q^3+q^4+q^6) - x^2(q^3+q^4+q^5+2q^6+q^7+q^8+q^9) \\
&\quad - x^3(q^7+q^9+q^{10}+q^{12}), \\
p_6(x,q) &= x^3(q^{11}+q^{13}) + x^4(q^{14}+q^{15}+q^{16}+q^{17}+q^{18}) + x^5(q^{19}+q^{21}),
\end{aligned}
$$

and

$$
p_9(x,q) = x^5q^{27} + x^6(q^{28}+q^{30}).
$$

In the same manner, we may find the q-difference equations for $H_6(x)$ and $H_7(x)$, and accordingly $G_{\mathscr{I}_{T_{\mathrm{I},2}}}(x)$ and $G_{\mathscr{I}_{T_{\mathrm{I},3}}}(x)$.

Theorem 6.3.3. *It holds that*

$$
\begin{aligned}
p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{I},2}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{I},2}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{I},2}}}(xq^6)& \\
+ p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{I},2}}}(xq^9) = 0,&
\end{aligned}
\tag{6.3.28}
$$

where

$$
\begin{aligned}
p_0(x,q) &= 1 + x(q^5+q^8), \\
p_3(x,q) &= -1 - x(q^2+q^3+q^4+q^5+q^8) \\
&\quad - x^2(2q^6+q^7+q^8+q^9+q^{10}+q^{11}+q^{12}) - x^3(q^{11}+2q^{14}+q^{17}), \\
p_6(x,q) &= x^3(q^{16}+q^{17}) + x^4(-q^{17}+q^{18}+q^{19}+q^{21}+q^{22}+q^{23}+q^{24}) \\
&\quad + x^5(q^{26}+q^{29}),
\end{aligned}
$$

and

$$
p_9(x,q) = x^5q^{33} + x^6(q^{35}+q^{38}).
$$

Theorem 6.3.4. *It holds that*

$$
\begin{aligned}
p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{I},3}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{I},3}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{I},3}}}(xq^6)& \\
+ p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{I},3}}}(xq^9) = 0,&
\end{aligned}
\tag{6.3.29}
$$

where

$$
p_0(x,q) = 1 + x(q^6+q^7),
$$

$$\begin{aligned} p_3(x,q) &= -1 - x(q^3+q^4+q^5+q^6+q^7) \\ &\qquad - x^2(q^6+q^8+2q^9+2q^{10}+q^{11}+q^{12}) - x^3(q^{12}+q^{13}+q^{15}+q^{16}), \\ p_6(x,q) &= x^3(q^{16}+q^{17}) + x^4(q^{20}+q^{21}+q^{22}+q^{23}+q^{24}) + x^5(q^{27}+q^{28}), \end{aligned}$$

and

$$p_9(x,q) = x^5q^{36} + x^6(q^{39}+q^{40}).$$

6.3.2 Partition Set of Type II

Recall that the partition set of type II is the set of partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is congruent to 2 modulo 3. In other words, if $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is in this partition set, then

(i) $\lambda_i - \lambda_{i+2} \geq 3$;

(ii) $\lambda_i - \lambda_{i+1} \leq 1$ implies $\lambda_i + \lambda_{i+1} \equiv 2 \pmod 3$.

Let $\mathscr{I}_{T_{\mathrm{II}}}$ denote the partition set of type II.

Claim 6.3.5. *$\mathscr{I}_{T_{\mathrm{II}}}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ where $S = 3$, and $\Pi = \{\pi_1, \pi_2, \ldots, \pi_7\}$ along with the linking sets given as follows.*

Π	*linking set*
$\pi_1 = \emptyset$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_2 = 1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_3 = 1+1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_4 = 3+1$	$\{\pi_1,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_5 = 2$	$\{\pi_1,\ \pi_2,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_6 = 3+2$	$\{\pi_1,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_7 = 3$	$\{\pi_1,\ \pi_5,\ \pi_6,\ \pi_7\}$

Similarly, let us denote by $H_i(x) = H_i(x,q)$ the generating function of partitions λ in $\mathscr{I}_{T_{\mathrm{II}}}$ with 3-tail equal to π_i for $i = 1, 2, \ldots, 7$ where the π_i's are as defined in Claim 6.3.5.

Following (6.1.7), we have

$$H_1(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3),$$

(6.3.30)

$$x^{-1}q^{-1}H_2(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.31}$$

$$x^{-2}q^{-2}H_3(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.32}$$

$$x^{-2}q^{-4}H_4(x) = H_1(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.33}$$

$$x^{-1}q^{-2}H_5(x) = H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.34}$$

$$x^{-2}q^{-5}H_6(x) = H_1(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3), \tag{6.3.35}$$

$$x^{-1}q^{-3}H_7(x) = H_1(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3). \tag{6.3.36}$$

Let $G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x) = G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x,q)$ (resp. $G_{\mathscr{I}_{T_{\mathrm{II},2}}}(x)$) denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{II}}}$ whose smallest part is at least 1 (resp. 2).

Let $G_{\mathscr{I}_{T_{\mathrm{II},a}}}(x)$ denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{II}}}$ where 1 appears at most once.

It follows that

$$\begin{aligned}
G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x) &= H_1(x) + H_2(x) + H_3(x) + H_4(x) + H_5(x) + H_6(x) + H_7(x) \\
&= H_1(xq^{-3}), && (6.3.37) \\
G_{\mathscr{I}_{T_{\mathrm{II},2}}}(x) &= H_1(x) + H_5(x) + H_6(x) + H_7(x) \\
&= x^{-1}H_7(xq^{-3}), && (6.3.38) \\
G_{\mathscr{I}_{T_{\mathrm{II},a}}}(x) &= H_1(x) + H_2(x) + H_4(x) + H_5(x) + H_6(x) + H_7(x) \\
&= x^{-1}qH_5(xq^{-3}). && (6.3.39)
\end{aligned}$$

We may deduce from (6.3.30), (6.3.31) and (6.3.32) that

$$H_2(x) = xqH_1(x), \tag{6.3.40}$$

$$H_3(x) = x^2q^2H_1(x), \tag{6.3.41}$$

and likewise from (6.3.33), (6.3.35) and (6.3.36) that

$$H_4(x) = xqH_7(x), \tag{6.3.42}$$
$$H_6(x) = xq^2H_7(x). \tag{6.3.43}$$

Hence, the system (6.3.30)–(6.3.36) can be rewritten as

$$H_1(x) = (1 + xq^4 + x^2q^8)H_1(xq^3) + H_5(xq^3) + (1 + xq^4 + xq^5)H_7(xq^3), \tag{6.3.44}$$
$$H_5(x) = (xq^2 + x^2q^6)H_1(xq^3) + xq^2H_5(xq^3) + (xq^2 + x^2q^6 + x^2q^7)H_7(xq^3), \tag{6.3.45}$$
$$H_7(x) = xq^3H_1(xq^3) + xq^3H_5(xq^3) + (xq^3 + x^2q^8)H_7(xq^3). \tag{6.3.46}$$

Using the algorithm in §6.2, we are able to prove the following q-difference equations for $G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x)$, $G_{\mathscr{I}_{T_{\mathrm{II},2}}}(x)$ and $G_{\mathscr{I}_{T_{\mathrm{II},a}}}(x)$, respectively.

Theorem 6.3.6. *It holds that*

$$\begin{aligned} p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{II},1}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{II},1}}}(xq^6)& \\ + p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{II},1}}}(xq^9) = 0,& \end{aligned} \tag{6.3.47}$$

where

$$\begin{aligned} p_0(x,q) &= 1 + x(q^4 + q^5), \\ p_3(x,q) &= -1 - x(q + q^2 + q^3 + q^4 + q^5) - x^2(q^2 + q^4 + 2q^5 + 2q^6 + q^7 + q^8) \\ &\quad - x^3(q^6 + q^7 + q^9 + q^{10}), \\ p_6(x,q) &= x^3(q^{10} + q^{11}) + x^4(q^{12} + q^{13} + q^{14} + q^{15} + q^{16}) + x^5(q^{17} + q^{18}), \end{aligned}$$

and

$$p_9(x,q) = x^5q^{26} + x^6(q^{27} + q^{28}).$$

Theorem 6.3.7. *It holds that*

$$p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{II},2}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{II},2}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{II},2}}}(xq^6)$$

$$+ p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{II},2}}}(xq^9) = 0, \tag{6.3.48}$$

where

$$\begin{aligned}
p_0(x,q) &= 1 + x(q^5+q^7),\\
p_3(x,q) &= -1 - x(q^2+q^3+q^4+q^5+q^7)\\
&\quad - x^2(q^5+q^6+q^7+2q^8+q^9+q^{10}+q^{11}) - x^3(q^{10}+q^{12}+q^{13}+q^{15}),\\
p_6(x,q) &= x^3(q^{14}+q^{16}) + x^4(q^{18}+q^{19}+q^{20}+q^{21}+q^{22}) + x^5(q^{24}+q^{26}),
\end{aligned}$$

and

$$p_9(x,q) = x^5q^{32} + x^6(q^{34}+q^{36}).$$

Theorem 6.3.8. *It holds that*

$$\begin{aligned}
p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{II},a}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{II},a}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{II},a}}}(xq^6)&\\
+ p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{II},a}}}(xq^9) = 0,&
\end{aligned} \tag{6.3.49}$$

where

$$\begin{aligned}
p_0(x,q) &= 1 + x(q^4+q^8),\\
p_3(x,q) &= -1 - x(q+q^2+q^3+q^4+q^8)\\
&\quad - x^2(q^4+2q^5+q^6+q^8+q^9+q^{10}+q^{11}) - x^3(q^9+q^{12}+q^{13}+q^{16}),\\
p_6(x,q) &= x^3(-q^{12}+q^{13}+q^{14}+q^{15})\\
&\quad + x^4(-q^{13}+q^{15}+q^{16}+q^{19}+q^{20}+q^{21}+q^{22}) + x^5(q^{23}+q^{27}),
\end{aligned}$$

and

$$p_9(x,q) = x^5q^{29} + x^6(q^{30}+q^{34}).$$

6.3.3 Partition Set of Type III

Recall that the partition set of type III is the set of partitions with difference at least 3 at distance 3 such that if parts at distance 2 differ by at most 1, then the sum of the two parts and their intermediate part is congruent to 1 modulo 3. In other words, if $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is in this partition set, then

(i) $\lambda_i - \lambda_{i+3} \geq 3$;

(ii) $\lambda_i - \lambda_{i+2} \leq 1$ implies $\lambda_i + \lambda_{i+1} + \lambda_{i+2} \equiv 1 \pmod 3$.

Let $\mathscr{I}_{T_{\mathrm{III}}}$ denote the partition set of type III.

Claim 6.3.9. *$\mathscr{I}_{T_{\mathrm{III}}}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ where $S = 3$, and $\Pi = \{\pi_1, \pi_2, \ldots, \pi_{15}\}$ along with the linking sets are given as follows.*

Π	*linking set*
$\pi_1 = \emptyset$	$\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_2 = 1$	$\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_3 = 1+1$	$\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_4 = 2$	$\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_5 = 2+1$	$\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_6 = 2+1+1$	$\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_7 = 2+2$	$\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_8 = 3$	$\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_9 = 3+1$	$\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_{10} = 3+1+1$	$\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_{11} = 3+2$	$\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_{12} = 3+2+1$	$\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_7, \pi_8, \pi_9, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{14}, \pi_{15}\}$
$\pi_{13} = 3+2+2$	$\{\pi_1, \pi_4, \pi_7, \pi_8, \pi_{11}, \pi_{13}, \pi_{14}\}$
$\pi_{14} = 3+3$	$\{\pi_1, \pi_2, \pi_4, \pi_8, \pi_9, \pi_{11}, \pi_{14}, \pi_{15}\}$
$\pi_{15} = 3+3+1$	$\{\pi_1, \pi_2, \pi_4, \pi_8, \pi_9, \pi_{11}, \pi_{14}, \pi_{15}\}$

Let us denote by $H_i(x) = H_i(x,q)$ the generating function of partitions λ in $\mathscr{I}_{T_{\mathrm{III}}}$ with 3-tail equal to π_i for $i = 1, 2, \ldots, 15$ where the π_i's are as defined in Claim 6.3.9.

Following (6.1.7), we have

$$\begin{aligned} H_1(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\ &\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), \end{aligned} \tag{6.3.50}$$

$$\begin{aligned} x^{-1}q^{-1}H_2(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\ &\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), \end{aligned} \tag{6.3.51}$$

$$\begin{aligned} x^{-2}q^{-2}H_3(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\ &\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), \end{aligned} \tag{6.3.52}$$

$$\begin{aligned} x^{-1}q^{-2}H_4(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \end{aligned}$$

$$
\begin{aligned}
&\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.53)\\
x^{-2}q^{-3}H_5(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3)\\
&\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3)\\
&\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.54)\\
x^{-3}q^{-4}H_6(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3)\\
&\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3)\\
&\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.55)\\
x^{-2}q^{-4}H_7(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_7(xq^3) + H_8(xq^3)\\
&\quad + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3)\\
&\quad + H_{15}(xq^3), && (6.3.56)\\
x^{-1}q^{-3}H_8(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_7(xq^3) + H_8(xq^3)\\
&\quad + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3)\\
&\quad + H_{15}(xq^3), && (6.3.57)\\
x^{-2}q^{-4}H_9(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_7(xq^3) + H_8(xq^3)\\
&\quad + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3)\\
&\quad + H_{15}(xq^3), && (6.3.58)\\
x^{-3}q^{-5}H_{10}(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_7(xq^3) + H_8(xq^3)\\
&\quad + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3)\\
&\quad + H_{15}(xq^3), && (6.3.59)\\
x^{-2}q^{-5}H_{11}(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_7(xq^3) + H_8(xq^3)\\
&\quad + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3)\\
&\quad + H_{15}(xq^3), && (6.3.60)\\
x^{-3}q^{-6}H_{12}(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_7(xq^3) + H_8(xq^3)\\
&\quad + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3)\\
&\quad + H_{15}(xq^3), && (6.3.61)\\
x^{-3}q^{-7}H_{13}(x) &= H_1(xq^3) + H_4(xq^3) + H_7(xq^3) + H_8(xq^3) + H_{11}(xq^3) + H_{13}(xq^3)\\
&\quad + H_{14}(xq^3), && (6.3.62)\\
x^{-2}q^{-6}H_{14}(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{11}(xq^3)\\
&\quad + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.63)\\
x^{-3}q^{-7}H_{15}(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{11}(xq^3)\\
&\quad + H_{14}(xq^3) + H_{15}(xq^3). && (6.3.64)
\end{aligned}
$$

This system may be simplified as

$$
\begin{aligned}
H_1(x) &= (1 + xq^4 + x^2q^8 + xq^5 + x^2q^9 + x^3q^{13})H_1(xq^3) \\
&\quad + (xq^4 + 1 + xq^4 + x^2q^8 + xq^5 + x^2q^9)H_8(xq^3) \\
&\quad + H_{13}(xq^3) + (1 + xq^4)H_{14}(xq^3), \qquad (6.3.65) \\
H_8(x) &= (xq^3 + x^2q^7 + x^2q^8 + x^3q^{12})H_1(xq^3) \\
&\quad + (x^2q^7 + xq^3 + x^2q^7 + x^2q^8 + x^3q^{12})H_8(xq^3) \\
&\quad + xq^3H_{13}(xq^3) + (xq^3 + x^2q^7)H_{14}(xq^3), \qquad (6.3.66) \\
H_{13}(x) &= (x^3q^7 + x^4q^{12})H_1(xq^3) + (x^4q^{11} + x^3q^7 + x^4q^{12})H_8(xq^3) \\
&\quad + x^3q^7H_{13}(xq^3) + x^3q^7H_{14}(xq^3), \qquad (6.3.67) \\
H_{14}(x) &= (x^2q^6 + x^3q^{10} + x^3q^{11})H_1(xq^3) + (x^2q^6 + x^3q^{10} + x^3q^{11})H_8(xq^3) \\
&\quad + (x^2q^6 + x^3q^{10})H_{14}(xq^3). \qquad (6.3.68)
\end{aligned}
$$

Let $G_{\mathscr{I}_{T_{\mathrm{III},1}}}(x) = G_{\mathscr{I}_{T_{\mathrm{III},1}}}(x,q)$ (resp. $G_{\mathscr{I}_{T_{\mathrm{III},2}}}(x)$) denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{III}}}$ whose smallest part is at least 1 (resp. 2).

Let $G_{\mathscr{I}_{T_{\mathrm{III},a}}}(x)$ denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{III}}}$ where 1 appears at most once.

It follows that

$$
\begin{aligned}
G_{\mathscr{I}_{T_{\mathrm{III},1}}}(x) &= H_1(x) + H_2(x) + H_3(x) + H_4(x) + H_5(x) + H_6(x) \\
&\quad + H_7(x) + H_8(x) + H_9(x) + H_{10}(x) + H_{11}(x) \\
&\quad + H_{12}(x) + H_{13}(x) + H_{14}(x) + H_{15}(x) \\
&= H_1(xq^{-3}), \qquad (6.3.69) \\
G_{\mathscr{I}_{T_{\mathrm{III},2}}}(x) &= H_1(x) + H_4(x) + H_7(x) + H_8(x) + H_{11}(x) + H_{13}(x) \\
&\quad + H_{14}(x) \\
&= x^{-3}q^2H_{13}(xq^{-3}), \qquad (6.3.70) \\
G_{\mathscr{I}_{T_{\mathrm{III},a}}}(x) &= H_1(x) + H_2(x) + H_4(x) + H_5(x) + H_7(x) + H_8(x) \\
&\quad + H_9(x) + H_{11}(x) + H_{12}(x) + H_{13}(x) + H_{14}(x) \\
&\quad + H_{15}(x) \\
&= x^{-1}H_8(xq^{-3}). \qquad (6.3.71)
\end{aligned}
$$

Likewise, we can use the algorithm in §6.2 to deduce the following q-difference equations for $G_{\mathscr{I}_{T_{\mathrm{III},1}}}(x)$, $G_{\mathscr{I}_{T_{\mathrm{III},2}}}(x)$ and $G_{\mathscr{I}_{T_{\mathrm{III},a}}}(x)$, respectively.

Theorem 6.3.10. *It holds that*

$$\begin{aligned}p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{III},1}}}(x)+p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{III},1}}}(xq^3)+p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{III},1}}}(xq^6)\\+p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{III},1}}}(xq^9)+p_{12}(x,q)G_{\mathscr{I}_{T_{\mathrm{III},1}}}(xq^{12})=0,\end{aligned}\tag{6.3.72}$$

where

$$\begin{aligned}p_0(x,q)={}&1+x(q^4+q^5+2q^7+q^9+q^{10})\\&+x^2(q^9+2q^{11}+q^{12}+q^{13}+2q^{14}+q^{15}+q^{16}+2q^{17}+q^{19})\\&+x^3(q^{16}+q^{18}+q^{19}+2q^{21}+q^{23}+q^{24}+q^{26}),\\p_3(x,q)={}&-1-x(q+q^2+q^3+q^4+q^5+2q^7+q^9+q^{10})\\&-x^2(q^2+q^3+2q^4+2q^5+3q^6+2q^7+3q^8+3q^9+3q^{10}+4q^{11}+3q^{12}+2q^{13}\\&\quad+2q^{14}+q^{15}+q^{16}+2q^{17}+q^{19})\\&-x^3(q^4+q^5+2q^6+4q^7+3q^8+5q^9+5q^{10}+6q^{11}+7q^{12}+8q^{13}+7q^{14}+6q^{15}\\&\quad+6q^{16}+4q^{17}+5q^{18}+4q^{19}+3q^{20}+3q^{21}+q^{22}+q^{23}+q^{24}+q^{26})\\&-x^4(q^8+2q^9+2q^{10}+5q^{11}+4q^{12}+6q^{13}+10q^{14}+8q^{15}+10q^{16}+11q^{17}+8q^{18}\\&\quad+10q^{19}+9q^{20}+8q^{21}+7q^{22}+6q^{23}+4q^{24}+3q^{25}+2q^{26}+2q^{27}+q^{28}\\&\quad+q^{29})\\&-x^5(q^{13}+3q^{15}+3q^{16}+4q^{17}+7q^{18}+6q^{19}+7q^{20}+10q^{21}+7q^{22}+9q^{23}+9q^{24}\\&\quad+6q^{25}+7q^{26}+5q^{27}+4q^{28}+3q^{29}+3q^{30}+q^{31}+q^{32})\\&-x^6(q^{20}+2q^{22}+2q^{23}+q^{24}+4q^{25}+2q^{26}+3q^{27}+4q^{28}+2q^{29}+3q^{30}+3q^{31}\\&\quad+q^{32}+2q^{33}+q^{34}+q^{36}),\\p_6(x,q)={}&x^4(q^{12}+q^{14}+2q^{16}+q^{18}+q^{20})\\&+x^5(q^{13}+2q^{15}+q^{16}+4q^{17}+3q^{18}+4q^{19}+4q^{20}+5q^{21}+5q^{22}+4q^{23}+4q^{24}\\&\quad+3q^{25}+4q^{26}+q^{27}+2q^{28}+q^{30})\\&+x^6(q^{17}+2q^{18}+3q^{19}+5q^{20}+5q^{21}+9q^{22}+9q^{23}+10q^{24}+12q^{25}+12q^{26}\\&\quad+14q^{27}+12q^{28}+12q^{29}+10q^{30}+9q^{31}+9q^{32}+5q^{33}+5q^{34}+3q^{35}\\&\quad+2q^{36}+q^{37})\\&+x^7(q^{22}+3q^{23}+4q^{24}+6q^{25}+7q^{26}+12q^{27}+12q^{28}+16q^{29}+18q^{30}+16q^{31}\\&\quad+19q^{32}+19q^{33}+16q^{34}+18q^{35}+16q^{36}+12q^{37}+12q^{38}+7q^{39}+6q^{40}\\&\quad+4q^{41}+3q^{42}+q^{43})\\&+x^8(q^{28}+2q^{29}+3q^{30}+6q^{31}+6q^{32}+9q^{33}+11q^{34}+11q^{35}+13q^{36}+16q^{37}\\&\quad+12q^{38}+16q^{39}+13q^{40}+11q^{41}+11q^{42}+9q^{43}+6q^{44}+6q^{45}+3q^{46}\\&\quad+2q^{47}+q^{48})\end{aligned}$$

$$
\begin{aligned}
&+ x^9(q^{35} + q^{36} + q^{37} + 3q^{38} + 2q^{39} + 3q^{40} + 5q^{41} + 3q^{42} + 5q^{43} + 5q^{44} + 3q^{45} \\
&\qquad + 5q^{46} + 3q^{47} + 2q^{48} + 3q^{49} + q^{50} + q^{51} + q^{52}),
\end{aligned}
$$

$$
\begin{aligned}
p_9(x,q) = &-x^6q^{30} - x^7(q^{31} + q^{32} + 2q^{34} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40}) \\
&- x^8(q^{33} + 2q^{35} + q^{36} + q^{37} + 2q^{38} + 2q^{39} + 3q^{40} + 4q^{41} + 3q^{42} + 3q^{43} + 3q^{44} \\
&\qquad + 2q^{45} + 3q^{46} + 2q^{47} + 2q^{48} + q^{49} + q^{50}) \\
&- x^9(q^{37} + q^{39} + q^{40} + q^{41} + 3q^{42} + 3q^{43} + 4q^{44} + 5q^{45} + 4q^{46} + 6q^{47} + 6q^{48} \\
&\qquad + 7q^{49} + 8q^{50} + 7q^{51} + 6q^{52} + 5q^{53} + 5q^{54} + 3q^{55} + 4q^{56} + 2q^{57} + q^{58} \\
&\qquad + q^{59}) \\
&- x^{10}(q^{45} + q^{46} + 2q^{47} + 2q^{48} + 3q^{49} + 4q^{50} + 6q^{51} + 7q^{52} + 8q^{53} + 9q^{54} + 10q^{55} \\
&\qquad + 8q^{56} + 11q^{57} + 10q^{58} + 8q^{59} + 10q^{60} + 6q^{61} + 4q^{62} + 5q^{63} + 2q^{64} + 2q^{65} \\
&\qquad + q^{66}) \\
&- x^{11}(q^{53} + q^{54} + 3q^{55} + 3q^{56} + 4q^{57} + 5q^{58} + 7q^{59} + 6q^{60} + 9q^{61} + 9q^{62} + 7q^{63} \\
&\qquad + 10q^{64} + 7q^{65} + 6q^{66} + 7q^{67} + 4q^{68} + 3q^{69} + 3q^{70} + q^{72}) \\
&- x^{12}(q^{60} + q^{62} + 2q^{63} + q^{64} + 3q^{65} + 3q^{66} + 2q^{67} + 4q^{68} + 3q^{69} + 2q^{70} + 4q^{71} \\
&\qquad + q^{72} + 2q^{73} + 2q^{74} + q^{76}),
\end{aligned}
$$

and

$$
\begin{aligned}
p_{12}(x,q) = &\; x^{12}q^{90} + x^{13}(q^{91} + q^{92} + 2q^{94} + q^{96} + q^{97}) \\
&+ x^{14}(q^{93} + 2q^{95} + q^{96} + q^{97} + 2q^{98} + q^{99} + q^{100} + 2q^{101} + q^{103}) \\
&+ x^{15}(q^{97} + q^{99} + q^{100} + 2q^{102} + q^{104} + q^{105} + q^{107}).
\end{aligned}
$$

Theorem 6.3.11. *It holds that*

$$
\begin{aligned}
p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{III},2}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{III},2}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{III},2}}}(xq^6) \\
+ p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{III},2}}}(xq^9) + p_{12}(x,q)G_{\mathscr{I}_{T_{\mathrm{III},2}}}(xq^{12}) = 0,
\end{aligned} \tag{6.3.73}
$$

where

$$
\begin{aligned}
p_0(x,q) = &\; 1 + x(q^5 + q^6 + 2q^8 + q^{10} + q^{11}) \\
&+ x^2(q^{11} + 2q^{13} + q^{14} + q^{15} + 2q^{16} + q^{17} + q^{18} + 2q^{19} + q^{21}) \\
&+ x^3(q^{19} + q^{21} + q^{22} + 2q^{24} + q^{26} + q^{27} + q^{29}), \\
p_3(x,q) = &-1 - x(q^2 + q^3 + q^4 + q^5 + q^6 + 2q^8 + q^{10} + q^{11}) \\
&- x^2(q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 2q^9 + 3q^{10} + 3q^{11} + 3q^{12} + 4q^{13} + 3q^{14} + 2q^{15} \\
&\qquad + 2q^{16} + q^{17} + q^{18} + 2q^{19} + q^{21}) \\
&- x^3(q^7 + q^8 + 2q^9 + 4q^{10} + 3q^{11} + 5q^{12} + 5q^{13} + 6q^{14} + 7q^{15} + 8q^{16} + 7q^{17} + 6q^{18} \\
&\qquad + 6q^{19} + 4q^{20} + 5q^{21} + 4q^{22} + 3q^{23} + 3q^{24} + q^{25} + q^{26} + q^{27} + q^{29})
\end{aligned}
$$

$$
\begin{aligned}
&- x^4(q^{12} + 2q^{13} + 2q^{14} + 5q^{15} + 4q^{16} + 6q^{17} + 10q^{18} + 8q^{19} + 10q^{20} + 11q^{21} + 8q^{22} \\
&\qquad + 10q^{23} + 9q^{24} + 8q^{25} + 7q^{26} + 6q^{27} + 4q^{28} + 3q^{29} + 2q^{30} + 2q^{31} + q^{32} + q^{33}) \\
&- x^5(q^{18} + 3q^{20} + 3q^{21} + 4q^{22} + 7q^{23} + 6q^{24} + 7q^{25} + 10q^{26} + 7q^{27} + 9q^{28} + 9q^{29} \\
&\qquad + 6q^{30} + 7q^{31} + 5q^{32} + 4q^{33} + 3q^{34} + 3q^{35} + q^{36} + q^{37}) \\
&- x^6(q^{26} + 2q^{28} + 2q^{29} + q^{30} + 4q^{31} + 2q^{32} + 3q^{33} + 4q^{34} + 2q^{35} + 3q^{36} + 3q^{37} \\
&\qquad + q^{38} + 2q^{39} + q^{40} + q^{42}),
\end{aligned}
$$

$$
\begin{aligned}
p_6(x,q) = {}& x^4(q^{16} + q^{18} + 2q^{20} + q^{22} + q^{24}) \\
&+ x^5(q^{18} + 2q^{20} + q^{21} + 4q^{22} + 3q^{23} + 4q^{24} + 4q^{25} + 5q^{26} + 5q^{27} + 4q^{28} + 4q^{29} \\
&\qquad + 3q^{30} + 4q^{31} + q^{32} + 2q^{33} + q^{35}) \\
&+ x^6(q^{23} + 2q^{24} + 3q^{25} + 5q^{26} + 5q^{27} + 9q^{28} + 9q^{29} + 10q^{30} + 12q^{31} + 12q^{32} \\
&\qquad + 14q^{33} + 12q^{34} + 12q^{35} + 10q^{36} + 9q^{37} + 9q^{38} + 5q^{39} + 5q^{40} + 3q^{41} \\
&\qquad + 2q^{42} + q^{43}) \\
&+ x^7(q^{29} + 3q^{30} + 4q^{31} + 6q^{32} + 7q^{33} + 12q^{34} + 12q^{35} + 16q^{36} + 18q^{37} + 16q^{38} \\
&\qquad + 19q^{39} + 19q^{40} + 16q^{41} + 18q^{42} + 16q^{43} + 12q^{44} + 12q^{45} + 7q^{46} + 6q^{47} \\
&\qquad + 4q^{48} + 3q^{49} + q^{50}) \\
&+ x^8(q^{36} + 2q^{37} + 3q^{38} + 6q^{39} + 6q^{40} + 9q^{41} + 11q^{42} + 11q^{43} + 13q^{44} + 16q^{45} \\
&\qquad + 12q^{46} + 16q^{47} + 13q^{48} + 11q^{49} + 11q^{50} + 9q^{51} + 6q^{52} + 6q^{53} + 3q^{54} \\
&\qquad + 2q^{55} + q^{56}) \\
&+ x^9(q^{44} + q^{45} + q^{46} + 3q^{47} + 2q^{48} + 3q^{49} + 5q^{50} + 3q^{51} + 5q^{52} + 5q^{53} + 3q^{54} \\
&\qquad + 5q^{55} + 3q^{56} + 2q^{57} + 3q^{58} + q^{59} + q^{60} + q^{61}),
\end{aligned}
$$

$$
\begin{aligned}
p_9(x,q) = {}& -x^6q^{36} - x^7(q^{38} + q^{39} + 2q^{41} + q^{43} + q^{44} + q^{45} + q^{46} + q^{47}) \\
&- x^8(q^{41} + 2q^{43} + q^{44} + q^{45} + 2q^{46} + 2q^{47} + 3q^{48} + 4q^{49} + 3q^{50} + 3q^{51} + 3q^{52} \\
&\qquad + 2q^{53} + 3q^{54} + 2q^{55} + 2q^{56} + q^{57} + q^{58}) \\
&- x^9(q^{46} + q^{48} + q^{49} + q^{50} + 3q^{51} + 3q^{52} + 4q^{53} + 5q^{54} + 4q^{55} + 6q^{56} + 6q^{57} \\
&\qquad + 7q^{58} + 8q^{59} + 7q^{60} + 6q^{61} + 5q^{62} + 5q^{63} + 3q^{64} + 4q^{65} + 2q^{66} + q^{67} \\
&\qquad + q^{68}) \\
&- x^{10}(q^{55} + q^{56} + 2q^{57} + 2q^{58} + 3q^{59} + 4q^{60} + 6q^{61} + 7q^{62} + 8q^{63} + 9q^{64} + 10q^{65} \\
&\qquad + 8q^{66} + 11q^{67} + 10q^{68} + 8q^{69} + 10q^{70} + 6q^{71} + 4q^{72} + 5q^{73} + 2q^{74} + 2q^{75} \\
&\qquad + q^{76}) \\
&- x^{11}(q^{64} + q^{65} + 3q^{66} + 3q^{67} + 4q^{68} + 5q^{69} + 7q^{70} + 6q^{71} + 9q^{72} + 9q^{73} + 7q^{74} \\
&\qquad + 10q^{75} + 7q^{76} + 6q^{77} + 7q^{78} + 4q^{79} + 3q^{80} + 3q^{81} + q^{83}) \\
&- x^{12}(q^{72} + q^{74} + 2q^{75} + q^{76} + 3q^{77} + 3q^{78} + 2q^{79} + 4q^{80} + 3q^{81} + 2q^{82} + 4q^{83} \\
&\qquad + q^{84} + 2q^{85} + 2q^{86} + q^{88}),
\end{aligned}
$$

and

$$
\begin{aligned}
p_{12}(x,q) &= x^{12}q^{102} + x^{13}(q^{104} + q^{105} + 2q^{107} + q^{109} + q^{110}) \\
&\quad + x^{14}(q^{107} + 2q^{109} + q^{110} + q^{111} + 2q^{112} + q^{113} + q^{114} + 2q^{115} + q^{117}) \\
&\quad + x^{15}(q^{112} + q^{114} + q^{115} + 2q^{117} + q^{119} + q^{120} + q^{122}).
\end{aligned}
$$

Theorem 6.3.12. *It holds that*

$$
\begin{aligned}
&p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{III},a}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{III},a}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{III},a}}}(xq^6) \\
&\qquad + p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{III},a}}}(xq^9) + p_{12}(x,q)G_{\mathscr{I}_{T_{\mathrm{III},a}}}(xq^{12}) = 0,
\end{aligned}
\tag{6.3.74}
$$

where

$$
\begin{aligned}
p_0(x,q) &= 1 + x(q^4 + q^5 + q^7 + q^8 + q^{10} + q^{11}) \\
&\quad + x^2(q^9 + q^{11} + 2q^{12} + q^{14} + 2q^{15} + q^{16} + 2q^{18} + q^{19} + q^{21}) \\
&\quad + x^3(q^{16} + q^{19} + q^{20} + q^{22} + q^{23} + q^{25} + q^{26} + q^{29}), \\
p_3(x,q) &= -1 - x(q + q^2 + q^3 + q^4 + q^5 + q^7 + q^8 + q^{10} + q^{11}) \\
&\quad - x^2(q^3 + 2q^4 + 2q^5 + 3q^6 + 2q^7 + 3q^8 + 3q^9 + 2q^{10} + 3q^{11} + 4q^{12} + 2q^{13} + 2q^{14} \\
&\qquad + 2q^{15} + q^{16} + 2q^{18} + q^{19} + q^{21}) \\
&\quad - x^3(q^6 + 3q^7 + 3q^8 + 4q^9 + 5q^{10} + 5q^{11} + 5q^{12} + 7q^{13} + 7q^{14} + 7q^{15} + 7q^{16} + 5q^{17} \\
&\qquad + 4q^{18} + 5q^{19} + 4q^{20} + 3q^{21} + 3q^{22} + 2q^{23} + q^{24} + q^{25} + q^{26} + q^{29}) \\
&\quad - x^4(q^{10} + 3q^{11} + 3q^{12} + 3q^{13} + 6q^{14} + 7q^{15} + 7q^{16} + 10q^{17} + 10q^{18} + 8q^{19} + 9q^{20} \\
&\qquad + 9q^{21} + 8q^{22} + 8q^{23} + 7q^{24} + 5q^{25} + 5q^{26} + 4q^{27} + 2q^{28} + 2q^{29} + q^{30} \\
&\qquad + q^{31} + q^{32}) \\
&\quad - x^5(q^{15} + q^{16} + q^{17} + 4q^{18} + 5q^{19} + 3q^{20} + 6q^{21} + 8q^{22} + 5q^{23} + 7q^{24} + 10q^{25} \\
&\qquad + 6q^{26} + 6q^{27} + 9q^{28} + 5q^{29} + 4q^{30} + 5q^{31} + 3q^{32} + 2q^{33} + 3q^{34} + q^{35} \\
&\qquad + q^{37}) \\
&\quad - x^6(q^{22} + q^{23} + q^{25} + 3q^{26} + q^{27} + q^{28} + 4q^{29} + 2q^{30} + q^{31} + 4q^{32} + 3q^{33} + 3q^{35} \\
&\qquad + 3q^{36} + q^{38} + 2q^{39} + q^{42}), \\
p_6(x,q) &= x^4(q^{16} + q^{18} + q^{19} + q^{20} + q^{21} + q^{23}) \\
&\quad + x^5(q^{17} + q^{19} + 3q^{20} + 3q^{21} + 3q^{22} + 4q^{23} + 5q^{24} + 4q^{25} + 4q^{26} + 5q^{27} + 4q^{28} \\
&\qquad + 3q^{29} + 3q^{30} + 3q^{31} + q^{32} + q^{34}) \\
&\quad + x^6(q^{21} + 2q^{22} + 2q^{23} + 4q^{24} + 6q^{25} + 8q^{26} + 8q^{27} + 10q^{28} + 10q^{29} + 11q^{30} \\
&\qquad + 13q^{31} + 13q^{32} + 11q^{33} + 10q^{34} + 10q^{35} + 8q^{36} + 8q^{37} + 6q^{38} \\
&\qquad + 4q^{39} + 2q^{40} + 2q^{41} + q^{42}) \\
&\quad + x^7(q^{26} + 3q^{27} + 3q^{28} + 4q^{29} + 7q^{30} + 8q^{31} + 11q^{32} + 14q^{33} + 14q^{34} + 15q^{35}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 17q^{36} + 17q^{37} + 17q^{38} + 17q^{39} + 15q^{40} + 14q^{41} + 14q^{42} + 11q^{43}\\
&\quad + 8q^{44} + 7q^{45} + 4q^{46} + 3q^{47} + 3q^{48} + q^{49})\\
&+ x^8(q^{31} + q^{33} + 4q^{34} + 3q^{35} + 4q^{36} + 8q^{37} + 8q^{38} + 7q^{39} + 12q^{40} + 12q^{41} + 10q^{42}\\
&\quad + 14q^{43} + 14q^{44} + 10q^{45} + 12q^{46} + 12q^{47} + 7q^{48} + 8q^{49} + 8q^{50} + 4q^{51}\\
&\quad + 3q^{52} + 4q^{53} + q^{54} + q^{56})\\
&+ x^9(q^{38} + 2q^{41} + 2q^{42} + 3q^{44} + 4q^{45} + q^{46} + 3q^{47} + 6q^{48} + 2q^{49} + 2q^{50} + 6q^{51}\\
&\quad + 3q^{52} + q^{53} + 4q^{54} + 3q^{55} + 2q^{57} + 2q^{58} + q^{61}),
\end{aligned}
$$

$$
\begin{aligned}
p_9(x,q) &= -x^6q^{36} - x^7(q^{37} + q^{38} + q^{40} + q^{41} + q^{43} + q^{44} + q^{45} + q^{46} + q^{47})\\
&- x^8(q^{39} + q^{41} + 2q^{42} + q^{44} + 2q^{45} + 2q^{46} + 2q^{47} + 4q^{48} + 3q^{49} + 2q^{50} + 3q^{51}\\
&\quad + 3q^{52} + 2q^{53} + 3q^{54} + 2q^{55} + 2q^{56} + q^{57})\\
&- x^9(q^{43} + q^{46} + q^{47} + q^{48} + 2q^{49} + 3q^{50} + 3q^{51} + 4q^{52} + 5q^{53} + 4q^{54} + 5q^{55}\\
&\quad + 7q^{56} + 7q^{57} + 7q^{58} + 7q^{59} + 5q^{60} + 5q^{61} + 5q^{62} + 4q^{63} + 3q^{64}\\
&\quad + 3q^{65} + q^{66})\\
&- x^{10}(q^{52} + q^{53} + q^{54} + 2q^{55} + 2q^{56} + 4q^{57} + 5q^{58} + 5q^{59} + 7q^{60} + 8q^{61} + 8q^{62}\\
&\quad + 9q^{63} + 9q^{64} + 8q^{65} + 10q^{66} + 10q^{67} + 7q^{68} + 7q^{69} + 6q^{70} + 3q^{71}\\
&\quad + 3q^{72} + 3q^{73} + q^{74})\\
&- x^{11}(q^{59} + q^{61} + 3q^{62} + 2q^{63} + 3q^{64} + 5q^{65} + 4q^{66} + 5q^{67} + 9q^{68} + 6q^{69} + 6q^{70}\\
&\quad + 10q^{71} + 7q^{72} + 5q^{73} + 8q^{74} + 6q^{75} + 3q^{76} + 5q^{77} + 4q^{78} + q^{79} + q^{80}\\
&\quad + q^{81})\\
&- x^{12}(q^{66} + 2q^{69} + q^{70} + 3q^{72} + 3q^{73} + 3q^{75} + 4q^{76} + q^{77} + 2q^{78} + 4q^{79} + q^{80}\\
&\quad + q^{81} + 3q^{82} + q^{83} + q^{85} + q^{86}),
\end{aligned}
$$

and

$$
\begin{aligned}
p_{12}(x,q) &= x^{12}q^{99} + x^{13}(q^{100} + q^{101} + q^{103} + q^{104} + q^{106} + q^{107})\\
&+ x^{14}(q^{102} + q^{104} + 2q^{105} + q^{107} + 2q^{108} + q^{109} + 2q^{111} + q^{112} + q^{114})\\
&+ x^{15}(q^{106} + q^{109} + q^{110} + q^{112} + q^{113} + q^{115} + q^{116} + q^{119}).
\end{aligned}
$$

6.3.4 Partition Set of Type IV

Recall that the partition set of type IV is the set of partitions with difference at least 3 at distance 3 such that if parts at distance 2 differ by at most 1, then the sum of the two parts and their intermediate part is congruent to 2 modulo 3. In other words, if $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is in this partition set, then

(i) $\lambda_i - \lambda_{i+3} \geq 3$;

(ii) $\lambda_i - \lambda_{i+2} \leq 1$ implies $\lambda_i + \lambda_{i+1} + \lambda_{i+2} \equiv 2 \pmod 3$.

Let $\mathscr{I}_{T_{\mathrm{IV}}}$ denote the partition set of type IV.

Claim 6.3.13. *$\mathscr{I}_{T_{\mathrm{IV}}}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ where $S = 3$, and $\Pi = \{\pi_1, \pi_2, \ldots, \pi_{15}\}$ along with the linking sets are given as follows.*

Π	*linking set*
$\pi_1 = \emptyset$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_2 = 1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_3 = 1+1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_4 = 2$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_5 = 2+1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_6 = 2+2$	$\{\pi_1,\ \pi_2,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7,\ \pi_8,\ \pi_9,\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_7 = 2+2+1$	$\{\pi_1,\ \pi_2,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7,\ \pi_8,\ \pi_9,\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_8 = 3$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_9 = 3+1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_{10} = 3+1+1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_8,\ \pi_9,\ \pi_{10},\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_{11} = 3+2$	$\{\pi_1,\ \pi_2,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_8,\ \pi_9,\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_{12} = 3+2+1$	$\{\pi_1,\ \pi_2,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_8,\ \pi_9,\ \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{14},\ \pi_{15}\}$
$\pi_{13} = 3+3$	$\{\pi_1,\ \pi_4,\ \pi_8,\ \pi_{11},\ \pi_{13},\ \pi_{15}\}$
$\pi_{14} = 3+3+1$	$\{\pi_1,\ \pi_4,\ \pi_8,\ \pi_{11},\ \pi_{13},\ \pi_{15}\}$
$\pi_{15} = 3+3+2$	$\{\pi_1,\ \pi_4,\ \pi_8,\ \pi_{11},\ \pi_{13},\ \pi_{15}\}$

Let us denote by $H_i(x) = H_i(x,q)$ the generating function of partitions λ in $\mathscr{I}_{T_{\mathrm{IV}}}$ with 3-tail equal to π_i for $i = 1, 2, \ldots, 15$ where the π_i's are as defined in Claim 6.3.13.

Following (6.1.7), we have

$$\begin{aligned} H_1(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\ &\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), \end{aligned} \tag{6.3.75}$$

$$\begin{aligned} x^{-1}q^{-1}H_2(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\ &\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), \end{aligned} \tag{6.3.76}$$

$$\begin{aligned} x^{-2}q^{-2}H_3(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\ &\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), \end{aligned} \tag{6.3.77}$$

$$\begin{aligned} x^{-1}q^{-2}H_4(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\ &\quad + H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\ &\quad + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), \end{aligned} \tag{6.3.78}$$

$$x^{-2}q^{-3}H_5(x) = H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3)$$

$$
\begin{aligned}
&+ H_7(xq^3) + H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) \\
&+ H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.79) \\
x^{-2}q^{-4}H_6(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3) \\
&+ H_8(xq^3) + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) \\
&+ H_{14}(xq^3) + H_{15}(xq^3), && (6.3.80) \\
x^{-3}q^{-5}H_7(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_7(xq^3) \\
&+ H_8(xq^3) + H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) \\
&+ H_{14}(xq^3) + H_{15}(xq^3), && (6.3.81) \\
x^{-1}q^{-3}H_8(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\
&+ H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) \\
&+ H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.82) \\
x^{-2}q^{-4}H_9(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\
&+ H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) \\
&+ H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.83) \\
x^{-3}q^{-5}H_{10}(x) &= H_1(xq^3) + H_2(xq^3) + H_3(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) \\
&+ H_8(xq^3) + H_9(xq^3) + H_{10}(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) \\
&+ H_{13}(xq^3) + H_{14}(xq^3) + H_{15}(xq^3), && (6.3.84) \\
x^{-2}q^{-5}H_{11}(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_8(xq^3) \\
&+ H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) \\
&+ H_{15}(xq^3), && (6.3.85) \\
x^{-3}q^{-6}H_{12}(x) &= H_1(xq^3) + H_2(xq^3) + H_4(xq^3) + H_5(xq^3) + H_6(xq^3) + H_8(xq^3) \\
&+ H_9(xq^3) + H_{11}(xq^3) + H_{12}(xq^3) + H_{13}(xq^3) + H_{14}(xq^3) \\
&+ H_{15}(xq^3), && (6.3.86) \\
x^{-2}q^{-6}H_{13}(x) &= H_1(xq^3) + H_4(xq^3) + H_8(xq^3) + H_{11}(xq^3) + H_{13}(xq^3) + H_{15}(xq^3), && (6.3.87) \\
x^{-3}q^{-7}H_{14}(x) &= H_1(xq^3) + H_4(xq^3) + H_8(xq^3) + H_{11}(xq^3) + H_{13}(xq^3) + H_{15}(xq^3), && (6.3.88) \\
x^{-3}q^{-8}H_{15}(x) &= H_1(xq^3) + H_4(xq^3) + H_8(xq^3) + H_{11}(xq^3) + H_{13}(xq^3) + H_{15}(xq^3). && (6.3.89)
\end{aligned}
$$

This system may be simplified as

$$
\boxed{
\begin{aligned}
H_1(x) &= (1 + xq^4 + x^2q^8 + xq^5 + x^2q^9)H_1(xq^3) + (1 + xq^4)H_6(xq^3) \\
&+ (1 + xq^4 + x^2q^8)H_8(xq^3) + (1 + xq^4)H_{11}(xq^3) \\
&+ (1 + xq^4 + xq^5)H_{13}(xq^3), && (6.3.90)
\end{aligned}
}
$$

$$
\begin{aligned}
H_6(x) &= (x^2q^4 + x^3q^8 + x^3q^9 + x^4q^{13})H_1(xq^3) + (x^2q^4 + x^3q^8)H_6(xq^3) \\
&\quad + (x^2q^4 + x^3q^8)H_8(xq^3) + (x^2q^4 + x^3q^8)H_{11}(xq^3) \\
&\quad + (x^2q^4 + x^3q^8 + x^3q^9)H_{13}(xq^3), \qquad (6.3.91) \\
H_8(x) &= (xq^3 + x^2q^7 + x^3q^{11} + x^2q^8 + x^3q^{12})H_1(xq^3) + xq^3H_6(xq^3) \\
&\quad + (xq^3 + x^2q^7 + x^3q^{11})H_8(xq^3) + (xq^3 + x^2q^7)H_{11}(xq^3) \\
&\quad + (xq^3 + x^2q^7 + x^2q^8)H_{13}(xq^3), \qquad (6.3.92) \\
H_{11}(x) &= (x^2q^5 + x^3q^9 + x^3q^{10} + x^4q^{14})H_1(xq^3) + x^2q^5H_6(xq^3) \\
&\quad + (x^2q^5 + x^3q^9)H_8(xq^3) + (x^2q^5 + x^3q^9)H_{11}(xq^3) \\
&\quad + (x^2q^5 + x^3q^9 + x^3q^{10})H_{13}(xq^3), \qquad (6.3.93) \\
H_{13}(x) &= (x^2q^6 + x^3q^{11})H_1(xq^3) + x^2q^6H_8(xq^3) + x^2q^6H_{11}(xq^3) \\
&\quad + (x^2q^6 + x^3q^{11})H_{13}(xq^3). \qquad (6.3.94)
\end{aligned}
$$

Let $G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(x) = G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x,q)$ denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{IV}}}$ whose smallest part is at least 1.

Let $G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(x)$ denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{IV}}}$ where 1 appears at most once.

Let $G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(x)$ denote the generating function of partitions in $\mathscr{I}_{T_{\mathrm{IV}}}$ where the smallest part is at least 2 with 2 appearing at most once.

It follows that

$$
\begin{aligned}
G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(x) &= H_1(x) + H_2(x) + H_3(x) + H_4(x) + H_5(x) + H_6(x) \\
&\quad + H_7(x) + H_8(x) + H_9(x) + H_{10}(x) + H_{11}(x) \\
&\quad + H_{12}(x) + H_{13}(x) + H_{14}(x) + H_{15}(x) \\
&= H_1(xq^{-3}), \qquad (6.3.95) \\
G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(x) &= H_1(x) + H_2(x) + H_4(x) + H_5(x) + H_6(x) + H_7(x) \\
&\quad + H_8(x) + H_9(x) + H_{11}(x) + H_{12}(x) + H_{13}(x) \\
&\quad + H_{14}(x) + H_{15}(x) \\
&= x^{-2}q^2H_6(xq^{-3}), \qquad (6.3.96) \\
G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(x) &= H_1(x) + H_4(x) + H_8(x) + H_{11}(x) + H_{13}(x) + H_{15}(x)
\end{aligned}
$$

$$= x^{-2}H_{13}(xq^{-3}). \tag{6.3.97}$$

Likewise, we can use the algorithm in §6.2 to deduce the following q-difference equations for $G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(x)$, $G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(x)$ and $G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(x)$, respectively.

Theorem 6.3.14. *It holds that*

$$\begin{aligned} p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(xq^6) \\ + p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(xq^9) + p_{12}(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(xq^{12}) = 0, \end{aligned} \tag{6.3.98}$$

where

$$\begin{aligned}
p_0(x,q) = {} & 1 + x(q^4+q^5+q^7+q^8+q^{10}+q^{11}) \\
& + x^2(q^9+q^{11}+2q^{12}+q^{14}+2q^{15}+q^{16}+2q^{18}+q^{19}+q^{21}) \\
& + x^3(q^{16}+q^{19}+q^{20}+q^{22}+q^{23}+q^{25}+q^{26}+q^{29}), \\
p_3(x,q) = {} & -1 - x(q+q^2+q^3+q^4+q^5+q^7+q^8+q^{10}+q^{11}) \\
& - x^2(q^2+q^3+2q^4+2q^5+3q^6+2q^7+2q^8+3q^9+2q^{10}+3q^{11}+4q^{12}+2q^{13} \\
& \qquad + 2q^{14}+2q^{15}+q^{16}+2q^{18}+q^{19}+q^{21}) \\
& - x^3(2q^5+2q^6+3q^7+4q^8+4q^9+4q^{10}+5q^{11}+6q^{12}+7q^{13}+7q^{14}+6q^{15} \\
& \qquad + 6q^{16}+5q^{17}+3q^{18}+4q^{19}+4q^{20}+3q^{21}+3q^{22}+2q^{23}+q^{24}+q^{25} \\
& \qquad + q^{26}+q^{29}) \\
& - x^4(2q^9+3q^{10}+2q^{11}+4q^{12}+6q^{13}+6q^{14}+8q^{15}+10q^{16}+8q^{17}+9q^{18}+9q^{19} \\
& \qquad + 7q^{20}+8q^{21}+8q^{22}+7q^{23}+6q^{24}+5q^{25}+3q^{26}+3q^{27}+2q^{28}+q^{29} \\
& \qquad + q^{30}+q^{31}+q^{32}) \\
& - x^5(q^{14}+q^{15}+2q^{16}+5q^{17}+3q^{18}+4q^{19}+8q^{20}+6q^{21}+5q^{22}+9q^{23}+8q^{24} \\
& \qquad + 6q^{25}+9q^{26}+6q^{27}+4q^{28}+6q^{29}+4q^{30}+2q^{31}+3q^{32}+2q^{33}+q^{34} \\
& \qquad + q^{35}) \\
& - x^6(q^{21}+q^{22}+2q^{24}+2q^{25}+q^{26}+3q^{27}+3q^{28}+q^{29}+3q^{30}+4q^{31}+q^{32}+2q^{33} \\
& \qquad + 3q^{34}+q^{35}+q^{36}+2q^{37}+q^{40}), \\
p_6(x,q) = {} & x^4(q^{12}+q^{14}+q^{16}+q^{17}+q^{19}+q^{21}) \\
& + x^5(q^{13}+q^{15}+2q^{16}+2q^{17}+3q^{18}+3q^{19}+4q^{20}+4q^{21}+4q^{22}+4q^{23}+4q^{24} \\
& \qquad + 4q^{25}+3q^{26}+3q^{27}+2q^{28}+2q^{29}+q^{30}+q^{32}) \\
& + x^6(q^{17}+q^{18}+2q^{19}+3q^{20}+4q^{21}+7q^{22}+7q^{23}+8q^{24}+8q^{25}+11q^{26}+11q^{27} \\
& \qquad + 12q^{28}+12q^{29}+11q^{30}+11q^{31}+8q^{32}+8q^{33}+7q^{34}+7q^{35}+4q^{36} \\
& \qquad + 3q^{37}+2q^{38}+q^{39}+q^{40}) \\
& + x^7(2q^{23}+3q^{24}+3q^{25}+4q^{26}+7q^{27}+9q^{28}+10q^{29}+14q^{30}+14q^{31}+16q^{32}
\end{aligned}$$

$$
\begin{aligned}
&\quad + 17q^{33} + 15q^{34} + 15q^{35} + 17q^{36} + 16q^{37} + 14q^{38} + 14q^{39} + 10q^{40} + 9q^{41} \\
&\quad + 7q^{42} + 4q^{43} + 3q^{44} + 3q^{45} + 2q^{46}) \\
&+ x^8(q^{29} + q^{30} + 3q^{31} + 4q^{32} + 4q^{33} + 8q^{34} + 8q^{35} + 7q^{36} + 11q^{37} + 13q^{38} + 10q^{39} \\
&\quad + 14q^{40} + 14q^{41} + 10q^{42} + 13q^{43} + 11q^{44} + 7q^{45} + 8q^{46} + 8q^{47} + 4q^{48} \\
&\quad + 4q^{49} + 3q^{50} + q^{51} + q^{52}) \\
&+ x^9(q^{36} + q^{38} + 2q^{39} + q^{40} + 2q^{41} + 4q^{42} + 2q^{43} + 3q^{44} + 5q^{45} + 3q^{46} + 3q^{47} \\
&\quad + 5q^{48} + 3q^{49} + 2q^{50} + 4q^{51} + 2q^{52} + q^{53} + 2q^{54} + q^{55} + q^{57}),
\end{aligned}
$$

$$
\begin{aligned}
p_9(x,q) = &-x^6q^{30} - x^7(q^{31} + q^{32} + q^{34} + q^{35} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41}) \\
&- x^8(q^{33} + q^{35} + 2q^{36} + q^{38} + 2q^{39} + 2q^{40} + 2q^{41} + 4q^{42} + 3q^{43} + 2q^{44} + 3q^{45} \\
&\quad + 2q^{46} + 2q^{47} + 3q^{48} + 2q^{49} + 2q^{50} + q^{51} + q^{52}) \\
&- x^9(q^{37} + q^{40} + q^{41} + q^{42} + 2q^{43} + 3q^{44} + 3q^{45} + 4q^{46} + 4q^{47} + 3q^{48} + 5q^{49} \\
&\quad + 6q^{50} + 6q^{51} + 7q^{52} + 7q^{53} + 6q^{54} + 5q^{55} + 4q^{56} + 4q^{57} + 4q^{58} + 3q^{59} \\
&\quad + 2q^{60} + 2q^{61}) \\
&- x^{10}(q^{46} + q^{47} + q^{48} + q^{49} + 2q^{50} + 3q^{51} + 3q^{52} + 5q^{53} + 6q^{54} + 7q^{55} + 8q^{56} \\
&\quad + 8q^{57} + 7q^{58} + 9q^{59} + 9q^{60} + 8q^{61} + 10q^{62} + 8q^{63} + 6q^{64} + 6q^{65} + 4q^{66} \\
&\quad + 2q^{67} + 3q^{68} + 2q^{69}) \\
&- x^{11}(q^{55} + q^{56} + 2q^{57} + 3q^{58} + 2q^{59} + 4q^{60} + 6q^{61} + 4q^{62} + 6q^{63} + 9q^{64} + 6q^{65} \\
&\quad + 8q^{66} + 9q^{67} + 5q^{68} + 6q^{69} + 8q^{70} + 4q^{71} + 3q^{72} + 5q^{73} + 2q^{74} + q^{75} + q^{76}) \\
&- x^{12}(q^{62} + 2q^{65} + q^{66} + q^{67} + 3q^{68} + 2q^{69} + q^{70} + 4q^{71} + 3q^{72} + q^{73} + 3q^{74} + 3q^{75} \\
&\quad + q^{76} + 2q^{77} + 2q^{78} + q^{80} + q^{81}),
\end{aligned}
$$

and

$$
\begin{aligned}
p_{12}(x,q) = &\, x^{12}q^{93} + x^{13}(q^{94} + q^{95} + q^{97} + q^{98} + q^{100} + q^{101}) \\
&+ x^{14}(q^{96} + q^{98} + 2q^{99} + q^{101} + 2q^{102} + q^{103} + 2q^{105} + q^{106} + q^{108}) \\
&+ x^{15}(q^{100} + q^{103} + q^{104} + q^{106} + q^{107} + q^{109} + q^{110} + q^{113}).
\end{aligned}
$$

Theorem 6.3.15. *It holds that*

$$
\begin{aligned}
p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(xq^6) \\
+ p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(xq^9) + p_{12}(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(xq^{12}) = 0,
\end{aligned}
\tag{6.3.99}
$$

where

$$
\begin{aligned}
p_0(x,q) = &\, 1 + x(q^4 + q^5 + q^6 + q^7 + q^8 + q^9) \\
&+ x^2(q^9 + q^{10} + q^{11} + 2q^{12} + 2q^{13} + 2q^{14} + q^{15} + q^{16} + q^{17}) \\
&+ x^3(q^{16} + q^{17} + q^{18} + q^{19} + q^{20} + q^{21} + q^{22} + q^{23}),
\end{aligned}
$$

$$
\begin{aligned}
p_3(x,q) = &-1 - x(q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9)\\
&- x^2(q^3+2q^4+2q^5+3q^6+3q^7+4q^8+4q^9+4q^{10}+3q^{11}+3q^{12}+2q^{13}+2q^{14}\\
&\qquad +q^{15}+q^{16}+q^{17})\\
&- x^3(q^5+q^6+2q^7+4q^8+5q^9+6q^{10}+8q^{11}+8q^{12}+8q^{13}+8q^{14}+8q^{15}+7q^{16}\\
&\qquad +6q^{17}+4q^{18}+3q^{19}+2q^{20}+q^{21}+q^{22}+q^{23})\\
&- x^4(q^9+2q^{10}+3q^{11}+5q^{12}+7q^{13}+8q^{14}+9q^{15}+11q^{16}+13q^{17}+13q^{18}\\
&\qquad +11q^{19}+10q^{20}+8q^{21}+6q^{22}+5q^{23}+4q^{24}+3q^{25}+q^{26})\\
&- x^5(q^{14}+2q^{15}+2q^{16}+4q^{17}+6q^{18}+8q^{19}+10q^{20}+10q^{21}+10q^{22}+9q^{23}\\
&\qquad +8q^{24}+8q^{25}+7q^{26}+5q^{27}+3q^{28}+q^{29}+q^{30}+q^{31})\\
&- x^6(q^{21}+2q^{22}+2q^{23}+3q^{24}+3q^{25}+3q^{26}+4q^{27}+4q^{28}+3q^{29}+2q^{30}+2q^{31}\\
&\qquad +q^{32}+q^{33}+q^{34}),\\
p_6(x,q) = &\, x^4(q^{16}+q^{17}+q^{18}+q^{19}+q^{20}+q^{21})\\
&+ x^5(q^{17}+2q^{18}+2q^{19}+3q^{20}+5q^{21}+6q^{22}+5q^{23}+5q^{24}+6q^{25}+5q^{26}+3q^{27}\\
&\qquad +2q^{28}+2q^{29}+q^{30})\\
&+ x^6(q^{20}+2q^{21}+5q^{22}+7q^{23}+8q^{24}+10q^{25}+13q^{26}+14q^{27}+15q^{28}+15q^{29}\\
&\qquad +14q^{30}+13q^{31}+10q^{32}+8q^{33}+7q^{34}+5q^{35}+2q^{36}+q^{37})\\
&+ x^7(q^{24}+2q^{25}+4q^{26}+7q^{27}+10q^{28}+13q^{29}+16q^{30}+19q^{31}+21q^{32}+21q^{33}\\
&\qquad +21q^{34}+21q^{35}+19q^{36}+16q^{37}+13q^{38}+10q^{39}+7q^{40}+4q^{41}+2q^{42}\\
&\qquad +q^{43})\\
&+ x^8(q^{29}+q^{30}+2q^{31}+5q^{32}+7q^{33}+10q^{34}+12q^{35}+14q^{36}+16q^{37}+16q^{38}\\
&\qquad +16q^{39}+16q^{40}+14q^{41}+12q^{42}+10q^{43}+7q^{44}+5q^{45}+2q^{46}+q^{47}\\
&\qquad +q^{48})\\
&+ x^9(q^{36}+q^{37}+2q^{38}+3q^{39}+3q^{40}+4q^{41}+5q^{42}+5q^{43}+5q^{44}+5q^{45}+4q^{46}\\
&\qquad +3q^{47}+3q^{48}+2q^{49}+q^{50}+q^{51}),\\
p_9(x,q) = &-x^6q^{36} - x^7(q^{37}+q^{38}+q^{39}+q^{40}+q^{41}+q^{42}+q^{43}+q^{44}+q^{45})\\
&- x^8(q^{39}+q^{40}+q^{41}+2q^{42}+2q^{43}+3q^{44}+3q^{45}+4q^{46}+4q^{47}+4q^{48}+3q^{49}\\
&\qquad +3q^{50}+2q^{51}+2q^{52}+q^{53})\\
&- x^9(q^{43}+q^{44}+q^{45}+2q^{46}+3q^{47}+4q^{48}+6q^{49}+7q^{50}+8q^{51}+8q^{52}+8q^{53}\\
&\qquad +8q^{54}+8q^{55}+6q^{56}+5q^{57}+4q^{58}+2q^{59}+q^{60}+q^{61})\\
&- x^{10}(q^{50}+3q^{51}+4q^{52}+5q^{53}+6q^{54}+8q^{55}+10q^{56}+11q^{57}+13q^{58}+13q^{59}\\
&\qquad +11q^{60}+9q^{61}+8q^{62}+7q^{63}+5q^{64}+3q^{65}+2q^{66}+q^{67})\\
&- x^{11}(q^{55}+q^{56}+q^{57}+3q^{58}+5q^{59}+7q^{60}+8q^{61}+8q^{62}+9q^{63}+10q^{64}+10q^{65}\\
&\qquad +10q^{66}+8q^{67}+6q^{68}+4q^{69}+2q^{70}+2q^{71}+q^{72})\\
&- x^{12}(q^{62}+q^{63}+q^{64}+2q^{65}+2q^{66}+3q^{67}+4q^{68}+4q^{69}+3q^{70}+3q^{71}+3q^{72}\\
&\qquad +2q^{73}+2q^{74}+q^{75}),
\end{aligned}
$$

and

$$\begin{aligned}p_{12}(x,q) &= x^{12}q^{93} + x^{13}(q^{94}+q^{95}+q^{96}+q^{97}+q^{98}+q^{99})\\ &\quad + x^{14}(q^{96}+q^{97}+q^{98}+2q^{99}+2q^{100}+2q^{101}+q^{102}+q^{103}+q^{104})\\ &\quad + x^{15}(q^{100}+q^{101}+q^{102}+q^{103}+q^{104}+q^{105}+q^{106}+q^{107}).\end{aligned}$$

Theorem 6.3.16. *It holds that*

$$\begin{aligned}&p_0(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(x) + p_3(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(xq^3) + p_6(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(xq^6)\\ &\qquad + p_9(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(xq^9) + p_{12}(x,q)G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(xq^{12}) = 0,\end{aligned} \tag{6.3.100}$$

where

$$\begin{aligned}p_0(x,q) &= 1 + x(q^5+q^6+q^7+q^8+q^9+q^{10})\\ &\quad + x^2(q^{11}+q^{12}+q^{13}+2q^{14}+2q^{15}+2q^{16}+q^{17}+q^{18}+q^{19})\\ &\quad + x^3(q^{19}+q^{20}+q^{21}+q^{22}+q^{23}+q^{24}+q^{25}+q^{26}),\\ p_3(x,q) &= -1 - x(q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10})\\ &\quad - x^2(q^5+2q^6+2q^7+3q^8+3q^9+4q^{10}+4q^{11}+4q^{12}+3q^{13}+3q^{14}+2q^{15}+2q^{16}\\ &\qquad + q^{17}+q^{18}+q^{19})\\ &\quad - x^3(q^8+q^9+2q^{10}+4q^{11}+5q^{12}+6q^{13}+8q^{14}+8q^{15}+8q^{16}+8q^{17}+8q^{18}\\ &\qquad + 7q^{19}+6q^{20}+4q^{21}+3q^{22}+2q^{23}+q^{24}+q^{25}+q^{26})\\ &\quad - x^4(q^{13}+2q^{14}+3q^{15}+5q^{16}+7q^{17}+8q^{18}+9q^{19}+11q^{20}+13q^{21}+13q^{22}\\ &\qquad + 11q^{23}+10q^{24}+8q^{25}+6q^{26}+5q^{27}+4q^{28}+3q^{29}+q^{30})\\ &\quad - x^5(q^{19}+2q^{20}+2q^{21}+4q^{22}+6q^{23}+8q^{24}+10q^{25}+10q^{26}+10q^{27}+9q^{28}\\ &\qquad + 8q^{29}+8q^{30}+7q^{31}+5q^{32}+3q^{33}+q^{34}+q^{35}+q^{36})\\ &\quad - x^6(q^{27}+2q^{28}+2q^{29}+3q^{30}+3q^{31}+3q^{32}+4q^{33}+4q^{34}+3q^{35}+2q^{36}+2q^{37}\\ &\qquad + q^{38}+q^{39}+q^{40}),\\ p_6(x,q) &= x^4(q^{20}+q^{21}+q^{22}+q^{23}+q^{24}+q^{25})\\ &\quad + x^5(q^{22}+2q^{23}+2q^{24}+3q^{25}+5q^{26}+6q^{27}+5q^{28}+5q^{29}+6q^{30}+5q^{31}+3q^{32}\\ &\qquad + 2q^{33}+2q^{34}+q^{35})\\ &\quad + x^6(q^{26}+2q^{27}+5q^{28}+7q^{29}+8q^{30}+10q^{31}+13q^{32}+14q^{33}+15q^{34}+15q^{35}\\ &\qquad + 14q^{36}+13q^{37}+10q^{38}+8q^{39}+7q^{40}+5q^{41}+2q^{42}+q^{43})\\ &\quad + x^7(q^{31}+2q^{32}+4q^{33}+7q^{34}+10q^{35}+13q^{36}+16q^{37}+19q^{38}+21q^{39}+21q^{40}\\ &\qquad + 21q^{41}+21q^{42}+19q^{43}+16q^{44}+13q^{45}+10q^{46}+7q^{47}+4q^{48}+2q^{49}\\ &\qquad + q^{50})\\ &\quad + x^8(q^{37}+q^{38}+2q^{39}+5q^{40}+7q^{41}+10q^{42}+12q^{43}+14q^{44}+16q^{45}+16q^{46}\end{aligned}$$

$$
\begin{aligned}
&\quad + 16q^{47} + 16q^{48} + 14q^{49} + 12q^{50} + 10q^{51} + 7q^{52} + 5q^{53} + 2q^{54} + q^{55}\\
&\quad + q^{56})\\
&+ x^9(q^{45} + q^{46} + 2q^{47} + 3q^{48} + 3q^{49} + 4q^{50} + 5q^{51} + 5q^{52} + 5q^{53} + 5q^{54} + 4q^{55}\\
&\quad + 3q^{56} + 3q^{57} + 2q^{58} + q^{59} + q^{60}),
\end{aligned}
$$

$$
\begin{aligned}
p_9(x,q) = &-x^6q^{42} - x^7(q^{44} + q^{45} + q^{46} + q^{47} + q^{48} + q^{49} + q^{50} + q^{51} + q^{52})\\
&- x^8(q^{47} + q^{48} + q^{49} + 2q^{50} + 2q^{51} + 3q^{52} + 3q^{53} + 4q^{54} + 4q^{55} + 4q^{56} + 3q^{57}\\
&\quad + 3q^{58} + 2q^{59} + 2q^{60} + q^{61})\\
&- x^9(q^{52} + q^{53} + q^{54} + 2q^{55} + 3q^{56} + 4q^{57} + 6q^{58} + 7q^{59} + 8q^{60} + 8q^{61} + 8q^{62}\\
&\quad + 8q^{63} + 8q^{64} + 6q^{65} + 5q^{66} + 4q^{67} + 2q^{68} + q^{69} + q^{70})\\
&- x^{10}(q^{60} + 3q^{61} + 4q^{62} + 5q^{63} + 6q^{64} + 8q^{65} + 10q^{66} + 11q^{67} + 13q^{68} + 13q^{69}\\
&\quad + 11q^{70} + 9q^{71} + 8q^{72} + 7q^{73} + 5q^{74} + 3q^{75} + 2q^{76} + q^{77})\\
&- x^{11}(q^{66} + q^{67} + q^{68} + 3q^{69} + 5q^{70} + 7q^{71} + 8q^{72} + 8q^{73} + 9q^{74} + 10q^{75} + 10q^{76}\\
&\quad + 10q^{77} + 8q^{78} + 6q^{79} + 4q^{80} + 2q^{81} + 2q^{82} + q^{83})\\
&- x^{12}(q^{74} + q^{75} + q^{76} + 2q^{77} + 2q^{78} + 3q^{79} + 4q^{80} + 4q^{81} + 3q^{82} + 3q^{83} + 3q^{84}\\
&\quad + 2q^{85} + 2q^{86} + q^{87}),
\end{aligned}
$$

and

$$
\begin{aligned}
p_{12}(x,q) = &\, x^{12}q^{105} + x^{13}(q^{107} + q^{108} + q^{109} + q^{110} + q^{111} + q^{112})\\
&+ x^{14}(q^{110} + q^{111} + q^{112} + 2q^{113} + 2q^{114} + 2q^{115} + q^{116} + q^{117} + q^{118})\\
&+ x^{15}(q^{115} + q^{116} + q^{117} + q^{118} + q^{119} + q^{120} + q^{121} + q^{122}).
\end{aligned}
$$

6.4 "Guessing" the Generating Functions

It is, of course, not easy to discover a closed form for each generating function directly from q-difference equations obtained in the previous section. However, Andrews' conjecture presented in the introduction shall give us enough clues.

Recall that Andrews' conjecture states as follows.

Conjecture 6.4.1. Every linked partition ideal $\mathscr{I}$ has a bivariate generating function $G_{\mathscr{I}}(x,q)$ of the form

$$\sum_{n_1,\ldots,n_r\geq 0} \frac{(-1)^{L_1(n_1,\ldots,n_r)} q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)} x^{L_3(n_1,\ldots,n_r)}}{(q^{B_1};q^{A_1})_{n_1}\cdots(q^{B_r};q^{A_r})_{n_r}}, \tag{6.4.1}$$

where L_1, L_2 and L_3 are linear forms in $n_1,\ldots,n_r$ and Q is a quadratic form in $n_1,\ldots,n_r$.

It also appears to be true that some "nice" subsets of a linked partition ideal enjoy a generating function of the form (6.4.1). One may investigate the second Rogers–Ramanujan identity as an example.

Hence, we may search from a number of multi-summations of the form (6.4.1) and compare the series expansions to find suitable candidates.

Theorem 6.4.1. *Let $G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x,q)$ (resp. $G_{\mathscr{I}_{T_{\mathrm{I},2}}}(x,q)$, $G_{\mathscr{I}_{T_{\mathrm{I},3}}}(x,q)$) denote the generating function of partitions of type I whose smallest part is at least* 1 *(resp.* 2, 3*). We have*

$$G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x,q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.4.2}$$

$$G_{\mathscr{I}_{T_{\mathrm{I},2}}}(x,q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2+n_1+3n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.4.3}$$

$$G_{\mathscr{I}_{T_{\mathrm{I},3}}}(x,q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2+2n_1+3n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{6.4.4}$$

Remark 6.4.1. Here (6.4.2), (6.4.3) and (6.4.4) are (3.1), (3.10) and (3.14) in [116]. They correspond to the Kanade–Russell conjectures I_1, I_2 and I_3, respectively.

Theorem 6.4.2. *Let $G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x,q)$ (resp. $G_{\mathscr{I}_{T_{\mathrm{II},2}}}(x,q)$) denote the generating function of partitions of type II whose smallest part is at least* 1 *(resp.* 2*) and let $G_{\mathscr{I}_{T_{\mathrm{II},a}}}(x,q)$ denote the generating function of partitions of type II where* 1 *appears at most once. We have*

$$G_{\mathscr{I}_{T_{\mathrm{II},1}}}(x,q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2-n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.4.5}$$

$$G_{\mathscr{I}_{T_{\mathrm{II},2}}}(x,q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2+n_1+2n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.4.6}$$

$$G_{\mathscr{I}_{T_{\mathrm{II},a}}}(x,q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2+2n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{6.4.7}$$

Remark 6.4.2. Here (6.4.6) is (3.15) in [116]. It corresponds to the Kanade–Russell conjecture I_4.

Theorem 6.4.3. *Let $G_{\mathscr{I}_{T_{\mathrm{III},1}}}(x,q)$ (resp. $G_{\mathscr{I}_{T_{\mathrm{III},2}}}(x,q)$) denote the generating function of partitions of type III whose smallest part is at least* 1 *(resp.* 2*) and let $G_{\mathscr{I}_{T_{\mathrm{III},a}}}(x,q)$ denote the generating function of partitions of type III where* 1 *appears at most once. We have*

$$G_{\mathscr{I}_{T_{\mathrm{III},1}}}(x,q)$$

$$= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}-n_2-\frac{n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}, \tag{6.4.8}$$

$$G_{\mathscr{I}_{T_{\mathrm{III},2}}}(x,q) = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{3n_1}{2}+n_2+\frac{5n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}, \tag{6.4.9}$$

$$G_{\mathscr{I}_{T_{\mathrm{III},a}}}(x,q) = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}+n_2+\frac{5n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}. \tag{6.4.10}$$

Remark 6.4.3. Here (6.4.10) is (47) (corrected: in the numerator of which the last term of the exponent of q should read $4k$ instead of $3k$) in [109]. It corresponds to the Kanade–Russell conjecture I_5.

Theorem 6.4.4. *Let $G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(x,q)$ denote the generating function of partitions of type IV whose smallest part is at least 1, let $G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(x,q)$ denote the generating function of partitions of type IV where 1 appears at most once and let $G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(x,q)$ denote the generating function of partitions of type IV where the smallest part is at least 2 with 2 appearing at most once. We have*

$$G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(x,q) = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}-n_2+\frac{n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}, \tag{6.4.11}$$

$$G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(x,q) = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}+n_2+\frac{n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}, \tag{6.4.12}$$

$$G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(x,q) = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{3n_1}{2}+3n_2+\frac{7n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}. \tag{6.4.13}$$

Remark 6.4.4. Here (6.4.13) is (51) in [109]. It corresponds to the Kanade–Russell conjecture I_6.

In the above theorems, we rediscover six generating function identities proved in [109] and [116] and obtain six new identities. We will provide an approach to prove these identities in the next section with the help of computer algebra.

Remark 6.4.5. It is, of course, fine to discover the above sum-like generating functions by trial and error with this tedious work left to a computer. But sometimes human observation might reduce the workload. Let us use (6.4.2) as an example. If we write

$$G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x,q)=\sum_{M\geq 0} g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)x^M,$$

then the q-difference equation in Theorem 6.3.2 gives the first several expressions of $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$:

$$\begin{aligned}
g_{\mathscr{I}_{T_{\mathrm{I},1}}}(0)&=1,\\
g_{\mathscr{I}_{T_{\mathrm{I},1}}}(1)&=\frac{q}{1-q},\\
g_{\mathscr{I}_{T_{\mathrm{I},1}}}(2)&=\frac{q^3+q^4+q^6}{(1-q^2)(1-q^3)},\\
g_{\mathscr{I}_{T_{\mathrm{I},1}}}(3)&=\frac{q^7}{(1-q)(1-q^2)(1-q^3)},\\
g_{\mathscr{I}_{T_{\mathrm{I},1}}}(4)&=\frac{q^{12}+q^{15}+q^{17}+q^{18}-q^{19}+q^{20}-q^{21}}{(1-q)(1-q^3)(1-q^4)(1-q^6)}.
\end{aligned}$$

Recall that the sum-like generating function is

$$\sum_{n_1,\ldots,n_r\geq 0}\frac{(-1)^{L_1(n_1,\ldots,n_r)}q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}x^{L_3(n_1,\ldots,n_r)}}{(q^{B_1};q^{A_1})_{n_1}\cdots(q^{B_r};q^{A_r})_{n_r}}. \tag{6.4.14}$$

We observe that the numerator of $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(2)$ has more than one term. Hence, the linear equation $L_3(n_1,\ldots,n_r)=2$ might have multiple nonnegative solutions $(n_1,\ldots,n_r)$. It is fair to guess that L_3 looks like $n_1+n_2+\cdots$ or $n_1+2n_2+\cdots$. Also, the denominators of $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$ indicate that there might be terms like $(q;q)_n$ and $(q^3;q^3)_n$ in the denominator of the summand in (6.4.14). Hence, one may first try multi-summations like

$$\sum_{n_1,n_2\geq 0}\frac{(-1)^{L_1(n_1,n_2)}q^{Q(n_1,n_2)+L_2(n_1,n_2)}x^{n_1+n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}$$

or

$$\sum_{n_1,n_2\geq 0}\frac{(-1)^{L_1(n_1,n_2)}q^{Q(n_1,n_2)+L_2(n_1,n_2)}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}.$$

If these expressions fail to be a candidate, then one could continue to modify them and carry on the searching procedure. However, it should be emphasized that in this remark we do not intend to assert that the sum-like generating function must contain some particular "magical" exponents and bases.

6.5 Computer Algebra Assistance

Proofs of the generating function identities in the previous section can be carried out by the same procedure. We only demonstrate (6.4.2) as an instance.

6.5.1 The Main Idea

If we write

$$G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x,q) = \sum_{M\geq 0} g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)x^M, \tag{6.5.1}$$

where $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) \in \mathbb{Q}(q)$, then we can translate the q-difference equation in Theorem 6.3.2 to a recurrence of $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$.

Definition 6.5.1. Let $\mathbb{K} = \mathbb{Q}(q)$ with q transcendental. A sequence (a_n) in $\mathbb{K}$ is called *q-holonomic* if there exist $p, p_0, \ldots, p_r \in \mathbb{K}[x]$, not all zero, such that

$$p_0(q^n)a_n + p_1(q^n)a_{n+1} + \cdots + p_r(q^n)a_{n+r} = p(q^n).$$

Hence, the sequence $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$ is q-holonomic.

On the other hand, if we write

$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} = \sum_{M\geq 0} \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)x^M, \tag{6.5.2}$$

we may also find a recurrence relation satisfied by $\tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$. Hence, $\tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$ is also q-holonomic.

A result of Kauers and Koutschan [110] states that if two sequences (a_n) and (b_n) are q-holonomic, so is their linear combination $(\alpha a_n + \beta b_n)$. Hence, we may find a recurrence relation satisfied by $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) - \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$. As long as $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) - \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) = 0$ for enough initial cases, we are safe to say that this difference is identical to 0 for all M and hence arrive at the desired generating function identity.

6.5.2 Two *Mathematica* Packages

To proceed with our proof, we require two *Mathematica* packages: `qMultiSum` [153] and `qGeneratingFunctions` [110]. These packages along with their instructions can be found on the webpage of Research Institute for Symbolic Computation (RISC) of Johannes Kepler University[2].

To begin with, we load the two packages after installing them.

```
<<RISC`qMultiSum`
<<RISC`qGeneratingFunctions`
```

6.5.3 Recurrence for $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$

For the polynomials $p_{3i}(x,q)$ $(i=0,\ldots,3)$ defined in Theorem 6.3.2, we write

$$p_{3i}(x,q)=\sum_{j=0}^{J_{3i}} p_{3i,j}(q)x^j.$$

Then with (6.5.1), one may rewrite (6.3.27) as

$$\begin{aligned}0&=\sum_{i=0}^{3} p_{3i}(x,q)G_{\mathscr{I}_{T_{\mathrm{I},1}}}(xq^{3i})\\&=\sum_{i=0}^{3}\sum_{j=0}^{J_{3i}}\sum_{m\geq 0} p_{3i,j}(q)g_{\mathscr{I}_{T_{\mathrm{I},1}}}(m)q^{3im}x^{m+j}\\&=\sum_{M\geq 0}\sum_{i=0}^{3}\sum_{m=\max(0,M-J_{3i})}^{M} q^{3im}p_{3i,M-m}(x,q)g_{\mathscr{I}_{T_{\mathrm{I},1}}}(m)x^M.\end{aligned}$$

Hence, for all $M\geq 0$,

$$\sum_{i=0}^{3}\sum_{m=\max(0,M-J_{3i})}^{M} q^{3im}p_{3i,M-m}(x,q)g_{\mathscr{I}_{T_{\mathrm{I},1}}}(m)=0, \tag{6.5.3}$$

from which we see that $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$ $(M\geq 1)$ is uniquely determined by $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(0)$. It is also trivial that $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(0)=1$.

In particular, for $M\geq 0$, we have the following recurrence

$$0=g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)\left((q^{28}+q^{30})q^{9M}\right)$$

[2]See `https://www3.risc.jku.at/research/combinat/software/ergosum/index.html`

$$
\begin{aligned}
&+ g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+1)\left((q^{19}+q^{21})q^{6(M+1)}+q^{27}q^{9(M+1)}\right)\\
&+ g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+2)\left((q^{14}+q^{15}+q^{16}+q^{17}+q^{18})q^{6(M+2)}\right)\\
&+ g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+3)\left(-(q^{7}+q^{9}+q^{10}+q^{12})q^{3(M+3)}+(q^{11}+q^{13})q^{6(M+3)}\right)\\
&+ g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+4)\left(-(q^{3}+q^{4}+q^{5}+2q^{6}+q^{7}+q^{8}+q^{9})q^{3(M+4)}\right)\\
&+ g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+5)\left((q^{4}+q^{6})-(q+q^{2}+q^{3}+q^{4}+q^{6})q^{3(M+5)}\right)\\
&+ g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+6)\left(1-q^{3(M+6)}\right). \qquad (6.5.4)
\end{aligned}
$$

6.5.4 Recurrence for $\tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$

Notice that for $M \geq 0$

$$\tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) = \sum_{n\leq \frac{M}{2}} \frac{q^{(M-2n)^2+3n^2+3n(M-2n)}}{(q;q)_{M-2n}(q^3;q^3)_n}.$$

The recurrence satisfied by $\tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$ can be computed automatically by the `qMultiSum` package with the following codes:

```
ClearAll[M];
summand = q^(3n^2+(M-2n)^2+3n(M-2 n))/(qPochhammer[q,q,
   M-2n] qPochhammer[q^3,q^3,n]);
stru = qFindStructureSet[summand, {M}, {n}, {1}, {2},
   {2}, qProtocol -> True]
rec = qFindRecurrence[summand, {M}, {n}, {1}, {2}, {2},
    qProtocol -> True, StructSet -> stru[[1]]]
sumrec = qSumRecurrence[rec]
```

This gives us, for $M \geq 0$,

$$
\begin{aligned}
0 &= \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)q^{9M+24}(1+2q^2+q^4+q^{3M+14})\\
&+ \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+1)q^{6M+21}(1+2q^2+q^4-q^{3M+8}-q^{3M+10}+q^{3M+11}+q^{3M+13}+q^{3M+14})\\
&+ \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+2)q^{6M+22}(1+q^2)(1+q^2+q^3+q^4+q^{3M+12})\\
&- \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+3)q^{3M+12}(1+q^2)(1-q+q^2)(1+q+q^2+q^3+q^{3M+12})\\
&- \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+4)q^{3M+12}(1-q+q^2)(1+q+q^2)(1+q+q^2+q^3+q^{3M+13})\\
&+ \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M+5)(1-q^{3M+15})(1+2q^2+q^4+q^{3M+11}). \qquad (6.5.5)
\end{aligned}
$$

6.5.5 Recurrence for $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) - \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$

Finally, we deduce the recurrence for $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) - \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$ from (6.5.4) and (6.5.5). This can be accomplished by the `QREPlus` function of the `qGeneratingFunctions` package.

We need the following codes, in which `sumrec1` records the recurrence relation for $g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$ and `sumrec2` records the recurrence relation for $\tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$.

```
ClearAll[M];
sumrec1 = {SUM[M] ((q^(28)+q^(30))q^(9M))
+ SUM[M+1] ((q^(19)+q^(21))q^(6(M+1))+q^(27)q^(9(M+1)))
+ SUM[M+2] ((q^(14)+q^(15)+q^(16)+q^(17)+q^(18))q^(6(M
   +2)))
+ SUM[M+3] (-(q^7+q^9+q^(10)+q^(12))q^(3(M+3))+(q^(11)+
   q^(13))q^(6(M+3)))
+ SUM[M+4] (-(q^3+q^4+q^5+2q^6+q^7+q^8+q^9)q^(3(M+4)))
+ SUM[M+5] ((q^4+q^6)-(q+q^2+q^3+q^4+q^6)q^(3(M+5)))
+ SUM[M+6] (1-q^(3(M+6)))
== 0};
sumrec2 = {SUM[M] q^(9M+24) (1+2q^2+q^4+q^(3M+14))
+ SUM[M+1] q^(6M+21) (1+2q^2+q^4-q^(3M+8)-q^(3M+10)+q
   ^(3M+11)+q^(3M+13)+q^(3M+14))
+ SUM[M+2] q^(6M+22) (1+q^2) (1+q^2+q^3+q^4+q^(3M+12))
- SUM[M+3] q^(3M+12) (1+q^2) (1-q+q^2) (1+q+q^2+q^3+q
   ^(3M+12))
- SUM[M+4] q^(3M+12) (1-q+q^2) (1+q+q^2) (1+q+q^2+q^3+q
   ^(3M+13))
+ SUM[M+5] (1-q^(3M+15)) (1+2q^2+q^4+q^(3M+11))
== 0};
QREPlus[sumrec1, sumrec2, SUM[M]]
```

The output gives us an order six recurrence. Hence, to show

$$g_{\mathscr{I}_{T_{\mathrm{I},1}}}(M) = \tilde{g}_{\mathscr{I}_{T_{\mathrm{I},1}}}(M)$$

for all $M \geq 0$, it suffices to show that the equality holds for $M = 0, \ldots, 5$. This can be checked easily.

We therefore arrive at

$$G_{\mathscr{I}_{T_{\mathrm{I},1}}}(x,q)=\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2+3n_2^2+3n_1n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}.$$

6.5.6 Other Identities

Similar to (6.5.1) and (6.5.2), let us write

$$G_{\mathscr{I}_{T_*}}(x,q)=\sum_{M\geq 0}g_{\mathscr{I}_{T_*}}(M)x^M$$

and the multiple summations on the right hand sides of (6.4.3)–(6.4.13) as

$$\sum_{M\geq 0}\tilde{g}_{\mathscr{I}_{T_*}}(M)x^M,$$

where "$*$" may be "$\mathrm{I},2$", "$\mathrm{I},3$", etc. We list the orders of recurrences satisfied by $g_{\mathscr{I}_{T_*}}(M)$, $\tilde{g}_{\mathscr{I}_{T_*}}(M)$ and $g_{\mathscr{I}_{T_*}}(M)-\tilde{g}_{\mathscr{I}_{T_*}}(M)$ in Table 6.1 for the reader's convenience.

Table 6.1. Orders of recurrences satisfied by $g_{\mathscr{I}_{T_*}}(M)$, $\tilde{g}_{\mathscr{I}_{T_*}}(M)$ and $g_{\mathscr{I}_{T_*}}(M)-\tilde{g}_{\mathscr{I}_{T_*}}(M)$

$*$	$\mathrm{I},2$	$\mathrm{I},3$	$\mathrm{II},1$	$\mathrm{II},2$	II,a	$\mathrm{III},1$	$\mathrm{III},2$	III,a	$\mathrm{IV},1$	IV,a	IV,b
$g_{\mathscr{I}_{T_*}}$	6	6	6	6	6	15	15	15	15	15	15
$\tilde{g}_{\mathscr{I}_{T_*}}$	5	5	5	5	5	4	4	4	4	4	4
$g_{\mathscr{I}_{T_*}}-\tilde{g}_{\mathscr{I}_{T_*}}$	6	6	6	6	6	15	15	15	15	15	15

6.6 Endnotes

In a very recent paper of Bringmann, Jennings-Shaffer and Mahlburg [43], the Kanade–Russell conjectures I_5 and I_6 were proved. Here the analytic forms of I_5 and I_6 read respectively as

$$\begin{aligned}G_{\mathscr{I}_{T_{\mathrm{III},a}}}(1,q)&=\sum_{n_1,n_2,n_3\geq 0}\frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}+n_2+\frac{5n_3}{2}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}\\&=\frac{1}{(q,q^3,q^4,q^6,q^7,q^{10},q^{11};q^{12})_\infty},\end{aligned}\tag{6.6.1}$$

$$G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(1,q)=\sum_{n_1,n_2,n_3\geq 0}\frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{3n_1}{2}+3n_2+\frac{7n_3}{2}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}$$

$$= \frac{1}{(q^2, q^3, q^5, q^6, q^7, q^8, q^{11}; q^{12})_\infty}. \tag{6.6.2}$$

The authors of [43] cleverly reformulated $G_{\mathscr{I}_{T_{\mathrm{III},a}}}(1,q)$ and $G_{\mathscr{I}_{T_{\mathrm{IV},b}}}(1,q)$ and then added a new parameter so that the new bivariate generating functions satisfy simpler q-difference equations, from which the authors deduced the above identities.

Following the proofs of (1.15) and (1.16) in [43], one may prove the following identities with no difficulty.

Theorem 6.6.1. *We have*

$$\begin{aligned} G_{\mathscr{I}_{T_{\mathrm{III},1}}}(1,q) &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}-n_2-\frac{n_3}{2}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &= (-q;q)_\infty(-q^3;q^6)_\infty\, {}_2\phi_1\left(\begin{matrix} q^{-1}, q \\ q^2 \end{matrix}; q^6, -q^3\right), \end{aligned} \tag{6.6.3}$$

$$\begin{aligned} G_{\mathscr{I}_{T_{\mathrm{III},2}}}(1,q) &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{3n_1}{2}+n_2+\frac{5n_3}{2}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &= (-q^2;q)_\infty(-q^3;q^6)_\infty\, {}_2\phi_1\left(\begin{matrix} q, q^5 \\ q^8 \end{matrix}; q^6, -q^3\right), \end{aligned} \tag{6.6.4}$$

$$\begin{aligned} G_{\mathscr{I}_{T_{\mathrm{IV},1}}}(1,q) &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}-n_2+\frac{n_3}{2}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &= (-q;q)_\infty(-q^3;q^6)_\infty\, {}_2\phi_1\left(\begin{matrix} q^{-1}, q \\ q^4 \end{matrix}; q^6, -q^3\right), \end{aligned} \tag{6.6.5}$$

$$\begin{aligned} G_{\mathscr{I}_{T_{\mathrm{IV},a}}}(1,q) &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}+n_2+\frac{n_3}{2}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &= (-q;q)_\infty(-q^3;q^6)_\infty\, {}_2\phi_1\left(\begin{matrix} q, q^5 \\ q^4 \end{matrix}; q^6, -q^3\right). \end{aligned} \tag{6.6.6}$$

Note that we shall use a refinement of Proposition 2.3 in [43], the proof of which comes from a slight modification of the original proof of Bringmann, Jennings-Shaffer and Mahlburg.

Proposition 6.6.2. *Suppose that* $A(x) = \sum_{n\geq 0} \alpha_n x^n$ *has positive radius of convergence* *and $A(x)$ satisfies*

$$\begin{aligned} A(x) &= (1+q^a+x^2q^b+x^2q^c)A(xq^3) \\ &\quad - q^a(1+x^2q^{b+c-a-d+6})(1+x^2q^d)A(xq^6), \end{aligned} \tag{6.6.7}$$

where $a \notin 3\mathbb{Z}$ if $a \leq -6$. Then

$$\begin{aligned} A(x) &= \alpha_0(-x^2q^{d-6};q^6)_\infty \sum_{n\geq 0} \frac{(q^{b-d+6},q^{c-d+6};q^6)_n(-1)^n q^{(d-6)n}}{(q^6,q^{a+6};q^6)_n} x^{2n} \\ &\quad + \alpha_1(-x^2q^{d-6};q^6)_\infty \sum_{n\geq 0} \frac{(q^{b-d+9},q^{c-d+9};q^6)_n(-1)^n q^{(d-6)n}}{(q^9,q^{a+9};q^6)_n} x^{2n+1}. \end{aligned} \tag{6.6.8}$$

Proof. We divide by $(-x^2q^d;q^6)_\infty$ on both sides of (6.6.7) and put

$$B(x) := \frac{A(x)}{(-x^2q^{d-6};q^6)_\infty},$$

then

$$\begin{aligned} (1+x^2q^{d-6})B(x) &= (1+q^a+x^2q^b+x^2q^c)B(xq^3) \\ &\quad - q^a(1+x^2q^{b+c-a-d+6})B(xq^6). \end{aligned}$$

Writing $B(x) = \sum_{n\geq 0} \beta_n x^n$, one has, after simplification,

$$\beta_n = -\frac{q^{d-6}(1-q^{3n+b-d})(1-q^{3n+c-d})}{(1-q^{3n})(1-q^{3n+a})}\beta_{n-2}.$$

Finally, noting that $\beta_0 = \alpha_0$ and $\beta_1 = \alpha_1$ yields the desired result. $\square$

Now we prove (6.6.3) as an example.

Proof of (6.6.3). We first rewrite $G_{\mathscr{I}_{T_{\mathrm{III},1}}}(1,q)$ as follows:

$$\begin{aligned} G_{\mathscr{I}_{T_{\mathrm{III},1}}}(1,q) &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}-n_2-\frac{n_3}{2}}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &= \sum_{n_2,n_3\geq 0} \frac{q^{3n_2^2+\frac{9n_3^2}{2}+6n_2n_3-n_2-\frac{n_3}{2}}}{(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \sum_{n_1\geq 0} \frac{q^{\frac{n_1^2}{2}+2n_1n_2+3n_3n_1+\frac{n_1}{2}}}{(q;q)_{n_1}}. \end{aligned}$$

For the inner summation, we apply the identity

$$(x;q)_\infty = \sum_{n\geq 0} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}.$$

It follows that

$$\begin{aligned} G_{\mathscr{I}_{T_{\mathrm{III},1}}}(1,q) &= \sum_{n_2,n_3\geq 0} \frac{q^{3n_2^2+\frac{9n_3^2}{2}+6n_2n_3-n_2-\frac{n_3}{2}}(-q^{2n_2+3n_3+1};q)_\infty}{(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &= (-q;q)_\infty \sum_{n_2,n_3\geq 0} \frac{q^{3n_2^2+\frac{9n_3^2}{2}+6n_2n_3-n_2-\frac{n_3}{2}}}{(-q;q)_{2n_2+3n_3}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}. \end{aligned}$$

Let us define an auxiliary function

$$H(x) := \sum_{n_2,n_3\geq 0} \frac{q^{3n_2^2+\frac{9n_3^2}{2}+6n_2n_3-n_2-\frac{n_3}{2}} x^{2n_2+2n_3}}{(-q;q)_{2n_2+3n_3}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}.$$

We also assume that $H(x) = \sum_{n\geq 0} h_n x^{2n}$.

One may use the *Mathematica* package `qZeil` [138] to find a recurrence satisfied by h_n through the following codes.

```
<< RISC`qZeil`
ClearAll[n2, n3, M]
n3 = M - n2;
summand =
q^(3 n2^2 + (9 n3^2)/2 + 6 n2*n3 - n2 - n3/
2)/(qPochhammer[-q, q, 2 n2 + 3 n3] qPochhammer
   [q^2, q^2,
n2] qPochhammer[q^3, q^3, n3]);
qZeil[summand, {n2, 0, Infinity}, M, 2]
```

The resulting recurrence is

$$(1-q^{6n-4}-q^{6n}+q^{12n-4})h_n - (q^{6n-4}+q^{6n-2}-q^{12n-9}-q^{12n-7})h_{n-1} + q^{12n-12}h_{n-2} = 0.$$

This recurrence then leads to

$$H(x) = (1+q^{-4}+x^2q^2+x^2q^4)H(xq^3) - q^{-4}(1+x^2q^7)(1+x^2q^9)H(xq^6).$$

Finally, we use Proposition 6.6.2 with $a = -4$, $b = 2$, $c = 4$ and $d = 9$. Then,

$$\begin{aligned} H(x) &= (-x^2q^3;q^6)_\infty \sum_{n\geq 0} \frac{(q^{-1},q;q^6)_n(-q^3)^n}{(q^6,q^2;q^6)_m} x^{2n} \\ &= (-x^2q^3;q^6)_\infty \; {}_2\phi_1 \left(\begin{matrix} q^{-1}, q \\ q^2 \end{matrix} ; q^6, -x^2q^3 \right). \end{aligned}$$

It follows that

$$G_{\mathscr{I}_{T_{\mathrm{III},1}}}(1,q) = (-q;q)_\infty H(1) = (-q;q)_\infty (-q^3;q^6)_\infty \; {}_2\phi_1 \left(\begin{matrix} q^{-1}, q \\ q^2 \end{matrix} ; q^6, -q^3 \right),$$

which is our desired identity. □

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Chapter 7

Span One Linked Partition Ideals: Directed Graphs and q-Multi-summations

This chapter comes from

- S. Chern, Linked partition ideals, directed graphs and q-multi-summations, *Electron. J. Combin.* **27** (2020), no. 3, Paper No. 3.33, 29 pp. (Ref. [54])

In the previous chapter, we have explained the definition of span one linked partition ideals. Given a span one linked partition ideal $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$, one crucial problem discussed is how to determine its generating function

$$\mathscr{G}(x) = \mathscr{G}(x, q) := \sum_{\lambda \in \mathscr{I}} x^{\sharp(\lambda)} q^{|\lambda|}.$$

It should be admitted that to derive an Andrews–Gordon type generating function identity, one has to obtain first a conjectural (x, q) sum-side. This then requires an extensive search using the general shape given by Andrews' Conjecture 6.1.2.

However, we could also start in the opposite direction. That is, if we are given a family of nice q-multi-summations, then we may try to use the approach in Section 7.4 to construct identities like (7.4.5) and (7.4.13), from which we may further construct some combinatorial objects, or even more luckily, a span one linked partition ideal and its subsets, such that the q-multi-summations correspond to their generating functions. One such instance is given in Theorem 7.4.4 and Corollary 7.4.5. This is indeed what we hope the framework in this chapter could provide.

7.1 Main Result

Assume that in $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$, Π is given by $\{\pi_1, \pi_2, \ldots, \pi_K\}$ where $\pi_1 = \emptyset$, the empty partition. We define a $(0,1)$-matrix $\mathscr{A} = \mathscr{A}(\langle \Pi, \mathcal{L}\rangle)$ by

$$\mathscr{A}_{i,j} = \begin{cases} 1 & \text{if } \pi_j \in \mathcal{L}(\pi_i), \\ 0 & \text{if } \pi_j \notin \mathcal{L}(\pi_i), \end{cases} \tag{7.1.1}$$

and a diagonal matrix $\mathscr{W}(x) = \mathscr{W}(\langle \Pi, \mathcal{L}\rangle \mid x, q)$ by

$$\mathscr{W}(x) = \begin{pmatrix} x^{\sharp(\pi_1)} q^{|\pi_1|} & & & \\ & x^{\sharp(\pi_2)} q^{|\pi_2|} & & \\ & & \ddots & \\ & & & x^{\sharp(\pi_K)} q^{|\pi_K|} \end{pmatrix}. \tag{7.1.2}$$

Theorem 7.1.1. *For each $1 \leq k \leq K$, we denote by $\mathscr{I}_k$ the subset of partitions λ in $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ whose S-tail is $\pi_k \in \Pi$. We further write*

$$\mathscr{G}_k(x) = \mathscr{G}_k(x,q) := \sum_{\lambda \in \mathscr{I}_k} x^{\sharp(\lambda)} q^{|\lambda|}.$$

Let $\mathscr{A}$ and $\mathscr{W}(x)$ be defined as in (7.1.1) *and* (7.1.2)*, respectively. Then, for $|q| < 1$ and $|x| < |q|^{-1}$,*

$$\begin{pmatrix} \mathscr{G}_1(x) \\ \mathscr{G}_2(x) \\ \vdots \\ \mathscr{G}_K(x) \end{pmatrix} = \mathscr{W}(x). \left(\lim_{M\to\infty} \prod_{m=1}^{M} \left(\mathscr{A}.\mathscr{W}(xq^{mS})\right)\right). \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7.1.3}$$

Remark 7.1.1. Recall that $\pi_1 = \emptyset$ (so that $\pi_1 \in \mathcal{L}(\pi)$ for all $\pi \in \Pi$) and $\mathcal{L}(\emptyset) = \Pi$. It follows that all entries in the first row and column of $\mathscr{A}$ are 1. Further, the first entry in $\mathscr{W}(x)$ is also $x^0 q^0 = 1$. When $|q| < 1$ and $|x| < |q|^{-1}$, we have

$$\lim_{M\to\infty} \mathscr{A}.\mathscr{W}(xq^{MS}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Throughout, $\prod_{m=1}^{M}(\mathscr{A}.\mathscr{W}(xq^{mS}))$ means

$$\mathscr{A}.\mathscr{W}(xq^{S}).\mathscr{A}.\mathscr{W}(xq^{2S}).\cdots.\mathscr{A}.\mathscr{W}(xq^{MS}). \tag{7.1.4}$$

Remark 7.1.2. We have

$$\mathscr{G}(x) = \sum_{k=1}^{K} \mathscr{G}_k(x),$$

but since $\mathcal{L}(\emptyset) = \Pi$, it is not hard to see that

$$\mathscr{G}_1(x) = \sum_{k=1}^{K} \mathscr{G}_k(xq^{S}).$$

Hence,

$$\mathscr{G}(x) = \mathscr{G}_1(xq^{-S}). \tag{7.1.5}$$

In the next section, we will consider our main result in a more general setting of graph-theoretic flavor.

7.2 Directed Graphs

Let $G = (V, E)$ be a directed graph where V is the set of vertices and E is the set of directed edges. Throughout, we allow loops (that is, directed edges connecting vertices with themselves) in G but for any two vertices u and v, not necessarily distinct, we allow at most one directed edge connecting u with v. Let $V = \{v_1, v_2, \ldots, v_K\}$. Let $\mathscr{A} = \mathscr{A}(G)$ be the adjacency matrix of G, that is,

$$\mathscr{A}_{i,j} = \begin{cases} 1 & \text{if there is a directed edge from } v_i \text{ with } v_j, \\ 0 & \text{if there are no directed edges from } v_i \text{ with } v_j. \end{cases} \tag{7.2.1}$$

We say that w is a walk of step M in G if w is a chain of $M+1$ vertices

$$\varpi_0 \to \varpi_1 \to \cdots \to \varpi_M$$

such that for each $1 \le m \le M$, there is an edge from ϖ_{m-1} to ϖ_m. Let $\mathcal{W}_M$ be the set of walks of step M in G.

7.2.1 Generating Function for Walks in a Directed Graph

To define the generating function for step M walks in a directed graph $G = (V, E)$, we assign two weights to each vertex v: one is called *length*, denoted by $\sharp(v) \in \mathbb{N}$, and the other is called *size*, denoted by $|v| \in \mathbb{N}$.

Let the *shift* S be a non-negative integer.

For any walk $w \in \mathcal{W}_M$,

$$w = \varpi_0 \to \varpi_1 \to \cdots \to \varpi_M, \tag{7.2.2}$$

we define its generating function by

$$\mathscr{G}(w \,|\, x, q) := x^{\sharp(\varpi_0)} q^{|\varpi_0|} \times (xq^S)^{\sharp(\varpi_1)} q^{|\varpi_1|} \times \cdots \times (xq^{MS})^{\sharp(\varpi_M)} q^{|\varpi_M|}. \tag{7.2.3}$$

Now we are able to define the generating function for step M walks from v_i to v_j for any $1 \le i, j \le K$:

$$\mathscr{G}_{i,j}(\mathcal{W}_M \,|\, x) = \mathscr{G}_{i,j}(\mathcal{W}_M \,|\, x, q) := \sum_{\substack{w \in \mathcal{W}_M \\ \varpi_0 = v_i \\ \varpi_M = v_j}} \mathscr{G}(w \,|\, x, q). \tag{7.2.4}$$

Let us define a diagonal matrix $\mathscr{W}(x) = \mathscr{W}(x, q)$ by

$$\mathscr{W}(x) = \begin{pmatrix} x^{\sharp(v_1)} q^{|v_1|} & & & \\ & x^{\sharp(v_2)} q^{|v_2|} & & \\ & & \ddots & \\ & & & x^{\sharp(v_K)} q^{|v_K|} \end{pmatrix}. \tag{7.2.5}$$

Theorem 7.2.1. *Let $\mathscr{A}$ be the adjacency matrix of G and let $\mathscr{W}(x)$ be as in* (7.2.5)*. Then $\mathscr{G}_{i,j}(\mathcal{W}_M \,|\, x)$ is the (i,j)-th entry of*

$$\mathscr{W}(x).\mathscr{A}.\mathscr{W}(xq^S).\mathscr{A}.\mathscr{W}(xq^{2S}).\cdots.\mathscr{A}.\mathscr{W}(xq^{MS}). \tag{7.2.6}$$

Remark 7.2.1. Let us set $x = q = 1$. Then $\mathscr{W}(1,1)$ is a $K \times K$ identity matrix and hence (7.2.6) becomes $\mathscr{A}^M$. Since $\mathscr{G}_{i,j}(\mathcal{W}_M \,|\, 1, 1)$ equals the number of walks of step M from vertex v_i to vertex v_j, Theorem 7.2.1 immediately leads to a well-known result in graph theory:

Corollary 7.2.2. *The number of walks of step M from vertex v_i to vertex v_j is the*

(i,j)*-th entry of* $\mathscr{A}^M$.

Proof of Theorem 7.2.1. We induct on M. When $M=0$, that is, the chain w of vertices in (7.2.2) contains only one vertex ϖ_0, it follows that

$$\mathscr{G}_{i,j}(\mathcal{W}_0 \,|\, x) = \begin{cases} x^{\sharp(v_i)} q^{|v_i|} & \text{if } i=j, \\ 0 & \text{if } i \neq j, \end{cases}$$

which is identical to the (i,j)-th entry of $\mathscr{W}(x)$.

Now let us assume that the theorem is true for some $M \geq 0$. We also write for convenience

$$\mathscr{M}(M) = \mathscr{W}(x).\mathscr{A}.\mathscr{W}(xq^S).\mathscr{A}.\mathscr{W}(xq^{2S}).\cdots.\mathscr{A}.\mathscr{W}(xq^{MS}).$$

Then $\mathscr{G}_{i,j}(\mathcal{W}_M \,|\, x) = \mathscr{M}(M)_{i,j}$. Further,

$$\begin{aligned} \mathscr{M}(M+1)_{i,j} &= \sum_{k=1}^{K} \mathscr{M}(M)_{i,k} \mathscr{A}_{k,j} (xq^{(M+1)S})^{\sharp(v_j)} q^{|v_j|} \\ &= \sum_{k=1}^{K} \mathscr{G}_{i,k}(\mathcal{W}_M \,|\, x) \mathscr{A}_{k,j} (xq^{(M+1)S})^{\sharp(v_j)} q^{|v_j|}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathscr{G}_{i,j}(\mathcal{W}_{M+1} \,|\, x) &= \sum_{\substack{w \in \mathcal{W}_{M+1} \\ \varpi_0 = v_i \\ \varpi_M = v_j}} \mathscr{G}(w \,|\, x, q) \\ &= \sum_{k=1}^{K} \left(\sum_{\substack{w \in \mathcal{W}_{M} \\ \varpi_0 = v_i \\ \varpi_M = v_k}} \mathscr{G}(w \,|\, x, q) \right) \mathscr{A}_{k,j} (xq^{(M+1)S})^{\sharp(v_j)} q^{|v_j|} \\ &= \sum_{k=1}^{K} \mathscr{G}_{i,k}(\mathcal{W}_M \,|\, x) \mathscr{A}_{k,j} (xq^{(M+1)S})^{\sharp(v_j)} q^{|v_j|}. \end{aligned}$$

Hence, $\mathscr{G}_{i,j}(\mathcal{W}_{M+1} \,|\, x) = \mathscr{M}(M+1)_{i,j}$, which is our desired result. ☐

7.2.2 Assigning an Empty Vertex

Let us assume that $v_1 \in V$ is an empty vertex, that is, its length and size are both 0:

$$\sharp(v_1) = 0 \quad \text{and} \quad |v_1| = 0. \tag{7.2.7}$$

We also assume that, for $2 \leq k \leq K$, $\sharp(v_k)$ and $|v_k|$ are both positive integers.

We require that, for each $1 \leq k \leq K$, there is an edge from vertex v_k to the empty vertex v_1. Hence, the entries in the first column of the adjacency matrix $\mathscr{A}$ are all 1.

We call such a modified directed graph $G^! = (V^!, E^!)$.

For any finite walk in $G^!$,

$$w = \varpi_0 \to \varpi_1 \to \cdots \to \varpi_M,$$

with $\varpi_M \neq v_1$, we may extend it to an infinite walk

$$w^\star = \varpi_0 \to \varpi_1 \to \cdots \to \varpi_M \to v_1 \to v_1 \to \cdots.$$

Conversely, for any infinite walk $w^\star$ in $G^!$ ending with $v_1 \to v_1 \to \cdots$, a series of empty vertex, we may find the last vertex, say ϖ_M, which is not empty, and reduce $w^\star$ to a finite walk $w = \varpi_0 \to \varpi_1 \to \cdots \to \varpi_M$. If there is no such ϖ_M, that is, if the infinite walk is $v_1 \to v_1 \to \cdots$, we reduce it to v_1.

It follows from the assumptions $\sharp(v_1) = 0$ and $|v_1| = 0$ that

$$\mathscr{G}(w^\star \,|\, x, q) = \mathscr{G}(w \,|\, x, q). \tag{7.2.8}$$

Also, for the infinite walk $v_1 \to v_1 \to \cdots$, we have

$$\mathscr{G}(v_1 \to v_1 \to \cdots \,|\, x, q) = \mathscr{G}(v_1 \,|\, x, q) = x^0 q^0 = 1.$$

Let $\mathcal{W}^\star$ denote the set of infinite walks in $G^!$ ending with $v_1 \to v_1 \to \cdots$, a series of empty vertex.

We are now in the position to define the generating function of $G^!$, by

$$\begin{aligned} \mathscr{G}(G^! \,|\, x, q) &:= \sum_{w^\star \in \mathcal{W}^\star} \mathscr{G}(w^\star \,|\, x, q) && (7.2.9) \\ &= \sum_{M \geq 0} \sum_{\substack{w \in \mathcal{W}_M \\ w_M \neq v_1}} \mathscr{G}(w \,|\, x, q). && (7.2.10) \end{aligned}$$

Theorem 7.2.3. *For each $1 \leq k \leq K$, let $\mathscr{G}_k(G^! \,|\, x) = \mathscr{G}_k(G^! \,|\, x, q)$ denote the generating function for infinite walks in $\mathcal{W}^\star$ starting at v_k. Let the shift S be a positive integer. Let $\mathscr{A}$ and $\mathscr{W}(x)$ be defined as in* (7.2.1) *and* (7.2.5)*, respectively. Then, for $|q| < 1$ and*

$|x| < |q|^{-1}$,

$$\begin{pmatrix} \mathscr{G}_1(G^! \,|\, x) \\ \mathscr{G}_2(G^! \,|\, x) \\ \vdots \\ \mathscr{G}_K(G^! \,|\, x) \end{pmatrix} = \mathscr{W}(x). \left(\lim_{M\to\infty} \prod_{m=1}^{M} (\mathscr{A}.\mathscr{W}(xq^{mS})) \right) . \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{7.2.11}$$

Proof. We simply observe that, for each $1 \le k \le K$,

$$\mathscr{G}_k(G^! \,|\, x) = \lim_{M\to\infty} \sum_{\substack{w\in\mathcal{W}_M \\ \varpi_0 = v_k \\ \varpi_M = v_1}} \mathscr{G}(w \,|\, x, q).$$

By Theorem 7.2.1, this is the $(k, 1)$-th entry of

$$\mathscr{W}(x).\mathscr{A}.\mathscr{W}(xq^S).\mathscr{A}.\mathscr{W}(xq^{2S}).\cdots = \mathscr{W}(x). \left(\lim_{M\to\infty} \prod_{m=1}^{M} (\mathscr{A}.\mathscr{W}(xq^{mS})) \right).$$

The desired result therefore follows. □

Remark 7.2.2. Results of the same flavor as Theorem 7.2.3 are available in literature for some other concrete identities; see [124, Section 3] for Gordon's identities, [74, Section 5] for the Andrews–Göllnitz–Gordon identities, and [107, Section 6] for the Andrews–Bressoud identities.

7.2.3 Proof of Theorem 7.1.1

To prove Theorem 7.1.1, let us define the *associated directed graph* of a span one linked partition ideal $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$.

We first define the set of vertices. Since $\Pi = \{\pi_1, \pi_2, \ldots, \pi_K\}$ is a finite set of partitions, we may treat each π_k as a vertex. We also define the length of π_k as the number of parts in π_k and the size of π_k as the sum of all parts in π_k. In particular, since π_1 is an empty partition so that $\sharp(\pi_1) = 0$ and $|\pi_1| = 0$, we may treat π_1 as an empty vertex.

We next define the directed edges in a natural way. For $1 \le i, j \le K$, if $\pi_j \in \mathcal{L}(\pi_i)$, then we say that there is an edge from vertex π_i to vertex π_j. Recall that for any $\pi \in \Pi$, its linking set $\mathcal{L}(\pi)$ is defined to contain the empty partition $\pi_1 = \emptyset$. Hence, for each $1 \le k \le K$, there is an edge from vertex π_k to vertex π_1.

We call this graph the associated directed graph of $\mathscr{I}$, denoted by $G^!(\mathscr{I}) = (V^!(\mathscr{I}), E^!(\mathscr{I}))$. In fact, $G^!(\mathscr{I})$ is a modified directed graph described in §7.2.2.

Recall from (6.1.3) that each partition λ in $\mathscr{I}$ can be uniquely decomposed as

$$\lambda = \lambda_0 \oplus \phi^S(\lambda_1) \oplus \phi^{2S}(\lambda_2) \oplus \cdots \oplus \phi^{KS}(\lambda_K) \oplus \phi^{(K+1)S}(\emptyset) \oplus \phi^{(K+2)S}(\emptyset) \oplus \cdots$$

so that $\lambda_K \neq \emptyset$ as long as $\lambda \neq \emptyset$. Hence, we have a natural bijection to infinite walks in $G^!(\mathscr{I})$ ending with $\pi_1 \to \pi_1 \to \cdots$:

$$w^\star(\lambda) = \lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots \to \lambda_K \to \pi_1 \to \pi_1 \to \cdots .$$

Further, if λ is an empty partition, then the resulted infinite walk is simply $\pi_1 \to \pi_1 \to \cdots$.

Now let us define S to be the shift. Then

$$x^{\sharp(\lambda)} q^{|\lambda|} = \mathscr{G}(w^\star(\lambda) \,|\, x, q). \tag{7.2.12}$$

Hence,

$$\mathscr{G}(x) = \sum_{\lambda \in \mathscr{I}} x^{\sharp(\lambda)} q^{|\lambda|} = \sum_{w^\star \in \mathcal{W}^\star} \mathscr{G}(w^\star \,|\, x, q).$$

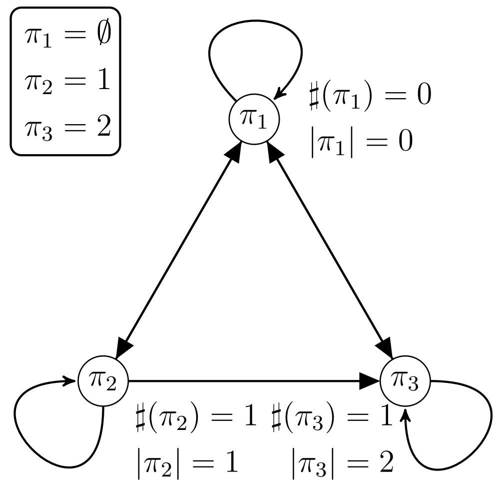
The rest follows directly from Theorem 7.2.3.

Example 7.2.1. It is shown in Example 6.1.1 that partitions with difference at least 2 at distance 1 form a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$ where $\Pi = \{\emptyset, 1, 2\}$, the linking sets are

$$\mathcal{L}(\emptyset) = \{\emptyset, 1, 2\}, \quad \mathcal{L}(1) = \{\emptyset, 1, 2\}, \quad \mathcal{L}(2) = \{\emptyset, 2\},$$

and $S = 2$. We represent its associated directed graph in Figure 7.1.

Figure 7.1. The associated directed graph in Example 7.2.1



7.3 q-Multi-summations

7.3.1 A q-Difference System and the Uniqueness of Solutions

Recall that in Theorem 7.1.1 we have shown that

$$\begin{pmatrix} \mathscr{G}_1(x) \\ \mathscr{G}_2(x) \\ \vdots \\ \mathscr{G}_K(x) \end{pmatrix} = \mathscr{W}(x). \left(\lim_{M\to\infty} \prod_{m=1}^{M} (\mathscr{A}.\mathscr{W}(xq^{mS})) \right). \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7.3.1}$$

Let us focus on

$$\begin{pmatrix} F_1^\star(x) \\ F_2^\star(x) \\ \vdots \\ F_K^\star(x) \end{pmatrix} := \left(\lim_{M\to\infty} \prod_{m=1}^{M} (\mathscr{A}.\mathscr{W}(xq^{mS})) \right). \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7.3.2}$$

Notice that

$$\begin{pmatrix} F_1^\star(x) \\ F_2^\star(x) \\ \vdots \\ F_K^\star(x) \end{pmatrix} = \left(\lim_{M\to\infty} \prod_{m=1}^{M} (\mathscr{A}.\mathscr{W}(xq^{mS})) \right). \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \mathscr{A}.\mathscr{W}(xq^S).\left(\lim_{M\to\infty}\prod_{m=1}^{M}(\mathscr{A}.\mathscr{W}(xq^Sq^{mS}))\right).\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$$

$$= \mathscr{A}.\mathscr{W}(xq^S).\begin{pmatrix}F_1^\star(xq^S)\\F_2^\star(xq^S)\\\vdots\\F_K^\star(xq^S)\end{pmatrix}.$$

If we further write $F_k(x) := F_k^\star(xq^{-S})$ for each k, then the column vector

$$\underline{\mathbf{F}}(x) := \begin{pmatrix}F_1(x)\\F_2(x)\\\vdots\\F_K(x)\end{pmatrix}$$

satisfies the q-difference system

$$\underline{\mathbf{F}}(x) = \mathscr{A}.\mathscr{W}(x).\underline{\mathbf{F}}(xq^S). \tag{7.3.3}$$

Remark 7.3.1. It follows from (7.3.3) that

$$\underline{\mathbf{F}}(x) = \mathscr{A}.\mathscr{W}(x).\begin{pmatrix}F_1^\star(x)\\F_2^\star(x)\\\vdots\\F_K^\star(x)\end{pmatrix} = \mathscr{A}.\begin{pmatrix}\mathscr{G}_1(x)\\\mathscr{G}_2(x)\\\vdots\\\mathscr{G}_K(x)\end{pmatrix}. \tag{7.3.4}$$

Recall that, we have defined in Theorem 7.1.1 that, for each $1 \le k \le K$, $\mathscr{I}_k$ denotes the subset of partitions in $\mathscr{I}(\langle\Pi,\mathcal{L}\rangle, S)$ whose S-tail is π_k. Further, $\mathscr{G}_k(x)$ is the generating function of $\mathscr{I}_k$. Since $\mathscr{A}$ is a $(0,1)$-matrix, it follows that $F_k(x) \in \mathbb{Z}[[q]][[x]]$ for each $1 \le k \le K$. More importantly, since the empty partition $\emptyset$ is contained in $\mathscr{I}_1$ but not in $\mathscr{I}_k$ for $2 \le k \le K$, we have $\mathscr{G}_1(0) = 1$ and $\mathscr{G}_k(0) = 0$ for $2 \le k \le K$. Since the entries in the first column of $\mathscr{A}$ are all 1, it follows that

$$F_1(0) = F_2(0) = \cdots = F_K(0) = 1. \tag{7.3.5}$$

We next show the uniqueness of solutions of (7.3.3).

Proposition 7.3.1. *In the q-difference system* (7.3.3)*, we assume that, for each* $1 \leq k \leq K$*,* $F_k(x) \in \mathbb{C}[[q]][[x]]$*. If* $F_1(0) = F_2(0) = \cdots = F_K(0)$*, then there exists a solution to* (7.3.3)*. Further, the solution is uniquely determined by* $\underline{\mathbf{F}}(0)$*.*

Proof. For each $1 \leq k \leq K$, let us write

$$F_k(x) = \sum_{n \geq 0} f_k(n) x^n,$$

where $f_k(n) \in \mathbb{C}[[q]]$ for $n \geq 0$. We also write for notational convenience that $f_k(n) = 0$ for $n < 0$. Then,

$$\begin{aligned}\sum_{n \geq 0} f_k(n) x^n &= \sum_{j=1}^{K} \mathscr{A}_{k,j} x^{\sharp(\pi_j)} q^{|\pi_j|} \sum_{n \geq 0} f_j(n) q^{nS} x^n \\ &= \sum_{n \geq 0} \left(\sum_{j=1}^{K} \mathscr{A}_{k,j} q^{|\pi_j| + (n - \sharp(\pi_j))S} f_j(n - \sharp(\pi_j)) \right) x^n.\end{aligned}$$

Recall that $\sharp(\pi_1) = |\pi_1| = 0$ and $\mathscr{A}_{k,1} = 1$ for all k. We have that, for $n \geq 0$,

$$f_k(n) = q^{nS} f_1(n) + \sum_{j=2}^{K} \mathscr{A}_{k,j} q^{|\pi_j| + (n - \sharp(\pi_j))S} f_j(n - \sharp(\pi_j)). \tag{7.3.6}$$

Setting $n = 0$ gives the requirement $F_1(0) = F_2(0) = \cdots = F_K(0)$. Also, $\underline{\mathbf{F}}(0) = (f_1(0), f_2(0), \ldots, f_K(0))^T$ uniquely determines $f_k(n)$ for all $1 \leq k \leq K$ and $n \geq 1$ by (7.3.6). $\square$

7.3.2 Two Examples

Recall that, for each $1 \leq k \leq K$, $\mathscr{I}_k$ denotes the subset of partitions in $\mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$ whose S-tail is π_k. Further,

$$\mathscr{G}_k(x) = \sum_{\lambda \in \mathscr{I}_k} x^{\sharp(\lambda)} q^{|\lambda|}.$$

7.3.2.1 Example 1

In the first example, we consider

"partitions with difference at least 2 at distance 1."

This partition set obviously corresponds to the Rogers–Ramanujan identities. In Example 6.1.1, we have shown that it is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ where $\Pi = \{\pi_1, \pi_2, \pi_3\}$ with $\pi_1 = \emptyset$, $\pi_2 = 1$ and $\pi_3 = 2$, the linking sets are

$$\mathcal{L}(\pi_1) = \{\pi_1, \pi_2, \pi_3\}, \quad \mathcal{L}(\pi_2) = \{\pi_1, \pi_2, \pi_3\}, \quad \mathcal{L}(\pi_3) = \{\pi_1, \pi_3\},$$

and $S = 2$.

Notice that the generating function for partitions with difference at least 2 at distance 1 is

$$\mathscr{G}_1(x) + \mathscr{G}_2(x) + \mathscr{G}_3(x) = \sum_{n\geq 0} \frac{q^{n^2} x^n}{(q;q)_n} \tag{7.3.7}$$

and that the generating function for partitions with difference at least 2 at distance 1 with the smallest part ≥ 2 is

$$\mathscr{G}_1(x) + \mathscr{G}_3(x) = \sum_{n\geq 0} \frac{q^{n^2+n} x^n}{(q;q)_n}. \tag{7.3.8}$$

We know from (7.3.4) that

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \mathscr{A} \cdot \begin{pmatrix} \mathscr{G}_1(x) \\ \mathscr{G}_2(x) \\ \mathscr{G}_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathscr{G}_1(x) \\ \mathscr{G}_2(x) \\ \mathscr{G}_3(x) \end{pmatrix}.$$

Hence, by (7.3.7) and (7.3.8), if we put

$$F_1(x) = F_2(x) = \sum_{n\geq 0} \frac{q^{n^2} x^n}{(q;q)_n} \tag{7.3.9}$$

and

$$F_3(x) = \sum_{n\geq 0} \frac{q^{n^2+n} x^n}{(q;q)_n}, \tag{7.3.10}$$

then we have the following relation from (7.3.3):

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & xq & \\ & & xq^2 \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \end{pmatrix}. \tag{7.3.11}$$

Conversely, if we are able to prove (7.3.11) directly (notice that $F_1(0) = F_2(0) = F_3(0) = 1$), then by Remark 7.3.1 and Proposition 7.3.1, we can compute that

$$\begin{aligned}\begin{pmatrix}\mathscr{G}_1(x)\\ \mathscr{G}_2(x)\\ \mathscr{G}_3(x)\end{pmatrix} &= \begin{pmatrix}1 & & \\ & xq & \\ & & xq^2\end{pmatrix} \cdot \begin{pmatrix}F_1^\star(x)\\ F_2^\star(x)\\ F_3^\star(x)\end{pmatrix}\\ &= \begin{pmatrix}1 & & \\ & xq & \\ & & xq^2\end{pmatrix} \cdot \begin{pmatrix}F_1(xq^2)\\ F_2(xq^2)\\ F_3(xq^2)\end{pmatrix}.\end{aligned}$$

Also, (7.3.7) and (7.3.8) can be deduced with no difficulty.

7.3.2.2 Example 2

In the second example, we consider

> "partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3."

This partition set corresponds to the Kanade–Russell conjectures I_1–I_3. It is shown in §6.3.1 that this partition set is a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, S)$ where $S = 3$, and $\Pi = \{\pi_1, \pi_2, \ldots, \pi_7\}$ along with the linking sets given as follows.

Π	linking set
$\pi_1 = \emptyset$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_2 = 1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_3 = 2+1$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_4 = 3+1$	$\{\pi_1,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_5 = 2$	$\{\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_6 = 3$	$\{\pi_1,\ \pi_5,\ \pi_6,\ \pi_7\}$
$\pi_7 = 3+3$	$\{\pi_1,\ \pi_6,\ \pi_7\}$

It is also shown in §6.3.1 that the generating function for such partitions is

$$\begin{aligned}\mathscr{G}_1(x)+\mathscr{G}_2(x)+\mathscr{G}_3(x)+\mathscr{G}_4(x)&\\ +\mathscr{G}_5(x)+\mathscr{G}_6(x)+\mathscr{G}_7(x)&= \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2} x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}},\end{aligned} \tag{7.3.12}$$

that the generating function for such partitions with the smallest part ≥ 2 is

$$\mathscr{G}_1(x)+\mathscr{G}_5(x)+\mathscr{G}_6(x)+\mathscr{G}_7(x)=\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2+3n_2^2+3n_1n_2+n_1+3n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{7.3.13}$$

and that the generating function for such partitions with the smallest part ≥ 3 is

$$\mathscr{G}_1(x)+\mathscr{G}_6(x)+\mathscr{G}_7(x)=\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2+3n_2^2+3n_1n_2+2n_1+3n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{7.3.14}$$

We know from (7.3.4) that

$$\begin{pmatrix}F_1(x)\\F_2(x)\\F_3(x)\\F_4(x)\\F_5(x)\\F_6(x)\\F_7(x)\end{pmatrix}=\begin{pmatrix}1&1&1&1&1&1&1\\1&1&1&1&1&1&1\\1&1&1&1&1&1&1\\1&0&0&0&1&1&1\\1&1&1&1&1&1&1\\1&0&0&0&1&1&1\\1&0&0&0&0&1&1\end{pmatrix}\cdot\begin{pmatrix}\mathscr{G}_1(x)\\\mathscr{G}_2(x)\\\mathscr{G}_3(x)\\\mathscr{G}_4(x)\\\mathscr{G}_5(x)\\\mathscr{G}_6(x)\\\mathscr{G}_7(x)\end{pmatrix}.$$

Hence, by (7.3.12), (7.3.13) and (7.3.14), if we put

$$F_1(x)=F_2(x)=F_3(x)=F_5(x)=\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2+3n_2^2+3n_1n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{7.3.15}$$

$$F_4(x)=F_6(x)=\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2+3n_2^2+3n_1n_2+n_1+3n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} \tag{7.3.16}$$

and

$$F_7(x)=\sum_{n_1,n_2\geq 0}\frac{q^{n_1^2+3n_2^2+3n_1n_2+2n_1+3n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{7.3.17}$$

then we have the following relation from (7.3.3):

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \\ F_6(x) \\ F_7(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & & & \\ & xq & & & & & \\ & & x^2q^3 & & & & \\ & & & x^2q^4 & & & \\ & & & & xq^2 & & \\ & & & & & xq^3 & \\ & & & & & & x^2q^6 \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^3) \\ F_2(xq^3) \\ F_3(xq^3) \\ F_4(xq^3) \\ F_5(xq^3) \\ F_6(xq^3) \\ F_7(xq^3) \end{pmatrix}. \quad (7.3.18)$$

Conversely, we are also able to recover

$$(\mathscr{G}_1(x), \mathscr{G}_2(x), \mathscr{G}_3(x), \mathscr{G}_4(x), \mathscr{G}_5(x), \mathscr{G}_6(x), \mathscr{G}_7(x))^T$$

as well as (7.3.12), (7.3.13) and (7.3.14) provided that we have proved (7.3.18) directly since $F_1(0) = F_2(0) = \cdots = F_7(0) = 1$.

7.3.3 A Matrix Factorization Problem

Motivated by (7.3.11) and (7.3.18), we turn our interest to a matrix factorization problem as follows.

Let R be a positive integer. Let $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathrm{Mat}_{R\times R}(\mathbb{N})$ be a fixed symmetric matrix. Let $\underline{\mathbf{A}} = (A_r) \in \mathbb{N}^R_{>0}$ and $\underline{\boldsymbol{\gamma}} = (\gamma_r) \in \mathbb{N}^R_{>0}$ be fixed.

Let $\mathfrak{F}$ be a set of q-multi-summations defined by

$$\mathfrak{F} := \Big\{ H(\underline{\boldsymbol{\beta}}) \,:\, \underline{\boldsymbol{\beta}} \in \mathbb{Z}^R \text{ and condition (7.3.21) is satisfied} \Big\}, \qquad (7.3.19)$$

where $H(\underline{\boldsymbol{\beta}}) = H(\beta_1, \ldots, \beta_R)$ is of the form

$$H(\underline{\boldsymbol{\beta}}) := \sum_{n_1,\ldots,n_R \geq 0} \frac{q^{\sum_{r=1}^R \alpha_{r,r} n_r(n_r-1)/2} q^{\sum_{1\leq i<j\leq R} \alpha_{i,j} n_i n_j} q^{\sum_{r=1}^R \beta_r n_r} x^{\sum_{r=1}^R \gamma_r n_r}}{(q^{A_1};q^{A_1})_{n_1} \cdots (q^{A_R};q^{A_R})_{n_R}} \qquad (7.3.20)$$

and the additional condition reads: for all $(n_1, \ldots, n_R) \in \mathbb{N}^R \backslash \{(0,0,\ldots,0)\}$,

$$\sum_{r=1}^R \frac{\alpha_{r,r} n_r(n_r-1)}{2} + \sum_{1\leq i<j\leq R} \alpha_{i,j} n_i n_j + \sum_{r=1}^R \beta_r n_r > 0. \qquad (7.3.21)$$

Now we consider a column functional vector

$$\underline{\mathbf{F}}_{\underline{\beta}}(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_K(x) \end{pmatrix} := \begin{pmatrix} H(\underline{\boldsymbol{\beta}}_1) \\ H(\underline{\boldsymbol{\beta}}_2) \\ \vdots \\ H(\underline{\boldsymbol{\beta}}_K) \end{pmatrix}, \tag{7.3.22}$$

where $H(\underline{\boldsymbol{\beta}}_k) \in \mathfrak{F}$ for all $1 \leq k \leq K$.

We expect $\underline{\mathbf{F}}_{\underline{\beta}}(x)$ to satisfy the following factorization property.

Factorization Property. Let $\mathscr{U}$ be a $(0,1)$-matrix such that all entries in the first row and column are 1. Let $\mathscr{V}$ be a diagonal matrix such that all (diagonal) entries are monic monomials in x and q with $\mathscr{V}_{1,1} = 1$. We say that $\underline{\mathbf{F}}_{\underline{\beta}}(x)$ satisfies the *Factorization Property* if

$$\underline{\mathbf{F}}_{\underline{\beta}}(x) = \mathscr{U}.\mathscr{V}.\underline{\mathbf{F}}_{\underline{\beta}}(xq^S) \tag{7.3.23}$$

for some positive integer S.

Example 7.3.1. In the example in §7.3.2.1, we have $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} 2 \end{pmatrix}$, $\underline{\boldsymbol{\gamma}} = (1)$, $\underline{\mathbf{A}} = (1)$ and

$$\underline{\mathbf{F}}_{\underline{\beta}}(x) = \begin{pmatrix} H(1) \\ H(1) \\ H(2) \end{pmatrix}.$$

Also, $S = 2$.

Example 7.3.2. In the example in §7.3.2.2, we have $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$, $\underline{\boldsymbol{\gamma}} = (1,2)$, $\underline{\mathbf{A}} = (1,3)$ and

$$\underline{\mathbf{F}}_{\underline{\beta}}(x) = \begin{pmatrix} H(1,3) \\ H(1,3) \\ H(1,3) \\ H(2,6) \\ H(1,3) \\ H(2,6) \\ H(3,6) \end{pmatrix}.$$

Also, $S = 3$.

7.4 Non-computer-assisted Proofs

The aim of this section is to prove Andrews–Gordon type generating function identities such as (7.3.12), (7.3.13) and (7.3.14) without computer assistance.

As we have seen in §7.3.2.2, to prove (7.3.12), (7.3.13) and (7.3.14), it suffices to show (7.3.18).

Our starting point is a recurrence relation enjoyed by $H(\beta_1, \ldots, \beta_R)$ defined in (7.3.20).

7.4.1 A Recurrence Relation

Recall that

$$\begin{aligned} &H(\beta_1, \ldots, \beta_R) \\ &\quad = \sum_{n_1,\ldots,n_R \geq 0} \frac{q^{\sum_{r=1}^R \alpha_{r,r} n_r(n_r-1)/2} q^{\sum_{1 \leq i<j \leq R} \alpha_{i,j} n_i n_j} q^{\sum_{r=1}^R \beta_r n_r} x^{\sum_{r=1}^R \gamma_r n_r}}{(q^{A_1}; q^{A_1})_{n_1} \cdots (q^{A_R}; q^{A_R})_{n_R}}. \end{aligned}$$

Lemma 7.4.1. *For $1 \leq r \leq R$, we have*

$$\begin{aligned} H(\beta_1, \ldots, \beta_r, \ldots, \beta_R) &= H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R) \\ &\quad + x^{\gamma_r} q^{\beta_r} H(\beta_1 + \alpha_{r,1}, \ldots, \beta_r + \alpha_{r,r}, \ldots, \beta_R + \alpha_{r,R}). \end{aligned} \tag{7.4.1}$$

Proof. We have (recall that $\underline{\boldsymbol{\alpha}}$ is a symmetric matrix so that $\alpha_{i,j} = \alpha_{j,i}$ for $1 \leq i, j \leq R$)

$$\begin{aligned} &H(\beta_1, \ldots, \beta_r, \ldots, \beta_R) - H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R) \\ &\quad = \sum_{n_1,\ldots,n_R \geq 0} \frac{q^{\sum_i \alpha_{i,i} n_i(n_i-1)/2} q^{\sum_{i<j} \alpha_{i,j} n_i n_j} q^{\sum_i \beta_i n_i} (1 - q^{n_r A_r}) x^{\sum_i \gamma_i n_i}}{(q^{A_1}; q^{A_1})_{n_1} \cdots (q^{A_r}; q^{A_r})_{n_r} \cdots (q^{A_R}; q^{A_R})_{n_R}} \\ &\quad = \sum_{\substack{n_1,\ldots,n_R \geq 0 \\ n_r \geq 1}} \frac{q^{\sum_i \alpha_{i,i} n_i(n_i-1)/2} q^{\sum_{i<j} \alpha_{i,j} n_i n_j} q^{\sum_i \beta_i n_i} x^{\sum_i \gamma_i n_i}}{(q^{A_1}; q^{A_1})_{n_1} \cdots (q^{A_r}; q^{A_r})_{n_r - 1} \cdots (q^{A_R}; q^{A_R})_{n_R}} \\ &\quad = x^{\gamma_r} q^{\beta_r} \sum_{n_1,\ldots,n_R \geq 0} \frac{q^{\sum_i \alpha_{i,i} n_i(n_i-1)/2} q^{\sum_{i<j} \alpha_{i,j} n_i n_j} q^{\sum_i (\beta_i + \alpha_{r,i}) n_i} x^{\sum_i \gamma_i n_i}}{(q^{A_1}; q^{A_1})_{n_1} \cdots (q^{A_r}; q^{A_r})_{n_r} \cdots (q^{A_R}; q^{A_R})_{n_R}} \\ &\quad = x^{\gamma_r} q^{\beta_r} H(\beta_1 + \alpha_{r,1}, \ldots, \beta_r + \alpha_{r,r}, \ldots, \beta_R + \alpha_{r,R}). \end{aligned}$$

The desired identity therefore follows. □

Remark 7.4.1. It is worth pointing out that the recurrence (7.4.1) and its relations to sum-like generating functions have connections with the theory of vertex operator algebras, especially in the context of principal subspaces of modules. For one recent example, see (4.71)–(4.81) in [139].

Remark 7.4.2. A recent paper of Ablinger and Uncu [1] also seems to outline some functionality regarding recurrences for q-multi-summations.

Recall that the Factorization Property says that

$$\underline{\mathbf{F}}_{\underline{\boldsymbol{\beta}}}(x) = \mathscr{U}.\mathscr{V}.\underline{\mathbf{F}}_{\underline{\boldsymbol{\beta}}}(xq^S).$$

Further, if $F(x) = H(\beta_1, \dots, \beta_R)$, then

$$F(xq^S) = H(\beta_1 + \gamma_1 S, \dots, \beta_R + \gamma_R S). \tag{7.4.2}$$

Probably, if we expect to apply Lemma 7.4.1 to deduce Andrews–Gordon type generating function identities, we need to attach some additional conditions to the Factorization Property.

Additional Conditions. For all $1 \le s \le R$:

(i). $\gamma_s S \in A_s\mathbb{Z}$;

(ii). for all $1 \le r \le R$, $\alpha_{r,s} \in A_s\mathbb{Z}$.

7.4.2 Proof of (7.3.11)

We first prove (7.3.11), which is relatively easy.

Theorem 7.4.2. *Let*

$$F_1(x) = F_2(x) = \sum_{n\ge 0} \frac{q^{n^2}x^n}{(q;q)_n} \tag{7.4.3}$$

and

$$F_3(x) = \sum_{n\ge 0} \frac{q^{n^2+n}x^n}{(q;q)_n}. \tag{7.4.4}$$

Then,

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & & \\ & xq & \\ & & xq^2 \end{pmatrix} . \begin{pmatrix} F_1(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \end{pmatrix} \tag{7.4.5}$$

We have shown in Example 7.3.1 that in this case $S = 2$, $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} 2 \end{pmatrix}$, $\underline{\boldsymbol{\gamma}} = (1)$, $\underline{\mathbf{A}} = (1)$ and

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} H(1) \\ H(1) \\ H(2) \end{pmatrix}.$$

Further, it follows from (7.4.2) that

$$F_1(xq^2) = F_2(xq^2) = H(3) \tag{7.4.6}$$

and

$$F_3(xq^2) = H(4). \tag{7.4.7}$$

To prove (7.4.5), it suffices to show that

$$F_1(x) = F_1(xq^2) + xqF_2(xq^2) + xq^2F_3(xq^2) \tag{7.4.8}$$

and

$$F_3(x) = F_1(xq^2) + xq^2F_3(xq^2). \tag{7.4.9}$$

It follows from Lemma 7.4.1 that

$$\begin{aligned} F_1(x) &= H(1) \\ &= H(1+1) + xqH(1+2) \\ &= H(2) + xqH(3) \\ &= \Big(H(2+1) + xq^2H(2+2)\Big) + xqH(3) \\ &= H(3) + xq^2H(4) + xqH(3) \\ &= F_1(xq^2) + xq^2F_3(xq^2) + xqF_2(xq^2). \end{aligned}$$

Also,

$$\begin{aligned} F_3(x) &= H(2) \\ &= H(2+1) + xq^2H(2+2) \\ &= H(3) + xq^2H(4) \end{aligned}$$

$$= F_1(xq^2) + xq^2F_3(xq^2).$$

Identities (7.4.8) and (7.4.9) are therefore proved.

7.4.3 Proof of (7.3.18)

We next prove (7.3.18).

Theorem 7.4.3. *Let*

$$F_1(x) = F_2(x) = F_3(x) = F_5(x) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{7.4.10}$$

$$F_4(x) = F_6(x) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2+n_1+3n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} \tag{7.4.11}$$

and

$$F_7(x) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_2^2+3n_1n_2+2n_1+3n_2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{7.4.12}$$

Then,

$$\begin{pmatrix} F_1(x)\\ F_2(x)\\ F_3(x)\\ F_4(x)\\ F_5(x)\\ F_6(x)\\ F_7(x)\end{pmatrix} = \begin{pmatrix} 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&0&0&0&1&1&1\\ 1&1&1&1&1&1&1\\ 1&0&0&0&1&1&1\\ 1&0&0&0&0&1&1\end{pmatrix} \cdot \begin{pmatrix} 1&&&&&&\\ &xq&&&&&\\ &&x^2q^3&&&&\\ &&&x^2q^4&&&\\ &&&&xq^2&&\\ &&&&&xq^3&\\ &&&&&&x^2q^6\end{pmatrix} \cdot \begin{pmatrix} F_1(xq^3)\\ F_2(xq^3)\\ F_3(xq^3)\\ F_4(xq^3)\\ F_5(xq^3)\\ F_6(xq^3)\\ F_7(xq^3)\end{pmatrix}. \tag{7.4.13}$$

We have shown in Example 7.3.2 that in this case $S = 3$, $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} 2 & 3\\ 3 & 6\end{pmatrix}$, $\underline{\boldsymbol{\gamma}} = (1,2)$, $\underline{\mathbf{A}} = (1,3)$ and

$$\begin{pmatrix} F_1(x)\\ F_2(x)\\ F_3(x)\\ F_4(x)\\ F_5(x)\\ F_6(x)\\ F_7(x)\end{pmatrix} = \begin{pmatrix} H(1,3)\\ H(1,3)\\ H(1,3)\\ H(2,6)\\ H(1,3)\\ H(2,6)\\ H(3,6)\end{pmatrix}.$$

Again, it follows from (7.4.2) that

$$F_1(xq^3) = F_2(xq^3) = F_3(xq^3) = F_5(xq^3) = H(4,9), \tag{7.4.14}$$
$$F_4(xq^3) = F_6(xq^3) = H(5,12) \tag{7.4.15}$$

and

$$F_7(xq^3) = H(6,12). \tag{7.4.16}$$

To prove (7.4.5), it suffices to show that

$$F_1(x) = \left\{ \begin{array}{l} F_1(xq^3) + xqF_2(xq^3) + x^2q^3F_3(xq^3) + x^2q^4F_4(xq^3) \\ +\, xq^2F_5(xq^3) + xq^3F_6(xq^3) + x^2q^6F_7(xq^3) \end{array} \right\}, \tag{7.4.17}$$
$$F_4(x) = F_1(xq^3) + xq^2F_5(xq^3) + xq^3F_6(xq^3) + x^2q^6F_7(xq^3) \tag{7.4.18}$$

and

$$F_7(x) = F_1(xq^2) + xq^3F_6(xq^3) + x^2q^6F_7(xq^3). \tag{7.4.19}$$

We will adopt the following notation to make our argument more transparent. First, a term in gray indicates that we will apply Lemma 7.4.1 to this term. Also, if Lemma 7.4.1 is applied to one coordinate, then that coordinate will be shown in boldface. Finally, the two underlined terms in the next line are deduced from the previous gray term by Lemma 7.4.1.

It follows from Lemma 7.4.1 that

$$\begin{aligned}
F_1(x) &= {\color{gray}H(1,\mathbf{3})} \\
&= \underline{{\color{gray}H(\mathbf{1},6)}} + \underline{x^2q^3H(4,9)} \\
&= \underline{{\color{gray}H(\mathbf{2},6)}} + \underline{xqH(3,9)} + x^2q^3H(4,9) \\
&= \underline{H(3,6)} + \underline{xq^2H(4,9)} + {\color{gray}xqH(\mathbf{3},9)} + x^2q^3H(4,9) \\
&= {\color{gray}H(3,\mathbf{6})} + xq^2H(4,9) + \underline{xqH(4,9)} + \underline{x^2q^4H(5,12)} + x^2q^3H(4,9) \\
&= \underline{{\color{gray}H(\mathbf{3},9)}} + \underline{x^2q^6H(6,12)} + xq^2H(4,9) + xqH(4,9) + x^2q^4H(5,12) \\
&\quad + x^2q^3H(4,9)
\end{aligned}$$

$$
\begin{aligned}
&= H(4,9) + xq^3H(5,12) + x^2q^6H(6,12) + xq^2H(4,9) + xqH(4,9) \\
&\quad + x^2q^4H(5,12) + x^2q^3H(4,9) \\
&= F_1(xq^3) + xq^3F_6(xq^3) + x^2q^6F_7(xq^3) + xq^2F_5(xq^3) + xqF_2(xq^3) \\
&\quad + x^2q^4F_4(xq^3) + x^2q^3F_3(xq^3).
\end{aligned}
$$

Also,

$$
\begin{aligned}
F_4(x) &= H(\mathbf{2},6) \\
&= H(3,\mathbf{6}) + xq^2H(4,9) \\
&= H(\mathbf{3},9) + x^2q^6H(6,12) + xq^2H(4,9) \\
&= H(4,9) + xq^3H(5,12) + x^2q^6H(6,12) + xq^2H(4,9) \\
&= F_1(xq^3) + xq^3F_6(xq^3) + x^2q^6F_7(xq^3) + xq^2F_5(xq^3).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
F_7(x) &= H(3,\mathbf{6}) \\
&= H(\mathbf{3},9) + x^2q^6H(6,12) \\
&= H(4,9) + xq^3H(5,12) + x^2q^6H(6,12) \\
&= F_1(xq^3) + xq^3F_6(xq^3) + x^2q^6F_7(xq^3).
\end{aligned}
$$

Identities (7.4.17), (7.4.18) and (7.4.19) are therefore proved.

Figure 7.2. Node $H(\beta_1,\dots,\beta_r,\dots,\beta_R)$ and its children

$$
\begin{array}{ccc}
 & H(\beta_1,\dots,\boldsymbol{\beta_r},\dots,\beta_R) & \\
\overset{1}{\swarrow} & & \overset{x^{\gamma_r}q^{\beta_r}}{\searrow} \\
H(\beta_1,\dots,\beta_r+A_r,\dots,\beta_R) & & H(\beta_1+\alpha_{r,1},\dots,\beta_r+\alpha_{r,r},\dots,\beta_R+\alpha_{r,R})
\end{array}
$$

7.4.4 Binary Trees

Interestingly, the previous two proofs can be represented nicely by binary trees. More precisely, all nodes are of the form $H(\beta_1,\dots,\beta_r,\dots,\beta_R)$. Then Lemma 7.4.1 gives two children of $H(\beta_1,\dots,\beta_r,\dots,\beta_R)$: the left child is $H(\beta_1,\dots,\beta_r+A_r,\dots,\beta_R)$, weighted by 1, and the right child is $H(\beta_1+\alpha_{r,1},\ldots,\beta_r+\alpha_{r,r},\ldots,\beta_R+\alpha_{r,R})$, weighted by $x^{\gamma_r}q^{\beta_r}$. See Figure 7.2.

Now the proofs of (7.3.11) and (7.3.18) can be illustrated by Figures 7.3 and 7.4, respectively.

Figure 7.3. The binary tree for (7.3.11)

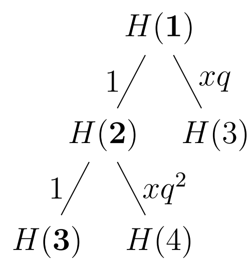
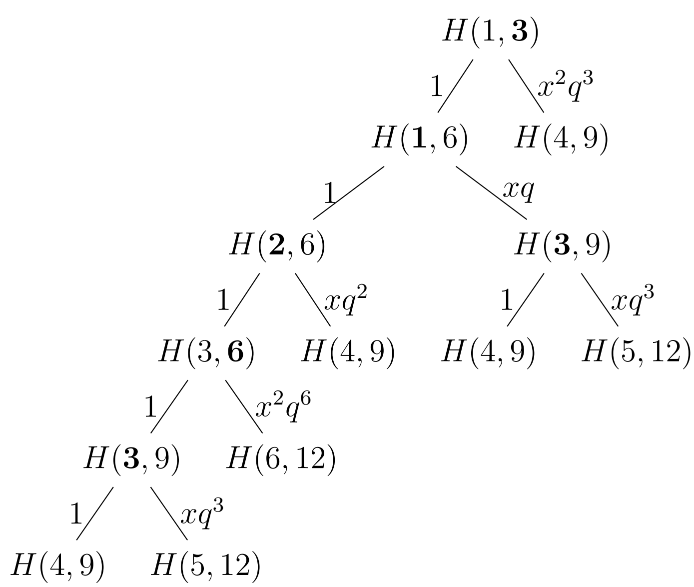


Figure 7.4. The binary tree for (7.3.18)



In fact, it is relatively easy to deduce other much more complicated identities of the same flavor as (7.3.11) and (7.3.18). For example, the next result follows from the binary tree in Figure 7.5.

Theorem 7.4.4. *Let*

$$F_1(x)=\cdots=F_6(x)$$

$$
= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{n_1}{2}-n_2-\frac{n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}, \tag{7.4.20}
$$

$$
\begin{aligned}
&F_7(x) = \cdots = F_{13}(x) \\
&\quad = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{3n_1}{2}+n_2+\frac{5n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}},
\end{aligned} \tag{7.4.21}
$$

$$
\begin{aligned}
&F_{14}(x) = \cdots = F_{21}(x) \\
&\quad = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{3n_1}{2}+3n_2+\frac{11n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}
\end{aligned} \tag{7.4.22}
$$

and

$$
\begin{aligned}
&F_{22}(x) = F_{23}(x) \\
&\quad = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{\frac{n_1^2}{2}+3n_2^2+\frac{9n_3^2}{2}+2n_1n_2+6n_2n_3+3n_3n_1+\frac{5n_1}{2}+3n_2+\frac{11n_3}{2}} x^{n_1+2n_2+3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}.
\end{aligned} \tag{7.4.23}
$$

Let

$$
\mathscr{A} = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0
\end{pmatrix}
$$

and

$$\begin{aligned}\mathscr{W}(x) = \mathrm{diag}(&1, xq^2, xq, x^2q^3, x^2q^2, x^3q^4,\\ &xq^3, x^2q^5, x^2q^4, x^3q^7, x^2q^4, x^3q^6, x^3q^5,\\ &x^2q^7, x^2q^6, x^3q^9, x^3q^8, x^3q^8, x^3q^7, x^4q^{10}, x^4q^9,\\ &x^3q^{10}, x^4q^{11}).\end{aligned}$$

Then,

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_{23}(x) \end{pmatrix} = \mathscr{A}.\mathscr{W}(x). \begin{pmatrix} F_1(xq^3) \\ F_2(xq^3) \\ \vdots \\ F_{23}(xq^3) \end{pmatrix}. \tag{7.4.24}$$

Remark 7.4.3. It is worth pointing out that the q-multi-summations in this theorem are similar to those appear in [109, (47) and (51)].

Proof. Let $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 9 \end{pmatrix}$, $\underline{\boldsymbol{\gamma}} = (1,2,3)$, $\underline{\mathbf{A}} = (1,2,3)$ and $S = 3$. We have

$$\begin{aligned} F_1(x) = \cdots = F_6(x) = H(1,2,4) \quad &\xrightarrow{x \mapsto xq^3} \quad H(4,8,13),\\ F_7(x) = \cdots = F_{13}(x) = H(2,4,7) \quad &\xrightarrow{x \mapsto xq^3} \quad H(5,10,16),\\ F_{14}(x) = \cdots = F_{21}(x) = H(2,6,10) \quad &\xrightarrow{x \mapsto xq^3} \quad H(5,12,19) \end{aligned}$$

and

$$F_{22}(x) = F_{23}(x) = H(3,6,10) \quad \xrightarrow{x \mapsto xq^3} \quad H(6,12,19).$$

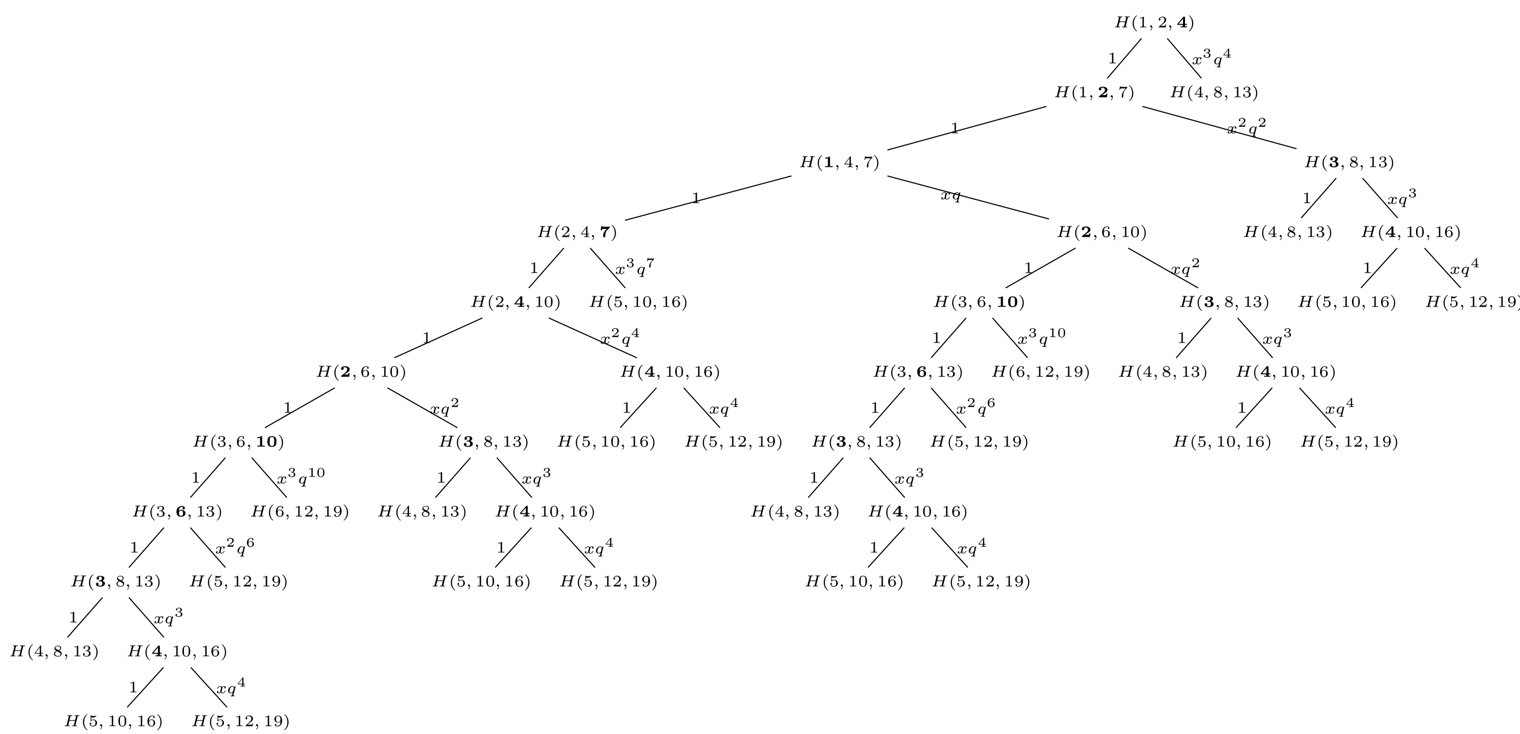
The rest follows from the binary tree in Figure 7.5. □

It looks like one cannot deduce a span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L} \rangle, S)$ from Theorem 7.4.4. This is because by (7.4.24), we need $S = 3$. But in the diagonal matrix $\mathscr{W}(x)$, there is a term x^2q^7, which induces a partition of size 7 that has two parts. This means that one of the parts is larger than 3. However, for a span one linked partition ideal, we require that all parts in partitions among Π must not exceed S.

On the other hand, we will show in the next corollary that Theorem 7.4.4 still corresponds to a partition set.

Recall that $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 9 \end{pmatrix}$, $\underline{\boldsymbol{\gamma}} = (1,2,3)$, $\underline{\mathbf{A}} = (1,2,3)$ and $S = 3$.

Figure 7.5. The binary tree for (7.4.24)



Corollary 7.4.5. *Let* $\Pi = \{\pi_1, \pi_2, \ldots, \pi_{23}\}$ *be a set of integer partitions where*

$$\left[\begin{array}{llll}
\pi_1 = \varnothing & \pi_2 = 2 & \pi_3 = 1 & \pi_4 = 2+1 \\
\pi_5 = 1+1 & \pi_6 = 2+1+1 & & \\
\hline
\pi_7 = 3 & \pi_8 = 3+2 & \pi_9 = 2+2 & \pi_{10} = 3+2+2 \\
\pi_{11} = 3+1 & \pi_{12} = 2+2+2 & \pi_{13} = 3+1+1 & \\
\hline
\pi_{14} = 4+3 & \pi_{15} = 3+3 & \pi_{16} = 3+3+3 & \pi_{17} = 3+3+2 \\
\pi_{18} = 4+2+2 & \pi_{19} = 3+2+2 & \pi_{20} = 3+3+2+2 & \pi_{21} = 3+2+2+2 \\
\hline
\pi_{22} = 4+3+3 & \pi_{23} = 3+3+3+2 & &
\end{array}\right].$$

Let $\mathcal{L} : \Pi \to P(\Pi)$ *where* $P(\Pi)$ *is the power set of* Π *be defined by*

$$\mathcal{L}(\pi_i) = \begin{cases} \{\pi_1, \pi_2, \ldots, \pi_{23}\} & \text{for } 1 \le i \le 6, \\ \{\pi_1, \pi_2, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{14}, \pi_{15}, \pi_{16}, \pi_{17}, \pi_{22}\} & \text{for } 7 \le i \le 13, \\ \{\pi_1, \pi_2, \pi_7, \pi_8, \pi_{14}, \pi_{15}, \pi_{16}, \pi_{22}\} & \text{for } 14 \le i \le 21, \\ \{\pi_1, \pi_7, \pi_{14}, \pi_{15}, \pi_{22}\} & \text{for } 22 \le i \le 23. \end{cases}$$

Let

$$\mathcal{C}_\lambda : \lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots \to \lambda_K \to \varnothing \to \varnothing \to \cdots$$

be a chain such that for all $i \ge 0$*,* $\lambda_i \in \Pi$ *and* $\lambda_{i+1} \in \mathcal{L}(\lambda_i)$*. Let* Φ_λ *be an integer partition induced from* $\mathcal{C}_\lambda$ *defined as in* (6.1.3) *with* $S = 3$*:*

$$\Phi_\lambda = \lambda_0 \oplus \phi^3(\lambda_1) \oplus \phi^6(\lambda_2) \oplus \cdots \oplus \phi^{3K}(\lambda_K) \oplus \phi^{3(K+1)}(\varnothing) \oplus \phi^{3(K+2)}(\varnothing) \oplus \cdots .$$

Let $\mathscr{S}$ *be the set of such partitions* Φ_λ*. Then,*

$$\sum_{v \in \mathscr{S}} x^{\sharp(v)} q^{|v|} = \sum_{n_1, n_2, n_3 \ge 0} \frac{q^{\frac{n_1^2}{2} + 3n_2^2 + \frac{9n_3^2}{2} + 2n_1n_2 + 6n_2n_3 + 3n_3n_1 + \frac{n_1}{2} - n_2 - \frac{n_3}{2}} x^{n_1 + 2n_2 + 3n_3}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}. \tag{7.4.25}$$

Proof. First, it is easy to see that given a chain $\mathcal{C}_\lambda$, the induced Φ_λ is indeed an integer partition. Now we claim that for any two chains $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$, we have $\Phi_\mu = \Phi_\nu$ if and only if $\mathcal{C}_\mu = \mathcal{C}_\nu$. Notice that the "if" part is trivial.

We show the "only if" part by contradiction. Namely, we assume that there are two chains $\mathcal{C}_\mu \neq \mathcal{C}_\nu$ such that $\Phi_\mu = \Phi_\nu$. Let ℓ be the index such that $\mu_\ell \neq \nu_\ell$ and $\mu_i = \nu_i$ for $0 \le i \le \ell - 1$. If neither μ_ℓ nor ν_ℓ contains a part of size 4, then the parts in Φ_μ of size up to $3(\ell+1)$ are given by $\oplus_{i=0}^{\ell} \phi^{3i}(\mu_i)$ and similarly the parts in Φ_ν of size up to $3(\ell+1)$ are given by $\oplus_{i=0}^{\ell} \phi^{3i}(\nu_i)$. Since $\Phi_\mu = \Phi_\nu$ and $\mu_i = \nu_i$ for $0 \le i \le \ell - 1$ as assumed, it follows that $\phi^{3\ell}(\mu_\ell) = \phi^{3\ell}(\nu_\ell)$ so that $\mu_\ell = \nu_\ell$. This contradicts the assumption that $\mu_\ell \neq \nu_\ell$. If 4 is a part in one of μ_ℓ and ν_ℓ, then without loss of generality, we assume that 4 is a part in μ_ℓ. Then $\mu_\ell \in \{\pi_{14}, \pi_{18}, \pi_{22}\}$. Apparently, if 4 is also a part in ν_ℓ, we must have $\nu_\ell = \mu_\ell$, which violates the assumption. Now let us assume that 4 is not a part in ν_ℓ. Since $\Phi_\mu = \Phi_\nu$, we know that 1 must be a part in $\nu_{\ell+1}$; otherwise, Φ_ν contains no parts of size $3\ell + 4$. Thus, $\nu_{\ell+1} \in \{\pi_3, \pi_4, \pi_5, \pi_6, \pi_{11}, \pi_{13}\}$. Since $\nu_{\ell+1} \in \mathcal{L}(\nu_\ell)$, we find that $\nu_\ell \in \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$ and also the parts in Φ_ν of size up to $3(\ell+1)$ are given by $(\oplus_{i=0}^{\ell-1} \phi^{3i}(\nu_i)) \oplus \phi^{3\ell}(\nu_\ell)$. On the other hand, since $\mu_\ell \in \{\pi_{14}, \pi_{18}, \pi_{22}\}$, the parts in Φ_μ of size up to $3(\ell+1)$ are $\oplus_{i=0}^{\ell-1} \phi^{3i}(\mu_i)$ plus one of $\phi^{3\ell}(3)$, $\phi^{3\ell}(2+2)$ or $\phi^{3\ell}(3+3)$ none of which could be $\phi^{3\ell}(\nu_\ell)$. This implies that $\Phi_\mu \neq \Phi_\nu$, which leads to a contradiction.

Once we have shown that the induced partitions Φ_λ are pairwise distinct, the rest is a simple application of the framework developed in this paper by first constructing the associated directed graph as in Section 7.2.3. We leave this as an exercise to the interested reader. $\square$

7.5 Open Problems

Our main concern is about the Factorization Property. Recall that $\mathscr{U}$ is a $(0,1)$-matrix such that all entries in the first row and column are 1, and $\mathscr{V}$ is a diagonal matrix such that all (diagonal) entries are monic monomials in x and q with $\mathscr{V}_{1,1} = 1$. The Factorization Property says that

$$\underline{\mathbf{F}}_{\underline{\beta}}(x) = \mathscr{U} . \mathscr{V} . \underline{\mathbf{F}}_{\underline{\beta}}(xq^S), \tag{7.5.1}$$

where S is a positive integer and

$$\underline{\mathbf{F}}_{\underline{\beta}}(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_K(x) \end{pmatrix} = \begin{pmatrix} H(\underline{\boldsymbol{\beta}}_1) \\ H(\underline{\boldsymbol{\beta}}_2) \\ \vdots \\ H(\underline{\boldsymbol{\beta}}_K) \end{pmatrix},$$

in which $H(\underline{\boldsymbol{\beta}}) = H(\beta_1, \ldots, \beta_R)$ is of the form

$$H(\underline{\boldsymbol{\beta}}) = \sum_{n_1,\ldots,n_R\geq 0} \frac{q^{\sum_{r=1}^{R}\alpha_{r,r}n_r(n_r-1)/2} q^{\sum_{1\leq i<j\leq R}\alpha_{i,j}n_i n_j} q^{\sum_{r=1}^{R}\beta_r n_r} x^{\sum_{r=1}^{R}\gamma_r n_r}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_R};q^{A_R})_{n_R}}.$$

Probably we also require the Additional Conditions: for all $1 \leq s \leq R$:

(i). $\gamma_s S \in A_s\mathbb{Z}$;

(ii). for all $1 \leq r \leq R$, $\alpha_{r,s} \in A_s\mathbb{Z}$.

Problem 7.5.1. For given $\mathscr{U}$ and $\mathscr{V}$, is it possible to determine if there exist $\mathbf{F}_{\underline{\boldsymbol{\beta}}}(x)$ and S such that (7.5.1) is true?

We have another problem from a different direction.

Problem 7.5.2. Are there any criteria of $\mathbf{F}_{\underline{\boldsymbol{\beta}}}(x)$ that we are always able to find $\mathscr{U}$, $\mathscr{V}$ and S such that (7.5.1) is true?

The last problem is probably simpler.

Problem 7.5.3. Can we construct a family of $\mathscr{U}$, $\mathscr{V}$, $\mathbf{F}_{\underline{\boldsymbol{\beta}}}(x)$ and S such that (7.5.1) holds?

If we are able to find such construction, then we may derive a family of span one linked partition ideals (or at least a family of modified directed graphs) with nice analytic generation functions.

7.6 References

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Chapter 8

Span One Linked Partition Ideals: Gleißberg's Identity

This chapter comes from

- S. Chern, On a Rogers–Ramanujan type identity of Gleißberg, preprint. (Ref. [63])

In the previous two chapters, a general theory on span one linked partition ideals is introduced. Now it is time to return to where the race starts. That is, we will truely prove a Rogers–Ramanujan type identity, instead of just some Andrews–Gordon type generating function identities.

8.1 Main Result

Recall that in 1926, Schur [160] proved the following Rogers–Ramanujan type identity.

Theorem 8.1.1 (Schur)**.** *Let $A(n)$ denote the number of partitions of n into distinct parts congruent to ± 1 modulo 3.*

Let $C(n)$ denote the number of partitions of n such that the difference between two consecutive parts is at least 3 and greater than 3 if the smaller part is a multiple of 3.

Then,

$$A(n) = C(n). \tag{8.1.1}$$

Two years later, in 1928, Gleißberg [84] further provided an extension of Schur's identity.

Theorem 8.1.2 (Gleißberg)**.** *Let m and r be positive integers with $r < m/2$. Let $A_{m,r}(n)$ denote the number of partitions of n into distinct parts congruent to $\pm r$ modulo m, and let $A_{m,r}(k,n)$ denote the number of partitions of n counted by $A_{m,r}(n)$ with k parts.*

Let $C_{m,r}(n)$ denote the number of partitions of n into parts congruent to 0 or $\pm r$ modulo m such that the difference between two consecutive parts is at least m and greater *than m if the smaller part is a multiple of m. Further , let $C_{m,r}(k,n)$ denote the number of partitions of n counted by $C_{m,r}(n)$ such that the number of parts plus the number of multiples of m among the parts equals k.*

Then,

$$A_{m,r}(k,n) = C_{m,r}(k,n). \tag{8.1.2}$$

The object of this chapter is to not only reprove Gleißberg's identity but also show the following analog.

Theorem 8.1.3. *Let m be a positive even integer and let r be a positive integer with $r < m/2$.*

Let $B_{m,r}(n)$ denote the number of partitions of n into parts congruent to $\pm r$ modulo m appearing exactly once and parts congruent to $\frac{m}{2}$ or 0 modulo m appearing exactly twice such that the difference between two consecutive parts that are distinct is at least m if the smaller part is congruent to $\pm r$ modulo m, at least $\frac{m}{2} + r$ if the smaller part is congruent to $\frac{m}{2}$ modulo m, and at least $m - r$ if the smaller part is congruent to 0 modulo m, and let $B_{m,r}(k,n)$ denote the number of partitions of n counted by $B_{m,r}(n)$ with k parts.

Then,

$$A_{m,r}(k,n) = B_{m,r}(k,n). \tag{8.1.3}$$

Remark. For partitions counted by $B_{m,r}$, we allow those like (r), $(m-r)$, $(\frac{m}{2}, \frac{m}{2})$ or (m,m). But partitions (r,r) and $(m-r, m-r)$ are not allowed since r and $m-r$ appear twice. Partitions $(\frac{m}{2})$, (m) and $(\frac{m}{2}, m)$ are not allowed since $\frac{m}{2}$ and m do not appear exactly twice, and in the last case the difference conditions are also not satisfied.

Examples. (i). Let $m = 4$ and $r = 1$. Partitions of 16 counted by $A_{4,1}(16)$ are $15+1$, $13+3$, $11+5$, $9+7$ and $7+5+3+1$. Partitions of 16 counted by $B_{4,1}(16)$ are $15+1$, $13+3$, $11+5$, $8+8$ and $6+6+2+2$. Hence, $A_{4,1}(2,16) = B_{4,1}(2,16) = 4$, $A_{4,1}(4,16) = B_{4,1}(4,16) = 1$ and $A_{4,1}(k,16) = B_{4,1}(k,16) = 0$ otherwise.

(ii). Let $m = 6$ and $r = 2$. Partitions of 22 counted by $A_{6,2}(22)$ are 22, $20+2$, $14+8$, $16+4+2$ and $10+8+4$. Partitions of 22 counted by $B_{6,2}(22)$ are 22, $20+2$, $14+8$, $16+3+3$ and $10+6+6$. Hence, $A_{6,2}(1,22) = B_{6,2}(1,22) = 1$, $A_{6,2}(2,22) = B_{6,2}(2,22) = 2$, $A_{6,2}(3,22) = B_{6,2}(3,22) = 2$ and $A_{6,2}(k,22) = B_{6,2}(k,22) = 0$ otherwise.

8.2 In the Setting of Span One Linked Partition Ideals

Let $\mathscr{B}_{m,r}(n)$ denote the set of partitions counted by $B_{m,r}(n)$ and let $\mathscr{B}_{m,r} = \cup_{n\geq 0}\mathscr{B}_{m,r}(n)$. We will interpret $\mathscr{B}_{m,r}$ in terms of span one linked partition ideals.

Claim 8.2.1. *$\mathscr{B}_{m,r}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi_B, \mathcal{L}_B\rangle, S)$ where $S = m$, and $\Pi_B = \{\pi_{B,1}, \pi_{B,2}, \pi_{B,3}, \pi_{B,4}, \pi_{B,5}\}$ along with the linking sets given as follows.*

Π_B	*linking set*
$\pi_{B,1} = \emptyset$	$\{\pi_{B,1}, \pi_{B,2}, \pi_{B,3}, \pi_{B,4}, \pi_{B,5}\}$
$\pi_{B,2} = (r)$	$\{\pi_{B,1}, \pi_{B,2}, \pi_{B,3}, \pi_{B,4}, \pi_{B,5}\}$
$\pi_{B,3} = (\frac{m}{2}, \frac{m}{2})$	$\{\pi_{B,1}, \pi_{B,2}, \pi_{B,3}, \pi_{B,4}, \pi_{B,5}\}$
$\pi_{B,4} = (m-r)$	$\{\pi_{B,1}, \pi_{B,4}, \pi_{B,5}\}$
$\pi_{B,5} = (m,m)$	$\{\pi_{B,1}, \pi_{B,4}, \pi_{B,5}\}$

Proof. A straightforward verification tells us that any partition in $\mathscr{I}(\langle \Pi_B, \mathcal{L}_B\rangle, S)$ is in $\mathscr{B}_{m,r}$.

On the other hand, given a partition $\lambda \in \mathscr{B}_{m,r}$, we decompose it as

$$\lambda_0 \oplus \phi^m(\lambda_1) \oplus \phi^{m\cdot 2}(\lambda_2) \oplus \cdots \oplus \phi^{mK}(\lambda_K).$$

Note that for $0 \leq k \leq K$, $\phi^{mk}(\lambda_k)$ is simply the collection of parts in λ of size between $mk+1$ and $mk+m$. First, to ensure the difference conditions, we must have $\lambda_k \in \Pi$ for all k. Now fix some $k \geq 0$. If $\lambda_k = \pi_{B,1} = \emptyset$, then there are no parts in λ of size between $mk+1$ and $mk+m$. Hence, λ_{k+1} can be any partition in Π_B so that $\lambda_{k+1} \in \mathcal{L}_B(\pi_{B,1})$. If $\lambda_k = \pi_{B,2} = (r)$, then λ has one part of size $mk+r$. Now to satisfy the difference conditions, we have five choices for λ_{k+1} (here we only enumerate parts of size between $m(k+1)+1$ and $m(k+1)+m$):

(i). λ has no parts of size between $m(k+1)+1$ and $m(k+1)+m$ and hence $\lambda_{k+1} = \emptyset = \pi_{B,1}$;

(ii). λ has only one part of size $m(k+1)+r$ and hence $\lambda_{k+1} = (r) = \pi_{B,2}$;

(iii). λ has only two parts of size $m(k+1)+\frac{m}{2}$ and hence $\lambda_{k+1} = (\frac{m}{2}, \frac{m}{2}) = \pi_{B,3}$;

(iv). λ has only one part of size $m(k+1)+m-r$ and hence $\lambda_{k+1} = (m-r) = \pi_{B,4}$;

(v). λ has only two parts of size $m(k+1)+m$ and hence $\lambda_{k+1} = (m,m) = \pi_{B,5}$.

Hence, $\lambda_{k+1} \in \mathcal{L}_B(\pi_{B,2})$. For other cases, one has similar arguments. Hence, λ is in $\mathscr{I}(\langle \Pi_B, \mathcal{L}_B \rangle, S)$.

Consequently, $\mathscr{B}_{m,r} = \mathscr{I}(\langle \Pi_B, \mathcal{L}_B \rangle, S)$. $\square$

Let $\mathscr{C}_{m,r}(n)$ denote the set of partitions counted by $C_{m,r}(n)$ and let $\mathscr{C}_{m,r} = \cup_{n \geq 0} \mathscr{C}_{m,r}(n)$.

Claim 8.2.2. *$\mathscr{C}_{m,r}$ is a span one linked partition ideal $\mathscr{I}(\langle \Pi_C, \mathcal{L}_C \rangle, S)$ where $S = m$, and $\Pi_C = \{\pi_{C,1}, \pi_{C,2}, \pi_{C,3}, \pi_{C,4}\}$ along with the linking sets given as follows.*

Π_C	*linking set*
$\pi_{C,1} = \emptyset$	$\{\pi_{C,1}, \pi_{C,2}, \pi_{C,3}, \pi_{C,4}\}$
$\pi_{C,2} = (r)$	$\{\pi_{C,1}, \pi_{C,2}, \pi_{C,3}, \pi_{C,4}\}$
$\pi_{C,3} = (m-r)$	$\{\pi_{C,1}, \pi_{C,3}, \pi_{C,4}\}$
$\pi_{C,4} = (m)$	$\{\pi_{C,1}\}$

Proof. The proof is analogous to that of Claim 8.2.1 and is therefore omitted. $\square$

8.3 A Refinement of Lemma 7.4.1

Let us turn to a refinement of Lemma 7.4.1.

As in Lemma 7.4.1, let R be a fixed positive integer. Let the symmetric matrix $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathrm{Mat}_{R\times R}(\mathbb{N})$ and the vector $\underline{\mathbf{A}} = (A_r) \in \mathbb{N}_{>0}^R$ be fixed. This time we will fix J vectors $\underline{\boldsymbol{\gamma_j}} = (\gamma_{j,r}) \in \mathbb{N}_{\geq 0}^R$ for $j = 1, 2, \ldots, J$. Let $x_1, x_2, \ldots, x_J$ and q be intermediates such that the following q-multi-summation $H(\underline{\boldsymbol{\beta}}) = H(\beta_1, \ldots, \beta_R)$ converges.

$$\begin{aligned} H(\underline{\boldsymbol{\beta}}) := \sum_{n_1,\ldots,n_R \geq 0} & \frac{q^{\sum_{r=1}^R \alpha_{r,r} n_r(n_r-1)/2} q^{\sum_{1\leq i<j\leq R} \alpha_{i,j} n_i n_j} q^{\sum_{r=1}^R \beta_r n_r}}{(q^{A_1}; q^{A_1})_{n_1} \cdots (q^{A_R}; q^{A_R})_{n_R}} \\ & \times x_1^{\sum_{r=1}^R \gamma_{1,r} n_r} \cdots x_J^{\sum_{r=1}^R \gamma_{J,r} n_r}. \end{aligned} \tag{8.3.1}$$

Lemma 8.3.1. *For $1 \leq r \leq R$, we have*

$$\begin{aligned} H(\beta_1, \ldots, \beta_r, \ldots, \beta_R) &= H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R) \\ &\quad + x_1^{\gamma_{1,r}} \cdots x_J^{\gamma_{J,r}} q^{\beta_r} H(\beta_1 + \alpha_{r,1}, \ldots, \beta_r + \alpha_{r,r}, \ldots, \beta_R + \alpha_{r,R}). \end{aligned} \tag{8.3.2}$$

Proof. We have (recall that $\underline{\boldsymbol{\alpha}}$ is symmetric so that $\alpha_{i,j} = \alpha_{j,i}$ for $1 \leq i, j \leq R$)

$$H(\beta_1, \ldots, \beta_r, \ldots, \beta_R) - H(\beta_1, \ldots, \beta_r + A_r, \ldots, \beta_R)$$

$$
\begin{aligned}
&= \sum_{n_1,\ldots,n_R\geq 0} \frac{q^{\sum_i \alpha_{i,i}n_i(n_i-1)/2} q^{\sum_{i<j}\alpha_{i,j}n_in_j} q^{\sum_i \beta_i n_i}(1-q^{n_rA_r})}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}\cdots(q^{A_R};q^{A_R})_{n_R}}\\
&\quad\times x_1^{\sum_i \gamma_{1,i}n_i}\cdots x_J^{\sum_i \gamma_{J,i}n_i}\\
&= \sum_{\substack{n_1,\ldots,n_R\geq 0\\ n_r\geq 1}} \frac{q^{\sum_i \alpha_{i,i}n_i(n_i-1)/2} q^{\sum_{i<j}\alpha_{i,j}n_in_j} q^{\sum_i \beta_i n_i}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r-1}\cdots(q^{A_R};q^{A_R})_{n_R}}\\
&\quad\times x_1^{\sum_i \gamma_{1,i}n_i}\cdots x_J^{\sum_i \gamma_{J,i}n_i}\\
&= x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r} \sum_{n_1,\ldots,n_R\geq 0} \frac{q^{\sum_i \alpha_{i,i}n_i(n_i-1)/2} q^{\sum_{i<j}\alpha_{i,j}n_in_j} q^{\sum_i (\beta_i+\alpha_{r,i}) n_i}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}\cdots(q^{A_R};q^{A_R})_{n_R}}\\
&\quad\times x_1^{\sum_i \gamma_{1,i}n_i}\cdots x_J^{\sum_i \gamma_{J,i}n_i}\\
&= x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r} H(\beta_1+\alpha_{r,1},\ldots,\beta_r+\alpha_{r,r},\ldots,\beta_R+\alpha_{r,R}).
\end{aligned}
$$

The desired identity therefore follows. $\square$

Like Figure 7.2, the recurrence relation (8.3.2) can be illustrated by a binary tree shown in Figure 8.1.

Figure 8.1. Node $H(\beta_1,\ldots,\beta_r,\ldots,\beta_R)$ and its children (refined)

$$
\begin{tikzcd}
 & H(\beta_1,\ldots,\boldsymbol{\beta_r},\ldots,\beta_R) \arrow[dl, "1"'] \arrow[dr, "x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r}"] & \\
H(\beta_1,\ldots,\beta_r+A_r,\ldots,\beta_R) & & H(\beta_1+\alpha_{r,1},\ldots,\beta_r+\alpha_{r,r},\ldots,\beta_R+\alpha_{r,R})
\end{tikzcd}
$$

8.4 Generating Functions

8.4.1 Partition Set $\mathscr{B}_{m,r}$

Let $\mathcal{B}(x)$ denote the bivariate generating function

$$\mathcal{B}(x) := \sum_{\lambda\in\mathscr{B}_{m,r}} x^{\sharp(\lambda)}q^{|\lambda|} = \sum_{n\geq 0}\sum_{k\geq 0} B_{m,r}(k,n)x^kq^n. \tag{8.4.1}$$

Further, for $i=1,2,\ldots,5$, we write $\mathcal{B}_i(x) := \sum x^{\sharp(\lambda)}q^{|\lambda|}$ where the sum runs through all partitions $\lambda\in\mathscr{B}_{m,r}$ whose m-tail is $\pi_{B,i}$. Noting that $\pi_{B,i}$ itself is also such a partition, hence for each i, we are able to write

$$\mathcal{B}_i(x) = x^{\sharp(\pi_{B,i})} q^{|\pi_{B,i}|} F_i^\star(x)$$

for some $F_i^\star(x) \in \mathbb{Z}[[q]][[x]]$ such that $F_i^\star(0) = 1$. Therefore,

$$\begin{pmatrix} \mathcal{B}_1(x) \\ \mathcal{B}_2(x) \\ \mathcal{B}_3(x) \\ \mathcal{B}_4(x) \\ \mathcal{B}_5(x) \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & xq^r & & & \\ & & x^2q^m & & \\ & & & xq^{m-r} & \\ & & & & x^2q^{2m} \end{pmatrix} \cdot \begin{pmatrix} F_1^\star(x) \\ F_2^\star(x) \\ F_3^\star(x) \\ F_4^\star(x) \\ F_5^\star(x) \end{pmatrix}. \tag{8.4.2}$$

Further, since $\mathscr{B}_{m,r}$ is a span one linked partition ideal as claimed in Claim 8.2.1, we have

$$\begin{pmatrix} \mathcal{B}_1(x) \\ \mathcal{B}_2(x) \\ \mathcal{B}_3(x) \\ \mathcal{B}_4(x) \\ \mathcal{B}_5(x) \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & xq^r & & & \\ & & x^2q^m & & \\ & & & xq^{m-r} & \\ & & & & x^2q^{2m} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{B}_1(xq^m) \\ \mathcal{B}_2(xq^m) \\ \mathcal{B}_3(xq^m) \\ \mathcal{B}_4(xq^m) \\ \mathcal{B}_5(xq^m) \end{pmatrix}. \tag{8.4.3}$$

Substituting (8.4.2) into (8.4.3), replacing x by xq^{-m} and putting $F_i(x) = F_i^\star(xq^{-m})$ for each i, we have a matrix equation as follows.

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & \\ & xq^r & & & \\ & & x^2q^m & & \\ & & & xq^{m-r} & \\ & & & & x^2q^{2m} \end{pmatrix} \cdot \begin{pmatrix} F_1(xq^m) \\ F_2(xq^m) \\ F_3(xq^m) \\ F_4(xq^m) \\ F_5(xq^m) \end{pmatrix}. \tag{8.4.4}$$

Further, we have $F_i(0) = F_i^\star(0) = 1$ for all i.

Theorem 8.4.1. *We have*

$$\begin{aligned} &F_1(x) = F_2(x) = F_3(x) \\ &= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+2n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}} \\ &\times q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+rn_1+(m-r)n_2+mn_3} \end{aligned} \tag{8.4.5}$$

and

$$\begin{aligned}&F_4(x) = F_5(x)\\&= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+2n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}}\\&\times q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+(m+r)n_1+(m-r)n_2+2mn_3}.\end{aligned} \tag{8.4.6}$$

Proof. We know from Proposition 7.3.1 that it suffices to verify that these triple summations satisfy (8.4.4) since the right-hand sides of (8.4.5) and (8.4.6) decay to 1 as x decays to 0.

We choose $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} m & m & m \\ m & m & m \\ m & m & 2m \end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}} = (1,1,2)$ and $\underline{\mathbf{A}} = (m,m,m)$ in (8.3.1). We also write $x_1 = x$. Then

$$\begin{aligned}&H(r,m-r,m)\\&= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+2n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}}\\&\times q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+rn_1+(m-r)n_2+mn_3}\end{aligned}$$

and

$$\begin{aligned}&H(m+r,m-r,2m)\\&= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+2n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}}\\&\times q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+(m+r)n_1+(m-r)n_2+2mn_3}.\end{aligned}$$

Further, taking $x \to xq^m$ in the above two summations respectively gives $H(m+r,2m-r,3m)$ and $H(2m+r,2m-r,4m)$.

Now it suffices to show that

$$\begin{pmatrix} H(r,m-r,m) \\ H(r,m-r,m) \\ H(r,m-r,m) \\ H(m+r,m-r,2m) \\ H(m+r,m-r,2m) \end{pmatrix} = \begin{pmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&0&0&1&1 \\ 1&0&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1 &&&& \\ & xq^r &&& \\ && x^2q^m && \\ &&& xq^{m-r} & \\ &&&& x^2q^{2m} \end{pmatrix} \cdot \begin{pmatrix} H(m+r,2m-r,3m) \\ H(m+r,2m-r,3m) \\ H(m+r,2m-r,3m) \\ H(2m+r,2m-r,4m) \\ H(2m+r,2m-r,4m) \end{pmatrix}. \tag{8.4.7}$$

But this can be illustrated by the binary tree displayed in Figure 8.2. $\square$

Finally, $\mathcal{B}(x)$ can be represented as follows.

Theorem 8.4.2. *We have*

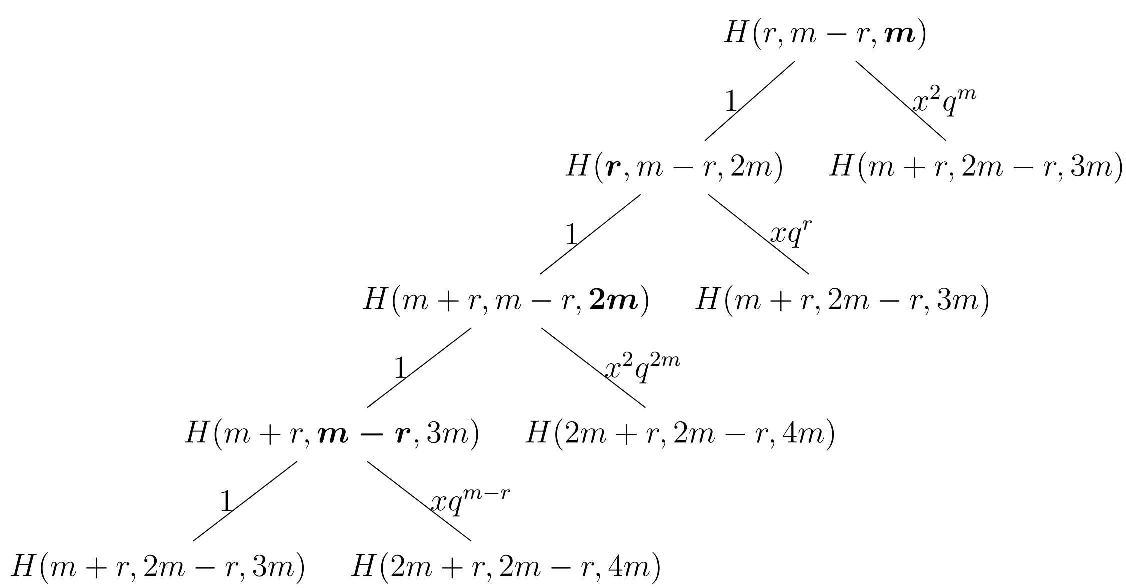
$$\mathcal{B}(x) = \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+2n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}} \times q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+rn_1+(m-r)n_2+mn_3}. \tag{8.4.8}$$

Proof. We have

$$\begin{aligned}
&\mathcal{B}(x) = \mathcal{B}_1(x) + \mathcal{B}_2(x) + \mathcal{B}_3(x) + \mathcal{B}_4(x) + \mathcal{B}_5(x)\\
&= F_1^\star(x) + xq^r F_2^\star(x) + x^2q^m F_3^\star(x) + xq^{m-r} F_4^\star(x) + x^2q^{2m} F_5^\star(x)\\
&= F_1(xq^m) + xq^r F_2(xq^m) + x^2q^m F_3(xq^m) + xq^{m-r} F_4(xq^m) + x^2q^{2m} F_5(xq^m).
\end{aligned}$$

It follows from (8.4.4) that the right-hand side is $F_1(x)$. Therefore, $\mathcal{B}(x) = F_1(x)$ and the theorem follows from (8.4.5). $\square$

Figure 8.2. The binary tree for (8.4.7)



8.4.2 Partition Set $\mathscr{C}_{m,r}$

Let $\mathcal{C}(x,y)$ denote the trivariate generating function

$$\mathcal{C}(x,y) := \sum_{\lambda\in\mathscr{C}_{m,r}} x^{\sharp(\lambda)} y^{\sharp_m(\lambda)} q^{|\lambda|}, \tag{8.4.9}$$

where $\sharp_m(\lambda)$ counts the number of parts in λ that is a multiple of m. Note that the definition of $C_{m,r}(k,n)$ indicates that

$$\mathcal{C}(x,x) = \sum_{n\geq 0}\sum_{k\geq 0} C_{m,r}(k,n) x^k q^n. \tag{8.4.10}$$

For $i = 1,2,3,4$, we write $\mathcal{C}_i(x) = \mathcal{C}_i(x,y) := \sum x^{\sharp(\lambda)} y^{\sharp_m(\lambda)} q^{|\lambda|}$ where the sum runs through all partitions $\lambda \in \mathscr{C}_{m,r}$ whose m-tail is $\pi_{C,i}$. Similarly, for each i, we are able to write

$$\mathcal{C}_i(x) = x^{\sharp(\pi_{C,i})} y^{\sharp_m(\pi_{C,i})} q^{|\pi_{C,i}|} G_i^\star(x)$$

for some $G_i^\star(x) \in \mathbb{Z}[[q]][[x,y]]$ such that $G_i^\star(0) = 1$. Therefore,

$$\begin{pmatrix} \mathcal{C}_1(x) \\ \mathcal{C}_2(x) \\ \mathcal{C}_3(x) \\ \mathcal{C}_4(x) \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & xq^r & & \\ & & xq^{m-r} & \\ & & & xyq^m \end{pmatrix} \cdot \begin{pmatrix} G_1^\star(x) \\ G_2^\star(x) \\ G_3^\star(x) \\ G_4^\star(x) \end{pmatrix}. \tag{8.4.11}$$

Also, if we write $G_i(x) = G_i^\star(xq^{-m})$ for each i, then Claim 8.2.2 yields the following matrix equation.

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq^r & & \\ & & xq^{m-r} & \\ & & & xyq^m \end{pmatrix} \cdot \begin{pmatrix} G_1(xq^m) \\ G_2(xq^m) \\ G_3(xq^m) \\ G_4(xq^m) \end{pmatrix}. \tag{8.4.12}$$

Again, we have that for all i, $G_i(x)$ decays to 1 as x decays to 0.

Theorem 8.4.3. *We have*

$$\begin{aligned} &G_1(x) = G_2(x) \\ &= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3} y^{n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}} \end{aligned}$$

$$\times\, q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+rn_1+(m-r)n_2+mn_3}, \tag{8.4.13}$$

$$\begin{aligned} &G_3(x) \\ &= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}} \\ &\times\, q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+(m+r)n_1+(m-r)n_2+mn_3} \end{aligned} \tag{8.4.14}$$

and

$$\begin{aligned} &G_4(x) \\ &= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}} \\ &\times\, q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+(m+r)n_1+(2m-r)n_2+2mn_3}. \end{aligned} \tag{8.4.15}$$

Proof. We choose $\underline{\boldsymbol{\alpha}} = \begin{pmatrix} m & m & m \\ m & m & m \\ m & m & 2m \end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}} = (1,1,1)$, $\underline{\boldsymbol{\gamma_2}} = (0,0,1)$ and $\underline{\mathbf{A}} = (m,m,m)$ in (8.3.1). We also write $x_1 = x$ and $x_2 = y$. To prove the desired result, it suffices to show that

$$\begin{pmatrix} H(r,m-r,m) \\ H(r,m-r,m) \\ H(m+r,m-r,m) \\ H(m+r,2m-r,2m) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & xq^r & & \\ & & xq^{m-r} & \\ & & & xyq^m \end{pmatrix} \cdot \begin{pmatrix} H(m+r,2m-r,2m) \\ H(m+r,2m-r,2m) \\ H(2m+r,2m-r,2m) \\ H(2m+r,3m-r,3m) \end{pmatrix}. \tag{8.4.16}$$

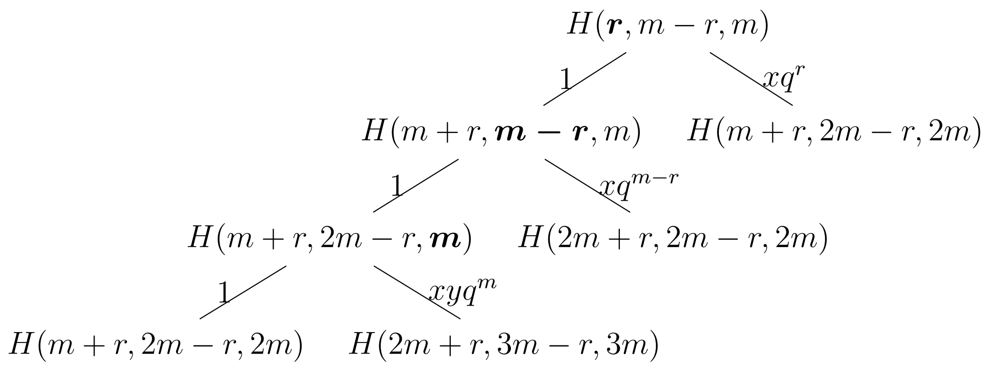
Finally, these identities can be verified with the help of the binary tree displayed in Figure 8.3. □

Analogously, it can be seen that $\mathcal{C}(x,y) = G_1(x)$. Hence, the following result holds.

Theorem 8.4.4. *We have*

$$\begin{aligned} &\mathcal{C}(x,y) \\ &= \sum_{n_1,n_2,n_3\geq 0} \frac{x^{n_1+n_2+n_3}y^{n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}} \\ &\times\, q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+rn_1+(m-r)n_2+mn_3}. \end{aligned} \tag{8.4.17}$$

Figure 8.3. The binary tree for (8.4.16)



8.5 Proof of Theorems 8.1.2 and 8.1.3

Let $\mathscr{A}_{m,r}$ denote the set of partitions into distinct parts congruent to $\pm r$ modulo m. We have

$$\mathcal{A}(x) := \sum_{\lambda \in \mathscr{A}_{m,r}} x^{\sharp(\lambda)} q^{|\lambda|} = \sum_{n \geq 0} \sum_{k \geq 0} A_{m,r}(k,n) x^k q^n = (-xq^r, -xq^{m-r}; q^m)_\infty. \tag{8.5.1}$$

Theorem 8.5.1. *We have*

$$\begin{aligned}&(-xq^r, -xq^{m-r}; q^m)_\infty \\ &= \sum_{n_1,n_2,n_3 \geq 0} \frac{x^{n_1+n_2+2n_3}}{(q^m;q^m)_{n_1}(q^m;q^m)_{n_2}(q^m;q^m)_{n_3}} \\ &\times q^{\frac{m}{2}n_1(n_1-1)+\frac{m}{2}n_2(n_2-1)+mn_3(n_3-1)+mn_1n_2+mn_2n_3+mn_3n_1+rn_1+(m-r)n_2+mn_3}.\end{aligned} \tag{8.5.2}$$

Proof. We know from (8.4.4) that

$$F_1(x) = \Big(1 + xq^r + x^2q^m\Big) F_1(xq^m) + \Big(xq^{m-r} + x^2q^{2m}\Big) F_4(xq^m) \tag{8.5.3}$$

and

$$F_4(x) = F_1(xq^m) + \Big(xq^{m-r} + x^2q^{2m}\Big) F_4(xq^m). \tag{8.5.4}$$

It turns out by subtracting (8.5.4) from (8.5.3) that

$$F_4(x) = F_1(x) - \Big(xq^r + x^2q^m\Big) F_1(xq^m). \tag{8.5.5}$$

Substituting (8.5.5) into (8.5.3) yields a recurrence relation satisfied by $F_1(x)$.

$$F_1(x) = \Big(1 + xq^r + x^2q^m + xq^{m-r} + x^2q^{2m}\Big)F_1(xq^m) \\ - \Big(xq^{m-r} + x^2q^{2m}\Big)\Big(xq^{m+r} + x^2q^{3m}\Big)F_1(xq^{2m}). \tag{8.5.6}$$

In the recurrence relation (8.5.6), expanding $F_1(x)$ as a series in x indicates that $F_1(x)$ is uniquely determined by $F_1(0)$. Note also that

$$\Big[(-xq^r, -xq^{m-r}; q^m)_\infty\Big]_{x=0} = 1 = F_1(0).$$

Hence, to show

$$(-xq^r, -xq^{m-r}; q^m)_\infty = F_1(x), \tag{8.5.7}$$

it suffices to show

$$\begin{aligned} &(-xq^r, -xq^{m-r}; q^m)_\infty \\ &= \Big(1 + xq^r + x^2q^m + xq^{m-r} + x^2q^{2m}\Big)(-xq^{m+r}, -xq^{2m-r}; q^m)_\infty \\ &\quad - \Big(xq^{m-r} + x^2q^{2m}\Big)\Big(xq^{m+r} + x^2q^{3m}\Big)(-xq^{2m+r}, -xq^{3m-r}; q^m)_\infty, \end{aligned}$$

or

$$\begin{aligned} &\Big(1 + xq^r\Big)\Big(1 + xq^{m-r}\Big)\Big(1 + xq^{m+r}\Big)\Big(1 + xq^{2m-r}\Big) \\ &= \Big(1 + xq^r + x^2q^m + xq^{m-r} + x^2q^{2m}\Big)\Big(1 + xq^{m+r}\Big)\Big(1 + xq^{2m-r}\Big) \\ &\quad - \Big(xq^{m-r} + x^2q^{2m}\Big)\Big(xq^{m+r} + x^2q^{3m}\Big), \end{aligned}$$

which is valid.

The desired identity then follows from (8.4.5). $\square$

It follows from Theorems 8.4.2, 8.4.4 and 8.5.1 that $\mathcal{A}(x) = \mathcal{B}(x) = \mathcal{C}(x, x)$. Therefore, we deduce from (8.5.1), (8.4.1) and (8.4.10) that $A_{m,r}(k, n) = B_{m,r}(k, n) = C_{m,r}(k, n)$ for any nonnegative integers n and k.

8.6 Endnotes

We are also able to demonstrate identities concerning certain q-multi-summations. Here we give one example.

Theorem 8.6.1. *We have*

$$\sum_{n_1,n_2\geq 0}\frac{q^{2n_1^2+4n_2^2+4n_1n_2-n_1}x^{n_1+2n_2}}{(q^2;q^2)_{n_1}(q^4;q^4)_{n_2}}$$
$$=\sum_{n_1,n_2,n_3\geq 0}\frac{q^{2n_1^2+2n_2^2+4n_3^2+4n_1n_2+4n_2n_3+4n_3n_1-n_1+n_2}x^{n_1+n_2+2n_3}}{(q^4;q^4)_{n_1}(q^4;q^4)_{n_2}(q^4;q^4)_{n_3}} \tag{8.6.1}$$

and

$$\sum_{n_1,n_2\geq 0}\frac{q^{2n_1^2+4n_2^2+4n_1n_2+n_1+4n_2}x^{n_1+2n_2}}{(q^2;q^2)_{n_1}(q^4;q^4)_{n_2}}$$
$$=\sum_{n_1,n_2,n_3\geq 0}\frac{q^{2n_1^2+2n_2^2+4n_3^2+4n_1n_2+4n_2n_3+4n_3n_1+3n_1+n_2+4n_3}x^{n_1+n_2+2n_3}}{(q^4;q^4)_{n_1}(q^4;q^4)_{n_2}(q^4;q^4)_{n_3}}. \tag{8.6.2}$$

Proof. We know from Theorem 8.4.1 with $m=4$ and $r=1$ that

$$\begin{pmatrix}\mathrm{RHS}(8.6.1)(x)\\ \mathrm{RHS}(8.6.1)(x)\\ \mathrm{RHS}(8.6.1)(x)\\ \mathrm{RHS}(8.6.2)(x)\\ \mathrm{RHS}(8.6.2)(x)\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&0&0&1&1\\1&0&0&1&1\end{pmatrix}\cdot\begin{pmatrix}1&&&&\\&xq&&&\\&&x^2q^4&&\\&&&xq^3&\\&&&&x^2q^8\end{pmatrix}\cdot\begin{pmatrix}\mathrm{RHS}(8.6.1)(xq^4)\\ \mathrm{RHS}(8.6.1)(xq^4)\\ \mathrm{RHS}(8.6.1)(xq^4)\\ \mathrm{RHS}(8.6.2)(xq^4)\\ \mathrm{RHS}(8.6.2)(xq^4)\end{pmatrix}. \tag{8.6.3}$$

Further, LHS(8.6.1)(0) = RHS(8.6.1)(0) = 1 and LHS(8.6.2)(0) = RHS(8.6.2)(0) = 1. Hence, to show (8.6.1) and (8.6.2), it suffices to prove

$$\begin{pmatrix}\mathrm{LHS}(8.6.1)(x)\\ \mathrm{LHS}(8.6.1)(x)\\ \mathrm{LHS}(8.6.1)(x)\\ \mathrm{LHS}(8.6.2)(x)\\ \mathrm{LHS}(8.6.2)(x)\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&0&0&1&1\\1&0&0&1&1\end{pmatrix}\cdot\begin{pmatrix}1&&&&\\&xq&&&\\&&x^2q^4&&\\&&&xq^3&\\&&&&x^2q^8\end{pmatrix}\cdot\begin{pmatrix}\mathrm{LHS}(8.6.1)(xq^4)\\ \mathrm{LHS}(8.6.1)(xq^4)\\ \mathrm{LHS}(8.6.1)(xq^4)\\ \mathrm{LHS}(8.6.2)(xq^4)\\ \mathrm{LHS}(8.6.2)(xq^4)\end{pmatrix}. \tag{8.6.4}$$

Let us choose $\underline{\boldsymbol{\alpha}}=\begin{pmatrix}4&4\\4&8\end{pmatrix}$, $\underline{\boldsymbol{\gamma_1}}=(1,2)$ and $\underline{\mathbf{A}}=(2,4)$, and write $x_1=x$ in (8.3.1).

Then (8.6.4) is equivalent to

$$\begin{pmatrix} H(1,4) \\ H(1,4) \\ H(1,4) \\ H(3,8) \\ H(3,8) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & \\ & xq & & & \\ & & x^2q^4 & & \\ & & & xq^3 & \\ & & & & x^2q^8 \end{pmatrix} \cdot \begin{pmatrix} H(5,12) \\ H(5,12) \\ H(5,12) \\ H(7,16) \\ H(7,16) \end{pmatrix}. \tag{8.6.5}$$

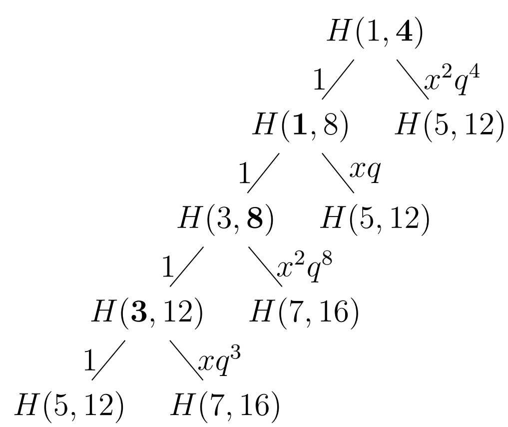
Finally, this matrix equation could be verified by the binary tree displayed in Figure 8.4. $\square$

In light of Theorem 8.5.1 with $m = 4$ and $r = 1$, we have the following corollary.

Corollary 8.6.2. *We have*

$$\begin{aligned} (-xq; q^2)_\infty &= \sum_{n_1,n_2 \geq 0} \frac{q^{2n_1^2+4n_2^2+4n_1n_2-n_1} x^{n_1+2n_2}}{(q^2;q^2)_{n_1}(q^4;q^4)_{n_2}} \\ &= \sum_{n_1,n_2,n_3 \geq 0} \frac{q^{2n_1^2+2n_2^2+4n_3^2+4n_1n_2+4n_2n_3+4n_3n_1-n_1+n_2} x^{n_1+n_2+2n_3}}{(q^4;q^4)_{n_1}(q^4;q^4)_{n_2}(q^4;q^4)_{n_3}}. \end{aligned} \tag{8.6.6}$$

Figure 8.4. The binary tree for (8.6.5)



8.7 References

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Chapter 9

Rogers–Ramanujan Type Identities: An Analytic Perspective

This chapter comes from

- C. Wang and S. Chern, Some basic hypergeometric transformations and Rogers–Ramanujan type identities, *Integral Transforms Spec. Funct.* **31** (2020), no. 11, 873–890. (Ref. [169])

9.1 Introduction

In the previous three chapters, we have mentioned identities of Rogers–Ramanujan type in a combinatorial perspective. Now we will turn our attention to analytic Rogers–Ramanujan type identities, which are generally of the form that a q-series infinite product equals a q-summation or q-multi-summation.

Our starting point is the following transformation formula.

Theorem 9.1.1. *Let A_n be a complex sequence. Then, under suitable convergence conditions, we have*

$$\begin{aligned}\sum_{m=0}^{\infty} &\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m \sum_{n=0}^{m}\frac{(q^{-m},q^m a/x,y;q)_n}{(y/x;q)_n}q^n A_n\\ &=\frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{2m})(a,y;q)_m(a/x;q)_{2m}}{(1-a)(q,aq/y;q)_m(axq;q)_{2m}}\left(\frac{x^2q}{y}\right)^m\sum_{n=0}^{m}(q^{-m},aq^m;q)_n q^n A_n.\end{aligned} \tag{9.1.1}$$

The derivation of this relation in [169] relies on a q-series expansion formula due to Liu [129, Theorem 9.1]. However, we will present a different proof at this place. Let us recall [83, (3.4.7)]:

$${}_2\phi_1\left(\begin{matrix}a,b\\ aq/b\end{matrix};q,\frac{qx}{b^2}\right)=\frac{(xq/b,aqx^2/b^2;q)_\infty}{(aqx/b,qx^2/b^2;q)_\infty}$$

$$\times {}_8\phi_7\left(\begin{matrix} ax/b, q(ax/b)^{\frac{1}{2}}, -q(ax/b)^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, x \\ (ax/b)^{\frac{1}{2}}, -(ax/b)^{\frac{1}{2}}, x(aq)^{\frac{1}{2}}/b, -x(aq)^{\frac{1}{2}}/b, xqa^{\frac{1}{2}}/b, -xqa^{\frac{1}{2}}/b, aq/b \end{matrix}; q, \frac{qx}{b^2}\right). \tag{9.1.2}$$

Proof of Theorem 9.1.1. We simply compare the coefficients of A_N for each $N \geq 0$ on both sides. First,

$$\begin{aligned}
&\text{Coefficient of } A_N \text{ on the LHS} \\
&= \sum_{m\geq N} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m}\left(\frac{x^2q}{y}\right)^m \frac{(q^{-m}, q^m a/x, y; q)_N}{(y/x; q)_N} q^N \\
&= \sum_{m\geq N} \frac{(a/x; q)_{m+N}(yq^N/x; q)_{m-N}(y; q)_N}{(q; q)_{m-N}(aq/y; q)_m}\left(\frac{x^2q}{y}\right)^m q^{N-mN+\binom{N}{2}} \\
&= \frac{(a/x; q)_{2N}(y; q)_N}{(aq/y; q)_N}\left(\frac{x^2q}{y}\right)^N q^{-\binom{N}{2}} {}_2\phi_1\left(\begin{matrix} aq^{2N}/x, yq^N/x \\ aq^{N+1}/y \end{matrix}; q, \frac{x^2}{yq^{N-1}}\right) \\
&= \frac{(a/x; q)_{2N}(y; q)_N}{(aq/y; q)_N}\left(\frac{x^2q}{y}\right)^N q^{-\binom{N}{2}} \frac{(aq^{2N+1}x, xq; q)_\infty}{(aq^{2N+1}, x^2q; q)_\infty} \\
&\quad \times \sum_{m\geq 0} \frac{(1-aq^{2N}q^{2m})(aq^{2N}, yq^N; q)_m (aq^{2N}/x; q)_{2m}}{(1-aq^{2N})(q, aq^{N+1}/y; q)_m (axq^{2N+1}; q)_{2m}}\left(\frac{x^2q^{1-N}}{y}\right)^m.
\end{aligned}$$

(by (9.1.2) with $a \mapsto aq^{2N}/x$, $b \mapsto yq^N/x$ and $x \mapsto yq^N$)

Also,

$$\begin{aligned}
&\text{Coefficient of } A_N \text{ on the RHS} \\
&= \frac{(axq, xq; q)_\infty}{(aq, x^2q; q)_\infty} \sum_{m\geq N} \frac{(1-aq^{2m})(a, y; q)_m (a/x; q)_{2m}}{(1-a)(q, aq/y; q)_m (axq; q)_{2m}}\left(\frac{x^2q}{y}\right)^m (q^{-m}, aq^m; q)_N q^N \\
&= \frac{(axq, xq; q)_\infty}{(aq, x^2q; q)_\infty} \sum_{m\geq N} \frac{(1-aq^{2m})(a; q)_{m+N}(y; q)_m (a/x; q)_{2m}}{(1-a)(aq/y; q)_m (axq; q)_{2m}(q; q)_{m-N}}\left(\frac{x^2q}{y}\right)^m q^{N-mN+\binom{N}{2}} \\
&= \frac{(axq, xq; q)_\infty}{(aq, x^2q; q)_\infty} \frac{(a; q)_{2N}(y; q)_N (a/x; q)_{2N}}{(aq/y; q)_N (axq; q)_{2N}}\left(\frac{x^2q}{y}\right)^N q^{-\binom{N}{2}} \\
&\quad \times \sum_{m\geq 0} \frac{(1-aq^{2(m+N)})(aq^{2N}, yq^N; q)_m (aq^{2N}/x; q)_{2m}}{(1-a)(q, aq^{N+1}/y; q)_m (axq^{2N+1}; q)_{2m}}\left(\frac{x^2q^{1-N}}{y}\right)^m.
\end{aligned}$$

It is easy to see that the two expressions are the same and therefore the desired result follows. $\square$

9.2 Some q-Transformations and Their Applications

In this section, we apply Theorem 9.1.1 to deduce several new q-transformations and exhibit their applications to identities of Rogers–Ramanujan type.

9.2.1 q-Transformations

9.2.1.1 Transformation I

Taking $(a, x, y, q) = (a^2, x^2, y^2, q^2)$ and

$$A_n = \frac{(-\lambda, -\lambda q; q^2)_n}{(q^2, -a, -aq, \lambda^2 q^2; q^2)_n}$$

in Theorem 9.1.1, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a^2/x^2, y^2/x^2; q^2)_m}{(q^2, a^2q^2/y^2; q^2)_m} \left(\frac{x^4q^2}{y^2}\right)^m {}_5\phi_4 \left(\begin{matrix} q^{-2m}, q^{2m}a^2/x^2, y^2, -\lambda, -\lambda q \\ -a, -aq, \lambda^2 q^2, y^2/x^2 \end{matrix} ; q^2, q^2 \right) \\
&= \frac{(a^2x^2q^2, x^2q^2; q^2)_\infty}{(a^2q^2, x^4q^2; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(1-a^2q^{4m})(a^2, y^2; q^2)_m (a^2/x^2; q^2)_{2m}}{(1-a^2)(q^2, a^2q^2/y^2; q^2)_m (a^2x^2q^2; q^2)_{2m}} \left(\frac{x^4q^2}{y^2}\right)^m \\
&\quad \times {}_4\phi_3 \left(\begin{matrix} q^{-2m}, q^{2m}a^2, -\lambda, -\lambda q \\ -a, -aq, \lambda^2 q^2 \end{matrix} ; q^2, q^2 \right).
\end{aligned}$$

Combining the above identity with a formula due to Verma and Jain [168, (5.3)]:

$${}_4\phi_3 \left(\begin{matrix} q^{-2n}, a^2q^{2n}, -\lambda, -q\lambda \\ -a, -qa, q^2\lambda^2 \end{matrix} ; q^2, q^2 \right) = \frac{(-q, a/\lambda; q)_n (-\lambda)^n}{(-a, q\lambda; q)_n},$$

we arrive at our first transformation formula after simplification.

Transformation 9.2.1.

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a^2/x^2, y^2/x^2; q^2)_m}{(q^2, a^2q^2/y^2; q^2)_m} \left(\frac{x^4q^2}{y^2}\right)^m {}_5\phi_4 \left(\begin{matrix} q^{-2m}, q^{2m}a^2/x^2, y^2, -\lambda, -\lambda q \\ -a, -aq, \lambda^2 q^2, y^2/x^2 \end{matrix} ; q^2, q^2 \right) \\
&= \frac{(a^2x^2q^2, x^2q^2; q^2)_\infty}{(a^2q^2, x^4q^2; q^2)_\infty} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(1-a^2q^{4m})(a, a/\lambda; q)_m (y^2; q^2)_m (a^2/x^2; q^2)_{2m}}{(1-a^2)(q, q\lambda; q)_m (a^2q^2/y^2; q^2)_m (a^2x^2q^2; q^2)_{2m}} \left(\frac{-x^4q^2\lambda}{y^2}\right)^m.
\end{aligned} \tag{9.2.1}$$

9.2.1.2 Transformation II

Taking

$$A_n = \frac{(\sqrt{\lambda}, -\sqrt{\lambda}; q)_n}{(q, \sqrt{aq}, -\sqrt{aq}, \lambda; q)_n}$$

in Theorem 9.1.1, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m} \left(\frac{x^2 q}{y}\right)^m {}_5\phi_4 \left(\begin{matrix} q^{-m}, q^m a/x, y, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{aq}, -\sqrt{aq}, \lambda, y/x \end{matrix} ; q, q \right) \\
&\quad = \frac{(axq, xq; q)_\infty}{(aq, x^2 q; q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{2m})(a, y; q)_m (a/x; q)_{2m}}{(1-a)(q, aq/y; q)_m (axq; q)_{2m}} \left(\frac{x^2 q}{y}\right)^m \\
&\qquad \times {}_4\phi_3 \left(\begin{matrix} q^{-m}, q^m a/x, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{aq}, -\sqrt{aq}, \lambda \end{matrix} ; q, q \right).
\end{aligned}$$

Applying the following identity due to Andrews [13, (4.6)]:

$${}_4\phi_3 \left(\begin{matrix} q^{-n}, aq^n, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{aq}, -\sqrt{aq}, \lambda \end{matrix} ; q, q \right) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{(q, aq/\lambda; q^2)_{n/2} (\lambda)^{n/2}}{(aq, \lambda q; q^2)_{n/2}} & \text{if } n \text{ is even,} \end{cases}$$

the second transformation formula follows.

Transformation 9.2.2.

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m} \left(\frac{x^2 q}{y}\right)^m {}_5\phi_4 \left(\begin{matrix} q^{-m}, q^m a/x, y, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{aq}, -\sqrt{aq}, \lambda, y/x \end{matrix} ; q, q \right) \\
&= \frac{(axq, xq; q)_\infty}{(aq, x^2 q; q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a, aq/\lambda; q^2)_m (y; q)_{2m} (a/x; q)_{4m}}{(1-a)(q^2, q\lambda; q^2)_m (aq/y; q)_{2m} (axq; q)_{4m}} \left(\frac{x^4 q^2 \lambda}{y^2}\right)^m. \qquad (9.2.2)
\end{aligned}$$

9.2.1.3 Transformation III

Taking

$$A_n = \frac{(a^{1/3}, a^{1/3} e^{2\pi i/3}, a^{1/3} e^{4\pi i/3}; q)_n}{(q, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_n}$$

in Theorem 9.1.1 gives

$$\sum_{m=0}^{\infty} \frac{(a/x, y/x; q)_m}{(q, aq/y; q)_m} \left(\frac{x^2 q}{y}\right)^m {}_6\phi_5 \left(\begin{matrix} q^{-m}, q^m a/x, y, a^{1/3}, a^{1/3} e^{2\pi i/3}, a^{1/3} e^{4\pi i/3} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, y/x \end{matrix} ; q, q \right)$$

$$
\begin{aligned}
&= \frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{2m})(a,y;q)_m(a/x;q)_{2m}}{(1-a)(q,aq/y;q)_m(axq;q)_{2m}}\left(\frac{x^2q}{y}\right)^m\\
&\quad\times {}_5\phi_4\left(\begin{matrix} q^{-m},aq^m,a^{1/3},a^{1/3}e^{2\pi i/3},a^{1/3}e^{4\pi i/3}\\ \sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq}\end{matrix};q,q\right).
\end{aligned}
$$

Our third transformation formula comes from the following identity of Andrews [13, (4.7)]:

$$
{}_5\phi_4\left(\begin{matrix} q^{-n},aq^n,a^{1/3},a^{1/3}e^{2\pi i/3},a^{1/3}e^{4\pi i/3}\\ \sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq}\end{matrix};q,q\right)=\begin{cases}0 & \text{if } 3\nmid n,\\ \dfrac{(a;q^3)_{n/3}(q;q)_n a^{n/3}}{(a;q)_n(q^3;q^3)_{n/3}} & \text{if } 3\mid n.\end{cases}
$$

Transformation 9.2.3.

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m {}_6\phi_5\left(\begin{matrix} q^{-m},q^m a/x,y,a^{1/3},a^{1/3}e^{2\pi i/3},a^{1/3}e^{4\pi i/3}\\ \sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},y/x\end{matrix};q,q\right)\\
&= \frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{6m})(a;q^3)_m(y;q)_{3m}(a/x;q)_{6m}}{(1-a)(q^3;q^3)_m(aq/y;q)_{3m}(axq;q)_{6m}}\left(\frac{ax^6q^3}{y^3}\right)^m. \qquad (9.2.3)
\end{aligned}
$$

9.2.1.4 Other Known Transformations

One could also apply our approach to deduce a handful of known transformation formulas. Let us present two instances that were first shown in [128].

We first take $(a,x,y,q)=(a^2,x^2,y^2,q^2)$ and

$$
A_n=\frac{(-\lambda,-\lambda q;q^2)_n}{(q^2,-aq,-aq^2,\lambda^2;q^2)_n}
$$

in Theorem 9.1.1. With the help of an identity due to Verma and Jain [168, (5.4)]:

$$
{}_4\phi_3\left(\begin{matrix} q^{-2n},a^2q^{2n},-\lambda,-\lambda q\\ -aq,-aq^2,\lambda^2\end{matrix};q^2,q^2\right)=\frac{(-q,qa/\lambda;q)_n(1+a)(-\lambda)^n}{(-a,\lambda;q)_n(1+aq^{2n})},
$$

we obtain a transformation formula that is equivalent to [128, (2.1)].

Transformation 9.2.4.

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(a^2/x^2,y^2/x^2;q^2)_m}{(q^2,a^2q^2/y^2;q^2)_m}\left(\frac{x^4q^2}{y^2}\right)^m {}_5\phi_4\left(\begin{matrix} q^{-2m},q^{2m}a^2/x^2,y^2,-\lambda,-\lambda q\\ -aq,-aq^2,\lambda^2,y^2/x^2\end{matrix};q^2,q^2\right)\\
&= \frac{(a^2x^2q^2,x^2q^2;q^2)_\infty}{(a^2q^2,x^4q^2;q^2)_\infty}
\end{aligned}
$$

$$\times \sum_{m=0}^{\infty} \frac{(1-aq^{2m})(a,aq/\lambda;q)_m(y^2;q^2)_m(a^2/x^2;q^2)_{2m}}{(1-a)(q,\lambda;q)_m(a^2q^2/y^2;q^2)_m(a^2x^2q^2;q^2)_{2m}}\left(\frac{-x^4q^2\lambda}{y^2}\right)^m. \tag{9.2.4}$$

On the other hand, we take $(a,x,y,q)=(a^3,x^3,y^3,q^3)$ and

$$A_n = \frac{(aq,aq^2,aq^3;q^3)_n}{(q^3,a^{3/2}q^3,-a^{3/2}q^3,a^{3/2}q^{3/2},-a^{3/2}q^{3/2};q^3)_n}$$

in Theorem 9.1.1. Then by an identity due to Andrews [13, (4.5)]:

$$\begin{aligned}
&{}_5\phi_4\left(\begin{matrix} q^{-n}, aq^n, a^{1/3}q^{1/3}, a^{1/3}q^{2/3}, a^{1/3}q \\ a^{1/2}q, -a^{1/2}q, a^{1/2}q^{1/2}, -a^{1/2}q^{1/2} \end{matrix}; q, q\right) \\
&\qquad = \frac{(1-a)(1-a^{1/3}q^{2n/3})(q;q)_n(a^{1/3};q^{1/3})_n(aq)^{n/3}}{(1-a^{1/3})(1-aq^{2n})(a;q)_n(q^{1/3};q^{1/3})_n},
\end{aligned}$$

we obtain a transformation formula that is equivalent to [128, (2.5)].

Transformation 9.2.5.

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a^3/x^3,y^3/x^3;q^3)_m}{(q^3,a^3q^3/y^3;q^3)_m}\left(\frac{x^6q^3}{y^3}\right)^m \\
&\times {}_6\phi_5\left(\begin{matrix} q^{-3m}, q^{3m}a^3/x^3, y^3, aq, aq^2, aq^3 \\ a^{3/2}q^3, -a^{3/2}q^3, a^{3/2}q^{3/2}, -a^{3/2}q^{3/2}, y^3/x^3 \end{matrix}; q^3, q^3\right) \\
&= \frac{(a^3x^3q^3,x^3q^3;q^3)_\infty}{(a^3q^3,x^6q^3;q^3)_\infty} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(1-aq^{2m})(a;q)_m(y^3;q^3)_m(a^3/x^3;q^3)_{2m}}{(1-a)(q;q)_m(a^3q^3/y^3;q^3)_m(a^3x^3q^3;q^3)_{2m}}\left(\frac{ax^6q^4}{y^3}\right)^m.
\end{aligned} \tag{9.2.5}$$

9.2.2 Rogers–Ramanujan Type Identities

We are ready to present a number of Rogers–Ramanujan type identities based on Transformations 9.2.1, 9.2.2 and 9.2.3.

Theorem 9.2.6. *We have*

$$2\sum_{s,t\geq 0} \frac{(q^{-1};q^2)_{2s}q^{8s^2+4t^2+8st}}{(q^4;q^4)_t(q^2;q^2)_{2s}(-1;q^2)_{2s}} = \frac{(q^7,q^{13},q^{20};q^{20})_\infty}{(q^4;q^4)_\infty} + \frac{(q^9,q^{11},q^{20};q^{20})_\infty}{(q^4;q^4)_\infty}, \tag{9.2.6}$$

$$\sum_{s,t\geq 0} \frac{(-1;q)_{2s}(-1)^s q^{3s^2+2t^2+4st+3s+2t}}{(q^2;q^2)_t(q^2;q^2)_s(-q;q)_{2s}(q^2;q^2)_{s+t}} = \frac{1}{(q^4;q^8)_\infty} \tag{9.2.7}$$

and

$$2\sum_{s,t\geq 0}\frac{(-q^{-1};q^2)_{2s}(-1)^s q^{6s^2+4t^2+8st}}{(q^4;q^4)_t(q^4;q^4)_s(-1;q^2)_{2s}(q^2;q^4)_{s+t}}=\frac{(q^7,q^9,q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}+\frac{(q^5,q^{11},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}. \tag{9.2.8}$$

Proof. We first set $x\to 0$, $y\to\infty$ and $\lambda=-a^{1/2}q^{-1/2}$ in (9.2.1) to obtain

$$\begin{aligned}&\sum_{s,t\geq 0}\frac{(a^{1/2}q^{-1/2};q)_{2s}a^{4s+2t}q^{4s^2+2t^2+4st}}{(q^2;q^2)_t(q^2;q^2)_s(aq;q^2)_s(-a;q)_{2s}}\\&\quad=\frac{1}{(a^2q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(1-a^2q^{4m})(a;q)_m}{(1-a^2)(q;q)_m}(-1)^m a^{\frac{9m}{2}}q^{5m^2-\frac{3m}{2}}.\end{aligned} \tag{9.2.9}$$

Replacing q by q^2 and taking $a=1$ in (9.2.9), we then arrive at (9.2.6) by utilizing the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}z^n=\prod_{n=1}^{\infty}(1-zq^{2n-1})(1-q^{2n-1}/z)(1-q^{2n}). \tag{9.2.10}$$

On the other hand, we choose $x\to 0$, $y=(aq)^{1/2}$ and $\lambda=a^{1/2}q^{-1/2}$ in (9.2.1). Then

$$\begin{aligned}&\sum_{s,t\geq 0}\frac{(-a^{1/2}q^{-1/2};q)_{2s}(-1)^s a^{3s+2t}q^{3s^2+2t^2+4st}}{(q^2;q^2)_t(q^2;q^2)_s(-a;q)_{2s}(aq;q^2)_{s+t}}\\&\quad=\frac{1}{(a^2q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(1-a^2q^{4m})(a;q)_m}{(1-a^2)(q;q)_m}(-1)^m a^{\frac{7m}{2}}q^{4m^2-\frac{3m}{2}}.\end{aligned} \tag{9.2.11}$$

Taking $a=q$ in (9.2.11), we obtain (9.2.7) with the help of (9.2.10). We further replace q by q^2 and take $a=1$ in (9.2.11). Then (9.2.8) follows. □

Remark 9.2.1. It is notable that by taking $a=q$, applying the Jacobi triple product identity (9.2.10) and replacing q^2 by q, one may recover the second Rogers–Ramanujan identity [83, (2.7.4)]:

$$\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}. \tag{9.2.12}$$

Theorem 9.2.7. *We have*

$$\sum_{s,t\geq 0}\frac{(-1;q^2)_s q^{2s^2+t^2+2st}}{(q;q)_t(q;q)_s(q;q^2)_s(-1;q)_s}=\frac{(q^9,q^{11},q^{20};q^{20})_\infty}{(q;q)_\infty}, \tag{9.2.13}$$

$$\sum_{s,t\geq 0}\frac{(-q;q^2)_s q^{2s^2+t^2+2st+4s+2t}}{(q;q)_t(q;q)_{2s+1}}=\frac{(q^2,q^{18},q^{20};q^{20})_\infty}{(q;q)_\infty}, \tag{9.2.14}$$

$$\sum_{s,t\geq 0}\frac{(-1;q^2)_s(-1)^s q^{\frac{3}{2}s^2+t^2+2st}}{(q;q)_t(q;q)_s(q;q^2)_s(-1;q)_s(q^{1/2+s};q)_t}=\frac{(q^7,q^9,q^{16};q^{16})_\infty}{(q;q)_\infty} \tag{9.2.15}$$

and

$$\sum_{s,t\geq 0}\frac{(-q;q^2)_s(-1)^s q^{\frac{3}{2}s^2+t^2+2st+3s+2t}}{(q;q)_t(q;q)_{2s+1}(q^{3/2+s};q)_t}=\frac{(q^2,q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty}. \tag{9.2.16}$$

Proof. In (9.2.2), setting $x\to 0$, $y\to\infty$ and $\lambda=-a^{1/2}$, we have

$$\begin{aligned}\sum_{s,t\geq 0}&\frac{(-a^{1/2};q^2)_s a^{2s+t}q^{2s^2+t^2+2st}}{(q;q)_t(q;q)_s(aq;q^2)_s(-a^{1/2};q)_s}\\ &=\frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m}(-1)^m a^{\frac{9m}{2}}q^{10m^2-m}.\end{aligned} \tag{9.2.17}$$

Taking $a=1$ and $a=q^2$ in (9.2.17), respectively, we arrive at (9.2.13) and (9.2.14) with the help of (9.2.10).

Also, one may take $x\to 0$, $y=(aq)^{1/2}$ and $\lambda=-a^{1/2}$ in (9.2.2). Then

$$\begin{aligned}\sum_{s,t\geq 0}&\frac{(-a^{1/2};q^2)_s(-1)^s a^{\frac{3}{2}s+t}q^{\frac{3}{2}s^2+t^2+2st}}{(q;q)_t(q;q)_s(aq;q^2)_s(-a^{1/2};q)_s(a^{1/2}q^{s+1/2};q)_t}\\ &=\frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m}(-1)^m a^{\frac{7m}{2}}q^{8m^2-m}.\end{aligned} \tag{9.2.18}$$

We have (9.2.15) and (9.2.16) by taking $a=1$ and $a=q^2$ respectively in (9.2.18) and then using (9.2.10). □

Theorem 9.2.8. *We have*

$$1+\sum_{\substack{s\geq 1\\ t\geq 0}}\frac{(q^3;q^3)_{s-1}q^{2s^2+t^2+2st}}{(q;q)_t(q;q)_s(q;q)_{2s-1}}=\frac{(q^{21},q^{24},q^{45};q^{45})_\infty}{(q;q)_\infty}, \tag{9.2.19}$$

$$\sum_{s,t\geq 0}\frac{(q^3;q^3)_s q^{2s^2+t^2+2st+6s+3t}}{(q;q)_t(q;q)_s(q;q)_{2s+2}}=\frac{(q^3,q^{42},q^{45};q^{45})_\infty}{(q;q)_\infty}, \tag{9.2.20}$$

$$1+\sum_{\substack{s\geq 1\\ t\geq 0}}\frac{(q^6;q^6)_{s-1}q^{3s^2+2t^2+4st}}{(q^2;q^2)_t(q^2;q^2)_s(q^2;q^2)_{2s-1}(-q^{2s+1};q^2)_t}=\frac{(q^{33},q^{39},q^{72};q^{72})_\infty}{(q^2;q^2)_\infty} \tag{9.2.21}$$

and

$$\sum_{s,t\geq 0}\frac{(q^3;q^3)_s q^{\frac{3}{2}s^2+t^2+2st+\frac{9}{2}s+3t}}{(q;q)_t(q;q)_s(q;q)_{2s+2}(-q^{s+2};q)_t}=\frac{(q^3,q^{33},q^{36};q^{36})_\infty}{(q;q)_\infty}. \tag{9.2.22}$$

Proof. In (9.2.3), we first set $x\to 0$ and $y\to\infty$ to obtain

$$\begin{aligned}&\sum_{s,t\geq 0}\frac{(a;q^3)_s a^{2s+t}q^{2s^2+t^2+2st}}{(q;q)_t(q;q)_s(a;q)_{2s}}\\&\quad=\frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{6m})(a;q^3)_m}{(1-a)(q^3;q^3)_m}(-1)^m a^{7m}q^{\frac{45m^2-3m}{2}}.\end{aligned} \tag{9.2.23}$$

Taking $a=1$ and $a=q^3$ in (9.2.23) respectively and using (9.2.10), we have (9.2.19) and (9.2.20).

Further, we take $x\to 0$ and $y=-(aq)^{1/2}$ in (9.2.3). Then

$$\begin{aligned}&\sum_{s,t\geq 0}\frac{(a;q^3)_s a^{\frac{3}{2}s+t}q^{\frac{3}{2}s^2+t^2+2st}}{(q;q)_t(q;q)_s(a;q)_{2s}(-a^{1/2}q^{s+\frac{1}{2}};q)_t}\\&\quad=\frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{6m})(a;q^3)_m}{(1-a)(q^3;q^3)_m}(-1)^m a^{\frac{11}{2}m}q^{18m^2-\frac{3m}{2}}.\end{aligned} \tag{9.2.24}$$

Letting $a=1$ (with q replaced by q^2) and $a=q^3$ in (9.2.24) respectively and using (9.2.10), we have (9.2.21) and (9.2.22). □

9.3 Generalized Transformations and Their Applications

In this section, we establish several generalized transformations based on Theorem 9.1.1 and some formulas due to Verma and Jain [168]. For convenience, we define

$$M_i:=\begin{cases}0 & \text{if } i=0 \text{ or } -1,\\ r_1+r_2+\cdots+r_i & \text{if } i\geq 1.\end{cases}$$

We also introduce the following compact notation [69]:

$$\Lambda_k^{(c)}\left[\begin{array}{c}\{x_i,y_i\}\\ [u,v]\end{array}\right]_q:=\prod_{i=u}^{v}\frac{(x_i,y_i;q)_k}{(c/x_i,c/y_i;q)_k}\left(\frac{c}{x_iy_i}\right)^k.$$

9.3.1 Multiple Rogers–Ramanujan Type Identities

Before introducing our generalized transformations, we first present some multiple Rogers–Ramanujan type identities to illustrate their power.

Theorem 9.3.1. *For $t \geq 1$, we have*

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t};q^2)_i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i}(-q^{2M_t};q)_i}$$
$$\times\frac{q^{2i^2+j^2+2ij+(8i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+9M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$=\frac{(q^{2t+9},q^{2t+11},q^{4t+20};q^{4t+20})_\infty}{(q;q)_\infty}, \tag{9.3.1}$$

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t+1};q^2)_i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i+1}(-q^{2M_t+1};q)_i}$$
$$\times\frac{q^{2i^2+j^2+2ij+4i+2j+(8i+4j+9)M_t+2(M_1^2+\cdots+M_{t-1}^2+M_1+\cdots+M_{t-1})+9M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$=\frac{(q^2,q^{4t+18},q^{4t+20};q^{4t+20})_\infty}{(q;q)_\infty}, \tag{9.3.2}$$

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t};q^2)_i(-1)^i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i}(-q^{2M_t};q)_i(q^{i+2M_t+1/2};q)_j}$$
$$\times\frac{q^{\frac{3}{2}i^2+j^2+2ij+(6i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+7M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$=\frac{(q^{2t+7},q^{2t+9},q^{4t+16};q^{4t+16})_\infty}{(q;q)_\infty} \tag{9.3.3}$$

and

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-q^{2M_t+1};q^2)_i(-1)^i}{(q;q)_i(q;q)_j(q;q^2)_{2M_t+i+1}(-q^{2M_t+1};q)_i(q^{i+2M_t+3/2};q)_{i+j}}$$
$$\times\frac{q^{\frac{3}{2}i^2+j^2+2ij+3i+2j+(6i+4j+7)M_t+2(M_1^2+\cdots+M_{t-1}^2+M_1+\cdots+M_{t-1})+7M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$=\frac{(q^2,q^{4t+14},q^{4t+16};q^{4t+16})_\infty}{(q;q)_\infty}. \tag{9.3.4}$$

Proof. Letting $x\to 0$, $y\to\infty$, $\lambda=-a^{1/2}$ and $c_1,d_1,\ldots,c_t,d_t\to\infty$ in (9.3.14), we have

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-a^{1/2}q^{2M_t};q^2)_i a^{2i+j+M_1+\cdots+M_{t-1}+\frac{9}{2}M_t}}{(q;q)_i(q;q)_j(aq;q^2)_{2M_t+i}(-a^{1/2}q^{2M_t};q)_i}$$

$$\times \frac{q^{2i^2+j^2+2ij+(8i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+9M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$= \frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m}(-1)^m a^{(\frac{9}{2}+t)m} q^{2(t+5)m^2-m}. \tag{9.3.5}$$

Taking $a=1$ and $a=q^2$ in (9.3.5) respectively and using the Jacobi triple product identity (9.2.10), we have (9.3.1) and (9.3.2).

On the other hand, we set $x \to 0$, $y=(aq)^{1/2}$, $\lambda = -a^{1/2}$ and $c_1, d_1, \ldots, c_t, d_t \to \infty$ in (9.3.14). Then

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(-a^{1/2}q^{2M_t};q^2)_i(-1)^i a^{\frac{3}{2}i+j+M_1+\cdots+M_{t-1}+\frac{7}{2}M_t}}{(q;q)_i(q;q)_j(aq;q^2)_{2M_t+i}(-a^{1/2}q^{2M_t};q)_i(a^{1/2}q^{i+2M_t+1/2};q)_j}$$
$$\times \frac{q^{\frac{3}{2}i^2+j^2+2ij+(6i+4j)M_t+2(M_1^2+\cdots+M_{t-1}^2)+7M_t^2}}{(q^2;q^2)_{r_1}(q^2;q^2)_{r_2}\cdots(q^2;q^2)_{r_t}}$$
$$= \frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{4m})(a;q^2)_m}{(1-a)(q^2;q^2)_m}(-1)^m a^{(\frac{7}{2}+t)m} q^{2(t+4)m^2-m}. \tag{9.3.6}$$

Taking $a=1$ and $a=q^2$ in (9.3.6), and then using (9.2.10), one has (9.3.3) and (9.3.4). $\square$

Theorem 9.3.2. *For $t \geq 1$, we have*

$$1+\sum_{i,j\geq 0}\sum_{\substack{r_1,r_2,\ldots,r_t\geq 0\\(i,r_1,\ldots,r_t)\neq 0}}\frac{(q^3;q^3)_{2M_t+i-1}}{(q;q)_i(q;q)_j(q;q)_{6M_t+2i-1}}$$
$$\times \frac{q^{2i^2+j^2+2ij+(12i+6j)M_t+3(M_1^2+\cdots+M_{t-1}^2)+21M_t^2}}{(q^3;q^3)_{r_1}(q^3;q^3)_{r_2}\cdots(q^3;q^3)_{r_t}}$$
$$= \frac{(q^{3t+21},q^{3t+24},q^{6t+45};q^{6t+45})_\infty}{(q;q)_\infty}, \tag{9.3.7}$$

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(q^3;q^3)_{2M_t+i}}{(q;q)_i(q;q)_j(q;q)_{6M_t+2i+2}}$$
$$\times \frac{q^{2i^2+j^2+2ij+6i+3j+(12i+6j+21)M_t+3(M_1^2+\cdots+M_{t-1}^2+M_1+\cdots+M_{t-1})+21M_t^2}}{(q^3;q^3)_{r_1}(q^3;q^3)_{r_2}\cdots(q^3;q^3)_{r_t}}$$
$$= \frac{(q^3,q^{6t+42},q^{6t+45};q^{6t+45})_\infty}{(q;q)_\infty}, \tag{9.3.8}$$

$$\sum_{i,j\geq 0}\sum_{r_1,r_2,\ldots,r_t\geq 0}\frac{(q^3;q^3)_{2M_t+i-1}a^{\frac{3}{2}i+j+M_1+\cdots+M_{t-1}+\frac{11}{2}M_t}}{(q;q)_i(q;q)_j(q;q)_{6M_t+2i-1}(-q^{3M_t+i+1/2};q)_j}$$
$$\times \frac{q^{\frac{3}{2}i^2+j^2+2ij+(9i+6j)M_t+3(M_1^2+\cdots+M_{t-1}^2)+\frac{33}{2}M_t^2}}{(q^3;q^3)_{r_1}(q^3;q^3)_{r_2}\cdots(q^3;q^3)_{r_t}}$$

$$= \frac{(q^{6t+33}, q^{6t+39}, q^{12t+72}; q^{12t+72})_\infty}{(q^2;q^2)_\infty} \tag{9.3.9}$$

and

$$\sum_{i,j\geq 0} \sum_{r_1,r_2,\ldots,r_t\geq 0} \frac{(q^3;q^3)_{2M_t+i}}{(q;q)_i(q;q)_j(q;q)_{6M_t+2i+2}(-q^{3M_t+i+2};q)_j}$$
$$\times \frac{q^{\frac{3}{2}i^2+j^2+2ij+\frac{9}{2}i+3j+(9i+6j+\frac{33}{2})M_t+3(M_1^2+\cdots+M_{p-1}^2+M_1+\cdots+M_{t-1})+\frac{33}{2}M_t^2}}{(q^3;q^3)_{r_1}(q^3;q^3)_{r_2}\cdots(q^3;q^3)_{r_t}}$$
$$= \frac{(q^3, q^{6t+33}, q^{6t+36}; q^{6t+36})_\infty}{(q^2;q^2)_\infty}. \tag{9.3.10}$$

Proof. We set $x \to 0$, $y \to \infty$ and $c_1, d_1, \ldots, c_t, d_t \to \infty$ in (9.3.15). Then

$$\sum_{i,j\geq 0} \sum_{r_1,r_2,\ldots,r_t\geq 0} \frac{(a;q^3)_{2M_t+i}a^{2i+j+M_1+\cdots+M_{t-1}+7M_t}}{(q;q)_i(q;q)_j(a;q)_{6M_t+2i}}$$
$$\times \frac{q^{2i^2+j^2+2ij+(12i+6j)M_t+3(M_1^2+\cdots+M_{t-1}^2)+21M_t^2}}{(q^3;q^3)_{r_1}(q^3;q^3)_{r_2}\cdots(q^3;q^3)_{r_t}}$$
$$= \frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty} \frac{(a;q^3)_n(1-aq^{6m})}{(q^3;q^3)_m(1-a)}(-1)^m a^{(t+7)m} q^{(\frac{45}{2}+3t)m^2-\frac{3}{2}m}. \tag{9.3.11}$$

Taking $a = 1$ and $a = q^3$ in (9.3.11), and using (9.2.10), we have (9.3.7) and (9.3.8).

Further, letting $x \to 0$, $y = (aq)^{1/2}$ and $c_1, d_1, \ldots, c_t, d_t \to \infty$ in (9.3.15), we have

$$\sum_{i,j\geq 0} \sum_{r_1,r_2,\ldots,r_t\geq 0} \frac{(a;q^3)_{2M_t+i}a^{\frac{3}{2}i+j+M_1+\cdots+M_{t-1}+\frac{11}{2}M_t}}{(q;q)_i(q;q)_j(a;q)_{6M_t+2i}(-a^{1/2}q^{3M_t+i+1/2};q)_j}$$
$$\times \frac{q^{\frac{3}{2}i^2+j^2+2ij+(9i+6j)M_t+3(M_1^2+\cdots+M_{t-1}^2)+\frac{33}{2}M_t^2}}{(q^3;q^3)_{r_1}(q^3;q^3)_{r_2}\cdots(q^3;q^3)_{r_t}}$$
$$= \frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty} \frac{(a;q^3)_n(1-aq^{6m})}{(q^3;q^3)_m(1-a)}(-1)^m a^{(t+\frac{11}{2})m} q^{(18+3t)m^2-\frac{3}{2}m}. \tag{9.3.12}$$

Taking $a = 1$ (with q replaced by q^2) and $a = q^3$ in (9.3.12), and using (9.2.10), one has (9.3.9) and (9.3.10). $\square$

9.3.2 Generalized q-Transformations

9.3.2.1 Transformation I

We first take

$$A_n=\sum_{k=0}^{[n/2]}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{k(2k-1)}}{(aq^{2k},q;q)_{2k}(aq^{4k+1},q;q)_{n-2k}(aq,q\lambda;q^2)_k}\Lambda_k^{(aq^2)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^2}$$

in Theorem 9.1.1. Then

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{n=0}^{m}\frac{(q^{-m},q^ma/x,y;q)_nq^n}{(y/x;q)_n}\\&\times\sum_{k=0}^{[n/2]}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{k(2k-1)}}{(aq^{2k},q;q)_{2k}(aq^{4k+1},q;q)_{n-2k}(aq,q\lambda;q^2)_k}\Lambda_k^{(aq^2)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^2}\\&=\frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{2m})(a,y;q)_m(a/x;q)_{2m}}{(1-a)(q,aq/y;q)_m(axq;q)_{2m}}\left(\frac{x^2q}{y}\right)^m\\&\quad\times\sum_{n=0}^{m}(q^{-m},aq^m;q)_nq^n\\&\quad\times\sum_{k=0}^{[n/2]}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{k(2k-1)}}{(aq^{2k},q;q)_{2k}(aq^{4k+1},q;q)_{n-2k}(aq,q\lambda;q^2)_k}\Lambda_k^{(aq^2)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^2}.\end{aligned}$$

Interchanging the last two summations on both sides yields

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{k=0}^{m}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{k(2k-1)}}{(aq^{2k},q;q)_{2k}(aq,q\lambda;q^2)_k}\\&\times\Lambda_k^{(aq^2)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^2}\sum_{n=2k}^{m}\frac{(q^{-m},q^ma/x,y;q)_nq^n}{(aq^{4k+1},q;q)_{n-2k}(y/x;q)_n}\\&=\frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{2m})(a,y;q)_m(a/x;q)_{2m}}{(1-a)(q,aq/y;q)_m(axq;q)_{2m}}\left(\frac{x^2q}{y}\right)^m\\&\quad\times\sum_{k=0}^{m}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{k(2k-1)}}{(aq^{2k},q;q)_{2k}(aq,q\lambda;q^2)_k}\Lambda_k^{(aq^2)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^2}\sum_{n=2k}^{m}\frac{(q^{-m},aq^m;q)_nq^n}{(aq^{4k+1},q;q)_{n-2k}}.\end{aligned}$$

It follows that

$$\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{k=0}^{m}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{2k(k+1)}}{(aq^{2k},q;q)_{2k}(aq,q\lambda;q^2)_k}\Lambda_k^{(aq^2)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^2}$$

$$\times \frac{(q^{-m},q^m a/x,y;q)_{2k}}{(y/x;q)_{2k}}\,{}_3\phi_2\left(\begin{matrix} q^{-m+2k},aq^{m+2k}/x,yq^{2k} \\ aq^{4k+1},yq^{2k}/x\end{matrix};q,q\right)$$
$$= \frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{2m})(a,y;q)_m(a/x;q)_{2m}}{(1-a)(q,aq/y;q)_m(axq;q)_{2m}}\left(\frac{x^2q}{y}\right)^m$$
$$\times \sum_{k=0}^{m}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{2k(k+1)}(q^{-m},aq^m;q)_{2k}}{(aq^{2k},q;q)_{2k}(aq,q\lambda;q^2)_k}\Lambda_k^{(aq^2)}\left[\begin{matrix}\{c_i,d_i\}\\ [1,t]\end{matrix}\right]_{q^2}$$
$$\times\,{}_2\phi_1\left(\begin{matrix} q^{-m+2k},aq^{m+2k} \\ a^2q^{4k+1}\end{matrix};q,q\right).$$

Applying the q-Pfaff–Saalschütz identity

$$ {}_3\phi_2\left(\begin{matrix} q^{-n},a,b \\ c,abc^{-1}q^{1-n}\end{matrix};q,q\right) = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}, \tag{9.3.13}$$

we have

$$ {}_3\phi_2\left(\begin{matrix} q^{-m+2k},aq^{m+2k}/x,yq^{2k} \\ aq^{4k+1},yq^{2k}/x\end{matrix};q,q\right)$$
$$= \frac{(xq^{-m+2k+1};q)_{m-2k}(aq^{2k+1}/y;q)_{m-2k}}{(aq^{4k+1};q)_{m-2k}(xq^{-m+1}/y;q)_{m-2k}}$$
$$= \frac{(1/x;q)_{m-2k}(aq^{2k+1}/y;q)_{m-2k}(yq^{2k})^{m-2k}}{(aq^{4k+1};q)_{m-2k}(yq^{2k}/x;q)_{m-2k}}.$$

Notice also that the inner summation on the right-hand side equals 0 when $m \neq 2k$. Hence,

$$\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{k=0}^{m}\frac{(q,aq/\lambda;q^2)_k\lambda^k q^{2k^2}}{(aq^{2k},q;q)_{2k}(aq,q\lambda;q^2)_k}\Lambda_k^{(aq^2)}\left[\begin{matrix}\{c_i,d_i\}\\ [1,t]\end{matrix}\right]_{q^2}$$
$$\times \frac{(q^{-m},q^m a/x,y;q)_{2k}}{(y/x;q)_{2k}}\frac{(1/x;q)_{m-2k}(aq^{2k+1}/y;q)_{m-2k}(yq^{2k})^{m-2k}}{(aq^{4k+1};q)_{m-2k}(yq^{2k}/x;q)_{m-2k}}$$
$$= \frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(1-aq^{4m})(a,aq/\lambda;q^2)_m(y;q)_{2m}(a/x;q)_{4m}}{(1-a)(q^2,q\lambda;q^2)_m(aq/y;q)_{2m}(axq;q)_{4m}}\left(\frac{x^4\lambda q^2}{y^2}\right)^m$$
$$\times \Lambda_m^{(aq^2)}\left[\begin{matrix}\{c_i,d_i\}\\ [1,t]\end{matrix}\right]_{q^2}.$$

After simplification, we have

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a/x, y/x;q)_m}{(q, aq/y;q)_m} \left(\frac{x^2 q}{y}\right)^m \frac{(1/x, aq/y;q)_m y^m}{(aq, y/x;q)_m} \\
&\times \sum_{k=0}^{m} \frac{(1-aq^{4k})(a, aq/\lambda;q^2)_k (q^m a/x, y, q^{-m};q)_{2k} \lambda^k q^{2k^2}}{(1-a)(q^2, q\lambda;q^2)_k (xq^{-m+1}, aq/y, aq^{m+1};q)_{2k}} \left(\frac{\lambda x^2 q^2}{y^2}\right)^k \\
&\times \Lambda_k^{(aq^2)} \left[\begin{array}{c} \{c_i, d_i\} \\ [1,t] \end{array} \right]_{q^2} \\
&= \frac{(axq, xq;q)_\infty}{(aq, x^2 q;q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a, aq/\lambda;q^2)_m (y;q)_{2m} (a/x;q)_{4m}}{(1-a)(q^2, q\lambda;q^2)_m (aq/y;q)_{2m} (axq;q)_{4m}} \left(\frac{x^4 \lambda q^2}{y^2}\right)^m \\
&\quad \times \Lambda_m^{(aq^2)} \left[\begin{array}{c} \{c_i, d_i\} \\ [1,t] \end{array} \right]_{q^2}.
\end{aligned}
$$

With the aid of [168, (4.3)], we arrive at the first generalized transformation.

Transformation 9.3.3.

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a/x, y/x;q)_m}{(q, aq/y;q)_m} \left(\frac{x^2 q}{y}\right)^m \sum_{r_1, r_2 \ldots, r_t \geq 0} \prod_{j=1}^{t} \frac{(\frac{aq^2}{c_j d_j};q^2)_{r_j} (c_j, d_j;q^2)_{M_{j-1}}}{(q^2;q^2)_{r_j} (\frac{aq^2}{c_j}, \frac{aq^2}{d_j};q^2)_{M_j}} \frac{(a^j q^{2j})^{r_{t-j+1}}}{(c_j d_j)^{M_{j-1}}} \\
&\times \frac{(aq/\lambda;q^2)_{M_t} (q^{-m}, aq^m/x, y;q)_{2M_t} (-\lambda)^{M_t} q^{M_t^2}}{(aq;q^2)_{2M_t} (\lambda q;q^2)_{M_t} (y/x;q)_{2M_t} a^{M_t}} \\
&\times {}_5\phi_4 \left(\begin{array}{c} q^{-m+2M_t}, aq^{m+2M_t}/x, yq^{2M_t}, \sqrt{\lambda} q^{M_t}, -\sqrt{\lambda} q^{M_t} \\ a^{1/2} q^{1/2+2M_t}, -a^{1/2} q^{1/2+2M_t}, \lambda q^{2M_t}, yq^{2M_t}/x \end{array} ; q, q \right) \\
&= \frac{(axq, xq;q)_\infty}{(aq, x^2 q;q)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{4m})(a, aq/\lambda;q^2)_m (y;q)_{2m} (a/x;q)_{4m}}{(1-a)(q^2, \lambda q;q^2)_m (aq/y;q)_{2m} (axq;q)_{4m}} \\
&\quad \times \Lambda_m^{(aq^2)} \left[\begin{array}{c} \{c_i, d_i\} \\ [1,t] \end{array} \right]_{q^2} \left(\frac{\lambda x^4 q^2}{y^2}\right)^m.
\end{aligned} \tag{9.3.14}
$$

Remark 9.3.1. Transformation (9.3.14) reduces to (9.2.2) when $t = 0$.

We prove the next two transformation formulas in a similar way to that of (9.3.14); the details will be omitted.

9.3.2.2 Transformation II

Taking

$$A_n=\sum_{k=0}^{[n/3]}\frac{(-1)^k q^{3k(3k-1)/2}(a;q^3)_k a^k}{(a;q)_{6k}(aq^{6k+1},q;q)_{n-3k}(q^3;q^3)_k}\Lambda_k^{(aq^3)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^3}$$

in Theorem 9.1.1, and applying [168, (4.5)], we obtain the second generalized transformation.

Transformation 9.3.4.

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(a/x,y/x;q)_m}{(q,aq/y;q)_m}\left(\frac{x^2q}{y}\right)^m\sum_{r_1,r_2\dots,r_t\geq 0}\prod_{j=1}^{t}\frac{(\frac{aq^3}{c_jd_j};q^3)_{r_j}(c_j,d_j;q^3)_{M_{j-1}}}{(q^3;q^3)_{r_j}(\frac{aq^3}{c_j},\frac{aq^3}{d_j};q^3)_{M_j}}\frac{(a^jq^{3j-3})^{r_{t-j+1}}}{(c_jd_j)^{M_{j-1}}}\\
&\times\frac{(a;q^3)_{2M_t}(q^{-m},aq^m/x,y;q)_{3M_t}q^{3M_t(M_t+1)}}{(a;q)_{6M_t}(y/x;q)_{3M_t}}\\
&\times{}_6\phi_5\left(\begin{array}{c}q^{-m+3M_t},aq^{m+3M_t}/x,yq^{3M_t},a^{1/3}q^{2M_t},a^{1/3}e^{2\pi i/3}q^{2M_t},a^{1/3}e^{4\pi i/3}q^{2M_t}\\ \sqrt{a}q^{3M_t},-\sqrt{a}q^{3M_t},\sqrt{aq}q^{3M_t},-\sqrt{aq}q^{3M_t},yq^{3M_t}/x\end{array};q,q\right)\\
&=\frac{(axq,xq;q)_\infty}{(aq,x^2q;q)_\infty}\sum_{m=0}^{\infty}\frac{(a;q^3)_m(1-aq^{6m})(y;q)_{3m}(a/x;q)_{6m}}{(q^3;q^3)_m(1-a)(aq/y;q)_{3m}(axq;q)_{6m}}\left(\frac{ax^6q^3}{y^3}\right)^m\\
&\quad\times\Lambda_m^{(aq^3)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_{q^3}.
\end{aligned}\tag{9.3.15}$$

Remark 9.3.2. When $t=0$, (9.3.15) reduces to (9.2.3). It is also necessary to point out that there are two typos in [168, (4.5)]: $(aq^3;q^3)_{2M_{p-4}}$ and $(aq;q)_{6M_{p-4}}$ should be $(a;q^3)_{2M_{p-4}}$ and $(a;q)_{6M_{p-4}}$, respectively.

9.3.2.3 Transformation III

Taking $(a,x,y,q)=(a^2,x^2,y^2,q^2)$ and

$$A_n=\sum_{k=0}^{n}\frac{(-q,aq/\lambda;q)_k(1+a)\lambda^kq^{k(k-1)}}{(a^2q^{2k},q^2;q^2)_k(a^2q^{4k+2},q^2;q^2)_{n-k}(-a,\lambda;q)_k(1+aq^{2k})}\Lambda_k^{(aq)}\left[\begin{array}{c}\{c_i,d_i\}\\ [1,t]\end{array}\right]_q$$

in Theorem 9.1.1 and applying [168, (4.1)], we obtain the following generalized transformation that is equivalent to [128, (2.3)].

Transformation 9.3.5.

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(a^2/x^2, y^2/x^2; q)_m}{(q^2, a^2q^2/y^2; q^2)_m} \left(\frac{x^4q^2}{y^2}\right)^m \sum_{r_1,r_2,\ldots,r_t\geq 0} \prod_{j=1}^{t} \frac{(\frac{aq}{c_j,d_j}; q)_{r_j} (c_j, d_j; q)_{M_j-1}}{(q;q)_j (aq/c_j, aq/d_j; q)_{M_j}} \frac{(a^j q^{j+1})^{r_{t-j+1}}}{(c_jd_j)^{M_{j-1}}} \\
&\times \frac{(aq/\lambda; q)_{M_t} (a^2q^{2m}/x^2, y^2, q^{-2m}; q^2)_{M_t} (-\lambda)^{M_t} q^{M_t(M_t-1)/2}}{(-aq;q)_{2M_t} (\lambda; q)_{M_t} (y^2/x^2; q^2)_{M_t} a^{M_t}} \\
&\times {}_5\phi_4 \left(\begin{matrix} q^{-2m+2M_t}, q^{2m+2M_t}a^2/x^2, y^2q^{2M_t}, -\lambda q^{M_t}, -\lambda q^{1+M_t} \\ -aq^{1+2M_t}, -aq^{2+2M_t}, \lambda^2 q^{2+2M_t}, q^{2M_t}y^2/x^2 \end{matrix} ; q^2, q^2 \right) \\
&= \frac{(a^2x^2q^2, x^2q^2; q^2)_\infty}{(a^2q^2, x^4q^2; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(1-aq^{2m})(a, aq/\lambda; q)_m (y^2;q^2)_m (a^2/x^2; q^2)_{2m}}{(1-a)(q,\lambda;q)_m (a^2q^2/y^2; q^2)_m (a^2x^2q^2; q^2)_{2m}} \\
&\quad \times \Lambda_m^{(aq)} \left[\begin{matrix} \{c_i, d_i\} \\ [1,t] \end{matrix} \right]_q \left(\frac{-\lambda x^4 q^2}{y^2}\right)^m. \qquad (9.3.16)
\end{aligned}
$$

Remark 9.3.3. Transformation (9.3.16) reduces to (9.2.4) when $t = 0$.

9.4 References

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Part III
Asymptotics

Outline

- Chapter 10 is devoted to a refined Meinardus-type method with its application to square-root partitions into distinct parts.
- Chapters 11–13 are devoted to asymptotics for coefficients in modular infinite products that concern either Dedekind eta function or Jacobi theta function with the assistance of Rademacher's circle method.
- Chapter 14 is devoted to nonmodular infinite products that arise from a conjecture of Seo and Yee.

Chapter 10

The Square-root Partition into Distinct Parts

This chapter comes from

- S. Chern, Note on the square-root partition into distinct parts, *Ramanujan J.* **54** (2021), no. 2, 449–461. (Ref. [55])

10.1 Introduction

The square-root partition, which is a partition into parts with the order $\{\lfloor\sqrt{1}\rfloor, \lfloor\sqrt{2}\rfloor, \lfloor\sqrt{3}\rfloor, \ldots, \lfloor\sqrt{k}\rfloor, \ldots\}$, was introduced by Balasubramanian and Luca [33]. For example, 1 has three square-root partitions: $\lfloor\sqrt{1}\rfloor$, $\lfloor\sqrt{2}\rfloor$, and $\lfloor\sqrt{3}\rfloor$. Let $r(n)$ be the number of square-root partitions of n. It is not hard to see that $r(n)$ has generating function:

$$\sum_{n\geq 0} r(n)q^n = \prod_{k\geq 1} \frac{1}{(1-q^k)^{2k+1}}.$$

In [131], Luca and Ralaivaosaona studied the asymptotic behavior of $r(n)$. They showed that, as $n\to\infty$,

$$\begin{aligned} r(n) = \Big(1+o(1)\Big)\, & 2^{5/18}3^{-1/2}\pi^{-1/2}\zeta(3)^{7/18}n^{-8/9} \\ & \times \exp\left(\frac{3\zeta(3)^{1/3}}{2^{1/3}}n^{2/3} + \frac{\zeta(2)}{2^{2/3}\zeta(3)^{1/3}}n^{1/3} - \frac{\zeta(2)^2}{24\zeta(3)} + 2\zeta'(-1) + \zeta'(0)\right), \end{aligned}$$

where as usual $\zeta(\cdot)$ is the Riemann zeta function.

In general, if we are given a prescribed ordered set of parts, then apart from partitions into parts in this set, we are often interested in partitions into distinct parts as well. In the square-root partition case, we will assume that, for instance, $\lfloor\sqrt{1}\rfloor$, $\lfloor\sqrt{2}\rfloor$, and $\lfloor\sqrt{3}\rfloor$ are different parts, although they have the same numerical value. Let $r_D(n)$ be the number of square-root partitions of n into distinct parts. One would see that the generating function of $r_D(n)$ is

$$F(q) := \sum_{n\geq 0} r_D(n)q^n = \prod_{k\geq 1}(1+q^k)^{2k+1} = \prod_{k\geq 1}\frac{(1-q^{2k})^{2k+1}}{(1-q^k)^{2k+1}}. \tag{10.1.1}$$

Similar to the asymptotic formula of $r(n)$, we will prove the following result.

Theorem 10.1.1. *As $n\to\infty$, we have that*

$$\begin{aligned} r_D(n) = \Big(1+o(1)\Big)\, & 2^{-7/6}3^{-1/3}\pi^{-1/2}\zeta(3)^{1/6}n^{-2/3} \\ & \times \exp\left(\frac{3^{4/3}\zeta(3)^{1/3}}{2}n^{2/3} + \frac{\zeta(2)}{2\cdot 3^{1/3}\zeta(3)^{1/3}}n^{1/3} - \frac{\zeta(2)^2}{72\zeta(3)}\right). \end{aligned} \tag{10.1.2}$$

We remark that for a general infinite product

$$\prod_{k\geq 1}\frac{1}{(1-q^k)^{a_k}},$$

where a_1, a_2, … is a "nice" sequence of non-negative integers, Meinardus' theorem [134] is a powerful tool to study the asymptotic behavior of its Taylor coefficients. A delicate presentation of Meinardus' approach is given in Chapter 6 of George Andrews' book *The theory of partitions* [12]. However, Meinardus' original theorem requires that the associated Dirichlet series of the sequence $(a_k)_{k\geq 1}$,

$$D(s) := \sum_{k\geq 1}\frac{a_k}{k^s},$$

has only one simple pole. But if $D(s)$ has multiple singularities, Meinardus' approach is still admissible, provided that we make suitable adjustments. A general result on such case was given by Granovsky and Stark [88], but the computation of coefficients is not explicit there. In this chapter, we are going to give a more transparent account of the generalization of Meinardus' approach, using $r_D(n)$ as a specific example.

10.2 Outline of the Proof

10.2.1 Cauchy's Integral Formula

Recall that Cauchy's integral formula indicates that

$$r_D(n) = \frac{1}{2\pi i}\int_{|q|=e^{-x}} \frac{F(q)}{q^{n+1}} dq.$$

Making the change of variables $q = e^{-\tau}$ with $\tau = x + 2\pi i y$, reversing the integral order and writing $f(\tau) = F(e^{-\tau})$, we obtain

$$r_D(n) = e^{nx}\int_{|y|\le\frac{1}{2}} f(\tau)e^{2\pi i n y}\, dy. \tag{10.2.1}$$

Note that here

$$f(\tau) = \prod_{k\ge 1}\frac{(1-e^{-2k\tau})^{2k+1}}{(1-e^{-k\tau})^{2k+1}}. \tag{10.2.2}$$

10.2.2 The Saddle Point Method

Let us begin with an estimate of $f(\tau)$, the proof of which will be given in §§10.3 and 10.4.

Theorem 10.2.1. *Let $\tau = x + 2\pi i y$. For $0 < x < 1/2$, we have that*

(i) *For $|\operatorname{Arg}(\tau)| \le \pi/4$,*

$$\log f(\tau) = \frac{3\zeta(3)}{2}\tau^{-2} + \frac{\zeta(2)}{2}\tau^{-1} - \frac{2\log 2}{3} + O(x^{1/2}). \tag{10.2.3}$$

(ii) *For $\pi/4 \le |\operatorname{Arg}(\tau)| \le \pi/2$,*

$$|f(\tau)| < f(x)e^{-\frac{1}{22x}}. \tag{10.2.4}$$

Now we apply the saddle point method to study the asymptotics of $r_D(n)$. To do so, we need to roughly minimize $e^{nx}f(\tau)$. In light of (10.2.3), it is enough to minimize

$$nx + \frac{3\zeta(3)}{2}x^{-2} + \frac{\zeta(2)}{2}x^{-1}.$$

Taking derivatives, setting to 0 and multiplying both sides by x^3, one has

$$nx^3 - \frac{\zeta(2)}{2}x - 3\zeta(3) = 0. \tag{10.2.5}$$

Let X be the unique positive root. Then

$$X = \Big(1 + o(1)\Big)\sqrt[3]{\frac{3\zeta(3)}{n}}. \tag{10.2.6}$$

Also, it can be computed that (see §10.5 for a proof):

$$X^{-1} = \frac{1}{(3\zeta(3))^{1/3}}n^{\frac{1}{3}} - \frac{\zeta(2)}{18\zeta(3)} + \frac{\zeta(2)^2}{36(3\zeta(3))^{5/3}}n^{-\frac{1}{3}} + O(n^{-\frac{2}{3}}). \tag{10.2.7}$$

Recall that

$$r_D(n) = e^{nX}\int_{|y|\le\frac{1}{2}} f(X+2\pi iy)e^{2\pi iny}\,dy.$$

Let us split the integral into two pieces.

$$\begin{aligned} r_D(n) &= e^{nX}\left(\int_{|y|\le X^{11/6}} + \int_{X^{11/6}\le|y|\le\frac{1}{2}}\right) f(X+2\pi iy)e^{2\pi iny}\,dy \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{10.2.8}$$

For convenience, we shall still write $\tau = X + 2\pi iy$.

10.2.3 The Main Term

Let us first compute the main term $\mathcal{I}_1$. Note that when $|y| \le X^{11/6}$, it follows from Part (i) of Theorem 10.2.1 that, as $n\to\infty$,

$$\log f(\tau) = \frac{3\zeta(3)}{2}\tau^{-2} + \frac{\zeta(2)}{2}\tau^{-1} - \frac{2\log 2}{3} + O(X^{1/2}).$$

We have the expansions

$$\tau^{-2} = \frac{1}{X^2} - \frac{4\pi iy}{X^3} - \frac{12\pi^2y^2}{X^4} + O(X^{1/2}) \tag{10.2.9}$$

and

$$\tau^{-1} = \frac{1}{X} - \frac{2\pi iy}{X^2} - \frac{4\pi^2y^2}{X^3} + O(X^{3/2}). \tag{10.2.10}$$

Hence,

$$f(\tau)e^{2\pi i n y} = \Big(1+o(1)\Big)\, 2^{-2/3} \exp\Bigg(\frac{3\zeta(3)}{2}\bigg(\frac{1}{X^2} - \frac{4\pi i y}{X^3} - \frac{12\pi^2 y^2}{X^4}\bigg)$$
$$+ \frac{\zeta(2)}{2}\bigg(\frac{1}{X} - \frac{2\pi i y}{X^2} - \frac{4\pi^2 y^2}{X^3}\bigg) + 2\pi i n y\Bigg)$$
$$= \Big(1+o(1)\Big)\, 2^{-2/3} \exp\Bigg(\bigg(\frac{3\zeta(3)}{2X^2} + \frac{\zeta(2)}{2X}\bigg) - \bigg(\frac{3\zeta(3)}{X^3} + \frac{\zeta(2)}{2X^2} - n\bigg)2\pi i y$$
$$- \bigg(\frac{9\zeta(3)}{2X^4} + \frac{\zeta(2)}{2X^3}\bigg)4\pi^2 y^2\Bigg).$$

It follows from (10.2.5) that

$$f(\tau)e^{2\pi i n y} = \Big(1+o(1)\Big)\, 2^{-2/3} \exp\Bigg(\bigg(\frac{3\zeta(3)}{2X^2} + \frac{\zeta(2)}{2X}\bigg) - \bigg(\frac{9\zeta(3)}{2X^4} + \frac{\zeta(2)}{2X^3}\bigg)4\pi^2 y^2\Bigg).$$

Hence,

$$\mathcal{I}_1 = e^{nX}\int_{|y|\le X^{11/6}} f(\tau)e^{2\pi i n y}\, dy$$
$$= \Big(1+o(1)\Big)\, 2^{-2/3} \exp\bigg(nX + \frac{3\zeta(3)}{2X^2} + \frac{\zeta(2)}{2X}\bigg)$$
$$\times \int_{|y|\le X^{11/6}} \exp\Bigg(-\bigg(\frac{9\zeta(3)}{2X^4} + \frac{\zeta(2)}{2X^3}\bigg)4\pi^2 y^2\Bigg)\, dy.$$

It follows from (10.2.5) and (10.2.7) that

$$\exp\bigg(nX + \frac{3\zeta(3)}{2X^2} + \frac{\zeta(2)}{2X}\bigg) = \exp\bigg(\frac{3\zeta(3)}{X^2} + \frac{\zeta(2)}{2X} + \frac{3\zeta(3)}{2X^2} + \frac{\zeta(2)}{2X}\bigg)$$
$$= \Big(1+o(1)\Big)\exp\bigg(\frac{3^{4/3}\zeta(3)^{1/3}}{2}n^{2/3} + \frac{\zeta(2)}{2\cdot 3^{1/3}\zeta(3)^{1/3}}n^{1/3} - \frac{\zeta(2)^2}{72\zeta(3)}\bigg).$$

Further, making the change of variables $u = \kappa y$ with

$$\kappa = 2\pi\sqrt{\frac{9\zeta(3)}{2X^4} + \frac{\zeta(2)}{2X^3}},$$

one has

$$\int_{|y|\le X^{11/6}} \exp\left(-\left(\frac{9\zeta(3)}{2X^4}+\frac{\zeta(2)}{2X^3}\right)4\pi^2y^2\right)\,dy = \kappa^{-1}\int_{|u|\le \kappa X^{11/6}} e^{-u^2}\,du. \tag{10.2.11}$$

Note that $\kappa X^{11/6} \gg X^{-1/6} \to \infty$ as $n \to \infty$ (so that $X \to 0$). Hence,

$$\int_{|u|\le \kappa X^{11/6}} e^{-u^2}\,du = \big(1+o(1)\big)\sqrt{\pi}. \tag{10.2.12}$$

Moreover,

$$\kappa^{-1} = \big(1+o(1)\big)\,2^{-1/2}3^{-1/3}\pi^{-1}\zeta(3)^{1/6}n^{-2/3}.$$

Hence,

$$\begin{aligned}\mathcal{I}_1 &= \big(1+o(1)\big)\,2^{-7/6}3^{-1/3}\pi^{-1/2}\zeta(3)^{1/6}n^{-2/3}\\ &\quad\times \exp\left(\frac{3^{4/3}\zeta(3)^{1/3}}{2}n^{2/3}+\frac{\zeta(2)}{2\cdot 3^{1/3}\zeta(3)^{1/3}}n^{1/3}-\frac{\zeta(2)^2}{72\zeta(3)}\right).\end{aligned}$$

10.2.4 The Error Term

The integral $\mathcal{I}_2$ contributes the error term. Note that for sufficiently large n, one has $X^{11/6} \le X/2\pi \le 1/2$. Now we separate $X^{11/6} \le |y| \le 1/2$ into two cases.

Case 1: $X^{11/6} \le |y| \le 1/2$ and $|\operatorname{Arg}(\tau)| \ge \pi/4$. Hence, $X/2\pi \le |y| \le 1/2$. We can see from the proof above that

$$\begin{aligned}e^{nX}f(X) &\ll \exp\left(nX+\frac{3\zeta(3)}{2X^2}+\frac{\zeta(2)}{2X}\right)\\ &\ll \exp\left(\frac{3^{4/3}\zeta(3)^{1/3}}{2}n^{2/3}+\frac{\zeta(2)}{2\cdot 3^{1/3}\zeta(3)^{1/3}}n^{1/3}\right).\end{aligned}$$

Further, it follows from Part (ii) of Theorem 10.2.1 that

$$\begin{aligned}e^{nX}\int_{\frac{X}{2\pi}\le|y|\le\frac12} f(\tau)e^{2\pi i n y}\,dy &\ll e^{nX}f(X)e^{-\frac{1}{22X}}\\ &\ll \exp\left(\frac{3^{4/3}\zeta(3)^{1/3}}{2}n^{2/3}+\frac{\zeta(2)}{2\cdot 3^{1/3}\zeta(3)^{1/3}}n^{1/3}-\frac{1+o(1)}{22\cdot 3^{1/3}\zeta(3)^{1/3}}n^{1/3}\right)\\ &= o(\mathcal{I}_1).\end{aligned}$$

Case 2: $X^{11/6} \leq |y| \leq 1/2$ and $|\operatorname{Arg}(\tau)| \leq \pi/4$. Hence, $X^{11/6} \leq |y| \leq X/2\pi$. It follows from the expansions (10.2.9) and (10.2.10) that there exist constants $c_1,\ c_2 > 0$ such that

$$\Re(\tau^{-2}) \leq X^{-2} - c_1 X^{-1/3}$$

and

$$\Re(\tau^{-1}) \leq X^{-1} - c_2 X^{2/3}.$$

It follows from Part (i) of Theorem 10.2.1 that there exists a constant $c > 0$ such that

$$|f(\tau)| \ll \exp\left(\frac{3\zeta(3)}{2X^2} + \frac{\zeta(2)}{2X} - \frac{c}{X^{1/3}}\right).$$

Hence,

$$\begin{aligned} e^{nX} \int_{X^{11/6} \leq |y| \leq \frac{X}{2\pi}} & f(\tau) e^{2\pi i n y}\, dy \\ & \ll \exp\left(nX + \frac{3\zeta(3)}{2X^2} + \frac{\zeta(2)}{2X} - \frac{c}{X^{1/3}}\right) \\ & = o(\mathcal{I}_1), \end{aligned}$$

where we use a similar argument to that in *Case 1*.

10.2.5 Remark on the Choice $X^{11/6}$

Let us briefly comment on why do we split the integral at $|y| = X^{11/6}$ in (10.2.8). Let us say the integral is split at $|y| = X^{\theta}$.

To obtain the truncated Gaussian integral in (10.2.11), one should expand τ^{-2} and τ^{-1} in (10.2.9) and (10.2.10) to the third term so that the error terms are $o(1)$. We can compute that the two errors are, respectively, $O(y^3/X^5) = O(X^{3\theta-5})$ and $O(y^3/X^4) = O(X^{3\theta-4})$. Hence, one should have $3\theta - 5 > 0$ so that $\theta > 5/3$.

On the other hand, to ensure that (10.2.12) is true, one should have $\kappa X^{\theta} \to \infty$ as $X \to 0$. This indicates that $\theta - 2 < 0$ so that $\theta < 2$.

Hence, we merely need to choose θ in $(5/3, 2)$.

10.3 Part (i) of Theorem 10.2.1

Recall that $|\operatorname{Arg}(\tau)| \leq \pi/4$.

10.3.1 Mellin Transform

It follows from (10.2.2) that

$$\begin{aligned}\log f(\tau) &= \sum_{k\geq 1}(2k+1)\Big(\log(1-e^{-2k\tau}) - \log(1-e^{-k\tau})\Big)\\ &= \sum_{k\geq 1}(2k+1)\sum_{\ell\geq 1}\left(\frac{e^{-k\ell\tau}}{\ell} - \frac{e^{-2k\ell\tau}}{\ell}\right).\end{aligned}$$

Recall that the Mellin transform maps e^{-t} to the Gamma function $\Gamma(s)$, that is, for $c > 0$,

$$e^{-t} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)t^{-s}\,ds.$$

Hence,

$$\begin{aligned}\log f(\tau) &= \sum_{k\geq 1}(2k+1)\sum_{\ell\geq 1}\frac{1}{\ell}\left(\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}\Gamma(s)\Big((k\ell\tau)^{-s} - (2k\ell\tau)^{-s}\Big)\,ds\right)\\ &= \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}\tau^{-s}(1-2^{-s})\Gamma(s)\left(\sum_{\ell\geq 1}\frac{1}{\ell^{s+1}}\right)\left(\sum_{k\geq 1}\frac{2k+1}{k^s}\right)ds\\ &= \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty}\tau^{-s}(1-2^{-s})\Gamma(s)\zeta(s+1)\Big(2\zeta(s-1)+\zeta(s)\Big)\,ds. \qquad (10.3.1)\end{aligned}$$

Here we may interchange the integral and summations as $\Gamma(s)$ decays rapidly to 0 as we integrate up the line $3+it$ and the summation over k and ℓ is absolutely convergent for s on this line.

Let us write

$$\Phi(s) := (1-2^{-s})\Gamma(s)\zeta(s+1)\Big(2\zeta(s-1)+\zeta(s)\Big).$$

10.3.2 Shifting the Path of Integration

Recall that a standard result on the Gamma function asserts that $|\Gamma(s)|$ dies away to 0 exponentially along any fixed vertical line. More precisely, for $s = \sigma + it$ with σ fixed, we have that, as $|t| \to \infty$,

$$|\Gamma(s)| = \exp\left(\left(-\frac{\pi}{2} + o(1)\right)|t|\right). \qquad (10.3.2)$$

On the other hand, $|\zeta(s)|$ has at most polynomial growth on fixed vertical lines.[1]

Note that $\Phi(s)$ has three poles $s = 0$, 1, and 2 in the stripe $-1/2 \le \Re(s) \le 3$. If we integrate $\tau^{-s}\Phi(s)$ over the rectangle with corners

$$3 - iT, \quad 3 + iT, \quad -\frac{1}{2} + iT, \quad -\frac{1}{2} - iT,$$

the residue theorem tells us that

$$\begin{aligned}
&\frac{1}{2\pi i}\left(\int_{3-iT}^{3+iT} + \int_{3+iT}^{-\frac{1}{2}+iT} + \int_{-\frac{1}{2}+iT}^{-\frac{1}{2}-iT} + \int_{-\frac{1}{2}-iT}^{3-iT}\right)\tau^{-s}\Phi(s)\, ds \\
&\quad = \operatorname{Res}_{s=0} \tau^{-s}\Phi(s) + \operatorname{Res}_{s=1} \tau^{-s}\Phi(s) + \operatorname{Res}_{s=2} \tau^{-s}\Phi(s). \qquad (10.3.3)
\end{aligned}$$

Further, the two integrals

$$\int_{3+iT}^{-\frac{1}{2}+iT} \quad \text{and} \quad \int_{-\frac{1}{2}-iT}^{3-iT}$$

die away to 0 as $T \to \infty$, following from the growth rates of $\Gamma(s)$ and $\zeta(s)$ along vertical lines.

We now bound the integral

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \tau^{-s}\Phi(s)\, ds.$$

Recall that $|\operatorname{Arg}(\tau)| \le \pi/4$. It follows from the relation $\tau = x + 2\pi i y$ that

$$|\tau| \le \sqrt{2}x.$$

Hence, along the line $s = -\frac{1}{2} + it$, we have

$$\begin{aligned}
|\tau^{-s}| &= |\exp(-s\log\tau)| \\
&= \exp\left(\frac{1}{2}|\tau| + t\operatorname{Arg}(\tau)\right)
\end{aligned}$$

[1]In fact, if we define $\mu(\sigma) := \inf\left\{m \in \mathbb{R} : \zeta(\sigma + it) = O(|t|^m)\right\}$, then

$$\mu(\sigma) = \begin{cases} 0 & \text{if } \sigma > 1, \\ \frac{1}{2}(1-\sigma) & \text{if } 0 \le \sigma \le 1, \\ \frac{1}{2} - \sigma & \text{if } \sigma < 0. \end{cases}$$

When $\sigma > 1$, it is trivial. When $\sigma < 0$, the result follows from the functional equation of the Riemann zeta function. When $0 \le \sigma \le 1$, the result can be deduced from the theorem of Phragmén–Lindelöf.

$$\leq |\tau|^{1/2} \exp\left(\frac{\pi}{4}|t|\right).$$

It follows from (10.3.2) that

$$|\tau^{-s}\Gamma(s)| \leq |\tau|^{1/2} \exp\left(\left(-\frac{\pi}{4} + o(1)\right)|t|\right).$$

Again, the exp($\cdot$) factor above decreases to 0 exponentially as $|t| \to \infty$. It follows from the polynomial growth of the Riemann zeta function that

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \tau^{-s}\Phi(s)\, ds \ll x^{1/2}.$$

It follows from (10.3.1) and (10.3.3) that

$$\log f(\tau) = \operatorname{Res}_{s=0} \tau^{-s}\Phi(s) + \operatorname{Res}_{s=1} \tau^{-s}\Phi(s) + \operatorname{Res}_{s=2} \tau^{-s}\Phi(s) + O(x^{1/2}). \qquad (10.3.4)$$

10.3.3 Residues

Let us compute the residues of $\Phi(s)$ at $s = 0$, 1, and 2. Recall that

$$\Phi(s) = (1 - 2^{-s})\Gamma(s)\zeta(s+1)\Big(2\zeta(s-1) + \zeta(s)\Big).$$

$\Phi(s)$ has simple poles at $s = 1$ and 2. Hence,

$$\begin{aligned}\operatorname{Res}_{s=1} \tau^{-s}\Phi(s) &= \tau^{-1}(1 - 2^{-1})\Gamma(1)\zeta(2) \operatorname{Res}_{s=1} \zeta(s) \\ &= \frac{\zeta(2)}{2}\tau^{-1}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Res}_{s=2} \tau^{-s}\Phi(s) &= 2\tau^{-2}(1 - 2^{-2})\Gamma(2)\zeta(3) \operatorname{Res}_{s=2} \zeta(s-1) \\ &= \frac{3\zeta(3)}{2}\tau^{-2}.\end{aligned}$$

For the pole at $s = 0$, we know that $1 - 2^{-s} = (\log 2)s + O(s^2)$, $\Gamma(s) = s^{-1} + O(1)$, and $\zeta(s+1) = s^{-1} + O(1)$. Hence,

$$\tau^{-s}\Phi(s) = \Big(1 + O(s)\Big)\Big((\log 2)s + O(s^2)\Big)\Big(s^{-1} + O(1)\Big)$$

$$\times \Big(s^{-1} + O(1)\Big)\Big((2\zeta(-1) + \zeta(0)) + O(s)\Big),$$

so that

$$\operatorname{Res}_{s=0} \tau^{-s}\Phi(s) = \Big(2\zeta(-1) + \zeta(0)\Big)\log 2 = -\frac{2\log 2}{3}.$$

It follows from (10.3.4) that

$$\log f(\tau) = \frac{3\zeta(3)}{2}\tau^{-2} + \frac{\zeta(2)}{2}\tau^{-1} - \frac{2\log 2}{3} + O(x^{1/2}).$$

10.4 Part (ii) of Theorem 10.2.1

Recall that $\pi/4 \le |\operatorname{Arg}(\tau)| \le \pi/2$. We have also assumed that $0 < x < 1/2$ and $|y| \le 1/2$. For convenience, let us put

$$a_k = \begin{cases} 2k+1 & \text{if } k \text{ is odd,} \\ k & \text{if } k \text{ is even.} \end{cases}$$

Note that $a_k \ge 1$ for all positive k. It follows from (10.2.2) that

$$f(\tau) = \prod_{k\ge1} \frac{(1-e^{-2k\tau})^{2k+1}}{(1-e^{-k\tau})^{2k+1}} = \prod_{k\ge1} \frac{1}{(1-e^{-k\tau})^{a_k}}.$$

Hence,

$$\begin{aligned}
\log \frac{|f(\tau)|}{f(x)} &= \Re\left(\sum_{k\ge1} a_k\Big(\log(1-e^{-kx}) - \log(1-e^{-k\tau})\Big)\right) \\
&= \sum_{k\ge1} a_k \sum_{\ell\ge1} \frac{e^{-k\ell x}}{\ell}\Big(\cos(2\pi k\ell y) - 1\Big) \\
&\le \sum_{k\ge1} a_k e^{-kx}\Big(\cos(2\pi k y) - 1\Big) \\
&\le \sum_{k\ge1} e^{-kx}\Big(\cos(2\pi k y) - 1\Big) \\
&= \Re\left(\frac{e^{-\tau}}{1-e^{-\tau}}\right) - \frac{e^{-x}}{1-e^{-x}} \\
&\le \frac{e^{-x}}{|1-e^{-\tau}|} - \frac{e^{-x}}{1-e^{-x}}.
\end{aligned}$$

Next, we observe that

$$|1-e^{-\tau}| = \sqrt{1-2e^{-x}\cos(2\pi y)+e^{-2x}}. \tag{10.4.1}$$

Since $\pi/4 \le |\operatorname{Arg}(\tau)| \le \pi/2$ and $|y| \le 1/2$, it follows that $x \le |2\pi y| \le \pi$. Hence for fixed x, the right-hand side of (10.4.1) is minimized when $2\pi y = \pm x$. That is,

$$|1-e^{-\tau}| \ge \sqrt{1-2e^{-x}\cos(x)+e^{-2x}}. \tag{10.4.2}$$

Now we show that when $0 < x < 1/2$,

$$1-2e^{-x}\cos(x)+e^{-2x} > 1.21(1-e^{-x})^2. \tag{10.4.3}$$

This is equivalent to

$$0.21(e^{-2x}+1) - 2.42e^{-x} + 2e^{-x}\cos(x) < 0. \tag{10.4.4}$$

It suffices to show that the left-hand side of (10.4.4) is a decreasing function for $x \in (0,1/2)$. Differentiating the left-hand side of (10.4.4), one has

$$2e^{-x}\Big(1.21 - \sqrt{2}\sin(x+\pi/4)\Big) - 0.42e^{-2x}.$$

To show the above function is < 0, it suffices to check

$$0.21e^{-x} + \sqrt{2}\sin(x+\pi/4) > 1.21. \tag{10.4.5}$$

Noting that $e^{-x} > 1-x$, we have

$$0.21e^{-x} + \sqrt{2}\sin(x+\pi/4) > 0.21(1-x) + \sqrt{2}\sin(x+\pi/4) > 1.21.$$

The last inequality is true since $0.21(1-x)+\sqrt{2}\sin(x+\pi/4)$ is increasing for $x \in (0,1/2)$.

It follows from (10.4.2) and (10.4.3) that, when $0 < x < 1/2$,

$$|1-e^{-\tau}| > 1.1(1-e^{-x}).$$

It turns out that

$$\log\frac{|f(\tau)|}{f(x)} < -\frac{1}{11}\frac{e^{-x}}{1-e^{-x}}.$$

It is not hard to verify that when $0 < x < 1/2$,

$$\frac{e^{-x}}{1-e^{-x}} > \frac{1}{2x}.$$

Hence,

$$\log \frac{|f(\tau)|}{f(x)} < -\frac{1}{22x},$$

so that

$$|f(\tau)| < f(x) e^{-\frac{1}{22x}}.$$

10.5 Expansion of X^{-1}

In this section, we give the expansion of X^{-1}. Recall from (10.2.5) that

$$aX^{-3} + bX^{-2} - n = 0, \tag{10.5.1}$$

where

$$a = 3\zeta(3) \quad \text{and} \quad b = \frac{\zeta(2)}{2}.$$

Let us write

$$\mu = n^{-\frac{1}{3}} \quad \text{and} \quad X^{-1} = a^{-\frac{1}{3}}\mu^{-1} + \xi.$$

Then (10.5.1) becomes

$$a\Big(a^{-\frac{1}{3}}\mu^{-1} + \xi\Big)^3 + b\Big(a^{-\frac{1}{3}}\mu^{-1} + \xi\Big)^2 - \mu^{-3} = 0,$$

so that by multiplying by $a^{\frac{2}{3}}\mu^2$ on both sides, one has

$$a\xi \cdot \left(\Big(1 + a^{\frac{1}{3}}\mu\xi\Big)^2 + \Big(1 + a^{\frac{1}{3}}\mu\xi\Big) + 1\right) + b\Big(1 + a^{\frac{1}{3}}\mu\xi\Big)^2 = 0. \tag{10.5.2}$$

Now we may treat $\xi := \xi(\mu)$ as an implicit function of μ defined by (10.5.2). Note that

$$\xi(0) = -\frac{b}{3a} = -\frac{\zeta(2)}{18\zeta(3)}.$$

The implicit function theorem ensures that we may write $\xi(\mu)$ as a power series in μ in a neighborhood of $\mu = 0$. We compute that

$$\xi'(0) = \frac{b^2}{9a^{5/3}} = \frac{\zeta(2)^2}{36(3\zeta(3))^{5/3}}.$$

Hence,

$$\xi(\mu) = -\frac{\zeta(2)}{18\zeta(3)} + \frac{\zeta(2)^2}{36(3\zeta(3))^{5/3}}\mu + O(\mu^2)$$

so that as $n \to \infty$,

$$X^{-1} = \frac{1}{(3\zeta(3))^{1/3}}n^{\frac{1}{3}} - \frac{\zeta(2)}{18\zeta(3)} + \frac{\zeta(2)^2}{36(3\zeta(3))^{5/3}}n^{-\frac{1}{3}} + O(n^{-\frac{2}{3}}).$$

10.6 References

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Chapter 11

The Method of Rademacher: Background and Preliminaries

This chapter comes from

- S. Chern, Asymptotics for the Fourier coefficients of eta-quotients, *J. Number Theory* **199** (2019), 168–191. (Ref. [53])
- S. Chern, S. Chern, Asymptotics for the Taylor coefficients of certain infinite products, to appear in *Ramanujan J.* (Ref. [57])

In this series of three chapters, Rademacher's method on asymptotics will be investigated. We will utilize this method to deduce two general results, one of which concerns infinite products involving the Dedekind eta function and the other of which concerns infinite products involving the Jacobi theta function.

11.1 Introduction

It is, more or less, reasonable to say that the prospering circle method was born when Hardy and Ramanujan decided to study the asymptotics of the partition function $p(n)$. By focusing on the asymptotics of the generating function $1/(q;q)_\infty$ near $q=1$ inside the unit disc $\mathbb{D} \subset \mathbb{C}$, Hardy and Ramanujan [96] showed that

$$p(n) \sim \frac{1}{4\sqrt{3}} n^{-1} e^{\frac{2\pi\sqrt{n}}{\sqrt{6}}}. \tag{11.1.1}$$

A couple of decades later, Rademacher [143] stepped further and proved an exact series for $p(n)$:

$$p(n) = \frac{1}{2\sqrt{2}\pi} \sum_{k\geq 1} A_k(n)\sqrt{k}\, \frac{d}{dn}\left(\frac{2}{\sqrt{n-\frac{1}{24}}}\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)\right), \tag{11.1.2}$$

where

$$A_k(n) = \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} e^{\pi i (s(h,k) - 2nh/k)}$$

with $s(h,k)$ being the Dedekind sum defined by

$$s(d,c) := \sum_{n \bmod c} \left(\!\!\left(\frac{dn}{c}\right)\!\!\right) \left(\!\!\left(\frac{n}{c}\right)\!\!\right) \tag{11.1.3}$$

where

$$(\!(x)\!) := \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Rademacher's approach is straightforward in essence, however delicate in detail. The basic idea is merely Cauchy's integral formula, but we need various techniques including Ford circles, Farey sequences, modular symmetry and the Dedekind eta-function.

One natural generalization that would come up to one's mind is the following general family of holomorphic functions on the open unit disk $\mathbb{D}$:

$$G(q) = \sum_{n \ge 0} g(n) q^n = \prod_{j=1}^{J} (q^{m_j}; q^{m_j})_\infty^{\delta_j}, \tag{11.1.4}$$

where $\mathbf{m} = (m_1, \ldots, m_J)$ is a sequence of J distinct positive integers and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_J)$ is a sequence of J non-zero integers. For some specific $G(q)$, the interested readers may refer to the work of Grosswald [89], Iseki [102, 103], Hagis Jr. [92–94], O-Y. Chan [46] and many others. For general $G(q)$ with $\sum_{r=1}^{R} \delta_r < 0$, the recent work of Sussman [166] presented a Rademacher-type formula. Sussman's result can in some sense be treated as a special case of the work of Bringmann and Ono [44], in which the coefficients of harmonic Maass forms are studied. On the other hand, Sills [162] provided an automatic algorithm when $\sum_{r=1}^{R} \delta_r = 0$. When $\sum_{r=1}^{R} \delta_r = 1$, a subclass of such $G(q)$ was studied by B. Kim [112].

Another direction that might be of one's interest is about infinite products under symmetric congruence conditions:

$$G(q) = \sum_{n \ge 0} g(n) q^n = \prod_{j=1}^{J} (q^{r_j}, q^{m_j - r_j}; q^{m_j})_\infty^{\delta_j}, \tag{11.1.5}$$

where $\mathbf{m} = (m_1, \ldots, m_J)$ and $\mathbf{r} = (r_1, \ldots, r_J)$ are two sequences of J positive integers satisfying $1 \le r_j < m_j$ for all $j = 1, \ldots, J$, and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_J)$ is a sequence of J

nonzero integers. Regarding such infinite products, the most famous examples arise from the Rogers–Ramanujan identities (Rogers [156], Ramanujan [145]). Recall that the first Rogers–Ramanujan identity states that (cf. Corollary 7.67 in [12])

$$\frac{1}{(q,q^4;q^5)_\infty}=\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}.$$

On the left-hand side, we have the generating function for partitions such that each part is congruent to ± 1 modulo 5. Let $p_{5,\pm 1}(n)$ be the number of such partitions of n. Its asymptotic formula was shown by Lehner [118]:

$$p_{5,\pm 1}(n)\sim\frac{\csc(\pi/5)}{4\cdot 3^{1/4}\cdot 5^{1/4}}n^{-3/4}\exp\left(2\pi\sqrt{\frac{n}{15}}\right). \tag{11.1.6}$$

The interested reader may also refer to Niven [136], Livingood [130], Petersson [140, 141], Subrahmanyasastri [165] and so forth for the asymptotic behaviors of other partition functions under symmetric congruence conditions. Next, the infinite product (11.1.5) may also be of number-theoretic interest. One example is the Rogers–Ramanujan continued fraction. Recall that the Rogers–Ramanujan continued fraction has an infinite product form

$$\frac{q^{1/5}}{1}\;+\;\frac{q}{1}\;+\;\frac{q^2}{1}\;+\;\frac{q^3}{1}\;+\cdots=\left(\begin{matrix}q,q^4\\q^2,q^3\end{matrix};q^5\right)_\infty.$$

Let us focus on the infinite product part and write

$$\sum_{n\geq 0}C(n)q^n=\left(\begin{matrix}q,q^4\\q^2,q^3\end{matrix};q^5\right)_\infty.$$

It is known from Richmond and Szekeres [152] that

$$C(n)\sim\frac{2^{1/2}}{5^{3/4}}\cos\left(\frac{4\pi}{5}\left(n+\frac{3}{20}\right)\right)n^{-3/4}\exp\left(\frac{4\pi}{5}\sqrt{\frac{n}{5}}\right). \tag{11.1.7}$$

Hence for sufficiently large n, $C(5n+0,2)>0$ and $C(5n+1,3,4)<0$. We also remark that in [152], Richmond and Szekeres indeed studied the asymptotic behavior of the Taylor coefficients of the general infinite product

$$\prod_{j=1}^{m-1}(q^j;q^m)^{-\zeta\chi(j)}$$

where m is a positive fundamental discriminant, $\chi(j) = (m|j)$ is the Kronecker symbol and ζ is either 1 or -1. Finally, in recent years, there are a number of papers [4,21,99,133,167] studying vanishing Taylor coefficients of certain infinite products. For instance, Tang [167] showed that the Taylor coefficients of

$$\sum_{n\geq 0} B(n)q^n = (-q^2,-q^3;q^5)_\infty^2 (q^2,q^8;q^{10})_\infty = \frac{(q^2,q^8;q^{10})_\infty (q^4;q^6;q^{10})_\infty^2}{(q^2,q^3;q^5)_\infty^2}$$

satisfy $B(5n+1) = 0$ for all $n \geq 0$. At the end of Tang's paper, he also provided numerical evidence of the inequalities $B(5n+0,2,3) > 0$ and $B(5n+4) < 0$ for sufficiently large n. Similar numerical evidences are also provided for inequalities of Taylor coefficients of other infinite products.

The aim of this series is to study the asymptotics for the Taylor coefficients in the infinite products (11.1.4) and (11.1.5). In this chapter, we will provide necessary preliminaries.

11.2 Dedekind Eta Function and Jacobi Theta Function

In this section, we introduce the Dedekind eta function and Jacobi theta function. All results here are standard, which can be found in, for example, [25] or [176].

Let $\tau \in \mathbb{H}$ and $\varsigma \in \mathbb{C}$. The Dedekind eta function is defined by

$$\eta(\tau) := q^{1/24}(q;q)_\infty \tag{11.2.1}$$

with $q := e^{2\pi i\tau}$. Further, the Jacobi theta function reads

$$\vartheta(\varsigma;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i\nu(\varsigma+\frac{1}{2})+\pi i\nu^2\tau}. \tag{11.2.2}$$

Notice that if we put $\zeta := e^{2\pi i\varsigma}$, then the Jacobi triple product identity indicates that

$$\vartheta(\varsigma;\tau) = -iq^{1/8}\zeta^{-1/2}(\zeta,\zeta^{-1}q,q;q)_\infty. \tag{11.2.3}$$

The Dedekind eta function and Jacobi theta function are of broad interest due to their transformation properties. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ where we assume that $c > 0$.

Recall that the Möbius transformation for $\tau \in \mathbb{H}$ is defined by

$$\gamma(\tau) := \frac{a\tau + b}{c\tau + d}.$$

Further, for the γ given above, we write for convenience

$$\gamma^*(\tau) := \frac{1}{c\tau + d}.$$

If

$$\chi(\gamma) = \exp\left(\pi i\left(\frac{a+d}{12c} - s(d,c) - \frac{1}{4}\right)\right),$$

where, again, $s(d,c)$ is the Dedekind sum, then

$$\eta(\gamma(\tau)) = \chi(\gamma)(c\tau + d)^{1/2}\eta(\tau) \tag{11.2.4}$$

and

$$\vartheta(\varsigma\gamma^*(\tau); \gamma(\tau)) = \chi(\gamma)^3(c\tau + d)^{1/2}e^{\frac{\pi i c\varsigma^2}{c\tau+d}}\vartheta(\varsigma; \tau). \tag{11.2.5}$$

Further, let α and β be integers. The Jacobi theta function also satisfies

$$\vartheta(\varsigma + \alpha\tau + \beta; \tau) = (-1)^{\alpha+\beta}e^{-\pi i\alpha^2\tau}e^{-2\pi i\alpha\varsigma}\vartheta(\varsigma; \tau). \tag{11.2.6}$$

11.3 Cauchy's Integral Formula and Farey Arcs

To study the asymptotics for the Taylor coefficients of a holomorphic function $G(q)$ inside the unit disk, we turn to the celebrated circle method due to Rademacher [143,144] whose idea originates from Hardy and Ramanujan [96]. We directly apply Cauchy's integral formula to deduce

$$g(n) = \frac{1}{2\pi i}\oint_{\mathcal{C}:|q|=r}\frac{G(q)}{q^{n+1}}\,dq,$$

where the contour integral is taken counter-clockwise. Now one puts $r = e^{-2\pi\varrho}$ with $\varrho = 1/N^2$ where N is a sufficiently large positive integer.

The next task is to study the asymptotics of $G(q)$ when q is close to a rational point $\exp(2\pi i h/k)$ on the unit circle. To do so, we dissect the circle $\mathcal{C}$ by Farey arcs. Let h/k

with $\gcd(h,k)=1$ be a Farey fraction of order N.[1] If we denote by $\xi_{h,k}$ the interval $[-\theta'_{h,k},\theta''_{h,k}]$ with $\theta'_{h,k}$ and $\theta''_{h,k}$ being the positive distances from h/k to its neighboring mediants, then $\mathbb{R}/\mathbb{Z}$ can be covered by intervals $\bigcup_{h,k}\xi_{h,k}$ where $0\le h<k\le N$ and $\gcd(h,k)=1$. For each q on the circle $\mathcal{C}$, we may find a Farey fraction h/k such that $\arg(q)=2\pi(h/k+\phi)$ with $\phi\in\xi_{h,k}$. Thus, we have $q=e^{2\pi i(h/k+i\varrho+\phi)}$ and hence,

$$g(n)=\sum_{1\le k\le N}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi inh}{k}}\int_{\xi_{h,k}} G\Big(e^{2\pi i(h/k+i\varrho+\phi)}\Big)e^{-2\pi in\phi}e^{2\pi n\varrho}\,d\phi.$$

Let $z=k(\varrho-i\phi)$. Making the change of variables $\tau=(h+iz)/k$ yields

$$g(n)=\sum_{1\le k\le N}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi inh}{k}}\int_{\xi_{h,k}} G\Big(e^{2\pi i\tau}\Big)e^{-2\pi in\phi}e^{2\pi n\varrho}\,d\phi. \tag{11.3.1}$$

11.4 Choosing a Suitable Matrix in $SL_2(\mathbb{Z})$

Another task we should finish is to construct a suitable matrix in $SL_2(\mathbb{Z})$ so that the infinite products (11.1.4) and (11.1.5) can be nicely reformulated around the Farey arc with respect to h/k through the transformation properties of the Dedekind eta function and Jacobi theta function.

Below we assume that $0\le h<k$ are integers such that $\gcd(h,k)=1$. Let m be a positive integer.

Let $d=\gcd(m,k)$. For convenience, we write $m=dm'$ and $k=dk'$. We put $\hbar_m(h,k)$ an integer such that

$$\hbar_m(h,k)\frac{mh}{\gcd(m,k)}\equiv -1 \pmod{\frac{k}{\gcd(m,k)}}.$$

Notice that one may always find such an integer since $\gcd(h,k)=1$. Let us put $b_{m'}=(\hbar_m(h,k)m'h+1)/k'$. It is straightforward to verify that the following matrix is in $SL_2(\mathbb{Z})$:

$$\gamma_{(m,h,k)}=\begin{pmatrix}\hbar_m(h,k) & -b_{m'}\\ k' & -m'h\end{pmatrix}. \tag{11.4.1}$$

[1]The *Farey sequence* of order N is the increasing sequence of irreducible fractions between 0 and 1 whose denominator is at most N. For example, $\{\frac{0}{1},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{1}{1}\}$ is the Farey sequence of order 6.

Since $\tau = (h+iz)/k = (h+iz)/dk'$, one may compute

$$\begin{aligned}
&\gamma_{(m,h,k)}(m\tau) \\
&\quad = \frac{\hbar_m(h,k)\cdot m\frac{h+iz}{dk'} - b_{m'}}{k'\cdot m\frac{h+iz}{dk'} - m'h} = \frac{\hbar_m(h,k)m'h + i\hbar_m(h,k)m'z - (\hbar_m(h,k)m'h+1)}{m'hk' + ik'm'z - m'hk'} \\
&\quad = \frac{\hbar_m(h,k)}{k'} + \frac{1}{m'k'z}i.
\end{aligned}$$

Thus,

$$\gamma_{(m,h,k)}(m\tau) = \frac{\hbar_m(h,k)\gcd(m,k)}{k} + \frac{\gcd^2(m,k)}{mkz}i. \tag{11.4.2}$$

On the other hand, we have

$$\gamma^*_{(m,h,k)}(m\tau) = \frac{1}{k'\cdot m\frac{h+iz}{dk'} - m'h} = -\frac{\gcd(m,k)}{mz}i$$

and hence for $r < m$,

$$r\tau\gamma^*_{(m,h,k)}(m\tau) = \frac{r\gcd(m,k)}{mk} - \frac{rh\gcd(m,k)}{mkz}i. \tag{11.4.3}$$

Further, if we put

$$\lambda_{m,r}(h,k) := \left\lceil \frac{rh}{\gcd(m,k)} \right\rceil$$

and

$$\lambda^*_{m,r}(h,k) := \lambda_{m,r}(h,k) - \frac{rh}{\gcd(m,k)},$$

then,

$$\begin{aligned}
&r\tau\gamma^*_{(m,h,k)}(m\tau) + \lambda_{m,r}(h,k)\gamma_{(m,h,k)}(m\tau) \\
&\quad = \frac{r\gcd(m,k)}{mk} + \lambda_{m,r}(h,k)\frac{\hbar_m(h,k)\gcd(m,k)}{k} + \lambda^*_{m,r}(h,k)\frac{\gcd^2(m,k)}{mkz}i.
\end{aligned} \tag{11.4.4}$$

11.5 Some Auxiliary Results

11.5.1 Necessary Bounds

Now we are going to present some useful bounds.

First, it is well known (cf. Chapter 3 in [97]) that for a Farey fraction h/k of order

N, one has
$$\frac{1}{2kN} \le \theta'_{h,k}, \theta''_{h,k} \le \frac{1}{kN}. \tag{11.5.1}$$
Let $|\xi_{h,k}|$ be the length of the interval $\xi_{h,k}$. By noticing that $|\xi_{h,k}| = \theta'_{h,k} + \theta''_{h,k}$, one has
$$\frac{1}{kN} \le |\xi_{h,k}| \le \frac{2}{kN}. \tag{11.5.2}$$

Next, since $z = k(\varrho - i\phi)$, it follows that
$$\Re(z) = k\varrho = \frac{k}{N^2}. \tag{11.5.3}$$
This implies that
$$|z| \ge \frac{k}{N^2}. \tag{11.5.4}$$
Further, one has
$$\Re\left(\frac{1}{z}\right) \ge \frac{k}{2} \tag{11.5.5}$$
since
$$\Re\left(\frac{1}{z}\right) = \frac{1}{k}\frac{\varrho}{\varrho^2+\phi^2} \ge \frac{1}{k}\frac{N^{-2}}{N^{-4}+k^{-2}N^{-2}} = \frac{k}{k^2N^{-2}+1} \ge \frac{k}{1+1} = \frac{k}{2},$$
where we use the fact $k \le N$ in the last inequality.

11.5.2 Some Partition-theoretic Results

In this section, η is a positive integer and δ is a nonzero integer. Let q be such that $|q| < 1$.

Let $p_\eta(n)$ denote the number of partition η-tuples of n. Then
$$\sum_{n\ge 0} p_\eta(n) q^n = \left(\frac{1}{(q;q)_\infty}\right)^\eta.$$
Further, if we write
$$\sum_{n\ge 0} d_\eta(n) q^n := (q;q)_\infty^\eta,$$
an easy partition-theoretic argument indicates that $|d_\eta(n)| \le p_\eta(n)$ for all $n \ge 0$. Also,

we have $d_\eta(0) = p_\eta(0) = 1$. In general, if we write

$$\sum_{n\geq 0} a_\delta(n) q^n := (q;q)_\infty^\delta,$$

then

$$a_\delta(n) = \begin{cases} p_{|\delta|}(n) & \text{if } \delta < 0, \\ d_{|\delta|}(n) & \text{if } \delta > 0, \end{cases}$$

and hence $|a_\delta(n)| \leq p_{|\delta|}(n)$ for all $n \geq 0$. Trivially, we also have

$$\begin{aligned} \left|(q;q)_\infty^\delta\right| &= \left|\sum_{n\geq 0} a_\delta(n) q^n\right| \\ &\leq \sum_{n\geq 0} p_{|\delta|}(n) |q|^n. \end{aligned}$$

Further, for real $0 \leq x < 1$, we have

$$\begin{aligned} \sum_{n\geq 0} p_1(n) x^n &= \frac{1}{(x;x)_\infty} \\ &= \exp\left(- \sum_{k\geq 1} \log(1-x^k)\right) \\ &\leq \exp\left(\frac{x}{(1-x)^2}\right). \end{aligned} \tag{11.5.6}$$

Likewise, let $p_\eta^*(s,t;n)$ denote the number of 2-colored (say, red and blue) partition η-tuples of n with s parts in total colored by red and t parts in total colored by blue. Here we allow 0 as a part. Let ζ and ξ be such that $|\zeta| < 1$ and $|\xi| < 1$. The following infinite triple summation

$$\sum_{n\geq 0}\sum_{s\geq 0}\sum_{t\geq 0} p_\eta^*(s,t;n) \zeta^s \xi^t q^n = \left(\frac{1}{(\zeta,\xi;q)_\infty}\right)^\eta$$

is absolutely convergent. Further, considering another absolutely convergent infinite triple summation

$$\sum_{n\geq 0}\sum_{s\geq 0}\sum_{t\geq 0} d_\eta^*(s,t;n) \zeta^s \xi^t q^n := (\zeta,\xi;q)_\infty^\eta,$$

we have that $(-1)^{s+t} d_\eta^*(s,t;n)$ denotes the number of 2-colored (again, red and blue) distinct partition (in which 0 is still allowed as a part) η-tuples of n with s parts in total colored by red and t parts in total colored by blue. An easy partition-theoretic argument indicates that $|d^*_\eta(s,t;n)| \leq p^*_\eta(s,t;n)$ for all $s,t,n \geq 0$. Also, we have $d^*_\eta(0,0;0) = p^*_\eta(0,0;0) = 1$. In general, if we write

$$\sum_{n\geq 0}\sum_{s\geq 0}\sum_{t\geq 0} a_\delta(s,t;n)\zeta^s\xi^t q^n := (\zeta,\xi;q)^\delta_\infty,$$

then

$$a_\delta(s,t;n) = \begin{cases} p^*_{|\delta|}(s,t;n) & \text{if } \delta < 0, \\ d^*_{|\delta|}(s,t;n) & \text{if } \delta > 0, \end{cases}$$

and hence $|a_\delta(s,t;n)| \leq p^*_{|\delta|}(s,t;n)$ for all $s,t,n \geq 0$. We also have

$$\begin{aligned} \left|(\zeta,\xi;q)^\delta_\infty\right| &= \left|\sum_{n\geq 0}\sum_{s\geq 0}\sum_{t\geq 0} a_\delta(s,t;n)\zeta^s\xi^t q^n\right| \\ &\leq \sum_{n\geq 0}\sum_{s\geq 0}\sum_{t\geq 0} p^*_{|\delta|}(s,t;n)|\zeta|^s|\xi|^t|q|^n. \end{aligned}$$

Further, for real $0 \leq \alpha, \beta, x < 1$, we have

$$\begin{aligned} \sum_{n\geq 0}\sum_{s\geq 0}\sum_{t\geq 0} p^*_1(s,t;n)\alpha^s\beta^t x^n &= \frac{1}{(\alpha,\beta;x)_\infty} \\ &= \exp\left(-\sum_{k\geq 0}\log(1-\alpha x^k) - \sum_{\ell\geq 0}\log(1-\beta x^\ell)\right) \\ &\leq \exp\left(\frac{\alpha}{1-\alpha} + \frac{\alpha x}{(1-x)^2} + \frac{\beta}{1-\beta} + \frac{\beta x}{(1-x)^2}\right). \end{aligned} \tag{11.5.7}$$

11.5.3 Evaluating an Integral

The last task in this chapter is to evaluate a useful integral.

Lemma 11.5.1. *Let $a \in \mathbb{R}_{>0}$, $b \in \mathbb{R}$ and $c \in \frac{1}{2}\mathbb{Z}_{\leq 0}$. Let $\gcd(h,k) = 1$. Define*

$$I := \int_{\xi_{h,k}} e^{\frac{\pi}{12k}\left(\frac{a}{z}+bz\right)} z^c e^{-2\pi i n\phi} e^{2\pi n\varrho}\, d\phi. \tag{11.5.8}$$

Then, for those positive integers n with $n > -b/24$, we have

$$I = \frac{2\pi}{k}\left(\frac{24n+b}{a}\right)^{-\frac{c+1}{2}} I_{-c-1}\left(\frac{\pi}{6k}\sqrt{a(24n+b)}\right) + E(I), \tag{11.5.9}$$

where

$$|E(I)| \le \frac{2^{-c}\pi^{-1}e^{\frac{\pi a}{3}}N^{-c}}{n+\frac{b}{24}}e^{2\pi\varrho\left(n+\frac{b}{24}\right)}. \tag{11.5.10}$$

Proof. We first put $w = z/k = \varrho - i\phi$ and reverse the integral order to obtain

$$I = \frac{1}{2\pi i}\int_{\varrho-i\theta''_{h,k}}^{\varrho+i\theta'_{h,k}} 2\pi e^{\frac{\pi a}{12k^2 w}} e^{2\pi w\left(n+\frac{b}{24}\right)}(kw)^c\, dw.$$

One may separate the integral into three parts

$$\begin{aligned} I &= \frac{1}{2\pi i}\left(\int_{\Gamma} - \int_{-\infty-i\theta''_{h,k}}^{\varrho-i\theta''_{h,k}} + \int_{-\infty+i\theta'_{h,k}}^{\varrho+i\theta'_{h,k}}\right) 2\pi e^{\frac{\pi a}{12k^2 w}} e^{2\pi w\left(n+\frac{b}{24}\right)}(kw)^c\, dw \\ &=: J_1 - J_2 + J_3, \end{aligned}$$

where

$$\Gamma := (-\infty - i\theta''_{h,k}) \to (\varrho - i\theta''_{h,k}) \to (\varrho + i\theta'_{h,k}) \to (-\infty + i\theta'_{h,k})$$

is a Hankel contour.

To compute the main term J_1, we make the following change of variables $t = wk\sqrt{(24n+b)/a}$ to obtain

$$J_1 = \frac{2\pi}{k}\left(\frac{24n+b}{a}\right)^{-\frac{c+1}{2}} \frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\frac{\pi}{12k}\sqrt{a(24n+b)}\left(t+\frac{1}{t}\right)} t^c\, dt.$$

Note that the new contour $\tilde{\Gamma}$ is still a Hankel contour. Recalling the contour integral representation of $I_s(x)$:

$$I_s(x) = \frac{1}{2\pi i}\int_{\Gamma} t^{-s-1} e^{\frac{x}{2}\left(t+\frac{1}{t}\right)}\, dt \quad (\Gamma \text{ is a Hankel contour}),$$

we conclude

$$J_1 = \frac{2\pi}{k}\left(\frac{24n+b}{a}\right)^{-\frac{c+1}{2}} I_{-c-1}\left(\frac{\pi}{6k}\sqrt{a(24n+b)}\right).$$

For the error term $E(I)$, which comes from J_2 and J_3, we put $w = x + i\theta$ with $-\infty \le x \le \varrho$ and $\theta \in \{\theta'_{h,k}, -\theta''_{h,k}\}$. We know that

$$\left|e^{2\pi w\left(n+\frac{b}{24}\right)}\right| = e^{2\pi x\left(n+\frac{b}{24}\right)},$$

$$\left|e^{\frac{\pi a}{12k^2 w}}\right| = e^{\frac{\pi a}{12k^2}\Re\left(\frac{1}{w}\right)} = e^{\frac{\pi a}{12k^2}\frac{x}{x^2+\theta^2}} \leq e^{\frac{\pi a}{12k^2}\frac{x}{\theta^2}} \leq e^{\frac{\pi a}{12k^2}\varrho(2kN)^2} = e^{\frac{\pi a}{3}},$$

and

$$|(kw)^c| = \left(|kw|^{-1}\right)^{-c} \leq \left(\frac{1}{k\sqrt{x^2+\theta^2}}\right)^{-c} \leq \left(\frac{1}{k|\theta|}\right)^{-c} \leq (2N)^{-c},$$

where we use the bound $\frac{1}{2kN} \leq |\theta| \leq \frac{1}{kN}$. Hence for $j = 2$ and 3, we have

$$\begin{aligned} |J_j| &\leq \frac{1}{2\pi}\int_{-\infty}^{\varrho} 2\pi e^{\frac{\pi a}{3}} e^{2\pi x\left(n+\frac{b}{24}\right)}\,(2N)^{-c}\,dx \\ &= \frac{2^{-c-1}\pi^{-1}e^{\frac{\pi a}{3}}N^{-c}}{n+\frac{b}{24}} e^{2\pi\varrho\left(n+\frac{b}{24}\right)}. \end{aligned}$$

This implies that

$$|E(I)| = |-J_2 + J_3| \leq |J_2| + |J_3| \leq \frac{2^{-c}\pi^{-1}e^{\frac{\pi a}{3}}N^{-c}}{n+\frac{b}{24}} e^{2\pi\varrho\left(n+\frac{b}{24}\right)},$$

which gives (11.5.10). □

11.6 References

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Chapter 12

The Method of Rademacher: Dedekind Eta Products

This chapter comes from

- S. Chern, Asymptotics for the Fourier coefficients of eta-quotients, *J. Number Theory* **199** (2019), 168–191. (Ref. [53])
- S. Chern, D. Tang, and L. Wang, Some inequalities for Garvan's bicrank function of 2-colored partitions, *Acta Arith.* **190** (2019), no. 2, 171–191. (Ref. [67])

12.1 Main Result

We will study the asymptotics for

$$G(q)=\sum_{n\geq 0} g(n)q^n=\prod_{j=1}^{J}(q^{m_j};q^{m_j})_\infty^{\delta_j}, \tag{12.1.1}$$

where $\mathbf{m}=(m_1,\ldots,m_J)$ is a sequence of J distinct positive integers and $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_J)$ is a sequence of J non-zero integers.

Let k and h be positive integers such that $\gcd(h,k)=1$. We define

$$\Sigma:=-\frac{1}{2}\sum_{j=1}^{J}\delta_j,$$

$$\Omega:=\sum_{j=1}^{J}\delta_j m_j,$$

$$\Pi_k:=\prod_{j=1}^{J}\left(\frac{m_j}{\gcd(m_j,k)}\right)^{-\frac{\delta_j}{2}},$$

$$\Delta(k) := -\sum_{j=1}^{J} \frac{\delta_j \gcd^2(m_j, k)}{m_j}$$

and

$$\omega_{h,k} := \exp\left(-\pi i \sum_{j=1}^{J} \delta_j \cdot s\left(\frac{m_j h}{\gcd(m_j, k)}, \frac{k}{\gcd(m_j, k)}\right)\right), \tag{12.1.2}$$

where $s(d, c)$ is the Dedekind sum.

Let $L = \operatorname{lcm}(m_1, \dots, m_R)$. We divide the set $\{1, 2, \dots, L\}$ into two disjoint subsets:

$$\mathcal{L}_{>0} := \{1 \le \ell \le L \ : \ \Delta(\ell) > 0\},$$
$$\mathcal{L}_{\le 0} := \{1 \le \ell \le L \ : \ \Delta(\ell) \le 0\}.$$

Our main result states as follows.

Theorem 12.1.1. *If $\Sigma \le 0$ and the inequality*

$$\min_{1 \le j \le J} \left(\frac{\gcd^2(m_j, \ell)}{m_j}\right) \ge \frac{\Delta(\ell)}{24} \tag{12.1.3}$$

holds for all $1 \le \ell \le L$, then for positive integers $n > -\Omega/24$, we have

$$g(n) = E(n) + 2\pi \sum_{\ell \in \mathcal{L}_{>0}} \Pi_\ell \left(\frac{24n + \Omega}{\Delta(\ell)}\right)^{-\frac{\Sigma+1}{2}} \times \sum_{\substack{1 \le k \le N^\star \\ k \equiv \ell \bmod L}} \frac{1}{k} I_{-\Sigma-1}\left(\frac{\pi}{6k}\sqrt{\Delta(\ell)(24n + \Omega)}\right) \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}} \omega_{h,k}, \tag{12.1.4}$$

where

$$N^\star = \left\lfloor \sqrt{2\pi \left(n + \frac{\Omega}{24}\right)} \right\rfloor, \tag{12.1.5}$$

$I_s(x)$ is the modified Bessel function of the first kind, and

$$E(n) \ll_{\mathbf{m}, \boldsymbol{\delta}} \Xi_\Sigma(n) := \begin{cases} 1 & \text{if } \Sigma = 0, \\ \left(n + \frac{\Omega}{24}\right)^{1/4} & \text{if } \Sigma = -\frac{1}{2}, \\ \left(n + \frac{\Omega}{24}\right)^{1/2} \log\left(n + \frac{\Omega}{24}\right) & \text{if } \Sigma = -1, \\ \left(n + \frac{\Omega}{24}\right)^{-\Sigma-1/2} & \text{if } \Sigma \le -\frac{3}{2}. \end{cases} \tag{12.1.6}$$

Remark 12.1.1. To better understand the asymptotic behavior of $g(n)$, one may apply the asymptotic expansion of $I_s(x)$ (cf. [2, p. 377, (9.7.1)]): for fixed s, when $|\arg x| < \frac{\pi}{2}$,

$$I_s(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4s^2-1}{8x} + \frac{(4s^2-1)(4s^2-9)}{2!(8x)^2} - \cdots\right). \tag{12.1.7}$$

12.2 A Transformation Formula

Let us define

$$P(\tau) := \frac{1}{(q;q)_\infty} = e^{\frac{\pi i \tau}{12}}\frac{1}{\eta(\tau)}, \tag{12.2.1}$$

where $q := e^{2\pi i\tau}$. Let m be a positive integer. Hence,

$$P(m\tau) = e^{\frac{\pi i m\tau}{12}}\frac{1}{\eta(m\tau)}.$$

Recall that $d = \gcd(m,k)$, $m = dm'$ and $k = dk'$. Recall also that $\tau = (h+iz)/k$. One has, from (11.2.4) with $\gamma = \gamma_{(m,h,k)}$ as in (11.4.1) and the fact $s(-m'h,k') = -s(m'h,k')$, that

$$\begin{aligned}P(m\tau) &= e^{\frac{\pi i m\tau}{12}}\chi(\gamma_{(m,h,k)})(\gamma^*_{(m,h,k)}(m\tau))^{-\frac{1}{2}}\frac{1}{\eta(\gamma_{(m,h,k)}(m\tau))}\\ &= \sqrt{\frac{mz}{d}}e^{\pi i s(m'h,k')}\exp\left(\frac{\pi}{12k}\left(-mz + \frac{d^2}{m}\frac{1}{z}\right)\right)P\Big(\gamma_{(m,h,k)}(m\tau)\Big).\end{aligned}$$

Consequently, we deduce the following transformation formula.

Lemma 12.2.1. *We have*

$$\begin{aligned}G(e^{2\pi i\tau}) &= \prod_{j=1}^{J} P^{-\delta_j}(m_j\tau)\\ &= z^{\Sigma}\,\omega_{h,k}\,\Pi_k\,\exp\left(\frac{\pi}{12k}(\Omega z + \Delta(k)z^{-1})\right)\prod_{j=1}^{J}P^{-\delta_j}\Big(\gamma_{(m_j,h,k)}(m_j\tau)\Big).\end{aligned} \tag{12.2.2}$$

12.3 Outline of the Proof

We know from (11.3.1) and (13.2.3) that

$$g(n) = \sum_{1\le k\le N}\ \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}}\int_{\xi_{h,k}} G\Big(e^{2\pi i\tau}\Big)e^{-2\pi i n\phi}e^{2\pi n\varrho}\,d\phi$$

$$= \sum_{1\le k\le N} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi inh}{k}} \omega_{h,k}\, \Pi_k$$
$$\times \int_{\xi_{h,k}} z^{\Sigma}\, \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(k) z^{-1})\right) \prod_{j=1}^{J} P^{-\delta_j}\Big(\gamma_{(m_j,h,k)}(m_j\tau)\Big) e^{-2\pi in\phi} e^{2\pi n\varrho}\, d\phi.$$

Let us fix a Farey fraction h/k. We first find the integer $1 \le \ell \le L$ such that $k \equiv \ell \pmod{L}$. For convenience, we write $\rho(k) := \ell$. It is not hard to observe that for all $j = 1, 2, \ldots, J$,

$$\gcd(m_j, k) = \gcd(m_j, \ell).$$

It turns out that $\Delta(k) = \Delta(\ell)$ and $\Pi_k = \Pi_\ell$. We now split $g(n)$ as follows.

$$g(n) = \sum_{1\le \ell\le L} \Pi_\ell \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi inh}{k}} \omega_{h,k}$$
$$\times \int_{\xi_{h,k}} z^{\Sigma}\, \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\ell) z^{-1})\right) \prod_{j=1}^{J} P^{-\delta_j}\Big(\gamma_{(m_j,h,k)}(m_j\tau)\Big) e^{-2\pi in\phi} e^{2\pi n\varrho}\, d\phi$$
$$=: \sum_{1\le \ell\le L} S_\ell.$$

Define

$$\Xi^*_\Sigma(N) := \begin{cases} 1 & \text{if } \Sigma = 0,\\ N^{1/2} & \text{if } \Sigma = -\frac{1}{2},\\ N\log N & \text{if } \Sigma = -1,\\ N^{-2\Sigma-1} & \text{if } \Sigma \le -\frac{3}{2}. \end{cases} \tag{12.3.1}$$

The minor arcs are those with respect to h/k with $\rho(k) \in \mathcal{L}_{\le 0}$. We have the following bound.

Theorem 12.3.1. *Let $\ell \in \mathcal{L}_{\le 0}$. If $\Sigma \le 0$, then for positive integers $n > -\Omega/24$, we have*

$$S_\ell \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} \Xi^*_\Sigma(N) \exp\left(\frac{2\pi}{N^2}\left(n + \frac{\Omega}{24}\right)\right).$$

In particular, if we take $N = \left\lfloor \sqrt{2\pi\left(n + \frac{\Omega}{24}\right)} \right\rfloor$, then $S_\ell \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} \Xi_\Sigma(n)$.

The arcs with respect to h/k with $\rho(k) \in \mathcal{L}_{>0}$ give us the main contribution.

Theorem 12.3.2. *Let $\ell \in \mathcal{L}_{>0}$. If $\Sigma \leq 0$ and the inequality*

$$\min_{1\leq j\leq J}\left(\frac{\gcd^2(m_j,\ell)}{m_j}\right) \geq \frac{\Delta(\ell)}{24} \tag{12.3.2}$$

holds, then for positive integers $n > -\Omega/24$, we have

$$S_\ell = E_\ell + \Pi_\ell \sum_{\substack{1\leq k\leq N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}} \omega_{h,k} \frac{2\pi}{k}\left(\frac{24n+\Omega}{\Delta(\ell)}\right)^{-\frac{\Sigma+1}{2}}$$
$$\times I_{-\Sigma-1}\left(\frac{\pi}{6k}\sqrt{\Delta(\ell)(24n+\Omega)}\right),$$

where

$$E_\ell \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} \Xi_\Sigma^*(N) e^{\frac{2\pi}{N^2}(n+\frac{\Omega}{24})} + \frac{N^{-\Sigma+2} e^{\frac{2\pi}{N^2}\left(n+\frac{\Omega}{24}\right)}}{n+\frac{\Omega}{24}}.$$

In particular, if we take $N = \left\lfloor \sqrt{2\pi\left(n+\frac{\Omega}{24}\right)} \right\rfloor$, then $E_\ell \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} \Xi_\Sigma(n)$.

Theorems 12.3.1 and 12.3.2 immediately imply the main result.

12.4 Minor Arcs

Let $\ell \in \mathcal{L}_{\leq 0}$, namely, $\Delta(\ell) \leq 0$. Notice that

$$|S_\ell| \leq \Pi_\ell \sum_{\substack{1\leq k\leq N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} \int_{\xi_{h,k}} |z|^{\Sigma} \exp\left(\frac{\pi}{12k}(\Omega\Re(z) + \Delta(\ell)\Re(z^{-1}))\right)$$
$$\times \left|\prod_{j=1}^{J} P^{-\delta_j}\left(\gamma_{(m_j,h,k)}(m_j\tau)\right)\right| e^{2\pi n\varrho}\ d\phi.$$

We now consider the Farey arcs with respect to h/k with $k \equiv \ell \pmod{L}$. Since $\Delta(\ell) \leq 0$, it follows from (11.5.3) and (11.5.5) that

$$\exp\left(\frac{\pi}{12k}(\Omega\Re(z) + \Delta(\ell)\Re(z^{-1}))\right) \leq \exp\left(\frac{\pi}{12k}\left(\Omega\frac{k}{N^2} + \Delta(\ell)\frac{k}{2}\right)\right)$$
$$= \exp\left(\frac{\pi\varrho\,\Omega}{12}\right)\exp\left(\frac{\pi\Delta(\ell)}{24}\right).$$

Also, it follows from (11.5.4) with the assumption $\Sigma \leq 0$ that

$$|z|^{\Sigma} \leq k^{\Sigma} N^{-2\Sigma}.$$

Now we write for short $\tilde{\tau}_j = \gamma_{(m_j,h,k)}(m_j\tau)$. It follows from (11.4.2) that

$$\Im(\tilde{\tau}_j) = \frac{\gcd^2(m_j,k)}{m_j k}\Re(z^{-1}) = \frac{\gcd^2(m_j,\ell)}{m_j k}\Re(z^{-1}).$$

Then,

$$\prod_{j=1}^{J} P^{-\delta_j}\Big(\gamma_{(m_j,h,k)}(m_j\tau)\Big) = \prod_{j=1}^{J} \frac{1}{(e^{2\pi i\tilde{\tau}_j}; e^{2\pi i\tilde{\tau}_j})_\infty^{-\delta_j}}.$$

As we have seen in §11.5.2,

$$\begin{aligned}
\left|\frac{1}{(e^{2\pi i\tilde{\tau}_j}; e^{2\pi i\tilde{\tau}_j})_\infty^{-\delta_j}}\right| &\leq \sum_{n\geq 0} p_{|\delta_j|}(n)|e^{2\pi i\tilde{\tau}_j}|^n \\
&= \sum_{n\geq 0} p_{|\delta_j|}(n)e^{-2\pi\Im(\tilde{\tau}_j)n} \\
&= \sum_{n\geq 0} p_{|\delta_j|}(n)\exp\left(-2\pi\frac{\gcd^2(m_j,\ell)}{m_j k}\Re(z^{-1})n\right) \\
&\leq \sum_{n\geq 0} p_{|\delta_j|}(n)\exp\left(-\pi\frac{\gcd^2(m_j,\ell)}{m_j}n\right),
\end{aligned}$$

where we use $\Re(z^{-1}) \geq k/2$. It follows from (11.5.7) that

$$\left|\frac{1}{(e^{2\pi i\tilde{\tau}_j}; e^{2\pi i\tilde{\tau}_j})_\infty^{-\delta_j}}\right| \ll 1.$$

Putting the above arguments together yields

$$\begin{aligned}
S_\ell &\ll \sum_{\substack{1\leq k\leq N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} \int_{\xi_{h,k}} k^{\Sigma}N^{-2\Sigma}e^{\frac{\pi\varrho\,\Omega}{12}}e^{2\pi n\varrho}\,d\phi \\
&\ll \sum_{\substack{1\leq k\leq N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} e^{2\pi\varrho(n+\frac{\Omega}{24})}k^{\Sigma-1}N^{-2\Sigma-1} \\
&\ll \Xi_{\Sigma}^{*}(N)e^{2\pi\varrho(n+\frac{\Omega}{24})} = \Xi_{\Sigma}^{*}(N)e^{\frac{2\pi}{N^2}(n+\frac{\Omega}{24})}.
\end{aligned}$$

12.5 Major Arcs

Let $\ell \in \mathcal{L}_{>0}$, namely, $\Delta(\ell) > 0$. For convenience, we write $\tilde{\tau}_j(h,k) = \gamma_{(m_j,h,k)}(m_j\tau)$. Recall that

$$S_\ell = \Pi_\ell \sum_{\substack{1\le k\le N \\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{-\frac{2\pi inh}{k}} \omega_{h,k} \times \int_{\xi_{h,k}} z^\Sigma \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\ell) z^{-1})\right) \prod_{j=1}^J P^{-\delta_j}\Big(\tilde{\tau}_j(h,k)\Big) e^{-2\pi in\phi} e^{2\pi n\varrho}\, d\phi.$$

We split $S_{\varkappa,\ell}$ into two parts Σ_1 and Σ_2 where

$$\Sigma_1 := \Pi_\ell \sum_{\substack{1\le k\le N \\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{-\frac{2\pi inh}{k}} \omega_{h,k} \times \int_{\xi_{h,k}} z^\Sigma \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\ell) z^{-1})\right) e^{-2\pi in\phi} e^{2\pi n\varrho}\, d\phi$$

and

$$\Sigma_2 := \Pi_\ell \sum_{\substack{1\le k\le N \\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{-\frac{2\pi inh}{k}} \omega_{h,k} \times \int_{\xi_{h,k}} z^\Sigma \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\ell) z^{-1})\right) \left(\prod_{j=1}^J P^{-\delta_j}\Big(\tilde{\tau}_j(h,k)\Big) - 1\right) e^{-2\pi in\phi} e^{2\pi n\varrho}\, d\phi.$$

We first show that Σ_2 is negligible. Let us fix h and k and write $\tilde{\tau}_j = \tilde{\tau}_j(h,k)$. Then,

$$|\Sigma_2| \le \Pi_\ell \sum_{\substack{1\le k\le N \\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} \int_{\xi_{h,k}} |z|^\Sigma \exp\left(\frac{\pi}{12k}(\Omega \Re(z) + \Delta(\ell)\Re(z^{-1}))\right) \times \left| \prod_{j=1}^J P^{-\delta_j}\Big(\tilde{\tau}_j\Big) - 1 \right| e^{2\pi n\varrho}\, d\phi.$$

As we have seen in §11.5.2,

$$\left| \prod_{j=1}^J P^{-\delta_j}\Big(\tilde{\tau}_j\Big) - 1 \right| = \left| \prod_{j=1}^J \frac{1}{(e^{2\pi i\tilde{\tau}_j}; e^{2\pi i\tilde{\tau}_j})_\infty^{-\delta_j}} - 1 \right|$$

$$\begin{aligned}
&\leq \sum_{\mathbf{n}:=(n_1,\dots,n_J)\in\mathbb{Z}_{\geq 0}^J} \prod_{j=1}^J p_{|\delta_j|}(n_j)|e^{2\pi i\tilde{\tau}_j}|^{n_j} - 1\\
&= \sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}^J\backslash(0,\dots,0)} \prod_{j=1}^J p_{|\delta_j|}(n_j)|e^{2\pi i\tilde{\tau}_j}|^{n_j}\\
&= \sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}^J\backslash(0,\dots,0)} \prod_{j=1}^J p_{|\delta_j|}(n_j)e^{-2\pi\Im(\tilde{\tau}_j)n_j}\\
&= \sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}^J\backslash(0,\dots,0)} \left(\prod_{j=1}^J p_{|\delta_j|}(n_j)\right)\exp\left(-2\pi\frac{\Re(z^{-1})}{k}\sum_{j=1}^J\frac{\gcd^2(m_j,\ell)}{m_j}n_j\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\exp\left(\frac{\pi\Delta(\ell)}{12k}z^{-1}\right)\left|P^{-\delta_j}\left(\tilde{\tau}_j\right)-1\right|\\
&\leq \sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}^J\backslash(0,\dots,0)} \left(\prod_{j=1}^J p_{|\delta_j|}(n_j)\right)\exp\left(-2\pi\frac{\Re(z^{-1})}{k}\left(-\frac{\Delta(\ell)}{24}+\sum_{j=1}^J\frac{\gcd^2(m_j,\ell)}{m_j}n_j\right)\right).
\end{aligned}$$

Since at least one coordinate of $\mathbf{n}=(n_1,\dots,n_J)$ is nonzero, under the condition (12.3.2), we know that

$$-\frac{\Delta(\ell)}{24}+\sum_{j=1}^J\frac{\gcd^2(m_j,\ell)}{m_j}n_j \geq -\frac{\Delta(\ell)}{24}+\min_{1\leq j\leq J}\left(\frac{\gcd^2(m_j,\ell)}{m_j}\right)\geq 0$$

for all $\mathbf{n}\in\mathbb{Z}_{\geq 0}^J\backslash(0,\dots,0)$. Recalling that $\Re(z^{-1})\geq k/2$, it follows that

$$\exp\left(\frac{\pi\Delta(\ell)}{12k}z^{-1}\right)\left|P^{-\delta_j}\left(\tilde{\tau}_j\right)-1\right|$$

is maximized when $\Re(z^{-1})=k/2$. Namely,

$$\begin{aligned}
&\exp\left(\frac{\pi\Delta(\ell)}{12k}z^{-1}\right)\left|P^{-\delta_j}\left(\tilde{\tau}_j\right)-1\right|\\
&\leq \exp\left(\frac{\pi\Delta(\ell)}{12k}z^{-1}\right)\sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}^J\backslash(0,\dots,0)} \left(\prod_{j=1}^J p_{|\delta_j|}(n_j)\right)\exp\left(-\pi\sum_{j=1}^J\frac{\gcd^2(m_j,\ell)}{m_j}n_j\right)\ll 1.
\end{aligned}$$

We conclude that

$$\begin{aligned}\Sigma_2 &\ll \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \int_{\xi_{h,k}} k^{\Sigma} N^{-2\Sigma} e^{\frac{\pi\varrho\,\Omega}{12}} e^{2\pi n\varrho}\, d\phi \\ &\ll \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{2\pi\varrho(n+\frac{\Omega}{24})} k^{\Sigma-1} N^{-2\Sigma-1} \\ &\ll \Xi_{\Sigma}^{*}(N) e^{2\pi\varrho(n+\frac{\Omega}{24})} = \Xi_{\Sigma}^{*}(N) e^{\frac{2\pi}{N^2}(n+\frac{\Omega}{24})}.\end{aligned}$$

Finally, we estimate the main contribution Σ_1. Recall that

$$\begin{aligned}\Sigma_1 := \Pi_\ell &\sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}} \omega_{h,k} \\ &\times \int_{\xi_{h,k}} z^{\Sigma} \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\ell) z^{-1})\right) e^{-2\pi i n\phi} e^{2\pi n\varrho}\, d\phi\end{aligned}$$

We simply apply Lemma 11.5.1. The main contribution to Σ_1 is

$$\Pi_\ell \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}} \omega_{h,k} \frac{2\pi}{k} \left(\frac{24n+\Omega}{\Delta(\ell)}\right)^{-\frac{\Sigma+1}{2}} I_{-\Sigma-1}\left(\frac{\pi}{6k}\sqrt{\Delta(\ell)(24n+\Omega)}\right).$$

The error term in Σ_1 is bounded by

$$\sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \frac{N^{-\Sigma} e^{2\pi\varrho\left(n+\frac{\Omega}{24}\right)}}{n+\frac{\Omega}{24}} \ll \frac{N^{-\Sigma+2} e^{\frac{2\pi}{N^2}\left(n+\frac{\Omega}{24}\right)}}{n+\frac{\Omega}{24}}.$$

12.6 An Application

As an application of Theorem 12.1.1, we show some inequalities for Garvan's bicrank function of 2-colored partitions.

A partition is called 2-colored if each part is receiving a color from the set of two prescribed colors. Let $p_{-2}(n)$ count the number of 2-colored partitions of n. Then,

$$\sum_{n=0}^{\infty} p_{-2}(n) q^n = \frac{1}{(q;q)_\infty^2}.$$

It is notable that $p_{-2}(n)$ also satisfies nice arithmetic properties. For example, Hammond and Lewis [95] proved that

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod 5. \tag{12.6.1}$$

To give a unified combinatorial proof of all three congruences in (12.6.1), in 2010, Garvan [81] introduced a bicrank function for 2-colored partitions (see [81] for the lengthy definition). Let $M^*(m,n)$ count the number of 2-colored partitions of n with bicrank m. Garvan showed that $M^*(m,n)$ has the following generating function (cf. [81, (6.17)]):

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M^*(m,n) z^m q^n = \frac{(q;q)_\infty^2}{(zq, z^{-1}q, z^2q, z^{-2}q; q)_\infty}, \tag{12.6.2}$$

from which he proved that, for any integer $n \geq 0$,

$$\begin{aligned}
M^*(0,5,5n+2) &= M^*(1,5,5n+2) = \cdots = M^*(4,5,5n+2) = \frac{p_{-2}(5n+2)}{5},\\
M^*(0,5,5n+4) &= M^*(1,5,5n+4) = \cdots = M^*(4,5,5n+4) = \frac{p_{-2}(5n+4)}{5},\\
M^*(0,5,5n+3) &\equiv M^*(1,5,5n+3) \equiv \cdots \equiv M^*(4,5,5n+3) \pmod 5,
\end{aligned}$$

where $M^*(j,k,n) := \sum_{m \equiv j \pmod k} M^*(m,n)$ is the number of 2-colored partitions of n with bicrank congruent to j modulo k.

On the other hand, the following inequalities were shown by Andrews and Lewis [24].

Theorem 12.6.1 (Andrews–Lewis)**.** *For $n \geq 0$,*

$$M(0,2,2n) > M(1,2,2n), \qquad M(0,2,2n+1) < M(0,2,2n+1),$$

where $M(r,m,n)$ counts the number of partitions of n with crank congruent to r modulo m.

Along this line, it would be interesting to study sign patterns for the bicrank function. First, taking $z = \zeta_3 = e^{2\pi i/3}$ in (12.6.2) yields

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(M^*(0,3,n) - M^*(1,3,n)\right) q^n &= \frac{(q;q)_\infty^2}{(\zeta_3 q, \zeta_3^{-1}q, \zeta_3^2 q, \zeta_3^{-2}q; q)_\infty}\\
&= \frac{(q;q)_\infty^4}{(q^3;q^3)_\infty^2}.
\end{aligned} \tag{12.6.3}$$

For the infinite product in (12.6.3), we have, in the setting of (12.1.1), $\mathbf{m} = \{1,3\}$

and $\boldsymbol{\delta} = \{4, -2\}$. Hence, $L = 3$ and $\mathcal{L}_{>0} = \{3\}$. Applying Theorem 12.1.1 yields an asymptotic formula as follows.

Theorem 12.6.2. *For $n \geq 1$,*

$$M^*(0,3,n) - M^*(1,3,n) \sim c(n)\, I_0\left(\frac{2\pi\sqrt{n-1/12}}{3\sqrt{3}}\right), \tag{12.6.4}$$

where

$$c(n) = \begin{cases} \dfrac{4\pi}{3}\cos\dfrac{2\pi}{9} & \textit{if } n \equiv 0 \pmod 3, \\ -\dfrac{4\pi}{3}\cos\dfrac{\pi}{9} & \textit{if } n \equiv 1 \pmod 3, \\ \dfrac{4\pi}{3}\sin\dfrac{\pi}{18} & \textit{if } n \equiv 2 \pmod 3. \end{cases}$$

Remark 12.6.1. If one treats the error term $E(n)$ more carefully, it can be shown that $E(n)$ is able to be bounded explicitly:

$$\begin{aligned} |E(n)| \leq 173.1\sqrt{n-\frac{1}{12}}\,\left(\log\sqrt{2\pi\left(n-\frac{1}{12}\right)}+1\right) + 74.3\sqrt{n-\frac{1}{12}} \\ + 2.8\sqrt{n-\frac{1}{12}}\,\exp\left(\frac{\pi\sqrt{n-1/12}}{3\sqrt{3}}\right). \end{aligned}$$

See [67] for details. It turns out through a short computation that the sign of $M^*(0,3,n) - M^*(1,3,n)$ is determined by the main term (and hence by $c(n)$) when $n \geq 114$. We therefor deduce the following inequalities.

Theorem 12.6.3. *For $n \geq 0$,*

$$\begin{aligned} M^*(0,3,n) &> M^*(1,3,n) \qquad \textit{if } n \equiv 0, 2 \pmod 3, \\ M^*(0,3,n) &< M^*(1,3,n) \qquad \textit{if } n \equiv 1 \pmod 3, \end{aligned}$$

except for $n = 5$.

Likewise, taking $z = i$ in (12.6.2) yields

$$\sum_{n=0}^{\infty}\left(M^*(0,4,n) - M^*(2,4,n)\right)q^n$$

$$= \frac{(q;q)_\infty^2}{(iq,-iq,-q,-q;q)_\infty} = \frac{(q;q)_\infty^2}{(-q^2;q^2)_\infty(-q;q)_\infty^2} = \frac{(q;q)_\infty^4}{(q^2;q^2)_\infty(q^4;q^4)_\infty}. \tag{12.6.5}$$

Analogously, we deduce an asymptotic formula for $M^*(0,4,n) - M^*(2,4,n)$.

Theorem 12.6.4. *For* $n \geq 1$,

$$\begin{aligned} &M^*(0,4,n) - M^*(2,4,n) \\ &\quad = c_1(n)\ I_0\left(\frac{\pi\sqrt{n-1/12}}{2\sqrt{3}}\right) + c_2(n)\ I_0\left(\frac{\pi\sqrt{n-1/12}}{4\sqrt{3}}\right) + E(n), \end{aligned} \tag{12.6.6}$$

where

$$c_1(n) = \begin{cases} -\pi & \text{if } n \equiv 1 \pmod 4, \\ \pi & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{otherwise}, \end{cases} \qquad c_2(n) = \begin{cases} \pi \sin\frac{\pi}{8} & \text{if } n \equiv 0 \pmod 8, \\ \pi \cos\frac{\pi}{8} & \text{if } n \equiv 2 \pmod 8, \\ -\pi \sin\frac{\pi}{8} & \text{if } n \equiv 4 \pmod 8, \\ -\pi \cos\frac{\pi}{8} & \text{if } n \equiv 6 \pmod 8, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\begin{aligned} |E(n)| \leq 224.2\sqrt{n-\frac{1}{12}}\ &\left(\log\sqrt{2\pi\left(n-\frac{1}{12}\right)}+1\right) + 55.6\sqrt{n-\frac{1}{12}} \\ &+ 2.4\sqrt{n-\frac{1}{12}}\ \exp\left(\frac{\pi\sqrt{n-1/12}}{6\sqrt{3}}\right). \end{aligned}$$

We also deduce from a short computation that the sign of $M^*(0,4,n) - M^*(2,4,n)$ is determined by the main term when $n \geq 2160$. The following inequalities therefore hold.

Theorem 12.6.5. *For* $n \geq 0$,

$$\begin{aligned} M^*(0,4,n) &> M^*(2,4,n) \qquad \text{if } n \equiv 0,\ 2,\ 3,\ 7 \pmod 8, \\ M^*(0,4,n) &< M^*(2,4,n) \qquad \text{if } n \equiv 1,\ 4,\ 5,\ 6 \pmod 8, \end{aligned}$$

except for $n = 4$ *and* 20.

12.7 References

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Chapter 13

The Method of Rademacher: Jacobi Theta Products

This chapter comes from

- S. Chern, Asymptotics for the Taylor coefficients of certain infinite products, to appear in *Ramanujan J.* (Ref. [57])

13.1 Main Result

We will study the asymptotics for

$$G(q)=\sum_{n\geq 0} g(n)q^n=\prod_{j=1}^{J}(q^{r_j},q^{m_j-r_j};q^{m_j})_\infty^{\delta_j}, \tag{13.1.1}$$

where $\mathbf{m}=(m_1,\ldots,m_J)$ and $\mathbf{r}=(r_1,\ldots,r_J)$ are two sequences of J positive integers satisfying $1\leq r_j<m_j$ for all $j=1,\ldots,J$, and $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_J)$ is a sequence of J nonzero integers.

Recall that we have defined in §11.4 that for $0\leq h<k$ with $\gcd(h,k)=1$,

$$\lambda_{m,r}(h,k):=\left\lceil \frac{rh}{\gcd(m,k)}\right\rceil$$

and

$$\lambda^*_{m,r}(h,k):=\lambda_{m,r}(h,k)-\frac{rh}{\gcd(m,k)}.$$

Also, $\hbar_m(h,k)$ is an integer such that

$$\hbar_m(h,k)\frac{mh}{\gcd(m,k)}\equiv -1 \pmod{\frac{k}{\gcd(m,k)}}.$$

Next, we define

$$\Omega := \sum_{j=1}^{J} \delta_j \left(2m_j - 12r_j + \frac{12r_j^2}{m_j}\right),$$

$$\Delta(h,k) := -\sum_{j=1}^{J} \delta_j \left(\frac{2\gcd^2(m_j,k)}{m_j} + \frac{12\gcd^2(m_j,k)}{m_j}(\lambda^{*2}_{m_j,r_j}(h,k) - \lambda^{*}_{m_j,r_j}(h,k))\right)$$

and

$$\omega_{h,k} := \exp\left(-\pi i \sum_{j=1}^{J} \delta_j \cdot s\left(\frac{m_j h}{\gcd(m_j,k)}, \frac{k}{\gcd(m_j,k)}\right)\right), \tag{13.1.2}$$

where $s(d,c)$ is the Dedekind sum. We also define

$$\text{Д}_{h,k} := \exp\left(\pi i \sum_{j=1}^{J} \delta_j \left(\frac{r_j h}{k} - \frac{r_j \gcd(m_j,k)}{m_j k} + \frac{2r_j \gcd(m_j,k)\lambda^{*}_{m_j,r_j}(h,k)}{m_j k} \right.\right.$$
$$\left.\left. + \frac{\hbar_{m_j}(h,k)\gcd(m_j,k)}{k}(\lambda^{2}_{m_j,r_j}(h,k) - \lambda_{m_j,r_j}(h,k))\right)\right).$$

One readily verifies that the choice of $\hbar_m(h,k)$ does not affect the value of $\text{Д}_{h,k}$. At last, we define

$$\Pi_{h,k} := \begin{cases} \displaystyle\prod_{j:\lambda^{*}_{m_j,r_j}(h,k)=0} \left(1 - \exp\left(2\pi i \frac{r_j \gcd(m_j,k) + r_j \hbar_{m_j}(h,k) m_j h}{m_j k}\right)\right)^{\delta_j} & \\ \qquad\qquad \text{if there exists } j \text{ such that } \lambda^{*}_{m_j,r_j}(h,k) = 0, & \\ 1 & \\ \qquad\qquad \text{otherwise.} & \end{cases}$$

Remark 13.3.1 tells us that the choice of $\hbar_m(h,k)$ also does not affect the value of $\Pi_{h,k}$. Also, Proposition 13.3.3 indicates that for any j with $\lambda^{*}_{m_j,r_j}(h,k) = 0$, we have

$$1 - \exp\left(2\pi i \frac{r_j \gcd(m_j,k) + r_j \hbar_{m_j}(h,k) m_j h}{m_j k}\right) \neq 0.$$

Hence the value $\Pi_{h,k}$ is well-defined and $\Pi_{h,k} \neq 0$.

Given a real $0 \le x < 1$, we define

$$\Upsilon(x) := \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } 0 < x \le 1/2, \\ 1 - x & \text{if } 1/2 < x < 1. \end{cases}$$

Let $L = \operatorname{lcm}(m_1, \ldots, m_R)$. We define two disjoint sets:

$$\begin{aligned} \mathcal{L}_{>0} &:= \{(\varkappa, \ell) \ : \ 1 \le \ell \le L,\ 0 \le \varkappa < \ell,\ \Delta(\varkappa, \ell) > 0\}, \\ \mathcal{L}_{\le 0} &:= \{(\varkappa, \ell) \ : \ 1 \le \ell \le L,\ 0 \le \varkappa < \ell,\ \Delta(\varkappa, \ell) \le 0\}. \end{aligned}$$

Our main result states as follows.

Theorem 13.1.1. *If the inequality*

$$\min_{1 \le j \le J} \left(\Upsilon\Big(\lambda^*_{m_j, r_j}(\varkappa, \ell)\Big) \frac{\gcd^2(m_j, \ell)}{m_j} \right) \ge \frac{\Delta(\varkappa, \ell)}{24} \tag{13.1.3}$$

holds for all $1 \le \ell \le L$ and $0 \le \varkappa < \ell$, then for positive integers $n > -\Omega/24$, we have

$$\begin{aligned} g(n) = E(n) + 2\pi i^{\sum_{j=1}^J \delta_j} & \sum_{\substack{1 \le \ell \le L \\ (\varkappa,\ell) \in \mathcal{L}_{>0}}} \sum_{0 \le \varkappa < \ell} \left(\frac{24n + \Omega}{\Delta(\varkappa, \ell)} \right)^{-\frac{1}{2}} \\ & \times \sum_{\substack{1 \le k \le N^\star \\ k \equiv \ell \bmod L}} \frac{1}{k} I_{-1} \left(\frac{\pi}{6k} \sqrt{\Delta(\varkappa, \ell)(24n + \Omega)} \right) \\ & \times \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1 \\ h \equiv \varkappa \bmod \ell}} e^{-\frac{2\pi i n h}{k}} (-1)^{\sum_{j=1}^J \delta_j \lambda_{m_j, r_j}(h,k)} \omega_{h,k}^2 \,\text{Д}_{h,k} \Pi_{h,k}, \end{aligned} \tag{13.1.4}$$

where

$$N^\star = \left\lfloor \sqrt{2\pi \left(n + \frac{\Omega}{24} \right)} \right\rfloor, \tag{13.1.5}$$

$I_s(x)$ is the modified Bessel function of the first kind, and

$$E(n) \ll_{\mathbf{m}, \mathbf{r}, \boldsymbol{\delta}} 1. \tag{13.1.6}$$

Remark 13.1.1. To better understand the asymptotic behavior of $g(n)$, one may again apply the asymptotic expansion of $I_s(x)$ (cf. [2, p. 377, (9.7.1)]): for fixed s, when

$|\arg x| < \frac{\pi}{2}$,

$$I_s(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4s^2-1}{8x} + \frac{(4s^2-1)(4s^2-9)}{2!(8x)^2} - \cdots\right). \tag{13.1.7}$$

13.2 A Transformation Formula

Let us define

$$Ж(\varsigma;\tau) := \ (\zeta, \zeta^{-1}q; q)_\infty, \tag{13.2.1}$$

where $q := e^{2\pi i\tau}$ and $\zeta := e^{2\pi i\varsigma}$. It follows from (11.2.3) that

$$Ж(\varsigma;\tau) = \ ie^{-\frac{\pi i\tau}{6}} e^{\pi i\varsigma} \frac{\vartheta(\varsigma;\tau)}{\eta(\tau)}. \tag{13.2.2}$$

Let $r < m$ be positive integers. Hence,

$$Ж(r\tau; m\tau) = ie^{-\frac{\pi i m\tau}{6}} e^{\pi i r\tau} \frac{\vartheta(r\tau; m\tau)}{\eta(m\tau)}.$$

Recall that $d = \gcd(m,k)$, $m = dm'$ and $k = dk'$. Recall also that $\tau = (h+iz)/k$. One has, from (11.2.4), (11.2.5), (11.2.6) with $\gamma = \gamma_{(m,h,k)}$ as in (11.4.1) and the fact $s(-m'h,k') = -s(m'h,k')$, that

$$\begin{aligned}
Ж(r\tau; m\tau) &= ie^{-\frac{\pi i m\tau}{6}} e^{\pi i r\tau} \chi(\gamma_{(m,h,k)})^{-2} e^{-\frac{\pi i k' r^2\tau^2}{k'm\tau - m'h}} \\
&\quad \times \frac{\vartheta(r\tau\gamma^*_{(m,h,k)}(m\tau); \gamma_{(m,h,k)}(m\tau))}{\eta(\gamma_{(m,h,k)}(m\tau))} \\
&= ie^{-\frac{\pi i m\tau}{6}} e^{\pi i r\tau} \chi(\gamma_{(m,h,k)})^{-2} e^{-\frac{\pi i k r^2\tau^2}{km\tau - mh}} (-1)^{\lambda_{m,r}(h,k)} \\
&\quad \times e^{\pi i \lambda^2_{m,r}(h,k)\gamma_{(m,h,k)}(m\tau)} e^{2\pi i \lambda_{m,r}(h,k) r\tau\gamma^*_{(m,h,k)}(m\tau)} \\
&\quad \times \frac{\vartheta(r\tau\gamma^*_{(m,h,k)}(m\tau) + \lambda_{m,r}(h,k)\gamma_{(m,h,k)}(m\tau); \gamma_{(m,h,k)}(m\tau))}{\eta(\gamma_{(m,h,k)}(m\tau))} \\
&= i(-1)^{\lambda_{m,r}(h,k)} e^{-2\pi i s(m'h,k')} \\
&\quad \times \exp\Bigg(\pi i\bigg(\frac{rh}{k} - \frac{rd}{mk} + \frac{2rd\lambda^*_{m,r}(h,k)}{mk} \\
&\qquad\qquad + \frac{\hbar_m(h,k)d}{k}(\lambda^2_{m,r}(h,k) - \lambda_{m,r}(h,k))\bigg)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\times \exp\left(\frac{\pi}{12k}\left(\left(2m-12r+\frac{12r^2}{m}\right)z\right.\right.\\
&\qquad\left.\left.-\left(\frac{2d^2}{m}+\frac{12d^2}{m}(\lambda^{*2}_{m,r}(h,k)-\lambda^{*}_{m,r}(h,k))\right)\frac{1}{z}\right)\right)\\
&\times Ж\Big(r\tau\gamma^{*}_{(m,h,k)}(m\tau)+\lambda_{m,r}(h,k)\gamma_{(m,h,k)}(m\tau);\gamma_{(m,h,k)}(m\tau)\Big).
\end{aligned}
$$

Consequently, we deduce the following transformation formula.

Lemma 13.2.1. *We have*

$$
\begin{aligned}
G(e^{2\pi i\tau}) &= \prod_{j=1}^{J} Ж^{\delta_j}(r_j\tau;m_j\tau)\\
&= i^{\sum_{j=1}^{J}\delta_j}(-1)^{\sum_{j=1}^{J}\delta_j\lambda_{m_j,r_j}(h,k)}\omega^2_{h,k}\,Д_{h,k}\\
&\quad\times \exp\left(\frac{\pi}{12k}(\Omega z+\Delta(h,k)z^{-1})\right)\\
&\quad\times \prod_{j=1}^{J} Ж^{\delta_j}\Big(r_j\tau\gamma^{*}_{(m_j,h,k)}(m_j\tau)+\lambda_{m_j,r_j}(h,k)\gamma_{(m_j,h,k)}(m_j\tau);\gamma_{(m_j,h,k)}(m_j\tau)\Big).
\end{aligned} \tag{13.2.3}
$$

Remark 13.2.1. It follows from (11.4.4) that for all $j=1,2,\ldots,J$,

$$
0\le \Im\Big(r_j\tau\gamma^{*}_{(m_j,h,k)}(m_j\tau)+\lambda_{m_j,r_j}(h,k)\gamma_{(m_j,h,k)}(m_j\tau)\Big)<\Im\Big(\gamma_{(m_j,h,k)}(m_j\tau)\Big).
$$

13.3 Outline of the Proof

We know from (11.3.1) and (13.2.3) that

$$
\begin{aligned}
g(n) &= \sum_{1\le k\le N}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}}\int_{\xi_{h,k}} G\Big(e^{2\pi i\tau}\Big)e^{-2\pi i n\phi}e^{2\pi n\varrho}\,d\phi\\
&= i^{\sum_{j=1}^{J}\delta_j}\sum_{1\le k\le N}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i n h}{k}}(-1)^{\sum_{j=1}^{J}\delta_j\lambda_{m_j,r_j}(h,k)}\omega^2_{h,k}\,Д_{h,k}\\
&\quad\times\int_{\xi_{h,k}}\exp\left(\frac{\pi}{12k}(\Omega z+\Delta(h,k)z^{-1})\right)\\
&\qquad\times\prod_{j=1}^{J} Ж^{\delta_j}\Big(r_j\tau\gamma^{*}_{(m_j,h,k)}(m_j\tau)+\lambda_{m_j,r_j}(h,k)\gamma_{(m_j,h,k)}(m_j\tau);\gamma_{(m_j,h,k)}(m_j\tau)\Big)\\
&\qquad\times e^{-2\pi i n\phi}e^{2\pi n\varrho}\,d\phi.
\end{aligned}
$$

Let us fix a Farey fraction h/k. We first find integers $1 \le \ell \le L$ and $0 \le \varkappa < \ell$ such that $k \equiv \ell \pmod{L}$ and $h \equiv \varkappa \pmod{\ell}$. For convenience, we write $\rho(h,k) := (\varkappa, \ell)$. It is not hard to observe that for all $j = 1, 2, \ldots, J$,

$$\gcd(m_j, k) = \gcd(m_j, \ell) \quad \text{and} \quad \lambda^*_{m_j,r_j}(h,k) = \lambda^*_{m_j,r_j}(\varkappa, \ell).$$

It turns out that $\Delta(h,k) = \Delta(\varkappa, \ell)$. We now split $g(n)$ as follows.

$$\begin{aligned} g(n) = i^{\sum_{j=1}^J \delta_j} & \sum_{1\le \ell \le L} \sum_{0 \le \varkappa < \ell} \sum_{\substack{1 \le k \le N \\ k \equiv \ell \bmod L}} \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1 \\ h \equiv \varkappa \bmod \ell}} e^{-\frac{2\pi i n h}{k}} \\ & \times (-1)^{\sum_{j=1}^J \delta_j \lambda_{m_j,r_j}(h,k)} \omega_{h,k}^2 \, \text{Д}_{h,k} \\ & \times \int_{\xi_{h,k}} \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\varkappa,\ell) z^{-1})\right) \\ & \quad \times \prod_{j=1}^J \text{Ж}^{\delta_j}\Big(r_j \tau \gamma^*_{(m_j,h,k)}(m_j\tau) + \lambda_{m_j,r_j}(h,k)\gamma_{(m_j,h,k)}(m_j\tau); \gamma_{(m_j,h,k)}(m_j\tau)\Big) \\ & \quad \times e^{-2\pi i n \phi} e^{2\pi n \varrho} \, d\phi \\ & =: i^{\sum_{j=1}^J \delta_j} \sum_{1 \le \ell \le L} \sum_{0 \le \varkappa < \ell} S_{\varkappa,\ell}. \end{aligned}$$

The minor arcs are those with respect to h/k with $\rho(h,k) \in \mathcal{L}_{\le 0}$. We have the following bound.

Theorem 13.3.1. *Let $(\varkappa, \ell) \in \mathcal{L}_{\le 0}$. For positive integers $n > -\Omega/24$, we have*

$$S_{\varkappa,\ell} \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} \exp\left(\frac{2\pi}{N^2}\left(n + \frac{\Omega}{24}\right)\right).$$

In particular, if we take $N = \left\lfloor \sqrt{2\pi\left(n + \frac{\Omega}{24}\right)} \right\rfloor$, then $S_{\varkappa,\ell} \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} 1$.

The arcs with respect to h/k with $\rho(h,k) \in \mathcal{L}_{>0}$ give us the main contribution.

Theorem 13.3.2. *Let $(\varkappa, \ell) \in \mathcal{L}_{>0}$. If the inequality*

$$\min_{1 \le j \le J} \left(\Upsilon\Big(\lambda^*_{m_j,r_j}(\varkappa,\ell)\Big) \frac{\gcd^2(m_j,\ell)}{m_j}\right) \ge \frac{\Delta(\varkappa,\ell)}{24} \tag{13.3.1}$$

holds, then for positive integers $n > -\Omega/24$, we have

$$S_{\varkappa,\ell} = E_{\varkappa,\ell} + \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}} e^{-\frac{2\pi inh}{k}} (-1)^{\sum_{j=1}^{J}\delta_j\lambda_{m_j,r_j}(h,k)} \omega_{h,k}^2 \text{Д}_{h,k}\Pi_{h,k}$$
$$\times \frac{2\pi}{k}\left(\frac{24n+\Omega}{\Delta(\varkappa,\ell)}\right)^{-\frac{1}{2}} I_{-1}\left(\frac{\pi}{6k}\sqrt{\Delta(\varkappa,\ell)(24n+\Omega)}\right),$$

where

$$E_{\varkappa,\ell} \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} e^{\frac{2\pi}{N^2}(n+\frac{\Omega}{24})} + \frac{N^2 e^{\frac{2\pi}{N^2}\left(n+\frac{\Omega}{24}\right)}}{n+\frac{\Omega}{24}}.$$

In particular, if we take $N = \left\lfloor\sqrt{2\pi\left(n+\frac{\Omega}{24}\right)}\right\rfloor$, then $E_{\varkappa,\ell} \ll_{\mathbf{m},\mathbf{r},\boldsymbol{\delta}} 1$.

Theorems 13.3.1 and 13.3.2 immediately imply the main result. Before presenting proofs of the two results respectively in §§13.4 and 13.5, we make the following preparations.

For fixed $\varkappa$ and ℓ with $1 \le \ell \le L$ and $0 \le \varkappa < \ell$, one may split the indices $\{1, 2, \dots, J\}$ into two disjoint parts:

$$\mathcal{J}^*_{\varkappa,\ell} = \{j_1^*, \dots, j_\alpha^*\} \quad \text{and} \quad \mathcal{J}^{**}_{\varkappa,\ell} = \{j_1^{**}, \dots, j_\beta^{**}\},$$

so that for $j^* \in \mathcal{J}^*_{\varkappa,\ell}$ we have $\lambda^*_{m_{j^*},r_{j^*}}(\varkappa,\ell) = 0$ and for $j^{**} \in \mathcal{J}^{**}_{\varkappa,\ell}$ we have $\lambda^*_{m_{j^{**}},r_{j^{**}}}(\varkappa,\ell) \neq 0$.

Proposition 13.3.3. *Let $j^* \in \mathcal{J}^*_{\varkappa,\ell}$. For any Farey fraction h/k such that $k \equiv \ell \pmod L$ and $h \equiv \varkappa \pmod \ell$, we have that*

$$r_{j^*}\tau\gamma^*_{(m_{j^*},h,k)}(m_{j^*}\tau) + \lambda_{m_{j^*},r_{j^*}}(h,k)\gamma_{(m_{j^*},h,k)}(m_{j^*}\tau)$$
$$= \frac{r_{j^*}\gcd(m_{j^*},k) + r_{j^*}\hbar_{m_{j^*}}(h,k)m_{j^*}h}{m_{j^*}k} \tag{13.3.2}$$

is a real noninteger. Further,

$$\left|1 - e^{\frac{2\pi i}{m_{j^*}}}\right| \le \left|1 - e^{2\pi i\left(r_{j^*}\tau\gamma^*_{(m_{j^*},h,k)}(m_{j^*}\tau) + \lambda_{m_{j^*},r_{j^*}}(h,k)\gamma_{(m_{j^*},h,k)}(m_{j^*}\tau)\right)}\right| \le 2. \tag{13.3.3}$$

Proof. In this proof, we write for short $m = m_{j^*}$ and $r = r_{j^*}$. We also write $d = \gcd(m,k)$, $m = dm'$ and $k = dk'$. Since $j^* \in \mathcal{J}^*_{\varkappa,\ell}$, we have $\lambda^*_{m,r}(h,k) = \lambda^*_{m,r}(\varkappa,\ell) = 0$. Hence d

divides rh and $\lambda_{m,r}(h,k) = rh/d$. We know from (11.4.4) that

$$\begin{aligned}
&r\tau\gamma^*_{(m,h,k)}(m\tau) + \lambda_{m,r}(h,k)\gamma_{(m,h,k)}(m\tau) \\
&\quad = \frac{rd}{mk} + \lambda_{m,r}(h,k)\frac{\hbar_m(h,k)d}{k} + \lambda^*_{m,r}(h,k)\frac{d^2}{mkz}i \\
&\quad = \frac{rd}{mk} + \lambda_{m,r}(h,k)\frac{\hbar_m(h,k)d}{k} \\
&\quad = \frac{rd}{mk} + \frac{rh}{d}\frac{\hbar_m(h,k)d}{k} \\
&\quad = \frac{r(1+\hbar_m(h,k)m'h)}{m'k} \\
&\quad = \frac{b_{m'}r}{m} = \frac{b_{m'}}{m'}\frac{r}{d},
\end{aligned}$$

where as in §11.4, we have put $b_{m'} = (\hbar_m(h,k)m'h + 1)/k'$. Hence it is a real number.

Notice that $d = \gcd(m,k)$. Since $\gcd(h,k) = 1$, $d \mid rh$ implies that $d \mid r$. Further, $b_{m'} = (\hbar_m(h,k)m'h+1)/k'$ implies that $\gcd(m', b_{m'}) = 1$. Hence, if $\frac{b_{m'}}{m'}\frac{r}{d}$ is an integer, then $m' \mid \frac{r}{d}$ so that $m = dm' \mid r$. This violates the assumption that $1 \le r \le m - 1$. Hence $r\tau\gamma^*_{(m,h,k)}(m\tau) + \lambda_{m,r}(h,k)\gamma_{(m,h,k)}(m\tau)$ is not an integer and (13.3.3) follows immediately. □

Remark 13.3.1. Recall that $\hbar_m(h,k)$ is defined to be an integer such that

$$\hbar_m(h,k)\frac{mh}{\gcd(m,k)} \equiv -1 \pmod{\frac{k}{\gcd(m,k)}}.$$

Let n be an integer. It turns out that

$$\begin{aligned}
&\exp\left(2\pi i \frac{r_{j^*}\gcd(m_{j^*},k) + r_{j^*}\left(\hbar_{m_{j^*}}(h,k) + n\frac{k}{\gcd(m_{j^*},k)}\right)m_{j^*}h}{m_{j^*}k}\right) \\
&\quad = \exp\left(2\pi i \frac{r_{j^*}\gcd(m_{j^*},k) + r_{j^*}\hbar_{m_{j^*}}(h,k)m_{j^*}h}{m_{j^*}k} + 2n\pi i\frac{r_{j^*}h}{\gcd(m_{j^*},k)}\right) \\
&\quad = \exp\left(2\pi i \frac{r_{j^*}\gcd(m_{j^*},k) + r_{j^*}\hbar_{m_{j^*}}(h,k)m_{j^*}h}{m_{j^*}k}\right),
\end{aligned}$$

since from the above proof we have $\gcd(m_{j^*},k) \mid r_{j^*}$. Hence the choice of $\hbar_{m_{j^*}}(h,k)$ does not affect the value of

$$\exp\left(2\pi i\left(r_{j^*}\tau\gamma^*_{(m_{j^*},h,k)}(m_{j^*}\tau)+\lambda_{m_{j^*},r_{j^*}}(h,k)\gamma_{(m_{j^*},h,k)}(m_{j^*}\tau)\right)\right)$$
$$=\exp\left(2\pi i\frac{r_{j^*}\gcd(m_{j^*},k)+r_{j^*}\hbar_{m_{j^*}}(h,k)m_{j^*}h}{m_{j^*}k}\right).$$

13.4 Minor Arcs

Let $(\varkappa,\ell)\in\mathcal{L}_{\le 0}$, namely, $\Delta(\varkappa,\ell)\le 0$. We write $\mathcal{J}^*=\mathcal{J}^*_{\varkappa,\ell}$ and $\mathcal{J}^{**}=\mathcal{J}^{**}_{\varkappa,\ell}$. Notice that

$$\begin{aligned}|S_{\varkappa,\ell}|\le &\sum_{\substack{1\le k\le N\\ k\equiv\ell \bmod L}}\ \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}}\int_{\xi_{h,k}}\exp\left(\frac{\pi}{12k}(\Omega\Re(z)+\Delta(\varkappa,\ell)\Re(z^{-1}))\right)\\ &\times\left|\prod_{j=1}^{J}\text{Ж}^{\delta_j}\left(r_j\tau\gamma^*_{(m_j,h,k)}(m_j\tau)+\lambda_{m_j,r_j}(h,k)\gamma_{(m_j,h,k)}(m_j\tau);\gamma_{(m_j,h,k)}(m_j\tau)\right)\right|\\ &\times e^{2\pi n\varrho}\,d\phi.\end{aligned}$$

We now consider the Farey arcs with respect to h/k with $k\equiv\ell \pmod L$ and $h\equiv\varkappa \pmod \ell$. Since $\Delta(\varkappa,\ell)\le 0$, it follows from (11.5.3) and (11.5.5) that

$$\begin{aligned}\exp\left(\frac{\pi}{12k}(\Omega\Re(z)+\Delta(\varkappa,\ell)\Re(z^{-1}))\right)&\le\exp\left(\frac{\pi}{12k}\left(\Omega\frac{k}{N^2}+\Delta(\varkappa,\ell)\frac{k}{2}\right)\right)\\ &=\exp\left(\frac{\pi\varrho\,\Omega}{12}\right)\exp\left(\frac{\pi\Delta(\varkappa,\ell)}{24}\right).\end{aligned}$$

For convenience, now we write $\lambda_j=\lambda_{m_j,r_j}(h,k)$ and $\lambda_j^*=\lambda^*_{m_j,r_j}(h,k)$. We also write for short $\tilde{\varsigma}_j=r_j\tau\gamma^*_{(m_j,h,k)}(m_j\tau)+\lambda_{m_j,r_j}(h,k)\gamma_{(m_j,h,k)}(m_j\tau)$ and $\tilde{\tau}_j=\gamma_{(m_j,h,k)}(m_j\tau)$. We know from (11.4.2) and (11.4.4) that

$$\Im(\tilde{\tau}_j)=\frac{\gcd^2(m_j,k)}{m_jk}\Re(z^{-1})=\frac{\gcd^2(m_j,\ell)}{m_jk}\Re(z^{-1})$$

and

$$\Im(\tilde{\varsigma}_j)=\lambda_j^*\frac{\gcd^2(m_j,k)}{m_jk}\Re(z^{-1})=\lambda_j^*\frac{\gcd^2(m_j,\ell)}{m_jk}\Re(z^{-1}).$$

Notice that

$$0 \le \Im(\tilde{\varsigma}_j) < \Im(\tilde{\tau}_j).$$

We write

$$\begin{aligned}\prod_{j=1}^{J} \text{Ӂ}^{\delta_j}(\tilde{\varsigma}_j;\tilde{\tau}_j) = &\prod_{j^*\in\mathcal{J}^*}(1-e^{2\pi i\tilde{\varsigma}_{j^*}})^{\delta_{j^*}}\\ &\times \prod_{j^*\in\mathcal{J}^*}(e^{2\pi i(\tilde{\tau}_{j^*}+\tilde{\varsigma}_{j^*})}, e^{2\pi i(\tilde{\tau}_{j^*}-\tilde{\varsigma}_{j^*})}; e^{2\pi i\tilde{\tau}_{j^*}})_\infty^{\delta_{j^*}}\\ &\times \prod_{j^{**}\in\mathcal{J}^{**}}(e^{2\pi i\tilde{\varsigma}_{j^{**}}}, e^{2\pi i(\tilde{\tau}_{j^{**}}-\tilde{\varsigma}_{j^{**}})}; e^{2\pi i\tilde{\tau}_{j^{**}}})_\infty^{\delta_{j^{**}}}.\end{aligned}$$

First, it follows from Proposition 13.3.3 that

$$\prod_{j^*\in\mathcal{J}^*}(1-e^{2\pi i\tilde{\varsigma}_{j^*}})^{\delta_{j^*}} \ll 1.$$

Further, as we have seen in §11.5.2, for $j^*\in\mathcal{J}^*$ (hence $\lambda^*_{j^*}=0$),

$$\begin{aligned}&\left|(e^{2\pi i(\tilde{\tau}_{j^*}+\tilde{\varsigma}_{j^*})}, e^{2\pi i(\tilde{\tau}_{j^*}-\tilde{\varsigma}_{j^*})}; e^{2\pi i\tilde{\tau}_{j^*}})_\infty^{\delta_{j^*}}\right|\\ &\le \sum_{n\ge0}\sum_{s\ge0}\sum_{t\ge0} p^*_{|\delta_{j^*}|}(s,t;n)|e^{2\pi i(\tilde{\tau}_{j^*}+\tilde{\varsigma}_{j^*})}|^s|e^{2\pi i(\tilde{\tau}_{j^*}-\tilde{\varsigma}_{j^*})}|^t|e^{2\pi i\tilde{\tau}_{j^*}}|^n\\ &= \sum_{n\ge0}\sum_{s\ge0}\sum_{t\ge0} p^*_{|\delta_{j^*}|}(s,t;n)e^{-2\pi\Im(\tilde{\tau}_{j^*}+\tilde{\varsigma}_{j^*})s}e^{-2\pi\Im(\tilde{\tau}_{j^*}-\tilde{\varsigma}_{j^*})t}e^{-2\pi\Im(\tilde{\tau}_{j^*})n}\\ &= \sum_{n\ge0}\sum_{s\ge0}\sum_{t\ge0} p^*_{|\delta_{j^*}|}(s,t;n)\exp\left(-2\pi\frac{\gcd^2(m_{j^*},\ell)}{m_{j^*}k}\Re(z^{-1})s\right)\\ &\quad\times\exp\left(-2\pi\frac{\gcd^2(m_{j^*},\ell)}{m_{j^*}k}\Re(z^{-1})t\right)\exp\left(-2\pi\frac{\gcd^2(m_{j^*},\ell)}{m_{j^*}k}\Re(z^{-1})n\right)\\ &\le \sum_{n\ge0}\sum_{s\ge0}\sum_{t\ge0} p^*_{|\delta_{j^*}|}(s,t;n)\exp\left(-\pi\frac{\gcd^2(m_{j^*},\ell)}{m_{j^*}}s\right)\\ &\quad\times\exp\left(-\pi\frac{\gcd^2(m_{j^*},\ell)}{m_{j^*}}t\right)\exp\left(-\pi\frac{\gcd^2(m_{j^*},\ell)}{m_{j^*}}n\right),\end{aligned}$$

where we use $\Re(z^{-1})\ge k/2$. It follows from (11.5.7) that

$$(e^{2\pi i(\tilde{\tau}_{j^*}+\tilde{\varsigma}_{j^*})}, e^{2\pi i(\tilde{\tau}_{j^*}-\tilde{\varsigma}_{j^*})}; e^{2\pi i\tilde{\tau}_{j^*}})_\infty^{\delta_{j^*}} \ll 1.$$

Likewise, for $j^{**} \in \mathcal{J}^{**}$,

$$
\begin{aligned}
&\left|(e^{2\pi i \tilde{\varsigma}_{j^{**}}}, e^{2\pi i(\tilde{\tau}_{j^{**}} - \tilde{\varsigma}_{j^{**}})}; e^{2\pi i \tilde{\tau}_{j^{**}}})_\infty^{\delta_{j^{**}}}\right| \\
&\quad \le \sum_{n\ge 0}\sum_{s\ge 0}\sum_{t\ge 0} p^*_{|\delta_{j^{**}}|}(s,t;n) \exp\left(-\pi\lambda^*_{j^{**}} \frac{\gcd^2(m_{j^{**}},\ell)}{m_{j^{**}}} s\right) \\
&\quad\quad \times \exp\left(-\pi(1-\lambda^*_{j^{**}})\frac{\gcd^2(m_{j^{**}},\ell)}{m_{j^{**}}} t\right) \exp\left(-\pi\frac{\gcd^2(m_{j^{**}},\ell)}{m_{j^{**}}} n\right) \\
&\quad \ll 1.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
S_{\varkappa,\ell} &\ll \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}} \int_{\xi_{h,k}} e^{\frac{\pi\varrho\,\Omega}{12}} e^{2\pi n\varrho}\ d\phi \\
&\ll \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}} e^{2\pi\varrho(n+\frac{\Omega}{24})} \frac{1}{kN} \\
&\ll e^{2\pi\varrho(n+\frac{\Omega}{24})} = e^{\frac{2\pi}{N^2}(n+\frac{\Omega}{24})}.
\end{aligned}
$$

13.5 Major Arcs

Let $(\varkappa,\ell)\in\mathcal{L}_{>0}$, namely, $\Delta(\varkappa,\ell)>0$. Again, we write $\mathcal{J}^* = \mathcal{J}^*_{\varkappa,\ell}$ and $\mathcal{J}^{**} = \mathcal{J}^{**}_{\varkappa,\ell}$. Let us consider the Farey arcs with respect to h/k with $k\equiv \ell \pmod L$ and $h\equiv \varkappa \pmod \ell$. For convenience, we write $\tilde{\varsigma}_j(h,k) = r_j\tau\gamma^*_{(m_j,h,k)}(m_j\tau) + \lambda_{m_j,r_j}(h,k)\gamma_{(m_j,h,k)}(m_j\tau)$ and $\tilde{\tau}_j(h,k) = \gamma_{(m_j,h,k)}(m_j\tau)$.

Recall that

$$
\begin{aligned}
S_{\varkappa,\ell} = &\sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}} e^{-\frac{2\pi i n h}{k}} (-1)^{\sum_{j=1}^J \delta_j\lambda_{m_j,r_j}(h,k)} \omega^2_{h,k}\ \text{Д}_{h,k} \\
&\times \int_{\xi_{h,k}} \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\varkappa,\ell) z^{-1})\right) \prod_{j=1}^J \text{ЖК}^{\delta_j}\Big(\tilde{\varsigma}_j(h,k);\tilde{\tau}_j(h,k)\Big) \\
&\times e^{-2\pi i n\phi} e^{2\pi n\varrho}\ d\phi.
\end{aligned}
$$

We split $S_{\varkappa,\ell}$ into two parts Σ_1 and Σ_2 where

$$\Sigma_1 := \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}} e^{-\frac{2\pi i n h}{k}}(-1)^{\sum_{j=1}^J \delta_j \lambda_{m_j,r_j}(h,k)}\omega_{h,k}^2\,Д_{h,k}$$
$$\times \int_{\xi_{h,k}} \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\varkappa,\ell)z^{-1})\right)\Pi_{h,k}e^{-2\pi i n\phi}e^{2\pi n\varrho}\,d\phi$$

and

$$\Sigma_2 := \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}} e^{-\frac{2\pi i n h}{k}}(-1)^{\sum_{j=1}^J \delta_j \lambda_{m_j,r_j}(h,k)}\omega_{h,k}^2\,Д_{h,k}$$
$$\times \int_{\xi_{h,k}} \exp\left(\frac{\pi}{12k}(\Omega z + \Delta(\varkappa,\ell)z^{-1})\right)\left(\prod_{j=1}^J Ж^{\delta_j}\Big(\tilde{\varsigma}_j(h,k);\tilde{\tau}_j(h,k)\Big) - \Pi_{h,k}\right)$$
$$\times\, e^{-2\pi i n\phi}e^{2\pi n\varrho}\,d\phi.$$

We first show that Σ_2 is negligible. Notice that by (11.5.3)

$$|\Sigma_2| \le \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}} e^{2\pi\varrho(n+\frac{\Omega}{24})}|\Pi_{h,k}|$$
$$\times \int_{\xi_{h,k}} \exp\left(\frac{\pi\Delta(\varkappa,\ell)}{12k}\Re(z^{-1})\right)\left|\frac{1}{\Pi_{h,k}}\prod_{j=1}^J Ж^{\delta_j}\Big(\tilde{\varsigma}_j(h,k);\tilde{\tau}_j(h,k)\Big) - 1\right|\,d\phi.$$

Let us fix h and k and write $\tilde{\varsigma}_j = \tilde{\varsigma}_j(h,k)$ and $\tilde{\tau}_j = \tilde{\tau}_j(h,k)$. We also write $\lambda_j^* = \lambda_{m_j,r_j}^*(h,k)$. Recalling the definition of $\Pi_{h,k}$ and Proposition 13.3.3, we have

$$\frac{1}{\Pi_{h,k}}\prod_{j=1}^J Ж^{\delta_j}\Big(\tilde{\varsigma}_j;\tilde{\tau}_j\Big) - 1 = \prod_{j^*\in\mathcal{J}^*}(e^{2\pi i(\tilde{\tau}_{j^*}+\tilde{\varsigma}_{j^*})}, e^{2\pi i(\tilde{\tau}_{j^*}-\tilde{\varsigma}_{j^*})}; e^{2\pi i\tilde{\tau}_{j^*}})_\infty^{\delta_{j^*}}$$
$$\times \prod_{j^{**}\in\mathcal{J}^{**}}(e^{2\pi i\tilde{\varsigma}_{j^{**}}}, e^{2\pi i(\tilde{\tau}_{j^{**}}-\tilde{\varsigma}_{j^{**}})}; e^{2\pi i\tilde{\tau}_{j^{**}}})_\infty^{\delta_{j^{**}}} - 1.$$

Let us write for short

$$\tilde{\varsigma}_j^{\mathrm{New}} = \begin{cases}\tilde{\tau}_j + \tilde{\varsigma}_j & \text{if } j\in\mathcal{J}^*,\\ \tilde{\varsigma}_j & \text{if } j\in\mathcal{J}^{**}.\end{cases}$$

It follows again from (11.4.2) and (11.4.4) that

$$\Im(\tilde{\tau}_j) = \frac{\gcd^2(m_j,\ell)}{m_j k}\Re(z^{-1}),$$

$$\Im(\tilde{\varsigma}_j) = \lambda_j^* \frac{\gcd^2(m_j,\ell)}{m_j k}\Re(z^{-1})$$

and

$$\Im(\tilde{\varsigma}_j^{\text{New}}) = \Phi(\lambda_j^*) \frac{\gcd^2(m_j,\ell)}{m_j k}\Re(z^{-1}),$$

where for real $0 \le x < 1$,

$$\Phi(x) := \begin{cases} 1 & \text{if } x = 0, \\ x & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}
&\left| \frac{1}{\Pi_{h,k}} \prod_{j=1}^{J} Ж^{\delta_j}\Big(\tilde{\varsigma}_j; \tilde{\tau}_j\Big) - 1 \right| \\
&\quad = \left| \prod_{j=1}^{J} (e^{2\pi i \tilde{\varsigma}_j^{\text{New}}}, e^{2\pi i (\tilde{\tau}_j - \tilde{\varsigma}_j)}; e^{2\pi i \tilde{\tau}_j})_\infty^{\delta_j} - 1 \right| \\
&\quad \le \sum_{\mathbf{n} := (n_1,\ldots,n_J) \in \mathbb{Z}_{\ge 0}^J} \; \sum_{\mathbf{s} := (s_1,\ldots,s_J) \in \mathbb{Z}_{\ge 0}^J} \; \sum_{\mathbf{t} := (t_1,\ldots,t_J) \in \mathbb{Z}_{\ge 0}^J} \\
&\qquad \prod_{j=1}^{J} p^*_{|\delta_j|}(s_j, t_j; n_j) |e^{2\pi i \tilde{\varsigma}_j^{\text{New}}}|^{s_j} |e^{2\pi i (\tilde{\tau}_j - \tilde{\varsigma}_j)}|^{t_j} |e^{2\pi i \tilde{\tau}_j}|^{n_j} - 1 \\
&\quad = \sum\sum\sum_{\mathbf{n}\times\mathbf{s}\times\mathbf{t} \in (\mathbb{Z}_{\ge 0}^J)^3 \setminus (0,\ldots,0)^3} \prod_{j=1}^{J} p^*_{|\delta_j|}(s_j, t_j; n_j) |e^{2\pi i \tilde{\varsigma}_j^{\text{New}}}|^{s_j} |e^{2\pi i (\tilde{\tau}_j - \tilde{\varsigma}_j)}|^{t_j} |e^{2\pi i \tilde{\tau}_j}|^{n_j} \\
&\quad = \sum\sum\sum_{\mathbf{n}\times\mathbf{s}\times\mathbf{t} \in (\mathbb{Z}_{\ge 0}^J)^3 \setminus (0,\ldots,0)^3} \prod_{j=1}^{J} p^*_{|\delta_j|}(s_j, t_j; n_j) e^{-2\pi \Im(\tilde{\varsigma}_j^{\text{New}}) s_j} e^{-2\pi \Im(\tilde{\tau}_j - \tilde{\varsigma}_j) t_j} e^{-2\pi \Im(\tilde{\tau}_j) n_j} \\
&\quad = \sum\sum\sum_{\mathbf{n}\times\mathbf{s}\times\mathbf{t} \in (\mathbb{Z}_{\ge 0}^J)^3 \setminus (0,\ldots,0)^3} \left(\prod_{j=1}^{J} p^*_{|\delta_j|}(s_j, t_j; n_j) \right) \\
&\qquad \times \exp\left(-2\pi \frac{\Re(z^{-1})}{k} \sum_{j=1}^{J} \frac{\gcd^2(m_j,\ell)}{m_j} \Big(\Phi(\lambda_j^*) s_j + (1 - \lambda_j^*) t_j + n_j \Big) \right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\exp\left(\frac{\pi\Delta(\varkappa,\ell)}{12k}\Re(z^{-1})\right)\left|\frac{1}{\Pi_{h,k}}\prod_{j=1}^{J}\text{Ж}^{\delta_j}\Big(\tilde{\varsigma}_j;\tilde{\tau}_j\Big)-1\right|\\
&\leq \sum\sum\sum_{\mathbf{n}\times\mathbf{s}\times\mathbf{t}\in(\mathbb{Z}_{\geq 0}^{J})^3\backslash(0,\ldots,0)^3}\left(\prod_{j=1}^{J}p^{*}_{|\delta_j|}(s_j,t_j;n_j)\right)\\
&\times\exp\left(-2\pi\frac{\Re(z^{-1})}{k}\bigg(-\frac{\Delta(\varkappa,\ell)}{24}+\sum_{j=1}^{J}\frac{\gcd^2(m_j,\ell)}{m_j}\Big(\Phi(\lambda_j^{*})s_j+(1-\lambda_j^{*})t_j+n_j\Big)\bigg)\right).
\end{aligned}
$$

Since at least one coordinate of $\mathbf{n}\times\mathbf{s}\times\mathbf{t}$ is nonzero, under the condition (13.3.1), we know that

$$
\begin{aligned}
&-\frac{\Delta(\varkappa,\ell)}{24}+\sum_{j=1}^{J}\frac{\gcd^2(m_j,\ell)}{m_j}\Big(\Phi(\lambda_j^{*})s_j+(1-\lambda_j^{*})t_j+n_j\Big)\\
&\qquad\geq -\frac{\Delta(\varkappa,\ell)}{24}+\min_{1\leq j\leq J}\left(\Upsilon(\lambda_j^{*})\frac{\gcd^2(m_j,\ell)}{m_j}\right)\geq 0
\end{aligned}
$$

for all $\mathbf{n}\times\mathbf{s}\times\mathbf{t}\in(\mathbb{Z}_{\geq 0}^{J})^3\backslash(0,\ldots,0)^3$. Recalling that $\Re(z^{-1})\geq k/2$, it follows that

$$
\exp\left(\frac{\pi\Delta(\varkappa,\ell)}{12k}\Re(z^{-1})\right)\left|\frac{1}{\Pi_{h,k}}\prod_{j=1}^{J}\text{Ж}^{\delta_j}\Big(\tilde{\varsigma}_j;\tilde{\tau}_j\Big)-1\right|
$$

is maximized when $\Re(z^{-1})=k/2$. Namely,

$$
\begin{aligned}
&\exp\left(\frac{\pi\Delta(\varkappa,\ell)}{12k}\Re(z^{-1})\right)\left|\frac{1}{\Pi_{h,k}}\prod_{j=1}^{J}\text{Ж}^{\delta_j}\Big(\tilde{\varsigma}_j;\tilde{\tau}_j\Big)-1\right|\\
&\quad\leq \exp\left(\frac{\pi\Delta(\varkappa,\ell)}{24}\right)\sum\sum\sum_{\mathbf{n}\times\mathbf{s}\times\mathbf{t}\in(\mathbb{Z}_{\geq 0}^{J})^3\backslash(0,\ldots,0)^3}\left(\prod_{j=1}^{J}p^{*}_{|\delta_j|}(s_j,t_j;n_j)\right)\\
&\quad\quad\times\exp\left(-\pi\sum_{j=1}^{J}\frac{\gcd^2(m_j,\ell)}{m_j}\Big(\Phi(\lambda_j^{*})s_j+(1-\lambda_j^{*})t_j+n_j\Big)\right)\\
&\quad\ll 1.
\end{aligned}
$$

Together with the fact $\prod_{h,k}\ll 1$ which follows from (13.3.3), we conclude that

$$
\Sigma_2\ll\sum_{\substack{1\leq k\leq N\\ k\equiv\ell \bmod L}}\sum_{\substack{0\leq h<k\\ \gcd(h,k)=1\\ h\equiv\varkappa \bmod \ell}}e^{2\pi\varrho(n+\frac{\Omega}{24})}\int_{\xi_{h,k}}1\ d\phi
$$

$$\ll \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv \varkappa \bmod \ell}} e^{2\pi\varrho(n+\frac{\Omega}{24})}\frac{1}{kN}$$
$$\ll e^{2\pi\varrho(n+\frac{\Omega}{24})} = e^{\frac{2\pi}{N^2}(n+\frac{\Omega}{24})}.$$

Finally, we estimate the main contribution Σ_1. Recall that

$$\Sigma_1 = \sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv \varkappa \bmod \ell}} e^{-\frac{2\pi inh}{k}}(-1)^{\sum_{j=1}^{J}\delta_j\lambda_{m_j,r_j}(h,k)}\omega_{h,k}^2\,Д_{h,k}\Pi_{h,k}$$
$$\times \int_{\xi_{h,k}} \exp\left(\frac{\pi}{12k}(\Omega z+\Delta(\varkappa,\ell)z^{-1})\right)e^{-2\pi in\phi}e^{2\pi n\varrho}\ d\phi.$$

We simply apply Lemma 11.5.1. The main contribution to Σ_1 is

$$\sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv \varkappa \bmod \ell}} e^{-\frac{2\pi inh}{k}}(-1)^{\sum_{j=1}^{J}\delta_j\lambda_{m_j,r_j}(h,k)}\omega_{h,k}^2\,Д_{h,k}\Pi_{h,k}$$
$$\times \frac{2\pi}{k}\left(\frac{24n+\Omega}{\Delta(\varkappa,\ell)}\right)^{-\frac{1}{2}} I_{-1}\left(\frac{\pi}{6k}\sqrt{\Delta(\varkappa,\ell)(24n+\Omega)}\right).$$

The error term in Σ_1 is bounded by

$$\sum_{\substack{1\le k\le N\\ k\equiv \ell \bmod L}} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ h\equiv \varkappa \bmod \ell}} \frac{e^{2\pi\varrho\left(n+\frac{\Omega}{24}\right)}}{n+\frac{\Omega}{24}} \ll \frac{N^2 e^{\frac{2\pi}{N^2}\left(n+\frac{\Omega}{24}\right)}}{n+\frac{\Omega}{24}}.$$

13.6 An Application

As an application, we confirm Tang's inequalities in [167] in the asymptotic sense. Here we will expand the infinite product as $\sum_{n\ge 0} g(n)q^n$.

In general, to obtain an explicit asymptotic formula of $g(n)$, we first compute $\mathcal{L}_{>0}$. Next, we find the largest number among $\{\sqrt{\Delta(\varkappa,\ell)}/k\}$ with $(\varkappa,\ell)\in\mathcal{L}_{>0}$ and $k\equiv \ell \pmod{L}$. Now one needs to check if the corresponding I-Bessel function vanishes for this choice. If it is nonvanishing, then the asymptotic formula shall be obtained from the I-Bessel term. Otherwise, we move to find the second largest number among $\{\sqrt{\Delta(\varkappa,\ell)}/k\}$ and carry out the same program. Notice that if there are multiple choices of $\varkappa$, ℓ and k giving the same value of $\sqrt{\Delta(\varkappa,\ell)}/k$, one should sum up all such I-Bessel terms and check if the summation vanishes or not.

Let

$$\sum_{n\geq 0} g(n)q^n = \frac{(q^2,q^8;q^{10})_\infty (q^4;q^6;q^{10})_\infty^2}{(q^2,q^3;q^5)_\infty^2}.$$

Then $\mathbf{m} = \{5,10,10\}$, $\mathbf{r} = \{2,2,4\}$ and $\boldsymbol{\delta} = \{-2,1,2\}$. Hence $L = 10$ and $\Omega = -8$. We compute that

$$\begin{aligned}\mathcal{L}_{>0} = \{&(0,1),(0,3),(1,3),(2,3),(0,5),(2,5),(3,5),(0,7),(1,7),(2,7),(3,7),\\ &(4,7),(5,7),(6,7),(0,9),(1,9),(2,9),(3,9),(4,9),(5,9),(6,9),(7,9),\\ &(8,9),(1,10),(2,10),(3,10),(4,10),(6,10),(7,10),(8,10),(9,10)\}.\end{aligned}$$

First, the assumption (13.1.3) is satisfied. We next find that the largest number among $\{\sqrt{\Delta(\varkappa,\ell)}/k\}$ with $(\varkappa,\ell)\in\mathcal{L}_{>0}$ and $k\equiv \ell \pmod{L}$ is $\frac{1}{\sqrt{5}}$. Here we have four choices:

$$(\varkappa,\ell,k) = (0,1,1),\ (0,5,5),\ (2,5,5),\ (3,5,5).$$

When $k=1$, the admissible (h,k) is $(0,1)$. We compute that the I-Bessel term is

$$\frac{\sqrt{2}\pi}{\sqrt{15}}\sin\left(\frac{\pi}{5}\right)\left(n-\frac{1}{3}\right)^{-1/2} I_{-1}\left(\frac{\sqrt{2}\pi}{\sqrt{15}}\sqrt{n-\frac{1}{3}}\right).$$

When $k=5$, the admissible (h,k) are $(2,5)$ and $(3,5)$. We compute that, in total, the I-Bessel term is

$$\frac{\sqrt{2}\pi}{\sqrt{15}}\sin\left(\frac{2\pi}{5}\big(2n+1\big)\right)\left(n-\frac{1}{3}\right)^{-1/2} I_{-1}\left(\frac{\sqrt{2}\pi}{\sqrt{15}}\sqrt{n-\frac{1}{3}}\right).$$

In total, we therefore have

$$\frac{\sqrt{2}\pi}{\sqrt{15}}\left(\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\big(2n+1\big)\right)\right)\left(n-\frac{1}{3}\right)^{-1/2} I_{-1}\left(\frac{\sqrt{2}\pi}{\sqrt{15}}\sqrt{n-\frac{1}{3}}\right).$$

Notice that $\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}(2n+1)\right)$ vanishes only if $n\equiv 1 \pmod{5}$. Hence, we have the following asymptotic formula.

Theorem 13.6.1. *For* $n \not\equiv 1 \pmod 5$,

$$g(n) \sim \frac{\sqrt{2}\pi}{\sqrt{15}}\left(\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\big(2n+1\big)\right)\right)\left(n-\frac{1}{3}\right)^{-1/2} I_{-1}\left(\frac{\sqrt{2}\pi}{\sqrt{15}}\sqrt{n-\frac{1}{3}}\right)$$
$$\sim \frac{1}{30^{1/4}}\left(\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\big(2n+1\big)\right)\right)n^{-3/4}\exp\left(\frac{\sqrt{2}\pi}{\sqrt{15}}\sqrt{n}\right).$$

It follows that $g(5n+0,2,3)>0$ and $g(5n+4)<0$ for sufficiently large n. If we further compute a number of lower I-Bessel terms, we still encounter the same vanishment for $n \equiv 1 \pmod 5$. This highly suggests that $g(5n+1)=0$, which is, indeed, proved by Tang using elementary techniques in [167].

All other inequalities conjectured by Tang can be proved in the same manner. We omit the details here.

13.7 References

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Chapter 14

Nonmodular Infinite Products and a Conjecture of Seo and Yee

This chapter comes from

- S. Chern, Nonmodular infinite products and a Conjecture of Seo and Yee, submitted. Available at arXiv:1912.10341. (Ref. [61])

14.1 Introduction

In the previous three chapters, we have discussed the asymptotics for coefficients in infinite products $G(q)$ that are modular. Let $q = e^{2\pi i\tau}$ with τ in the upper half complex plane. If τ is replaced by $T(\tau)$ where T is a transformation belonging to some subgroup of finite index of the modular group, then the resulting function remains essentially invariant according to the modularity. This allows us to study the asymptotics of $G(q)$ when q is close to rational points on the unit circle. However, the story is different if the infinite product is no longer modular.

The motivation of this chapter is a recent conjecture of Seo and Yee [161] in their study of seaweed algebras. They proved that an earlier conjecture of Coll, A. Mayers and N. Mayers [70] is equivalent to the following nonnegativity conjecture.

Conjecture 14.1.1. The series expansion of

$$\frac{1}{(q,-q^3;q^4)_\infty} \tag{14.1.1}$$

has nonnegative coefficients.

Notice that the above infinite product is no more modular. Hence, a Rademacher-type proof fails. Also, if we rewrite this product as

$$\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty(q^6;q^8)_\infty},$$

then the numerator $(q^3;q^4)_\infty$ causes the expiration of Meinardus' powerful approach [134]. One of the few works about asymptotics of nonmodular infinite products is due to Grosswald [89], who absorbed ideas from Lehner [118] and Livingood [130]. In his paper, the infinite product

$$\frac{1}{(q^a;q^M)_\infty} \tag{14.1.2}$$

with a prime modulus M is considered. However, a closer examination of Grosswald's paper reveals several mistakes, among which at least the calculation of the residue R_3 on page 119 of [89] is not robust. Also, a natural question is about the case where the modulus is composite.

Let M be a positive integer and a be any of $1,2,\ldots,M$. The first goal of this chapter is to investigate the asymptotic behavior of

$$\Phi_{a,M}(q) := \log\left(\frac{1}{(q^a;q^M)_\infty}\right) \tag{14.1.3}$$

when the complex variable q with $|q|<1$ approaches the unit circle.

Theorem 14.1.1. *Let X be a sufficiently large positive number. Let*

$$q = e^{-\tau+2\pi i h/k} \tag{14.1.4}$$

where $1 \le h \le k \le \lfloor\sqrt{2\pi X}\rfloor =: N$ with $(h,k)=1$ (throughout, (m,n) denotes the greatest common divisor of integers m and n) and $\tau = X^{-1}+2\pi i Y$ with $|Y| \le 1/(kN)$. Let M be a positive integer and a be any of $1,2,\ldots,M$. If we denote by b the unique integer between 1 and (k,M) such that $b \equiv -ha \pmod{(k,M)}$ and write

$$b^* = \begin{cases} (k,M)-b & \text{if } b \neq (k,M), \\ (k,M) & \text{if } b = (k,M), \end{cases}$$

then

$$\log\left(\frac{1}{(q^a;q^M)_\infty}\right) = \frac{1}{\tau}\frac{(k,M)^2}{k^2M}\left(\pi^2\left(\frac{b^2}{(k,M)^2}-\frac{b}{(k,M)}+\frac{1}{6}\right)\right.$$
$$\left.+2\pi i\left(-\zeta'\left(-1,\frac{b}{(k,M)}\right)+\zeta'\left(-1,\frac{b^*}{(k,M)}\right)\right)\right)+E \qquad (14.1.5)$$

where

$$|\Re(E)| \ll_{a,M} X^{1/2}\log X. \qquad (14.1.6)$$

Remark 14.1.1. Let $\mathcal{Q}_{h/k}$ be the set of q with respect to h/k defined in Theorem 14.1.1. For any q with $|q| = e^{-1/X}$, we are always able to find an h/k such that $q \in \mathcal{Q}_{h/k}$. This is a direct consequence of the theory of Farey fractions. In fact, if h/k is a Farey fraction of order N and ξ_+ (resp. ξ_-) denotes the distance from h/k to its right (resp. left) neighboring mediant, then

$$\frac{1}{2kN} \le \xi_\pm \le \frac{1}{kN}.$$

Hence, $\mathbb{R}/\mathbb{Z}$ can be covered by intervals

$$\bigcup_{\substack{1\le h\le k\le N\\(h,k)=1}}\left[\frac{h}{k}-\frac{1}{kN},\frac{h}{k}+\frac{1}{kN}\right].$$

Equipped with Theorem 14.1.1, we almost arrive at a proof of Conjecture 14.1.1.

Theorem 14.1.2. *Let*

$$G(q) := \sum_{n\ge 0} g(n)q^n = \frac{1}{(q,-q^3;q^4)_\infty}. \qquad (14.1.7)$$

We have, as $n\to\infty$,

$$g(n) \sim \frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}}I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right)+(-1)^n\frac{\pi^{3/4}\Gamma(3/4)}{2^{11/4}3^{5/8}n^{5/8}}I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) \qquad (14.1.8)$$

where $I_s(x)$ is the modified Bessel function of the first kind. Further, when $n \ge 2.4\times 10^{14}$, we have $g(n) > 0$.

Unfortunately, my personal laptop did not support me to verify the coefficients $g(n)$ up to $n = 2.4\times 10^{14}$. But I deeply believe the validity of their nonnegativity after computing the first 10,000 terms.

Throughout, $\zeta(s)$ and $\zeta(s,a)$ are respectively Riemann zeta function and Hurwitz zeta function. We denote by $\zeta'(s,a)$ the partial derivative of Hurwitz zeta function with respect to s, namely,

$$\zeta'(s,a) = \frac{\partial}{\partial s}\zeta(s,a).$$

Finally, $\Gamma(s)$ is the gamma function and γ is the Euler–Mascheroni constant.

14.2 Theorem 14.1.1: Preparation

Recall that

$$\Phi_{a,M}(q) = \log\left(\frac{1}{(q^a;q^M)_\infty}\right) = \sum_{\substack{m\geq 1\\ m\equiv a \bmod M}} \sum_{\ell\geq 1} \frac{q^{\ell m}}{\ell}. \tag{14.2.1}$$

Throughout, let us assume $X \geq 16$ and $N = \left\lfloor\sqrt{2\pi X}\right\rfloor$. As in Theorem 14.1.1, we put

$$q = e^{-\tau}e^{2\pi i h/k} \tag{14.2.2}$$

where $1 \leq h \leq k \leq N$ with $(h,k) = 1$ and

$$\tau = X^{-1} + 2\pi i Y \tag{14.2.3}$$

with the restriction

$$|Y| \leq \frac{1}{kN}. \tag{14.2.4}$$

Now we are going to collect some bounds that will be frequently used in the sequel. First, the assumptions of X and N imply that

$$0.9\sqrt{2\pi X} \leq N \leq \sqrt{2\pi X}. \tag{14.2.5}$$

Further, $N \leq \sqrt{2\pi X}$ implies that

$$\frac{1}{X} \leq \frac{2\pi}{N^2} \leq \frac{2\pi}{kN}.$$

Hence,

$$|\tau| \leq \frac{2\sqrt{2}\pi}{kN}. \tag{14.2.6}$$

Finally,

$$\Re\left(\frac{1}{\tau}\right) \geq 0.07k^2. \tag{14.2.7}$$

This is because

$$\begin{aligned}\Re\left(\frac{1}{k^2\tau}\right) &= \frac{X^{-1}}{k^2(X^{-2}+4\pi^2Y^2)}\\ &\geq \frac{X^{-1}}{k^2(X^{-2}+4\pi^2k^{-2}N^{-2})}\\ &= \frac{X^{-1}}{k^2X^{-2}+4\pi^2N^{-2}}\\ &\geq \frac{X^{-1}}{N^2X^{-2}+4\pi^2N^{-2}}\\ &\geq \frac{X^{-1}}{(0.9\sqrt{2\pi X})^2X^{-2}+4\pi^2(0.9\sqrt{2\pi X})^{-2}}\\ &\geq 0.07.\end{aligned}$$

Given any positive integer k, we write

$$K = k\frac{M}{(k,M)}. \tag{14.2.8}$$

Notice that $M \mid K$. Write in (14.2.1)

$$\ell = bk+\mu \qquad (1 \leq \mu \leq k)$$

and

$$m = cK+\lambda \qquad (1\leq \lambda \leq K,\ \lambda \equiv a \bmod M).$$

Then

$$\Phi_{a,M}(q) = \sum_{\substack{1\leq\lambda\leq K\\ \lambda\equiv a \bmod M}} \sum_{1\leq\mu\leq k} e^{\frac{2\pi ih\mu\lambda}{k}} \sum_{b,c\geq 0} \frac{1}{bk+\mu} e^{-(bk+\mu)(cK+\lambda)\tau}.$$

Applying the Mellin transform further gives

$$\begin{aligned}\Phi_{a,M}(q) &= \sum_{\substack{1\leq\lambda\leq K\\ \lambda\equiv a \bmod M}} \sum_{1\leq\mu\leq k} e^{\frac{2\pi ih\mu\lambda}{k}} \sum_{b,c\geq 0} \frac{1}{2\pi i}\int_{(3/2)} \frac{\Gamma(s)}{bk+\mu}\frac{ds}{(bk+\mu)^s(cK+\lambda)^s\tau^s}\\ &= \sum_{\substack{1\leq\lambda\leq K\\ \lambda\equiv a \bmod M}} \sum_{1\leq\mu\leq k} e^{\frac{2\pi ih\mu\lambda}{k}} \frac{1}{2\pi i}\int_{(3/2)} \frac{\Gamma(s)}{\tau^s k^{s+1}K^s}\zeta\left(s,\frac{\lambda}{K}\right)\zeta\left(1+s,\frac{\mu}{k}\right)ds.\end{aligned}$$

Here the path of integration (α) is from $\alpha - i\infty$ to $\alpha + i\infty$.

Recall the functional equation of Hurwitz zeta function:

$$\zeta\left(s, \frac{\lambda}{k}\right) = 2\Gamma(1-s)(2\pi k)^{s-1}\left(\sin\frac{\pi s}{2}\sum_{1\le\nu\le k}\cos\frac{2\pi\lambda\nu}{k}\,\zeta\left(1-s, \frac{\nu}{k}\right)\right.$$
$$\left. + \cos\frac{\pi s}{2}\sum_{1\le\nu\le k}\sin\frac{2\pi\lambda\nu}{k}\,\zeta\left(1-s, \frac{\nu}{k}\right)\right). \tag{14.2.9}$$

If we further put

$$z = \frac{\tau k}{2\pi}, \tag{14.2.10}$$

then

$$\begin{aligned}\Phi_{a,M}(q) = {} & \frac{1}{4\pi i k K}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi h\mu\lambda}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta(1+s,\frac{\mu}{k})\,\zeta(1-s,\frac{\nu}{K})}{z^s\cos\frac{\pi s}{2}}ds\\ & + \frac{1}{4\pi i k K}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\cos\frac{2\pi h\mu\lambda}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta(1+s,\frac{\mu}{k})\,\zeta(1-s,\frac{\nu}{K})}{z^s\sin\frac{\pi s}{2}}ds\\ & + \frac{1}{4\pi k K}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\sin\frac{2\pi h\mu\lambda}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta(1+s,\frac{\mu}{k})\,\zeta(1-s,\frac{\nu}{K})}{z^s\sin\frac{\pi s}{2}}ds\\ & + \frac{1}{4\pi k K}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}}\sin\frac{2\pi h\mu\lambda}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)}\frac{\zeta(1+s,\frac{\mu}{k})\,\zeta(1-s,\frac{\nu}{K})}{z^s\cos\frac{\pi s}{2}}ds.\end{aligned} \tag{14.2.11}$$

Notice that $1 \le \lambda \le K$. If $h\lambda_1 \equiv h\lambda_2 \pmod k$, then by recalling $h_1 \equiv h_2 \equiv a \pmod M$ and the fact that $(h,k) = 1$, we conclude that $\lambda_1 \equiv \lambda_2 \pmod K$. Hence, the $h\lambda$'s give

$$\frac{K}{M} = \frac{k}{(k,M)}$$

residue classes modulo k. For each λ, we denote by $\rho = \rho(\lambda)$ the unique integer between 1 and k such that

$$\rho \equiv -h\lambda \pmod k. \tag{14.2.12}$$

Then the ρ's are pairwise distinct. Further, if we put

$$M^* = (k, M),$$

then for all ρ,

$$\rho \equiv -ha \pmod{M^*}. \tag{14.2.13}$$

Let us choose h' so that

$$hh' \equiv -1 \pmod{k}.$$

This is always possible since $(h,k)=1$. Notice that $\lambda \equiv a \pmod{M}$. Hence, we have the system

$$\begin{cases} \lambda \equiv h'\rho & \pmod{k} \\ \lambda \equiv a & \pmod{M} \end{cases}. \tag{14.2.14}$$

This system is solvable whenever $h'\rho \equiv a \pmod{M^*}$. But this can be ensured by (14.2.13) and the fact that $hh' \equiv -1 \pmod{M^*}$. We next find, using Euclid's algorithm, integers α and β such that

$$\alpha k + \beta M = M^*. \tag{14.2.15}$$

We therefore have (notice that $\mathrm{lcm}(k,M)=K$)

$$\lambda \equiv a + \beta M \frac{h'\rho - a}{M^*} = \beta h' \frac{M}{M^*}\rho + \alpha a \frac{k}{M^*} \pmod{K}. \tag{14.2.16}$$

In (14.2.11), replacing s by $-s$, reversing the direction of integration path and shifting the path back to $(3/2)$, one has, with $h\lambda$ replaced by $-\rho$,

$$\begin{aligned}
\Phi_{a,M}(q) = {} & \frac{1}{4\pi i k K} \sum_{\substack{1\le\lambda\le K \\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k \\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k} \cos\frac{2\pi\nu\lambda}{K} \int_{(3/2)} \frac{\zeta(1-s,\frac{\mu}{k})\,\zeta(1+s,\frac{\nu}{K})}{z^{-s}\cos\frac{\pi s}{2}} ds \\
& - \frac{1}{4\pi i k K} \sum_{\substack{1\le\lambda\le K \\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k \\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k} \sin\frac{2\pi\nu\lambda}{K} \int_{(3/2)} \frac{\zeta(1-s,\frac{\mu}{k})\,\zeta(1+s,\frac{\nu}{K})}{z^{-s}\sin\frac{\pi s}{2}} ds \\
& + \frac{1}{4\pi k K} \sum_{\substack{1\le\lambda\le K \\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k \\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k} \sin\frac{2\pi\nu\lambda}{K} \int_{(3/2)} \frac{\zeta(1-s,\frac{\mu}{k})\,\zeta(1+s,\frac{\nu}{K})}{z^{-s}\sin\frac{\pi s}{2}} ds \\
& - \frac{1}{4\pi k K} \sum_{\substack{1\le\lambda\le K \\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k \\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k} \cos\frac{2\pi\nu\lambda}{K} \int_{(3/2)} \frac{\zeta(1-s,\frac{\mu}{k})\,\zeta(1+s,\frac{\nu}{K})}{z^{-s}\cos\frac{\pi s}{2}} ds \\
& - 2\pi i(R_1+R_2+R_3+R_4) \\
=: {} & \Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4 - 2\pi i(R_1+R_2+R_3+R_4)
\end{aligned} \tag{14.2.17}$$

where R_* comes from the sum of residues of the corresponding integrand inside the stripe $-3/2 < \Re(s) < 3/2$.

In the next two sections, we shall evaluate the integrals Υ_* and the residues R_*, respectively. One may conclude Theorem 14.1.1 directly from (14.2.17) and the estimations (14.3.15), (14.4.11), (14.4.13), (14.4.15), (14.4.17), (14.4.22) and (14.4.24).

14.3 Theorem 14.1.1: The Integrals

14.3.1 An Auxiliary Function

Let us define an auxiliary function

$$\Psi_{a,M}(q) := \log\left(\prod_{\substack{m\geq 1\\ m\equiv -ha \bmod M^*}} \frac{1}{1-e^{2\pi i\alpha a/M}q^m}\right). \tag{14.3.1}$$

where α is defined in (14.2.15). We further write

$$m = bk+\rho \qquad (1\leq \rho\leq k,\ \rho\equiv -ha \bmod M^*).$$

Also, we put

$$q^* := \exp\left(\frac{2\pi i\beta h'}{k} - \frac{2\pi}{Kz}\right) \tag{14.3.2}$$

where β is again defined in (14.2.15). Then

$$\Psi_{a,M}(q^*) = -\sum_{\substack{1\leq\rho\leq k\\ \rho\equiv -ha \bmod M^*}} \sum_{b\geq 0} \log\left(1-\exp\left(\frac{2\pi i\beta h'}{k}\rho - \frac{2\pi}{Kz}(bk+\rho)+\frac{2\pi i\alpha a}{M}\right)\right).$$

It follows from (14.2.16) that

$$\exp\left(\frac{2\pi i\lambda}{K}\right) = \exp\left(\frac{2\pi i\beta h' M}{KM^*}\rho + \frac{2\pi i\alpha a k}{KM^*}\right) = \exp\left(\frac{2\pi i\beta h'}{k}\rho + \frac{2\pi i\alpha a}{M}\right).$$

Hence,

$$\begin{aligned}
\Psi_{a,M}(q^*) &= -\sum_{\substack{1\leq\rho\leq k\\ \rho\equiv -ha \bmod M^*}} \sum_{b\geq 0} \log\left(1-\exp\left(-\frac{2\pi}{Kz}(bk+\rho)+\frac{2\pi i\lambda}{K}\right)\right)\\
&= \sum_{\substack{1\leq\rho\leq k\\ \rho\equiv -ha \bmod M^*}} \sum_{1\leq\nu\leq K} \sum_{b,c\geq 0} \frac{1}{cK+\nu}\\
&\quad\times \exp\left((cK+\nu)\left(-\frac{2\pi}{Kz}(bk+\rho)+\frac{2\pi i\lambda}{K}\right)\right)\\
&= \sum_{\substack{1\leq\rho\leq k\\ \rho\equiv -ha \bmod M^*}} \sum_{1\leq\nu\leq K} e^{\frac{2\pi i\nu\lambda}{K}} \sum_{b,c\geq 0} \frac{1}{cK+\nu} e^{-(bk+\rho)(cK+\nu)\frac{2\pi}{Kz}}.
\end{aligned}$$

If we substitute ρ back to λ and apply Mellin transform and the functional equation of Hurwitz zeta function to $\Psi_{a,M}(q^*)$, then

$$\begin{aligned}
\Psi_{a,M}(q^*) = {} & \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
&+ \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&+ \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\sin\frac{\pi s}{2}}ds\\
&+ \frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k}\sin\frac{2\pi\nu\lambda}{K}\int_{(3/2)} \frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\cos\frac{\pi s}{2}}ds\\
=:{} & J_1 + J_2 + J_3 + J_4. \qquad (14.3.3)
\end{aligned}$$

Notice that

$$\Upsilon_1 = J_1 \qquad \text{and} \qquad \Upsilon_3 = J_3.$$

Further,

$$2(J_1 + J_3) = \Psi_{a,M}(q^*) + \Psi_{M-a,M}(q^*). \qquad (14.3.4)$$

14.3.2 Estimations Concerning Hurwitz Zeta Function

Recall (see, for instance, [27, (25.11.9)]) that for $\Re(s) > 1$ and $0 < \alpha \le 1$,

$$\zeta(1-s,\alpha) = \frac{2\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{1}{n^s}\cos\left(\frac{1}{2}\pi s - 2n\pi\alpha\right).$$

This implies that for $0 < \alpha \le 1$, we have a uniform bound

$$|\zeta(-0.5+it,\alpha)| \le \frac{2\Gamma(3/2)\zeta(3/2)\cosh(\pi|t|/2)}{(2\pi)^{3/2}}. \qquad (14.3.5)$$

It also follows from [26, Theorem 12.23] with some simple calculations that, uniformly for $|t| \ge 3$ and $0 < \alpha \le 1$,

$$|\zeta(-0.5+it,\alpha)| \le 11|t|^{3/2}. \qquad (14.3.6)$$

Finally, we have, for $0 < \alpha \le 1$,

$$|\zeta(2.5+it,\alpha)\,| \le \alpha^{-5/2} + \zeta(5/2). \tag{14.3.7}$$

Lemma 14.3.1. *Let z be a complex number with $\Re(z) > 0$. Let $0 < \alpha, \beta \le 1$. Define integrals*

$$\mathcal{I}_+(z) := \int_{(3/2)} z^s \zeta(1+s,\alpha)\,\zeta(1-s,\beta)\left(\frac{1}{\cos\frac{\pi s}{2}} + \frac{1}{i\sin\frac{\pi s}{2}}\right) ds \tag{14.3.8}$$

and

$$\mathcal{I}_-(z) := \int_{(3/2)} z^s \zeta(1+s,\alpha)\,\zeta(1-s,\beta)\left(\frac{1}{\cos\frac{\pi s}{2}} - \frac{1}{i\sin\frac{\pi s}{2}}\right) ds. \tag{14.3.9}$$

Then if $\Im(z) \le 0$, we have

$$|\mathcal{I}_+(z)| \le 7.23|z|^{3/2}\Big(\alpha^{5/2} + \zeta(5/2)\Big), \tag{14.3.10}$$

while if $\Im(z) \ge 0$, we have

$$|\mathcal{I}_-(z)| \le 7.23|z|^{3/2}\Big(\alpha^{5/2} + \zeta(5/2)\Big). \tag{14.3.11}$$

Proof. Let us write $s = 3/2 + it$ as the path of integration is the vertical line $\Re(s) = 3/2$. We have

$$|z^s| = |z|^{3/2} e^{-\operatorname{Arg}(z)t}.$$

Also,

$$\left|\frac{1}{\cos\frac{\pi s}{2}} + \frac{1}{i\sin\frac{\pi s}{2}}\right| = \frac{2e^{-\frac{\pi}{2}t}}{|\sin(\pi s)|}.$$

Hence, for z with $\Im(z) \le 0$ (recall that $\Re(z) > 0$ so that $-\pi/2 < \operatorname{Arg}(z) \le 0$), we have

$$|z^s|\left|\frac{1}{\cos\frac{\pi s}{2}} + \frac{1}{i\sin\frac{\pi s}{2}}\right| \le 2|z|^{3/2}\frac{e^{\frac{\pi}{2}|t|}}{|\sin(\pi s)|}.$$

It follows that

$$\begin{aligned}|\mathcal{I}_+(z)| &\le 2|z|^{3/2}\Big((\alpha^{5/2} + \zeta(5/2)\Big)\int_{-\infty}^{\infty} |\zeta(-0.5-it,\beta)|\,\frac{e^{\frac{\pi}{2}|t|}}{|\sin(\pi(1.5+it))|}dt \\ &\le 7.23|z|^{3/2}\Big(\alpha^{5/2} + \zeta(5/2)\Big).\end{aligned}$$

Similar arguments also apply to $\mathcal{I}_-(z)$ if $\Im(z) \geq 0$. $\square$

14.3.3 Bounding the Integrals

Recall that

$$z = \frac{\tau k}{2\pi}.$$

For Υ_2 and Υ_4, we define

$$\Upsilon_* \pm J_* := \begin{cases} \Upsilon_* + J_* & \text{if } \Im(z) \geq 0, \\ \Upsilon_* - J_* & \text{if } \Im(z) < 0. \end{cases} \tag{14.3.12}$$

It follows from Lemma 14.3.1 that

$$\begin{aligned} |\Upsilon_* \pm J_*| &\leq \frac{1}{4\pi kK} \frac{kK}{M} \sum_{1 \leq \nu \leq K} 7.23\, |z|^{3/2} \left(\left(\frac{K}{\nu}\right)^{5/2} + \zeta\left(\frac{5}{2}\right) \right) \\ &\leq \frac{1}{4\pi M} \cdot 7.23\, |z|^{3/2} \cdot 2\zeta(5/2) K^{5/2} \\ &\leq \frac{7.23\, \zeta(5/2)}{2\pi M} \left| \frac{\tau k}{2\pi} \right|^{3/2} \left(k \frac{M}{(k, M)} \right)^{5/2} \\ &\leq \frac{7.23\, \zeta(5/2)}{2\pi M} \left(\frac{\sqrt{2}}{N} \right)^{3/2} \left(\frac{M}{(k, M)} \right)^{5/2} N^{5/2} && \text{(by (14.2.6))} \\ &\leq \frac{7.23\, \zeta(5/2)\, 2^{3/4}}{2\pi M} \left(\frac{M}{(k, M)} \right)^{5/2} \sqrt{2\pi X} \\ &\leq 6.51 \frac{M^{3/2}}{(k, M)^{5/2}} X^{1/2} \\ &\ll X^{1/2}. \end{aligned} \tag{14.3.13}$$

Finally, we bound

$$\begin{aligned} |\Re(\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4)| &\leq |\Re(\Upsilon_1 + \Upsilon_3)| + |\Re(\Upsilon_2 + \Upsilon_4)| \\ &\leq |\Re(J_1 + J_3)| + |\Re(J_2 + J_4)| + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4| \\ &\leq |\Re(\Psi_{a,M}(q^*))| + 2|\Re(J_1 + J_3)| + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4| \\ &\leq 2|\Re(\Psi_{a,M}(q^*))| + |\Re(\Psi_{M-a,M}(q^*))| \\ &\quad + |\Upsilon_2 \pm J_2| + |\Upsilon_4 \pm J_4|. \end{aligned}$$

Recall from (14.3.2) that

$$q^* = \exp\left(\frac{2\pi i\beta h'}{k} - \frac{2\pi}{Kz}\right).$$

Hence,

$$|q^*| = \exp\left(\Re\left(-\frac{2\pi}{Kz}\right)\right) = \exp\left(-4\pi^2\frac{(k,M)}{M}\Re\left(\frac{1}{k^2\tau}\right)\right).$$

By (14.2.7), we have

$$|q^*| \le \exp\left(-4\pi^2\frac{(k,M)}{M}\cdot 0.07\right) \ll 1. \tag{14.3.14}$$

We further have, by some simple partition-theoretic arguments that, for any $a = 1, 2, \ldots, M$,

$$\begin{aligned} e^{|\Re(\Psi_{a,M}(q^*))|} &\le \prod_{\substack{m\ge 1\\ m\equiv -ha \bmod M^*}} \frac{1}{1-|q^*|^m} \le \frac{1}{(|q^*|;|q^*|)_\infty} \\ &= \exp\left(-\sum_{\ell\ge 1}\log(1-|q^*|^\ell)\right) = \exp\left(\sum_{\ell\ge 1}\sum_{m\ge 1}\frac{|q^*|^{\ell m}}{m}\right) \\ &\le \exp\left(\sum_{n\ge 1} n|q^*|^n\right) = \exp\left(\frac{|q^*|}{(1-|q^*|)^2}\right). \end{aligned}$$

In consequence,

$$|\Re(\Psi_{a,M}(q^*))| \le \frac{e^{-0.28\pi^2\frac{(k,M)}{M}}}{\left(1-e^{-0.28\pi^2\frac{(k,M)}{M}}\right)^2} \ll 1.$$

It turns out that

$$\begin{aligned} |\Re(\Upsilon_1+\Upsilon_2+\Upsilon_3+\Upsilon_4)| &\le 3\cdot\frac{e^{-0.28\pi^2\frac{(k,M)}{M}}}{\left(1-e^{-0.28\pi^2\frac{(k,M)}{M}}\right)^2} + 2\cdot 6.51\frac{M^{3/2}}{(k,M)^{5/2}}X^{1/2} \\ &\le \frac{3e^{-0.28\pi^2\frac{(k,M)}{M}}}{\left(1-e^{-0.28\pi^2\frac{(k,M)}{M}}\right)^2} + 13.02\frac{M^{3/2}}{(k,M)^{5/2}}X^{1/2} \\ &\ll X^{1/2}. \end{aligned} \tag{14.3.15}$$

14.4 Theorem 14.1.1: The Residues

14.4.1 Some Lemmas

We first require some finite summation formulas of Hurwitz zeta function, which follow from the first two aligned formulas on page 587 of [38].

Lemma 14.4.1. *For any* $\theta = 1, 2, \ldots, k$,

$$\sum_{1\leq\alpha\leq k} \cos\frac{2\pi\alpha\theta}{k}\zeta\Big(0, \frac{\alpha}{k}\Big) = -\frac{1}{2} \tag{14.4.1}$$

and

$$\sum_{1\leq\alpha\leq k} \cos\frac{2\pi\alpha\theta}{k}\zeta\Big(2, \frac{\alpha}{k}\Big) = \frac{\pi^2}{6}(6\theta^2 - 6k\theta + k^2). \tag{14.4.2}$$

For any $\theta = 1, 2, \ldots, k-1$,

$$\sum_{1\leq\alpha\leq k} \sin\frac{2\pi\alpha\theta}{k}\zeta\Big(0, \frac{\alpha}{k}\Big) = \frac{1}{2\pi}\left(\frac{\Gamma'}{\Gamma}\Big(1-\frac{\theta}{k}\Big) - \frac{\Gamma'}{\Gamma}\Big(\frac{\theta}{k}\Big)\right) = \frac{1}{2}\cot\frac{\pi\theta}{k} \tag{14.4.3}$$

and

$$\sum_{1\leq\alpha\leq k} \sin\frac{2\pi\alpha\theta}{k}\zeta\Big(2, \frac{\alpha}{k}\Big) = 2\pi k^2\left(\zeta'\left(-1, \frac{\theta}{k}\right) - \zeta'\left(-1, 1-\frac{\theta}{k}\right)\right). \tag{14.4.4}$$

We also need three finite summation formulas of the digamma function due to Gauß (cf. [164]).

Lemma 14.4.2. *For any* $\theta = 1, 2, \ldots, k-1$,

$$\sum_{1\leq\alpha\leq k} \cos\frac{2\pi\alpha\theta}{k}\frac{\Gamma'}{\Gamma}\Big(\frac{\alpha}{k}\Big) = k\log\left(2\sin\frac{\pi\theta}{k}\right) \tag{14.4.5}$$

and

$$\sum_{1\leq\alpha\leq k} \sin\frac{2\pi\alpha\theta}{k}\frac{\Gamma'}{\Gamma}\Big(\frac{\alpha}{k}\Big) = \frac{\pi}{2}(2\theta - k). \tag{14.4.6}$$

Further,

$$\sum_{1\le\alpha\le k}\frac{\Gamma'}{\Gamma}\left(\frac{\alpha}{k}\right)=-k(\gamma+\log k). \tag{14.4.7}$$

Next, it is easy to compute that

$$\begin{aligned}\sum_{|\Re(s)|\le 3/2}\operatorname{Res}_s\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\cos\frac{\pi s}{2}}&=\operatorname{Res}_{s=0}(*)+\operatorname{Res}_{s=-1}(*)+\operatorname{Res}_{s=1}(*)\\ =-\log z-\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right)&+\frac{2\zeta\left(2,\frac{\mu}{k}\right)\zeta\left(0,\frac{\nu}{K}\right)}{\pi z}-\frac{2z\zeta\left(0,\frac{\mu}{k}\right)\zeta\left(2,\frac{\nu}{K}\right)}{\pi}\end{aligned}$$

and

$$\begin{aligned}\sum_{|\Re(s)|\le 3/2}\operatorname{Res}_s\frac{\zeta\left(1-s,\frac{\mu}{k}\right)\zeta\left(1+s,\frac{\nu}{K}\right)}{z^{-s}\sin\frac{\pi s}{2}}&=\operatorname{Res}_{s=0}(*)\\ =-\frac{\pi}{12}-\frac{(\log z)^2}{\pi}&-\frac{2\log z}{\pi}\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)+\frac{2\log z}{\pi}\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right)\\ +\frac{2}{\pi}\frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right)\frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right)&+\frac{2}{\pi}\gamma_1\left(\frac{\mu}{k}\right)+\frac{2}{\pi}\gamma_1\left(\frac{\nu}{K}\right)\end{aligned}$$

where $\gamma_1(\alpha)$ is the generalized Stieltjes constant.

Finally, recall from (14.2.12) that ρ is the unique integer between 1 and k such that $\rho\equiv -h\lambda \pmod{k}$. Hence,

$$\lambda=K \iff \rho=k. \tag{14.4.8}$$

Further, (14.2.13) says $\rho\equiv -ha \pmod{M^*}$. Recall also that b is the unique integer between 1 and M^* such that

$$b\equiv -ha \pmod{M^*}. \tag{14.4.9}$$

Then the following two summations represent the same thing:

$$\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}(*)\equiv\sum_{\substack{1\le\rho\le k\\ \rho\equiv b \bmod M^*}}(*).$$

14.4.2 Evaluation of R_1

We have

$$R_1 = \frac{1}{4\pi ikK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \cos\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}$$
$$\times\left(\frac{2\zeta\left(2,\frac{\mu}{k}\right)\zeta\left(0,\frac{\nu}{K}\right)}{\pi z} - \frac{2z\zeta\left(0,\frac{\mu}{k}\right)\zeta\left(2,\frac{\nu}{K}\right)}{\pi}\right).$$

First,

$$\begin{aligned}
R_{11} &:= \frac{1}{z}\frac{1}{2i\pi^2 kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} \cos\frac{2\pi\mu\rho}{k}\zeta\left(2,\frac{\mu}{k}\right) \sum_{1\le\nu\le K}\cos\frac{2\pi\nu\lambda}{K}\zeta\left(0,\frac{\nu}{K}\right)\\
&= \frac{1}{z}\frac{1}{2i\pi^2 kK} \sum_{\substack{1\le\rho\le k\\ \rho\equiv b \bmod M^*}} \frac{\pi^2}{6}(6\rho^2-6k\rho+k^2)\cdot\left(-\frac{1}{2}\right)\\
&= \frac{1}{z}\frac{1}{2i\pi^2 kK}\cdot\frac{\pi^2}{6}\frac{k}{M^*}(6b^2-6bM^*+(M^*)^2)\cdot\left(-\frac{1}{2}\right)\\
&= -\frac{2\pi}{\tau k}\frac{1}{24iKM^*}(6b^2-6bM^*+(M^*)^2)\\
&= -\frac{2\pi}{\tau k}\frac{1}{24ikM}(6b^2-6bM^*+(M^*)^2). \qquad (14.4.10)
\end{aligned}$$

Hence,

$$-2\pi i R_{11} = \frac{1}{\tau}\frac{\pi^2}{6k^2M}(6b^2-6b(k,M)+(k,M)^2). \qquad (14.4.11)$$

Also,

$$\begin{aligned}
R_{12} &:= -z\frac{1}{2i\pi^2 kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} \cos\frac{2\pi\mu\rho}{k}\zeta\left(0,\frac{\mu}{k}\right) \sum_{1\le\nu\le K}\cos\frac{2\pi\nu\lambda}{K}\zeta\left(2,\frac{\nu}{K}\right)\\
&= -z\frac{1}{2i\pi^2 kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \left(-\frac{1}{2}\right)\cdot\frac{\pi^2}{6}(6\lambda^2-6k\lambda+k^2)\\
&= -z\frac{1}{2i\pi^2 kK}\cdot\left(-\frac{1}{2}\right)\cdot\frac{\pi^2}{6}\frac{K}{M}(6a^2-6aM+M^2)\\
&= \frac{\tau k}{2\pi}\frac{1}{24ikM}(6a^2-6aM+M^2)\\
&= \tau\frac{1}{48i\pi M}(6a^2-6aM+M^2). \qquad (14.4.12)
\end{aligned}$$

Hence, recalling that $a = 1, 2, \ldots, M$, we have

$$\begin{aligned}
|-2\pi i R_{12}| &= |\tau| \frac{|6a^2 - 6aM + M^2|}{24M} \\
&\leq \frac{2\sqrt{2}\pi}{kN} \cdot \frac{M^2}{24M} \\
&\leq \frac{2\sqrt{2}\pi}{k \cdot 0.9\sqrt{2\pi X}} \cdot \frac{M}{24}.
\end{aligned}$$

In consequence,

$$|-2\pi i R_{12}| \leq 0.17 \frac{M}{k} X^{-1/2} \ll X^{-1/2}. \tag{14.4.13}$$

14.4.3 Evaluation of R_2

We have

$$R_2 = -\frac{1}{4\pi i k K} \sum_{\substack{1 \leq \lambda \leq K \\ \lambda \equiv a \bmod M}} \sum_{\substack{1 \leq \mu \leq k \\ 1 \leq \nu \leq K}} \cos \frac{2\pi\mu\rho}{k} \sin \frac{2\pi\nu\lambda}{K} \cdot \frac{2}{\pi} \frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right) \frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right).$$

Hence, with (14.4.8),

$$\begin{aligned}
R_2 &= -\frac{1}{2i\pi^2 kK} \sum_{\substack{1 \leq \lambda < K \\ \lambda \equiv a \bmod M}} \sum_{1 \leq \mu \leq k} \cos \frac{2\pi\mu\rho}{k} \frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right) \sum_{1 \leq \nu \leq K} \sin \frac{2\pi\nu\lambda}{K} \frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right) \\
&= -\frac{1}{2i\pi^2 kK} \sum_{\substack{1 \leq \rho < k \\ \rho \equiv b \bmod M^*}} k \log\left(2 \sin \frac{\pi\rho}{k}\right) \cdot \frac{\pi}{2}(2\lambda - K) \\
&= -\frac{1}{4i\pi K} \sum_{\substack{1 \leq \rho < k \\ \rho \equiv b \bmod M^*}} (2\lambda - K) \log\left(2 \sin \frac{\pi\rho}{k}\right).
\end{aligned} \tag{14.4.14}$$

Notice that for $0 \leq x \leq \pi/2$, we have

$$|\log(2 \sin x)| \leq \frac{\pi \log 2}{2x}.$$

Hence,

$$\begin{aligned}
|-2\pi i R_2| &\leq \frac{1}{2K} \cdot 2 \sum_{1 \leq \rho < k} K \cdot \frac{\pi \log 2}{2} \frac{k}{\pi\rho} \\
&= \frac{\log 2}{2} k \sum_{1 \leq \rho < k} \frac{1}{\rho}
\end{aligned}$$

$$\leq \frac{\log 2}{2} k(\log k + \gamma)$$
$$\leq \frac{\log 2}{2} N(\log N + \gamma)$$
$$\leq \frac{\log 2}{2} \sqrt{2\pi X}(\log \sqrt{2\pi X} + \gamma).$$

In consequence,

$$|-2\pi i R_2| \leq 1.3X^{1/2} + 0.44X^{1/2} \log X \ll X^{1/2} \log X. \tag{14.4.15}$$

14.4.4 Evaluation of R_3

We have

$$R_3 = \frac{1}{4\pi kK} \sum_{\substack{1 \leq \lambda \leq K \\ \lambda \equiv a \bmod M}} \sum_{\substack{1 \leq \mu \leq k \\ 1 \leq \nu \leq K}} \sin \frac{2\pi\mu\rho}{k} \sin \frac{2\pi\nu\lambda}{K} \cdot \frac{2}{\pi} \frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right) \frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right).$$

Hence, with (14.4.8),

$$\begin{aligned} R_3 &= \frac{1}{2\pi^2 kK} \sum_{\substack{1 \leq \lambda < K \\ \lambda \equiv a \bmod M}} \sum_{1 \leq \mu \leq k} \sin \frac{2\pi\mu\rho}{k} \frac{\Gamma'}{\Gamma}\left(\frac{\mu}{k}\right) \sum_{1 \leq \nu \leq K} \sin \frac{2\pi\nu\lambda}{K} \frac{\Gamma'}{\Gamma}\left(\frac{\nu}{K}\right) \\ &= \frac{1}{2\pi^2 kK} \sum_{\substack{1 \leq \rho < k \\ \rho \equiv b \bmod M^*}} \frac{\pi}{2}(2\rho - k) \cdot \frac{\pi}{2}(2\lambda - K) \\ &= \frac{1}{8kK} \sum_{\substack{1 \leq \rho < k \\ \rho \equiv b \bmod M^*}} (2\rho - k)(2\lambda - K). \end{aligned} \tag{14.4.16}$$

In consequence,

$$|-2\pi i R_3| \leq 2\pi \cdot \frac{1}{8kK} \cdot \frac{k}{M^*} kK = \frac{\pi k}{4M^*} \leq \frac{\pi N}{4M^*} \leq \frac{\pi\sqrt{2\pi X}}{4M^*}.$$

Namely,

$$|-2\pi i R_3| \leq \frac{1.97}{(k, M)} X^{1/2} \ll X^{1/2}. \tag{14.4.17}$$

14.4.5 Evaluation of R_4

We have

$$R_4 = -\frac{1}{4\pi kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{\substack{1\le\mu\le k\\ 1\le\nu\le K}} \sin\frac{2\pi\mu\rho}{k}\cos\frac{2\pi\nu\lambda}{K}$$
$$\times\left(\frac{2\zeta\left(2,\frac{\mu}{k}\right)\zeta\left(0,\frac{\nu}{K}\right)}{\pi z} - \frac{2z\zeta\left(0,\frac{\mu}{k}\right)\zeta\left(2,\frac{\nu}{K}\right)}{\pi}\right).$$

First,

$$\begin{aligned} R_{41} &:= -\frac{1}{z}\frac{1}{2\pi^2 kK} \sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}} \sum_{1\le\mu\le k} \sin\frac{2\pi\mu\rho}{k}\zeta\left(2,\frac{\mu}{k}\right) \sum_{1\le\nu\le K}\cos\frac{2\pi\nu\lambda}{K}\zeta\left(0,\frac{\nu}{K}\right)\\ &= -\frac{1}{z}\frac{1}{2\pi^2 kK} \sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}} 2\pi k^2\left(\zeta'\left(-1,\frac{\rho}{k}\right) - \zeta'\left(-1,1-\frac{\rho}{k}\right)\right)\cdot\left(-\frac{1}{2}\right)\\ &= \frac{1}{z}\frac{k}{2\pi K} \sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}} \left(\zeta'\left(-1,\frac{\rho}{k}\right) - \zeta'\left(-1,1-\frac{\rho}{k}\right)\right). \end{aligned}$$

If $b = M^*$, then both ρ and $k-\rho$ run through all multiples of M^* within the range $[1,k)$, and hence

$$R_{41} = 0. \tag{14.4.18}$$

We further notice that if $d \mid k$ and $1 \le c \le d$, then for any $s \ne 1$,

$$\sum_{\substack{1\le\ell\le k\\ \ell\equiv c \bmod d}} \zeta\left(s,\frac{\ell}{k}\right) = \left(\frac{k}{d}\right)^s \zeta\left(s,\frac{c}{d}\right) \tag{14.4.19}$$

Hence,

$$\sum_{\substack{1\le\ell<k\\ \ell\equiv c \bmod d}} \zeta'\left(s,\frac{\ell}{k}\right) = \left(\frac{k}{d}\right)^s \zeta\left(s,\frac{c}{d}\right)\log(k/d) + \left(\frac{k}{d}\right)^s \zeta'\left(s,\frac{c}{d}\right). \tag{14.4.20}$$

Since $M^* = (k,M)$ divides k, it follows that if $b \ne M^*$ (and hence $\rho \ne k$), then

$$R_{41} = \frac{1}{z}\frac{k}{2\pi K}\left(\left(\frac{M^*}{k}\right)\zeta\left(-1,\frac{b}{M^*}\right)\log\frac{k}{M^*} + \left(\frac{M^*}{k}\right)\zeta'\left(-1,\frac{b}{M^*}\right)\right.$$

$$- \left(\frac{M^*}{k}\right)\zeta\left(-1, \frac{M^*-b}{M^*}\right)\log\frac{k}{M^*} - \left(\frac{M^*}{k}\right)\zeta'\left(-1, \frac{M^*-b}{M^*}\right)\Bigg)$$

$$= \frac{1}{z}\frac{(k,M)^2}{M}\frac{1}{2\pi k}\left(\zeta'\left(-1, \frac{b}{M^*}\right) - \zeta'\left(-1, \frac{M^*-b}{M^*}\right)\right)$$

$$= \frac{1}{\tau}\frac{(k,M)^2}{M}\frac{1}{k^2}\left(\zeta'\left(-1, \frac{b}{(k,M)}\right) - \zeta'\left(-1, \frac{(k,M)-b}{(k,M)}\right)\right). \tag{14.4.21}$$

It turns out that

$$-2\pi i R_{41} = \begin{cases} 0 & \text{if } b = (k,M), \\ -\frac{1}{\tau}\frac{(k,M)^2}{M}\frac{2\pi i}{k^2}\left(\zeta'\left(-1, \frac{b}{(k,M)}\right) - \zeta'\left(-1, \frac{(k,M)-b}{(k,M)}\right)\right) & \text{if } b \neq (k,M). \end{cases} \tag{14.4.22}$$

Also,

$$R_{42} := z\frac{1}{2\pi^2 kK}\sum_{\substack{1\le\lambda\le K\\ \lambda\equiv a \bmod M}}\sum_{1\le\mu\le k}\sin\frac{2\pi\mu\rho}{k}\zeta\left(0, \frac{\mu}{k}\right)\sum_{1\le\nu\le K}\cos\frac{2\pi\nu\lambda}{K}\zeta\left(2, \frac{\nu}{K}\right)$$

$$= z\frac{1}{2\pi^2 kK}\sum_{\substack{1\le\lambda<K\\ \lambda\equiv a \bmod M}}\frac{1}{2}\cot\frac{\pi\rho}{k}\cdot\frac{\pi^2}{6}(6\lambda^2 - 6K\lambda + K^2)$$

$$= z\frac{1}{24kK}\sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}}(6\lambda^2 - 6K\lambda + K^2)\cot\frac{\pi\rho}{k}$$

$$= \tau\frac{1}{48\pi K}\sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}}(6\lambda^2 - 6K\lambda + K^2)\cot\frac{\pi\rho}{k}. \tag{14.4.23}$$

Notice that for $1 \le \lambda \le K$,

$$|6\lambda^2 - 6K\lambda + K^2| \le K^2$$

and for $0 < x \le \pi/2$,

$$|\cot x| \le \frac{1}{x}.$$

Hence,

$$|-2\pi i R_{42}| = |\tau|\frac{1}{24K}\left|\sum_{\substack{1\le\rho<k\\ \rho\equiv b \bmod M^*}}(6\lambda^2 - 6K\lambda + K^2)\cot\frac{\pi\rho}{k}\right|$$

$$
\begin{aligned}
&\le \frac{2\sqrt{2}\pi}{kN}\frac{1}{24K}\cdot 2K^2\sum_{1\le \ell<k}\frac{k}{\pi\ell}\\
&= \frac{\sqrt{2}}{6N}\frac{kM}{(k,M)}(\log k+\gamma)\\
&\le \frac{\sqrt{2}}{6N}\frac{NM}{(k,M)}(\log N+\gamma)\\
&\le \frac{\sqrt{2}}{6}\frac{M}{(k,M)}(\log \sqrt{2\pi X}+\gamma).
\end{aligned}
$$

In consequence,

$$
|-2\pi i R_{42}| \le 0.12\frac{M}{(k,M)}\log X + 0.36\frac{M}{(k,M)} \ll \log X. \tag{14.4.24}
$$

14.5 Explicit Bounds of $G(q)$

Recall that

$$
G(q) = \frac{(q^3;q^4)_\infty}{(q;q^4)_\infty(q^6;q^8)_\infty}.
$$

The goal of this section is the following uniform bound of $|G(q)|$ when q is away from ± 1.

Theorem 14.5.1. *Let $\mathcal{Q}_{h/k}$ be as in Remark 14.1.1. For any q (with $|q| = e^{-1/X}$) not in $\mathcal{Q}_{1/1}$ and $\mathcal{Q}_{1/2}$, we have, if $X \ge 3.4\times 10^7$, then*

$$
|G(q)| \le \exp\left(\left(\frac{\pi^2}{48}-\frac{1}{100}\right)X\right). \tag{14.5.1}
$$

Further, if $q = e^{-\tau+2\pi i h/k}$ with $\tau = X^{-1}+2\pi i Y$ is in $\mathcal{Q}_{1/1}$ or $\mathcal{Q}_{1/2}$, then (14.5.1) *still holds under the assumption $X \ge 3.4\times 10^7$ provided that $|Y| \ge 1/(2\pi X)$.*

Notice that $\tau = X^{-1}+2\pi i Y$. Hence,

$$
\tau^{-1} = \frac{X^{-1}}{X^{-2}+4\pi^2Y^2} - i\frac{2\pi Y}{X^{-2}+4\pi^2Y^2}. \tag{14.5.2}
$$

In the sequel, we write b as $b(h,a,k,M)$ to avoid confusion. We also write for convenience

$$
\mathfrak{M}_{a,M} := \frac{1}{\tau}\frac{(k,M)^2}{k^2M}\left(\pi^2\left(\frac{b^2}{(k,M)^2}-\frac{b}{(k,M)}+\frac{1}{6}\right)\right.
$$

$$+ 2\pi i\left(-\zeta'\left(-1, \frac{b}{(k,M)}\right) + \zeta'\left(-1, \frac{b^*}{(k,M)}\right)\right)\Bigg), \tag{14.5.3}$$

which is the main term in (14.1.5). Further,

$$\mathfrak{M}_G := \mathfrak{M}_{1,4} - \mathfrak{M}_{3,4} + \mathfrak{M}_{6,8} \tag{14.5.4}$$

denotes the main term of $\log G(q)$ whereas

$$E_G := \log G(q) - \mathfrak{M}_G \tag{14.5.5}$$

denotes the error term.

14.5.1 Case 1: $k \in 2\mathbb{Z} + 1$

Notice that $(k, 4) = 1$. Hence, we always have $b(h, 1, k, 4) = b(h, 3, k, 4) = 1$. Also, $(k, 8) = 1$. Then $b(h, 6, k, 8) = 1$. It is not hard to compute that

$$\mathfrak{M}_G = \frac{1}{\tau}\frac{\pi^2}{48k^2}. \tag{14.5.6}$$

It follows from (14.5.2) that

$$\Re(\mathfrak{M}_G) \leq \frac{\pi^2}{48k^2}X. \tag{14.5.7}$$

We may also compute from the bounds (14.3.15), (14.4.13), (14.4.15), (14.4.17) and (14.4.24) that

$$|\Re(E_G)| \leq 1.32X^{1/2}\log X + 512.74X^{1/2} + 1.92\log X + 42.74 + 2.72X^{-1/2}. \tag{14.5.8}$$

14.5.2 Case 2: $k \in 4\mathbb{Z} + 2$

Notice that $(k, 4) = 2$. Since $(h, k) = 1$, so h is odd. Hence, we always have $b(h, 1, k, 4) = b(h, 3, k, 4) = 1$. Also, $(k, 8) = 1$. We have $b(h, 6, k, 8) = 2$. It is not hard to compute that

$$\mathfrak{M}_G = \frac{1}{\tau}\frac{\pi^2}{12k^2}. \tag{14.5.9}$$

It follows from (14.5.2) that

$$\Re(\mathfrak{M}_G) \leq \frac{\pi^2}{12k^2}X. \tag{14.5.10}$$

For the error term E_G, we have

$$|\Re(E_G)| \leq 1.32X^{1/2}\log X + 95.77X^{1/2} + 0.96\log X + 11.61 + 2.72X^{-1/2}. \tag{14.5.11}$$

14.5.3 Case 3: $k \in 8\mathbb{Z} + 4$

Notice that $(k,4) = 4$. If $h \equiv 1 \pmod 4$, then $b(h,1,k,4) = 3$ and $b(h,3,k,4) = 1$. If $h \equiv 3 \pmod 4$, then $b(h,1,k,4) = 1$ and $b(h,3,k,4) = 3$. Hence,

$$\mathfrak{M}_{1,4} - \mathfrak{M}_{3,4} = \frac{1}{\tau}\frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)$$

where

$$\chi(h) = \begin{cases} 1 & \text{if } h \equiv 1 \pmod 4, \\ -1 & \text{if } h \equiv 3 \pmod 4. \end{cases}$$

Also, $(k,8) = 4$. Since $(h,k) = 1$, so h is odd. Hence, we have $b(h,6,k,8) = 2$. It follows that

$$\mathfrak{M}_{6,8} = -\frac{1}{\tau}\frac{\pi^2}{6k^2}.$$

Hence,

$$\mathfrak{M}_G = \frac{1}{\tau}\left(-\frac{\pi^2}{6k^2} + \frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)\right). \tag{14.5.12}$$

It follows from (14.5.2) that

$$\begin{aligned}\Re(\mathfrak{M}_G) &= -\frac{\pi^2}{6k^2}\frac{X^{-1}}{X^{-2}+4\pi^2Y^2} \\ &\quad + \frac{16\pi\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)\frac{2\pi Y}{X^{-2}+4\pi^2Y^2} \\ &\leq \frac{1}{k^2}\cdot\frac{-\frac{\pi^2}{6}X^{-1} + 16\pi\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)2\pi|Y|}{X^{-2}+4\pi^2|Y|^2} \\ &= \frac{\pi^2}{6k^2}\cdot\frac{-X^{-1} + 192\left(\zeta'\left(-1,\frac{1}{4}\right) - \zeta'\left(-1,\frac{3}{4}\right)\right)|Y|}{X^{-2}+4\pi^2|Y|^2}.\end{aligned}$$

We next show that

$$\Re(\mathfrak{M}_G) \leq \frac{2.94}{k^2}X. \tag{14.5.13}$$

It suffices to prove that

$$\frac{\pi^2}{6k^2}\cdot\frac{-X^{-1}+192\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)|Y|}{X^{-2}+4\pi^2|Y|^2}\le\frac{2.94}{k^2}X.$$

Namely,

$$70.56X|Y|^2-192\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)|Y|+\left(\frac{17.64}{\pi^2}+1\right)X^{-1}\ge 0.$$

Notice that on the left-hand side if we replace $|Y|$ by t and treat it as a quadratic function of real t, then it reaches the minimum when

$$t=\frac{192\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)}{2\times 70.56X}.$$

Further, the minimum is

$$-70.56X\times\left(\frac{192\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)}{2\times 70.56X}\right)^2+\left(\frac{17.64}{\pi^2}+1\right)X^{-1}\ge 0.01X^{-1}\ge 0.$$

Hence, (14.5.13) holds.

For the error term E_G, we have

$$|\Re(E_G)|\le 1.32X^{1/2}\log X+21.1X^{1/2}+0.48\log X+3.22+2.72X^{-1/2}. \tag{14.5.14}$$

14.5.4 Case 4: $k\in 8\mathbb{Z}$

As in *Case 3*, we still have

$$\mathfrak{M}_{1,4}-\mathfrak{M}_{3,4}=\frac{1}{\tau}\frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right).$$

Also, $(k,8)=8$. If $h\equiv 1 \pmod 4$, then $b(h,6,k,8)=2$. If $h\equiv 3 \pmod 4$, then $b(h,6,k,8)=6$. Hence,

$$\mathfrak{M}_{6,8}=\frac{1}{\tau}\left(-\frac{\pi^2}{6k^2}-\frac{16\pi i\chi(h)}{k^2}\left(\zeta'\left(-1,\frac{1}{4}\right)-\zeta'\left(-1,\frac{3}{4}\right)\right)\right).$$

In consequence,

$$\mathfrak{M}_G=-\frac{1}{\tau}\frac{\pi^2}{6k^2}. \tag{14.5.15}$$

Further,

$$\Re(\mathfrak{M}_G) < 0. \tag{14.5.16}$$

For the error term E_G, we have

$$|\Re(E_G)| \leq 1.32X^{1/2}\log X + 13.27X^{1/2} + 0.36\log X + 1.73 + 2.72X^{-1/2}. \tag{14.5.17}$$

Proof of Theorem 14.5.1. We have

$$\log|G(q)| = \Re(\log G(q)) \leq \Re(\mathfrak{M}_G) + |\Re(E_G)|.$$

The first part simply follows from some direct computation by taking into account of the bounds for $\Re(\mathfrak{M}_G)$ and $|\Re(E_G)|$. For the second part, we notice by (14.5.2) that, when $|Y| \geq 1/(2\pi X)$,

$$\Re(\tau^{-1}) \leq \frac{X}{2}.$$

Whenever q is in $\mathcal{Q}_{1/1}$ or $\mathcal{Q}_{1/2}$, we apply (14.5.6) and (14.5.9) to obtain the bound

$$\Re(\mathfrak{M}_G) \leq \frac{\pi^2}{48}\frac{X}{2}.$$

Hence, (14.5.1) follows by inserting the contribution of the error term and carrying on the routine computation. □

14.6 Precise Approximations of $G(q)$ Near the Dominant Poles

Recall that

$$G(q) = \frac{1}{(q, -q^3; q^4)_\infty}. \tag{14.6.1}$$

From the analysis in the previous section, we know that $G(q)$ indeed has dominant poles at $q = \pm 1$. In fact, if $q = e^{-\tau + 2\pi i h/k}$ is in $\mathcal{Q}_{1/1}$ or $\mathcal{Q}_{1/2}$, then (14.5.6) and (14.5.9) tell us that $\log G(q)$ is dominated by $\pi^2/(48\tau)$ while the coefficient $\pi^2/48$ is the largest comparing with that for other $\mathcal{Q}_{h/k}$. Now we want to give some more precise approximations of $\log G(q)$ near the dominant poles.

Theorem 14.6.1. *Let $\tau = X^{-1} + 2\pi i Y$ with $|Y| \leq 1/(2\pi X)$. Then*

$$\log G(e^{-\tau}) = \frac{\pi^2}{48}\frac{1}{\tau} - \frac{1}{4}\log\tau - \frac{3}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma\left(\frac{1}{4}\right) + E_+ \tag{14.6.2}$$

where

$$|E_+| \leq 0.66 X^{-3/4}. \tag{14.6.3}$$

Further,

$$\log G(-e^{-\tau}) = \frac{\pi^2}{48}\frac{1}{\tau} + \frac{1}{4}\log\tau - \frac{1}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma\left(\frac{3}{4}\right) + E_- \tag{14.6.4}$$

where

$$|E_-| \leq 0.82 X^{-3/4}. \tag{14.6.5}$$

Proof. We deduce from (14.6.1) with the help of Mellin transform that

$$\begin{aligned}
\log G(e^{-\tau}) &= \sum_{m\geq 0}\sum_{\ell\geq 1}\left(\frac{e^{-(4m+1)\ell\tau}}{\ell} + \frac{(-1)^\ell e^{-(4m+3)\ell\tau}}{\ell}\right)\\
&= \frac{1}{2\pi i}\int_{(3/2)} \tau^{-s}\Gamma(s)\sum_{m\geq 0}\sum_{\ell\geq 1}\ell^{-s-1}\left(\frac{1}{(4m+1)^s} + \frac{(-1)^\ell}{(4m+3)^s}\right)ds\\
&= \frac{1}{2\pi i}\int_{(3/2)} (4\tau)^{-s}\Gamma(s)\zeta(s+1)\left(\zeta\left(s,\frac{1}{4}\right) - (1-2^{-s})\zeta\left(s,\frac{3}{4}\right)\right)ds\\
&=: \frac{1}{2\pi i}\int_{(3/2)} \Theta_+(s)ds.
\end{aligned}$$

Now one may shift the path of integration to $(-3/4)$ by taking into consideration of the residues of $\Theta_+(s)$ inside the stripe $-3/4 < \Re(s) < 3/2$. Hence,

$$\log G(e^{-\tau}) = \sum_{-3/4<\Re(s)<3/2} \operatorname{Res}_s \Theta_+(s) + \frac{1}{2\pi i}\int_{(-3/4)} \Theta_+(s)ds.$$

Notice that $\Theta_+(s)$ has two singularities respectively at $s = 0$ and 1 when $-3/4 < \Re(s) < 3/2$. We compute that

$$\operatorname{Res}_{s=1}\Theta_+(s) = \frac{\pi^2}{48}\frac{1}{\tau}$$

and

$$\begin{aligned}
\operatorname{Res}_{s=0}\Theta_+(s) &= -\frac{1}{4}\log(4\tau) + \zeta'\left(0,\frac{1}{4}\right) - (\log 2)\zeta\left(0,\frac{3}{4}\right)\\
&= -\frac{1}{4}\log(4\tau) + \log\Gamma\left(\frac{1}{4}\right) - \frac{1}{2}\log(2\pi) + \frac{1}{4}\log 2\\
&= -\frac{1}{4}\log\tau - \frac{3}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma\left(\frac{1}{4}\right).
\end{aligned}$$

Further, recalling that $\tau = X^{-1} + 2\pi i Y$ where $|Y| \leq 1/(2\pi X)$, we have $|\operatorname{Arg}(\tau)| \leq \pi/4$. Since for $\Re(s) = -3/4$,

$$|\tau^{-s}| = \exp\left(\frac{3}{4}\log|\tau| + \Im(s)\operatorname{Arg}(\tau)\right) \leq |\tau|^{3/4} e^{|\Im(s)|\pi/4},$$

it follows that

$$\begin{aligned}
|E_+| &= \left|\frac{1}{2\pi i}\int_{(-3/4)} \Theta_+(s)ds\right| \\
&= \left|\frac{1}{2\pi i}\int_{(-3/4)} (4\tau)^{-s}\Gamma(s)\zeta(s+1)\left(\zeta\Big(s,\frac{1}{4}\Big) - (1-2^{-s})\zeta\Big(s,\frac{3}{4}\Big)\right)ds\right| \\
&\leq |\tau|^{3/4} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} 4^{3/4} e^{|t|\pi/4}\left|\Gamma\left(-\frac{3}{4}+it\right)\right|\left|\zeta\left(\frac{1}{4}+it\right)\right| \\
&\quad \times \left(\left|\zeta\Big(-\frac{3}{4}+it,\frac{1}{4}\Big)\right| + (1+2^{3/4})\left|\zeta\Big(-\frac{3}{4}+it,\frac{3}{4}\Big)\right|\right)dt \\
&\leq 0.507|\tau|^{3/4}.
\end{aligned}$$

We also have

$$|\tau| = \sqrt{X^{-2} + 4\pi^2 Y^2} \leq \sqrt{2}X^{-1}.$$

Hence,

$$|E_+| \leq 0.66 X^{-3/4}.$$

For $\log G(-e^{-\tau})$, we simply notice that

$$\log G(-e^{-\tau}) = \frac{1}{2\pi i}\int_{(3/2)} (4\tau)^{-s}\Gamma(s)\zeta(s+1)\left(\zeta\Big(s,\frac{3}{4}\Big) - (1-2^{-s})\zeta\Big(s,\frac{1}{4}\Big)\right)ds.$$

The rest follows from similar calculations. □

14.7 Applying the Circle Method

The proof of Theorem 14.1.2 is simply an exercise of the circle method. We first put

$$X = \sqrt{\frac{48n}{\pi^2}}. \tag{14.7.1}$$

Since it is assumed that $X \geq 3.4 \times 10^7$ as in Theorem 14.5.1, one has

$$n \geq 2.4 \times 10^{14}. \tag{14.7.2}$$

Recall that Cauchy's integral formula indicates that

$$\begin{aligned} g(n) &= \frac{1}{2\pi i} \int_{|q|=e^{-1/X}} \frac{G(q)}{q^{n+1}} dq \\ &= e^{n/X} \int_{-\frac{1}{2\pi X}}^{1-\frac{1}{2\pi X}} G\Big(e^{-(X^{-1}+2\pi i t)}\Big) e^{2\pi i n t}\, dt. \end{aligned} \tag{14.7.3}$$

Now we separate the interval $[-\frac{1}{2\pi X},\ 1-\frac{1}{2\pi X}]$ into three (disjoint) subintervals:

$$\begin{aligned} I_1 &:= \left[-\frac{1}{2\pi X}, \frac{1}{2\pi X}\right], \\ I_2 &:= \left[\frac{1}{2} - \frac{1}{2\pi X}, \frac{1}{2} + \frac{1}{2\pi X}\right] \end{aligned}$$

and

$$I_3 := \left[-\frac{1}{2\pi X},\ 1 - \frac{1}{2\pi X}\right] - I_1 - I_2.$$

Before evaluating (14.7.3) for each subinterval, we fix the notation that $\mathfrak{O}(x)$ means an expression E such that $|E| \leq x$. We also write for $j = 1, 2, 3$,

$$g_j(n) := e^{n/X} \int_{I_j} G\Big(e^{-(X^{-1}+2\pi i t)}\Big) e^{2\pi i n t}\, dt.$$

First,

$$\begin{aligned} g_1(n) &= e^{n/X} \int_{-\frac{1}{2\pi X}}^{\frac{1}{2\pi X}} G\Big(e^{-(X^{-1}+2\pi i t)}\Big) e^{2\pi i n t}\, dt \\ &= \frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} e^{n\tau} G(e^{-\tau})\, d\tau. \end{aligned}$$

Notice that for $|x| \leq 1$,

$$e^x = 1 + \mathfrak{O}(2|x|).$$

Applying (14.6.2) yields

$$g_1(n) = \frac{\Big(1 + \mathfrak{O}(1.32X^{-3/4})\Big)\Gamma(1/4)}{2^{3/4}\pi^{1/2}} \frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right)\, d\tau. \tag{14.7.4}$$

We then separate the integral as

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau}+n\tau\right)\, d\tau \\
&\quad = \frac{1}{2\pi i}\left(\int_{\Gamma} - \int_{-\infty-i\frac{1}{X}}^{\frac{1}{X}-i\frac{1}{X}} + \int_{-\infty+i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}}\right)\tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau}+n\tau\right)\, d\tau \\
&\quad =: J_{11}+J_{12}+J_{13}
\end{aligned}$$

where

$$\Gamma := (-\infty - iX^{-1}) \to (X^{-1}-iX^{-1}) \to (X^{-1}+iX^{-1}) \to (-\infty+iX^{-1}) \tag{14.7.5}$$

is a Hankel contour. To evaluate J_{11}, we make the change of variables

$$\tau = \sqrt{\frac{\pi^2}{48n}}w.$$

Then

$$J_{11} = \left(\frac{\pi^2}{48n}\right)^{3/8} \frac{1}{2\pi i}\int_{\tilde{\Gamma}} w^{-\frac{1}{4}} \exp\left(\sqrt{\frac{\pi^2 n}{48}}\left(\frac{1}{w}+w\right)\right)\, dw$$

where $\tilde{\Gamma}$ is the new contour. Recalling the contour integral representation of $I_s(x)$:

$$I_s(x) = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} w^{-s-1} e^{\frac{x}{2}\left(w+\frac{1}{w}\right)}\, dw,$$

we conclude

$$J_{11} = \frac{\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).$$

To bound J_{12}, we put $\tau = x - iX^{-1}$. Then

$$J_{12} = \frac{1}{2\pi i}\int_{-\infty}^{X^{-1}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau}+n\tau\right)\, dx.$$

Since $|\tau| \geq X^{-1}$, we have

$$|\tau|^{-1/4} \leq X^{1/4}.$$

Also,

$$|e^{n\tau}| = e^{nx}.$$

Further,

$$\left| e^{\frac{\pi^2}{48}\frac{1}{\tau}} \right| = e^{\frac{\pi^2}{48}\frac{x}{x^2+X^{-2}}} \le e^{\frac{\pi^2}{96}X}.$$

Hence,

$$\begin{aligned}
|J_{12}| &\le \frac{1}{2\pi} \cdot X^{1/4} e^{\frac{\pi^2}{96}X} \int_{-\infty}^{X^{-1}} e^{nx}\, dx \\
&= \frac{1}{2\pi} \cdot X^{1/4} e^{\frac{\pi^2}{96}X} \cdot \frac{1}{n} e^{n/X} \\
&= \frac{3^{1/8}}{2^{1/2}\pi^{5/4} n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right).
\end{aligned}$$

One may carry out a similar argument to obtain

$$|J_{13}| \le \frac{3^{1/8}}{2^{1/2}\pi^{5/4} n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right).$$

In consequence,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{-\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right)\, d\tau &= \frac{\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) \\
&\quad + \mathfrak{O}\left(\frac{2^{1/2}3^{1/8}}{\pi^{5/4}n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right)\right).
\end{aligned}$$

Recalling (14.7.4), we have

$$g_1(n) = \frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + E_{g_1} \tag{14.7.6}$$

where

$$\begin{aligned}
|E_{g_1}| &\le \frac{\Gamma(1/4)}{2^{3/4}\pi^{1/2}} \left(\frac{1.32\pi^{3/2}}{2^3 3^{3/4} n^{3/4}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) \right. \\
&\qquad \left. + \left(1 + \frac{1.32\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}}\right) \frac{2^{1/2}3^{1/8}}{\pi^{5/4}n^{7/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right) \right) \\
&\ll n^{-3/4} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).
\end{aligned} \tag{14.7.7}$$

On the other hand,

$$g_2(n) = (-1)^n e^{n/X} \int_{-\frac{1}{2\pi X}}^{\frac{1}{2\pi X}} G\left(-e^{-(X^{-1}+2\pi i t)}\right) e^{2\pi i n t}\, dt$$

$$= \frac{(-1)^n}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} e^{n\tau} G(-e^{-\tau})\, d\tau.$$

It follows from (14.6.4) that

$$g_2(n) = (-1)^n \frac{\Big(1+\mathfrak{O}(1.64X^{-3/4})\Big)\Gamma(3/4)}{2^{1/4}\pi^{1/2}} \frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right)\, d\tau. \quad (14.7.8)$$

Similarly, we separate the integral as

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\frac{1}{X}-i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}} \tau^{\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right)\, d\tau \\
&\quad = \frac{1}{2\pi i}\left(\int_\Gamma - \int_{-\infty-i\frac{1}{X}}^{\frac{1}{X}-i\frac{1}{X}} + \int_{-\infty+i\frac{1}{X}}^{\frac{1}{X}+i\frac{1}{X}}\right) \tau^{\frac{1}{4}} \exp\left(\frac{\pi^2}{48}\frac{1}{\tau} + n\tau\right)\, d\tau \\
&\quad =: J_{21} + J_{22} + J_{23}
\end{aligned}$$

where the Hankel contour Γ is as in (14.7.5). One may compute by the same argument that

$$J_{21} = \frac{\pi^{5/4}}{2^{5/2}3^{5/8}n^{5/8}} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).$$

To bound J_{22}, we still put $\tau = x - iX^{-1}$. Noticing that

$$|\tau|^{1/4} = (x^2 + X^{-2})^{1/8} \le |x|^{1/4} + X^{-1/4},$$

we have

$$\begin{aligned}
|J_{22}| &\le \frac{1}{2\pi} \cdot e^{\frac{\pi^2}{96}X} \int_{-\infty}^{X^{-1}} e^{nx}\Big(|x|^{1/4} + X^{-1/4}\Big)\, dx \\
&\le \frac{1}{2\pi} \cdot e^{\frac{\pi^2}{96}X} \int_{-\infty}^{0} e^{nx}(-x)^{1/4}\, dx + \frac{1}{2\pi} \cdot e^{\frac{\pi^2}{96}X} \int_{-\infty}^{X^{-1}} e^{nx} \cdot 2X^{-1/4}\, dx \\
&= \frac{\Gamma(5/4)}{2\pi n^{5/4}} \exp\left(\frac{\pi}{8}\sqrt{\frac{n}{3}}\right) + \frac{1}{2^{1/2}3^{1/8}\pi^{3/4}n^{9/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right).
\end{aligned}$$

Likewise,

$$|J_{23}| \le \frac{\Gamma(5/4)}{2\pi n^{5/4}} \exp\left(\frac{\pi}{8}\sqrt{\frac{n}{3}}\right) + \frac{1}{2^{1/2}3^{1/8}\pi^{3/4}n^{9/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right).$$

In consequence,

$$g_2(n) = (-1)^n \frac{\pi^{3/4}\Gamma(3/4)}{2^{11/4}3^{5/8}n^{5/8}} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) + E_{g_2} \tag{14.7.9}$$

where

$$\begin{aligned} |E_{g_2}| \leq \frac{\Gamma(3/4)}{2^{1/4}\pi^{1/2}} &\Bigg(\frac{1.64\pi^2}{2^4 3^1 n} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right) \\ &+ \left(1 + \frac{1.64\pi^{3/4}}{2^{3/2}3^{3/8}n^{3/8}}\right) \\ &\times 2\left(\frac{\Gamma(5/4)}{2\pi n^{5/4}} \exp\left(\frac{\pi}{8}\sqrt{\frac{n}{3}}\right) + \frac{1}{2^{1/2}3^{1/8}\pi^{3/4}n^{9/8}} \exp\left(\frac{3\pi}{8}\sqrt{\frac{n}{3}}\right)\right)\Bigg) \\ \ll n^{-1} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right). \end{aligned} \tag{14.7.10}$$

Remark 14.7.1. It is necessary to point out that $g_2(n)$ has an absolute size of

$$\text{constant} \times n^{-5/8} I_{-5/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right),$$

while from (14.7.7),

$$E_{g_1} \ll n^{-3/4} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right).$$

Since the two I-Bessel functions have the same order, we conclude that E_{g_1} is negligible comparing with $g_2(n)$.

Finally,

$$g_3(n) := e^{n/X} \int_{I_3} G\left(e^{-(X^{-1}+2\pi it)}\right) e^{2\pi int}\, dt.$$

Hence, by Theorem 14.5.1,

$$\begin{aligned} |g_3(n)| &\leq e^{n/X} \int_{I_3} \exp\left(\left(\frac{\pi^2}{48} - \frac{1}{100}\right)X\right) dt \\ &\leq \exp\left(\frac{n}{X} + \left(\frac{\pi^2}{48} - \frac{1}{100}\right)X\right). \end{aligned}$$

Namely,

$$|g_3(n)| \leq \exp\left(\frac{\pi}{2}\sqrt{\frac{n}{3}} - \frac{\sqrt{3n}}{25\pi}\right). \tag{14.7.11}$$

The asymptotic formula (14.1.8) follows from (14.7.6), (14.7.9) and (14.7.11). Further, a simple calculation reveals that when $n \geq 2.4 \times 10^{14}$, the sign of $g(n)$ depends only on the leading term

$$\frac{\pi^{1/4}\Gamma(1/4)}{2^{9/4}3^{3/8}n^{3/8}} I_{-3/4}\left(\frac{\pi}{2}\sqrt{\frac{n}{3}}\right),$$

which is of course positive.

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Part IV The World of *p*atterns

Outline

- Chapter 15 is devoted to Lin and Ma's conjecture on 0012-avoiding inversion sequences.
- Chapter 16 is devoted to Lin's conjecture on the avoidance of triples of binary relations with the ascent statistic considered.

Chapter 15

Lin and Ma's Conjecture on 0012-Avoiding Inversion Sequences

This chapter comes from

- S. Chern, On 0012-avoiding inversion sequences and a Conjecture of Lin and Ma, submitted. Available at arXiv:2006.04318. (Ref. [62])

15.1 Introduction

Our starting point is a recent paper of Yan and Lin [174], in which they proved a conjecture due to Martinez and Savage [132] that claims

$$|\mathbf{I}_n(021,120)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}. \tag{15.1.1}$$

This sequence is registered as A279561 in OEIS [163]. Lin and Yan also showed that this sequence as well enumerates $|\mathbf{I}_n(102,110)|$ and $|\mathbf{I}_n(102,120)|$. This therefore establishes the Wilf-equivalences

$$\mathbf{I}_n(021,120) \sim \mathbf{I}_n(102,110) \sim \mathbf{I}_n(102,120). \tag{15.1.2}$$

At the end of [174], a conjecture of Zhicong Lin and Jun Ma discovered in 2019 is recorded.

Conjecture 15.1.1 (Lin and Ma)**.** For $n \geq 1$,

$$|\mathbf{I}_n(0012)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}. \tag{15.1.3}$$

In other words, one may extend the balanced Wilf-equivalences (15.1.2) as the following unbalanced ones:

$$\mathbf{I}_n(0012) \sim \mathbf{I}_n(021,120) \sim \mathbf{I}_n(102,110) \sim \mathbf{I}_n(102,120).$$

It is also worth pointing out that the consideration of length-four pattern avoidance in inversion sequences appears novel in the literature.

The object of this chapter is to confirm the above conjecture of Lin and Ma.

Theorem 15.1.1. *Conjecture 15.1.1 is true.*

Let us fix some notation. Given $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n(0012)$, we define

$$\mathcal{R}(e) := \{m : \exists\, i \neq j \text{ such that } e_i = e_j = m\}.$$

In other words, $\mathcal{R}(e)$ is the set of letters that appear more than once in e. We further define

$$\operatorname{srpt}(e) := \min \mathcal{R}(e),$$

that is, the smallest number in $\mathcal{R}(e)$ — here srpt stands for "smallest repeated." Notice that there is only one sequence $01\cdots(n-1)$ in which none of the letters repeat. For this sequence, we assign that

$$\operatorname{srpt}(01\cdots(n-1)) := n-1.$$

Finally, we define

$$\operatorname{last}(e) := e_n,$$

the last entry of e.

15.2 Combinatorial Observations

We collect some combinatorial observations about inversion sequences in $\mathbf{I}_n(0012)$.

Lemma 15.2.1. *For $n \geq 1$ and $e \in \mathbf{I}_n(0012)$, if $\operatorname{srpt}(e) = k$, then for $1 \leq i \leq k+1$, we have $e_i = i-1$.*

Proof. If $\operatorname{srpt}(e) = n-1$, then $e = 01\cdots(n-1)$ and hence the lemma is true. Let $\operatorname{srpt}(e) \neq n-1$. If in this case the lemma is not true, then since $0 \leq e_i \leq i-1$ for each i, there must exist some $k_1 < k = \operatorname{srpt}(e)$ that appears more than once among $e_1, e_2, \ldots, e_{k+1}$. This violates the assumption that $\operatorname{srpt}(e) = k$. $\square$

Lemma 15.2.2. *For $n \geq 2$ and $e = e_1e_2\cdots e_n \in \mathbf{I}_n(0012)$, let $\gamma(e) = e_1e_2\cdots e_{n-1}$. We further assume that $e \neq 01\cdots(n-1)$. Then*

(a). *if* $\mathrm{last}(e) > \mathrm{srpt}(\gamma(e))$, *then*

$$\mathrm{srpt}(e) = \mathrm{srpt}(\gamma(e));$$

(b). *if* $\mathrm{last}(e) \leq \mathrm{srpt}(\gamma(e))$, *then*

$$\mathrm{srpt}(e) = \mathrm{last}(e).$$

Proof. A simple observation is that $\gamma(e) \in \mathbf{I}_{n-1}(0012)$. Below let us assume that $\mathrm{last}(e) = \ell$, $\mathrm{srpt}(e) = k$ and $\mathrm{srpt}(\gamma(e)) = k'$.

First, if $\mathcal{R}(\gamma(e)) = \emptyset$, then for each $0 \leq i \leq n-1$, $e_i = i-1$. Since $e \neq 01\cdots(n-1)$, we have $\mathrm{last}(e) = \ell \leq n-2 = \mathrm{srpt}(\gamma(e))$. This fits into Case (b). Further, we find that $\mathcal{R}(e) = \{\ell\}$ and hence $\mathrm{srpt}(e) = \ell$. This implies that $\mathrm{srpt}(e) = \mathrm{last}(e)$.

Now we assume that $\mathcal{R}(\gamma(e)) \neq \emptyset$. Notice that Case (a) is trivial. For Case (b), we first deduce from $\mathcal{R}(\gamma(e)) \neq \emptyset$ that $k' \leq n-3$. By Lemma 15.2.1, we find that for $1 \leq i \leq k'+1$, $e_i = i-1$. If $\mathrm{last}(e) = \ell \leq k'$, then we know that $e_{\ell+1} = \ell = e_n$. Also, we notice that the indices satisfy $\ell+1 \leq k'+1 \leq n-2 < n$. Hence, $\ell \in \mathcal{R}(e)$. Therefore, $\mathrm{srpt}(e) = \min\{\ell, k'\} = \ell = \mathrm{last}(e)$. □

Corollary 15.2.3. *For $e \in \mathbf{I}_n(0012)$,*

$$0 \leq \mathrm{srpt}(e) \leq \mathrm{last}(e) \leq n-1.$$

Proof. If $e = 01\cdots(n-1)$, the above inequalities are trivial since $\mathrm{srpt}(e) = \mathrm{last}(e) = n-1$. If $e \neq 01\cdots(n-1)$, the inequalities are direct consequences of Lemma 15.2.2 and the fact that $\mathrm{srpt}(e) \geq 0$ and $\mathrm{last}(e) \leq n-1$. □

Lemma 15.2.4. *For $n \geq 2$ and $e = e_1e_2\cdots e_n \in \mathbf{I}_n(0012)$, let e be such that* $\mathrm{srpt}(e) = \mathrm{last}(e) = k$ *with* $0 \leq k \leq n-2$. *Then*

(a). *for* $1 \leq i \leq k+1$,

$$e_i = i-1;$$

(b). *if we denote $e' = e'_1e'_2\cdots e'_{n-k}$ by the sequence obtained via $e'_i = e_{k+i} - k$ for each $1 \leq i \leq n-k$, then $e' \in \mathbf{I}_{n-k}(0012)$ such that*

$$\mathrm{srpt}(e') = \mathrm{last}(e') = 0.$$

Proof. Part (a) simply comes from Lemma 15.2.1. Also, we know from Part (a) that for $k+1 \leq i \leq n$, it holds that $e_i \geq k$. On the other hand, $e_i \leq i-1$. Hence, e' is still an inversion sequence. Further, it is trivial to see that e' still avoids the pattern 0012. Finally, we have $e'_1 = e_{k+1} - k = k - k = 0$ and $\mathrm{last}(e') = e'_{n-k} = e_n - k = k - k = 0$. Since $n-k \geq 2 > 1$, we have $0 \in \mathcal{R}(e')$ and hence $\mathrm{srpt}(e') = 0$. $\square$

15.3 Recurrences

Let

$$f_n(k,\ell) := \left\{ \begin{matrix} \text{the number of sequences } e \in \mathbf{I}_n(0012) \text{ with} \\ \mathrm{srpt}(e) = k \text{ and } \mathrm{last}(e) = \ell \end{matrix} \right\}.$$

We will establish the following recurrences.

Lemma 15.3.1. *We have*

(a). *for* $n \geq 1$,

$$f_n(n-1, n-1) = 1;$$

(b). *for* $n \geq 2$,

$$f_n(n-2, n-1) = 0;$$

(c). *for* $n \geq 2$ *and* $0 \leq k \leq n-3$,

$$f_n(k, n-1) = \sum_{k'=k}^{n-2} f_{n-1}(k', n-2);$$

(d). *for* $n \geq 2$ *and* $0 \leq \ell \leq n-2$,

$$f_n(\ell,\ell) = \sum_{\ell'=\ell}^{n-2} \sum_{k'=\ell}^{\ell'} f_{n-1}(k',\ell');$$

(e). *for* $n \geq 2$ *and* $0 \leq k < \ell \leq n-2$,

$$f_n(k,\ell) = \sum_{k'=k}^{\ell} f_{n-1}(k',\ell) + \sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell').$$

Proof. Cases (a) and (b) are trivial. In particular, Case (a) enumerates the only inversion sequence $01\cdots(n-1)$ in which none of the letters repeat. Below we always assume that $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n(0012)$. Let $\gamma(e)$ be as in Lemma 15.2.2.

For Case (c), let e be such that $\mathrm{srpt}(e) = k \leq n-3$ and $\mathrm{last}(e) = n-1$. We first notice that $e_{n-1} = \mathrm{last}(\gamma(e)) \geq \mathrm{srpt}(\gamma(e))$ by Corollary 15.2.3. Also, it is easy to see that $\mathrm{srpt}(\gamma(e)) = \mathrm{srpt}(e) = k$ since $\mathrm{last}(e) = n-1 > k$. Now we claim that $e_{n-1} = k$. Otherwise, namely, if $e_{n-1} > k$, we may find $i < j < n-1$ such that $e_i = e_j = k$. Hence, $e_i e_j e_{n-1} e_n$ has the reduction 0012, which contradicts the assumption that $e \in \mathbf{I}_n(0012)$. We therefore have a bijection

$$e = e_1 e_2 \cdots e_{n-2}(k)(n-1) \longleftrightarrow e_1 e_2 \cdots e_{n-2}(n-2) = e'.$$

Notice that e' is still an inversion sequence avoiding the pattern 0012. Also, $\mathrm{srpt}(e') \geq k$. Otherwise, there exists some $k' < k$ that appears more than once among $e_1, e_2, \ldots, e_{n-2}$ and therefore $\mathrm{srpt}(e) < k$, which leads to a contradiction. Finally, to prove Case (c), it suffices to show that e' could be any inversion sequence in $\mathbf{I}_{n-1}(0012)$ with $\mathrm{last}(e') = n-2$ (which is of course true) and $\mathrm{srpt}(e') \geq k$. Let e' be such a sequence and assume that $\mathrm{srpt}(e') = k' \geq k$. By Lemma 15.2.1, we have $e_{k+1} = k$. Pulling back to e, we have $e_{k+1} = e_{n-1} = k$ with the indices $k+1 \leq n-2 < n-1$. Therefore, for this e, we have $k \in \mathcal{R}(e)$ and hence $\mathrm{srpt}(e) = \min\{k', k\} = k$.

For Case (d), let e be such that $\mathrm{srpt}(e) = \mathrm{last}(e) = \ell$ with $0 \leq \ell \leq n-2$. We first find that $\mathrm{srpt}(\gamma(e)) \geq \mathrm{srpt}(e) = \ell$. On the other hand, let $e' = e'_1 e'_2 \cdots e'_{n-1} \in \mathbf{I}_{n-1}(0012)$ be such that $\mathrm{srpt}(e') \geq \ell$. By Lemma 15.2.1, $e'_{\ell+1} = \ell$. Hence, by appending ℓ to the end of e', we obtain a sequence with both srpt and last equal to ℓ. We therefore arrive at a bijection between e and e',

$$e = e_1 e_2 \cdots e_{n-1}(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-1} = e',$$

and the desired relation follows.

For Case (e), let e be such that $\mathrm{srpt}(e) = k$ and $\mathrm{last}(e) = \ell$ with $0 \leq k < \ell \leq n-2$. Notice that $e_{n-1} \geq k$. Otherwise, we assume that $e_{n-1} = k' < k$. Then by Lemma 15.2.1, $e_{k'+1} = k' = e_{n-1}$. However, $k'+1 < k+1 < n-1$ and hence $k' \in \mathcal{R}(e)$. But this violates the fact that $k = \min \mathcal{R}(e)$. Now we have two cases.

- $e_{n-1} < e_n$. We claim that $e_{n-1} = k$. Otherwise, we may find $i < j < n-1$ such that $e_i = e_j = k$. Hence, $e_i e_j e_{n-1} e_n$ has the reduction 0012, which violates the assumption that $e \in \mathbf{I}_n(0012)$. Now we have a bijection between e and $e' \in \mathbf{I}_{n-1}(0012)$ such that $\mathrm{srpt}(e') \geq k$ and $\mathrm{last}(e') = \ell$ by

$$e = e_1 e_2 \cdots e_{n-2}(k)(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-2}(\ell) = e'.$$

The argument is similar to that for Case (c). This bijection leads to the first term in the right-hand side of the recurrence relation in Case (e).

- $e_{n-1} \geq e_n$. We have a bijection between e and $e' \in \mathbf{I}_{n-1}(0012)$ such that $\operatorname{srpt}(e') = k$ and $\operatorname{last}(e') \geq \ell$ by

$$e = e_1 e_2 \cdots e_{n-1}(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-1} = e'.$$

The argument is similar to that for Case (d). This bijection leads to the second term in the right-hand side of the recurrence relation in Case (e).

The proof of the lemma is therefore complete. □

We may therefore determine the support of $f_n(k, \ell)$.

Corollary 15.3.2. *For $n \geq 1$, $f_n(k, \ell)$ is supported on*

$$\{(k, \ell) \in \mathbb{N}^2 : 0 \leq k \leq \ell \leq n-1\} \backslash \{(n-2, n-1)\}.$$

Proof. By Corollary 15.2.3, $f_n(k, \ell) = 0$ if

$$(k, \ell) \notin \{(k, \ell) \in \mathbb{N}^2 : 0 \leq k \leq \ell \leq n-1\}.$$

Also, $f_n(n-2, n-1) = 0$ by Lemma 15.3.1(b). Finally, for the remaining (k, ℓ), we have $f_n(k, \ell) \neq 0$ with the help of the recurrences in Lemma 15.3.1. □

Finally, we have another recurrence.

Lemma 15.3.3. *We have, for $n \geq 2$ and $0 \leq k \leq n-2$,*

$$f_n(k, k) = f_{n-k}(0, 0).$$

Proof. This is an immediate consequence of Lemma 15.2.4. □

In the sequel, we require three auxiliary functions with q *within a sufficiently small neighborhood of* 0:

$$\mathcal{L}(x; q) = \sum_{n \geq 1} L_n(x) q^n := \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} f_n(k, n-1) x^k \right) q^n,$$

$$\mathcal{D}(x; q) = \sum_{n \geq 1} D_n(x) q^n := \sum_{n \geq 1} \left(\sum_{\ell=0}^{n-2} f_n(\ell, \ell) x^\ell \right) q^n,$$

$$\mathcal{F}(x,y;q)=\sum_{n\geq 1}F_n(x,y)q^n:=\sum_{n\geq 1}\left(\sum_{\ell=0}^{n-1}\sum_{k=0}^{\ell}f_n(k,\ell)x^ky^\ell\right)q^n.$$

Notice that $L_1(x)=1$, $D_1(x)=0$ and $F_1(x,y)=1$. Also, since $f_n(n-1,n-1)=1$, we have

$$\sum_{\ell=0}^{n-1}f_n(\ell,\ell)x^\ell=D_n(x)+x^{n-1}.$$

15.4 001-Avoidance and a Result of Corteel et al.

The following result on 001-avoidance was shown by Corteel et al. [73].

Theorem 15.4.1 (Corteel et al.)**.** *For $n\geq 1$,*

$$|\mathbf{I}_n(001)|=2^{n-1}. \tag{15.4.1}$$

One readily observes that, for $n\geq 2$, there is a natural bijection between 001-avoiding inversion sequences of length $n-1$ and 0012-avoiding inversion sequences of length n in which the last entry equals $n-1$. Such a bijection could be simply constructed by appending $n-1$ to the end of the 001-avoiding inversion sequences. Therefore, we have an enumeration result as follows.

Corollary 15.4.2. *For $n\geq 1$,*

$$|\{e\in\mathbf{I}_n(0012):\mathrm{last}(e)=n-1\}|=\begin{cases}1 & \text{if } n=1,\\ 2^{n-2} & \text{if } n\geq 2.\end{cases} \tag{15.4.2}$$

Notice that Corollary 15.4.2 is equivalent to

$$\begin{aligned}\mathcal{L}(1;q)&=\sum_{n\geq 1}\left(\sum_{k=0}^{n-1}f_n(k,n-1)\right)q^n\\&=q+q^2+2q^3+4q^4+8q^5+16q^6+\cdots\\&=\frac{q(1-q)}{1-2q}.\end{aligned}$$

Now we prove a bivariate strengthening of the above that will be utilized in our proof of Theorem 15.1.1.

Theorem 15.4.3. *We have*

$$\mathcal{L}(x;q) = \frac{q(1-q)^2}{(1-2q)(1-xq)}. \tag{15.4.3}$$

Proof. For $n \geq 2$, it follows from (a), (b) and (c) of Lemma 15.3.1 that

$$\begin{aligned}
\sum_{k=0}^{n-1} f_n(k,n-1)x^k &= x^{n-1} + \sum_{k=0}^{n-3}\sum_{k'=k}^{n-2} f_{n-1}(k',n-2)x^k \\
&= x^{n-1} + \sum_{k'=0}^{n-3} f_{n-1}(k',n-2)\sum_{k=0}^{k'} x^k + f_{n-1}(n-2,n-2)\sum_{k=0}^{n-3} x^k \\
&= x^{n-1} + \sum_{k'=0}^{n-3} f_{n-1}(k',n-2)\frac{1-x^{k'+1}}{1-x} + \frac{1-x^{n-2}}{1-x}.
\end{aligned}$$

Therefore,

$$L_n(x) = x^{n-1} + \frac{1}{1-x}\Big(L_{n-1}(1) - xL_{n-1}(x)\Big) - \frac{1-x^{n-1}}{1-x} + \frac{1-x^{n-2}}{1-x}.$$

Multiplying the above by q^n and summing over $n \geq 2$, we have

$$\mathcal{L}(x;q) - q = \frac{q}{1-x}\mathcal{L}(1;q) - \frac{xq}{1-x}\mathcal{L}(x;q) - \frac{q^2(1-x)}{1-xq},$$

or

$$(1-xq)(1-x+xq)\mathcal{L}(x;q) = q(1-xq)\mathcal{L}(1;q) + q(1-q)(1-x). \tag{15.4.4}$$

Applying the kernel method yields

$$\begin{cases} 1-x+xq = 0, \\ q(1-xq)\mathcal{L}(1;q) + q(1-q)(1-x) = 0. \end{cases}$$

Solving the first equation of the system for x gives

$$x = \frac{1}{1-q}.$$

Substituting the above into the second equation of the system, we have

$$\mathcal{L}(1;q) = \frac{q(1-q)}{1-2q}.$$

Substituting the above back to (15.4.4), we arrive at (15.4.3). □

15.5 Proof of Theorem 15.1.1

We first establish two relations concerning $\mathcal{D}(x;q)$.

Lemma 15.5.1. *We have*

$$\begin{aligned}\mathcal{D}(x;q) &= \frac{1}{1-xq}\mathcal{D}(0;q) && (15.5.1)\\ &= \frac{q}{1-xq}\mathcal{F}(1,1;q). && (15.5.2)\end{aligned}$$

Proof. We know from Lemma 15.3.3 that

$$\begin{aligned}\sum_{n\geq 2}\sum_{k=0}^{n-2} f_n(k,k)x^k q^n &= \sum_{n\geq 2}\sum_{k=0}^{n-2} f_{n-k}(0,0)x^k q^n\\ {\scriptstyle(\text{with } n'=n-k)} &= \sum_{n'\geq 2}\sum_{n\geq n'} f_{n'}(0,0)x^{n-n'}q^n\\ &= \sum_{n'\geq 2} f_{n'}(0,0)x^{-n'}\sum_{n\geq n'}(xq)^n\\ &= \frac{1}{1-xq}\sum_{n'\geq 2} f_{n'}(0,0)q^{n'}.\end{aligned}$$

Noticing that $D_1(x)=0$, we have

$$\mathcal{D}(x;q) = \frac{1}{1-xq}\mathcal{D}(0;q),$$

which is the first part of the lemma. For the second part, we deduce from Lemma 15.3.1(d) that

$$\begin{aligned}\mathcal{D}(0;q) &= \sum_{n\geq 2} f_n(0,0)q^n\\ &= \sum_{n\geq 2}\sum_{\ell'=0}^{n-2}\sum_{k'=0}^{\ell'} f_{n-1}(k',\ell')q^n\\ &= q\mathcal{F}(1,1;q).\end{aligned}$$

Therefore, (15.5.2) follows. □

Next, we show a relation between $\mathcal{F}(x,1;q)$ and $\mathcal{F}(1,1;q)$.

Lemma 15.5.2. *We have*

$$\mathcal{F}(x,1;q) = \frac{1-q}{1-xq}\mathcal{F}(1,1;q). \tag{15.5.3}$$

Proof. For $n \geq 2$, it follows from Lemma 15.3.1(d) that

$$\begin{aligned}
D_n(x) &= \sum_{\ell=0}^{n-2} f_n(\ell,\ell)x^\ell \\
&= \sum_{\ell=0}^{n-2}\sum_{\ell'=\ell}^{n-2}\sum_{k'=\ell}^{\ell'} f_{n-1}(k',\ell')x^\ell \\
&= \sum_{\ell'=0}^{n-2}\sum_{k'=0}^{\ell'} f_{n-1}(k',\ell')\sum_{k=0}^{k'} x^\ell \\
&= \sum_{\ell'=0}^{n-2}\sum_{k'=0}^{\ell'} f_{n-1}(k',\ell')\frac{1-x^{k'+1}}{1-x} \\
&= \frac{1}{1-x}\Big(F_{n-1}(1,1) - xF_{n-1}(x,1)\Big).
\end{aligned}$$

Therefore,

$$\mathcal{D}(x;q) = \frac{q}{1-x}\Big(\mathcal{F}(1,1;q) - x\mathcal{F}(x,1;q)\Big).$$

Substituting (15.5.2) into the above yields

$$\frac{q}{1-xq}\mathcal{F}(1,1;q) = \frac{q}{1-x}\Big(\mathcal{F}(1,1;q) - x\mathcal{F}(x,1;q)\Big),$$

from which (15.5.3) follows. □

We then construct a functional equation for $\mathcal{F}(x,y;q)$.

Lemma 15.5.3. *We have*

$$\begin{aligned}
&\left(1+\frac{xq}{1-x}+\frac{yq}{1-y}\right)\mathcal{F}(x,y;q) \\
&\quad = \frac{q}{1-x}\mathcal{F}(1,y;q) + \frac{q(1-q)}{(1-y)(1-xyq)}\mathcal{F}(1,1;q) + \frac{q(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)}.
\end{aligned} \tag{15.5.4}$$

Proof. We first observe that

$$\sum_{\ell=0}^{n-2} f_n(\ell,\ell)x^\ell y^\ell + \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_n(k,\ell)x^k y^\ell = F_n(x,y) - \sum_{k=0}^{n-1} f_n(k,n-1)x^k y^{n-1}$$

$$= F_n(x,y) - y^{n-1}L_n(x). \tag{15.5.5}$$

Notice also that

$$\sum_{\ell=0}^{n-2} f_n(\ell,\ell)x^\ell y^\ell = D_n(xy). \tag{15.5.6}$$

Now, by Lemma 15.3.1(e), we may separate

$$\begin{aligned}\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_n(k,\ell)x^k y^\ell &= \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell} f_{n-1}(k',\ell)x^k y^\ell \\ &\quad + \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell')x^k y^\ell.\end{aligned}$$

We further notice that the first term on the right-hand side can be separated as

$$\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell} f_{n-1}(k',\ell)x^k y^\ell = \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell-1} f_{n-1}(k',\ell)x^k y^\ell + \sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_{n-1}(\ell,\ell)x^k y^\ell.$$

We have

$$\begin{aligned}&\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{k'=k}^{\ell-1} f_{n-1}(k',\ell)x^k y^\ell \\ &\quad = \sum_{\ell=1}^{n-2}\sum_{k'=0}^{\ell-1} f_{n-1}(k',\ell)y^\ell \sum_{k=0}^{k'} x^k \\ &\quad = \sum_{\ell=1}^{n-2}\sum_{k'=0}^{\ell-1} f_{n-1}(k',\ell)y^\ell \frac{1-x^{k'+1}}{1-x} \\ &\quad = \sum_{\ell=0}^{n-2}\sum_{k'=0}^{\ell} f_{n-1}(k',\ell)y^\ell \frac{1-x^{k'+1}}{1-x} - \sum_{\ell=0}^{n-2} f_{n-1}(\ell,\ell)y^\ell \frac{1-x^{\ell+1}}{1-x} \\ &\quad = \frac{1}{1-x}\Big(F_{n-1}(1,y) - xF_{n-1}(x,y)\Big) \\ &\qquad - \frac{1}{1-x}\Big(D_{n-1}(y) + y^{n-2} - xD_{n-1}(xy) - x^{n-1}y^{n-2}\Big).\end{aligned}$$

Also,

$$\begin{aligned}\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_{n-1}(\ell,\ell)x^k y^\ell &= \sum_{\ell=1}^{n-2} f_{n-1}(\ell,\ell)y^\ell \frac{1-x^\ell}{1-x} \\ &= \sum_{\ell=0}^{n-2} f_{n-1}(\ell,\ell)y^\ell \frac{1-x^\ell}{1-x}\end{aligned}$$

$$= \frac{1}{1-x}\Big(D_{n-1}(y) + y^{n-2} - D_{n-1}(xy) - x^{n-2}y^{n-2}\Big).$$

On the other hand,

$$\begin{aligned}
\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1}\sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell')x^k y^\ell &= \sum_{\ell'=1}^{n-2}\sum_{k=0}^{\ell'-1} f_{n-1}(k,\ell')x^k \sum_{\ell=k+1}^{\ell'} y^\ell \\
&= \sum_{\ell'=1}^{n-2}\sum_{k=0}^{\ell'-1} f_{n-1}(k,\ell')x^k \frac{y^{k+1}-y^{\ell'+1}}{1-y} \\
&= \sum_{\ell'=0}^{n-2}\sum_{k=0}^{\ell'} f_{n-1}(k,\ell')x^k \frac{y^{k+1}-y^{\ell'+1}}{1-y} \\
&= \frac{y}{1-y}\Big(F_{n-1}(xy,1) - F_{n-1}(x,y)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\ell=1}^{n-2}\sum_{k=0}^{\ell-1} f_n(k,\ell)x^k y^\ell \\
&\quad = \frac{1}{1-x}\Big(F_{n-1}(1,y) - xF_{n-1}(x,y)\Big) + \frac{y}{1-y}\Big(F_{n-1}(xy,1) - F_{n-1}(x,y)\Big) \\
&\qquad - D_{n-1}(xy) - x^{n-2}y^{n-2}.
\end{aligned} \tag{15.5.7}$$

It follows from (15.5.5), (15.5.6) and (15.5.7) that

$$\begin{aligned}
&F_n(x,y) - y^{n-1}L_n(x) \\
&\quad = D_n(xy) + \frac{1}{1-x}\Big(F_{n-1}(1,y) - xF_{n-1}(x,y)\Big) \\
&\qquad + \frac{y}{1-y}\Big(F_{n-1}(xy,1) - F_{n-1}(x,y)\Big) - D_{n-1}(xy) - x^{n-2}y^{n-2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{F}(x,y;q) - y^{-1}\mathcal{L}(x;yq) \\
&\quad = \mathcal{D}(xy;q) + \frac{q}{1-x}\Big(\mathcal{F}(1,y;q) - x\mathcal{F}(x,y;q)\Big) \\
&\qquad + \frac{yq}{1-y}\Big(\mathcal{F}(xy,1;q) - \mathcal{F}(x,y;q)\Big) - q\mathcal{D}(xy;q) - \frac{q^2}{1-xyq}.
\end{aligned}$$

Applying (15.4.3), (15.5.2) and (15.5.3) gives the desired result. □

With the assistance of the kernel method, we may deduce a functional equation satisfied by $\mathcal{F}(1,y;q)$.

Lemma 15.5.4. *We have*

$$\mathcal{F}(1,y;q)=\frac{q}{1-y+y^2q}\mathcal{F}(1,1;q)+\frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)(1-y+y^2q)}. \tag{15.5.8}$$

Proof. We multiply both sides of (15.5.4) by $(1-x)(1-y)$. Then

$$\begin{aligned}\Big((1-y+yq)-x(1-y-q+2yq)\Big)&\mathcal{F}(x,y;q)\\ &=q(1-y)\mathcal{F}(1,y;q)+\frac{q(1-q)(1-x)}{1-xyq}\mathcal{F}(1,1;q)\\ &\quad+\frac{q(1-x)(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)}.\end{aligned}$$

We treat the kernel polynomial as a function in x and solve

$$(1-y+yq)-x(1-y-q+2yq)=0$$

so that

$$x=\frac{1-y+yq}{1-y-q+2yq}.$$

Substituting the above into

$$\begin{aligned}0=q(1-y)\mathcal{F}(1,y;q)&+\frac{q(1-q)(1-x)}{1-xyq}\mathcal{F}(1,1;q)\\ &+\frac{q(1-x)(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)},\end{aligned}$$

we arrive at (15.5.8) after simplification. □

Finally, we are ready to complete the proof of Theorem 15.1.1.

Proof of Theorem 15.1.1. It is known that (cf. [163, A279561])

$$1+\sum_{n\geq 1}\left(1+\sum_{i=1}^{n-1}\binom{2i}{i-1}\right)q^n=\frac{1-4q+(1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)}. \tag{15.5.9}$$

We then rewrite (15.5.8) as

$$(1-y+y^2q)\mathcal{F}(1,y;q)=q\mathcal{F}(1,1;q)+\frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)}.$$

We treat the kernel polynomial as a function in y and solve

$$1 - y + y^2 q = 0.$$

Then

$$y_{1,2} = \frac{1 \mp \sqrt{1-4q}}{2q}.$$

We choose the solution

$$y_1 = \frac{1 - \sqrt{1-4q}}{2q}$$

since $y_1 = 1 + q + O(q^2)$ as $q \to 0$. Substituting $y = y_1$ into

$$0 = q\mathcal{F}(1,1;q) + \frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)},$$

we find that

$$\begin{aligned}\mathcal{F}(1,1;q) &= \frac{-(1-2q)(1-4q)+(1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)}\\ &= \frac{1-4q+(1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)} - 1. \end{aligned} \tag{15.5.10}$$

This implies that for $n \geq 1$,

$$1 + \sum_{i=1}^{n-1} \binom{2i}{i-1} = \sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_n(k,\ell) = |\mathbf{I}_n(0012)|.$$

Therefore, Conjecture 15.1.1 is true. □

15.6 References

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Chapter 16

Lin's Conjecture on Inversion Sequences Avoiding Patterns of Relation Triples

This chapter comes from

- G. E. Andrews and S. Chern, A proof of Lin's conjecture on inversion sequences avoiding patterns of relation triples, *J. Combin. Theory Ser. A* **179** (2021), 105388, 20 pp. (Ref. [22])

16.1 Introduction

Apart from the usual pattern avoidance with fixed patterns, Martinez and Savage [132] also considered the following variation.

Definition 16.1.1 (Martinez and Savage [132]). We denote by $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ where $\rho_1, \rho_2, \rho_3 \in \{<, >, \leq, \geq, =, \neq, -\}$ the set of inversion sequences $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n$ such that there are no indices $1 \leq i < j < k \leq n$ with

$$e_i \, \rho_1 \, e_j, \quad e_j \, \rho_2 \, e_k \quad \text{and} \quad e_i \, \rho_3 \, e_k.$$

Here the binary relation "$-$" stands for "no restriction", that is, if $e_i - e_j$, then we assume that there is no restriction on the order of e_i and e_j.

Since the work of Martinez and Savage, the enumerations of such sequences have been investigated extensively. In particular, a handful of Wilf equivalences among the 343 possible sets of inversion sequences avoiding patterns of relation triples were conjectured in [132] and proved later in [37, 39, 45, 90, 113, 126, 127, 175].

A further direction for the study of pattern avoidance is to take account of various statistics and investigate their distribution over pattern avoiding sequences; see, for instance, [45, 113, 127, 132]. Along this road, in [127], Lin conjectured a curious identity concerning the ascent statistic over $\mathbf{I}_n(\geq, \neq, >)$ and $\mathbf{I}_n(>, \neq, \geq)$.

We first recall that the *ascent* statistic is defined as follows.

Definition 16.1.2. Let $e = e_1e_2\cdots e_n \in \mathbf{I}_n$. We define, $\operatorname{asc}(e) := \#\{i \in [n-1] : e_i < e_{i+1}\}$, that is, the number of *ascents* of e.

Conjecture 16.1.1 (Lin [127, Conjecture 2.4])**.** For $n \geq 1$,

$$\sum_{e\in\mathbf{I}_n(\geq,\neq,>)} z^{\operatorname{asc}(e)} = \sum_{e\in\mathbf{I}_n(>,\neq,\geq)} z^{n-1-\operatorname{asc}(e)}. \tag{16.1.1}$$

Below are the expressions of (16.1.1) for $1 \leq n \leq 6$:

$$\begin{gathered}1,\\ 1+z,\\ 1+4z+z^2,\\ 1+10z+11z^2+z^3,\\ 1+20z+55z^2+25z^3+z^4,\\ 1+35z+188z^2+220z^3+50z^4+z^5.\end{gathered}$$

It is notable that the Wilf equivalence of $\mathbf{I}_n(\geq,\neq,>)$ and $\mathbf{I}_n(>,\neq,\geq)$ was first conjectured by Martinez and Savage [132] and later proved bijectively by Lin [127]. However, Lin's bijection, although preserves other statistics, does not imply his conjecture.

Our objective of this chapter is to confirm Conjecture 16.1.1. More precisely, what we are going to show is the following equivalent form.

Theorem 16.1.1. *For $n \geq 1$,*

$$\sum_{e\in\mathbf{I}_n(>,\neq,\geq)} z^{\operatorname{asc}(e)} = \sum_{e\in\mathbf{I}_n(\geq,\neq,>)} z^{n-1-\operatorname{asc}(e)}. \tag{16.1.2}$$

One will see that by replacing z with z^{-1} in (16.1.2) and then multiplying z^{n-1} on both sides, the identity (16.1.1) follows.

Our proof of Theorem 16.1.1 is algebraic with the application of the kernel method. But as commented in [127], a bijective proof of Conjecture 16.1.1 would be more intriguing. Such a proof still remains mysterious.

16.2 Sequences in $\mathbf{I}_n(>,\neq,\geq)$ and $\mathbf{I}_n(\geq,\neq,>)$

In this section, we prove some combinatorial properties of sequences in $\mathbf{I}_n(>,\neq,\geq)$ and $\mathbf{I}_n(\geq,\neq,>)$. In particular, we are interested in the behavior of the subsequence from the left-most appearance of the largest entry to the last entry. The study of such subsequences will lead to useful recurrences concerning these inversion sequences which will be presented in the next section.

Definition 16.2.1. Let $e = e_1 e_2 \cdots e_n$ be a sequence of natural numbers in which e_ℓ is the left-most appearance of the largest entry. We call the subsequence $e_\ell e_{\ell+1} \cdots e_n$ the *tail* of e, denoted by $\tau(e)$. For example,

$$\tau(0,1,0,3,1,3,5,3,3,3,6,5,7,8,8,6,8,6,8) = (8,8,6,8,6,8).$$

Definition 16.2.2. We use $a_{\geq k}$ to denote a sequence of consecutive a's appearing at least k times, that is,

$$\underbrace{aa \cdots a}_{\geq k \text{ times}}.$$

16.2.1 Sequences in $\mathbf{I}_n(>, \neq, \geq)$

Lemma 16.2.1. *Let $e \in \mathbf{I}_n(>, \neq, \geq)$. Then the tail of e has the form*

$$a_{\geq 1}\, b_{\geq 0} \qquad (\textit{with } a > b),$$

that is, a sequence of at least one a followed by several b's while the subsequence of b might be empty.

Proof. Recall that for any $e \in \mathbf{I}_n(>, \neq, \geq)$, we cannot find indices $i < j < k$ such that

$$e_i > e_j, \quad e_j \neq e_k \quad \text{and} \quad e_i \geq e_k. \tag{16.2.1}$$

Let $e_\ell = a$ be the left-most appearance of the largest entry in e. We first claim that among $e_{\ell+1}, \ldots, e_n$, there do not exist two distinct entries both of which are smaller than a. Otherwise, if we have such two entries e_j and e_k (with $\ell + 1 \leq j < k \leq n$), then $e_\ell e_j e_k$ satisfies (16.2.1), which is not allowed. The above indicates that $e_{\ell+1}, \ldots, e_n \in \{a, b\}$ for some $b < a$.

Further, if we have $e_{\ell'} = a$ for some $\ell + 1 \leq \ell' \leq n$, then we must have $e_j = a$ for all $\ell + 1 \leq j \leq \ell'$. Otherwise, if there exists one such index j with $e_j = b$, then $e_\ell e_j e_{\ell'} = aba$ satisfies (16.2.1).

The desired lemma therefore follows. □

Equipped with Lemma 16.2.1, we may categorize $\mathbf{I}_n(>, \neq, \geq)$ into four disjoint types. (Below we always assume that $a > b$.)

- Type I.
  The tail of $e \in \mathbf{I}_n(>, \neq, \geq)$ is of the form
$$a_{\geq 2} \quad \text{or} \quad a_{\geq 1}\, b_{\geq 2};$$

- Type II.
  The tail of e is of the form
$$a;$$

- Type III.
  The tail of e is of the form
$$a_{\geq 2}\, b;$$

- Type IV.
  The tail of e is of the form
$$a\, b.$$

16.2.2 Sequences in $\mathbf{I}_n(\geq, \neq, >)$

Lemma 16.2.2. *Let $e \in \mathbf{I}_n(\geq, \neq, >)$. Then the tail of e has the form*

$$a\, b_{\geq 0}\, a_{\geq 0} \qquad (\textit{with } a > b),$$

that is, a sequence of one a followed by several b's and then by several a's while the subsequence of b and the second subsequence of a might be empty.

Proof. Recall that for any $e \in \mathbf{I}_n(\geq, \neq, >)$, we cannot find indices $i < j < k$ such that

$$e_i \geq e_j, \quad e_j \neq e_k \quad \text{and} \quad e_i > e_k. \tag{16.2.2}$$

Let $e_\ell = a$ be the left-most appearance of the largest entry in e. We first claim that among $e_{\ell+1}, \ldots, e_n$, there do not exist two distinct entries both of which are smaller than

a. Otherwise, if we have such two entries e_j and e_k (with $\ell+1 \le j < k \le n$), then $e_\ell e_j e_k$ satisfies (16.2.1), which is not allowed. The above indicates that $e_{\ell+1}, \dots, e_n \in \{a, b\}$ for some $b < a$.

Further, if we have $e_{\ell'} = b$ for some $\ell+1 \le \ell' \le n$, then we must have $e_j = b$ for all $\ell+1 \le j \le \ell'$. Otherwise, if there exists one such index j with $e_j = a$, then $e_\ell e_j e_{\ell'} = aab$ satisfies (16.2.1).

Also, if we have $e_{\ell''} = a$ for some $\ell+1 \le \ell'' \le n$, then we must have $e_k = a$ for all $\ell'' \le k \le n$. Otherwise, if there exists one such index k with $e_k = b$, then $e_\ell e_{\ell''} e_k = aab$ satisfies (16.2.1).

The desired lemma therefore follows. □

Analogously, we categorize $\mathbf{I}_n(\ge, \ne, >)$ into four disjoint types. (Below we also assume that $a > b$.)

- Type I.
  The tail of $e \in \mathbf{I}_n(\ge, \ne, >)$ is of the form

$$a_{\ge 2} \quad \text{or} \quad a\, b_{\ge 2} \quad \text{or} \quad a\, b_{\ge 1}\, a_{\ge 2};$$

- Type II.
  The tail of e is of the form

$$a;$$

- Type III.
  The tail of e is of the form

$$a\, b_{\ge 1}\, a;$$

- Type IV.
  The tail of e is of the form

$$a\, b.$$

16.3 Recurrences and Generating Functions

16.3.1 Recurrences

For $1 \le i \le 4$, let

$$\mathbf{I}_{n,i}(>,\neq,\geq) := \{e \in \mathbf{I}_n(>,\neq,\geq) : e \text{ is of Type } i\}$$

and

$$\mathbf{I}_{n,i}^{(\Lambda)}(>,\neq,\geq) := \{e \in \mathbf{I}_{n,i}(>,\neq,\geq) : \text{the largest entry of } e \text{ is } \Lambda\}.$$

We further write

$$f_i(n,\Lambda) := \sum_{e \in \mathbf{I}_{n,i}^{(\Lambda)}(>,\neq,\geq)} z^{\operatorname{asc}(e)}. \tag{16.3.1}$$

Notice that the initial values of the f_i's are

$$f_1(1,\Lambda) = f_3(1,\Lambda) = f_4(1,\Lambda) = 0 \quad \text{for all } \Lambda \ge 0, \tag{16.3.2}$$

$$f_2(1,\Lambda) = \begin{cases} 1 & \text{for } \Lambda = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{16.3.3}$$

and

$$f_3(2,\Lambda) = 0 \quad \text{for all } \Lambda \ge 0. \tag{16.3.4}$$

Lemma 16.3.1. *For $n \ge 2$, we have*

(a). *for $\Lambda \ge 0$,*

$$f_1(n,\Lambda) = f_1(n-1,\Lambda) + f_2(n-1,\Lambda) + f_3(n-1,\Lambda) + f_4(n-1,\Lambda);$$

(b). *for $\Lambda = 0$ and $\Lambda \ge n$,*

$$f_2(n,\Lambda) = 0,$$

and for $1 \le \Lambda \le n-1$,

$$f_2(n,\Lambda) = \sum_{0\le\Lambda'<\Lambda} \Big(zf_1(n-1,\Lambda') + zf_2(n-1,\Lambda') + zf_3(n-1,\Lambda') + zf_4(n-1,\Lambda') \Big);$$

(c). *for* $\Lambda \ge 0$,

$$f_3(n,\Lambda) = f_3(n-1,\Lambda) + f_4(n-1,\Lambda);$$

(d). *for* $\Lambda = 0$ *and* $\Lambda \ge n-1$,

$$f_4(n,\Lambda) = 0,$$

and for $1 \le \Lambda \le n-2$,

$$f_4(n,\Lambda) = \sum_{0\le\Lambda'<\Lambda} \Big(zf_1(n-1,\Lambda') + f_2(n-1,\Lambda') + zf_3(n-1,\Lambda') + zf_4(n-1,\Lambda') \Big).$$

Further,

(b'). *for* $1 \le \Lambda \le n-1$,

$$f_2(n,\Lambda) - f_2(n,\Lambda-1) = zf_1(n,\Lambda-1);$$

(d'). *for* $1 \le \Lambda \le n-2$,

$$f_4(n,\Lambda) - f_4(n,\Lambda-1) = f_2(n-1,\Lambda) + zf_3(n,\Lambda-1).$$

Proof. To prove (a), (b), (c) and (d) of the lemma, we need to bijectively construct sequences in the desired subset of $\mathbf{I}_{n-1}(>,\neq,\ge)$ for each type of sequences in $\mathbf{I}_n^{(\Lambda)}(>,\neq,\ge)$. Such constructions will be presented explicitly below by deleting one paticular element from each sequence in $\mathbf{I}_n^{(\Lambda)}(>,\neq,\ge)$ of a fixed type. The inverse constructions from the desired subset of $\mathbf{I}_{n-1}(>,\neq,\ge)$ to each type of sequences in $\mathbf{I}_n^{(\Lambda)}(>,\neq,\ge)$ will not be explicitly given but they are simply done by adding the paticular element to where it is deleted. In the sequel, we always write $e = e_1e_2\cdots e_n \in \mathbf{I}_n^{(\Lambda)}(>,\neq,\ge)$.
Case 1. If e is of Type I, then we observe that $e_{n-1} = e_n$. By deleting the last entry e_n, we obtain an inversion sequence e' of length $n-1$. Apparently, $e' \in \mathbf{I}_{n-1}(>,\neq,\ge)$. Also, we claim that e' can be any of the four types. For example, if $\tau(e) = \Lambda\,\Lambda$, then $\tau(e') = \Lambda$

and hence e' is of Type II. For other cases, we may carry on similar arguments. Further, the largest entry in e' is still Λ. Finally, we observe that $\mathrm{asc}(e) = \mathrm{asc}(e')$.

Case 2. If e is of Type II, then by deleting the last entry $e_n = \Lambda$, we obtain an inversion sequence e' of length $n-1$. Again, we notice that $e' \in \mathbf{I}_{n-1}(>, \neq, \geq)$ can be any of the four types. However, in this case, the largest entry in e' is smaller than Λ. This is because $e_n = \Lambda$ is the only largest entry in e, but it is deleted. Finally, we observe that $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.

Case 3. If e is of Type III, then $\tau(e)$ is of the form $\Lambda_{\geq 2}\, b$ for some $b < \Lambda$. By deleting one of the Λ's, the resulting sequence e' is in $\mathbf{I}_{n-1}^{(\Lambda)}(>, \neq, \geq)$ with largest entry still equal to Λ. Also, $\tau(e')$ is either of the form $\Lambda_{\geq 2}\, b$ or of the form $\Lambda\, b$. Therefore, e' is of either Type III or Type IV. Finally, we observe that $\mathrm{asc}(e) = \mathrm{asc}(e')$.

Case 4. If e is of Type IV, then $\tau(e) = \Lambda\, b$ for some $b < \Lambda$. We delete Λ from e to get e'. It is not hard to verify that $e' \in \mathbf{I}_{n-1}(>, \neq, \geq)$ with largest entry smaller than Λ. We have three subcases as follows.

- $b = e_n = e_{n-2}$. Then e' is of Type I and in this case $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.
- $b = e_n > e_{n-2}$. Then $b > \max\{e_1, e_2, \ldots, e_{n-2}\}$. Otherwise, there exists some e_i in $e_1 e_2 \cdots e_{n-3}$ such that $e_i \geq b = e_n$. Now the subsequence $e_i e_{n-2} e_n$ satisfies $e_i \geq e_n > e_{n-2}$, which is not allowed. It is then obvious that e' is of Type II and $\mathrm{asc}(e) = \mathrm{asc}(e')$.
- $b = e_n < e_{n-2}$. Then $\tau(e_1 e_2 \cdots e_{n-2})$ must be of the form $a_{\geq 1}$. Otherwise, there exists some $e_i > e_{n-2}$ with $i < n-2$. Hence, the subsequence $e_i e_{n-2} e_n$ satisfies $e_i > e_{n-2} > e_n$ and thus satisfies (16.2.1). But this is not allowed. Now if $\tau(e_1 e_2 \cdots e_{n-2})$ is of the form $a_{\geq 2}$, then e' is of Type III and $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$; if $\tau(e_1 e_2 \cdots e_{n-2})$ is of the form a, then e' is of Type IV and as well $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.

Now (a), (b), (c) and (d) of the lemma are proved. Next, we show (b') and (d'). For (b'), we simply notice that

$$\begin{aligned}
&f_2(n, \Lambda) - f_2(n, \Lambda - 1)\\
&\quad = z f_1(n-1, \Lambda-1) + z f_2(n-1, \Lambda-1) + z f_3(n-1, \Lambda-1) + z f_4(n-1, \Lambda-1)\\
&\quad = z f_1(n, \Lambda - 1),
\end{aligned}$$

where we make use of (a) in the last equality. For (d'),

$$f_4(n, \Lambda) - f_4(n, \Lambda - 1)$$

$$
\begin{aligned}
&= zf_1(n-1,\Lambda-1) + f_2(n-1,\Lambda-1) + zf_3(n-1,\Lambda-1) + zf_4(n-1,\Lambda-1)\\
&= \Big(zf_1(n-1,\Lambda-1) + f_2(n-1,\Lambda-1)\Big) + z\Big(f_3(n-1,\Lambda-1) + f_4(n-1,\Lambda-1)\Big)\\
&= f_2(n-1,\Lambda) + zf_3(n,\Lambda-1),
\end{aligned}
$$

where we utilize (b') and (c) in the last equality. □

Proposition 16.3.2. *For $n \geq 1$,*

$$
\begin{cases}
f_1(n,\Lambda) = 0, & \text{if } \Lambda > n-2,\\
f_2(n,\Lambda) = 0, & \text{if } \Lambda > n-1,\\
f_3(n,\Lambda) = 0, & \text{if } \Lambda > n-3,\\
f_4(n,\Lambda) = 0, & \text{if } \Lambda > n-2.
\end{cases}
$$

Proof. The equalities for f_2 and f_4 come from Lemma 16.3.1(b) and (d). The equalities for f_1 and f_3 can be proved jointly by a simple induction on n. □

On the other hand, for $1 \leq i \leq 4$, let

$$\mathbf{I}_{n,i}(\geq,\neq,>) := \{e \in \mathbf{I}_n(\geq,\neq,>) : e \text{ is of Type } i\}$$

and

$$\mathbf{I}_{n,i}^{(\Lambda)}(\geq,\neq,>) := \{e \in \mathbf{I}_{n,i}(\geq,\neq,>) : \text{the largest entry of } e \text{ is } \Lambda\}.$$

We further write

$$h_i(n,\Lambda) := \sum_{e \in \mathbf{I}_{n,i}^{(\Lambda)}(\geq,\neq,>)} z^{\mathrm{asc}(e)}. \tag{16.3.5}$$

Notice that the initial values of the h_i's are

$$
\begin{aligned}
h_1(1,\Lambda) = h_3(1,\Lambda) = h_4(1,\Lambda) = 0 \quad \text{for all } \Lambda \geq 0,\\
h_2(1,\Lambda) = \begin{cases} 1 & \text{for } \Lambda = 0,\\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

and

$$h_3(2, \Lambda) = 0 \quad \text{for all } \Lambda \geq 0.$$

Lemma 16.3.3. *For $n \geq 2$, we have*

(a). *for $\Lambda \geq 0$,*

$$h_1(n, \Lambda) = h_1(n-1, \Lambda) + h_2(n-1, \Lambda) + h_3(n-1, \Lambda) + h_4(n-1, \Lambda);$$

(b). *for $\Lambda = 0$ and $\Lambda \geq n$,*

$$h_2(n, \Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-1$,

$$\begin{aligned} h_2(n, \Lambda) = \sum_{0 \leq \Lambda' < \Lambda} \Big(& zh_1(n-1, \Lambda') + zh_2(n-1, \Lambda') \\ & + zh_3(n-1, \Lambda') + zh_4(n-1, \Lambda') \Big); \end{aligned}$$

(c). *for $\Lambda \geq 0$,*

$$h_3(n, \Lambda) = h_3(n-1, \Lambda) + zh_4(n-1, \Lambda);$$

(d). *for $\Lambda = 0$ and $\Lambda \geq n-1$,*

$$h_4(n, \Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-2$,

$$\begin{aligned} h_4(n, \Lambda) = \sum_{0 \leq \Lambda' < \Lambda} \Big(& zh_1(n-1, \Lambda') + h_2(n-1, \Lambda') \\ & + h_3(n-1, \Lambda') + zh_4(n-1, \Lambda') \Big). \end{aligned}$$

Further,

(b'). *for $1 \leq \Lambda \leq n-1$,*

$$h_2(n, \Lambda) - h_2(n, \Lambda - 1) = zh_1(n, \Lambda - 1);$$

(d'). *for $1 \leq \Lambda \leq n-2$,*

$$h_4(n, \Lambda) - h_4(n, \Lambda - 1) = h_2(n-1, \Lambda) + h_3(n, \Lambda - 1).$$

Proof. In analogy to the proof of Lemma 16.3.1, we construct bijective maps between each type of sequences in $\mathbf{I}_n^{(\Lambda)}(\geq, \neq, >)$ and the desired subset of $\mathbf{I}_{n-1}(\geq, \neq, >)$ while still only one side of the maps will be explicitly stated. For (a) and (b), we use the same way as that for Lemma 16.3.1(a) and (b) to reduce $e \in \mathbf{I}_n^{(\Lambda)}(\geq, \neq, >)$ to $e' \in \mathbf{I}_{n-1}(\geq, \neq, >)$ and hence the details are omitted. Now let us treat the rest two cases. We as well write $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n^{(\Lambda)}(\geq, \neq, >)$.

Case 3. If e is of Type III, then $\tau(e)$ is of the form $\Lambda\, b_{\geq 1}\, \Lambda$ for some $b < \Lambda$. We distinguish it into two subcases. It should be pointed out in advance that the largest entry of the resulting sequence e' in both cases is still Λ.

- $\tau(e) = \Lambda\, b\, \Lambda$. Then we delete the last Λ to get e'. We see that $e' \in \mathbf{I}_{n-1}^{(\Lambda)}(\geq, \neq, >)$ is of Type IV. Also $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.

- τ is of the form $\Lambda\, b_{\geq 2}\, \Lambda$. Then we delete one of the b's to get some $e' \in \mathbf{I}_{n-1}^{(\Lambda)}(\geq, \neq, >)$. This time e' is of Type III and $\mathrm{asc}(e) = \mathrm{asc}(e')$.

Case 4. If e is of Type IV, then $\tau(e) = \Lambda\, b$ for some $b < \Lambda$. We as well delete Λ from e to get e'. Notice that we also have $e' \in \mathbf{I}_{n-1}(>, \neq, \geq)$ with largest entry smaller than Λ. We have three subcases as follows.

- $b = e_n = e_{n-2}$. Then e' is of Type I and in this case $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.

- $b = e_n > e_{n-2}$. Then $b \geq \max\{e_1, e_2, \ldots, e_{n-2}\}$. Otherwise, there exists some e_i in $e_1 e_2 \cdots e_{n-3}$ such that $e_i > b = e_n$. Now the subsequence $e_i e_{n-2} e_n$ satisfies $e_i > e_n > e_{n-2}$, which is not allowed. If $b > \max\{e_1, e_2, \ldots, e_{n-2}\}$, then e' is of Type II and $\mathrm{asc}(e) = \mathrm{asc}(e')$; if $b = \max\{e_1, e_2, \ldots, e_{n-2}\}$, then $\tau(e_1 e_2 \cdots e_{n-2})$ must be of the form $b\, c_{\geq 1}$ where $c = e_{n-2} < b$ and hence e' is of Type III and $\mathrm{asc}(e) = \mathrm{asc}(e')$.

- $b = e_n < e_{n-2}$. Then $\tau(e_1 e_2 \cdots e_{n-2})$ must be of the form a. Otherwise, there exists some $e_i \geq e_{n-2}$ with $i < n-2$. Hence, the subsequence $e_i e_{n-2} e_n$ satisfies $e_i \geq e_{n-2} > e_n$ and thus satisfies (16.2.2). But this is not allowed. Thus, e' is of Type IV and $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.

The proofs of (b') and (d') are also similar to those for Lemma 16.3.1. $\square$

Finally, we define, for $1 \leq i \leq 4$,

$$g_i(n, \Lambda) := \sum_{e \in \mathbf{I}_{n,i}^{(\Lambda)}(\geq, \neq, >)} z^{n-1-\mathrm{asc}(e)}. \tag{16.3.6}$$

Then in view of (16.3.5),

$$g_i(n,\Lambda) = z^{n-1}\Big[h_i(n,\Lambda)\Big]_{z\mapsto z^{-1}}$$

and conversely,

$$h_i(n,\Lambda) = z^{n-1}\Big[g_i(n,\Lambda)\Big]_{z\mapsto z^{-1}}.$$

Thus, the initial values of the g_i's are

$$g_1(1,\Lambda) = g_3(1,\Lambda) = g_4(1,\Lambda) = 0 \quad \text{for all } \Lambda \geq 0,$$
$$g_2(1,\Lambda) = \begin{cases} 1 & \text{for } \Lambda = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_3(2,\Lambda) = 0 \quad \text{for all } \Lambda \geq 0.$$

Also, the recurrences for the g_i's can be translated with no difficulty from those for the h_i's.

Lemma 16.3.4. *For $n \geq 2$, we have*

(a). *for $\Lambda \geq 0$,*

$$g_1(n,\Lambda) = zg_1(n-1,\Lambda) + zg_2(n-1,\Lambda) + zg_3(n-1,\Lambda) + zg_4(n-1,\Lambda);$$

(b). *for $\Lambda = 0$ and $\Lambda \geq n$,*

$$g_2(n,\Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-1$,

$$g_2(n,\Lambda) = \sum_{0\leq \Lambda' < \Lambda} \Big(g_1(n-1,\Lambda') + g_2(n-1,\Lambda') + g_3(n-1,\Lambda') + g_4(n-1,\Lambda')\Big);$$

(c). *for $\Lambda \geq 0$,*

$$g_3(n,\Lambda) = zg_3(n-1,\Lambda) + g_4(n-1,\Lambda);$$

(d). *for* $\Lambda = 0$ *and* $\Lambda \geq n-1$,

$$g_4(n, \Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-2$,

$$g_4(n, \Lambda) = \sum_{0 \leq \Lambda' < \Lambda} \Big(g_1(n-1, \Lambda') + z g_2(n-1, \Lambda') \\ + z g_3(n-1, \Lambda') + g_4(n-1, \Lambda') \Big).$$

Further,

(b'). *for* $1 \leq \Lambda \leq n-1$,

$$g_2(n, \Lambda) - g_2(n, \Lambda - 1) = z^{-1} g_1(n, \Lambda - 1);$$

(d'). *for* $1 \leq \Lambda \leq n-2$,

$$g_4(n, \Lambda) - g_4(n, \Lambda - 1) = z g_2(n-1, \Lambda) + g_3(n, \Lambda - 1).$$

Similar to Proposition 16.3.2, we have the following equalities.

Proposition 16.3.5. *For* $n \geq 1$,

$$\begin{cases} g_1(n, \Lambda) = 0, & \textit{if } \Lambda > n-2, \\ g_2(n, \Lambda) = 0, & \textit{if } \Lambda > n-1, \\ g_3(n, \Lambda) = 0, & \textit{if } \Lambda > n-3, \\ g_4(n, \Lambda) = 0, & \textit{if } \Lambda > n-2. \end{cases}$$

16.3.2 Generating Functions

Let

$$\mathcal{F}_1(t) = \mathcal{F}_1(t; q) := \sum_{n \geq 2} \sum_{\Lambda = 0}^{n-2} f_1(n, \Lambda) t^{n-2-\Lambda} q^n,$$
$$\mathcal{F}_2(t) = \mathcal{F}_2(t; q) := \sum_{n \geq 2} \sum_{\Lambda = 0}^{n-1} f_2(n, \Lambda) t^{n-1-\Lambda} q^n,$$
$$\mathcal{F}_3(t) = \mathcal{F}_3(t; q) := \sum_{n \geq 3} \sum_{\Lambda = 0}^{n-3} f_3(n, \Lambda) t^{n-3-\Lambda} q^n,$$

$$\mathcal{F}_4(t) = \mathcal{F}_4(t;q) := \sum_{n\geq 2}\sum_{\Lambda=0}^{n-2} f_4(n,\Lambda)t^{n-2-\Lambda}q^n.$$

It is easy to translate the recurrences of the f_i's in Lemma 16.3.1 to functional equations of $\mathcal{F}_1(t)$, $\mathcal{F}_2(t)$, $\mathcal{F}_3(t)$ and $\mathcal{F}_4(t)$.

Lemma 16.3.6. *We have*

$$\begin{cases} \mathcal{F}_1(t) - q^2 = tq\mathcal{F}_1(t) + q\mathcal{F}_2(t) + t^2q\mathcal{F}_3(t) + tq\mathcal{F}_4(t), \\ t\mathcal{F}_2(t) - \Big(\mathcal{F}_2(t) - \mathcal{F}_2(0)\Big) = zt\mathcal{F}_1(t), \\ \mathcal{F}_3(t) = tq\mathcal{F}_3(t) + q\mathcal{F}_4(t), \\ t\mathcal{F}_4(t) - \Big(\mathcal{F}_4(t) - \mathcal{F}_4(0)\Big) = tq\mathcal{F}_2(t) + zt\mathcal{F}_3(t). \end{cases} \tag{16.3.7}$$

Proof. We show the first and second equations as instances. The proof of the third one is similar to the first and the proof of the fourth one resembles the second.

First, by Lemma 16.3.1(a) and Proposition 16.3.2, we have

$$\begin{aligned} \mathcal{F}_1(t) &= \sum_{n\geq 2}\sum_{\Lambda=0}^{n-2} f_1(n,\Lambda)t^{n-2-\Lambda}q^n \\ &= \sum_{n\geq 2}\sum_{\Lambda=0}^{n-2}\Big(f_1(n-1,\Lambda) + f_2(n-1,\Lambda) + f_3(n-1,\Lambda) + f_4(n-1,\Lambda)\Big)t^{n-2-\Lambda}q^n \\ &= \sum_{n\geq 1}\sum_{\Lambda=0}^{n-1}\Big(f_1(n,\Lambda) + f_2(n,\Lambda) + f_3(n,\Lambda) + f_4(n,\Lambda)\Big)t^{n-1-\Lambda}q^{n+1} \\ &= tq\sum_{n\geq 1}\sum_{\Lambda=0}^{n-2} f_1(n,\Lambda)t^{n-2-\Lambda}q^n + q\sum_{n\geq 1}\sum_{\Lambda=0}^{n-1} f_2(n,\Lambda)t^{n-1-\Lambda}q^n \\ &\quad + t^2q\sum_{n\geq 1}\sum_{\Lambda=0}^{n-3} f_3(n,\Lambda)t^{n-3-\Lambda}q^n + tq\sum_{n\geq 1}\sum_{\Lambda=0}^{n-2} f_4(n,\Lambda)t^{n-2-\Lambda}q^n. \end{aligned}$$

The first equation follows by recalling the initial values (16.3.2), (16.3.3) and (16.3.4).

For the second equation, we apply Lemma 16.3.1(b') and Proposition 16.3.2. Then

$$\begin{aligned} \mathcal{F}_2(t) &= \sum_{n\geq 2}\sum_{\Lambda=0}^{n-1} f_2(n,\Lambda)t^{n-1-\Lambda}q^n \\ &= \sum_{n\geq 2}\sum_{\Lambda=1}^{n-1} f_2(n,\Lambda)t^{n-1-\Lambda}q^n \qquad \text{(since } f_2(n,0) = 0 \text{ for } n\geq 2 \text{ by Lemma 16.3.1(b))} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 2}\sum_{\Lambda=1}^{n-1}\Big(zf_1(n,\Lambda-1)+f_2(n,\Lambda-1)\Big)t^{n-1-\Lambda}q^n \\
&= \sum_{n\geq 2}\sum_{\Lambda=0}^{n-2}\Big(zf_1(n,\Lambda)+f_2(n,\Lambda)\Big)t^{n-2-\Lambda}q^n \\
&= z\sum_{n\geq 2}\sum_{\Lambda=0}^{n-2} f_1(n,\Lambda)t^{n-2-\Lambda}q^n + t^{-1}\sum_{n\geq 2}\sum_{\Lambda=0}^{n-2} f_2(n,\Lambda)t^{n-1-\Lambda}q^n \\
&= z\mathcal{F}_1(t) + t^{-1}\Big(\mathcal{F}_2(t)-\mathcal{F}_2(0)\Big),
\end{aligned}
$$

which is essentially the second equation. □

We treat $\mathcal{F}_1(t)$, $\mathcal{F}_2(t)$, $\mathcal{F}_3(t)$ and $\mathcal{F}_4(t)$ as unknowns and solve the above system so that they are expressed in terms of $\mathcal{F}_2(0)$, $\mathcal{F}_4(0)$, z, q and t. In particular, we have the following expression for $\mathcal{F}_4(t)$.

Lemma 16.3.7. *We have*

$$
K_f(t)\mathcal{F}_4(t) = (1-qt)P_f(t), \tag{16.3.8}
$$

where

$$
\begin{aligned}
P_f(t) = \mathcal{F}_4(0) &- \Big(q\mathcal{F}_2(0)+\mathcal{F}_4(0)+q\mathcal{F}_4(0)-zq\mathcal{F}_4(0)\Big)t \\
&+ \Big(zq^3+q^2\mathcal{F}_2(0)+q\mathcal{F}_4(0)\Big)t^2
\end{aligned} \tag{16.3.9}
$$

and

$$
\begin{aligned}
K_f(t) = 1 &- (2+2q-2zq)t + (1+4q-2zq+q^2-2zq^2+z^2q^2)t^2 \\
&- (2q+2q^2-zq^2)t^3 + q^2t^4.
\end{aligned} \tag{16.3.10}
$$

Analogously, we define

$$
\begin{aligned}
\mathcal{G}_1(t) = \mathcal{G}_1(t;q) &:= \sum_{n\geq 2}\sum_{\Lambda=0}^{n-2} g_1(n,\Lambda)t^{n-2-\Lambda}q^n, \\
\mathcal{G}_2(t) = \mathcal{G}_2(t;q) &:= z\sum_{n\geq 2}\sum_{\Lambda=0}^{n-1} g_2(n,\Lambda)t^{n-1-\Lambda}q^n, \\
\mathcal{G}_3(t) = \mathcal{G}_3(t;q) &:= \sum_{n\geq 3}\sum_{\Lambda=0}^{n-3} g_3(n,\Lambda)t^{n-3-\Lambda}q^n,
\end{aligned}
$$

$$\mathcal{G}_4(t) = \mathcal{G}_4(t;q) := \sum_{n\geq 2}\sum_{\Lambda=0}^{n-2} g_4(n,\Lambda)t^{n-2-\Lambda}q^n.$$

By the recurrences of the g_i's in Lemma 16.3.4, the following system holds true.

Lemma 16.3.8. *We have*

$$\begin{cases} \mathcal{G}_1(t) - zq^2 = ztq\mathcal{G}_1(t) + q\mathcal{G}_2(t) + zt^2q\mathcal{G}_3(t) + ztq\mathcal{G}_4(t), \\ t\mathcal{G}_2(t) - \Big(\mathcal{G}_2(t) - \mathcal{G}_2(0)\Big) = t\mathcal{G}_1(t), \\ \mathcal{G}_3(t) = ztq\mathcal{G}_3(t) + q\mathcal{G}_4(t), \\ t\mathcal{G}_4(t) - \Big(\mathcal{G}_4(t) - \mathcal{G}_4(0)\Big) = tq\mathcal{G}_2(t) + t\mathcal{G}_3(t). \end{cases} \tag{16.3.11}$$

We may also solve the above system for $\mathcal{G}_1(t)$, $\mathcal{G}_2(t)$, $\mathcal{G}_3(t)$ and $\mathcal{G}_4(t)$. In particular, we have the following expression for $\mathcal{G}_4(t)$.

Lemma 16.3.9. *We have*

$$K_g(t)\mathcal{G}_4(t) = (1-zqt)P_g(t), \tag{16.3.12}$$

where

$$\begin{aligned} P_g(t) = \mathcal{G}_4(0) &- \Big(q\mathcal{G}_2(0) + \mathcal{G}_4(0) - q\mathcal{G}_4(0) + zq\mathcal{G}_4(0)\Big)t \\ &+ \Big(zq^3 + zq^2\mathcal{G}_2(0) + zq\mathcal{G}_4(0)\Big)t^2 \end{aligned} \tag{16.3.13}$$

and

$$\begin{aligned} K_g(t) = 1 - (2-2q+2zq)t &+ (1-2q+4zq+q^2-2zq^2+z^2q^2)t^2 \\ &- (2zq - zq^2 + 2z^2q^2)t^3 + (z^2q^2 - zq^3 + z^2q^3)t^4. \end{aligned} \tag{16.3.14}$$

Remark 16.3.1. We could, of course, derive kernel equations for $\mathcal{F}_2(t)$ and $\mathcal{G}_2(t)$ instead of $\mathcal{F}_4(t)$ and $\mathcal{G}_4(t)$. But such changes will not make any essential difference after the application of the kernel method; we are still led to Theorem 16.4.1.

16.4 Proof of Theorem 16.1.1

The objective of this section is to apply the kernel method to establish the following surprising relations, one of which will lead to a proof of Theorem 16.1.1.

Theorem 16.4.1. *We have*

$$\begin{cases} f_1(n,n-2) = g_1(n,n-2) & \text{for } n \geq 3, \\ f_2(n,n-1) = zg_2(n,n-1) & \text{for } n \geq 2, \\ f_3(n,n-3) = g_3(n,n-3) & \text{for } n \geq 3, \\ f_4(n,n-2) = g_4(n,n-2) & \text{for } n \geq 2. \end{cases}$$

16.4.1 Roots of the Kernel Polynomials

Before applying the kernel method to $\mathcal{F}_4(t)$ and $\mathcal{G}_4(t)$, let us first investigate properties of the roots of the two kernel polynomials $K_f(t)$ and $K_g(t)$.

Lemma 16.4.2. *Let r_1, r_2, r_3 and r_4 be the four roots of the quartic polynomial $K_f(t)$. Then the quartic polynomial $K_g(t)$ has roots s_1, s_2, s_3 and s_4 such that for $1 \leq i \leq 4$,*

$$s_i = \frac{r_i}{1-(1-z)qr_i}. \tag{16.4.1}$$

Proof. We have

$$K_f(t) = q^2(t-r_1)(t-r_2)(t-r_3)(t-r_4).$$

Since $K_f(t)$ has constant term 1, we know that the quartic polynomial $K_f^\star(t) := t^4 K_f(t^{-1})$ is monic. Further,

$$K_f^\star(t) = (t-r_1^{-1})(t-r_2^{-1})(t-r_3^{-1})(t-r_4^{-1}).$$

Similarly, if $K_g^\star(t) := t^4 K_g(t^{-1})$, then

$$K_g^\star(t) = (t-s_1^{-1})(t-s_2^{-1})(t-s_3^{-1})(t-s_4^{-1}).$$

Therefore, to obtain the desired relations, it suffices to show

$$K_g^\star(t) = K_f^\star\Big(t+(1-z)q\Big),$$

which is easy to verify. □

For the sake of simplicity when utilizing the general formula for roots of quartic equations, we assume that $0 < q < 1$ and $z > 0$. It can be computed that as $q \to 0^+$,

$K_f(t)$ has four roots

$$
\begin{aligned}
r_1 &= 1 + \left(z + \sqrt{z}\right)q + O_z(q^2),\\
r_2 &= 1 + \left(z - \sqrt{z}\right)q + O_z(q^2),\\
r_3 &= q^{-1} + \sqrt{z}q^{-1/2} - \frac{z}{2} + O_z(q^{1/2}),\\
r_4 &= q^{-1} - \sqrt{z}q^{-1/2} - \frac{z}{2} + O_z(q^{1/2}).
\end{aligned}
$$

Let s_i be as in Lemma 16.4.2 so that they are roots of $K_g(t)$. Then

$$
\begin{aligned}
s_1 &= 1 + \left(1 + \sqrt{z}\right)q + O_z(q^2),\\
s_2 &= 1 + \left(1 - \sqrt{z}\right)q + O_z(q^2),\\
s_3 &= \frac{1}{z}q^{-1} + \frac{1}{z^{3/2}}q^{-1/2} + \frac{2-3z}{2z^2} + O_z(q^{1/2}),\\
s_4 &= \frac{1}{z}q^{-1} - \frac{1}{z^{3/2}}q^{-1/2} + \frac{2-3z}{2z^2} + O_z(q^{1/2}).
\end{aligned}
$$

16.4.2 Applying the Kernel Method

To apply the kernel method, we need to choose roots of $K_f(t)$ and $K_g(t)$ that can be expanded as a formal power series in q. So only r_1, r_2 and s_1, s_2 are admissible, respectively. Recall (16.3.8):

$$K_f(t)\mathcal{F}_4(t) = (1 - qt)P_f(t).$$

We substitute the roots $t = r_1$ and r_2 into the above and arrive at $P_f(t) = 0$. Then recalling (16.3.9) yields the system

$$
\begin{cases}
0 = \mathcal{F}_4(0) - \left(q\mathcal{F}_2(0) + \mathcal{F}_4(0) + q\mathcal{F}_4(0) - zq\mathcal{F}_4(0)\right)r_1\\
\qquad + \left(zq^3 + q^2\mathcal{F}_2(0) + q\mathcal{F}_4(0)\right)r_1^2,\\
0 = \mathcal{F}_4(0) - \left(q\mathcal{F}_2(0) + \mathcal{F}_4(0) + q\mathcal{F}_4(0) - zq\mathcal{F}_4(0)\right)r_2\\
\qquad + \left(zq^3 + q^2\mathcal{F}_2(0) + q\mathcal{F}_4(0)\right)r_2^2.
\end{cases}
$$

Solving the above system for $\mathcal{F}_2(0)$ and $\mathcal{F}_4(0)$ gives

$$\begin{cases} \mathcal{F}_2(0) = \dfrac{zq^2\Big(r_1 + r_2 - (1 + (1-z)q)r_1 r_2\Big)}{1 - q(r_1 + r_2) + (1-z)q^2 r_1 r_2}, \\ \mathcal{F}_4(0) = \dfrac{zq^3 r_1 r_2}{1 - q(r_1 + r_2) + (1-z)q^2 r_1 r_2}. \end{cases}$$

Likewise, we substitute the roots $t = s_1$ and s_2 into $P_g(t) = 0$ and use (16.3.13) to obtain a similar system, which leads to the solution

$$\begin{cases} \mathcal{G}_2(0) = \dfrac{zq^2\Big(s_1 + s_2 - (1 - (1-z)q)s_1 s_2\Big)}{1 - zq(s_1 + s_2) - z(1-z)q^2 s_1 s_2}, \\ \mathcal{G}_4(0) = \dfrac{zq^3 s_1 s_2}{1 - zq(s_1 + s_2) - z(1-z)q^2 s_1 s_2}. \end{cases}$$

Finally, making use of the relations

$$\begin{cases} s_1 = \dfrac{r_1}{1 - (1-z)qr_1}, \\ s_2 = \dfrac{r_2}{1 - (1-z)qr_2}, \end{cases}$$

we find that

$$\begin{cases} \mathcal{F}_2(0) = \mathcal{G}_2(0), \\ \mathcal{F}_4(0) = \mathcal{G}_4(0). \end{cases}$$

Further, by (16.3.7) and (16.3.11), we have

$$\begin{cases} \mathcal{F}_1(0) = q\mathcal{F}_2(0) + q^2, \\ \mathcal{G}_1(0) = q\mathcal{G}_2(0) + zq^2, \end{cases}$$

and

$$\begin{cases} \mathcal{F}_3(0) = q\mathcal{F}_4(0), \\ \mathcal{G}_3(0) = q\mathcal{G}_4(0). \end{cases}$$

Theorem 16.4.1 therefore follows.

16.4.3 Proof of Theorem 16.1.1

Recalling the definition of the f_i's, we find that, for $n \geq 1$,

$$\begin{aligned}\sum_{e \in \mathbf{I}_n(>,\neq,\geq)} z^{\operatorname{asc}(e)} &= \sum_{\Lambda=0}^{n-1} \Big(f_1(n,\Lambda) + f_2(n,\Lambda) + f_3(n,\Lambda) + f_4(n,\Lambda) \Big) \\ &= z^{-1} f_2(n+1,n),\end{aligned}$$

where we make use of Lemma 16.3.1(b). Similarly, we have, for $n \geq 1$,

$$\begin{aligned}\sum_{e \in \mathbf{I}_n(\geq,\neq,>)} z^{n-1-\operatorname{asc}(e)} &= \sum_{\Lambda=0}^{n-1} \Big(g_1(n,\Lambda) + g_2(n,\Lambda) + g_3(n,\Lambda) + g_4(n,\Lambda) \Big) \\ &= g_2(n+1,n),\end{aligned}$$

where Lemma 16.3.4(b) is utilized. By the second relation in Theorem 16.4.1, we find that for $n \geq 1$,

$$z^{-1} f_2(n+1,n) = g_2(n+1,n),$$

and therefore complete the proof of (16.1.2).

16.5 References

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